Chapter 9
On Hodges and Lehmann's "6/π Result"

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9.1 Introduction

The Pitman asymptotic relative efficiency $\text{ARE}_f(\phi_1/\phi_2)$ under density f of a test ϕ_1 with respect to a test ϕ_2 is defined as the limit (when it exists), as n_1 tends to infinity, of the ratio $n_{2;f}(n_1)/n_1$ of the number $n_{2;f}(n_1)$ of observations it takes for the test ϕ_2, under density f, to match the local performance of the test ϕ_1 based on n_1 observations. That concept was first proposed by Pitman in the unpublished lecture notes (Pitman 1949) he prepared for a 1948–1949 course at Columbia University. The first published rigorous treatment of the subject was by Noether (1955). A similar definition applies to point estimation; see, for instance, Hallin (2012) for a more precise definition. An in-depth treatment of the concept can be found in Chap. 10 of Serfling (1980), Chap. 14 of van der Vaart (1998), or in the monograph by Nikitin (1995).

The study of the AREs of rank tests and R-estimators with respect to each other or with respect to their classical Gaussian counterparts has produced a number of interesting and sometimes surprising results. Considering the van der Waerden or normal-score two-sample location rank test ϕ_{vdW} and its classical normal-theory

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competitor, the two-sample Student test $\phi_{\mathcal{N}}$, Chernoff and Savage in (1958) established the rather striking fact that, under any density f satisfying very mild regularity assumptions,

$$\mathrm{ARE}_f(\phi_{\mathrm{vdW}}/\phi_{\mathcal{N}}) \geq 1, \tag{9.1}$$

with equality holding at the Gaussian density $f = \phi$ only. That result implies that rank tests based on Gaussian scores (that is, the two-sample rank-based tests for location, but also the one-sample signed-rank ones, traditionally associated with the names of van der Waerden, Fraser, Fisher, Yates, Terry and/or Hoeffding—for simplicity, in the sequel, we uniformly call them *van der Waerden tests*)—asymptotically outperform the corresponding everyday practice Student t test; see Chernoff and Savage (1958). That result readily extends to one-sample symmetric and m-sample location, regression, and analysis of variance models with independent noise.

Another celebrated bound is the one obtained in 1956 by Hodges and Lehmann, who proved that, denoting by ϕ_{W} the Wilcoxon test (same location and regression problems as above),

$$\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathcal{N}}) \geq 0.864, \tag{9.2}$$

which implies that the price to be paid for using rank-rank or signed-rank tests of the Wilcoxon type (that is, logistic-score-based rank tests) instead of the traditional Student ones never exceeds 13.6 % of the total number of observations. That bound moreover is sharp, being reached under the Epanechnikov density f. On the other hand, the benefits of considering Wilcoxon rather than Student can be arbitrarily large, as it is easily shown that the supremum over f of $\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathcal{N}})$ is infinite; see Hodges and Lehmann (1956).

Both (9.1) and (9.2) created quite a surprise in the statistical community of the late 1950s, and helped dispelling the wrong idea, by then quite widespread, that rank-based methods, although convenient and robust, could not be expected to compete with the efficiency of traditional parametric procedures.

Chernoff–Savage and Hodges–Lehmann inequalities since then have been extended to a variety of more general settings. In the elliptical context, optimal rank-based procedures for location (one and m-sample case), regression, and scatter (one and m-sample cases) have been constructed in a series of papers by Hallin and Paindaveine (2002a, 2006, and 2008b), based on a multivariate concept of signed ranks. The Gaussian competitors there are of the Hotelling, Fisher, or Lagrange multiplier forms. For all those tests, Chernoff–Savage result, similar to (9.1) have been established (see also Paindaveine 2004, 2006). Hodges–Lehmann results also have been obtained, with bounds that, quite interestingly, depend on the dimension of the observation space: see Hallin and Paindaveine (2002a).

Another type of extension is into the direction of time series and linear rank statistics of the serial type. Hallin (1994) extended Chernoff and Savage's result (9.1) to the serial context by showing that the serial van der Waerden rank tests also uniformly dominate their Gaussian competitors (of the correlogram-based portmanteau, Durbin–Watson or Lagrange multiplier forms). Similarly, Hallin and Tribel

(2000) proved that the 0.864 upper bound in (9.2) no longer holds for the AREs of the Wilcoxon serial rank test with respect to their Gaussian competitors, and is to be replaced by a slightly lower 0.854 one. Elliptical versions of those results are derived in Hallin and Paindaveine (2002a, 2004, 2005).

Now, AREs with respect to Gaussian procedures such as t-tests are not always the best evaluations of the asymptotic performances of rank-based tests. Their existence indeed requires the Gaussian procedures to be valid under the density f under consideration, a condition which places restrictions on f that may not be satisfied. When the Gaussian tests are no longer valid, one rather may like to consider AREs of the form

$$\mathrm{ARE}_f(\phi_J/\phi_K) = 1/\mathrm{ARE}_f(\phi_K/\phi_J) \tag{9.3}$$

comparing the asymptotic performances (under f) of two rank-based tests ϕ_J and ϕ_K, based on score-generating functions J and K, respectively. Being distribution-free, rank-based procedures indeed do not impose any validity conditions on f, so that $\mathrm{ARE}_f(\phi_J/\phi_K)$ in general exists under much milder requirements on f; see, for instance, Hallin et al. (2011) and Hallin (2013), where AREs of the form (9.3) are provided for rank-based methods in linear models with stable errors under which Student tests are not valid.

Obtaining bounds for $\mathrm{ARE}_f(\phi_J/\phi_K)$, in general, is not as easy as for AREs of the form $\mathrm{ARE}_f(\phi_J/\phi_{\mathcal{N}})$. The first result of that type was established in 1961 by Hodges and Lehmann, who in (Hodges and Lehmann 1961) show that

$$0 \le \mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}}) \le 6/\pi \approx 1.910 \tag{9.4}$$

or, equivalently,

$$0.524 \approx \pi/6 \le \mathrm{ARE}_f(\phi_{\mathrm{vdW}}/\phi_{\mathrm{W}}) \le \infty \tag{9.5}$$

for all f in some class $\mathcal{F}$ of density functions satisfying weak differentiability conditions. Hodges and Lehmann moreover exhibit a parametric family of densities $\mathcal{F}_{\mathrm{HL}} = \{f_\alpha \,|\, \alpha \in [0,\infty)\}$ for which the function $\alpha \mapsto \mathrm{ARE}_{f_\alpha}(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$ achieves any value in the open interval $(0, 6/\pi)$ ($\alpha \mapsto \mathrm{ARE}_{f_\alpha}(\phi_{\mathrm{vdW}}/\phi_{\mathrm{W}})$ achieves any value in the open interval $(\pi/6, \infty)$). The lower and upper bounds in (9.4) and (9.5) thus are *sharp* in the sense that they are the best possible ones. The same result was extended and generalized by Gastwirth (1970).

Note that, in case f has finite second-order moments (so that $\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathcal{N}})$ is well defined), since $\mathrm{ARE}_f(\phi_{\mathrm{vdW}}/\phi_{\mathcal{N}}) = \mathrm{ARE}_f(\phi_{\mathrm{vdW}}/\phi_{\mathrm{W}}) \times \mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathcal{N}})$, Hodges and Lehmann's "6/π result" implies that the ARE of the van der Waerden tests with respect to the Student ones, which by the Chernoff–Savage inequality is larger than or equal to one, actually can be arbitrarily large, and that this happens for the same types of densities as for the Wilcoxon tests. This is an indication that, when Wilcoxon is quite significantly outperforming Student, that performance is shared by a broad class of rank-based tests and R-estimators, which includes the van der Waerden ones.

In Sect. 9.2, we successively consider the traditional case of *nonserial* rank statistics used in the context of location and regression models with independent observations, and the case of *serial* rank statistics; the latter involve ranks at time t and $t-k$, say, and aim at detecting serial dependence among the observations. Serial rank statistics typically involve two score functions and, instead of (9.3), yield AREs of the form

$$\mathrm{ARE}^*_f(\phi_{J_1,J_2}/\phi_{J_3,J_4}). \tag{9.6}$$

To start with, in Sect. 9.2.1, we revisit Gastwirth's classical nonserial results. More precisely, we provide (Proposition 2) a slightly different proof of the main proposition in Gastwirth (1970), with some further illustrations in the case of Student scores. In Sect. 9.2.2, we turn to the serial case, with special attention for the so-called Wilcoxon–Wald–Wolfowitz, Kendall, and van der Waerden rank autocorrelation coefficients. Serial AREs of the form (9.6) typically are the product of two factors to which the nonserial techniques of Sect. 9.2.1 separately apply; this provides bounds which, however, are not sharp. Therefore, in Sect. 9.3, we restrict to a few parametric families—the Student family (indexed by the degrees of freedom), the power-exponential family, or the Hodges–Lehmann family $\mathcal{F}_{\mathrm{HL}}$—for which numerical values are displayed.

9.2 Asymptotic Relative Efficiencies of Rank-Based Procedures

The asymptotic behavior of rank-based test statistics under local alternatives, since Hájek and Šidák (1967), is obtained via an application of Le Cam's Third Lemma (see, for instance, Chap. 13 of van der Vaart 1998). Whether the statistic is of the serial or the nonserial type, the result, under a density f with distribution function F involves integrals of the form

$$\mathcal{K}(J) := \int_0^1 J^2(u)\mathrm{d}u \qquad \mathcal{K}(J,f) := \int_0^1 J(u)\varphi_f(F^{-1}(u))\mathrm{d}u,$$

and, in the serial case,

$$\mathcal{J}(J,f) := \int_0^1 J(u)F^{-1}(u)\mathrm{d}u$$

where, assuming that f admits a weak derivative f', $\varphi_f := -f'/f$ is such that the Fisher information for location $\mathcal{I}(f) := \int_0^1 \varphi_f^2(F^{-1}(u))\mathrm{d}u$ is finite. Denote by $\mathcal{F}$ the class of such densities. If local alternatives, in the serial case, are of the ARMA type, f is further restricted to the subset $\mathcal{F}_2$ of densities $f \in \mathcal{F}$ having finite second-order moments. Differentiability in quadratic mean of $f^{1/2}$ is the standard assumption here, see Chap. 7 of van der Vaart (1998); but absolute continuity of f in the traditional sense, with a.e. derivative f', is sufficient for most purposes. We refer to Hájek and Šidák (1967) and Hallin and Puri (1994) for details in the nonserial and the serial case, respectively.

9.2.1 The Nonserial Case

In location or regression problems, or, more generally, when testing linear constraints on the parameters of a linear model (this includes ANOVA etc.), the ARE, under density $f \in \mathcal{F}$, of a rank-based test ϕ_{J_1} based on the square-summable score-generating function J_1 with respect to another rank-based test ϕ_{J_2} based on the square-summable score-generating function J_2 takes the form

$$\mathrm{ARE}_f\left(\phi_{J_1}/\phi_{J_2}\right) = \frac{\mathcal{K}(J_2)}{\mathcal{K}(J_1)} C_f^2(J_1, J_2), \quad \text{with} \quad C_f(J_1, J_2) := \frac{\mathcal{K}(J_1, f)}{\mathcal{K}(J_2, f)}, \tag{9.7}$$

provided that J_1 and J_2 are monotone, or the difference between two monotone functions. Those ARE values readily extend to the m-sample setting, and to R-estimation problems. In a time-series context with innovation density $f \in \mathcal{F}_2$, and under slightly more restrictive assumptions on the scores, they also extend to the partly rank-based tests and R-estimators considered by Koul and Saleh in (1993) and (1995).

Gastwirth (1970) has based his analysis of (9.7) on an integration by parts of the integral in the definition of $\mathcal{K}(J, f)$. If both J_1 and J_2 are differentiable, with derivatives J_1' and J_2', respectively, and provided that f is such that

$$\lim_{x\to\infty} J_1(F(x))f(x) = 0 = \lim_{x\to\infty} J_2(F(x))f(x),$$

integration by parts in those integrals yields, for (9.7),

$$\mathrm{ARE}_f\left(\phi_{J_1}/\phi_{J_2}\right) = \frac{\mathcal{K}(J_2)}{\mathcal{K}(J_1)} \left(\frac{\int_{-\infty}^{\infty} J_1'(F(x))f^2(x)\mathrm{d}x}{\int_{-\infty}^{\infty} J_2'(F(x))f^2(x)\mathrm{d}x}\right)^2. \tag{9.8}$$

In view of the Chernoff–Savage result (9.1), the van der Waerden score-generating function

$$J_2(u) = J_{\mathrm{vdW}}(u) = \Phi^{-1}(u) \tag{9.9}$$

(with $u \mapsto \Phi^{-1}(u)$ the standard normal quantile function) may appear as a natural benchmark for ARE computations. From a technical point of view, under this integration by parts approach, the Wilcoxon score-generating function

$$J_2(u) = J_{\mathrm{W}}(u) = u - 1/2 \tag{9.10}$$

(the Spearman–Wald–Wolfowitz score-generating function in the serial case) is more appropriate, though. Convexity arguments indeed will play an important role, and, being linear, J_{W} is both convex and concave. Since $J_{\mathrm{W}}'(u) = 1$ and $\mathcal{K}(J_{\mathrm{W}}) = 1/12$, Eq. (9.8) yields

$$12\,\mathrm{ARE}_f\left(\phi_{J_1}/\phi_{\mathrm{W}}\right) = \frac{1}{\mathcal{K}(J_1)} \left(\frac{\int_{-\infty}^{\infty} J_1'(F(x))f^2(x)\mathrm{d}x}{\int_{-\infty}^{\infty} f^2(x)\mathrm{d}x}\right)^2. \tag{9.11}$$

Bounds on $J_1'(F(x))$ then readily yield bounds on AREs, irrespective of f.

That property of Wilcoxon scores is exploited in Propositions 2 and 3 for nonserial AREs, in Proposition 4 for the serial ones; those bounds are mainly about AREs of, or with respect to, Wilcoxon (Spearman–Wald–Wolfowitz) procedures, but not exclusively so.

Assume that $f \in \mathcal{F}_0 := \{f \in \mathcal{F} | \lim_{x\to\pm\infty} f(x) = 0\}$. Then, integration by parts is possible in the definition of $\mathcal{K}(J_W, f)$, yielding

$$\mathcal{K}(J_W, f) = \int_{-\infty}^{\infty} f^2(x)\mathrm{d}x.$$

Assume, furthermore, that the square-integrable score-generating function J_1 (the difference of two monotone increasing functions) is differentiable, with derivative J_1', and that

$$f \in \mathcal{F}_{J_1} := \{f \in \mathcal{F}_0 | \lim_{x\to\pm\infty} J_1(F(x))f(x) = 0\},$$

so that (9.8) holds. Finally, assume that J_1 is skew-symmetric about $1/2$. Defining the (possibly infinite) constants

$$\kappa_J^+ := \sup_{u\geq 1/2} \left|J'(u)\right| \quad \text{and} \quad \kappa_{\mathrm{J}}^- := \inf_{\mathrm{u}\geq 1/2} \left|J'(\mathrm{u})\right|,$$

we can always write

$$12\,\mathrm{ARE}_f\left(\phi_{J_1}/\phi_{\mathrm{W}}\right) \leq (\kappa_{J_1}^+)^2/\mathcal{K}(J_1) \tag{9.12}$$

while, if J_1 is non-decreasing (hence J_1' is non-negative), we further have

$$(\kappa_{J_1}^-)^2/\mathcal{K}(J_1) \leq 12\,\mathrm{ARE}_f\left(\phi_{J_1}/\phi_{\mathrm{W}}\right) \leq (\kappa_{J_1}^+)^2/\mathcal{K}(J_1). \tag{9.13}$$

The quantities appearing in (9.12) and (9.13) often can be computed explicitly, yielding ARE bounds which are, moreover, sharp under certain conditions.

For example, if J_1 is convex on $[1/2, 1)$, its derivative J_1' is non-decreasing over $[1/2, 1)$, so that

$$\kappa_{J_1}^- = J_1'(1/2) \geq 0 \quad \text{and} \quad \kappa_{\mathrm{J}_1}^+ = \lim_{\mathrm{u}\to 1} J_1'(\mathrm{u}) \leq +\infty. \tag{9.14}$$

It follows that, under the assumptions made,

$$(J_1'(1/2))^2/\mathcal{K}(J_1) \leq 12\,\mathrm{ARE}_f\left(\phi_{J_1}/\phi_{\mathrm{W}}\right) \leq (\lim_{u\to 1} J_1'(u))^2/\mathcal{K}(J_1). \tag{9.15}$$

The lower bound in (9.15) is established in Theorem 2.1 of Gastwirth (1970).

The double inequality (9.15) holds, for instance (still, under $f \in \mathcal{F}_{J_1}$), when the scores $J_1 = \varphi_g \circ G^{-1}$ are the optimal scores associated with some symmetric and *strongly unimodal* density g with distribution function G; such densities indeed are log-concave and have monotone increasing, convex over $[1/2, 1)$ score functions. Symmetric log-concave densities take the form

$$g(x) = Ke^{-\mu(x)}, \qquad K^{-1} = \int_{-\infty}^{\infty} e^{-\mu(x)}\mathrm{d}x \tag{9.16}$$

with $x \mapsto \mu(x)$ a convex, even (that is, $\mu(x) = \mu(-x)$) function; assume it to be twice differentiable, with derivatives μ' and μ''. Then, $\varphi_g(x) = \mu'(x)$, so that

$$J_1(u) := \varphi_g(G^{-1}(u)) = \mu'(G^{-1}(u)), \qquad \mathcal{K}(J_1) = \int_{-\infty}^{\infty} (\mu'(x))^2\, g(x)\mathrm{d}x = \mathcal{I}(g)$$

where $\mathcal{I}(g)$ the Fisher information of g (which we assume to be finite), and

$$J_1'(u) = \mu''(G^{-1}(u))/g(G^{-1}(u)), \quad \text{hence} \quad J_1'(1/2) = \frac{\mu''(0)}{g(0)} = \frac{\mu''(0)}{K}.$$

Specializing (9.15) to this situation, we obtain the following proposition.

Proposition 1. *If the square-integrable score-generating function J_1 is of the form $\varphi_g \circ G^{-1}$ with g given by (9.16), μ even, convex, and twice differentiable, then, under any $f \in \mathcal{F}_{J_1}$,*

$$\left(\frac{\mu''(0)}{K}\right)^2 \leq 12\,\mathcal{I}(g) ARE_f(\phi_{J_1}/\phi_{\mathrm{W}}) \leq (\lim_{u\to 1} J_1'(u))^2 = (\lim_{x\to\infty} (\mu''(x)/g(x))^2. \tag{9.17}$$

With $\mu(x) = x^2/2$ (so that $K^{-1} = \sqrt{2\pi}$) in (9.16), g is the standard Gaussian density; $\mu''(0) = 1, \mathcal{I}(g) = 1$, and the lower bound in (9.17) becomes $(\mu''(0)/K)^2 = 2\pi$, whereas the upper bound is trivially infinite. This yields the Hodges–Lehmann result (9.4).

Turning back to (9.12) and (9.13), but with J_1 concave (and still nondecreasing) on $[1/2, 1)$, J_1' is nonincreasing, so that $\kappa^+_{J_1} = J_1'(1/2)$ and

$$12\,\mathrm{ARE}_f\left(\phi_{J_1}/\phi_{\mathrm{W}}\right) \leq (J_1'(1/2))^2/\mathcal{K}(J_1). \tag{9.18}$$

Not much can be said on the lower bound, though, without further assumptions on the behavior of J_1 around $u = 1$.

Replacing, for various score-generating functions J_1 and densities f, the quantities appearing in (9.12), (9.15) or (9.18) with their explicit values provides a variety of bounds of the Hodges–Lehmann type. Below, we consider the van der Waerden tests ϕ_{vdW}, based on the score-generating function (9.9) and the Cauchy-score rank tests ϕ_{Cauchy}, based on the score-generating function

$$J_{\mathrm{Cauchy}}(u) = \sin(2\pi(u - 1/2)). \tag{9.19}$$

Proposition 2. *For all symmetric densities f in $\mathcal{F}_{\mathrm{vdW}}$, $\mathcal{F}_{\mathrm{Cauchy}}$ and $\mathcal{F}_{\mathrm{vdW}} \bigcap \mathcal{F}_{\mathrm{Cauchy}}$, respectively,*

(1) $\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}}) \leq 6/\pi$;
(2) $\mathrm{ARE}_f(\phi_{\mathrm{Cauchy}}/\phi_{\mathrm{W}}) \leq 2\pi^2/3$;
(3) $\mathrm{ARE}_f(\phi_{\mathrm{Cauchy}}/\phi_{\mathrm{vdW}}) \leq 4\pi$.

Proof. The van der Waerden score (9.9) is strictly increasing, and convex over[1/2, 1). One readily obtains

$$\mathcal{K}(J_{\mathrm{vdW}}) = 1 \quad \text{and} \quad \mathrm{J}'_{\mathrm{vdW}}(\mathrm{u}) = \sqrt{2\pi}\,\exp\{(\Phi^{-1}(\mathrm{u}))^2/2\},$$

hence $\kappa^-_{\mathrm{vdW}} = J'_{\mathrm{vdW}}(1/2) = \sqrt{2\pi}$. Plugging this into the left-hand side inequality of (9.15) yields (1). Alternatively one can directly apply (9.17).

The Cauchy score is concave over $[1/2, 1)$, but not monotone (being of bounded variation, however, it is the difference of two monotone function). Direct inspection of (9.19) nevertheless reveals that

$$\mathcal{K}(J_{\mathrm{Cauchy}}) = 1/2 \quad \text{and} \quad \mathrm{J}'_{\mathrm{Cauchy}}(\mathrm{u}) = 2\pi\cos\left(2\pi(\mathrm{u} - 1/2)\right),$$

hence $\kappa^+_{\mathrm{Cauchy}} = J'_{\mathrm{Cauchy}}(1/2) = 2\pi$. Substituting this in (9.12) yields (2). The product of the upper bounds in (1) and (2) yields (3). □

Remarkably, those three bounds are sharp. Indeed, numerical evaluation shows that they can be approached arbitrarily well by taking extremely heavy-tails such as those of stable densities f_α with tail index $\alpha \to 0$, Student densities with degrees of freedom $\nu \to 0$, or Pareto densities with $\alpha \to 0$; see also the family $\mathcal{F}_{\mathrm{HL}}$ of densities $f_{a,\epsilon}(x)$ defined in Eq. (9.24).

Figure 9.1 provides plots of $\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$ and $\mathrm{ARE}_f(\phi_{\mathrm{Cauchy}}/\phi_{\mathrm{vdW}})$ for various densities. Inspection of those graphs shows that both AREs are decreasing as the tails become lighter; the sharpness of bounds (1) and (3), hence also that of bound (2), is graphically confirmed.

The bounds proposed in Proposition 2 are not new, and have been obtained already in Gastwirth (1970). One would like to see similar bounds for other score functions, such as the Student ones

$$\begin{aligned} J_{t_\nu}(u) &= (\nu + 1)F^{-1}_{t_\nu}(u)/(\nu + F^{-1}_{t_\nu}(u)^2) && 0 < u < 1 \\ &= \frac{1+\nu}{\sqrt{\nu}}\sqrt{-1 + \frac{1}{\mathrm{IB}_\nu(1-2u)}}\mathrm{IB}_\nu(1-2u) && 1/2 \le u < 1 \end{aligned} \tag{9.20}$$

where $\mathrm{IB}_\nu(v)$ denotes the inverse of the regularized incomplete beta function evaluated at $(1, v, \nu/2, 1/2)$ and $F^{-1}_{t_\nu}$ stands for the Student quantile function with ν degrees of freedom. Note that $\lim_{\nu\to -1}\mathrm{IB}_\nu(v) = 0$, so that $\lim_{u\to 1} J_{t_\nu}(u) = 0$. Since $J_{t_\nu}(1/2) = 0$ and $J'_{t_\nu}(1/2) > 0$, this means that, on $[1/2, 1)$, J_{t_ν} is a redescending function; in general, it is neither convex nor concave on $[1/2, 1)$.

Differentiating (9.20), we get, for $u \ge 1/2$,

$$J'_{t_\nu}(u) = \frac{\sqrt{\pi}(\nu+1)\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\nu}\Gamma\left(\frac{\nu+1}{2}\right)}\left(-1 + 2\mathrm{IB}_\nu(1-2u)\right)\mathrm{IB}_\nu(1-2u)^{\frac{1-\nu}{2}}, \tag{9.21}$$

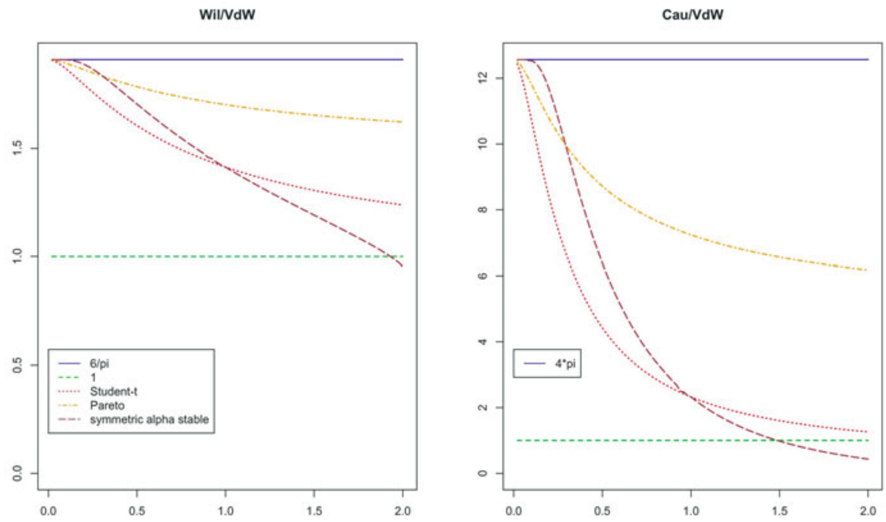

Fig. 9.1 $\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$ and $\mathrm{ARE}_f(\phi_{\mathrm{Cauchy}}/\phi_{\mathrm{vdW}})$ under various families of densities: symmetric stable (indexed by their tail parameter α), Student-t (indexed by their degrees of freedom ν) or Pareto (indexed by their shape parameter α)

from which we deduce that

$$\lim_{u\to 1} J'_{t_\nu}(u) = \begin{cases} 0 & 0<\nu<1 \\ -2\pi & \nu = 1 \\ -\infty & 1<\nu \end{cases}.$$

Except for the $\nu = 1$ case, which is covered by (2) and (3) in Proposition 2, these values do not provide exploitable values for κ^+. For $\nu < 1$, however, one can check from (9.21) that $\max_{u\geq 1/2}|J'(x)| = J'(1/2)$, so that

$$\kappa^+_{J_{t_\nu}} = -\sqrt{\pi}(\nu+1)\Gamma\left(\frac{\nu}{2}\right)/\sqrt{\nu}\,\Gamma\left(\frac{\nu+1}{2}\right).$$

Elementary, though somewhat tedious, algebra yields

$$\mathcal{K}(J_{t_\nu}) = (\nu+1)/(\nu+3).$$

Plugging this into (9.12), we obtain, for $\nu \leq 1$, the following additional bounds.

Proposition 3. *For all $0 < \nu \leq 1$ and all symmetric density f in $\mathcal{F}_{J_{t_\nu}}$ and $\mathcal{F}_{J_{t_\nu}} \bigcap \mathcal{F}_{J_{\mathrm{vdW}}}$, respectively,*

(4) $\mathrm{ARE}_f(\phi_{t_\nu}/\phi_{\mathrm{W}}) \leq \pi\Gamma^2(\frac{\nu}{2})(\nu+3)(\nu+1)/12\nu\Gamma^2(\frac{\nu+1}{2})$, and
(5) $\mathrm{ARE}_f(\phi_{t_\nu}/\phi_{\mathrm{vdW}}) \leq \Gamma^2(\frac{\nu}{2})(\nu+3)(\nu+1)/2\nu\Gamma^2(\frac{\nu+1}{2})$.

Inequality (4) is sharp, the bound being achieved, in the limit, under very heavy tails (stable densities with $\alpha \downarrow 0$, or Student-t_μ densities with $\mu \downarrow 0$). Since this is also the case, under the same sequences of densities, for inequality (1) in Proposition 2, inequality (5) is sharp as well. The upper bounds (4) and (5) are both decreasing functions of the tail index ν; both are unbounded at the origin, and both converge to the corresponding Cauchy values as $\nu \to 1$.

9.2.2 *The Serial Case*

Until the early 1980s, and despite some forerunning time-series applications such as Wald and Wolfowitz (1943) (published as early as 1943—two years before Frank Wilcoxon's pathbreaking 1945 paper), rank-based methods had been essentially limited to statistical models involving univariate independent observations. Therefore, the traditional ARE bounds (Hodges and Lehmann 1956, 1961), Chernoff–Savage (1958) or Gastwirth (1970), as well as the classical monographs (Hájek and Šidáak 1967; Randles and Wolfe 1979; Puri and Sen 1985, to quote only a few) mainly deal with univariate location and single-output linear (regression) models with independent observations. The situation since then has changed, and rank-based procedures nowadays have been proposed for a much broader class of statistical models, including time-series problems, where serial dependencies are the main features under study.

In this section, we focus on the linear rank statistics of the serial type involving two square-integrable score functions. Those statistics enjoy optimality properties in the context of linear time series (ARMA models; see Hallin and Puri 1994 for details). Once adequately standardized, those statistics yield the so-called *rank-based autocorrelation coefficients* that are denoted by $R^{(n)}{}_1, \ldots, R^{(n)}{}_n$, the ranks in a triangular array $X^{(n)}{}_1, \ldots, X^{(n)}{}_n$ of observations. *Rank autocorrelations* (with lag k) are linear serial rank statistics of the form

$$\underset{\sim J_1 J_2;k}{tr^{(n)}} := \left[(n-k)^{-1} \sum_{t=k+1}^{n} J_1\Big(\frac{R_t^{(n)}}{n+1}\Big) J_2\Big(\frac{R_{t-k}^{(n)}}{n+1}\Big) - m_{J_1 J_2}^{(n)}\right] (s_{J_1 J_2}^{(n)})^{-1},$$

where J_1 and J_2 are (square-integrable) score-generating functions, whereas $m^{(n)}_{J_1 J_2}$ and $s^{(n)}_{J_1 J_2} := s^{(n)}_{J_1 J_2;k}$ denote the exact mean of $J_1\big(\frac{R_t^{(n)}}{n+1}\big) J_2\big(\frac{R_{t-k}^{(n)}}{n+1}\big)$ and the exact standard error of $(n-k)^{-\frac{1}{2}} \sum_{t=k+1}^{n} J_1\big(\frac{R_t^{(n)}}{n+1}\big) J_2\big(\frac{R_{t-k}^{(n)}}{n+1}\big)$ under the assumption of i.i.d. $X_t^{(n)}$'s (more precisely, exchangeable $R_t^{(n)}$'s), respectively; we refer to pages 186 and 187 of Hallin and Puri (1994) for explicit formulas. *Signed-rank autocorrelation coefficients* are defined similarly; see Hallin and Puri (1992) or Hallin and Puri (1994).

Rank and signed-rank autocorrelations are measures of serial dependence offering rank-based alternatives to the usual autocorrelation coefficients, of the form

$$r_k^{(n)} := \sum_{t=k+1}^{n} X_t X_{t-k} \Big/ \sum_{t=1}^{n} X_t^2,$$

which consitute the Gaussian reference benchmark in this context. Of particular interest are

(i) the *van der Waerden autocorrelations* (Hallin and Puri 1988)

$$\underset{\sim \mathrm{vdW};k}{r^{(n)}} := \Big[(n-k)^{-1}\sum_{t=k+1}^{n}\Phi^{-1}\Big(\frac{R_t^{(n)}}{n+1}\Big)\Phi^{-1}\Big(\frac{R_{t-k}^{(n)}}{n+1}\Big)-m_{\mathrm{vdW}}^{(n)}\Big](s_{\mathrm{vdW}}^{(n)})^{-1},$$

(ii) the *Wald-Wolfowitz* or *Spearman autocorrelations* (Wald and Wolfowitz 1943)

$$\underset{\sim \mathrm{SWW};k}{r^{(n)}} := \Big[(n-k)^{-1}\sum_{t=k+1}^{n}R_t^{(n)}R_{t-k}^{(n)}-m_{\mathrm{SWW}}^{(n)}\Big](s_{\mathrm{SWW}}^{(n)})^{-1},$$

(iii) and the *Kendall autocorrelations* (Ferguson et al. 2000, where explicit values of $m_{\mathrm{K}}^{(n)}$ and $s_{\mathrm{K}}^{(n)}$ are provided)

$$\underset{\sim \mathrm{K};k}{r^{(n)}} := \Big[1-\frac{4D_k^{(n)}}{(n-k)(n-k-1)}-m_{\mathrm{K}}^{(n)}\Big](s_{\mathrm{K}}^{(n)})^{-1}$$

with $D_k^{(n)}$ denoting the number of discordances at lag k, that is, the number of pairs $(R_t^{(n)}, R_{t-k}^{(n)})$ and $(R_s^{(n)}, R_{s-k}^{(n)})$ that satisfy either

$$R_t^{(n)} < R_s^{(n)} \quad \text{and} \quad R_{t-k}^{(n)} > R_{s-k}^{(n)}, \quad \text{or} \quad R_t^{(n)} > R_s^{(n)} \quad \text{and} \quad R_{t-k}^{(n)} < R_{s-k}^{(n)};$$

more specifically, $D_k^{(n)} := \sum_{t=k+1}^{n}\sum_{s=t+1}^{n} I(R_t^{(n)} < R_s^{(n)},\ R_{t-k}^{(n)} > R_{s-k}^{(n)})$.

The van der Waerden autocorrelations are optimal—in the sense that they allow for *locally optimal* rank tests in the case of ARMA models with normal innovation densities. The Spearman and Kendall autocorrelations are serial versions of Spearman's *rho* and Kendall's *tau*, respectively, and are asymptotically equivalent under the null hypothesis of independence; although they are never optimal for any ARMA alternative, they achieve excellent overall performance. Signed rank autocorrelations are defined in a similar way.

Let J_i, $i = 1, \ldots, 4$ denote four square-summable score functions, and assume that they are monotone increasing, or the difference between two monotone increasing functions (that assumption tacitly will be made in the sequel each time AREs are to be computed). Recall that $\mathcal{F}_2$ denotes the subclass of densities $f \in \mathcal{F}$ having finite moments of order two. The asymptotic relative efficiency, under innovation density $f \in \mathcal{F}_2$, of the rank-based tests $\phi^r_{J_1 J_2}$ based on the autocorrelations $\underset{\sim J_1 J_2;k}{r^{(n)}}$ with respect to the rank-based tests $\phi^r_{J_3 J_4}$ based on the autocorrelations $\underset{\sim J_3 J_4;k}{r^{(n)}}$ is

$$\begin{aligned}
&\mathrm{ARE}^*_f(\phi^r_{J_1 J_2}/\phi^r_{J_3 J_4})\\
&= \frac{\mathcal{K}(J_3)}{\mathcal{K}(J_1)}\left(\frac{\int_0^1 J_1(v)\varphi_f(F^{-1}(v))\mathrm{d}v}{\int_0^1 J_3(v)\varphi_f(F^{-1}(v))\mathrm{d}v}\right)^2\frac{\mathcal{K}(J_4)}{\mathcal{K}(J_2)}\left(\frac{\int_0^1 J_2(v)F^{-1}(v)\mathrm{d}v}{\int_0^1 J_4(v)F^{-1}(v)\mathrm{d}v}\right)^2\\
&= \frac{\mathcal{K}(J_3)}{\mathcal{K}(J_1)}\,C_f^2(J_1, J_3)\frac{\mathcal{K}(J_4)}{\mathcal{K}(J_2)}\,D_f^2(J_2, J_4) \qquad (9.22)
\end{aligned}$$

with $C_f(J_1, J_3) := \mathcal{K}(J_1, f)/\mathcal{K}(J_3, f)$ and $D_f(J_2, J_4) := \mathcal{J}(J_2, f)/\mathcal{J}(J_4, f)$.

The C_f ratios have been studied in Sect. 9.2.1, and the same conclusions apply here; as for the D_f ratios, they can be treated by similar methods.

Denote by $\phi^r_{\mathrm{vdW}}, \phi^r_{\mathrm{SWW}}, \ldots$ the tests based on $\underset{\sim}{r}^{(n)}_{\mathrm{vdW};k}, \underset{\sim}{r}^{(n)}_{\mathrm{SWW};k}$, etc. The serial counterpart of $\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{J_1})$ is $\mathrm{ARE}^*_f(\phi^r_{\mathrm{SWW}}/\phi^r_{J_1J_2})$, for which the following result holds.

Proposition 4. *Let the score functions J_1 and J_2 be monotone increasing, skew-symmetric about $1/2$, and differentiable, with strictly positive $J_1'(1/2)$ and $J_2'(1/2)$. Suppose that $f \in \mathcal{F}_2 \bigcap \mathcal{F}_{J_1} \bigcap \mathcal{F}_{J_2}$ is a symmetric probability density function. Then,*

(1) if J_1 and J_2 are convex on $[1/2, 1)$,

$$\mathrm{ARE}^*_f(\phi^r_{\mathrm{SWW}}/\phi^r_{J_1J_2}) = \mathrm{ARE}^*_f(\phi^r_{\mathrm{K}}/\phi^r_{J_1J_2}) \leq 144\frac{\mathcal{K}(J_1)\mathcal{K}(J_2)}{(J_1'(1/2)\ J_2'(1/2))^2};$$

(2) if J_1 and J_2 are concave on $[1/2, 1)$,

$$\mathrm{ARE}^*_f(\phi^r_{J_1J_2}/\phi^r_{\mathrm{SWW}}) = \mathrm{ARE}^*_f(\phi^r_{J_1J_2}/\phi^r_{\mathrm{K}}) \leq \frac{1}{144}\frac{(J_1'(1/2)\ J_2'(1/2))^2}{\mathcal{K}(J_1)\mathcal{K}(J_2)}.$$

Proof. In view of (9.7), we have

$$\mathrm{ARE}^*_f(\phi^r_{\mathrm{SWW}}/\phi^r_{J_1J_2}) = \mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{J_1})\frac{\mathcal{K}(J_2)}{\mathcal{K}(J_W)}\left(\frac{\int_0^1 (v-1/2)F^{-1}(v)\mathrm{d}v}{\int_0^1 J_2(v)F^{-1}(v)\mathrm{d}v}\right)^2.$$

Consider part (1) of the proposition. It follows from (9.13) that

$$\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{J_1}) \leq 12\,\mathcal{K}(J_1)/(J_1'(1/2))^2.$$

Since J_2 is convex over $[1/2, 1)$, $J_2(u) \geq J_2'(1/2)(u-1/2)$ for all $u \in [1/2, 1)$, so that

$$\int_0^1 J_2(v)F^{-1}(v)\mathrm{d}v = 2\int_{1/2}^1 J_2(v)F^{-1}(v)\mathrm{d}v \geq J_2'(1/2)\int_{1/2}^1 (v-1/2)F^{-1}(v)\mathrm{d}v.$$

It follows that

$$\frac{\mathcal{K}(J_2)}{\mathcal{K}(J_W)}\left(\frac{\int_0^1 (v-1/2)F^{-1}(v)\mathrm{d}v}{\int_0^1 J_2(v)F^{-1}(v)\mathrm{d}v}\right)^2 \leq \frac{12\,\mathcal{K}(J_2)}{(J_2'(1/2))^2},$$

where the assumption of finite variance is used. Part (1) of the result follows. A similar argument holds (with reversed inequalities) if J_2 is concave, yielding part (2).

Applying this result to the score functions $J_1(u) = J_2(u) = \Phi^{-1}(u)$ (convex over $[1/2, 0)$) for which $J_1'(1/2) = J_2'(1/2) = \sqrt{2\pi}$ and $\mathcal{K}(J_1) = \mathcal{K}(J_2) = 1$, we readily obtain the following serial extension of Hodges and Lehmann's "$6/\pi$ result":

$$\mathrm{ARE}^*_f(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}}) = \mathrm{ARE}^*_f(\phi^r_{\mathrm{K}}/\phi^r_{\mathrm{vdW}}) \leq (6/\pi)^2. \tag{9.23}$$

Table 9.1 Numerical values of C_f, D_f, $ARE_f = ARE_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$, and $ARE^*_f = ARE^*_f(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}})$ under densities $f_{a,\epsilon}$ in the Hodges–Lehmann family $\mathcal{F}_{\mathrm{HL}}$ (see (9.24)), for various values of ε and $a \to 0$

ε	C_f	D_f	ARE_f	ARE^*_f
0	0.398942	0.282070	1.90986	1.82346
0.2	0.396313	0.276619	1.88476	1.73062
0.4	0.388772	0.271848	1.81372	1.60844
0.6	0.377291	0.271061	1.70818	1.50608
1	0.348213	0.287973	1.45503	1.44796
2	0.294160	0.303085	1.03836	1.14461
3	0.282852	0.285646	0.960064	0.940023
10	0.282095	0.282095	0.954930	0.911891
100	0.282095	0.282095	0.954930	0.911891

An important difference, though, is that the bound in (9.23) is unlikely to be sharp. Section 9.3 provides some numerical evidence of that fact, which is hardly surprising; while the ratio $C_f(J_{\mathrm{vdW}}, J_{\mathrm{W}})$ is maximized for densities putting all their weight about the origin, this no longer holds true for $D_f(J_{\mathrm{vdW}}, J_{\mathrm{W}})$. In particular, the sequences of densities considered in Hodges and Lehmann (1961) or Gastwirth (1970) along which $C_f(J_{\mathrm{vdW}}, J_{\mathrm{W}})$ tends to its upper bound typically are not the same as those along which $D_f(J_{\mathrm{vdW}}, J_{\mathrm{W}})$ does.

9.3 Some Numerical Results

In this final section, we provide numerical values of $ARE_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$ (denoted as ARE_f in the sequel) and $ARE^*_f(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}})$ (denoted as ARE^*_f in the sequel) under various families of distributions.

First, let us give some ARE values under Gaussian densities: if $f = \phi$, we obtain

$$C_\phi(J_{\mathrm{W}}, J_{\mathrm{vdW}}) = D_\phi(J_{\mathrm{W}}, J_{\mathrm{vdW}}) = \frac{1}{2\sqrt{\pi}} \approx 0.28209$$

so that

$$ARE_\phi(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}}) = \frac{3}{\pi} \approx 0.95493$$

and

$$ARE^*_\phi(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}}) = \frac{9}{\pi^2} \approx 0.91189.$$

Tables 9.1, 9.2, and 9.3 provide numerical values of ARE_f and ARE^*_f under

(1) (Table 9.1) The two-parameter family $\mathcal{F}_{\mathrm{HL}}$ of densities $f_{a,\epsilon}$ associated with the distribution functions

$$F_{a,\epsilon}(x) = \begin{cases} \Phi(x) & \text{if } 0 \le x \le \epsilon \\ \Phi(\epsilon + a(x-\epsilon)) & \text{if } \epsilon < x \end{cases} \tag{9.24}$$

Table 9.2 Numerical values of C_f, D_f, $ARE_f = ARE_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$, and $ARE_f^* = ARE_f^*(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}})$ under Student-t densities with various degrees of freedom ν

ν	C_f	D_f	ARE_f	ARE_f^*
0.1	0.394451	–	1.86710	–
1	0.343120	–	1.41277	–
2	0.321212	0.243196	1.23813	0.878736
4	0.304695	0.269173	1.11407	0.968623
6	0.297953	0.274541	1.06531	0.963551
8	0.294303	0.276784	1.03937	0.955507
10	0.292017	0.278005	1.02329	0.949042
100	0.283146	0.281737	0.962059	0.916370

Table 9.3 Numerical values of C_f, D_f, $ARE_f = ARE_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$, and $ARE_f^* = ARE_f^*(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}})$ under Student-t densities with various degrees of freedom ν

α	C_f	D_f	ARE_f	ARE_f^*
0.1	0.393903	0.175222	1.86191	0.685991
1	0.313329	0.2720600	1.1781	1.046388
2	0.282095	0.2820950	0.954930	0.911893
10	0.222095	0.2934363	0.591916	0.611600
100	0.168549	0.2953577	0.340904	0.356871

where $F_{a,\epsilon}(x)$ is defined by symmetry for $x \leq 0$ (this family of distributions, which has been used by Hodges and Lehmann (1961), is such that the nonserial $6/\pi$ bound is achieved, in the limit, as both a and ϵ go to zero),

(2) (Table 9.2) The family $\mathcal{F}_{\mathrm{Student}}$ of Student densities with degrees of freedom $\nu > 0$, and

(3) (Table 9.3) The family $\mathcal{F}_e$ of power-exponential densities, of the form

$$f_\alpha(x) := \frac{e^{-|x|^\alpha}}{2\Gamma(1+1/\alpha)} \qquad x \in \mathbb{R},\ \ \alpha > 0. \tag{9.25}$$

All tables seem to confirm the same findings: both the serial and the nonerial AREs are monotone in the size of the tails, with the nonserial ARE$_f$ attaining its maximal value ($6/\pi \approx 1.90986$) under heavy-tailed f densities, while the maximal value for the serial ARE$^*_f$ lies somewhere around $(6/\pi)(3/\pi) \approx 1.82346$. Inspection of Table 9.1 reveals that, although the limit of C_f as $a \to 0$ is monotone in the parameter ϵ, the ratio D_f is not; from Table 9.3, the highest values of D_f under the distribution (9.24) are attained for $a \to \infty$ and $\epsilon \approx 0$.

Under Student densities $f = f_{t_\nu}$, the nonserial ARE$_f$ is decreasing with ν, taking value 1.41277 at the Cauchy ($\nu = 1$), value one about $\nu = 15.42$ (a value of ν that is not shown in the figure; Wilcoxon is thus outperforming van der Waerden up to $\nu = 15$ degrees of freedom, with van der Waerden taking over from $\nu = 16$ on), and tending to the Gaussian value 0.95493 as $\nu \to \infty$; the serial ARE$^*_f$ is undefined for $\nu \leq 2$, increasing for small values of ν, from an infimum of 0.878736 (obtained as $\nu \downarrow 2$) up to a maximum of 0.968852 (reached about $\nu = 4.24$), then slowly decreasing to the Gaussian value 0.911891 as $\nu \to \infty$. Sperman–Wald–Wolfowitz and Kendall thus never outperform van der Waerden autocorrelations under Student densities.

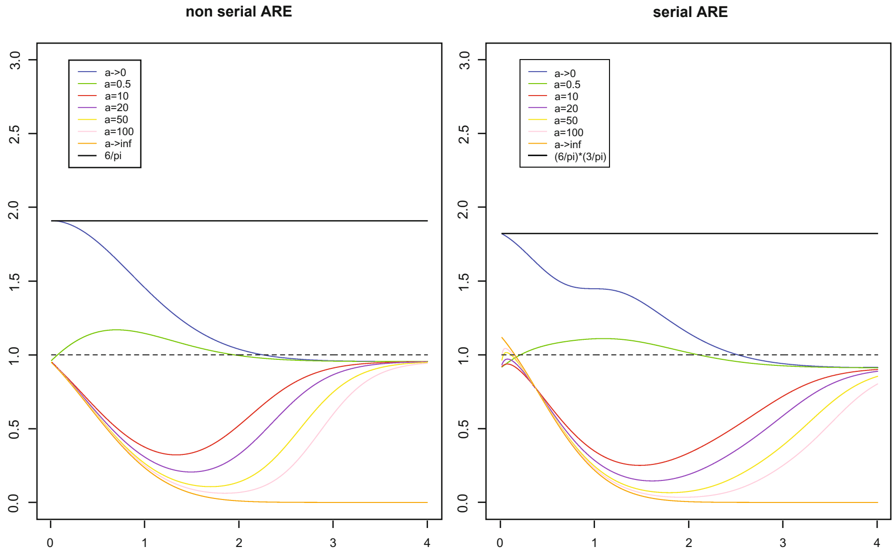

Fig. 9.2 Nonserial ARE$_f$ $= ARE_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$ (*left plot*) and serial ARE$^{\star}_f$ $= ARE^{*}_f(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}})$ (*right plot*) under densities $f_{a,\epsilon}$ in the Hodges–Lehmann family $\mathcal{F}_{\mathrm{HL}}$ (see 9.24), as a function of $\epsilon \in [0,4]]$, for various choices of the parameter a

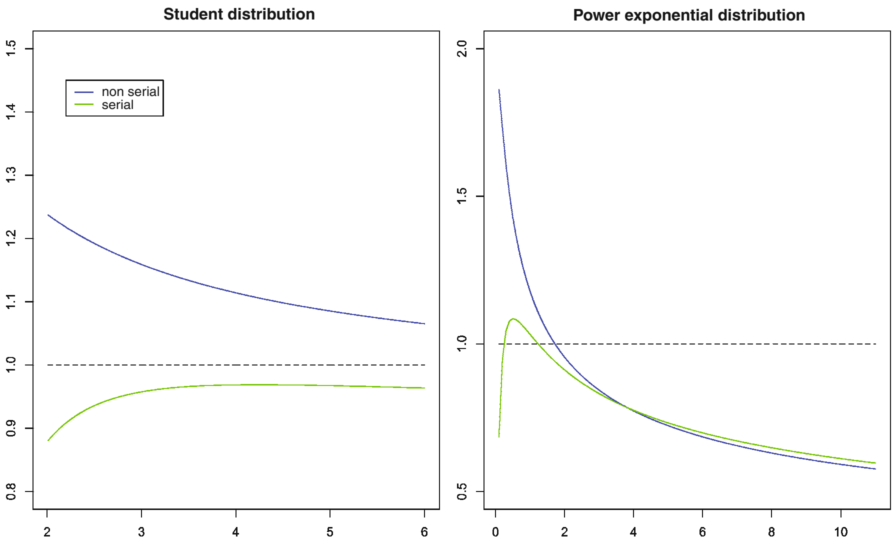

Fig. 9.3 *Left* plot: ARE$_{f_\nu}(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$ and ARE$^{\star}_{f_\nu}(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}})$ for f_ν the Student distribution, as a function of the degrees of freedom $\nu \in [2,6]$. *Right* plot: ARE$_{f_\alpha}$ and ARE$^{\star}_{f_\alpha}$ for the power exponential densities f_α (9.25), as a function of the shape parameter $\alpha \in [0,11]$

Under the double exponential densities $f = f_\alpha$, the nonserial ARE_f is decreasing with α, with a supremum of $6/\pi$ (the Hodges–Lehmann bound, obtained as $\alpha \downarrow 0$), and reaches value one about $\alpha = 1.7206$ (similar local asymptotic performances of Wilcoxon and van der Waerden, thus, occur at power-exponentials with parameter $\alpha = 1.7206$); the serial ARE^*_f is quite bad as $\alpha \downarrow 0$, then rapidly increasing for small values of α, with a maximum of 1.08552 about $\alpha = 0.510$, then deteriorating again as $\alpha \to \infty$; for α larger than 3, the serial and nonserial AREs roughly coincide (See figs. 9.2 and 9.3).

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