Heyting-Brouwer Rough Set Logic

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Abstract. A rough set logic based on Heyting-Brouwer algebras *HBRSL* is proposed as a basis for reasoning about rough information. It is an extension of Düntsch's logic with intuitionistic implication, and is seen as a variant of Heyting-Brouwer logic. A Kripke semantics and natural deduction for the logic are presented and the completeness theorem is proved.

Keywords: rough set logic, regular double Stone algebra, Heyting-Brouwer logic, Kripke semantics, natural deduction.

1 Introduction

In 1982, Pawlak proposed a *rough set* to represent coarse (rough) information; see Pawlak [5, 6]. The formalization of rough information has been the subject of investigation in rough set theory which is closely related to other areas. In formal logic, it is very important to develop a logic for rough sets.

Initial work in this direction has been done in Orlowska [3, 4]. The most significant is probably due to Düntsch [2] who proposed a propositional logic for rough sets with an algebraic semantics based on *regular double Stone algebras*.

It is a famous fact that the collection of all subsets of a set constitutes a Boolean algebra and that its logic is exactly the classical propositional logic. J. Pomykala

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and J.A. Pomykala [7] showed that the collection of rough sets of an approximation space forms a *regular double Stone algebra*. Based on their results, Düntsch succeed in developing a logic for rough sets.

There are, however, two problems with Düntsch's logic. The first problem is that he did not give a Kripke-type relational semantics. This means that we cannot intuitively understand his logic. The second problem is the lack of proof theory. Indeed in his presentation the Hilbert system is implicit, but it is not adequate for practical inferences.

The purpose of this paper is to develop another rough set logic which extends Düntsch's logic with intuitionistic implication. Our approach starts with *Heyting-Brouwer logic*, also known as bi-intuitionistic logic, which was proposed by Rauszer [8], and the idea leads interesting proof-theoretic and semantical characterization of the new rough set logic.

The structure of this paper is as follows. In section 2, we briefly review rough sets. In section 3, we present an exposition of Düntsch's logic for rough sets. In section 4, we introduce a new rough set logic called *Heyting-Brouwer rough set logic* with Kripke semantics and natural deduction. We also prove the completeness theorem based on a canonical model. The final section makes some conclusions with the discussion on future work.

2 Rough Set

The concept of *rough set* was proposed by Pawlak [5]; also see Pawlak [6]. A rough set can be seen as an approximation of a set denoted by a pair of sets, called the lower and upper approximation of the set to deal with reasoning from imprecise data.

We here sketch the background of rough sets. Let U be a non-empty finite set, called the *universe* of objects in question. Then, any subset $X \subseteq U$ is called a *concept* in U and any family of concepts in U is called *knowledge* about U. If R be the equivalence relation on U, then U/R denotes the family of all equivalence classes of R (or *classification* about U), called *categories* or *concepts* of R. We write $[x]_R$ for a category in R containing an element $x \in U$. If $P \subseteq R$ and $P \neq \emptyset$, then $\cap P$ is also an equivalence relation called *indiscernibility relation* on P, designated as $IND(P)$.

An *approximation space* is a pair $\langle U, R \rangle$. Then, for each subset $X \subseteq U$ and equivalence relation R, we associate two subsets, i.e.,

$$\underline{R}X = \{x \in U : [x]_R \subseteq X\}, \quad \overline{R}X = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

Here, $\underline{R}X$ is called the *lower approximation* of X, and $\overline{R}X$ is called the *upper approximation* of X, respectively. A *rough set* is designated as the pair $\langle \underline{R}X, \overline{R}X \rangle$. Intuitively, $\underline{R}X$ is the set of all elements of U which can be certainly classified as elements of X in the knowledge R, and $\overline{R}X$ is the set of elements which can be possibly classified as elements of X in the knowledge R. Then, we can define three types of sets, i.e.,

$POS_R(X) = \underline{R}X$ (R-positive region of X),
$NEG_R(X) = U - \overline{R}X$ (R-negative region of X),
$BN_R(X) = \overline{R}X - \underline{R}X$ (R-boundary region of X).

These sets enable us to classify our knowledge. Pawlak [6] contains comprehensive account of rough sets.

3 Rough Set Logic

Düntsch [2] developed a propositional logic for rough sets inspired by the topological construction of rough sets using Boolean algebras. His work is based on the fact that the collection of all subsets of a set forms a Boolean algebra under the set-theoretic operation, and that the collection of rough sets of an approximation space is a regular double Stone algebra. Thus, we can assume that regular double Stone algebras give a semantics for a logic for rough sets.

Here, we need to survey Düntsch's work. To understand his logic, we need some concepts. A *double Stone algebra DSA* is denoted by $\langle L, +, \cdot, *.+, 0, 1 \rangle$ with the type $\langle 2, 2, 1, 1, 0, 0 \rangle$ satisfying the following conditions:

(1) $\langle L, +, \cdot, 0, 1 \rangle$ is a bounded distributed lattice.
(2) x^* is the pseudocomplement of x, i.e.,

$$y \leq x^* \Leftrightarrow y \cdot x = 0.$$

(3) x^+ is the dual pseudocomplement of x, i.e.,

$$y \geq x^+ \Leftrightarrow y + x = 1.$$

(4) $x^* + x^{**} = 1, x^+ \cdot x^{++} = 0$

DSA is called *regular* if it satisfies the additional condition: $x \cdot x^+ \leq x + x^*$. Let B be a Boolean algebra, F be a filter on B, and

$$\langle B, F \rangle = \{ \langle a, b \rangle \mid a, b \in B, a \leq b, a + (-b) \in F \}$$

We define the following operations on $\langle B, F \rangle$ as follows:

$$\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle,$$
$$\langle a, b \rangle \cdot \langle c, d \rangle = \langle a \cdot c, b \cdot d \rangle,$$
$$\langle a, b \rangle^* = \langle -b, -b \rangle,$$
$$\langle a, b \rangle^+ = \langle -a, -a \rangle.$$

If $\langle U, R \rangle$ is an approximation space, the classes o R can be viewed as a complete subalgebra of the Boolean algebra $B(U)$. Conversely, any atomic complete subalgebra B of $B(U)$ yields an equivalence relation R on U by the relation:

$xRy \Leftrightarrow x$ and y are contained in the same atom of B,

and this correspondence is bijective. If $\{a\} \in B$, then for every $X \subseteq U$ we have:

If $a \in \underline{R}X$, then $a \in X$, and the rough sets of the corresponding approximation space are the elements of the regular double Stone algebra $\langle B, F \rangle$, where F is the filter of B which is generated by the union of the singleton elements of B.

Based on the construction of regular double Stone algebras, Düntsch proposed a propositional rough set logic *RSL*. The language $\mathscr{L}$ of *RSL* has two binary connectives $\wedge$ (conjunction), $\vee$ (disjunction), two unary connectives $^*, ^+$ for two types of negation, and the logical constant $\top$ for truth.

Let P be a non-empty set of propositional variables. Then, the set **Fml** of formulas with the logical operators constitutes an absolutely free algebra with a type $\langle 2, 2, 1, 1, 0 \rangle$.

Let W be a set and $B(W)$ be a Boolean algebra based on W. Then, a *model M* of L is seen as a pair (W, v), where $v : P \to B(W) \times B(W)$ is the *valuation function* for all $p \in P$ satisfying:

$$\text{if } v(p) = \langle A, B \rangle, \text{ then } A \subseteq B.$$

Here, $v(p) = \langle A, B \rangle$ states that p holds at all states of A and does not hold at any state outside B.

Düntsch relates the valuation to Lukasiewicz's three-valued logic by the following construction. For each $p \in P$, let $v_p : W \to \mathbf{3} = \{0, \frac{1}{2}, 1\}$. $v : P \to B(W) \times B(W)$ is defined as follows:

$$v(p) = \langle \{w \in W : v_p(w) = 1\}, \{w \in W : v_p(w) \neq 0\} \rangle.$$

In addition, Düntsch connected the valuation and rough sets as follows:

$$v_p(w) = 1 \text{ if } w \in A,$$
$$v_p(w) = \tfrac{1}{2} \text{ if } w \in B \setminus A,$$
$$v_p(w) = 0 \text{ otherwise.}$$

Given a model $\mathtt{M} = (W, v)$, the *meaning function* $\mathtt{mng} : \mathbf{Fml} \to \mathtt{B}(W) \times \mathtt{B}(W)$ is defined to give a valuation of arbitrary formulas in the following way:

$$\mathtt{mng}(\top) = \langle W, W \rangle,$$
$$\mathtt{mng}(p) = v(p) \text{ for } p \in P.$$

If $\mathtt{mng}(\phi) = \langle A, B \rangle$ and $\mathtt{mng}(\psi) = \langle C, D \rangle$, then

$$\mathtt{mng}(\phi \wedge \psi) = \langle A \cap C, B \cap D \rangle,$$
$$\mathtt{mng}(\phi \vee \psi) = \langle A \cup C, B \cup D \rangle,$$
$$\mathtt{mng}(\phi^*) = \langle -B, -B \rangle,$$
$$\mathtt{mng}(\phi^+) = \langle -A, -A \rangle.$$

Here, $-A$ denotes the complement of A in $B(W)$. We can understand that the meaning function assigns the meaning to formulas.

A formula A *holds* in a model $M = \langle W, v \rangle$, written $M \models A$, if $\mathtt{mng}(A) = \langle W, W \rangle$. A set Γ of sentences *entails* a formula A, written $\Gamma \vdash A$, if every model of Γ is a model of A. Düntsch proved that *RSL* is sound and complete with respect to the above semantics, where he seemed to assume a Hilbert system as a proof theory.

4 Heyting-Brouwer Rough Set Logic

As noted in section 1, Düntsch did not provide a Kripke semantics for his logic. In addition, his Hilbert system seems to be abstractly presented. To overcome these difficulties, we introduce a new logic, i.e., Heyting-Brouwer rough logic denoted *HBRSL* whose language is the one of *RSL* with intuitionistic implication $\rightarrow$. The addition of $\rightarrow$ is essential in that we can construct a rough set logic as a variant of Heyting-Brouwer logic.

Heyting-Brouwer logic is the system founded on Heyting algebras and Brouwerian algebras. *Heyting algebra* $\langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a lattice with the bottom 0, the top 1, and the binary operation called *implication* $\rightarrow$ satisfying $a \wedge b \leq c \Leftrightarrow a \leq b \rightarrow c$. $\neg a = a \rightarrow 0$ is called the *pseudocomplement* of a. Heyting algebra is an algebraic model for intuitionistic logic in which Heyting (intuitionistic) negation is denoted by $\neg$.

The dual of Heyting algebra is called *Brouwerian algebra* $\langle L, \vee, \wedge, -\!\!<, 0, 1\rangle$ is a lattice with 0 and 1, and the binary operation called *dual implication* $-\!\!<$ satisfying $x -\!\!< y \leq z \ \Leftrightarrow \ x \leq y \vee z$. $-a = 1 -\!\!< a$ is called the *dual pseudocomplement* of a. Brouwerian algebra is an algebraic model for dual intuitionistic logic in which Brouwerian (dual intuitionistic) negation is denoted by $-$.

Heyting-Brouwer logic *HBL* is an extension of positive intuitionistic logic with implication and dual implication (and intuitionistic negation and dual intuitionistic negation, if needed). Rauszer [7] extensively studied proof and model theory for Heyting-Brouwer logic.

We are now ready to turn to an exposition of a new rough set logic called *Heyting-Brouwer rough set logic* denoted by *HBRSL*. The language of *HBRSL* is that of *RSL* with intuitionistic implication $\rightarrow$, truth $\top$ and falsity $\bot$. We write atomic formula by $p, q, r, \ldots$ and arbitrary formula by $A, B, C, \ldots$, respectively.

Note that $*$ and $+$ denote intuitionistic-like negation and dual intuitionistic-like negation, respectively. We here say intuitionistic-like and dual intuitionistic-like negation, because they do not correspond to intuitionistic and dual intuitionistic negation in the sense of Rauszer. The addition of $\rightarrow$ is essential in that it enables us to work out an elegant theoretical foundation for *HBRSL*. Of course, one could also add $-\!\!<$ to *HBRSL*, but its addition may not be important for practical purposes.

We start with a *Kripke semantics* for *HBRSL*, which is a modification of that for *HBL* in Rauszer [7]. A *Kripke model* for *HBRSL* is a tuple $\mathscr{M} = \langle W, R, V\rangle$. Here, W is a non-empty set of *worlds*. R is a binary relation on W, which is reflexive and transitive, and directed, i.e., $\exists v \forall w (wRv)$, dual directed, i.e., $\exists v \forall w (vRw)$, and bridged, i.e., $\forall w \forall v (wRv \ \Rightarrow w = v$ or $\forall u (wRu \ \Rightarrow w = u))$ for $w, v, u \in W$. V is a valuation function from $W \times At$ to $\{0, 1\}$, where At is a set of atomic formulas, satisfying that $V(w, \top) = 1$ and $V(w, \bot) = 0$ for any $w \in W$.

Then, we define the truth relation $\models$ such that $V(w, p) = 1 \ \Leftrightarrow \ w \models p$ and $V(w, p) = 0 \ \Leftrightarrow \ w \not\models p$. Here, $w \models p$ reads "p is true at w" and $w \not\models p$ reads "p is not true at w", respectively. The truth relation $\models$ is then defined for any formula A, B as follows.

$$\begin{array}{ll} w \models A \wedge B & \Leftrightarrow w \models A \text{ and } w \models B \\ w \models A \vee B & \Leftrightarrow w \models A \text{ or } w \models B \\ w \models A \to B & \Leftrightarrow \forall v(wRv \text{ and } v \models A \;\Rightarrow\; v \models B) \\ w \models *A & \Leftrightarrow \forall v(wRv \;\Rightarrow\; v \not\models A) \\ w \models +A & \Leftrightarrow \exists v(vRw \text{ and } v \not\models A) \end{array}$$

Although *HBRSL* has no dual implication $-\!<$, it can be added to *HBRSL* and interpreted as follows:

$$w \models A -\!< B \Leftrightarrow \exists v(vRw \text{ and } v \models A \text{ and } v \not\models B)$$

In the Kripke model, both persistency (P) and dual-persistency (DP) with respect to $\models$ and $\not\models$ hold:

(P) $\forall w \forall v(w \models p \text{ and } wRv \;\Rightarrow v \models p)$
(DP) $\forall w \forall v(w \not\models p \text{ and } vRw \;\Rightarrow v \not\models p)$

for any atomic p. We write $w \models \Gamma$ to mean that for all formulas in Γ are true at w. We say that a formula A is *valid*, written $\models A$, if it is true for all worlds for all models.

Lemma 1. *For any formula A, both* (P) *and* (DP) *hold:*

$\forall w \forall v(w \models A \text{ and } wRv \;\Rightarrow v \models A)$,
$\forall w \forall v(w \not\models A \text{ and } vRw \;\Rightarrow v \not\models A)$.

Proof. By induction A.

Next, we describe a proof theory of *HBRSL* denoted *NHBRSL* using *natural deduction* in a sequential form. *NHBRSL* is formalized by *axiom* and *rule*. Let Γ, Δ be sets of formulas, A, B, C, D be formulas. An expression of the form $\Gamma \vdash A$ is called a *sequent*. If $\Gamma = \{A_1, ..., A_n\}$, then $\Gamma \vdash B$ iff $(A_1 \wedge ... \wedge A_n) \to B$. When Γ is empty, $\Gamma \vdash A$ is written as $\vdash A$. Then, a rule is of the form:

$$\frac{\Gamma_1 \vdash A_1 .... \Gamma_i \vdash A_i}{\Delta \vdash B}$$

which says that if $\Gamma_1 \vdash A_1, ... \Gamma_i \vdash A_n$ (premises) holds then $\Delta \vdash B$ (consequent) holds. An axiom can be regarded as the rule without premises.

There are two types of rules, i.e., *introduction rule* and *elimination rule*. An introduction rule introduces a logical symbol in the consequent, and an elimination rule eliminates a logical symbol in the consequent. We denote, for example, the introduction rule for $\wedge$ by $(\wedge I)$ and the elimination rule for $\wedge$ by $(\wedge E)$, respectively. Additionally, we use some special rules. A *proof* is constructed as a tree in which all leaves are axioms, and in this case the formula in the root is a formula to be proved. We write $\vdash_{NHBRSL} A$ when A is provable in *NHBRSL*.

Below are axioms and rules for *NHBRSL*. Γ is a (possibly empty) set of formulas and A, B, C, D are formulas, respectively.

Natural Deduction System *NHBRSL*

Axioms

$(A1)\ \Gamma, A \vdash A$ $(A2)\ \Gamma, \bot \vdash A$
$(A3)\ \Gamma \vdash \top$ $(A4)\ \vdash *A \vee **A$
$(A5)\ +A \wedge ++A \vdash$ $(A6)\ \Gamma, A \wedge +A \vdash A \vee *A$
$(A7)\ *+A \vdash A$

Rules

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}(\wedge I) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \ \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}(\wedge E)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}(\vee I) \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}(\vee E)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}(\to I) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}(\to E)$$

$$\frac{\Gamma, A \vdash}{\Gamma \vdash *A}(*I) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash *A}{\Gamma \vdash \bot}(*E)$$

$$\frac{D \vdash \top \quad A \vdash C}{D \vdash +A}(+I) \qquad \frac{\Gamma \vdash +A \quad \Gamma, \top \vdash A}{\Gamma \vdash B}(+E)$$

Here, we can dispense with rules for $*$, since $*A$ is defined as $A \to \bot$. Observe that the condition in $(+I)$, namely D in the premise $D \vdash \top$ and in the consequent $D \vdash +A$ and A in the premise $A \vdash C$ must be a single formula, not a set of formulas, is crucial to our formalization. One could also describe the rules for dual implication $-\!<$ as follows:

$$\frac{D \vdash A \quad B \vdash C}{D \vdash A -\!< B}(-\!< I) \quad \frac{\Gamma \vdash A -\!< B \quad \Gamma, A \vdash B}{\Gamma \vdash C}(-\!< E)$$

The natural deduction system with axioms $(A1)-(A3)$ and rules for $\wedge, \vee, *$ is for intuitionistic propositional logic *Int*, in which $*$ can be identified with intuitionistic negation $\neg$.

If we delete the axiom $(A6), (A7)$ and rules for $(\to)$, the natural deduction system $NHBRSL_0$ for the logic based on double Stone algebras is available. A natural deduction system $NHBRSL_1$ is obtainable from $NHBRSL_0$ by adding $(A6)$.

Lemma 2. *The following formulas are provable in NHBRSL.*

(i) $\vdash_{NHBRSL} A \to \top$

(ii) $\vdash_{NHBRSL} \bot \to A$

(iii) $\vdash_{NHBRSL} *(+A \wedge ++A)$

(iv) $\vdash_{NHBRSL} (A \wedge +A) \to (A \vee *A)$

(v) $\vdash_{NHBRSL} A \leftrightarrow *+A$

Here, $A \leftrightarrow B$ *abbreviates* $(A \to B) \wedge (B \to A)$.

Next, we present the soundness result of *HBRSL*. As noted above, $*$ is intuitionistic-like negation and $+$ is dual intuitionistic-like negation. The fact is technically important here. To validate $(A4)$, we need the condition of directedness. Logics stronger than intuitionistic logic but weaker than classical logic are called the *intermediate logics* or *superintuitionistic logics*.

Intuitionistic logic with the axiom called *the weak law of excluded middle*: $\neg A \vee \neg\neg A$ is the intermediate logic often denoted by *LQ*; e.g., see Akama [1]. Similarly, the condition of dual directedness is needed to validate $(A5)$. The intermediate extensions of dual intuitionistic logic did not seem to be fully studied in the literature. The condition of bridge is added for the validity of $(A6)$.

Theorem 1 (soundness). $\vdash_{NHBRSL} A \;\Rightarrow\; \models_{NHBRSL} A$

Proof. It can be proved by checking that all axioms are valid and all rules preserve validity. Most cases are immediate from the soundness proof for intuitionistic logic. Checking of rules is trivial and omitted. For axioms, the validity of $(A1),(A2),(A3)$, and $(A7)$ are obvious. Thus, we here only consider $(A4),(A5)$ and $(A6)$.
$(A4)$: Suppose $(A4)$ is not valid. Then, there is a Kripke model satisfying that $w \not\models *A \vee **A$ for some w. From the truth definition of $*$, we have that $w \not\models *A$ and $w \not\models **A$ which is equivalent to the following:

$$\exists v(wRv \text{ and } v \models A) \text{ and } \exists v(wRv \text{ and } \forall u(vRu \;\Rightarrow\; u \not\models A)).$$

From the first conjunct, $v \models A$ holds. Since R is directed, $\exists u \forall v(vRu)$. By persistency (P), we have $u \models A$. The second conjunct says that $u \not\models A$, which contradicts $u \models A$ from the first conjunct. Consequently, $(A4)$ is shown to be sound.
$(A5)$: Suppose $(A5)$ is not valid. Then, we have a Kripke model satisfying that $w \models +A \wedge ++A$ for some w. From the truth definition of $+$, we have:

$$\exists v(wRv \text{ and } v \not\models A) \text{ and } \exists v(wRw \text{ and } \forall u(uRv \;\Rightarrow\; u \models A)).$$

From the first conjunct, $v \not\models A$ holds. By the dual directedness of R, i.e., $\exists u \forall v(uRv)$, together with the dual persistency (DP), $u \not\models A$ is derived. But it contradicts with the second conjunct $u \models A$. Consequently, $(A5)$ is a sound rule.
$(A6)$: It suffices to see the validity of $A \wedge +A \vdash A \vee *A$. Suppose $A \wedge +A \vdash A \vee *A$ is not valid. Then, there is a Kripke model satisfying that $w \models A \wedge +A$ but $w \not\models A \vee *A$ for some w, which is equivalent to the following:

$$w \models A \text{ and } \exists v(vRw \text{ and } v \not\models A) \text{ and } w \not\models A \text{ and } \exists u(wRu \text{ and } u \models A).$$

Since R is bridged, $\forall w \forall v(wRv \;\Rightarrow w = v \text{ or } \forall u(wRu \;\Rightarrow w = u))$ holds. We must consider two cases. First, if the condition $(wRv \;\Rightarrow\; w = v)$ in the bridge condition holds, then the first and second conjuncts give contradiction. Second, if the condition $(wRu \;\Rightarrow w = u)$ in the bridge condition holds, then the third and fourth conjuncts give contradiction. From these considerations, we obtain the fact that $(A6)$ is sound.

Next, we prove the completeness of *HBRSL* by means of *canonical model*. Our method for proving completeness is a suitable modification of the one used in Kripke semantics for intuitionistic logic. We need some preliminary definitions. If $\Gamma = \{A_1,...,A_n\}, \Delta = \{B_1,...,B_m\}$, then we set $\bigwedge \Gamma = A_1 \wedge ... \wedge A_n, \bigvee \Delta = B_1 \vee ... \vee B_m$. The pair (Γ, Δ) is *consistent* iff there are no finite subsets $\Gamma_0 \subset \Gamma$ and $\Delta_0 \subset \Delta$ such that $\vdash_{NHBRSL} \bigwedge \Gamma_0 \to \bigvee \Delta_0$, where $\bigwedge \emptyset = \top, \bigvee \emptyset = \bot$. Γ is *consistent* iff $(\Gamma, \emptyset)$ is consistent. A pair (Γ', Δ') is an *extension* of a pair (Γ, Δ) iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

A set Γ of formulas is *saturated* if the following conditions hold: (i) Γ is consistent, (ii) $\Gamma \vdash_{NHBRSL} A \ \Rightarrow A \in \Gamma$, (iii) $\Gamma \vdash_{NHBRSL} A \vee B \ \Rightarrow \ \Gamma \vdash_{NHBRSL} A$ or $\Gamma \vdash_{NHBRSL}$. If $\Gamma \not\vdash_{NHBRSL} A$, then Γ can be extended to saturated $\Gamma' \supset \Gamma$ such that $\Gamma' \not\vdash_{NHBRSL} A$ by standard construction.

For our setting, we generalize the notion of saturated set for the pair of sets of formulas defined above. Let T, S be sets of formulas of the language of *HBRSL*. The pair (T, S) is *saturated* iff the following hold:

$T \cap S = \emptyset$
$A \wedge B \in T \ \Rightarrow \ A \in T$ and $B \in T$
$A \vee B \in T \ \Rightarrow \ A \in T$ or $B \in T$
$A \to B \in T \ \Rightarrow \ A \in S$ or $B \in T$
$*A \in T \Rightarrow \ A \in S$ or $\bot \in T$
$+A \in T \ \Rightarrow \ \top \in T$ and $A \in S$
$A \wedge B \in S \ \Rightarrow \ A \in S$ or $B \in S$
$A \vee B \in S \ \Rightarrow \ A \in S$ and $B \in S$
$A \to B \in S \ \Rightarrow A \in T$ and $B \in S$
$*A \in S \ \Rightarrow \ A \in T$ and $\bot \in S$
$+A \in S \ \Rightarrow \ \top \in S$ or $A \in T$

We are now ready to define a canonical model $\mathscr{M}^c = \langle W^c, R^c, V^c \rangle$. Here, W^c is a set of all sets $\Gamma = T \cup S$ where (T, S) is a saturated pair. R^c is $\subseteq$ satisfying $\exists \Delta \forall \Gamma (\Gamma \subseteq \Delta), \exists \Sigma \forall \Gamma (\Sigma \subseteq \Gamma)$, and $\forall \Gamma \forall \Delta (\Gamma \subseteq \Delta \ \Rightarrow \ \Gamma = \Delta$ or $\forall \Sigma (\Gamma \subseteq \Sigma \ \Rightarrow \ \Gamma = \Sigma))$. $V^c(\Gamma, p) \ \Leftrightarrow \ p \in \Gamma$ for atomic p. We can then define $\models^c$ for any formula as described before.

Lemma 3 is a key lemma to prove completeness.

Lemma 3. *For any $\Gamma \in W^c$ and any formula: we have:*

$$\Gamma \models^c A \ \Leftrightarrow \ A \in \Gamma$$

Proof. The cases in which A is of the form $B \wedge C, B \vee C, B \to C$ or $*B$ are proved as in intuitionistic logic. It thus suffices to only consider the case in which $A = +B$.

$$\begin{aligned} \Gamma \models^c +B &\Leftrightarrow \exists \Delta (\Delta \subseteq \Gamma \text{ and } \Delta \not\models^c B) \\ &\Leftrightarrow \exists \Delta (\Delta \subseteq \Gamma \text{ and } B \notin \Delta) \\ &\Leftrightarrow +B \in \Gamma \end{aligned}$$

Here, we must prove that $\exists \Delta (\Delta \subseteq \Gamma$ and $B \notin \Delta) \ \Leftrightarrow \ +B \in \Gamma$. For $(\Leftarrow)$, suppose $+B \in \Gamma$. Then, by the definition of saturated pair, we have that $\top \in T$ and $B \in S$. In $\Gamma = T \cup S$ set $T = \Delta$, then $\Delta \subseteq \Gamma$ follows. Since $T \cap S = \emptyset$, from $B \in S$ we have $B \notin \Delta = T$. Then, $\Delta \subseteq \Gamma$ and $B \notin \Delta$ hold.

For $(\Rightarrow)$, suppose $\exists \Delta (\Delta \subseteq \Gamma$ and $B \notin \Delta)$. Set $\Gamma = \{D\}$, then $D \vdash_{NHBRSL} \top$ by the axiom $(A3)$. As $B \notin \{D\}$, $B \vdash_{NHBSRL} \bot$ by saturatedness. By applying $(I \to)$ to the axiom $(A2)$ $B, \bot \vdash_{NHBRSL} C$, we have that $B \vdash_{NHBRSL} \bot \to C$. By $(\to E)$, $B \vdash C$ follows. Using the rule $(+I)$ enables us to obtain $D \vdash +B$. Thus, $+B \in \Gamma$ holds.

Then, we can conclude the completeness of *HBRSL*:

Theorem 2 (completeness). $\Gamma \vdash_{NHBRSL} A \Leftrightarrow \Gamma \models A$.

Proof. For soundness ($\Rightarrow$), $\Gamma \vdash_{NHBRSL} A$ iff $\vdash_{NHBRSL} \Gamma \rightarrow A$. Then, applying theorem 1 to it leads the soundness.

For completeness ($\Leftarrow$), we use contrapositive argument. Assume that $\Gamma \not\vdash_{NHBRSL} A$. Then, there is a saturated pair $\Gamma' = (T, S)$. Thus, the completeness follows by Lemma 3.

We can similarly establish completeness results of $HBRSL_0$ and $HBRSL_1$ by considering the corresponding conditions of Kripke models. Our result also implies the Kripke completeness of Düntsch's logic. If we add dual intuitionistic implication to *HBRSL*, we can show the completeness proof for Heyting-Brouwer logic and some of its extensions.

5 Conclusion

We proposed a rough set logic called the Heyting-Brouwer rough set logic *HBRSL*, which extends Düntsch's rough set logic with intuitionistic implication. A model theory was supplied by a Kripke model to give an intuitive semantics, and proof theory based on natural deduction is presented. We established a completeness result by means of a canonical model. Thus, an alternative foundation for rough set logics was outlined in this paper. We believe that our logic can serve as a logical framework for reasoning about rough information.

There are some interesting research topics related to our logic. Although we use a natural deduction system as a proof theory, other proof methods like sequent calculus and tableau calculus can be explored. In particular, cut-free sequent formulation seems important to advance a practical proof method.

It would be also possible to investigate other types of logics based on double Stone algebras with Kripke or algebraic semantics. For instance, introducing different types of implications is one of the important problems.

Another line of work in this research will be the modal and three-valued characterizations of rough set logics. A modal approach would be promising because many connections of intuitionistic (and intermediate) logic and modal logic are known. A three-valued approach is also interesting. For example, a model theory based on three-valued Lukasiewicz algebra appears to provide some extensions of rough set logics.

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References

1. Akama, S.: The Gentzen-Kripke construction of the intermediate logic *LQ*. Notre Dame Journal of Formal Logic 33, 148–153 (1992)
2. Düntsch, I.: A logic for rough sets. Theoretical Computer Science 179, 427–436 (1997)

3. Orlowska, E.: Modal logics in the theory of information systems. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 30, 213–222 (1988)
4. Orlowska, E.: Logic for reasoning about knowledge. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 35, 559–572 (1989)
5. Pawlak, Z.: Rough sets. International Journal of Computer and Information Sciences 11, 341–356 (1982)
6. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning about Data. Kluwer, Dordrecht (1990)
7. Pomykala, J., Pomykala, J.A.: The stone algebra of rough sets. Bulletin of Polish Academy of Science, Mathematics 36, 495–508 (1988)
8. Rauszer, C.: Semi-Boolean algebras and their applications to intuitionistic logic with dual operations. Fundamenta Mathematicae 83, 219–249 (1974)