Chapter 8
Boundary Conditions, Time Reversal and Measurements

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Abstract This contribution is divided into two parts. In the first one, we argue that the idea of time reversal in Quantum Mechanics is considerably more subtle than generally thought. For example, it is not even possible to make sense of Feynman's reinterpretation of the Heisenberg uncertainty principle without a good grasp of it. In the second part, more speculative, we discuss the importance of "randomizing" some times, in Quantum Mechanics, as a preliminary step before the expected conciliation with General Relativity.

1 The (Deterministic) Time We Know

There are basically two levels of analysis, in theoretical physics, of the issue of time-reversal (TR) symmetry:

(A) It is a trivial issue.
(B) It is one of the most vexing issues of Theoretical Physics.

There is no need to allude to the devastating problems associated with the Wave Function of the Universe to see how limited is the first opinion. As a matter of fact, it is sufficient to pick the most offensively trivial system of classical mechanics: the one dimensional free particle (of mass 1), whose second order (Newton's) dynamical law is

$$\frac{d^2}{dt^2}q = 0 \tag{1}$$

According to (A) there is no more in the statement that this law is invariant (or symmetric) under time reversal than the trivial observation:

"If $q(t)$ solves (1) so does $\hat{q}(t) = q(-t)$, $\forall t \in \mathbb{R}$".

One can as well define a time-reversal operator T, acting on the state of the system, here $\xi = (q, p) \in S = \mathbb{R}^2$ by $T(q, p) = (q, -p)$. Then, since the Hamiltonian flow

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(with Hamiltonian $h(q, p) = \frac{1}{2}p^2$) is given by $U_t : S \to S, (q, p) \mapsto (q + pt, p)$, the fact that $T\hat{q}(t) = q(t)$ and $T\hat{p}(t) = -p(t)$ can be rewritten as

$$U_{-t} = T^{-1}U_t T \tag{2}$$

Of course, the time reversed $(\hat{q}(\cdot), \hat{p}(\cdot))$ are not really used physically. Instead, the previous formula allows us to extend the dynamical information available about the future, i.e. $t \in [0, \infty[$, into the past $t \in]-\infty, 0]$, given the initial condition, i.e the state, at $t = 0$. (The initial time is, of course, arbitrary.)

This way to think about time symmetry of physical laws of nature is, in fact, universal since it is thought that (almost) all fundamental laws are invariant under time reversal for the appropriate operation T, which depends on the considered domain of physics (for instance, in classical electrodynamics, if $(\vec{E}, \vec{B})$ denotes, respectively, the electric and magnetic fields then $T(\vec{E}, \vec{B}) = (\vec{E}, -\vec{B})$.

A substantial part of the discussions on physical interpretations of the time-reversal symmetry amounts to ponder over the operational meaning, if any, of the mathematical procedure given before. Is it physically realistic to transfer our dynamical information from the future to the past. (Or the other way around!) What is the meaning of such a transfer in the lab?

In any time reversal, initial conditions become final ones and this may easily conflict with our naïve (intuitive) concept of causality. It is a trivial observation that initial boundary conditions are, practically speaking, more easy to deal with than final ones. But one tends to use excessively this argument to eliminate (or ignore) some solutions of the laws of motion which are precisely needed to show the invariance of the theory under TR! An example is the propagation of classical waves where we tend to ignore the advanced solution and retain only the retarded one, more in accordance with "causality".

We can, of course, give at once boundary conditions at two different times but the associated boundary value problem is, in general, considerably more subtle than the traditional (Cauchy) problem. Consistency conditions are needed between those data, and we may easily loose the existence and uniqueness of the solution.

Let us come back to our trivial mechanical example, but regarded now as a boundary value problem. Since nothing in it depends on the choice of initial instant we shall consider any time interval $I = [s, u]$ and pick a reference time t in between. According to the classical Hamilton–Jacobi theory, we have now a dual description of the dynamics on I, when the boundary data of (1) become

$$q(s) = x \quad \text{and} \quad q(u) = z \tag{3}$$

According to the first description, say the "causal" one, we have to consider a family of solutions of the (free) Hamiltonian equations with (past) boundary conditions:

$$q(s) = x, \qquad p(s) = \nabla S_s^*(x) \tag{4}$$

where S_s^* is regular enough to define an initial Lagrangian manifold in phase space (we shall need, in fact, singular manifold for our example). This family of solutions is described by the action with initial condition, regarded as function of the final point (q,t):

$$S_L^*(q,t) = S_s^*(x) + \int_{x,s}^{q,t} L\,d\tau \tag{5}$$

for any t in I, where L is the Lagrangian of the system (for our Hamiltonian $h(q,p) = \frac{p^2}{2}$, $L(q,\dot{q})$ reduces to $\frac{1}{2}|\dot{q}|^2$ and the integral is computed along the characteristics connecting x and $q(t) = q$, q being regarded as variable). As a function, S_L^* solves the Hamilton–Jacobi (HJ) equation

$$\begin{cases} \frac{\partial S_L^*}{\partial t} + h\big(q, \nabla S_L^*\big) = 0 \quad t \in I \\ S_L^*(q,s) = S_s^*(q) \end{cases} \tag{6}$$

Clearly, this first order equation chooses definitely an arrow of time. But how come, since the resulting free dynamics does not? Even stranger, the time-symmetric Newton's equation results from the gradient of the "irreversible" HJ equation! There is no paradox here, however, but the explanation may be more interesting and general than expected. In the Hamilton–Jacobi framework, we had to ignore half of the boundary conditions (3), the future one. But we could have done a symmetric selection and keep the future information of (3). Then the relevant family of solutions would be described by an action with this final condition and regarded as a function of the initial point (q,t):

$$S_L(q,t) = S_u(z) + \int_{q,t}^{z,u} L\,d\tau \tag{7}$$

that is, the solution of

$$\begin{cases} -\frac{\partial S_L}{\partial t} + h(q, -\nabla S_L) = 0 \quad t \in I \\ S_L(q,u) = S_u(q) \end{cases} \tag{8}$$

This HJ equation can be regarded as the time reversed of (6) on I, because $dS_L^* = L\,dt$ and $dS_L = -L\,dt$. But, since our trivial boundary value dynamical system (1) and (3) has clearly an unique solution $t \mapsto q(t)$, $\forall t \in I$, some consistency condition is needed between (6) and (8). It is the following.

For any $t \in]s,u[$ along this solution

$$p_*\big(q(t),t\big) = \frac{\partial S_L}{\partial q}(x,s,q,t)\bigg|_{q=q(t)} = -\frac{\partial S_L}{\partial q}(q,t,z,u)\bigg|_{q=q(t)} = p\big(q(t),t\big) \tag{9}$$

expressing the smoothness of the trajectory, $\forall t \in I$. Notice that because our conditions (3) at the boundary ∂I are trivial, here, we can just use Hamilton's principal function and, then, drop the $*$ on the l.h.s. action without ambiguity.

So our trivial (time homogeneous) boundary value problem (1) and (3) involves, in the Hamilton–Jacobi perspective, two distinct momenta needed to take the arbitrary given data at ∂I into consideration. And our second order homogeneous problem can be solved via two time dependent first order problems. Since the Hamilton

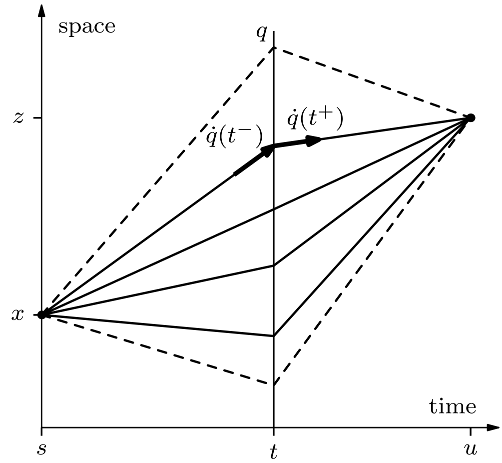

Fig. 1 The two distinct momenta (or velocities) at a given time t

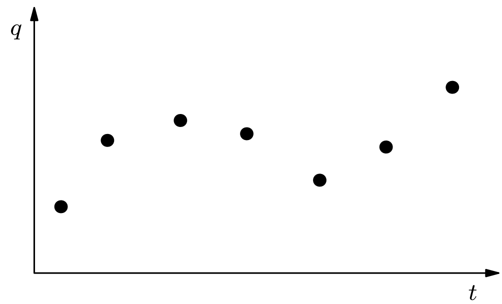

Fig. 2 The track of a quantum particle according to Heisenberg [1]

principal function reduces here to $S(q_1,t_1,q_2,t_2) = \frac{1}{2}\frac{|q_2-q_1|^2}{t_2-t_1}$, they can be written as a left hand differential $d_* q(\tau) = \dot q(\tau^-)\, d\tau$ where (cf. Fig. 1)

$$\begin{cases} d_* q = \frac{q-x}{\tau-s}\, d\tau = p_*^x(q,\tau)\, d\tau \quad s \le \tau < t \\ q(t) = q \end{cases} \tag{10}$$

and a right hand differential $dq(\tau) = \dot q(\tau^+)\, d\tau$:

$$\begin{cases} dq = \frac{z-q}{u-\tau}\, d\tau = p^z(q,\tau)\, d\tau \quad t < \tau \le u \\ q(t) = q \end{cases} \tag{11}$$

The consistency relation (9) determines uniquely the solution of (1) and (3).

According to Heisenberg we are not allowed to preserve any such space-time view for the quantum version of our trivial dynamical system, i.e for the one dimensional quantum free particle. We are even told why this is impossible; because the track of our quantum particle looks like (see [1]) Fig. 2.

But, 20 years after Heisenberg, Feynman has shown that this radicalism was not necessary [2]. One should just relax the classical hypothesis of smoothness of the trajectories. The building block of the Feynman reinterpretation is the concept of transition element (or amplitude) on I:

$$\begin{aligned}\langle\varphi|\mathbb{I}\psi\rangle_{S_L} &= \int\int \psi_s(x)K(x,u-s,z)\bar{\varphi}_u(z)\,dx\,dz \\ &= \int\int_{\Omega_x^z}\int \psi_s(x)e^{\frac{i}{\hbar}S_L[\omega(\cdot);u-s]}\bar{\varphi}_u(z)\,\mathcal{D}\omega\,dx\,dz \end{aligned}\tag{12}$$

where S_L is the classical action, regarded now as a functional along Feynman's quantum paths $\omega\in\Omega_x^z=\{\omega\in C([s,u];\mathbb{R})$ such that $\omega(s)=x,\ \omega(u)=z\}$, $\mathbb{I}$ denotes the identity operator, $\hbar$ is Planck's quantum of action and $\mathcal{D}\omega$ denotes the symbolic product $\prod_{s\le\tau\le u} d\omega(\tau)$.

The definition (12) involves boundary conditions $\{\psi_s,\varphi_u\}$, two states in $L^2(\mathbb{R})$ at two different times. When those states are arbitrary, the transition element has no probabilistic interpretation; it is just a (complex) scalar product of vectors. But we can, in particular, propagate a single state ψ_s to its future value by $\varphi_u=\exp(-\frac{i}{\hbar}(u-s)H)\psi_s=\int\psi_s(x)K(x,u-s,z)\,dx$, where H is the quantization of the Hamiltonian h. Then the integrand of (12) reduces to Born's probability density of the initial (or final) wave function. The integral kernel propagating forward (causally!) the initial probability $|\varphi_s(x)|^2$ in I is

$$P_F(s,x,t,dz)=\big(\bar{\varphi}_s(x)\big)^{-1}K(x,u-s,z)\bar{\varphi}_u(z)\,dz \tag{13}$$

for all x s.t. $\varphi_s(x)\neq 0$. But we could as well propagate backward in time Born's final probability density $|\psi_u(z)|^2$, via the kernel

$$P_B(s,dx,u,z)=\psi_s(x)K(x,u-s,z)\big(\psi_u(z)\big)^{-1}\,dx \tag{14}$$

Notice that if we were allowed to regard $t\mapsto\omega(t)$ as a well defined Markovian (stochastic) process, then, using (13) and (14), the integrand of Feynman's transition element (12) would satisfy a "detailed balance condition", one of the statistical expressions of equilibrium:

$$dx\big|\varphi_s(x)\big|^2P_F(s,x,u,dz)=P_B(s,dx,u,z)\big|\psi_u(z)\big|^2\,dz \tag{15}$$

Of course, now, the classical consistency condition (9) in I cannot be true anymore since it means that the realized (extremum) trajectory is smooth everywhere in I. Moreover it uses a (dual) concept of momentum apparently obsolete in the quantum context.

Still a quantum deformation of (9) is available. It has been discovered by Feynman, in a time discretized way, as the following kinematical property (see [3]):

$$\left\langle\omega(t)\left(\frac{\omega(t)-\omega(t-\Delta t)}{\Delta t}\right)\right\rangle_{S_L}-\left\langle\left(\frac{\omega(t+\Delta t)-\omega(t)}{\Delta t}\right)\omega(t)\right\rangle_{S_L}=i\hbar \tag{16}$$

where $\langle\cdot\rangle_{S_L}$ denotes the "expectation" with respect to the above-mentioned "process". If we were allowed to take the limit $\Delta t \to 0$ in (16) then the first time derivative in (16) should be a left hand one, like $p_*(q,t)$ before, and the second one a right hand derivative like $p(q,t)$. It is, therefore intuitively clear that the only way the difference on the l.h.s. of (16) could be non-zero, for our free quantum dynamics, is when $t \mapsto \omega(t)$ becomes very irregular. This is, indeed Feynman's way to show that the quantum trajectories are Brownian like. The beauty of (16) is that it is the space-time version of $QP - PQ = i\hbar$, i.e. of Heisenberg's uncertainty principle motivated by Fig. 2!

What is definitely missing for a probabilistic understanding of Feynman's ideas is the stochastic process itself and, therefore, the expectation $\langle\cdot\rangle_{S_L}$. But using (13) and (14) it is a simple exercise to find its profile: $\omega(t)$ should be a diffusion process (for Hamiltonians like the one considered here) with drift, or mean velocity, $i\hbar\frac{\nabla\bar{\psi}_t}{\bar{\psi}_t}$ (or $-i\hbar\frac{\nabla\psi_t}{\psi_t}$) and diffusion constant $i\hbar$, like the r.h.s. of (16). Following St. Anselm, however, we regard the existence as an important part of the perfection and so we feel compelled to look for what can really makes sense in Feynman's point of view.

Besides the existence problem there is another one showing us the way: to give boundary conditions at ∂I is not usual in the classical theory of stochastic processes. The future data excludes, for instance, the basic class of processes with independent increments (like Brownian or Poisson processes). On the other hand, the separation between past and future is sharp, here; this suggests that the process should still be Markovian. Coming back to our trivial example, we shall keep the classical drifts of (10) and (11) and just add a mathematically decent noise to Feynman's picture, namely, for $t \in I$

$$\begin{cases} d_*X(t) = \sqrt{\hbar}\, d_*W_*(t) + p_*^x\big(X(t),t\big)\,dt \\ X(u) = z \end{cases} \tag{17}$$

and

$$\begin{cases} dX(t) = \sqrt{\hbar}\, dW(t) + p^z\big(X(t),t\big)\,dt \\ X(s) = x \end{cases} \tag{18}$$

where W_* and W denote, respectively, Brownian motions adapted to our dual description. The diffusion coefficient $\sqrt{\hbar}$ is imposed by the above mentioned profile. Anyone of these (Itô's stochastic) differential equations can be solved explicitly. Their common solution is a Gaussian process, whose mean solves our classical boundary value problem (1) and (3). Its covariance is the one computed by Feynman using $\omega(t)$, after the substitution $t \mapsto it$ (the "Euclidean" or "Wick" rotation). The ("Bernstein") process $X(t)$ is Markovian, not of independent increment, but invariant under time reversal in the same sense as (1). The probabilistic counterpart of Feynman's kinematical property (16) in terms of the well defined expectation $E[\cdot]$ of $X(t)$ is

$$E\Big[X(t)\cdot p_*^x\big(X(t),t\big) - p^z\big(X(t),t\big)\cdot X(t)\Big] = \hbar \tag{16'}$$

If we relax the boundary conditions δ_x, δ_z at ∂I and give, instead, a pair of (strictly positive) probability densities at time s and u, the construction survives and provides all the well defined processes realizing Feynman's idea (12) of transition element on I. This is also the case if our starting classical particle is not free anymore but subjected to a force $F(q) = -\nabla V(q)$, for most of the potentials V of physical interest (see [4]).

It is interesting to reconsider Feynman's approach to the one-slit experiment in this new perspective. The introduction of a slit in the picture corresponds to a measurement of position of the (free) particle. To say that a particle, starting originally form the origin, has to be localized in the slit at a given time T in the future is a conditioning, in the traditional probabilistic sense [5, 6]. Then one verifies that this conditioning introduces indeed an irreversibility in an otherwise perfectly time-symmetric framework [7].

In general, any such process $X(t)$, $t \in I$, associated with an Hamiltonian H as before can be found in an interval A with the probability $P(X(t) \in A) = \int_A \eta^* \eta(q,t)\, dq$ where η and η^* are positive solutions of

$$\begin{cases} -\hbar \frac{\partial \eta^*}{\partial t} = H\eta^* \\ \eta^*(q,s) = \eta_s^*(q) \end{cases} \quad \text{and} \quad \begin{cases} +\hbar \frac{\partial \eta}{\partial t} = H\eta \\ \eta(q,u) = \eta_u(q) \end{cases} \tag{19}$$

One checks easily that the drifts of $X(t)$ are the Euclidean translation of Feynman's ones. This is not a surprise since its above probability constitutes manifestly the Euclidean counterpart of (Born's) probabilistic interpretation of the state ψ_t. In this sense, our boundary value problem (19) mimics the way probability arises in quantum theory. Is it accidental?

A crucial theoretical test is to look for symmetries. Here, this means that knowing the pair (η, η^*) determining $X(t)$ we look for another one $(\eta_\alpha, \eta_{\alpha^*})$ determining $X^\alpha(t)$, for any α in $\mathbb{R}$. But then, clearly, we should have, $\forall t \in I$,

$$1 = \int_{\mathbb{R}} \eta\eta^*\, dq = \int_{\mathbb{R}} \eta\eta^* \frac{\eta_\alpha}{\eta} \frac{\eta^*_\alpha}{\eta^*}\, dq \equiv E\big[h^\alpha h^\alpha_*\big(X(t), t\big)\big]$$

The probabilists are familiar with such transformations $X(t) \to X^\alpha(t)$. They are called Doob's h-transforms (our notations in the last expectation are not arbitrary) and allow us to produce a large collection of Euclidean counterpart of quantum unitary transformations. The first integrals associated with those symmetries are martingales of $X(t)$. The concept of martingale is the closest analogue of constant of motion for a stochastic process. It is also, interestingly enough, the cornerstone of the mathematical theory of stochastic processes [5, 6].

The good surprise of this way to interpret Feynman is that it enables us to guess new quantum symmetries. Let us consider again our free particle. A particular one-parameter family of solutions of the second equation (19), for instance, is

$$\eta_\alpha(q,t) = e^{\frac{1}{\hbar}(\alpha q - \frac{\alpha^2}{2} t)} \eta(q - \alpha t, t), \quad \forall \alpha \in \mathbb{R} \tag{20}$$

The simplest free solution is $\eta = 1$. Then the drift of Feynman's associated "diffusion" is zero, so we know that he is talking really about the Brownian motion. The relation (20) can be understood as $\eta_\alpha = e^{-\alpha N} 1$, for $N = t\frac{\partial}{\partial q} - \frac{q}{\hbar}$. $\eta_\alpha(q,t)$ is what the probabilists call the "exponential martingale". So

$$h_\alpha(q,t) = \eta_\alpha(q,t) = e^{\frac{1}{\hbar}(\alpha q - \frac{\alpha^2}{2}t)} = 1 + \frac{\alpha}{\hbar}q + \frac{\alpha^2}{2\hbar^2}\left(q^2 - \hbar t\right) + \frac{\alpha^3}{3!\hbar^3}\left(q^3 - 3\hbar t q\right) + \cdots$$

By successive differentiations with respect to α, at $\alpha = 0$, we find the collection of martingales of the Brownian motion. The quantum translation of this observation is that

$$Q(t), \quad Q^2(t) + i\hbar t, \quad Q^3(t) + 3i\hbar t Q(t), \quad \text{etc.} \tag{21}$$

for $Q(t)$ the position observable, in the sense of Heisenberg representation, are constants of the free quantum motion. Trivial as it is, this remark if far from being common knowledge.

The perspective sketched here (cf. [7] for more about this "Euclidean Quantum Mechanics" founded on Schrödinger's suggestion in [8], forgotten until the mid-1980s but periodically rediscovered since then: cf. L. Schulman's contribution in this volume, for example) suggests that it is indeed possible to think about quantum physics in probabilistic terms but that this is a rather subtle exercise. In part because, after A.N. Kolmogorov, the theory of stochastic processes itself has developed with an arrow of time in it, which is not natural in a quantum perspective. But the subtle exercise in question can be illuminating, for this reason, in probability theory and in quantum physics, since it leads us to question some generally accepted ideas.

One of the rewards of such a line of thought is precisely the fact that, on the Euclidean side, the problem belongs to regular statistical mechanics. It has been shown long ago (cf. [4] and references therein) that the unique difference with the "usual" construction of Markovian processes like $X(t)$ lies in our boundary conditions. As said before, to determine $X(t)$, $t \in I$ we need, in general, to give a probability density ρ_s at $t = s$ and another one, ρ_u at $t = u$. From this follows, indeed, a quantum-like structure suggesting, as we said, new results on the physical side. Is it a modest expression of the "Eternal Universe" mentioned by C. von der Malsburg? Or is it that, somehow, to understand better the structure of the probabilistic interpretation of quantum mechanics, one needs to think about a classical experiment already done, in our past? After all, it is not true that, for such a finished experiment, the nonlocality is much less shocking?

2 The (Random) Time We Would Like to Know

This section will be more speculative but will try to touch upon the heart of our subject matter: not only the direction of time, but its own nature.

It is not necessary, here, to elaborate on the fact that the two pillars of Modern Physics, i.e. General Relativity and Quantum Physics are irreconcilable. In fact, short after the heroic period of creation of the second theory, it was frequent to read very critical comments about the status of time in elementary Quantum Mechanics. For example, E. Schrödinger:

> Cette notion (beaucoup trop classique) de temps est un grave manque de conséquence dans la mécanique quantique … abstraction faite des postulats de relativité. [8, p. 293]

or J. von Neumann emphasizing the:

> Chief weakness of Quantum Mechanics: its non-relativistic character. While the space coordinate is represented by an operator, the time is an ordinary number parameter. [9, p. 354]

It may seem strange that, 70 years after, this issue is manifestly not regarded anymore as worrying by most scientists (but cf. [10–12], for example). Is it, as suggested by T. Kuhn, that Theoretical Physics did not leave, yet, one of these long periods of "normal science" where the community tends to ignore difficulties seriously challenging accepted theories?

As well known, the difficulty in question is already obvious if one tries to understand the possible interpretations of Heisenberg's uncertainty principle when the canonically conjugate observables of position Q and momentum P are replaced by time T and energy H.

It was shown by W. Pauli, in his famous 1958 Encyclopedia of Physics article, that since the first version of uncertainty relation requires the spectra of both Q and P to be unlimited and the one of H should be, realistically, bounded below, T cannot be an observable in von Neumann sense.

Although the names of some famous scientists are associated with various attempts to puzzle out Pauli's observation, it is fair to say that no indisputable progress has been made on this basic issue.

But what about Feynman's formulation of Heisenberg's uncertainty principle? It is revealing that the father of path integral does not have anything like (16) to suggest as counterpart of the informal (Hilbert space) time-energy commutation relation. And, indeed, he complains that his framework "does not exhibit the important relationship between the Hamiltonian and time ([3, Sect. 7.7]).

Taking for granted that (16') is the mathematically consistent version of (16) it is clear that, to make sense of such a time-energy relation, we should have some random times to start with.

There is little hope to ever construct those directly in the Hilbert space framework of Quantum Theory, for two kinds of reasons. The first one is that we do not know at all where to look for observables which are not (denselly defined) self-adjoint operators in Hilbert space, i.e. von Neumann's observables.

The second one is related with the very shaky status of probability theory in Quantum Physics. This framework is supposed to describe quantitatively the ultimate kind of unpredictable phenomena, only accessible to a statistical analysis. And, indeed, the theory does this quite well, with a remarkable level of a precision in its statistical predictions. But, as far as probability theory is concerned, Quantum Mechanics in an embarrassing mystery: all the ingredients needed to construct a decent mathematical model of random experiments are missing. The above-mentioned existence problem with Feynman's "stochastic process" is, unfortunately, typical. The situation gets only worse when more complicated quantum systems are considered.

On the other hand, when the stochastic processes make sense, the concept of random time is a tool immediately available. According to Kai Lai Chung, in point of fact, "this is the single tool that separates probabilistic methods from others, without which the theory of Markov processes would lose much of its strength and depth" ([7, p. 80]).

Feynman is by far the theoretical physicist who tried hardest to turn Quantum Mechanics and Quantum Field Theory into theories involving fundamentally the tools of Stochastic Analysis instead of the ones of elementary (Newton–Leibniz) calculus [3, 13]. The failure of his "probabilistic" approach (the "Path Integral" approach) is very relative. Relative, in particular, to the scientific community in charge of its assessment. Many physicists do not understand why an approach allowing systematically to guess new results is not taken more seriously by some mathematicians. Those, however, would invariably answer that none of Feynman's path integrals (or processes) do exist.

Our hunch is that, using the well defined counterpart of Feynman's approach sketched in the first section, it will be possible to construct specific random times, corresponding to realistic experimental conditions.

Now, of course, such times would not be the quantum times we would like to know. Our Euclidean counterpart is only an analogue of Quantum Mechanics. But it seems to be a pretty good analogue; for example the "new" quantum constants of the free motion listed in (21) have been discovered directly via our probabilistic analogy. As a matter of fact, they are a very special case of a quantum Theorem of Noether providing systematically richer informations on quantum symmetries than the textbooks results on that matter (cf. [7, 14]). The same should happen with random times. Although such times are, indeed, immediately available on the theoretical (Euclidean) side, the algorithms involved in their computations are sophisticated, plunging into the heart of the theory of Markov processes and properties of their trajectories. Nothing, certainly, that Hilbert spaces should help us to discover.

If, as expected, a natural randomization of some specific times is possible, this new breach into determinism could open the way to the more radical ones needed to think simultaneously about Quantum Physics and General Relativity.

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