Can K-12 Math Teachers Train Students to Make Valid Logical Reasoning?

A Question Affecting 21st Century Skills

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Abstract Valid logic is the mortar that binds all building blocks of critical thinking and analytical thinking. It is the common factor and foundation of three 21st century cognitive skills—critical thinking, analytical thinking and problem solving. According to the Common Core Standards of K-12 mathematics in the United States, proposition and basic first order predicate logic is embedded as a small topic in geometry courses. However, its applications crosscut almost all topics in STEAM (Sciences, Technology, Engineering, Arts and Mathematics) subjects. We inspected how logic is expected to be taught according to the USA K-12 Common Core State Standards and compared with the Singapore curriculum. We found that the difference is not about what should be taught, but how it is taught. The question of this chapter is: How can our K-12 teachers effectively teach logic lessons so that our students can use basic logic to connect concepts and recognize typical logical fallacies? We argued that the answer to that question affects the 21st century skills of the workforce. We hope that our analysis may shed light on the difficult problem in reforming math education and promoting 21st century skills in the workforce.

Keywords Logical reasoning · Critical thinking · Problem solving · Analytical thinking · K-12 Common core state standards · Deep learning

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X. Ge et al. (eds.), *Emerging Technologies for STEAM Education,*
Educational Communications and Technology: Issues and Innovations,
DOI 10.1007/978-3-319-02573-5_18

Introduction

Formal logic is a mathematical subject that studies the forms and complexity of symbolic computations based on symbolic constructs, inference rules and tautologies. We use the term Informal Logic or Logical Reasoning to refer to the broader applications of formal logic such as inductive and deductive reasoning that are used in scientific methods as well as abductive reasoning as an inference to the best explanation (but not necessarily valid) in everyday language. Valid logical reasoning patterns can be symbolically expressed as tautologies (Truth), and logic fallacies are contradictions (False) in formal logic. Hence, formal logic provides the theoretic framework for valid logical reasoning. There is not a clear line between where informal logic ends and formal logic begins. The two only differ in their forms of expression and contexts of application.

Mathematics is based on logical reasoning (Bakó, 2002). In our calculus courses, we observed that an increasing number of first year college students could not identify the logical connections between the examples and counter-examples used for explaining concepts and theorems. Instead, they only focused on checking the calculations of the examples without paying attention to the purposes of the examples (Epp, 1987; Liu & Raghavan, 2009; Raghavan Sena, Bethelmy & Liu, 2008). Consequently, their learning is limited to algorithmic mimicking and demonstrates little understanding. Many of them did not know which one among the inverse, converse, and contrapositive of an if-then conditional statement is equivalent to the conditional statement (Epp, 2003; Hawthorne & Rasmussen, 2014). This deficiency in logical reasoning can seriously restrict students' critical thinking and analytical thinking ability. If college graduates fail to understand deductive reasoning logic, we cannot expect them to comprehend complex mathematical deductions, write correct branching statements for their programs, or draw sound conclusions from scientific experiments.

According to the Common Core State Standards (CCSS) of USA K-12 curricula (CCSSO, 2010; Charles A. Dana Center, 2014), only one chapter of basic formal logic topics is covered in a Geometry course. There is no indication that the content is mandatory to high school students. Logic education only constitutes in a small component of K-12 education, but its applications crosscut almost all subjects, as it a foundation of critical and analytical thinking. We will argue that it is one of the most critical components in K-12 education based on the Common Core State Standards and the general education of college curricula for improving competences in 21st century skills.

The authors realize that the question in our title is too fundamental, and the scope of the problem is too great for us to provide any convincing answers. Though we will provide some technical recommendations, we do not think that our recommendations can be silver bullets to help solve the macroscopic problem. In the following five sections, we divide the question in the title into the following five questions: Section "Why does logic education matter to 21st century skills?", why does logic education matter to 21st century skills? Section "What is broken in k-12 logic education?", what is broken in logic education? Section "Why is logic education challenging to both students and teachers?", why is logic education challenging to both students and teachers? Section "What logic topics should be taught?", what logic topics should be taught in K-12 schools according to standards in the U.S. and other developed countries such as Singapore? Section "How can we use emerging technology to promote logic education?", how can we use emerging technologies to promote logic education? We hope that our analysis can help the US K-12 educational policy makers recognize the urgency of promoting fundamental logical reasoning education. We also recommend using emerging technologies to effectively train K-12 mathematics teachers and K-12 students in making sound logic deductions.

Why Does Logic Education Matter to 21st Century Skills?

21st cognitive skills include critical thinking, analytical thinking and problem solving (Finegold & Notabartolo, 2011). Logical reasoning is a common factor and foundation of these three skills. We will examine two dichotomous roles logical reasoning plays—the rigorous logical thinking that is a necessary ingredient to judicious decision making, and the logical fallacies that may lead to poor decisions. We will present examples to demonstrate that logical reasoning crosscuts almost all topics in STEAM (Science, Technology, Engineering, Arts and Mathematics). Difficulties with logical reasoning skills make it harder for students to adapt to the emerging technologies and keep pace of the changes with necessary skills for the 21st century.

Rational Thinking, Critical Thinking, and Analytical Thinking

Rational thinking, critical thinking, and analytical thinking are three concepts that are sometimes confused with one another and with logical reasoning. We would like to clarify these terms before we explain why we are investigating STEAM education issues from a logical reasoning point of view.

Formal logic is a branch in mathematics, and logical reasoning is taught in mathematics, sciences, and philosophy (Devlin, 2000). Rational thinking, on the other hand, is a concept in psychology (Kahneman, 2011). Someone who is acting irrationally might be considered emotional. Rigorous mathematics education can train students to be logical, but not necessarily rational. A math genius may make irrational decisions frequently based on unchecked emotions. On that other hand, psychological consulting services can help their clients to be rational about some incidents, but not necessarily make them logical. Thus, this chapter focuses on logical thinking, not rational thinking.

There are some confusing overlaps between critical thinking and analytical thinking, as well. While critical thinking involves dialogical short-term mental processing, analytical thinking is a problem solving skill that takes a more systematic research approach. Critical thinking needs to be sensitive to where the facts end and opinions start. Analytical thinking is derived from mathematical analysis and defined as the abstract separation of a whole into its constituent parts in order to study the parts and their relations. The difference between critical thinking and analytical thinking is that the former is based on qualitative mental models, and the latter is based on quantitative mathematical models (Warner, 2014). For example, when a Boy Scout troop discusses strategies to win a competition in a national camporee, the boys need to use critical thinking skills to debate the pros and cons of all recommended strategies. The boys may need to lay out possible competition scenarios, strategies and counter strategies (if, and then) based on their previous experiences and the possible logical sequences of activities. When the camporee organizers propose the schedule and charge of the events, they need to use analytical thinking to conduct basic statistical analysis so that the event may start with sufficient participants and ends with a balanced budget.

We can rank the relationships among logical reasoning, critical thinking, and analytical thinking according to the levels of complexity and the intensity of the mental effort involved. The dependent relationship of the three types of thinking is in reverse order: Sound critical thinking depends on valid logical reasoning, and rigorous analytical thinking depends on sound critical thinking. Critical thinking is not only the ability to reason logically but the ability to find relevant material in memory and deploy attention when needed. Analytical thinking further demands a problem solver or decision maker to have the ability to decompose complex problem to its constituent components, to find the causality of components, and to evaluate the available options based on observed data and processed information.

The "Mortar" Role of Logic in Learning, Problem Solving, and Decision-Making

In his book on *Object-Process Methodology*, Dori (2002) defined the informatics hierarchy from the bottom to the top as follows: data, information, knowledge, understanding, expertise, wisdom, and ingenuity. Dori's definitions of informatics hierarchy will be useful for us to measure the depth and complexity of education materials and testing problems. It can be used to assess learning from the perspective of the complexity and depth of content (Pirnay-Dummer, 2010; Spector, 2010). Such an assessment compensates the assessment of learning based on Bloom's taxonomy of learning, which focuses only on the learning depth of the learner, but neglects the informatics complexity of the content to be learned. The first two levels of informatics require only memorization, and a computer program can save and process the low levels of informatics. It primarily depends on human intelligence to organize and digest knowledge. To gain knowledge, a learner has to invoke logical thinking activities to identify the logic relationship of the information entities in order to absorb, assimilate, process, and analyse information. Understanding and expertise are an exclusive human mental capacity. The critical thinking activities must be invoked to build a mental model (Johnson-Laird, 1983) so that it can reveal the deep cause and effect chains of the problem under concern, and this revelation can be used to predict future events. Learning is to seek understanding. A mental model is a sense-making framework that recognizes the qualitative relationship among constituents such as data, events, cause and effects. One is called as an expert in a domain if one has understood relevant constituent knowledge in the domain and is capable of applying analytical thinking skills to solve new problems in the domain using good judgment and making smart decisions. In this process, it typically requires an expert to build mathematical models that reflect the quantitative relationship of the essential components (de Jong & Van Joolingen, 2008; Spector, 2010). The models can be evaluated and validated based on an understanding of previously solved problems in the domain and used to provide the justifiable solutions, judgments and decisions. Knowledge, understanding, and expertise are three levels of informatics hierarchy that are ranked by the depth and breadth of the informatics structures.

Roughly speaking, logical reasoning can transform information to knowledge; critical thinking can transform knowledge to understanding, and analytical thinking can transforms understanding to expertise. Though such strict matches are impossible in reality due to the ambiguity and context sensitive nature of the terms, the rough matches help us to clearly illustrate the integral role of logical reasoning. In a metaphorical manner, let us map data and information to bricks and concrete blocks; knowledge, understanding and expertise to walls, rooms, and houses; and *logic as mortar*. We can imagine that logical reasoning is binding bricks or blocks to a wall; critical thinking is joining walls to rooms; and analytical thinking is connecting rooms to a house. We use logical reasoning, critical thinking and analytical thinking to learn, to make decisions and to solve problems. Both sound critical thinking and analytical thinking are based on valid logical reasoning. In conclusion, valid logical reasoning is a necessary condition for students to learn how to make judicious decisions and solve problems.

Logic and its Interdependent Relationship to STEAM

Gries and Shneider (1994) described logic as *the glue* in the preface of their book *A Logical Approach to Discrete Math*:

> Logic is the glue that binds together methods of reasoning, in all domains. The traditional proof methods—for example, proof by assumption, contradiction, mutual implication, and induction—have their basis in formal logic. Thus, whether proofs are to be presented formally or informally, a study of logic can provide understanding.

Galileo Galilee called mathematics the language of science. In his book *The Assayer* (1623), he described mathematics as follows:

> Philosophy is written in this grand book, the universe which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.

Two attributes of mathematics make it the language of science and distinguish it from other natural languages: (1) it is based on valid logical inferences and (2) its symbols and the interpretations of its symbols must be unambiguous. Making sound logical inferences is the foundation of Mathematics and Sciences. Engineering and Technologies are the disciplinary fields that apply sciences and mathematics to solve industrial and business problems. Logical reasoning, as the foundation of mathematics and sciences, is also the foundation of Engineering and Technology.

In medieval universities, the seven liberal arts of classical antiquity consisted of the lower level *trivium*: Grammar, Logic, and Rhetoric, which in turn were the foundation for the upper level *quadrivium*: Arithmetic, Geometry, Music and Astronomy. In modern universities, liberal arts education refers to certain areas of literature, languages, art history, music history, philosophy, history, mathematics, psychology, and science. Hence, either in terms of medieval universities or modern universities, logic is not an only interdependent component of liberal arts education, but also an essential foundation of liberal arts education.

Based on the definitions of medieval and modern liberal arts education, we will argue that logical reasoning is also a crucial element in traditional Arts subjects. In Joseph (2002), Sister Miriam Joseph thus described the Trivium of classical antiquity:

> Grammar is the art of inventing symbols and combining them to express thought; logic is the art of thinking; and rhetoric [is] the art of communicating thought from one mind to another; the adaptation of language to circumstance….Grammar is concerned with the thing as-it-is-symbolized, Logic is concerned with the thing as-it-is-known, and Rhetoric is concerned with the thing as-it-is-communicated.

Euclid of Alexandria is often referred to as the “Father of Geometry.” Roughly 2400 years ago, he wrote Elements, a collection of 13 books containing information about algebra, number theory, perspective conics, geometry and logic. Euclid’s work on logic reasoning impacted not only all mathematicians and scientists, but also many great politicians and artists. (Nicolay & Hay, 2015). One of America’s greatest leaders, President Abraham Lincoln, once stated:

> In the course of my law reading I constantly came upon the word “demonstrate”. I thought at first that I understood its meaning, but soon became satisfied that I did not. I said to myself, ‘What do I do when I demonstrate more than when I reason or prove? How does demonstration differ from any other proof?’ I consulted Webster’s Dictionary. They told of ‘certain proof,’ ‘proof beyond the possibility of doubt;’ but I could form no idea of what sort of proof that was. I thought a great many things were proved beyond the possibility of doubt, without recourse to any such extraordinary process of reasoning as I understood ‘demonstration’ to be. I consulted all the dictionaries and books of reference I could find, but with no better results. You might as well have defined blue to a blind man. At last I said, ‘Lincoln, you never can make a lawyer if you do not understand what demonstrate means;’ and I left my situation in Springfield, went home to my father’s house, and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what ‘demonstrate’ means, and went back to my law studies.

In his biography of Lincoln, his law partner Billy Herndon tells how late at night Lincoln would lie on the floor studying Euclidean geometry by lamplight. Lincoln’s logical speeches and some of his phrases such as “dedicated to the proposition” in the Gettysburg address are accredited to his understanding of the *Elements*.

Grammar, Logic and Rhetoric are three necessary interdependent components of effective communication and all subjects of Arts depend on effective communication, which consequently depends on sound logical reasoning. In the beginning of the article "The History of Mental Models," Johnson-Laird (2005) quoted from C.S. Peirce about deduction:

> Deduction is that the mode of reasoning which examines the state of things asserted in the premises, forms a diagram of that state of things, perceives in the parts of the diagram relations not explicitly mentioned in the premises, satisfies itself by mental experiments upon the diagram that these relations would always subsist, or at least would do so in a certain proportion of cases, and conclusion their necessary, or probably, truth. (C.S. Peirce, 1931–1958, 1.66)

The definition of (logical) deduction given by C.S. Peirce pointed out the crucial function of logical deduction—to make inferences in our thinking and reasoning. A valid logical inference makes a valid conclusion from the premises, and invalid logic inference makes an invalid conclusion from the premises. If college students cannot distinguish valid logical inferences, or if they maintain logical fallacies, they are building their academics on sinking sands because *Logic*, unlike any other subject, crosscuts all subjects in STEAM education.

Logic Education and 21st Century Skills

Critical thinking, analytical thinking, and problem solving abilities are important 21st century skills (Finegold & Notabartolo, 2011). Partnership for 21st century Skills (www.p21.org) defined Learning and Innovation Skills in terms of Critical Thinking, Communication, Collaboration, and Creation. Additionally, the first set of skills listed by the ETS CPPI (Collegiate Personal Potential Index) are Critical Thinking and Problem Solving. The second item is to test the ability to form opinions based on logic and facts, (i.e. logical reasoning ability).

We argue that logical reasoning is a *common factor and core foundation* of critical thinking, analytical thinking, and problem solving skills. Improving logical reasoning education at the middle school level may thus help improve K-12 math education and 21st century skills. Though we look at the issues from different levels of granularity and perspectives, we will argue that focusing on this core competency has the following benefits:

1. One may measure and compare the logical reasoning abilities of students in different countries based on existent data.
2. Logical reasoning is a relatively small component such that we can propose a feasible plan to improve it.
3. We can assess the long-term performance on STEM topics between a control group of students and a group of students who have taken intensive and effective logic education.

Students may have difficulty with critical thinking and analytical thinking because they are not free from the troubles of the fundamental logical thinking. Hence, instead of focusing on critical thinking and analytical thinking directly as others have done (Finegold & Notabartolo, 2011), we focus on foundational instructions in logical reasoning to improve STEAM education and prepare students for 21st century skills.

What is Broken in K-12 Logic Education?

We begin with a literature review of difficulties people have with logical reasoning tasks. Then we will present our results of testing a logic task (Wason Selection) in four calculus classes. Lastly, we will use an example to demonstrate the chain effect of invalid logic education in calculus.

Literature Review

Deductive logic appears deceptively simple if we look at it in a nutshell (Guha, 2014). The two mathematics forms in Formal Language are: Modus Ponens ($p \rightarrow q$, p; $\therefore$ q) and Modus Tolens ($p \rightarrow q$, $\sim q$; $\therefore$ $\sim p$). Few people admit that they have any problems about the two forms of deduction. However, the British psychologist P. N. Johnson-Laird put it in 1975: "It has become a truism that whatever formal logic may be, it is not a model of how people make inferences." (p. 7.) A common estimate is that under 5 % of people use "correct" logic spontaneously (Epp, 1986; Johnson-Laird, 1975). Epp (1986) observed that very few of her students have an intuitive feel for the equivance begween a statement and its contrapositive or realized the converse of a true statement could be false. Futhermore, Epp (2003) and Bakó (2002) observed that many students of pure mathematics could not write proofs properly. It is mainly because they failed to see the logical moves that underlie the steps of a mathematical proof (Guha, 2014).

Martin and Harel, (1989) asked 39 prospective elementary teachers to judge the mathematical correctness of inductive and deductive verification statements. 52 % of the tested students accepted incorrect deductive arguments as valid for unfamiliar statements, and more than 33 % of the students did not understand why a counterexample may satisfy the conditions of a conjecture but violate the conclusion. 18 % believed that only one counterexample is not sufficient to disprove a statement. Martin and Harel found that people consider a proof as "what convinces me." (pp. 41–42).

Senk (1985) tested 1520 students on proof writing in geometry classes. Her conclusion was that only 30 % of the students taking geometry courses that teach proof were able to reach a 75 % mastery level in proof writing. Brumfield (1973) tested 52 students that had taken accelerated geometry course in the previous year and have been placed in an AP Calculus course. 81 % did not attempt to write a proof, and less than 10% were correct in writing a proof. The conclusion of the testing was that even students in advanced courses get insufficient meaningful mathematics out of the traditional proof-oriented geometry course. Driscoll (1983) concluded that students needed to be properly instructed to understand the nature of proof, and how it is different from common (everyday) argumentation.

Wason Selection Task to College Students

The Wason Selection Task (http://www.philosophyexperiments.com/wason/) tests the correct application of conditional statements (e.g. the equivalence of contrapositive statements). Wason's original research in 1966 indicated that not even 10% of subjects found the correct solution in this task. This result was replicated in Evans, Newstead and Byrne (1993).

The first author gave an abstract version and an intuitive version of the Wason Selection Test to four calculus classes that have 112 students. Both versions are based on the same conditional statements and a participant can get correct answers if he or she can identify the original statement into its equivalent contrapostive statement. The abstract version is to check if the four cards follow a rule given as a conditional statement. When this abstract version was given, only seven students found the correct answer. The intuitive version is to check two people or two photo IDs if the law about legal drinking age is followed. Almost everybody got it correctly. The testing result confirmed the discovery in Wason (1966) and Evans et al. (1993) that less than 10% of students could find the correct answer. When the same test was given to the college juniors and senior in an upper level elective class in Mathematical Modelling and Simulation (Liu & Raghavan, 2009), five out of 16 of them answered the abstract version correctly. The test confirms what Johnson-laird claimed that "correct" logic is often not spontaneously used (Johnson-Laird, 1975; Epp, 1986). Scientists and philosophies can make mistakes sometimes when giving hasty answers without pen and paper (Kahneman, 2011).

Chain Effect of Invalid Logic Deduction in Calculus

The example given below illustrates the chain effects of invalid abductive logical reasoning. Similar observations about the college students' confusion between necessary and sufficient conditions can be found in Epp (1986). After students learned the concepts of infinite sequences and series, convergence and divergence, they were taught a theorem as follows:

$$\begin{aligned}&\textit{Theorem}:\\&(\textit{Statement } A): \textit{If a series} \sum_{n=1}^{\infty} a_n - \textit{is convergent}\\&\textit{then}, (\textit{Statement } B): \lim_{n\to\infty} a_n = 0.\end{aligned}$$

The theorem says that the statement B is necessary for the statement A. Students learned Statement B1 is a True Statement as an instance of B.

$$\textit{Statement B1}: \lim_{n \to \infty} 1/n = 0.$$

Now, we polled the classes for the question whether the Statement A1 must be true.

$$\textit{Statement A1}: \sum_{n=1}^{\infty} 1/n \ (\textit{called harmonic series}) \textit{ is convergent}.$$

Unfortunately, the majority of our students agreed that the Statement A1 must be true. They believed that the theorems above can deduce the conclusion that the harmonic series must be convergent, which is not correct. This is a very typical case of misunderstanding where students are confused between a conditional statement and its converse statement. We know that if our students are insensitive to the difference of sufficient conditions and necessary conditions of theorems, confused about the converse, inverse and contrapositive statements, it will be very hard to explain to them about the logical connections of theorems and concepts even when we give them many examples and counterexamples. This leads to a chain effect: if students fail to recognize the difference between deductive and abductive logic inferences, a valid logical inference and a logical fallacy, they cannot even understand basic calculus knowledge about the relationship between convergent sequences and series.

Abductive reasoning is a form of logical inference that goes from an observation to a hypothesis that accounts for the observation, ideally seeking to find the simplest and most likely explanation. In abductive reasoning, unlike in deductive reasoning, the premise is necessary, but probably insufficient to entail the conclusion. That is to say the conclusion is not guaranteed from the premise. One can understand abductive reasoning as *"inference to the best explanation."* In summary, students who made the wrong conclusion applied the invalid *abductive* logical reasoning. The mistake can be more clearly illustrated by the following simplified statements and everyday example.

Major Premise: If A is true, then B is true.
Minor Premise: A1 is an instance of A, B1 is an instance of B, and B1 is true.
Conclusion: A1 is true.
A: It is raining in a local community,
B: The lawns in the community are wet
Theorem: If A is true, then B is true.
B1: My lawn is wet, (B1 is an instance of B)
A1: It is raining outside my house (A1 is an instance of A).

Contrary to a previous logically equivalent math problem, most students could realize that *my lawn can be wet (B1) because my sprinkling system is on*. A1 (*it is raining*) is just one of the many valid explanations of B1 *(My lawn is wet*) and A1 is not necessary true even the statement B1 is true.

Why is Logic Education Challenging to Both Students and Teachers?

In this section, we review why logic education is challenging to K-12 students and teachers. Next, we take a hard look at the status quo implementation of logic education in the United States, and why most people do not develop valid logical thinking naturally. Finally, we scrutinize the negative feedback system of logic education in USA.

Challenges in Teaching Proofs in Geometry

Two Dutch educators, Dina van Hiele-Geldof and Pierre van Hiele, claimed that their students performed poorly in secondary geometry, which focuses on deductive reasoning and proof, because students did not develop the required high level logical thinking during the elementary grades (Mason, 2014). They developed a model, known as the van Hiele levels, that identifies the "levels of thinking" and suggested the five recurring instructional "phases":

1. Level 0 "Visualization": the learners should be able to visual recognize the shapes by their appearance as a whole
2. Level 1 "Analysis": the learners should be able to analyse and describe the geometrical shapes by using their properties
3. Level 2 "Abstraction": the learners start using deductive reasoning to answer to the question "why…", by making logical connections and understanding the relationships between the properties
4. Level 3 "Deduction": the learners should be able to combine simple proofs to form a system of formal proof. This is the level of understanding the Euclidian geometry.
5. Level 4 "Rigor": the learners are able to work with abstract geometric systems. This is the level of understanding the Non-Euclidian geometry.

The van Hiele levels have five properties: Fixed sequence, Adjacency, Distinction, Separation, and Attainment. Van Hiele believed that the property of separation was one of the main reasons for failure in geometry. A teacher who is reasoning at one level cannot be understood by a student reasoning a lower level. Even if the teacher believes that they are expressing themselves clearly and logically, the teacher may not understand how the student is reasoning, and the learner-teacher connection is not established.

Why Doesn't Valid Logical Reasoning Come Naturally?

In the book *Thinking Fast and Thinking Slow* (2011), the renowned psychologist and Nobel laureate Daniel Kahneman defined two types of thinking systems as follows: "System I operated automatically and quickly, with little or no effort and no sense of voluntary control. System II allocated attention to the effortful mental activities that demand it, including complex computations. The operations of system II are often associated with the subjective experiences of agency, choices and concentration (p. 18.)". System I includes some components used for everyday reasoning such as associative priming based on resemblance and continuity in time and space. Rational thinking that includes slow and deep logical reasoning is part of System II. A statement such as "since the lawn is wet, it must have rained" is the type of abductive inference System I might make based on everyday experiences and patterns of the converse type: "if it rained, then the lawn is wet." However, the first statement is not valid logically. The lawn can be wet because the sprinkler was on. A student who has been effectively taught basic logic can be vigilant to the possible falsehood of a converse of a valid entailment statement. This abductive logic example is a typical example of System I thinking, using the wrong logic based on the logic converse association.

It is clear that it takes the effort of the System II thinking to identify the logic mistakes of abductive reasoning. As Kahneman emphasized in this book, our System II is a lazy system and it does not automatically work unless we identify the need to invoke the system. Most mathematical problems, especially the math courses above college algebra, require system II thinking to gain understanding. However, too many students get used to the easy learning style such as simply checking calculations between steps, remembering formulas, and mimicking procedures, with little to no conceptual understanding. An important question for us is: can we make our students respond to our teaching and queries with more "surprise" in classes and cultivate them to ask more "why" questions? The students who are trained to observe everything under a logical reasoning lens will certainly ask why frequently.

Negative Feedback Loop Discourages Logic Learning

Several factors may discourage the effective teaching and learning of logical thinking skills in K-12 schools and universities:

1. Many students and many teachers cannot identify typical logical fallacies.
2. Treating student feedback as a customer satisfaction score discourages teachers from teaching topics that are more difficult and require more logical thinking effort on the part of students.
3. Teachers have to "teach to the test," and the tests may not adequately assess logical thinking and other critical reasoning skills.

A negative feedback system develops in which the poorer a student is in logical reasoning, the more likely he or she will choose subjects that require little logical reasoning to succeed in the courses. The more students are poor in logical reasoning in a school, the more likely the teachers in the school will teach shallow knowledge that has little challenges in logical reasoning.

When students take a calculus course with deficiency in algebra (e.g. do not know how to solve quadratic equations), they may eventually catch the missing knowledge in calculus course. However, it is not likely that they can automatically catch their logical fallacies in an upper level course that requires sophisticated logical reasoning. The ability to conduct logical reasoning in daily life reflects the competences of the person. This ability is deep rooted in one's thinking habit. It has to be cultivated gradually in school, in family, and in other learning environments.

What Logic Topics Should be Taught?

In this section, we will inspect the logic education components of several curricula based on the USA Common Core State Standards (CCSS) and the Core Plus Mathematics of CCSS, and then compare the topics with those in Singapore curricula. Based on the inspection of CCSS and Singapore curricula, we will make some suggestions.

Logic Education Based on the Common Core Standards in USA

The High School Integrated Model Course Sequence Alignment to Core-Plus Mathematics by Achieve, Inc. (CPMP, 2008) described the objectives covered in section C of Fundamentals of Logic as follows:

> This relatively short unit formalizes the vocabulary and methods of reasoning that form the foundation for logical arguments in mathematics. Examples should be taken from numeric and algebraic branches of mathematics as well as from everyday reasoning and argument. While this unit emphasizes the application of reasoning in a broad spectrum of contexts, the following unit will mainly apply logical thinking to geometric contexts.

These two logic education objectives of CPMP, 2008, their corresponding learning outcomes and covered course units are given in table 1:

Core Plus Mathematic of CCSS (CPM CCSS, 2011) provides the main page references in Core-Plus Mathematics Courses 1–3 and Course 4: Preparation for Calculus for each of the CCSS mathematical content standard. The CMP CCSS, 2011 emphasizes that the students should be capable of reasoning abstractly and quantitatively, constructing viable arguments and critiquing the reasoning of others.

Table 1 CPMP, 2008, course units covers the logic education objectives

Objectives	Learning outcomes	Course units
Use mathematical notation, terminology, syntax, and logic; use and interpret the vocabulary of logic to describe statements and the relationship between statements	Identify and give examples of definitions, conjectures, theorems, proofs, and counterexamples	Course 2 unit 3 and course 3 units 1 and 3
	Describe logical statements using such terms as assumption, hypothesis, conclusion, converse, and contrapositive	Course 3 units 1 and unit 3
	Recognize syllogisms, tautologies, and circular reasoning and use them to assess the validity of an argument, is not addressed	*Not addressed*
Make, test, and confirm or refute conjectures using a variety of methods	Distinguish between inductive and deductive reasoning; explain and illustrate the importance of generalization in mathematics	Course 3 unit 1
	Construct simple logical arguments and proofs; determine simple counterexamples	Course 3 units 1 & 3, course 2 unit 3
	Demonstrate through example or explanation how indirect reasoning can be used to establish a claim	Course 3 unit 1
	Recognize and avoid flawed reasoning; recognize flaws or gaps in the reasoning used to support an argument	Course 3 unit 1

Comparison with Singapore Curriculum

We compare how logic topics are taught in Singapore and several states in the United States. The PISA test of Reading, Mathematics, and Science Literacy (Fisher, 2010) showed the U.S. now ranks 25th in math, 17th in science, and 14th in reading, while Shanghai China ranks number one, and Singapore ranks number two in all three categories (also see Carnoy & Rothstein, 2013).

The Singapore curriculum aligns with US standards but has different teaching methods (Singapore Academy, 2011). In the US Curriculum, a large number of math concepts are covered and revisited each year. In this way, the students that did not acquire mastery in a concept would have the chance to understand the topic in the next year. Unfortunately, this creates differences in the level of understanding of a concept, and in the same grade level there are students which do not fully internalize the concepts and see this spiralling process only as a repetition and do not sense the full complexity of it. By contrast, the Singapore Math Curriculum covers a less number of concepts each year, but allocates a greater amount of time per each concept to be taught. They are taught to an increasing depth and give students the possibility to reach mastery before introducing new concepts.

Barry Garelick described some other unique features of Singapore Math in the *San Diego Jewish Journal* (2011) as follows:

> … the books (referring to Singapore textbooks) are noticeably short on explicit narrative instruction. The books provide pictures and worked out examples and excellent problems; the topics are ordered in a logical sequence so that material mastered in the various lessons builds upon itself and is used to advance to more complex applications. But what is assumed in Singapore is that teachers know how to teach the material—the teacher's manuals contain very little guidance. Singapore's strength is the logical consistency of the development of mathematical concepts. And much to the chagrin of educators who may have learned differently, mastery of number facts and arithmetic procedures is part and parcel to conceptual understanding.

Besides the math curricula in Singapore, we also compared the math curricula of P. R. China with that of the United States. There are no significant differences about which logical reasoning topics are included in the curricula and textbooks among the three countries. However, we found that whenever it is proper to apply, the Chinese textbooks include mathematics proofs in both lessons and homework exercises. We checked the sample test problems from some Singapore schools as well as Chinese schools and found that their tests emphasize mathematics proofs far more than those in the United States. But for the most part, *the differences between the three countries are not about what should be taught, but how it is taught*.

We propose to start logic education in early grades and insert small logic education modules for middle school (Bakó, 2002). Similar to the Singapore curriculum, each module takes about 2–3 weeks and offers adequate drills and exercises to assure deep learning. We need to increase mathematical proof exercises in homework and add more weight of mathematical proofs in math exams. In addition, logic should be taught by a combination of intuitively visual expressions and rigorous formal expressions.

Critical Topics for Logical Reasoning Education

Geometry provides a rich context for the early development of mathematical thinking. It builds up the thinking process progressively: starting with lower order thinking processes, such as identifying the shapes, and advancing to higher level thinking processes, such as investigating properties of shapes and then solving geometric problems and creating patterns. In addition to the relevant topics of logic education included in K-12 Common Cores above and State standard in appendix 1–4, we would like to recommend and emphasize the following three components as mandatory:

1. Learn how to justify each step of deduction either in algebra and geometry and give plenty of exercises that apply deductive reasoning, for example, direct and indirect (proof by contradiction) proves in Euclidean Geometry and algebra (Agile Mind, 2011; Moore, 1994; Gift of Logic, http://www.giftoflogic.com/faq.html)
2. Identify logical fallacies (https://en.wikipedia.org/wiki/List_of_fallacies).
3. Learn abductive reasoning in a basic statistics context (http://en.wikipedia.org/w/index.php?title=Abductive_reasoning&oldid=636486476).

Critical School Years for Logical Reasoning Education

A critical transition in a child's cognitive development occurs during the middle school years. According to Piaget's four stages of cognitive development, children at age 11 or 12 enter the fourth and final stage of cognitive development, called the formal operational stage. Children at pre-adolescence age from 7 to 11 are at the third stage of cognitive development called concrete operational stages. At the third stage, children mostly use inductive reasoning, drawing general conclusions from personal experiences and specific facts, adolescents become capable of deductive reasoning, in which they draw specific conclusions from abstract concepts using logic. This capability results from their capacity to think hypothetically. Children at the third stage are able to incorporate inductive reasoning, but struggle with deductive reasoning, which involves using a generalized principle in order to try to predict the outcome of an event. During the final stage at middle school years, children show significant growth in their ability to think abstractly, use advanced reasoning skills, make hypotheses and inferences, and draw logical conclusions. Ideally, the middle school years provide educators with great opportunities to foster good logical thinking and mathematical practices.

How can we Use Emerging Technology to Promote Logic Education?

Logical thinking crosscuts all disciplinary fields in STEAM. Applying rigorous logical reasoning to prove mathematics theorems, draw scientific conclusions, and make sound arguments in debate is one of most important 21st century skills for future American workforce. We argued in Section “What logic topics should be taught?” that the middle school years are the critical time for foster the children in sound logic education. In this section, we will focus on the training of both middle school mathematics teachers as well as middle school students directly. We selected the following three tasks as our top list to promote logical reasoning education.

1. Develop innovative training materials for formal logic education and mathematics proofs, including high tech virtual classroom simulations to train pre-service and in-service K-12 math teachers.
2. Design different logic training games for middle school boys and girls that are not only attractive to them according to their gender differences, but also seamlessly synergize visual intuitiveness and mathematics rigor to foster logical thinking. For example, design games that analyse and using one given concept in its algebraic form, geometric (visual) form, application form, etc. Exposing the player to different angles towards the same concept builds connectivity.
3. Offer short-term summer workshops and online training to help K-12 teachers and students improve their logical reasoning skills.

Many colleagues in K-12 education research or practice have probably been working on those tasks for years or decades. Our perspective of the following discussion is how emerging technology can be used to make the teachers’ training more effective and students’ learning more engaged.

Use Education Technology to Offer Easily Accessible Logic Courses for Middle School Teachers

The quality of teacher training will be crucial to the success of the teaching logical reasoning to achieve new CCSS in math. If we cannot assure that our math teachers are well trained in logical reasoning and mathematics proofs, it will be in vain to promote logical reasoning education for the students in K-12 schools. As we argued in the section above, middle school years are the most critical time to properly train the students in logical thinking, and the training for middle school mathematics teachers is paramount. However, three conditions are necessary to make the training or retraining of middle school math teachers successful: (1) Grants and other resource for covering the cost of the training including the stipends to teachers; (2) Training materials in formal logic and mathematics proofs that have been proven effective and feasible; (3) Facility and proven education technology to be used to train and test the readiness of the teachers to teach real middle school students.

It requires an interdisciplinary team of mathematicians, cognitive scientists, experts in pedagogy and teacher education to work together to develop quality training materials and offer effective training programs to middle school teachers. Federal and state government grants (NSF, NIH, HHMI, etc.) are the primary sources of funding to support these types of projects. Those grants have attracted many college teachers in to collaborate with K-12 teachers to train new K-12 teachers and provide summer STEAM workshops for K-12 students. The sponsored projects have already made noticeable impacts to motivate college bounded students, especially minority students to make STEAM career choices. We observed that the government grants helped to make abundant STEAM education materials, mostly online multimedia materials freely available to K-12 teachers. Unfortunately, those materials are underused because there is inadequate training and incentive for K-12 teachers to use them.

Design and Develop Attractive Games for Middle School Boys and Girls to Foster Logical thinking

We all know that computer games are popular with the post-millennial generation. Most children in this generation have learned how to play computer games before they learned how to write. The testing results from two different versions of Wason Selection Tasks (one concrete version and the other abstract version) indicate that we can effectively teach logical reasoning by offering intuitive examples to help students understand equivalent abstract examples. Visual animations in video games can scaffold learning to abstract mathematics ideas if the game is properly designed. If the education content can mix seamlessly with the entertainment, the game can both attract the children and learn valuable knowledge.

There are so many different games to attract the different age groups of boys and girls. An interesting research problem is to find how many effective educational games were designed for middle school geometry and algebra and how much (if any) content was targeted at formal or informal logic education. We divide the education games into two basic categories in terms of the targeted users: the first category is for unsupervised learning that targets massive independent student users; and the second category is supervised learning that targets primarily to trained instructors besides students. We found that most of logical educational games of the first category are bipolarized: one subtype is the game oriented websites that have the fun of games but lack of rigorous educational contents; the other subtype is education-oriented websites that have the rigorous educational lessons, but are short of the attraction of real games. The contents of game-oriented websites (http://www.mathsisfun.com/ and www.learninggamesforkids.com) are mostly puzzles. On the contrary, the contents of the education-oriented websites (https://www.brainpop.com/ and www.shmmoop.com) include formal lessons for logical reasoning and proofs. Those websites use some funny animations and stories relevant to middle school kids to attract their attention. This approach, unfortunately, has only limited success to attract students like real games. The problem of the second category of educational games is their limited accessibility due to the instructional costs. For example, the SUCCEED workshops of Shodor (www.shodor.org) provide K-12 students the opportunities to investigate forensics and scientific computing through logical reasoning and discovery. The program emphasizes hands on investigations and agent-based simulations (Agentsheets and NetLogo). GUTS (code.org/curriculum/mss), another computer science education program provides logic training lessons to middle kids by using the agent-based programming platform StarLoGo (http://education.mit.edu/projects/starlogo-tng). The learning environment of peers, the context of intuitive applications and the timely feedback of these programs contribute to the success of engaged learning.

It is much more challenging and expensive to design and develop effective education games for unsupervised student users than those for trained instructors. The former requires the collaboration of mathematicians, cognitive scientists, experts in pedagogy and teacher education, computer programmers and software engineering. However, the impact of the former is much broader than that of the latter in terms of the number of users. A successful education game of the first category can potentially help millions of middle school children to use deductive correct logic correctly. We cannot understate its impact to STEAM education.

Offer Problem-Based Learning Summer Programs and After School Programs to Foster Logical Thinking

Educational technologies and online and blended courses have started to transform the paradigm of education in both colleges and K-12 schools (New Media Consortium, 2014). The Florida Virtual School has become an attractive option for K-12 students to take their favourite subjects online. Now, K-12 students can access thousands of video tutorials for their math and science courses through websites such as the Khan Academy. At the current stage of educational technology, we do not think that most of middle school children can gain valuable education from online material without adult supervision.

Parents play a paramount role in their children's education, and they should be the owners of their own children's education. If parents cannot schedule time to serve as tutors or hire tutors, there are plenty of free or inexpensive after school academic enrichment programs to help children. Other education technologies such as adaptive learning environments, intelligent tutoring programs, etc. can be used to reduce the cost of personalized learning. Courtney (2014) predicted that intelligent computer aided education systems can make personalized education available to more and more students in the next 10 years.

Many prestigious research universities offer summer academic enrichment programs (see Duke TIP Talent Search, http://tip.duke.edu/) to talented K-12 students. Those summer programs provide excellent problem-based learning opportunities for the participants. Many of them provide effective logical reasoning training under fun application context. Unfortunately, those programs are very selective based on the applicants' academic competence and only small percentage of families can afford. We hope more regional universities, liberal arts colleges, and community colleges offer similar problem-based summer programs with focus on logical reasoning education to middle school children in their local communities. In order to provide equal learning opportunities, those summer programs should apply for government grants to subsidize partial or total tuition so that children from low income families can participate. The United States has about 5000 colleges. If most of those colleges will contribute their spare facilities and faculties to offer summer academic camps, the collective impact to US K-12 education, especially CCSS, is immeasurable.

In the summer of 2013, Dr. Andrei Ludu and the first author offered a summer SeaPerch underwater robotics camp to 22 commuting middle school kids. Computer Animation and hands-on experiments were used to illustrate how Archimedes used physics principles and logic to discover the crown mystery. The kids first learned *if the crown is made of pure gold, then its volume should the same as the original gold bar that the king gave*. They understood that the same amount of water should flow out when the gold bar and crown were submerged into two equal-sized vessels full of water. The students got a sense of eureka when they saw that the crown expelled more water out of the vessel. They instantly identified the contrapositive logic claim. *Since the volumes of the crown and gold bar are not equal, the crown is not made of pure gold*.

Conclusion

In summary, this book chapter analysed the relationship between logical reasoning, critical thinking and analytical thinking. We argued that the soundness of upper level critical thinking and analytical thinking depends on valid logical reasoning. To address the problem of logic education deficiencies, we investigated the issue from a microscopic level based on our first-hand observations in our own math classes. In order to find feasible remedies, we also compared how logic is taught in the K-12 math curricula of other countries, especially Asian countries such as Singapore and P.R. China. We found that the difference is not about what logical reasoning topics should be taught, but how those topics are taught. Our observation is that inadequate logical reasoning education is at the root of unsatisfactory overall USA K-12 mathematics education. The question in our title is fundamental and the scope of the problem is nationwide. It is beyond the scope of this book chapter and our ability to provide any convincing answers to the questions that we brought forth. We hope that our analysis may shed light on the difficult problem in reforming American math education and promoting 21st century skills in the American workforce.

Acknowledgements The authors would like to thank Dr. Michael Spector at University of North Texas for our casual conversation about logic education and its impact to American math education inspired us to write this book chapter.

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