Uniqueness of Solutions of a Class of Quasilinear Subelliptic Equations

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Dedicated to Ermanno Lanconelli on the occasion of his 70th birthday

Abstract We study the uniqueness problem of the equation,

$$-\Delta_{L,p}u + |u|^{q-1}u = h \quad on \quad \mathbb{R}^N,$$

where $q > p - 1 > 0$. and $N > p$. Uniqueness results proved in this paper hold for equations associated to the mean curvature type operators as well as for more general quasilinear coercive subelliptic problems.

Keywords A priori estimates • Carnot groups • Comparison and uniqueness • Quasilinear elliptic inequalities

AMS Subject Classifications: 35B45, 35B51, 35B53, 35J62, 35J70, 35R03

1 Introduction

Nonlinear elliptic problems of coercive type is still an interesting subject for scholars of nonlinear partial differential equations.

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In [3] the authors studied, among other things, one of the simplest canonical quasilinear coercive problem with non regular data, namely,

$$-\Delta_p u + |u|^{q-1}u = h \quad on \quad \mathbb{R}^N, \tag{1}$$

where $q > p-1 > 0$ and $h \in L^1_{loc}(\mathbb{R}^N)$.

An earlier contribution to this problem in the case $p = 2$ was obtained in [5]. Among other things in [5] it was proved that for the semilinear equation (1), for any $h \in L^1_{loc}(\mathbb{R}^N)$ there exists a unique distributional solution $u \in L^q_{loc}(\mathbb{R}^N)$.

Later on in [3] the Authors studied the general case $p > 1$.

By using an approximation procedure they proved that if $q > p-1$ and $p > 2 - \dfrac{1}{N}$, then for any $h \in L^1_{loc}(\mathbb{R}^N)$ the Eq. (1) possesses a solution belonging to the space

$$X = W^{1,1}_{loc}(\mathbb{R}^N) \cap W^{1,p-1}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N).$$

No general results about uniqueness of solutions were claimed in that paper.

In this work, we shall study the uniqueness problem of solutions of general quasilinear equations of the type

$$-\operatorname{div}_L (\mathscr{A}(x, u(x), \nabla_L u(x))) + \psi^\ell |u|^{q-1}u = h \quad on \quad \mathbb{R}^N, \tag{2}$$

and related qualitative properties in the subelliptic setting (see Sect. 2 for details). The main goal of this paper is to show that the ideas introduced in [10] and developed [11] apply to this more general setting as well.

In this regards we observe that the Eq. (2) contains a weight function ψ which is related to subellipticity of the operator appearing in (2) and may vanish on some unbounded negligible set. Problems containing this kind of degeneracy were not studied in [11].

By using the notations introduced in Sect. 2, we shall prove the uniqueness of solutions of (2) in the space

$$W^{1,p}_{L,loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N) = \{u \in L^p_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N) : |\nabla_L u| \in L^p_{loc}(\mathbb{R}^N)\}.$$

To this end, first we set up two essential tools which are of independent interest.

Namely, the regularity of weak solutions of (2) in the space $W^{1,p}_{L,loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ and comparison principles on $\mathbb{R}^N$. Further we shall derive some properties of the solutions of the problems under consideration.

Our efforts here is to apply an approach that can be useful when dealing with more general operators and related equations or inequalities.

Canonical cases of the main results proved in this paper are the following.

Theorem 1.1 *Let* $1 < p < 2$, $0 \leq \ell \leq p$, $q \geq 1$, $h \in L^1_{loc}(\mathbb{R}^N)$, *then the problem*

$$-\operatorname{div}_L\left(|\nabla_L u|^{p-2}\nabla_L u\right) + \psi^\ell |u|^{q-1}u = h \quad on \quad \mathbb{R}^N,$$

has at most one weak solution $v \in W^{1,p}_{L,loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. *Moreover,*

$$\inf_{\mathbb{R}^N} \frac{h}{\psi^\ell} \leq |v|^{q-1}v \leq \sup_{\mathbb{R}^N} \frac{h}{\psi^\ell}.$$

In the semilinear case we have,

Theorem 1.2 *Let* $0 \leq \ell \leq 2$, $q > 1$, $h \in L^1_{loc}(\mathbb{R}^N)$, *then the problem*

$$-\operatorname{div}_L(\nabla_L u) + \psi^\ell |u|^{q-1}u = h \quad on \quad \mathbb{R}^N,$$

has at most one weak solution $v \in W^{1,2}_{L,loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. *Moreover,*

$$\inf_{\mathbb{R}^N} \frac{h}{\psi^\ell} \leq |v|^{q-1}v \leq \sup_{\mathbb{R}^N} \frac{h}{\psi^\ell}.$$

Theorem 1.3 *Let* $q \geq 1$, $0 \leq \ell \leq 1$, $h \in L^1_{loc}(\mathbb{R}^N)$ *then the problem,*

$$-\operatorname{div}_L\left(\frac{\nabla_L u}{\sqrt{1+|\nabla_L u|^2}}\right) + \psi^\ell |u|^{q-1}u = h \quad on \quad \mathbb{R}^N,$$

has at most one weak solution $v \in W^{1,1}_{L,loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. *Moreover,*

$$\inf_{\mathbb{R}^N} \frac{h}{\psi^\ell} \leq |v|^{q-1}v \leq \sup_{\mathbb{R}^N} \frac{h}{\psi^\ell}.$$

When considering the case $\ell > 1$, we need to look at solutions that belong to a functional space which is smaller than $W^{1,1}_{L,loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$.

We have the following.

Theorem 1.4 *Let* $1 < \ell \leq 2$, $q \geq 1$, $q > \ell - 1$, $h \in L^1_{loc}(\mathbb{R}^N)$ *then the problem,*

$$-\operatorname{div}_L\left(\frac{\nabla_L u}{\sqrt{1+|\nabla_L u|^2}}\right) + \psi^\ell |u|^{q-1}u = h \quad on \quad \mathbb{R}^N,$$

has at most one weak solution $v \in W^{1,\ell}_{L,loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. *Moreover,*

$$\inf_{\mathbb{R}^N} \frac{h}{\psi^\ell} \leq |v|^{q-1}v \leq \sup_{\mathbb{R}^N} \frac{h}{\psi^\ell}.$$

Our uniqueness results concern solutions that belong to the class $W^{1,p}_{L,loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. Of course, this set in the canonical Euclidean case is contained in the space X considered in [3]. We point out that when dealing with uniqueness results additional regularity is required by several Authors. See for instance [1]. Indeed, in that work the Authors obtain the existence of solutions of problem (1) belonging to a certain space $T^{1,p}_0$. Uniqueness of solutions proved in [1] concerns entropy solutions.

The paper is organized as follow. In the next section we describe the setting and the notations. In Sect. 3 we prove some general a priori estimates on the solutions of the problems under consideration.

In Sect. 4 we prove some comparison results and derive some consequences.

Finally in Sect. 5 we discuss an open question and we point out its solution in a special case.

In this paper an important role is played by the **M**-p-**C** operators (see below for the definition). For easy reference, in Sect. 6 we recall some inequalities proved in [11]. These inequalities are of independent interest and will be used throughout the paper when checking that an operator satisfies the **M**-p-**C** property.

2 Notations and Definitions

In this paper ∇ and $|\cdot|$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm.

Let $\mu \in \mathscr{C}(\mathbb{R}^N;\mathbb{R}^l)$ be a matrix $\mu := (\mu_{ij})$, $i = 1,\dots,l$, $j = 1,\dots,N$ and assume that for any $i = 1,\dots,l$, $j = 1,\dots,N$ the derivative $\frac{\partial}{\partial x_j}\mu_{ij} \in \mathscr{C}(\mathbb{R}^N)$. For $i = 1,\dots,l$, let X_i and its formal adjoint X_i^* be defined as

$$X_i := \sum_{j=1}^{N} \mu_{ij}(\xi)\frac{\partial}{\partial \xi_j}, \qquad X_i^* := -\sum_{j=1}^{N} \frac{\partial}{\partial \xi_j}\left(\mu_{ij}(\xi)\cdot\right), \tag{3}$$

and let ∇_L be the vector field defined by

$$\nabla_L := (X_1,\dots,X_l)^T = \mu\nabla,$$

and

$$\nabla_L^* := (X_1^*,\dots,X_l^*)^T.$$

For any vector field $h = (h_1,\dots,h_l)^T \in \mathscr{C}^1(\mathbb{R}^N,\mathbb{R}^l)$, we shall use the following notation $\mathrm{div}_L(h) := \mathrm{div}(\mu^{Th})$, that is

$$\mathrm{div}_L(h) = -\sum_{i=1}^{l} X_i^* h_i = -\nabla_L^* \cdot h.$$

We suppose that the vector fields satisfy the following assumption. Let $\delta := (\delta_1, \ldots, \delta_N)$ be an N-uple of positive real number. We shall denote by δ_R the function $\delta_R : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\delta_R(x) = \delta_R(x_1, \ldots, x_N) := (R^{\delta_1}x_1, \ldots, R^{\delta_N}x_N). \tag{4}$$

We require that ∇_L *is* δ_R*-homogeneous*, that is, there exists $\delta = (\delta_1, \ldots, \delta_N)$ such that ∇_L is pseudo homogeneous of degree 1 with respect to dilation δ_R, namely

$$\nabla_L(\phi(\delta_R(\cdot))) = R(\nabla_L\phi)(\delta_R(\cdot)) \textit{ for } R > 0 \textit{ and } \phi \in \mathscr{C}^1(\mathbb{R}^N).$$

Notice that in the Euclidean framework we have $\mu = I_N$, the identity matrix on $\mathbb{R}^N$. Examples of vector fields satisfying our assumptions are the usual gradient acting on $l(\leq N)$ variables, vector fields related to Bouendi–Grushin operator, Heisenberg–Kohn sub-Laplacian, Heisenberg–Greiner operator, sub-Laplacian on Carnot Groups.

A nonnegative continuous function $S : \mathbb{R}^N \to \mathbb{R}_+$ is called a δ_R*-homogeneous norm* on $\mathbb{R}^N$, if $S(\xi^{-1}) = S(\xi), S(\xi) = 0$ if and only if $\xi = 0$, and it is homogeneous of degree 1 with respect to δ_R (i.e. $S(\delta_R(\xi)) = RS(\xi)$).

An example of smooth homogeneous norm is

$$S(\xi) := \left(\sum_{i=1}^{N}(\xi_i^r)^{\frac{d}{\delta_i}}\right)^{\frac{1}{rd}}, \tag{5}$$

where $d := \delta_1\delta_2\cdots\delta_N$ and r is the lowest even integer such that $r \geq \max\{\delta_1/d, \ldots, \delta_N/d\}$.

Notice that if S is a homogeneous norm differentiable a.e., then $|\nabla_L S|$ is homogeneous of degree 0 with respect to δ_R; hence $|\nabla_L S|$ is bounded.

Throughout this paper we assume that $|\cdot|_L \in \mathscr{C}^1(\mathbb{R}^N \setminus \{0\})$ is a general, however fixed, homogeneous norm.

We denote by B_R the open ball generated by $|\cdot|_L$, that is $B_R := \{\xi \in \mathbb{R}^N : |\xi|_L < R\}$. Since the Jacobian of the map δ_R is $J(\delta_R) = R^Q$ with $Q := \delta_1 + \delta_2 + \ldots \delta_N$, we have $|B_R| = R^Q|B_1|$,

We define $\psi := |\nabla_L|\xi|_L|$ and assume that the set where ψ vanishes is negligible.

The function ψ is bounded and may vanish at some point. For instance in the Euclidean setting, if $|\cdot|_L$ is the Euclidean norm, then $\psi \equiv 1$. If we endow $\mathbb{R}^N$ with the Heisenberg group structure with $\mathbb{R}^N \approx \mathbb{H}^n = \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}_t$, ∇_L is the Heisenberg gradient and $|\cdot|_L$ is the gauge of the canonical sublaplacian, then $\psi^2(\xi) = (|x|^2 + |y|^2)/|\xi|_L^2$ with $\xi = (x, y, t)$.

In what follows we shall assume that $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ is a Carathéodory function, that is for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^l$ the function $\mathscr{A}(\cdot, t, \xi)$ is measurable; and for a.e. $x \in \mathbb{R}^N$, $\mathscr{A}(x, \cdot, \cdot)$ is continuous.

We consider operators L "generated" by $\mathscr{A}$, that is

$$L(u)(x) = \mathrm{div}_L \left(\mathscr{A}(x, u(x), \nabla_L u(x))\right). \tag{6}$$

Our canonical model cases are the p-Laplacian operator, the mean curvature operator and some related generalizations. See Examples 2.1 below.

Definition 2.1 Let $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ be a Carathéodory function. The function $\mathscr{A}$ is called *weakly elliptic* if it generates a weakly elliptic operator L i.e.

$$\begin{aligned} &\mathscr{A}(x,t,\xi)\cdot\xi \geq 0 \ \textit{for each}\ x\in\mathbb{R}^N,\ t\in\mathbb{R},\ \xi\in\mathbb{R}^l,\\ &\mathscr{A}(x,0,\xi)=0 \ \textit{ or } \ \mathscr{A}(x,t,0)=0. \end{aligned} \tag{WE}$$

Let $p \geq 1$, the function $\mathscr{A}$ is called **W**-p-**C** (weakly-p-coercive) (see [2]), if $\mathscr{A}$ is (WE) and it generates a weakly-p-coercive operator L, i.e. if there exists a constant $k_2 > 0$ such that

$$(\mathscr{A}(x,t,\xi)\cdot\xi)^{p-1} \geq k_2^p|\mathscr{A}(x,t,\xi)|^p \text{ for each } x\in\mathbb{R}^N,\ t\in\mathbb{R},\ \xi\in\mathbb{R}^l. \tag{W-p-C}$$

Let $p > 1$, the function $\mathscr{A}$ is called **S-**p**-C** (strongly-p-coercive) (see [2, 13, 14]), if there exist $k_1, k_2 > 0$ constants such that

$$(\mathscr{A}(x,t,\xi)\cdot\xi) \geq k_1|\xi|^p \geq k_2^{p'}|\mathscr{A}(x,t,\xi)|^{p'} \text{ for each } x\in\mathbb{R}^N,\ t\in\mathbb{R},\ \xi\in\mathbb{R}^l. \tag{S-p-C}$$

We look for solution in the space $W^{1,p}_{L,loc}(\Omega)$ defined as

$$W^{1,p}_{L,loc}(\Omega) := \{u \in L^p_{loc}(\Omega) : |\nabla_L u| \in L^p_{loc}(\Omega)\}.$$

Definition 2.2 Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Carathéodory function. Let $p \geq 1$. We say that $u \in W^{1,p}_{L,loc}(\Omega)$ is a *weak solution* of

$$\mathrm{div}_L\left(\mathscr{A}(x,u,\nabla_L u)\right) \geq f(x,u,\nabla_L u) \qquad \textit{on } \ \Omega,$$

if $\mathscr{A}(\cdot,u,\nabla_L u) \in L^{p'}_{loc}(\Omega)$, $f(\cdot,u,\nabla_L u) \in L^1_{loc}(\Omega)$, and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have

$$-\int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L\phi \geq \int_\Omega f(x,u,\nabla_L u)\phi.$$

Example 2.1

1. Let $p > 1$. The p-Laplacian operator defined on suitable functions u by,

$$\Delta_p u = \mathrm{div}_L\left(|\nabla_L u|^{p-2}\nabla_L u\right)$$

is an operator generated by $\mathscr{A}(x,t,\xi) := |\xi|^{p-2}\xi$ which is **S**-p-**C**.

2. If $\mathscr{A}$ is of mean curvature type, that is $\mathscr{A}$ can be written as $\mathscr{A}(x,t,\xi) := A(|\xi|)\xi$ with $A : \mathbb{R} \to \mathbb{R}$ a positive bounded continuous function (see [2, 12]), then $\mathscr{A}$ is **W-2-C**.
3. The mean curvature operator in non parametric form

$$Tu := \operatorname{div}_L \left(\frac{\nabla_L u}{\sqrt{1 + |\nabla_L u|^2}} \right),$$

is generated by $\mathscr{A}(x,t,\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}$. In this case $\mathscr{A}$ is **W**-p-**C** with $1 \le p \le 2$ and of mean curvature type but it is not **S-2-C**.
4. Let $m > 1$. The operator

$$T_m u := \operatorname{div}_L \left(\frac{|\nabla_L u|^{m-2} \nabla_L u}{\sqrt{1 + |\nabla_L u|^m}} \right)$$

is **W**-p-**C** for $m \ge p \ge m/2$.

Definition 2.3 Let $\mathscr{A} : \mathbb{R}^N \times \mathbb{R}^l \to \mathbb{R}^l$ be a Charateodory function. We say that $\mathscr{A}$ is monotone if

$$(\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)) \cdot (\xi - \eta) \ge 0 \qquad \text{for } \xi, \eta \in \mathbb{R}^l. \tag{7}$$

Let $p \ge 1$. We say that $\mathscr{A}$ is **M**-p-**C** (monotone p-coercive) if $\mathscr{A}$ is monotone and if there exists $k_2 > 0$ such that

$$((\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)) \cdot (\xi - \eta))^{p-1} \ge k_2^p |\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)|^p. \tag{8}$$

Example 2.2

1. Let $1 < p \le 2$ the function $\mathscr{A}(\xi) := |\xi|^{p-2}\xi$ is **M**-p-**C** (see Sect. 6 for details).
2. The mean curvature operator is **M**-p-**C** with $1 \le p \le 2$ (see Sect. 6).

In what follows we shall use a special family of test functions that we call *cut-off functions*. More precisely, let $\varphi_1 \in \mathscr{C}_0^1(\mathbb{R})$ be such that $0 \le \varphi_1 \le 1$, $\varphi_1(t) = 0$ if $|t| \ge 2$ and $\varphi_1(t) = 1$ if $|t| \le 1$. Next, for $R > 0$ by *cut-off function* we mean the function φ_R defined as $\varphi_R(x) = \varphi_1(|x|_L/R)$.

Finally, if not otherwise stated, the integrals are considered on the whole space $\mathbb{R}^N$.

3 A Priori Estimates

The following is a slight variation of a result proved in [10]. For easy reference we shall include its detailed proof.

Consider the following inequality,

$$\operatorname{div}_L(\mathscr{A}(x,v,\nabla_L v)) - f \ge \operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) - g \qquad \text{on } \mathbb{R}^N. \tag{9}$$

We have,

Theorem 3.1 *Let $p \geq 1$ and let $\mathscr{A} : \mathbb{R}^N \times \mathbb{R}^l \to \mathbb{R}^l$ be **M**-p-**C**. Let $f, g \in L^1_{loc}(\mathbb{R}^N)$ and let (u, v) be weak solution of (9). Set $w := (v-u)^+$ and let $s > 0$ and $p \geq \ell \geq 0$. If $(f-g)w \geq 0$ and*

$$w^{s+p-1}\psi^\ell \in L^1(B_{2R} \setminus B_R) \qquad \textit{for } R \textit{ large}, \tag{10}$$

then

$$(f-g)w^s, \ \ (\mathscr{A}(x, \nabla_L v) - \mathscr{A}(x, \nabla_L u)) \cdot \nabla_L w \, w^{s-1} \chi_{\{w>0\}} \in L^1_{loc}(\mathbb{R}^N). \tag{11}$$

Moreover, for any nonnegative $\phi \in \mathscr{C}^1_0(\mathbb{R}^N)$ we have,

$$\int (f-g)w^s\phi + c_1 s \int (\mathscr{A}(x, \nabla_L v) - \mathscr{A}(x, \nabla_L u)) \cdot \nabla_L w \, w^{s-1}\phi \leq c_2 s^{1-p} \int w^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}}, \tag{12}$$

where $c_1 = 1 - \frac{p-1}{p}\left(\frac{\epsilon}{k_2}\right)^{\frac{p}{p-1}} > 0$, $c_2 = \frac{p^p}{p\epsilon^p}$ and $\epsilon > 0$ is sufficiently small for $p > 1$ and $c_1 = 1$ and $c_2 = 1/k_2$ for $p = 1$.

Remark 3.1

i) Notice that from the above result it follows that if $u, v \in W^{1,p}_{L,loc}(\mathbb{R}^N)$ is a weak solution of (9), then $(f-g)\, w \in L^1_{loc}(\mathbb{R}^N)$.
ii) The above lemma still holds if we replace the function $f - g \in L^1_{loc}(\mathbb{R}^N)$ with a regular Borel measure on $\mathbb{R}^N$.
iii) The right hand side in (12) could be divergent since we know only that $w^{s+p-1}\psi^\ell \in L^1_{loc}(\mathbb{R}^N)$.
iv) If in Theorem 3.1 we consider the case $\ell = 0$, then Theorem 3.1 can be restated for inequalities (9) on a open set Ω by replacing $\mathbb{R}^N$ with Ω and requiring that $w^{s+p-1} \in L^1_{loc}(\Omega)$.
v) If (u, v) is a weak solution of (9) and u is a constant i.e. $u \equiv const$, then Theorem 3.1 still holds even for **W**-p-**C** operators. See the following Lemma 3.1.

Lemma 3.1 *Let $p \geq 1$ and let $\mathscr{A}$ be **W**-p-**C**. Let $f, g \in L^1_{loc}(\mathbb{R}^N)$ and let $v \in W^{1,p}_{L,loc}(\mathbb{R}^N)$ be a weak solution of*

$$\mathrm{div}_L\,(\mathscr{A}(x, u, \nabla_L u)) \geq f - g, \qquad \textit{on } \mathbb{R}^N. \tag{13}$$

Let $k > 0$ and set $w := (v-k)^+$ and let $s > 0$, $p \geq \ell \geq 0$. If $(f-g)w \geq 0$ and (10) holds, then

$$(f-g)w^s, \ \ \mathscr{A}(x, v, \nabla_L v) \cdot \nabla_L w \, w^{s-1} \chi_{\{w>0\}} \in L^1_{loc}(\mathbb{R}^N) \tag{14}$$

and for any nonnegative $\phi \in \mathscr{C}_0^1(\mathbb{R}^N)$ *we have,*

$$\int (f-g)w^s\phi + c_1 s \int \mathscr{A}(x,v,\nabla_L v)\cdot\nabla_L w\, w^{s-1}\phi \le c_2 s^{1-p} \int w^{s+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}, \qquad (15)$$

where c_1 *and* c_2 *are as in Theorem 3.1.*

The above lemma lies on the following result proved in [10, Theorem 2.7].

Theorem 3.2 ([10]) *Let* $\mathscr{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ *be a monotone Carathéodory function. Let* $f, g \in L^1_{loc}(\Omega)$ *and let* u, v *be weak solution of*

$$\mathrm{div}_L\,(\mathscr{A}(x,v,\nabla_L v)) - f \ge \mathrm{div}_L\,(\mathscr{A}(x,u,\nabla_L u)) - g \qquad \text{on } \Omega. \qquad (16)$$

Let $\gamma \in \mathscr{C}^1(\mathbb{R})$ *be such that* $0 \le \gamma(t), \gamma'(t) \le M$*, then*

$$-\int_\Omega (\mathscr{A}(x,v,\nabla_L v) - \mathscr{A}(x,u,\nabla_L u))\cdot\nabla_L\phi\;\gamma(v-u) \ge \qquad (17)$$

$$\ge \int_\Omega \gamma'(v-u)\,(\nabla_L v - \nabla_L u)\cdot(\mathscr{A}(x,v,\nabla_L v) - \mathscr{A}(x,u,\nabla_L u)\,\phi \qquad (18)$$

$$+\int_\Omega \phi\gamma(v-u)(f-g) \quad \text{on } \Omega. \qquad (19)$$

Hence

$$\mathrm{div}_L\,(\gamma(v-u)(\mathscr{A}(x,v,\nabla_L v) - \mathscr{A}(x,u,\nabla_L u))) \ge \gamma(v-u)(f-g) \quad \text{on } \Omega.$$

Moreover[1]

$$\mathrm{div}_L\left(\mathrm{sign}^+(v-u)(\mathscr{A}(x,v,\nabla_L v) - \mathscr{A}(x,u,\nabla_L u))\right) \ge \mathrm{sign}^+(v-u)(f-g) \quad \text{on } \Omega. \qquad (20)$$

Proof (of Theorem 3.1) Let $\gamma \in \mathscr{C}^1(\mathbb{R})$ be a bounded nonnegative function with bounded nonnegative first derivative and let $\phi \in \mathscr{C}_0^1(\Omega)$ be a nonnegative test function.

For simplicity we shall omit the arguments of $\mathscr{A}$. So we shall write $\mathscr{A}_u$ and $\mathscr{A}_v$ instead of $\mathscr{A}(x,\nabla_L u)$ and $\mathscr{A}(x,\nabla_L v)$ respectively.

[1] We recall that the function sign^+ is defined as $\mathrm{sign}^+(t) := 0$ if $t \le 0$ and $\mathrm{sign}^+(t) := 1$ otherwise.

Applying Lemma 3.2, we obtain

$$\int (f-g)\gamma(w)\phi + \int (\mathscr{A}_v - \mathscr{A}_u)\cdot\ \nabla_L w\, \gamma'(w)\phi \le -\int (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla_L \phi\, \gamma(w) \le \int |\mathscr{A}_v - \mathscr{A}_u|\ |\nabla_L\phi|\gamma(w). \tag{21}$$

Let $p > 1$. From (21) we have

$$\int (f-g)\gamma(w)\phi + \int (\mathscr{A}_v - \mathscr{A}_u)\cdot\ \nabla_L w\, \gamma'(w)\phi \le$$
$$\le \left(\int |\mathscr{A}_v - \mathscr{A}_u|^{p'}\gamma'(w)\phi\right)^{1/p'} \left(\int \frac{\gamma(w)^p}{\gamma'(w)^{p-1}}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}\right)^{1/p}$$
$$\le \frac{\epsilon^{p'}}{p'k_2^{p'}}\int (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla_L w\, \gamma'(w)\phi + \frac{1}{p\epsilon^p}\int \frac{\gamma(w)^p}{\gamma'(w)^{p-1}}\frac{|\nabla_L\phi|^p}{\phi^{p-1}},$$

where $\epsilon > 0$ and all integrals are well defined provided $\frac{\gamma(w)^p}{\gamma'(w)^{p-1}} \in L^1_{loc}(\Omega)$. With a suitable choice of $\epsilon > 0$, for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ and $\gamma \in \mathscr{C}^1(\mathbb{R})$ as above such that $\frac{\gamma(w)^p}{\gamma'(w)^{p-1}} \in L^1_{loc}(\Omega)$, it follows that,

$$\int (f-g)\gamma(w)\phi + c_1\int (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla_L w\, \gamma'(w)\phi \le \frac{1}{p\epsilon^p}\int \frac{\gamma(w)^p}{\gamma'(w)^{p-1}}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}. \tag{22}$$

Now for $s > 0$, $1 > \delta > 0$ and $n \ge 1$, define

$$\gamma_n(t) := \begin{cases} (t+\delta)^s & \textit{if } 0 \le t < n-\delta, \\ cn^s - \dfrac{s}{\beta-1}n^{\beta+s-1}(t+\delta)^{1-\beta} & \textit{if } t \ge n-\delta, \end{cases} \tag{23}$$

where $c := \frac{\beta-1+s}{\beta-1}$ and $\beta > 1$ will be chosen later. Clearly $\gamma_n \in \mathscr{C}^1$,

$$\gamma_n'(t) = \begin{cases} s(t+\delta)^{s-1} & \textit{if } 0 \le t < n-\delta, \\ sn^{\beta+s-1}(t+\delta)^{-\beta} & \textit{if } t \ge n-\delta, \end{cases}$$

and γ_n, γ_n' are nonnegative and bounded with $||\gamma_n||_\infty = cn^s$ and $||\gamma_n'||_\infty = sn^{s-1}$. Moreover

$$\frac{\gamma_n(t)^p}{\gamma_n'(t)^{p-1}} = \begin{cases} s^{1-p}(t+\delta)^{s+p-1} & \textit{for } t < n-\delta, \\ \theta(t,n) & \textit{for } t \ge n-\delta, \end{cases}$$

where

$$\theta(t,n) := \frac{(cn^s - \frac{s}{\beta-1} n^{\beta+s-1}(t+\delta)^{1-\beta})^p}{(sn^{\beta+s-1}(t+\delta)^{-\beta})^{p-1}} \le (cn^s)^p s^{1-p} n^{-(\beta+s-1)(p-1)} \, (t+\delta)^{\beta(p-1)}.$$

Choosing $\beta := \frac{s+p-1}{p-1}$ we have $c = p$, and

$$\theta(t,n) \le p^p s^{1-p} n^{sp-(\beta+s-1)(p-1)} (t+\delta)^{s+p-1} = p^p s^{1-p} (t+\delta)^{s+p-1}.$$

Therefore, for $t \ge 0$ we have,

$$\frac{\gamma_n(t)^p}{\gamma_n'(t)^{p-1}} \le p^p s^{1-p} (t+\delta)^{s+p-1}.$$

Since by assumption $w^{s+p-1} \in L^1_{loc}(\Omega)$, from (22) with $\gamma = \gamma_n$, it follows that

$$\int (f-g)\gamma_n(w)\phi + c_1 \int (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla_L w \, \gamma_n'(w)\phi \le \frac{p^p s^{1-p}}{p\epsilon^p} \int (w+\delta)^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}}.$$

Now, noticing that $\gamma_n(t) \to (t+\delta)^s$ and $\gamma_n'(t) \to s(t+\delta)^{s-1}$ as $n \to +\infty$, $(f-g)(\gamma_n(w) - \gamma_n(0) \ge 0$ and $\mathscr{A}$ is monotone (that is $(\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla_L w \ge 0$), by Fatou's Lemma theorem we obtain

$$\int (f-g)\,(w+\delta)^s \phi + c_1 s \int (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla_L w \, (w+\delta)^{s-1}\phi \le c_2 s^{1-p} \int (w+\delta)^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}}.$$

By letting $\delta \to 0$ in the above inequality, we have the inequality (12).

Next, we choose $R > 0$ large enough and $\phi := \varphi_R^p$ with φ_R a cut off function, that is

$$\phi(x) := (\varphi_R(x))^p := (\varphi_1(|x|_L/R))^p.$$

With these choice we have

$$\frac{|\nabla_L \phi|^p}{\phi^{p-1}} = p^p \psi^p R^{-p} |\varphi_1'|^p \left(\frac{|x|_L}{R}\right) \le p^p ||\psi||^{p-\ell} ||\varphi_1'||^p_\infty R^{-p} \psi^\ell =: c_3 \psi^\ell,$$

and from (12) we deduce

$$\int_{B_R} (f-g)\, w^s + c_1 s \int_{B_R} (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla_L w \, w^{s-1} \le c_2 s^{1-p} c_3 \int_{B_{2r}\setminus B_R} w^{s+p-1} \psi^\ell,$$

which completes the proof of the claim in the case $p > 1$.

Let $p = 1$. From (21) and the fact that $\mathscr{A}_v - \mathscr{A}_u$ is bounded, the estimate (22) holds provided we replace p with 1 and ϵ with k_2. The remaining argument is similar to the case $p > 1$, hence we shall omit it.

Lemma 3.2 *Let $p \geq 1$ and let $\mathscr{A} : \mathbb{R}^N \times \mathbb{R}^l \to \mathbb{R}^l$ be **M**-p-**C**. Let $f, g \in L^1_{loc}(\mathbb{R}^N)$ and let (u, v) be weak solution of (9). Set $w := (v-u)^+$. If $(f-g)w \geq 0$ and $w^q\psi^\ell \in L^1(B_{2R} \setminus B_R)$ for $q > p-1$, $p \geq \ell \geq 0$ and $R > 0$ large, then*

$$(f-g)w^{q-p+1}, \quad ((\mathscr{A}(x, \nabla_L v) - \mathscr{A}(x, \nabla_L u)) \cdot \nabla_L w \, w^{q-p} \chi_{\{w>0\}} \in L^1_{loc}(\mathbb{R}^N), \tag{24}$$

and for any $\varphi_R \in \mathscr{C}^1_0(\mathbb{R}^N)$ cut-off function, for R large enough, we have,

$$\int (f-g) \operatorname{sign}^+(w) \, \varphi_R^\sigma \leq c \left(\int_{B_{2R} \setminus B_R} w^q \psi^\ell \varphi_R^\sigma \right)^{\frac{p-1}{q}} R^{Q(\frac{q-p+1}{q})-p}, \tag{25}$$

where $c = c(\sigma, k_2, p, q, ||\psi||_\infty, \ell)$ and $\sigma \geq \frac{pq}{q-p+1-s}$, $0 < s < \min\{1, q-p+1\}$.

Proof The claim (24) follows from Theorem 3.1.

Let $s > 0$ be such that $q \geq s + p - 1$. From Lemma 3.1, for any nonnegative $\phi \in \mathscr{C}^1_0(\mathbb{R}^N)$, we have

$$\int (f-g)w^s\phi + c_1 s \int (\mathscr{A}_v - \mathscr{A}_u) \cdot \nabla_L w \, w^{s-1}\phi \leq c_2 s^{1-p} \int_S w^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}}, \tag{26}$$

where, as in the proof of Theorem 3.1, we write $\mathscr{A}_v$ and $\mathscr{A}_v$ for $\mathscr{A}(x, \nabla_L v)$ and $\mathscr{A}(x, \nabla_L u)$ respectively and S is the support of $|\nabla_L \phi|$.

Next, an application of Theorem 3.2 gives (20). That is

$$\operatorname{div}_L \left(\operatorname{sign}^+(v-u)(\mathscr{A}(x, v, \nabla_L v) - \mathscr{A}(x, u, \nabla_L u)) \right) \geq \operatorname{sign}^+(v-u)(f-g) \quad \text{on } \mathbb{R}^N. \tag{27}$$

Now we consider the case $p > 1$. Let $0 < s < \min\{1, q-p+1\}$. By definition of weak solution and Hölder's inequality with exponent p', taking into account that $\mathscr{A}$ is **M**-p-**C** and from (26) we get,

$$\int \operatorname{sign}^+ w(f-g)\,\phi \leq \int_S |\mathscr{A}_v - \mathscr{A}_u||\nabla_L \phi| \operatorname{sign}^+ w \tag{28}$$

$$= \int_S |\mathscr{A}_v - \mathscr{A}_u| w^{\frac{s-1}{p'}} \phi^{\frac{1}{p'}} \, |\nabla_L \phi| w^{\frac{1-s}{p'}} \phi^{-\frac{1}{p'}} \tag{29}$$

$$\leq \frac{1}{k_2} \left(\int_S (\mathscr{A}_v - \mathscr{A}_u) \cdot \nabla_L w \, w^{s-1} \phi \right)^{1/p'} \left(\int_S w^{(1-s)(p-1)} \frac{|\nabla_L \phi|^p}{\phi^{p-1}} \right)^{1/p} \tag{30}$$

$$\leq \frac{1}{k_2} \left(\frac{c_2}{c_1 s^p} \right)^{1/p'} \left(\int_S w^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}} \right)^{1/p'} \left(\int_S w^{(1-s)(p-1)} \frac{|\nabla_L \phi|^p}{\phi^{p-1}} \right)^{1/p}. \tag{31}$$

Since $q > s + p - 1$ and $q > p - 1$, applying Hölder inequality to (31) with exponents $\chi := \frac{q}{s+p-1}$ and $y := \frac{q}{(1-s)(p-1)}$, we obtain

$$\int \operatorname{sign}^+ w\,(f-g)\phi \le c_3' \left(\int_S w^q \psi^\ell \phi\right)^\delta \left(\int_S \frac{|\nabla_L \phi|^{p\chi'}}{\psi^{\chi'-1}\phi^{p\chi'-1}}\right)^{\frac{1}{p'\chi'}} \left(\int_S \frac{|\nabla_L \phi|^{py'}}{\chi^{y'-1}\phi^{py'-1}}\right)^{\frac{1}{py'}}, \tag{32}$$

where

$$\delta := \frac{1}{\chi p'} + \frac{1}{yp} = \frac{p-1}{q}, \qquad c_3' := \left(\frac{c_2}{c_1 s^p}\right)^{1/p'} \frac{1}{k_2}.$$

Next for $\sigma \ge p\chi'$ (notice that $p\chi' > py'$ implies $\sigma > py'$), we choose $\phi := \varphi_R^\sigma$. From (32) it follows that $S = B_{2R} \setminus B_R$ and

$$\int \operatorname{sign}^+ w\,(f-g)\varphi_R^\sigma \le c_3'\sigma^p \left(\int_S w^q \psi^\ell \varphi_R^\sigma\right)^\delta \times$$

$$\times \left(\int_S \psi^p R^{-p\chi'} |\varphi_1'|^{p\chi'} (\frac{|x|_L}{R})\right)^{\frac{1}{p'\chi'}} \left(\int_S \psi^p R^{-py'} |\varphi_1'|^{py'} (\frac{|x|_L}{R})\right)^{\frac{1}{py'}}$$

$$\le c_3'\sigma^p \left(\int_S w^q \psi^\ell \varphi_R^\sigma\right)^\delta ||\psi||_\infty^{\frac{p}{p'\chi'}+\frac{p}{py'}} R^{-\frac{p\chi'}{p'\chi'}-\frac{py'}{py'}} ||\varphi_1'||_\infty^{\frac{p\chi'}{p'\chi'}+\frac{py'}{py'}} |B_{2R} \setminus B_R|^{\frac{1}{p'\chi'}+\frac{1}{py'}}$$

$$\le c \left(\int_S w^q \psi^\ell \varphi_R^\sigma\right)^\delta R^{Q(1-\delta)-p},$$

completing the proof of (25).

Now, we assume that $p = 1$. From (28), with the choice $\phi := \varphi_R^\sigma$, we have

$$\int \operatorname{sign}^+ w\,(f-g)\varphi_R^\sigma \le \frac{\sigma}{k_2} \int_S |\nabla_L \varphi_R| \le cR^{Q-1},$$

which completes the proof.

4 Comparison and Uniqueness

In this section we prove a comparison principle and its implication on the uniqueness property.

Consider the following inequality,

$$\operatorname{div}_L (\mathscr{A}(x, \nabla_L v)) - \psi^\ell |v|^{q-1} v \ge \operatorname{div}_L (\mathscr{A}(x, \nabla_L u)) - \psi^\ell |u|^{q-1} u \qquad on\ \mathbb{R}^N. \tag{33}$$

As preliminary result we have the following.

Lemma 4.1 *Let $p \geq 1$, let $\mathscr{A}$ be **M**-p-**C**, $q \geq 1$ and $q > p-1$.*

Let (u, v) be weak solution of (33) with $p \geq \ell \geq 0$. Then $((v-u)^+)^r \psi^\ell \in L^1_{loc}(\mathbb{R}^N)$ for any $r < +\infty$.

Proof Let (u, v) be a solution of (33) and set $w := (v-u)^+$. By using the well known inequality

$$|t|^{q-1}t - |s|^{q-1}s \geq c_q(t-s)^q, \quad \textit{for } \ t \geq s \qquad (q \geq 1), \tag{34}$$

we deduce that $w^q\psi^\ell \in L^1_{loc}(\mathbb{R}^N)$. From this it follows that we are in the position to apply Theorem 3.1, with $s = q-p+1$ obtaining $w^{q_1}\psi^\ell \in L^1_{loc}(\mathbb{R}^N)$ with $q_1 := 2q-p+1$. Applying again Theorem 3.1, with $s = q_1 - p + 1$, we get $w^{q_2}\psi^\ell \in L^1_{loc}(\mathbb{R}^N)$ with $q_2 := q_1 + q - p + 1 = q + 2(q-p+1)$. Iterating j times we have that $w^{q_j}\psi^\ell \in L^1_{loc}(\mathbb{R}^N)$ with $q_j := q + j(q-p+1)$. By choosing j sufficiently large we get the claim.

Theorem 4.1 *Let $p \geq 1$, let $\mathscr{A}$ be **M**-p-**C**, $q \geq 1$, $q > p-1$ and $p \geq \ell \geq 0$. Let (u, v) be a weak solution of*

$$\operatorname{div}_L(\mathscr{A}(x, \nabla_L v)) - \psi^\ell |v|^{q-1}v \geq \operatorname{div}_L(\mathscr{A}(x, \nabla_L u)) - \psi^\ell |u|^{q-1}u \qquad \textit{on } \mathbb{R}^N. \tag{35}$$

Then $v \leq u$ a.e. on $\mathbb{R}^N$.

In particular if (u, v) be a weak solution of

$$\operatorname{div}_L(\mathscr{A}(x, \nabla_L v)) - \psi^\ell |v|^{q-1}v = \operatorname{div}_L(\mathscr{A}(x, \nabla_L u)) - \psi^\ell |u|^{q-1}u \qquad \textit{on } \mathbb{R}^N, \tag{36}$$

then $u \equiv v$ a.e. on $\mathbb{R}^N$.

Proof Let (u, v) be a solution of (35) and set $w := (v-u)^+$. From Lemma 4.1 we know that $w^r\psi^\ell \in L^1_{loc}(\mathbb{R}^N)$ for any r, and hence we are in the position to apply Theorem 3.1 with s large enough. Thus, from (34) and (12) we get $w^{q+s}\psi^\ell \in L^1_{loc}(\mathbb{R}^N)$ and

$$\int w^{q+s}\psi^\ell\phi \leq c(s, q, p)\int w^{s+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}.$$

Applying the Hölder inequality with exponent $x := \frac{q+s}{s+p-1} > 1$ we have

$$\int w^{q+s}\psi^\ell\phi \leq c(s, q, p)\int \psi^{p(1-x')}\frac{|\nabla_L\phi|^{px'}}{\phi^{px'-1}}.$$

By the same choice of ϕ we made in the proof of Theorem 3.1, that is $\phi = \varphi_R$ a cut off functions, it follows that

$$\int_{B_R} w^{q+s}\psi^\ell \le cR^{Q-px'} = cR^{Q-p(q+s)/(q-p+1)}.$$

Choosing s large enough and letting $R \to +\infty$, we have that $w \equiv 0$ a.e. on $\mathbb{R}^N$. This completes the proof.

Corollary 4.1 *Let $p \ge 1$, let $\mathscr{A}$ be **W**-p-**C** such that $\mathscr{A}(x,0) = 0$. Let q and ℓ be as in Theorem 4.1. Let $h \in L^1_{loc}(\mathbb{R}^N)$. Let v be a weak solution of the problem*

$$-\operatorname{div}_L(\mathscr{A}(x,\nabla_L v)) + \psi^\ell |v|^{q-1}v = h. \tag{37}$$

Then,

$$\inf_{\mathbb{R}^N}\frac{h}{\psi^\ell} \le |v|^{q-1}v \le \sup_{\mathbb{R}^N}\frac{h}{\psi^\ell}.$$

In particular, if $h \ge 0$ [resp. ≤ 0], then $v \ge 0$ [resp. ≤ 0] and if $\frac{h}{\psi^\ell} \in L^\infty(\mathbb{R}^N)$, then $v \in L^\infty(\mathbb{R}^N)$.

Proof We shall prove only the estimate

$$|v|^{q-1}v \le \sup_{\mathbb{R}^N}\frac{h}{\psi^\ell},$$

the proof of the other inequality being similar. If $\sup_{\mathbb{R}^N}\frac{h}{\psi^\ell} = +\infty$ there is nothing to prove. Let $M := \sup_{\mathbb{R}^N}\frac{h}{\psi^\ell} < +\infty$. We define $u := \operatorname{sign}(M)|M|^{1/q}$. Then

$$\operatorname{div}_L(\mathscr{A}(x,\nabla_L v)) - \psi^\ell|v|^{q-1}v + h = 0 \ge h - \psi^\ell M = \operatorname{div}_L(\mathscr{A}(x,\nabla_L u)) - \psi^\ell|u|^{q-1}u + h,$$

that is (u, v) satisfy (35) with u constant. In this case all the previous estimates still hold since in this case the operator can be seen as it were **M**-p-**C**. See also Remark 3.1 and Lemma 3.1.

Thus the claim follows from Theorem 4.1.

Corollary 4.2 *Let $p \ge 1$ and let $\mathscr{A}$ be **M**-p-**C**. Let q and ℓ be as in Theorem 4.1. Let $h \in L^1_{loc}(\mathbb{R}^N)$. Then the possible weak solution of the problem (37) is unique.*

Moreover if $\mathscr{A}(x,0) = 0$ and v is a solution of (37), then

$$\inf_{\mathbb{R}^N}\frac{h}{\psi^\ell} \le |v|^{q-1}v \le \sup_{\mathbb{R}^N}\frac{h}{\psi^\ell}.$$

Proof Uniqueness. Let u and v two solutions of (37). Then (u, v) solves

$$\operatorname{div}_L(\mathscr{A}(x, \nabla_L v)) - \psi^\ell |v|^{q-1} v = \operatorname{div}_L(\mathscr{A}(x, \nabla_L u)) - \psi^\ell |u|^{q-1} u \qquad \text{on } \mathbb{R}^N,$$

and applying Theorem 4.1 we conclude that $u \equiv v$.

The remaining claim follows from Corollary 4.1.

5 Further Applications

5.1 Symmetry Results

An application of Theorem 4.1 to the symmetry of solutions is the following.

Proposition 5.1 *Let $p \geq 1$. Let $\mathscr{A}$ be **M**-p-**C** and Let L be the operator generated by $\mathscr{A}$, see (6). Let q be as in Theorem 4.1.*

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a map which leaves L invariant, that is

$$\operatorname{div}_L(\mathscr{A}(x, \nabla_L(\phi(\Phi(x))))) = \operatorname{div}_L(\mathscr{A}(\cdot, \nabla_L(\phi(\cdot)))(\Phi(x)) \quad \text{for any } \phi \in \mathscr{C}^2(\mathbb{R}^N).$$

i.e.

$$L(\phi(\Phi(x))) = L(\phi)(\Phi(x)) \quad \text{for any } \phi \in \mathscr{C}^2(\mathbb{R}^N).$$

Let $h \in L^1_{loc}(\mathbb{R}^N)$ be a Φ-invariant function, that is $h(\Phi(x)) = h(x)$ for a.e. $x \in \mathbb{R}^N$.

If v is a solution of

$$-\operatorname{div}_L(\mathscr{A}(x, \nabla_L v)) + |v|^{q-1} v = h, \tag{38}$$

then v is Φ-invariant.

If ψ is Φ-invariant, $p \geq \ell \geq 0$ and v is a solution of

$$-\operatorname{div}_L(\mathscr{A}(x, \nabla_L v)) + \psi^\ell |v|^{q-1} v = h, \tag{39}$$

then v is Φ-invariant.

Proof Set $v_\Phi(x) := v(\Phi(x))$. We have that

$$\begin{aligned} -L(v)(x) + \psi^\ell(x)|v|^{q-1}v(x) &= h(x) = h(\Phi(x)) \\ &= -L(v)(\Phi(x)) + \psi^\ell((\Phi(x))|v|^{q-1}v((\Phi(x)) \\ &= -L(v_\Phi)(x) + \psi^\ell(x)|v_\Phi|^{q-1}v_\Phi(x) \end{aligned}$$

and by the uniqueness of the solution we have the claim.

In the Heisenberg group examples of map which leaves the p-laplacian invariant are the following, $\Phi(\xi) = -\xi$, $\Phi(x, y, t) = (-x, y, t)$ and $\Phi(x, y, t) = (2\mu - x, y, -t - 4\mu y)$ for any $\mu \in \mathbb{R}$.

Proposition 5.2 *Let $q > 1$, $2 \geq \ell \geq 0$ and $h \in L^1_{loc}(\mathbb{R}^N)$. Let Δ_H be the Heisenberg Laplacian on the Heisenberg group $\mathbb{H}^n$ and let $|\cdot|_L$ the gauge related to Δ_H. Then the problem*

$$- \Delta_H v + \psi^\ell |v|^{q-1} v = h \tag{40}$$

has at most one solution.

Moreover, let v be a solution of (40) we have

i) If h is cylindrical, then v is cylindrical.
ii) Let $\ell = 0$. If h does not depend on t, then v is independent on t and it solves the problem

$$- \Delta v + |v|^{q-1} v = h \quad \text{on } \mathbb{R}^{2n}. \tag{41}$$

5.2 Some Applications to Systems

Another consequence of Theorem 4.1 is the following.

Theorem 5.1 *Let $p \geq 1$, let $\mathscr{A}$ be **M**-p-**C** and odd, that is $\mathscr{A}(x, -\xi) = -\mathscr{A}(x, \xi)$ for any $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^l$. Let $q \geq 1$, $q > p - 1$ and $p \geq \ell \geq 0$. Let $h_1, h_2 \in L^1_{loc}(\mathbb{R}^N)$. Let (u, v) be a weak solution of*

$$\begin{cases} \operatorname{div}_L (\mathscr{A}(x, \nabla_L u)) \geq \psi^\ell |v|^{q-1} v + h_1 & \text{on } \mathbb{R}^N, \\ \operatorname{div}_L (\mathscr{A}(x, \nabla_L v)) \geq \psi^\ell |u|^{q-1} u + h_2 & \text{on } \mathbb{R}^N. \end{cases} \tag{42}$$

If $h_1 + h_2 \geq 0$, then $u + v \leq 0$ a.e. on $\mathbb{R}^N$.

Moreover, if (u, v) solves also the equation in (42) and $h_1 = -h_2$, then $u = -v$ and u solves

$$- \operatorname{div}_L (\mathscr{A}(x, \nabla_L u)) = |u|^{q-1} u.$$

Proof Let $w := -u$. Summing up the inequalities, we have that (w, v) is a solution of (35). Hence by Theorem 4.1 it follows that $v \leq w$. This completes the first part of the proof.

Now, if (u, v) is a solution of (42) with equality sign, then $(-u, -v)$ solves the same equations. By the first part of this claim we deduce that $-u - v \leq 0$, thereby concluding the proof.

Corollary 5.1 *Let $p \geq 1$, let $\mathscr{A}$ be $\boldsymbol{M}$-p-$\boldsymbol{C}$ and odd. Let $q \geq 1$, $q > p-1$ and $p \geq \ell \geq 0$ and let (u, v) be a weak solution of*

$$\begin{cases} -\operatorname{div}_L(\mathscr{A}(x, \nabla_L u)) = \psi^\ell |v|^{q-1} v & \text{on } \mathbb{R}^N, \\ -\operatorname{div}_L(\mathscr{A}(x, \nabla_L v)) = \psi^\ell |u|^{q-1} u & \text{on } \mathbb{R}^N. \end{cases} \tag{43}$$

Then $u = v$ a.e. on $\mathbb{R}^N$.

Proof The claim follows by observing that $(-u, v)$ solves the system (42) with equality signs and $h_1 = h_1 \equiv 0$. Hence the claim follows from Theorem 5.1.

The above Theorem 5.1 and Corollary 5.1 were proved in a weaker form by the first author in [7].

5.3 An Interesting Question

We the point out the following challenging question.

If $p = 1$ and $q \geq 1$, from the results proved in the preceding sections it follows that uniqueness and comparison principles for problem (37) hold.

A natural open question is whether in the case $0 < q < 1$ the same results hold. In these respects, the following partial results may give some indication that this problem has an affirmative answer.

Theorem 5.2 *Let $p = 1$, let $\mathscr{A}$ be $\boldsymbol{M}$-p-$\boldsymbol{C}$, $q > 0$ and $p \geq \ell \geq 0$. If (u, v) is a bounded weak solution of*

$$\operatorname{div}_L(\mathscr{A}(x, \nabla_L v)) - \psi^\ell |v|^{q-1} v \geq \operatorname{div}_L(\mathscr{A}(x, \nabla_L u)) - \psi^\ell |u|^{q-1} u \qquad \textit{on } \mathbb{R}^N, \tag{44}$$

then $v \leq u$ a.e. on $\mathbb{R}^N$.

Proof It is easy to see that

$$|t|^{q-1} t - |s|^{q-1} s \geq c_q (t - s), \quad \textit{for } M \geq t \geq s \geq -M. \tag{45}$$

Therefore by the argument used in the proof of Theorem 4.1, the claim follows.

Corollary 5.2 *Let $p = 1$, let $\mathscr{A}$ be $\boldsymbol{M}$-p-$\boldsymbol{C}$, $q > 0$, $p \geq \ell \geq 0$ and let $h \in L^1_{loc}(\mathbb{R}^N)$. Then the possible bounded solution of (37) is unique.*

Looking at one of the model case, the p-Laplacian, one can easily realize that, for $p > 2$ the p-Laplacian operator in not **M**-p-**C**. In this direction some efforts have been made in [11]. However, even if the technique developed in the present paper shows that it is possible to study equations associated to general operators satisfying appropriated structural assumptions, the uniqueness problem for the equation

$$-\Delta_p u + |u|^{q-1}u = h \qquad on\ \mathbb{R}^N,$$

for $h \in L^1_{loc}(\mathbb{R}^N)$ and $u \in W^{1,p}_{loc}(\mathbb{R}^N)$, with $q > p-1$ and $p > 2$ remains still open.

Clearly, looking for nonnegative solution with $h \leq 0$ several results are known see [13] for the Euclidean setting and [6] for the degenerate and anisotropic case. The interested reader may refer also to [8–10] and [11].

6 Inequalities and M-*p*-C Operators

Here, we recall some fundamental elementary inequalities proved in [11] that we use throughout the paper.

In what follows we shall assume that $\mathscr{A}$ has the form

$$\mathscr{A}(x,\xi) = A(|\xi|)\xi,$$

where $\mathscr{A} : \mathbb{R}_+ \to \mathbb{R}$. We set $\phi(t) := A(t)t$.

Theorem 6.1 *Let A be nonincreasing and bounded function such that*

$$\phi(0) = 0, \ \ \phi(t) > 0 \text{ for } t > 0, \phi \textit{ is nondecreasing}. \tag{46}$$

Then $\mathscr{A}$ is ***M****-p-****C*** *with $p = 2$.*

Theorem 6.2 *Let $1 < p \leq 2$. Let ϕ be increasing, concave function satisfying (46) and such that there exist positive constants $c_p, c_\phi > 0$ such that*

$$\phi(t) \leq c_p t^{p-1} \tag{47}$$

and

$$\phi'(s)s \leq c_\phi \phi(s). \tag{48}$$

Then $\mathscr{A}$ is ***M****-p-****C****.*

Remark 6.1 We notice that (47) is a necessary condition on $\mathscr{A}$ to be an **M**-*p*-**C** operator. Indeed, if $\mathscr{A}$ is **M**-*p*-**C**, by taking $\eta = 0$, then it follows that $\mathscr{A}$ is **W**-*p*-**C**, and (47) holds by Hölder inequality.

7 Examples

Example 7.1 Let $l \leq N$ be a positive natural number and let $\mu^l \in \mathscr{C}^1(\mathbb{R}^N; \mathbb{R}^l)$ be the matrix defined as

$$\mu^l := \left(I_l \; 0\right).$$

The corresponding vector field ∇^l is the usual gradient acting only on the first l variables

$$\nabla^l = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_l}).$$

Clearly $\nabla^N = \nabla$ and ∇^l is homogeneous with respect to dilation

$$\delta_R(x) = (Rx_1, Rx_2, \ldots, Rx_l, R^{\delta_{l+1}} x_{l+1}, \ldots, R^{\delta_N} x_N)$$

with $\delta_{l+1}, \ldots, \delta_N$ are arbitrary real positive numbers.

Example 7.2 (Baouendi-Grushin Type Operator) Let $\xi = (x, y) \in \mathbb{R}^n \times \mathbb{R}^k (= \mathbb{R}^N)$. Let $\gamma \geq 0$ and let μ be the following matrix

$$\begin{pmatrix} I_n & 0 \\ 0 & |x|^\gamma I_k \end{pmatrix}. \tag{49}$$

The corresponding vector field is given by $\nabla_\gamma = (\nabla_x, |x|^\gamma \nabla_y)^T$ and the linear operator $L = \mathrm{div}_L(\nabla_L \cdot) = \Delta_x + |x|^{2\gamma} \Delta_y$ is the so-called Baouendi-Grushin operator. Notice that if $k = 0$ or $\gamma = 0$, then L coincides with the usual Laplacian operator. The vector field ∇_γ is homogeneous with respect to dilation $\delta_R(x) = (Rx_1, \ldots, Rx_n, R^{1+\gamma} y_1, \ldots, R^{1+\gamma} y_k)$.

Example 7.3 (Heisenberg-Kohn Operator) Let $\xi = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{H}^n$ and let μ be defined as

$$\begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}.$$

The corresponding vector field ∇_H is the Heisenberg gradient on the Heisenberg group $\mathbb{H}^n$. The vector field ∇_H is homogeneous with respect to $\delta_R(\xi) = (Rx, Ry, R^2 t)$ and $Q = 2n + 2$.

In $\mathbb{H}^1$ the corresponding vector fields are $X = \partial_x + 2y\partial_t$, $Y = \partial_y - 2x\partial_t$. In this case $Q = 4$.

In $\mathbb{H}^n$ a canonical homogeneous norm, called *gauge*, is defined as

$$|\xi|_H := \left(\left(\sum_{i=1}^n x_i^2 + y_i^2 \right)^2 + t^2 \right)^{1/4}.$$

Example 7.4 (Heisenberg-Greiner Operator) Let $\xi = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $r := |(x, y)|$, $\gamma \geq 1$ and let μ be defined as

$$\begin{pmatrix} I_n & 0 & 2\gamma y r^{2\gamma-2} \\ 0 & I_n & -2\gamma x r^{2\gamma-2} \end{pmatrix}. \tag{50}$$

The corresponding vector fields are $X_i = \partial_{x_i} + 2\gamma y_i r^{2\gamma-2}\partial_t$, $Y_i = \partial_{y_i} - 2\gamma x_i r^{2\gamma-2}\partial_t$ for $i = 1, \ldots, n$.

For $\gamma = 1$ $L = \mathrm{div}_L(\nabla_L \cdot)$ is the sub-Laplacian Δ_H on the Heisenberg group $\mathbb{H}^n$. If $\gamma = 2, 3, \ldots$, L is a Greiner operator. The vector field associated to μ is homogeneous with respect to $\delta_R(\xi) = (Rx, Ry, R^{2\gamma}t)$ and $Q = 2n + 2\gamma$.

Example 7.5 Let $\mathbb{R}^N$ be splitted as

$$\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r} \ni (x^1, x^2, \ldots, x^r),$$

and let $\alpha_2, \alpha_3, \ldots, \alpha_r > 0$ be fixed.

Let $g_2 : \mathbb{R}^{n_1} \to \mathbb{R}$ be an homogeneous function of degree α_2.

Let $g_3 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be an homogeneous function of degree α_3 with respect to dilation $\delta_R(x^1, x^2) = (Rx^1, R^{\alpha_2+1}x^2)$, that is $g_3(Rx^1, R^{\alpha_2+1}x^2) = R^{\alpha_3}g_3(x^1, x^2)$.

Let $g_4 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \to \mathbb{R}$ be an homogeneous function of degree α_4 with respect to dilation $\delta_R(x^1, x^2, x^3) = (Rx^1, R^{\alpha_2+1}x^2, R^{\alpha_3+1}x^3)$.

We iterate the procedure by choosing analogously other homogeneous functions g_j up to $g_r : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_{r-1}} \to \mathbb{R}$ a homogeneous function of degree α_r with respect to dilation $\delta_R(x^1, x^2, \ldots x^{r-1}) = (Rx^1, R^{\alpha_2+1}x^2, \ldots, R^{\alpha_{r-1}+1}x^{r-1})$.

Next we define the matrix μ as

$$\begin{pmatrix} I_{n_1} & 0 & & & \\ 0 & g_2(x_1)I_{n_2} & 0 & & \cdots \\ \cdots & 0 & g_3(x^1, x^2)I_{n_3} & & \\ & & & & \\ & & \cdots & & \cdots \\ & & 0 & & g_r(x^1, x^2, \ldots, x^{r-1})I_{n_r} \end{pmatrix}. \tag{51}$$

We have that the vector field $\mu\nabla$ satisfies the assumption of Sect. 2. Indeed it is homogeneous with respect to $\delta_R(x) = (Rx^1, R^{\alpha_2+1}x^2, \ldots, R^{\alpha_r+1}x^r)$. This example generalizes the Example 7.2.

Example 7.6 (Carnot Groups) On a Carnot group the horizontal gradient can be written in the form $\mu\nabla$ as in Sect. 2 and it satisfies our assumptions. We refer the reader to [4] for more detailed information on this subject. Special examples of Carnot groups are the Euclidean spaces $\mathbb{R}^N$. The simplest nontrivial example of a Carnot group is the Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3$. See Example 7.3. Several other examples can be found in the book [4].

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