Hölder Regularity of the Gradient for Solutions of Fully Nonlinear Equations with Sub Linear First Order Term

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Dedicated to Ermanno Lanconelli and his imperishable mathematical enthusiasm on the occasion of his 70th birthday

Abstract Using an improvement of flatness Lemma, we prove Hölder regularity of the gradient of solutions with higher order term a uniformly elliptic fully nonlinear operator and with Hamiltonian which is sub-linear. The result is based on some general compactness results.

Keywords Holder regularity • Fully nonlinear

Mathematical Subject Classification: 35J25, 35J60, 35P30

1 Introduction

In this paper we shall establish some regularity results of solutions of a class of fully nonlinear equations, with a first order term which is sub-linear; it is a natural continuation of [5, 12]. Precisely we shall consider the following family of equations

$$F(D^2u) + b(x)|\nabla u|^{\beta} = f(x) \text{ in } \Omega \subset \mathbb{R}^N. \tag{1}$$

See also [1] for related recent results.

Theorem 1.1 *Suppose that F is uniformly elliptic, that $\beta \in (0,1)$, f and b are in $\mathscr{C}(\overline{\Omega})$. For any u, bounded viscosity solution of (1) and for any $r < 1$, there exist $\gamma \in (0,1)$ depending on ellipticity constants of F, $\|b\|_\infty$, $\omega(b)$ and β and $C = C(\gamma)$*

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such that

$$\|u\|_{\mathscr{C}^{1,\gamma}(B_r(x_o))} \le C\left(\|u\|_\infty + \|b\|_\infty^{\frac{1}{1-\beta}} + \|f\|_\infty\right),$$

as long as $B_1(x_o) \subset \Omega$.

Answering a question that we raised in [4], Imbert and Silvestre in [12] proved an interior Hölder regularity for the gradient of the solutions of

$$|\nabla u|^\alpha F(D^2 u) = f(x)$$

when $\alpha \ge 0$. Their proof relies on a priori Lipschitz bounds, rescaling and an improvement of flatness Lemma, in this way they are lead to use the classical regularity results of Caffarelli, and Evans [7, 8, 11] for uniformly elliptic equations.

Following their breakthrough, in [5], we proved the same interior regularity when $\alpha \ge 0$ in the presence of lower order terms. We also proved $C^{1,\gamma}$ regularity up to the boundary if the boundary datum is sufficiently smooth. Our main motivation to investigate the regularity of these solutions i.e. the simplicity of the first eigenvalue associated to the Dirichlet problem for $|\nabla u|^\alpha F(D^2u)$, required continuity of the gradient up to the boundary.

When $\alpha \in (-1, 0)$, in [4] we proved $\mathscr{C}^{1,\gamma}$ regularity for solutions of the Dirichlet problem, using a fixed point argument which required global Dirichlet conditions on the whole boundary. So one of the question left open was: is the local regularity valid for $\alpha < 0$?

Theorem 1.1 answers to this question since the following holds:

Proposition 1.1 *Suppose that, for* $\alpha \in (-1, 0)$, *u is a viscosity solution of*

$$|\nabla u|^\alpha F(D^2u) = f(x) \text{ in } \Omega$$

then u is a viscosity solution of

$$F(D^2u) - f(x)|\nabla u|^{-\alpha} = 0 \text{ in } \Omega.$$

The proof is postponed to the appendix, but recall that singular equations require a special definition of viscosity solutions.

Theorem 1.1 concerns continuous viscosity solutions of (1); we should point out that in the case of L^p viscosity solutions (see [9]) it is possible to use a different strategy. Indeed one could prove first, using the argument below, that the solutions are Lipschitz continuous. By Rademacher theorem they are almost everywhere differentiable and hence they will be an L^p viscosity solution of

$$F(D^2u) = g(x)$$

with $g \in L^\infty$. The classical result of Caffarelli [7] implies that the solution are $C^{1,\alpha}$. But this is a different result from ours, since continuous viscosity solutions are L^p viscosity solutions only when g is continuous, which somehow is what we want to prove.

In turn the $C^{1,\alpha}$ regularity implies that g is Hölder continuous, so further regularities can be obtained (see e.g. [6, 14]).

Even for $F(D^2u) = \Delta u$ it would be impossible to mention all the work that has been done on equation of the form

$$F(D^2u) + |\nabla u|^p = f(x).$$

Interestingly most of the literature is concerned with the case $p > 1$. In particular the so called natural growth i.e. $p = 2$ has been much studied in variational contexts and the behaviours are quite different when $p \geq 2$ or $1 < p < 2$. We will just mention the fundamental papers of Lasry and Lions [13] and Trudinger [15]. And more recently the papers of Capuzzo Dolcetta et al. [10] and Barles et al. [2]. In the latter the Hölder regularity of the solution is proved for non local uniformly elliptic operators, and with lower order terms that may be sublinear.

Remark 1.1 Observe that the operator is not Lipschitz continuous with respect to ∇u. This implies that in general uniqueness of the Dirichlet problem does not hold. For example, when Ω is the ball of radius 1, then $u \equiv 0$ and $u(x) = C(1 - |x|^\gamma)$ with $\gamma = \frac{2-\beta}{1-\beta}$ and $C = \gamma^{-1}(\gamma + N - 2)^{\frac{1}{\beta-1}}$ are both solutions of equation

$$\begin{cases} \Delta u + |\nabla u|^\beta = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2 Interior Regularity Results

Let S^N denote the symmetric $N \times N$ matrices. In the whole paper F indicates a uniformly elliptic operator i.e. F satisfies $F(0) = 0$ and, for some $0 < \lambda \leq \Lambda$,

$$\lambda \mathrm{tr} N \leq F(M + N) - F(M) \leq \Lambda \mathrm{tr} N$$

for any $M \in S^N$ and any $N \in S^N$ such that $N \geq 0$. The constants appearing in the estimates below often depend on λ and Λ, but we will not specify them explicitly when it happens.

We recall that we want to prove

Theorem 2.1 *Let f and b continuous in $\overline{B}_1 \subset \Omega$. For any u, bounded viscosity solution of (1) in B_1, and for any $r < 1$ there exist*

$$\gamma = \gamma(\|f\|_\infty, \|b\|_\infty, \beta, \omega_b(\delta)) \textit{ and } C = C(\gamma)$$

such that

$$\|u\|_{\mathscr{C}^{1,\gamma}(B_r)} \leq C\left(\|u\|_\infty + \|b\|_\infty^{\frac{1}{1-\beta}} + \|f\|_\infty\right).$$

Before proving Theorem 2.1, we shall prove a local Lipschitz continuity result.

Lemma 2.1 *Suppose that $H : B_1 \times \mathbb{R}^N \to \mathbb{R}$ is such that*
$H(.,0)$ is bounded in B_1 and there exist $C > 0$ such that for all $q \in \mathbb{R}^N$,

$$|H(x,q) - H(x,0)| \leq C(|q|^\beta + |q|).$$

Then there exists C_o such that if $C < C_o$, any bounded solution u of

$$F(D^2u) + H(x, \nabla u) = f(x) \text{ in } B_1$$

is Lipschitz continuous in B_r, for $r < 1$ with some Lipschitz constant depending on r, $\|f\|_\infty$, C_o and $\|H(.0)\|_\infty$.

Proof of Lemma 2.1 The proof proceeds as in [5, 12]. We outline it here, in order to indicate the changes that need to be done.

Let $r < r' < 1$ and $x_o \in B_r$, we consider on $B_{r'} \times B_{r'}$ the function

$$\Phi(x,y) = u(x) - u(y) - L^2\omega(|x-y|) - L|x-x_o|^2 - L|y-x_o|^2$$

where the continuous function ω is given by $\omega(s) = s - w_o s^{\frac{3}{2}}$ for $s \leq (2/3w_o)^2$ and constant elsewhere; here w_o is chosen in order that $(2/3w_o)^2 > 1$.

The scope is to prove that, for L independent of x_o, chosen large enough,

$$\Phi(x,y) \leq 0 \text{ on } B_r^2. \tag{2}$$

This will imply that u is Lipschitz continuous on B_r by taking $x = x_o$, and letting x_o vary.

So we begin to choose $L > \frac{8 \sup u}{(r'-r)^2}$. Suppose by contradiction that $\Phi(\bar{x},\bar{y}) = \sup \Phi(x,y) > 0$. By the hypothesis on L, $(\bar{x},\bar{y})$ is in the interior of B_r^2. Proceeding in the calculations as in [2] (see also [3, 12]) we get that if (2) is not true then there exist X and Y such that

$$(q_x, X) \in J^{2,+}u(\bar{x}), (q_y, -Y) \in J^{2,-}u(\bar{y})$$

where $\overline{J}^{2,+}, \overline{J}^{2,-}$ are the standard semi-jets, while $q_x = L^2\omega'(|x-y|)\frac{x-y}{|x-y|} + 2L(x-x_o)$ and $q_y = L^2\omega'(|x-y|)\frac{x-y}{|x-y|} - 2L(y-x_o)$.

Then, there exist constant κ_1, κ_2 depending only on $\lambda, \Lambda, \omega_o$ such that

$$\mathscr{M}^+(X+Y) \leq -\kappa_1 L^2$$

and $|q_x|, |q_y| \leq \kappa_2 L^2$.

Using the equation,

$$\begin{aligned}
f(\bar{x}) &\leq H(\bar{x}, q_x) + F(X) \\
&\leq H(\bar{x}, q_x) + F(-Y) + \mathscr{M}^+(X+Y) \\
&\leq f(\bar{y}) - \kappa_1 L^2 \\
&\quad + \|H(.,0)\|_\infty + C(|q_x|^\beta + |q_y|^\beta + |q_x| + |q_y|).
\end{aligned}$$

The term $\|H(.,0)\|_\infty$ is $o(L^2)$, while for $C_o \leq \frac{\kappa_1}{16\kappa_2}$

$$\begin{aligned}
C(|q_x|^\beta + |q_y|^\beta + |q_x| + |q_y|) &\leq \frac{\kappa_1 L^2}{2} + 4C_o(1 + \kappa_2 L^2) \\
&\leq \frac{3\kappa_1 L^2}{4} + 4C_o.
\end{aligned}$$

In conclusion we have obtained that $f(\bar{x}) \leq f(\bar{y}) - \frac{\kappa_1 L^2}{4} + o(L^2)$. This is a contradiction for L large.

Corollary 2.1 *Suppose that $(f_n)_n$ and $(H_n(\cdot, 0))_n$ are sequences converging uniformly respectively to f_∞ and H_∞ on any compact subset of B_1, such that for all $q \in \mathbb{R}^N$,*

$$|H_n(x, q) - H_n(x, 0)| \leq \epsilon_n(|q|^\beta + |q|) \tag{3}$$

with $\epsilon_n \to 0$. Let u_n be a sequence of solutions of

$$F(D^2 u_n) + H_n(x, \nabla u_n) = f_n(x) \text{ in } B_1.$$

If $\|u_n\|_\infty$ is a bounded sequence, then up to subsequences, u_n converges, in any compact subset of B_1, to u_∞ a solution of the limit equation

$$F(D^2 u_\infty) + H_\infty(x) = f_\infty(x) \text{ in } B_1.$$

2.1 Holder Regularity of the Gradient: Main Ingredients

We will follow the line of proof in [5, 12]. The modulus of continuity of a function g is defined by $\omega_g(\delta) = \sup_{[x-y| \leq \delta} |g(x) - g(y)|$. In the following, ω will denote some continuous increasing function on $[0, \delta_o]$ such that $\omega(0) = 0$.

Lemma 2.2 (Improvement of Flatness) *There exist $\epsilon_o \in (0,1)$ and there exists $\rho \in (0,1)$ depending on $(\beta, N, \lambda, \Lambda, \omega)$ such that : for any $\epsilon < \epsilon_o$, for any $p \in \mathbb{R}^N$ and for any f and b such that $\|f\|_\infty \leq \epsilon$, $\|b\|_\infty \leq \epsilon$ and such that*

$\omega_b(\delta) \leq \|b\|_\infty \omega(\delta)$, *if u is a solution of*

$$F(D^2u) + b(x)|\nabla u + p|^\beta = f(x) \text{ in } B_1$$

with $\operatorname{osc}_{B_1} u \leq 1$ *, then there exists* $q^\star \in \mathbb{R}^N$ *such that*

$$\operatorname*{osc}_{B_\rho}(u - q^\star \cdot x) \leq \frac{1}{2}\rho.$$

Proof of Lemma 2.2 We argue by contradiction i.e. we suppose that, for any $n \in \mathbb{N}$, there exist $p_n \in \mathbb{R}^N$, and u_n a solution of

$$F(D^2u_n) + b_n(x)|\nabla u_n + p_n|^\beta = f_n(x) \text{ in } B_1$$

with $\operatorname{osc}_{B_1} u_n \leq 1$ and such that, for any $\rho \in (0, 1)$ and any $q^\star \in \mathbb{R}^N$,

$$\operatorname*{osc}_{B_\rho}(u_n - q^\star \cdot x) \geq \frac{1}{2}\rho.$$

Observe that $u_n - u_n(0)$ satisfies the same equation as u_n, it has oscillation 1 and it is bounded, we can then suppose that the sequence (u_n) is bounded. Suppose first that $|p_n|$ is bounded, so it converges, up to subsequences. Let $v_n(x) = u_n(x) + p_n \cdot x$, which is a solution of

$$F(D^2v_n) + b_n(x)|\nabla v_n|^\beta = f_n(x).$$

We can apply Corollary 2.1 with $H_n(x, q) = b_n(x)|q|^\beta$, since (3) holds.

Hence v_n converges uniformly to v_∞, a solution of the limit equation

$$F(D^2v_\infty) = 0 \text{ in } B_1.$$

Furthermore v_∞ satisfies, for any $\rho \in (0, 1)$ and any $q^\star \in \mathbb{R}^N$,

$$\operatorname*{osc}_{B_\rho}(v_\infty - q^\star \cdot x) \geq \frac{1}{2}\rho. \tag{4}$$

This contradicts the classical $\mathscr{C}^{1,\alpha}$ regularity results, see Evans [11] and Caffarelli [7].

We suppose now that $|p_n|$ goes to infinity. There are two cases, suppose first that $|p_n|^\beta \|b_n\|_\infty$ is bounded. Let $H_n(x, q) = b_n(x)|q + p_n|^\beta$. Since $\omega_{|p_n|^\beta b_n}(\delta) \leq |p_n|^\beta \|b_n\|_\infty \omega(\delta)$, $H_n(x, 0)$ is equicontinuous and up to a subsequence, it converges uniformly to some function $H_\infty(x)$, while $(u_n)_n$ is a uniformly bounded sequence of solutions of

$$F(D^2u_n) + H_n(x, \nabla u_n) = f_n(x).$$

We can apply Corollary 2.1 and up to a subsequence, u_n converges to u_∞ which is a solution of

$$F(D^2u_\infty) + H_\infty(x) = 0.$$

Furthermore u_∞ satisfies (4), for any $\rho \in (0, 1)$ and any $q^\star \in \mathbb{R}^N$. As in the case p_n bounded, this contradicts the classical $\mathscr{C}^{1,\gamma}$ regularity results cited above.

We are left to treat the case where $a_n = |p_n|^\beta \|b_n\|_\infty$ is unbounded. Hence, up to a subsequence, it goes to $+\infty$. We divide the equation by a_n, so $v_n := \frac{u_n}{a_n}$ satisfies

$$F(D^2v_n) + \frac{b_n(x)}{a_n}|a_n\nabla v_n + p_n|^\beta = \frac{f_n(x)}{a_n}.$$

We can apply Corollary 2.1 with

$$H_n(x, q) = b_n(x)a_n^{\beta-1}|q + a_n^{-1}p_n|^\beta.$$

Observe that, $H_n(x, 0) = b_n(x)a_n^{-1}|p_n|^\beta$ is equicontinuous, of L^∞ norm 1 and up to a subsequence, it converges uniformly to some function $H_\infty(x)$.

Passing to the limit one gets that the limit equation is

$$F(0) + H_\infty(x) = 0.$$

This yields a contradiction, since H_∞ has norm 1 and it ends the proof of Lemma 2.2.

The next step is an iteration process which is needed in order to prove Theorem 2.1.

Lemma 2.3 *Given ϵ_o, ω and ρ as in Lemma 2.2. Let b and f be such that $\|f\|_\infty, \|b\|_\infty \le \epsilon_o$ and such that $\omega_b(\delta) \le \|b\|_\infty\omega(\delta)$. Suppose that u is a viscosity solution of*

$$F(D^2u) + b(x)|\nabla u|^\beta = f(x) \text{ in } B_1 \tag{5}$$

and, $\operatorname{osc}_{B_1} u \le 1$. *Then, there exists $\gamma \in (0, 1)$, such that for all $k > 1$, $k \in \mathbb{N}$ there exists $p_k \in \mathbb{R}^N$ such that*

$$\operatorname*{osc}_{B_{r_k}}(u(x) - p_k \cdot x) \le r_k^{1+\gamma} \tag{6}$$

where $r_k := \rho^k$.

The proof is by induction and rescaling. For $k = 0$ just take $p_k = 0$. Suppose now that, for a fixe k, (6) holds with some p_k. Choose $\gamma \in (0, 1)$ such that $\rho^\gamma > \frac{1}{2}$.

Define the function $u_k(x) = r_k^{-1-\gamma}\left(u(r_k x) - p_k \cdot (r_k x)\right)$. By the induction hypothesis, p_k is such that $\text{osc}_{B_1} u_k \leq 1$ and u_k is a solution of

$$F(D^2 u_k) + r_k^{1-\gamma} b(r_k x)|r_k^\gamma(\nabla u_k + p_k r_k^{-\gamma})|^\beta = r_k^{1-\gamma} f(r_k x).$$

Denoting by b_k the function $b_k(x) = r_k^{1-\gamma(1-\beta)} b(r_k x)$ which satisfies $\omega_{b_k}(\delta) = r_k^{1-\gamma(1-\beta)}\omega_b(r_k\delta) \leq r_k^{1-\gamma(1-\beta)}\|b\|_\infty \omega(r_k\delta) \leq \|b_k\|_\infty \omega(\delta)$, the equation above can be written as

$$F(D^2 u_k) + b_k(x)|\nabla u_k + p_k r_k^{-\gamma}|^\beta = r_k^{1-\gamma} f(r_k x).$$

Since the L^∞ norm of $f_k = r_k^{1-\gamma} f(r_k \cdot)$ is less than ϵ, we can conclude that there exists q_k such that

$$\underset{B_\rho}{\text{osc}}(u_k(x) - q_k \cdot x) \leq \frac{1}{2}\rho.$$

So that, for $p_{k+1} = p_k + q_k r_k^{\gamma+1}$,

$$\underset{B_{r_{k+1}}}{\text{osc}}(u(x) - p_{k+1} \cdot x) \leq \frac{\rho}{2} r_k^{1+\gamma} \leq r_{k+1}^{1+\gamma}.$$

This ends the proof of Lemma 2.3.

2.2 Holder Regularity of the Gradient: Conclusion

Lemma 2.4 *Suppose that for any r, there exists p_r such that*

$$\underset{B_r}{\text{osc}}(u(x) - p_r \cdot x) \leq C r^{1+\gamma}$$

then u is $\mathscr{C}^{1,\gamma}$ in 0.

Proof It is clear that it is sufficient to prove that p_r converges when r goes to 0.

We will prove that the sequence $p_{2^{-k}}$ converges and then conclude for the whole sequence. Let $r_k = \frac{1}{2^k}$, since $r_{k+1} < r_k$ for x, y in $\overline{B_{r_{k+1}}}$

$$|u(x) - u(y) - p_{k+1} \cdot (x - y)| \leq C r_{k+1}^{1+\gamma}$$

and

$$|u(x) - u(y) - p_k \cdot (x - y)| \leq C r_k^{1+\gamma}.$$

Subtracting

$$|(p_{k+1} - p_k \cdot x - y)| \leq C(r_{k+1}^{1+\gamma} + r_k^{1+\gamma}).$$

Then, choosing $x = \frac{p_{k+1}-p_k}{|p_{k+1}-p_k|} r_{k+1} = -y$, one gets

$$2|p_{k+1} - p_k| r_{k+1} \leq C(r_{k+1}^{1+\gamma} + r_k^{1+\gamma})$$

which implies

$$|p_{k+1} - p_k| \leq C2r_k^{\gamma}.$$

This proves that the series of general term $(p_{k+1} - p_k)$ converges; hence so does the sequence p_k.

We deduce the convergence of the whole sequence p_ρ when ρ goes to zero. Let k be such that $r^{k+1} \leq \rho \leq r^k$. Then for all $x \in B_\rho$

$$(u(x) - p_\rho \cdot x) \leq C\rho^{1+\gamma} \leq Cr_k^{1+\gamma}$$

and also, since $x \in B_{r^k}$,

$$(u(x) - p_{r^k} \cdot x) \leq Cr_k^{1+\gamma}.$$

Hence, by subtracting, $(p_\rho - p_{r^k}) \cdot x \leq 2Cr_k^{1+\gamma}$. Then, taking $x = \frac{p_\rho - p_{r^k}}{|p_\rho - p_{r^k}|}\rho$, we get $|p_\rho - p_{r^k}| \leq C\frac{r_k^{1+\gamma}}{\rho} \leq C\frac{r_k^{1+\gamma}}{r_{k+1}} = 2Cr_k^{\gamma}$. This implies that p_ρ has the same limit as p_k. This ends the proof of Lemma 2.4.

Suppose now that u is a bounded solution of (5), for general f bounded in L^∞, and b continuous. The function $v(x) = \epsilon u(x)$ with $\epsilon^{-1} = \operatorname{osc} u + \frac{1}{\epsilon_o}(\|f\|_\infty + \|b\|_\infty^{\frac{1}{1-\beta}})$ satisfies the equation

$$F(D^2 v) + b(x)\epsilon^{1-\beta}|\nabla v|^\beta = \epsilon f(x).$$

Our choice of ϵ implies that we are under the conditions of Lemma 2.3, so v is in $\mathscr{C}^{1,\gamma}$, by Lemma 2.4, and so is u.

Appendix

Proof of Proposition 1.1 We assume that $\alpha \in (-1, 0)$ and that u is a supersolution of

$$|\nabla u|^\alpha F(D^2 u) = f(x) \text{ in } \Omega \tag{7}$$

i.e. we suppose that for any $x_o \in \Omega$ either u is locally constant in a neighbourhood of x_o and then $0 \le f$ in that neighbourhood, or, if it is not constant, for any φ test function that touches u by below at x_o and such that $\nabla\varphi(x_o) \neq 0$, we require that

$$|\nabla\varphi(x_o)|^\alpha F(D^2\varphi(x_o)) \le f(x_o).$$

We need to prove that this implies that u is a supersolution of

$$F(D^2u) - f(x)|\nabla u|^{-\alpha} = 0 \text{ in } \Omega, \tag{8}$$

in the usual viscosity sense. Without loss of generality we let $x_o = 0$. If u is constant around 0, $D^2u(0) = 0$ and $Du(0) = 0$, so the conclusion is immediate. If φ is some test function by below at zero such that $\nabla\varphi(0) \neq 0$, the conclusion is also immediate. We then suppose that there exists $M \in S$ such that

$$u(x) \ge u(0) + \frac{1}{2}\langle Mx, x\rangle + o(|x|^2). \tag{9}$$

We want to prove that

$$F(M) \le 0.$$

Let us observe first that one can suppose that M is invertible, since if it is not, it can be replaced by $M_n = M - \frac{1}{n}I$ which satisfies (9) and tends to M.

Let $k > 2$ and $R > 0$ such that

$$\inf_{|x|<R}\left(u(x) - \frac{1}{2}\langle Mx, x\rangle + |x|^k\right) = u(0)$$

where the infimum is strict. We choose $\delta < R$ such that $k(2\delta)^{k-2} < \frac{1}{2}\inf_i|\lambda_i(M)|$. Let ϵ be such that

$$\inf_{\delta<|x|<R}\left(u(x) - \frac{1}{2}\langle Mx, x\rangle + |x|^k\right) = u(0) + \epsilon$$

and let $\delta_2 < \delta$ and such that $k(2\delta)^{k-1}\delta_2 + \|M\|_\infty(\delta_2^2 + 2\delta_2\delta) < \frac{\epsilon}{4}$. Then, for x such that $|x| < \delta_2$,

$$\begin{aligned}\inf_{|y|\le\delta}\{u(y) - \frac{1}{2}\langle M(y-x), y-x\rangle + |y-x|^k\} &\le \inf_{|y|\le\delta}\{u(y) - \frac{1}{2}\langle My, y\rangle + |y|^k\} + \frac{\epsilon}{4}\\ &= u(0) + \frac{\epsilon}{4}\end{aligned}$$

and on the opposite

$$\inf_{R>|y|>\delta}\{u(y)-\frac{1}{2}\langle M(y-x),y-x\rangle+|y-x|^k\}$$
$$\geq \inf_{|y|>\delta}\{u(y)-\frac{1}{2}\langle My,y\rangle+|y|^k\}-\frac{\epsilon}{4}>u(0)+3\frac{\epsilon}{4}.$$

Since the function u is supposed to be non locally constant, there exist x_δ and y_δ in $B(0,\delta_2)$ such that

$$u(x_\delta)>u(y_\delta)-\frac{1}{2}\langle M(x_\delta-y_\delta),x_\delta-y_\delta\rangle+|x_\delta-y_\delta|^k$$

and then the infimum $\inf_{y,|y|\leq\delta}\{u(y)-\frac{1}{2}\langle M(x_\delta-y),x_\delta-y\rangle+|x_\delta-y|^k\}$ is achieved on some point z_δ different from x_δ. This implies that the function

$$\varphi(z):=u(z_\delta)+\frac{1}{2}\langle M(x_\delta-z),x_\delta-z\rangle-|x_\delta-z|^k+\frac{1}{2}\langle M(x_\delta-z_\delta),x_\delta-z_\delta\rangle+|x_\delta-z_\delta|^k$$

touches u by below at the point z_δ. But

$$\nabla\varphi(z_\delta)=M(z_\delta-x_\delta)-k|x_\delta-z_\delta|^{k-2}(z_\delta-x_\delta)\neq 0,$$

indeed, if it was equal to zero, $z_\delta-x_\delta$ would be an eigenvector corresponding to the eigenvalue $k|x_\delta-z_\delta|^{k-2}$ which is supposed to be strictly less than any eigenvalue of M.

Since u is a super-solution of (7), multiplying by $|\nabla\varphi(z_\delta)|^{-\alpha}$, we get

$$F\left(M-\frac{d^2}{dz^2}(|x_\delta-z|^k)(z_\delta)\right)\leq f(z_\delta)|\nabla\varphi(z_\delta)|^{-\alpha}.$$

By passing to the limit for $\delta\to 0$ we obtain the desired conclusion i.e. $F(M)\leq 0$.

We would argue in the same manner for sub-solutions.

References

1. Araujo, D., Ricarte, G., Teixeira, E.: Optimal gradient continuity for degenerate elliptic equations. (2013). Preprint [arXiv:1206.4089]
2. Barles, G., Chasseigne, E., Imbert, C.: Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations. J. Eur. Math. Soc. **13**, 1–26 (2011)
3. Birindelli, I., Demengel, F.: Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators. Commun. Pure Appl. Anal. **6**, 335–366 (2007)
4. Birindelli, I., Demengel, F.: Regularity and uniqueness of the first eigenfunction for singular fully non linear operators. J. Differ. Equ. **249**, 1089–1110 (2010)

5. Birindelli, I., Demengel, F.: $\mathscr{C}^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations. Control Optim. Calc. Var. **20**, 1009–1024 (2014)
6. Cabré, X., Caffarelli, L.: Regularity for viscosity solutions of fully nonlinear equations $F(D^2u) = 0$. Topol. Meth. Nonlinear Anal. **6**, 31–48 (1995)
7. Caffarelli, L.: Interior a priori estimates for solutions of fully nonlinear equations. Ann. Math. 2nd Ser. **130**, 189–213 (1989)
8. Caffarelli, L., Cabré, X.: Fully-Nonlinear Equations Colloquium Publications, vol. 43. American Mathematical Society, Providence (1995)
9. Caffarelli, L., Crandall, M.G., , Kocan, M., Swięch, A.: On viscosity solutions of fully nonlinear equations with measurable ingredients. Commun. Pure Appl. Math. **49**(4), 365–397 (1996)
10. Capuzzo Dolcetta, I., Leoni, F., Porretta, A.: Hölder estimates for degenerate elliptic equations with coercive Hamiltonians. Trans. Am. Math. Soc. **362**(9), 4511–4536 (2010)
11. Evans, L.C.: Classical solutions of fully nonlinear, convex, second-order elliptic equations. Commun. Pure Appl. Math. **25**, 333–363 (1982)
12. Imbert, C., Silvestre, L.: $C^{1,\alpha}$ regularity of solutions of degenerate fully non-linear elliptic equations. Adv. Math. **233**, 196–206 (2013)
13. Lasry, J.M., Lions, P.L.: Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints, I. The model problem. Math. Ann. **283**(4), 583–630 (1989)
14. Teixeira, E.V.: Universal moduli of continuity for solutions to fully nonlinear elliptic equations. Arch. Ration. Mech. Anal. **211**(3), 911–927 (2014)
15. Trudinger, N.S.: Fully nonlinear, uniformly elliptic equations under natural structure conditions. Trans. Am. Math. Soc. **278**, 751–769 (1983)