

Chapter 7

Two Arrows of Time in Nonlocal Particle Dynamics

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Abstract Considering what the world would be like if backwards causation were possible is usually mind-bending. Here I discuss something that is easier to study: a toy model that incorporates a very restricted sort of backwards causation. It defines particle world lines by means of a kind of differential delay equation with negative delay. The model presumably prohibits signaling to the past and superluminal signaling, but allows nonlocality while being fully covariant. And that is what constitutes the model's value: it is an explicit example of the possibility of Lorentz-invariant nonlocality. That is surprising in so far as many authors thought that nonlocality, in particular nonlocal laws for particle world lines, must conflict with relativity. The development of this model was inspired by the search for a fully covariant version of Bohmian mechanics.

Keywords Bohmian mechanics · Relativity · Quantum nonlocality · Backwards causation · Differential delay equations

In this paper I will introduce to you a dynamical system—a law of motion for point particles—that has been invented [5] as a toy model based on Bohmian mechanics. Bohmian mechanics is a version of quantum mechanics with particle trajectories; see [4] for an introduction and overview. What makes this toy model remarkable is that it has two arrows of time, and that precisely its having two arrows of time is what allows it to perform what it was designed for: to have effects travel faster than light from their causes (in short, *nonlocality*) without breaking Lorentz invariance. Why should anyone desire such a behavior of a dynamical system? Because Bell's nonlocality theorem [1] teaches us that any dynamical system violating Bell's inequality must be nonlocal in this sense. And Bell's inequality is, after all, violated in nature.

It is easy to come up with a nonlocal theory if one assumes that one of the Lorentz frames is preferred to the others: simply assume a mechanism of cause and effect

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(an interaction in the widest sense) that operates *instantaneously* in the preferred frame. That is what nonrelativistic theories usually do. In other frames, these non-local effects will either travel at a superluminal ($>c$) but finite velocity or precede their causes by a short time span. Of course, causal loops cannot arise since in the preferred frame effects never precede causes; yet the entire notion of a preferred frame is against the spirit of relativity. Without a preferred frame, to find a nonlocal law of motion is tricky, and much agonizing has been spent on this. About one way to achieve this you will learn below.

Let us come back first to the two arrows of time. They are opposite arrows, in fact. But unlike the arrows considered in Lawrence Schulman's contribution to this volume, they are not both thermodynamic arrows. One of the two is the thermodynamic arrow. Let us call it Θ . It arises, as emphasized first by Ludwig Boltzmann and in this conference by Schulman, not from whichever asymmetry in the microscopic laws of motion, but from boundary conditions. That is, from the condition that the initial state of the universe be taken from a particular subset of phase space (corresponding to, say, a certain low entropy macrostate), while the final state is not subjected to any such conditions—except in some scenarios studied by Schulman. The dynamical laws considered in discussions of the thermodynamic arrow of time are usually time reversal invariant. But not so ours! It explicitly breaks time symmetry, and that is how another arrow of time comes in: an arrow of microscopic time asymmetry, let us call it C . Such an arrow must be assumed before writing down the equation of motion, which will be (6) below. In addition, the equation of motion is easier to solve in the direction C than in the other direction. Does not it seem ugly and unnatural to introduce a time asymmetry? Sure, but we will see it buys us something: Lorentz-invariant nonlocality.

Recall that such an arrow is simply absent in Newtonian mechanics and other time symmetric theories. So it is not surprising that the microscopic arrow C is not the source of the macroscopic time arrow Θ , even more, the direction of Θ is completely independent of the direction of C . Θ depends on boundary conditions, and not on the details of the microscopic law of motion. In our case, Θ will indeed be opposite to C . Since inhabitants of a hypothetical universe will regard the thermodynamic arrow as their natural time arrow, related to macroscopic causation, to memory, and to apparent free will, you should always think of Θ as pointing towards the future, whereas C is pointing to what we call the past.

It is time to say what the equation of motion is. The equation is intended to be as close to Bohmian mechanics as possible, to be an immediate generalization, and to have Bohmian mechanics as its nonrelativistic limit. To remind you of how Bohmian mechanics works, you take the wave function (which is supposed to evolve according to Schrödinger's equation—without ever having to collapse), plug in the positions of all the particles (here is where a notion of simultaneity comes in), and from that you compute the velocity of any particle by applying a certain formula, Bohm's law of motion, which amounts to dividing the probability current by the probability density. Now, for a Lorentz-invariant version, we first have to worry about the wave function.

There are three respects in which the wave function of nonrelativistic quantum mechanics (or Bohmian mechanics, for that matter) conflicts with relativity:

(a) the dispersion relation $E = p^2/2m$ at the basis of the Schrödinger equation is nonrelativistic, (b) the wave function is a function of $3N$ position coordinates but only one time coordinate, (c) the collapse of the wave function is supposed instantaneous. While (a) has long been solved by means of the Klein–Gordon or Dirac equation, it is too early for enthusiasm since we still face (b) and (c). We will worry about (c) later, and focus on (b) now. The obvious answer is to introduce a wave function ψ of $4N$ coordinates, that is, one time coordinate for each particle, in other words ψ is a function on $(\text{space-time})^N$. You get back the nonrelativistic function of $3N + 1$ coordinates after picking a frame and setting all time coordinates equal. Such multi-time wave functions were first considered by Dirac et al. in 1932 [2], but what they did not mention was that the N time evolution equations

$$i\hbar \frac{\partial \psi}{\partial t_i} = H_i \psi \quad \text{for } i \in \{1, \dots, N\} \quad (1)$$

needed for determining ψ from initial data at $t = 0$ do not always possess solutions. They are usually inconsistent. They are only consistent if the following condition is satisfied:

$$[H_i, H_j] = 0 \quad \text{for } i \neq j. \quad (2)$$

This is easy to achieve for non-interacting particles and tricky in the presence of interaction. Indeed, to my knowledge it has never been attempted to write down consistent multi-time equations for many interacting particles, although this would seem an obvious and highly relevant problem if one desires a manifestly covariant formulation of relativistic quantum mechanics. We will here, however, stay on the easy side and simply consider a system of non-interacting particles. We take the multi-time equations to be Dirac equations in an external field A_μ ,

$$\mathbf{1} \otimes \dots \otimes \underbrace{\gamma^\mu}_{i\text{th place}} \otimes \dots \otimes \mathbf{1} \left(i \frac{\partial}{\partial x_i^\mu} - e A_\mu(x_i) \right) \psi = m \psi \quad (3)$$

where $\psi : (\text{space-time})^N \rightarrow (\mathbb{C}^4)^{\otimes N}$, and e and m are charge and mass, respectively. The corresponding Hamiltonians commute trivially since the derivatives act on different coordinates and the matrices on different indices.

Such a multi-time Dirac wave function naturally defines a tensor field

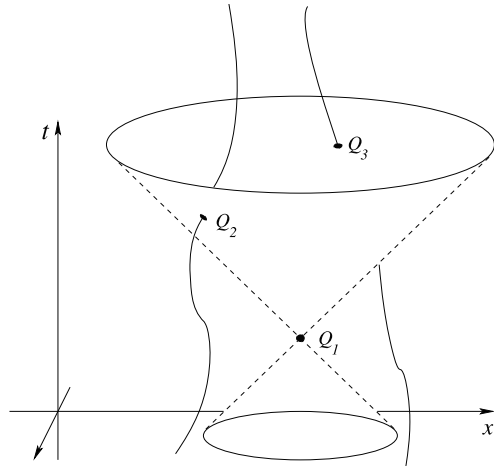
$$J^{\mu_1 \dots \mu_N} := \bar{\psi} \gamma^{\mu_1} \otimes \dots \otimes \gamma^{\mu_N} \psi, \quad (4)$$

and according to the original Bohmian law of motion (for Dirac wave functions), the 4-velocity of particle i is, in the preferred frame,

$$\frac{dQ_i^\mu}{ds} \propto J^{0 \dots \overset{i}{\mu} \dots 0}(Q_1, \dots, Q_N) \quad (5)$$

where only the i th index of J is nonzero, and $Q_i^\mu(s)$ is the world line parameterized by proper time, or indeed by any other parameter since a law of motion need only

Fig. 1 How to choose the N space-time points where to evaluate the wave function, as described in the text



(and (5) does only) specify the *direction* in space-time of the tangent to the world line. The coordinates taken for the other particles are their positions *at the same time*, $Q_j^0 = Q_i^0$. Instead of a Lorentz frame, one can take any foliation of space-time into spacelike hypersurfaces for the purpose of defining simultaneity-at-a-distance [3]. The theory I am about to describe, in contrast, uses the hypersurfaces naturally given by the Lorentzian structure on space-time: the light cones. More precisely: the future light cones—and that is how the time asymmetry comes in.

So here are the steps: first solve (3), so you know ψ on (space-time) N . Then, compute the tensor field J on (space-time) N according to (4). For determining the velocity of particle i at space-time point Q_i , find the points Q_j , $j \neq i$, where the other particles cross the future light cone of Q_i , as depicted in Fig. 1. Plug these N space-time points into the field J and get a single tensor. Find out what the 4-velocities u_j^{μ} of the other particles at Q_j , $j \neq i$, are. Use these to contract all but one index of J . We postulate that the resulting vector is, up to an irrelevant proportionality factor, the 4-velocity we have been looking for:

$$\frac{dQ_i^{\mu}}{ds} \propto J^{\mu_1 \dots \mu_N}(Q_1, \dots, Q_N) \prod_{j \neq i} u_{j\mu_j}(Q_j). \quad (6)$$

One can show [5] that this 4-velocity is always timelike or null.

This law of motion is what can be called an ordinary differential equation with advanced arguments, or a differential delay equation with negative delay, because the velocity depends on the positions (and velocities) of other particles at future times, indeed with a *variable* delay span $Q_j^0 - Q_i^0$. It may seem to complicate things considerably that what happens here depends on the *future* rather than past behavior of the other particles, but that is an artifact of perspective: look at the equation of motion (6) in the other time direction, that is, in the direction C , and notice it now has only *retarded* arguments. That is a more familiar sort of differential delay equation that gives rise to no logical or causal problems. So this theory, although

involving a mechanism of backwards causation, is provably paradox free, since no causal loops can arise: first solve the wave equation for ψ in the usual direction Θ , then solve the equation of motion in the opposite direction C .

Unfortunately, there is no obvious probability measure on the set of solutions to (6). This is different from the situation in Bohmian mechanics, where the $|\psi|^2$ distribution is conserved, a fact crucial for the probability predictions of that theory. The lack of such a measure for the model considered here makes it impossible to say whether or not this theory violates Bell's inequality, which is a relation between probabilities. But this law of motion takes what is perhaps the biggest hurdle on the way towards a fully covariant law of motion conserving the $|\psi|^2$ distribution, by fulfilling what Bell's theorem says is a necessary condition: nonlocality. I should add that in the nonrelativistic limit, the future light cone approaches the hyperplane $t = \text{const.}$ and the law of motion approaches the original Bohmian law of motion (5), conserving $|\psi|^2$.

How does nonlocality come about in this model? That has to do with the two arrows of time, pointing in opposite directions. Had we chosen them to point in the same direction, the theory would have been local, because what happens at Q_i would only depend on (what we call) the past light cone. But in this model, we evaluate ψ on the future light cone of Q_i , which means ψ has, in its multi-time evolution, gone through all the external fields at spacelike separation from Q_i . And that is how the velocity at Q_i may be influenced by the field imposed by an experimenter at spacelike separation from Q_i .

And what is the story then about problem (c) above, the instantaneous collapse? The first thing to say is that collapse is not among the basic rules of this model, or any Bohmian theory. That simply disposes of problem (c). But something more should be said, since the collapse rule can be derived in Bohmian mechanics: even if the wave function of Schrödinger's cat remains forever a superposition, *the cat itself* (formed by the particles) is either dead or alive, with probabilities determined by $|\psi|^2$. Moreover, since the wave packet of the dead cat (i.e., the corresponding term in the superposition) and that of the live cat have disjoint supports in configuration space, the wave packet of the dead cat does not influence the motion of the live cat (nor vice versa). In the model we are concerned with here, everything just said still applies, except that the model does not define any probabilities.

The model thus shows that a relativistic theory of particle world lines can indeed be nonlocal. Let me also point to another consequence: It has often been claimed that Bell's nonlocality proof excludes relativistic Bohm-type theories. This claim has always been inappropriate because Bell's proof actually shows that *any* serious version of quantum mechanics, Bohm-like or not, must be nonlocal; now we see that the claim is also inappropriate in another way, as nonlocality actually does not imply a conflict with relativity. Finally, let me add that a fully covariant version has been developed for a different quantum theory without observers, the GRW theory [6]. Also this model uses time-asymmetric laws, but not backwards causation.

To this day, thinking about time, time's arrows, and relativity remains a source of the unexpected.

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References

1. Bell, J.S.: *Speakable and Unsayable in Quantum Mechanics*. Cambridge University Press, Cambridge (1987)
2. Dirac, P.A.M., Fock, V.A., Podolsky, B.: On quantum electrodynamics. *Phys. Z. Sowjetunion* **2**, 468 (1932). Reprinted in Schwinger, J. (ed.): *Quantum Electrodynamics*. Dover Publishing, New York (1958)
3. Dürr, D., Goldstein, S., Münch-Berndl, K., Zanghì, N.: Hypersurface Bohm–Dirac models. *Phys. Rev. A* **60**, 2729 (1999). [quant-ph/9801070](#)
4. Goldstein, S.: Bohmian mechanics. In: Zalta, E.N. (ed.) *Stanford Encyclopedia of Philosophy* (2001). Published online by Stanford University. <http://plato.stanford.edu/entries/qm-bohm/>
5. Goldstein, S., Tumulka, R.: Opposite arrows of time can reconcile relativity and nonlocality. *Class. Quantum Gravity* **20**, 557–564 (2003). [quant-ph/0105040](#)
6. Tumulka, R.: A relativistic version of the Ghirardi–Rimini–Weber model. *J. Stat. Phys.* **125**, 821–840 (2006). [quant-ph/0406094](#)