Research in Mathematics Education
Series Editors: Jinfa Cai · James Middleton

Jane-Jane Lo
Keith R. Leatham
Laura R. Van Zoest *Editors*

Research Trends in Mathematics Teacher Education

Springer

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Foreword

The book series *Research in Mathematics Education* is being launched with this volume. We have designed the solicitation, review, and revision process of volumes in the series to produce thematic volumes, allowing researchers to access numerous studies on a theme in a single, peer-reviewed source. Our intent for this series is to publish the latest research in the field in a timely fashion. This design is particularly geared toward highlighting the work of promising graduate students and junior faculty working in conjunction with senior scholars. The audience for this monograph series consists of those in the intersection between researchers and mathematics education leaders—people who need the highest quality research, methodological rigor, and potentially transformative implications ready at hand to help them make decisions regarding the improvement of teaching, learning, policy, and practice. With this vision, our mission of this book series is:

1. To support the sharing of critical research findings among members of the mathematics education community;
2. To support graduate students and junior faculty and induct them into the research community by pairing them with senior faculty in the production of the highest quality peer-reviewed research papers;
3. To support the usefulness and widespread adoption of research-based innovation.

We are grateful for the support of Melissa James, Miriam Kamil, and Clemens Heine from Springer in developing and publishing this book series.

This volume, *Research Trends in Mathematics Teacher Education,* constitutes the first book in the series. There is consensus about the importance of teacher quality for ambitious teaching and learning, but there is no consensus about what kinds of quality are needed to ensure ambitious teaching and learning. This volume provides some structure to this argument, showing that quality teaching is multifaceted, bringing to bear multiple aspects of mathematical and pedagogical knowledge on the crafting of tasks, orchestration of practice, and assessment of student learning and performance.

All of the lead authors in this volume are teacher educators. Their research presented here is informed by and tested through practice. They present their most recent findings about the nature and function of teachers' mathematical knowledge for teaching, but also their beliefs and knowledge about themselves as learners and as part of the larger system of mathematics education. Moreover, their work points to key ways to support productive teacher learning in both preservice and inservice settings. Thus, this first volume in the series is not only timely to show current research trends in mathematics teacher education, but also quite appropriate to set the stage for the book series as a tool for informed decision-making.

We greatly appreciate the efforts of the editors (Lo, Leatham, and Van Zoest) and all of the authors who have contributed to this book. Congratulations to you all for such excellent work!

Jinfa Cai
James Middleton
Co-Editors-in-Chief, RME Book Series

Preface

From November 1–4, 2012, nearly 500 faculty and graduate students in mathematics education converged on Kalamazoo, Michigan, for the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA). Throughout and following the conference, many participants noted the high quality and richness of the scholarly work and of the intellectual exchanges at the sessions. An invitation from Jinfa Cai and James Middleton, the series editors for the Springer monograph series *Research in Mathematics Education*, to the conference co-organizers, Jane-Jane Lo and Laura Van Zoest, to submit a proposal that included expansions of select papers presented at the conference provided the opportunity to further share and expand on some of these important ideas.

The first task was to identify a theme for the monograph. Out of 319 research report and brief research report proposals submitted for review, over one third had focused on themes related to mathematics teacher education. This heightened interest in mathematics teacher education at PME-NA prompted the choice of mathematics teacher education as the theme. At this point, Keith Leatham was invited to join the editorial team to contribute his expertise in mathematics teacher education research.

To select the authors, the editors identified major themes related to mathematics teacher education in the 2012 PME-NA Proceedings. Three main themes emerged: (1) Mathematics Knowledge for Teaching, (2) Beliefs and Identities, and (3) Tools and Techniques to Support Mathematics Teacher Learning. These three themes thus became the three sections of this book. We then reviewed more carefully a set of 34 research reports and brief research reports that related to these themes and narrowed down the collection to 13 papers. We contacted the authors of these 13 papers to explore their interests in expanding their papers to chapters. The authors from 11 of the papers accepted our invitation. The other papers had already been expanded and submitted to journals for review. One set of authors, however, had written a related paper for the 2013 PME-NA, which we reviewed and found to fit nicely in the collection. This is how we arrived at the collection of 12 core chapters included in this volume.

We then set out to identify colleagues with research expertise in mathematics teacher education to serve as external reviewers and to write commentary chapters. Mark Hoover Thames, Denise Spangler, and Randolph Philipp accepted the invitation to become the commentary authors for the sections on Mathematics Knowledge for Teaching, Beliefs and Identities, and Tools and Techniques for Supporting mathematics Teacher Learning, respectively. Each section commentary chapter contains a brief review of the studies in that particular section, a discussion of major themes that cut across the studies, and suggestions for future directions in that particular area of research. Furthermore, Olive Chapman, the editor of the *Journal of Mathematics Teacher Education*, accepted the invitation to write a commentary for the entire collection to situate the findings of these 12 studies in the landscape of mathematics teacher education, and to offer her thoughts on future directions.

Each editor assumed the role of editor for one section: Laura for Mathematical Knowledge for Teaching, Keith for Beliefs and Identities, and Jane-Jane for Tools and Techniques to Support Teacher Learning. In so doing, we became the primary contact person for the authors of that section. We then went through a two-stage process to expand and refine the papers. During the first stage, we each read and reviewed all the papers from our own sections and two more from each of the other two sections. Section editors compiled this feedback and provided a list of suggestions for the authors to consider as they expanded their papers. During the second stage, each paper was read by the section commentary author, by the section editor, and again by one other member of the editorial team. Again, a list of suggestions, along with a marked manuscript with edits and comments, was sent to the authors for another round of revisions. Section editors then worked with authors in an iterative fashion to create final drafts. We also provided feedback to the commentary authors to help them clarify the main ideas in their papers.

We thank all the authors of this volume for their dedication in meeting the extremely tight deadlines involved in bringing this book together. We thank Hope Smith for her dedication to the technical details for the final preparation of the manuscripts, and series editors Jinfa Cai and James Middleton for their support and encouragement. We are pleased to present this volume as a timely and important resource for the mathematics teacher education research community.

Western Michigan University — Jane-Jane Lo

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About the Editors

Dr. Jane-Jane Lo earned a PhD in Curriculum and Instruction with a specialization in mathematics education from Florida State University under the direction of Grayson G. Wheatley. She is an associate professor of mathematics education at Western Michigan University in Kalamazoo, Michigan. She has a long-term research interest in studying the process of mathematical learning and concept development. This focus has been pursued in three complementary areas: rational number concepts, curriculum analysis, and international comparative studies both in the contexts of K-8 and teacher education. She has published in both research and practitioner journals, including the *Journal for Research in Mathematics Education, Educational Studies in Mathematics, Journal of Mathematics Teacher Education, Teaching Children Mathematics, Mathematics Teaching in the Middle School, and Mathematics Teacher,* and is on the editorial board for the *International Journal of Science and Mathematics Education*. She, Laura Van Zoest and James Kratky co-edited the 2012 PME-NA conference proceedings.

Dr. Keith Leatham earned a PhD in Mathematics Education at the University of Georgia and conducted his dissertation research under the direction of Tom Cooney. He is associate professor of mathematics education at Brigham Young University in Provo, Utah, where he has been working since 2003. His research focuses on understanding how preservice teachers learn to facilitate student mathematics learning. In particular he studies how preservice teachers learn to use technology in teaching and learning mathematics, how they learn to recognize and use students' mathematical thinking, and how their beliefs about mathematics, its teaching and learning are related to the teaching and learning-to-teach process. He has published in both research and practitioner journals, including the *Journal of Mathematics Teacher Education, Contemporary Issues in Technology and Teacher Education, Instructional Science, Teaching Children Mathematics, and Mathematics Teaching in the Middle School*. He has been attending and presenting at PME-NA since 1998. He served as the associate editor for the *Journal for Research in Mathematics Education* from 2004 to 2008 and currently serves on its editorial board.

Dr. Laura R. Van Zoest earned a PhD in Mathematics Education at Illinois State University under the direction of Jane Swafford. She is a full professor of mathematics education at Western Michigan University in Kalamazoo, Michigan. She specializes in secondary mathematics teacher education, focusing specifically on the process of becoming an effective mathematics teacher and ways university coursework can accelerate that process. Lines of research have included investigating the effect of reform curriculum materials on teacher development, the use of practice-based materials in university methods courses, and the cultivation of productive norms in teacher education. Her current work involves developing a theory of productive use of student mathematical thinking. She has served as the principal investigator for research and professional development projects funded at over two million dollars. She has published in research and practitioner journals, including the *Journal for Research in Mathematics Education, Journal of Mathematics Teacher Education, Teacher and Teacher Education, Mathematics Teacher Educator* and the *Mathematics Teacher*. She was editor of *Teachers Engaged in Research: Inquiry into Mathematics Practice, 9–12*, and guest co-editor of the *ZDM: The International Journal on Mathematics Education* focus issue *Theoretical frameworks in research on and with mathematics teachers*.

Part I
Mathematical Knowledge for Teaching

Understanding Preservice Teachers' Curricular Knowledge

Tonia J. Land and Corey Drake

Understanding Preservice Teachers' Curricular Knowledge

Mathematics curriculum materials are ubiquitous and often mandated in elementary classrooms, yet the field of mathematics education has few tools for developing and measuring teachers' knowledge related to using these materials in productive ways. Prior research suggests that teachers' use of curriculum materials can be characterized by patterns in the ways they read, evaluate, and adapt materials (Sherin and Drake 2009). Teachers' patterns of reading curriculum materials include particular approaches to reading curriculum materials as well as when and what teachers read. Evaluating includes the types of evaluative stances a teacher might have before, during, and after curriculum enactment. Adapting describes ways in which a teacher might modify materials, with adaptations ranging from something as simple as a slight omission to something as complex as an addition of entirely new activities. Brown and Edelson (2003) also describe the ways teachers might use materials—by offloading, adapting, or improvising. Offloading refers to a teacher who offloads "instructional responsibility onto the materials" (p. 6). Adapting occurs when a teacher adopts some of the curriculum design from the materials, but makes changes. Improvising is when a teacher contributes largely to the design of the lesson, with the materials contributing very little.

We understand these practices—described by Sherin and Drake (2009) as well as Brown and Edelson (2003)—to be part of a larger construct of expert curriculum

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use that incorporates many of the aspects of curricular knowledge described by Shulman (1986). Ultimately, we would like to develop a learning trajectory from initial curriculum use (beginning of the methods course) to expert curriculum use that documents increasingly sophisticated ways of using curriculum materials. The literature helps us understand the expert end of the trajectory, as described below. The study reported here helps us begin to understand the PST end of the trajectory by piloting a set of questions designed to document PSTs' knowledge related to reading, evaluating, and adapting a *Standards*-based[1] curriculum lesson. These questions were based on what we know about the knowledge and practices of expert teachers as they use curriculum materials. A *Standards*-based lesson was used because *Standards*-based materials explicitly include features intended for teacher learning, as well as student learning.

Our definition of expert curriculum use draws from a substantial body of work that has been conducted in the past several years, including the work of Remillard (2005); Remillard and Bryans (2004); Brown (2009); Sherin and Drake (2009); and Taylor (2010). Remillard (2005) examined 25 years of research investigating teachers' use of mathematics curricula. Over that time period, research findings and theoretical foundations were quite varied. Remillard classified these differences into four perspectives in order to "illuminate the varied and sometimes conflicting assumptions underlying research on curriculum use and to consider the implications of the variation" (p. 216). The four perspectives are as follows: "curriculum use as following or subverting the text (Remillard, 2005, p. 216)," "curriculum use as drawing on the text (p. 218)," "curriculum use as interpretation of the text (p. 219)," and "curriculum use as participation with the text (p. 221)." It is this last perspective that we most associate with expert curriculum use as it treats curriculum use as a collaboration between the teacher and the materials. This perspective implies that understanding curriculum use requires an exploration of the ways in which individual teachers interact with particular curricular resources.

In seeking to understand how curricular resources support teaching and learning, Remillard and Bryans (2004) found that the eight teachers in their study, all of whom used *Investigations* (TERC 1998), had different orientations—defined as a collection of perspectives toward mathematics teaching and learning as well as toward curriculum—toward their materials. Teachers were "adherent and trusting (Remillard and Bryans, 2004, p. 366)," "quietly resistant (p. 369)," "skeptical (p. 371)," "explorative (p. 378)," "piloting (p. 378)," "adaptive (p. 376)," or "actively focused (p. 367)." Differing orientations toward the curriculum affected the ways teachers used the materials and their opportunities for learning.

Taken as a set, this prior research suggests that teachers' curriculum use is a dynamic, interpretive, and interactive process in which both teachers and materials contribute resources to the design and enactment of instruction. Based on this literature, we define "expert" curriculum use as including (a) curriculum vision—an understanding of the goals of the curriculum, as well as strategies for using the curriculum materials to reach those goals (Drake et al. 2009; Drake and Sherin 2005);

[1] The term *Standards*-based curriculum refers to the curriculum materials that were developed to align with the NCTM *Standards* (1989, 2000) and were funded by the National Science Foundation.

(b) particular strategies for reading, evaluating, and adapting curriculum materials in productive ways (Sherin and Drake 2009); (c) practices for using curriculum materials to accomplish instructional goals (Brown 2009); and (d) strategies for "systematically" adapting curriculum materials to meet the needs of students (Taylor 2010). This chapter focuses on the second set of practices—reading, evaluating, and adapting curriculum materials—as a starting point for developing a clearer understanding of PSTs' curriculum knowledge and use at the beginning of the curriculum use learning trajectory. In this trajectory, our ultimate goal is to include all of these components of expert curriculum use, as well as additional features of curricular knowledge as described by Shulman (1986), such as knowing when to use "particular curriculum or program materials in particular circumstances" (p. 10).

Because curriculum use is a dynamic process between the teacher and the materials, it is important to understand how materials might contribute to that process. Davis and Krajcik (2005) summarized Ball and Cohen's (1996) "high-level guidelines" for the ways in which materials could be designed to support teacher learning: (a) "help teachers learn how to anticipate and interpret what learners may think about or do in response to instructional activities (p. 5);" (b) "support teachers' learning of subject matter knowledge including facts, concepts, and disciplinary practices (p. 5);" (c) "help teachers consider ways to relate units during the year (p. 5);" (d) "make visible the developers' pedagogical judgments (p. 5);" and (e) "promote pedagogical design capacity (p. 5, the ability to perceive and mobilize existing resources to achieve instructional goals (Brown and Edelson 2003))." In our study, we have focused not only on how PSTs read those elements of the curriculum materials explicitly designed to be educative for teachers (e.g., descriptions of student strategies and mathematical content), but also on how they read and understand elements that are designed to describe the features of the particular lesson, including the goals and activities of the lesson.

Methods and Data Sources

Participants

As part of the development of our measure, we piloted survey items with 34 PSTs enrolled in a small liberal arts university located in the Midwest. Thirty-one participants were female; three were male. Seven took the survey at the beginning of a semester-long elementary mathematics methods course that included a focus on the use of *Standards*-based curriculum materials, and the remaining PSTs took the survey at the end of the course. For this study, responses from the beginning and end of the semester were combined into a single set of responses in order to identify and describe the range of possible PST responses to the questions.

Description of Lesson

The curriculum use questions focus on a first-grade lesson from *Math Trailblazers* (University of Illinois at Chicago 2008) titled “Counting One Hundred Seventy-two.” We chose this lesson for several reasons: it came from a *Standards*-based curriculum series, had educative features, covered easily accessible mathematical content for PSTs, and was relatively short and focused. “Counting One Hundred Seventy-two” begins by presenting several numbers (e.g., 125) to students and asking them what those numbers mean. In the materials, anticipated student responses are listed (e.g., 5 groups of 25, 12 groups of 10 with 5 left over). During student exploration time, students consider the number 172 and represent it in any way they choose. Next, they share their representations with a partner, and finally, a whole group discussion occurs.

Data Collection and Analysis

The survey consists of 18 questions (see Appendix A). Our primary purpose was to capture the ways in which PSTs read, evaluated, and thought about adapting the lesson. Thus, several questions were created within each domain. For the purpose of this chapter, we share PSTs’ responses to the nine questions that provided the most interesting insights into PSTs’ approaches to reading, evaluating, and adapting the lessons. Below, we list the nine questions and our rationales for including those questions in the measure.

Reading

1. As a teacher, what would be your specific goal(s) for your students’ learning with this lesson?
2. What kinds of solutions do you think a teacher might see?
3. On page 37 in the first bullet point under the assessment heading, the lesson plan states, “Even though counting by ones is an inefficient strategy, it works if done carefully.” What does that mean?

The *Framework for Analyzing Teaching* (Hiebert et al. 2007; Morris et al. 2009) designated one of four key skills for learning how to teach as being able to set “learning goals for students” (Hiebert et al. 2007, p. 49). These same researchers found that PSTs struggle with the development of this skill (and the others). Thus, we wanted to understand how PSTs developed a learning goal with the aid of curriculum materials. Regarding the question about possible student solutions, Davis and Krajcik (2005) proposed design principles for educative materials that included the idea that curriculum materials should help teachers anticipate and interpret what students might do in response to the task. Because the “Counting One Hundred Seventy-two” lesson (University of Illinois at Chicago 2008) included possible student responses, we wanted to see how PSTs took those up. The third question assessed PSTs' reading of a specific phrase of text rather than a more general overview. We selected that particular phrase, as it seemed to be one that could be (and was) interpreted in a variety of ways.

Evaluating

4. When thinking about student learning, what are the strengths and weaknesses of this lesson?[2]
5. From the perspective of the teacher who has to prepare for and enact this lesson, what are the strengths and weaknesses of this lesson plan?
6. Does this lesson have multiple entry points? In other words, is the task accessible to a wide-range of learners? Explain.

Questions 4 and 5 were asked to understand how PSTs distinguished between student learning and teacher learning within the context of curriculum materials and to determine the strengths and weakness from both viewpoints (student and teacher). Educative curriculum materials are meant for both student and teacher learning. We wanted PSTs to be able to appropriately evaluate tasks and implement those tasks for student learning, but we also wanted PSTs to notice the educative features of the materials and/or how the materials provide support. Question 6 addressed PSTs' *evaluation* of the materials by asking them to assess the lesson against the concept of "multiple entry points." PSTs had discussed the idea of multiple entry points in class; therefore, we wanted to grasp how they understood and used that construct in their evaluation of curriculum materials.

Adapting

7. Would you make any changes to the lesson before teaching it? If so, what would they be?
8. Another pair of students represented 172 with 6 groups of 25 and had 22 left over. What would you say to or ask these students after they have shared their solution?
9. During this lesson, these were the correct solutions that you saw and were shared:
   1. 172 individual tallies
   2. 34 groups of 5 with 2 left over
   3. 6 groups of 25 with 22 left over
   4. 17 groups of 10 with 2 left over

   You still have 15 min of class after students share their representations of 172. What would you do with the remainder of class time? Why?

[2] We have used *lesson* and *lesson plan* interchangeably.

The last three questions allowed PSTs to describe the ways in which they might *adapt* the lesson after having read and evaluated the curriculum materials as well as respond to a particular solution strategy. Question 7 was broad in asking what changes PSTs might make to the lesson. We were interested in the different ways PSTs might think about adapting curriculum materials. The eighth question was designed to understand how PSTs might respond to a particular solution. This solution was not given as an example in the materials, but is still a strategy that students might devise. Jacobs et al. (2010) found that teachers with and without professional development on children's mathematical thinking struggled with responding to children's thinking in ways that build on that thinking. Finally, the last question was twofold. First, the question asked PSTs to respond to children's mathematical thinking. Second, the materials provided some guidance for teachers regarding what to do with the remaining time: "If time permits, try other numbers with the class" (University of Illinois at Chicago 2008, p. 36). We were interested in PSTs' ideas about how to use this "extra" time as a way of adapting the materials.

Each survey question was analyzed separately through a process of open and emergent coding (Strauss and Corbin 1998). Through this process, a set of codes was generated that illustrated the type of survey responses. Questions were split between the two authors and analyzed by the authors individually. Reliability was not established due to the pilot nature of the study. Codes are presented in our results section. Our goal at this point is to understand how PSTs read, evaluate, and adapt a *Standards*-based lesson.

Results

We present results from the nine survey questions, organized by sections for reading, evaluating, and adapting curriculum materials.

Reading Curriculum Materials

PSTs' goals (Question 1) for teaching the 172 lesson were categorized using three primary codes. Responses were categorized as *procedural* if they focused on counting/grouping; as *conceptual* if they focused on the understanding/meaning of number and/or place value; and as *with connections*[3] if there was explicit mention of making connections across multiple strategies and/or representations. Responses could be any combination or all of the above three codes, which led to six types of goal responses. Table 1 summarizes the responses to Question 1.

In the curriculum materials, the goals were listed as the following:

[3] Here, we are using "with connections" differently than Stein and Smith (1998). We are referring to PSTs having a procedural and/or conceptual goal with connections across strategies and/or representations.

Table 1 Responses to Question 1 (goal question)

Number of PSTs	Type of response	Example
7	Procedural	Counting and grouping objects that are greater than 100
11	Procedural with connections	Students will be able to represent numbers greater than 100 using manipulatives and words. Students will be able to group objects by ones, tens, and hundreds
3	Conceptual	My goals for this lesson would be for the students to understand what three-digit numbers mean and to be able to talk about and explain them
1	Conceptual with connections	To have students talk about the meaning of a number, represent a number by a picture, and use different objects to represent a number
5	Procedural and conceptual	As the teacher, my specific goal for this lesson would be that the students group numbers between 101 and 199 in a way that shows that they understand place value
6	Procedural and conceptual with connections	Understanding place value, hundreds, tens, and ones. Being able to break numbers into parts and recognize they belong to a whole representing numbers with pictures or symbols grouping and counting objects by ones, tens, and hundreds

- Representing numbers greater than 100 using manipulatives, pictures, symbols, and words
- Grouping and counting objects by ones, tens, and hundreds. (University of Illinois at Chicago 2008, p. 33)

Using our coding scheme, the goals given in the curriculum materials would be categorized as procedural with connections, which makes these results particularly interesting to us. We conjecture that the curriculum authors had a conceptual purpose in mind when writing these goals, but that purpose was not explicit in the materials. Identifying a conceptual goal for the lesson required a significant amount of interpretive work while reading the lesson, and we found that many of the PSTs (15/33) did engage in that work.

In Question 2, PSTs were asked to consider solutions that a teacher might see as the result of this lesson. In the "Counting One-Hundred Seventy-two" lesson, example student solutions for representing 125 were provided. We wanted to see if PSTs used that information to anticipate student solutions for representing 172. Therefore, we applied the solutions given in the textbook for 125 and used them to anticipate solutions for 172, which we provide in Fig. 1. In Table 2, we provide counts of which solutions given in Fig. 1 were anticipated by PSTs, as well as other strategies anticipated by PSTs.

As Table 2 indicates, many of the 34 PSTs mentioned three of the four solutions applied from the lesson plan text, with "seventeen groups of 10 and one group of two" as the solution mentioned the most. Interestingly, no one mentioned the "72 more than 100" solution. In addition to the solutions provided by the text, PSTs mentioned several other solutions. Individual beans or tallies were mentioned by

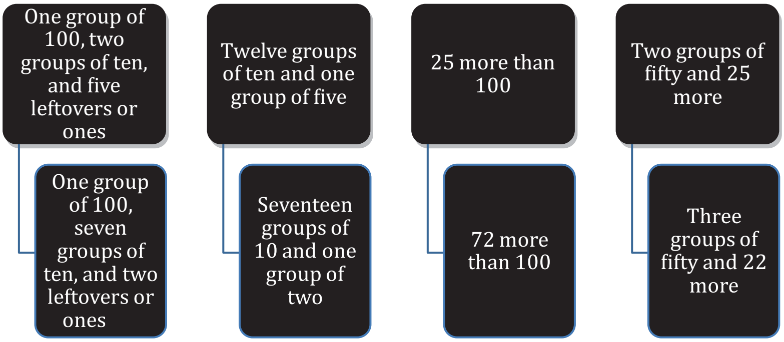

Fig. 1 Solutions for 125 (*first row*) and anticipated solutions for 172 (*second row*)

Table 2 Responses to Question 2 (student solutions)

Solutions applied from lesson plan text	Frequency of PSTs
One group of 100, seven groups of ten, and two leftovers or ones	15
17 groups of ten and one group of two	25
72 more than 100	0
Three groups of 50 and 22 more	11
Other solutions	*Frequency of PSTs*
172 individual beans or tallies	15
Groups of 5	11
Groups of 25	7
Groups of 2	4
Groups of 20	2
Other (e.g., 2 groups of 86, 100+50+10+10+1+1)	17

almost half the PSTs, which was not too surprising given that the text provided information about that solution path in a different section (see Question 3). In looking at how many PSTs mentioned solution paths applied from the lesson plan text, other solutions, or both, we found that seven PSTs mentioned solution paths applied from the text only, four mentioned other solutions only, and 23 mentioned both.

There was a wide range of responses for Question 3, which asked PSTs to interpret the following statement: *Even though counting by ones is an inefficient strategy, it works if done carefully.* Each response was coded with one or more of the following codes: *better ways* if the PST stated there were "better ways" to count or represent 172 than counting by ones; *specific limitation(s)* if the PST stated one or more limitations of counting by ones; *another strategy* if the PST suggested another strategy that students should use; *it works if careful* if the PST stated the strategy works, but students need to be careful; *count by ones* if the PST discussed that it is expected that some students will count by ones to represent 172; and *acceptable strategy* if the PST seemingly took a stance for students who wanted to use that strategy. Table 3, along with text after the table, summarizes the responses to Question 3.

Table 3 Responses to Question 3 (interpreting a phrase)

Number of PSTs	Type of response	Example
21	Specific limitations	This means that it takes longer to count three-digit numbers by ones, and is more prone to mistakes because of the tediousness of the strategy
7	Better ways	Counting by ones is not the quickest way to assess a large number of items. It does work, but there are better ways to do it
6	Counts by ones	It is expected that some students will count by ones even though the number is so large. When it states, "it works if done carefully," I think they are saying that it is ok that students do that
4	Acceptable strategy	The purpose of this lesson is counting 172, not grouping 172. If the student's method is counting by ones and they are getting the correct answer, then they are completing the lesson. From this foundation, you can build with them an understanding of grouping and they may change their method of counting as they grow older

Of the 21 PSTs who stated one or more specific limitations of the counting-by-one strategy, five mentioned another strategy students could use (e.g., grouping larger numbers) and six mentioned that the strategy works if it is done carefully. Of the seven PSTs who stated that there are better ways to count 172, three also mentioned specific limitations of the strategy, while another mentioned the strategy was acceptable. Of the six PSTs who thought it would be expected for students to count by ones, one PST also said that it was okay to do. In this set of findings, PSTs tended to focus on the limitations of the strategy—the first phrase in the selected text: *Even though counting by ones is an inefficient strategy*—rather than on the second part of the sentence, which states that the strategy works if done carefully.

Evaluating Curriculum Materials

In Question 4, PSTs were asked to list the strengths and weaknesses of the 172 lesson for student learning. For this response, we developed a set of 12 codes that categorized ideas listed as strengths or weaknesses. Table 4 lists the code and how many times it was mentioned as a strength and weakness. (The parenthetical comments describe why a PST thought the code was a weakness. For instance, two PSTs thought lack of teacher direction was a weakness.)

Although the PSTs listed a wide range of strengths and weaknesses, we can identify some important themes in looking across their evaluations of the lesson. First, the PSTs focused a great deal on the students' role in the lesson, with "student directed" and "multiple strategies" as the most common strengths. The most

Table 4 Responses to Question 4 (strengths and weaknesses for students)

Code	Frequency as strength	Frequency as weakness
Student directed	14	4
Multiple strategies	15	1
Lesson structure	3	10
Interactions	10	2
Too challenging or not challenging enough	0	11
Concrete	8	1 (Abstract)
Differentiation—lack of/can be/cannot be	7	2
Assessment	4	4
Number choice	1	3
Teacher directed	2	2 (lack of)
Affective (e.g., enjoyable, comfortable)	3	0
Connection to real life	1	2 (lack of)
Connection to more advanced mathematics	1	0

Table 5 Responses to Question 5 (strengths and weaknesses for teacher)

Code	Frequency as strength	Frequency as weakness
Beneficial (e.g., engaging)/unsuitable for students (e.g., low-level task) in some way	19	15
Preparation	13	5
Easy to follow/not enough detail	11	8
Teacher can observe or has freedom/too much freedom	8	2
Differentiation	1	2
Other teacher aspects (e.g., hard to factor time, no model)	0	3

commonly noted weaknesses were in the structure of the lesson (e.g., the lack of an opening routine, the use of worksheets) and that the lesson was perceived as too challenging for some students and not challenging enough for others. Finally, many aspects of the lesson that were viewed as a strength by some PSTs were also viewed as a weakness by other PSTs, suggesting that PSTs vary widely in their evaluations of lessons.

PSTs were also asked to list the strengths and weaknesses of the lesson from the perspective of the teacher (Question 5). Similar to Question 4, we developed a set of codes that categorized ideas listed as strengths or weaknesses (Table 5).

The most frequently reported ideas were ones that focused on student learning and not on teacher features within the lesson plan. Due to the focus on student learning, there were limited responses about teacher aspects. Of those, however, we can see that the PSTs focused mostly on the preparation required for the lesson and whether the lesson was easy to follow.

Responses to the sixth question about whether the 172 lesson had multiple entry points were first sorted into yes or no categories. Eight PSTs did not think the lesson had multiple entry points and all eight stated that this was because only one number was given for students, although some of the eight noted that this number could

Table 6 Responses to Question 6 (multiple entry points)

Number of PSTs	Yes/no	Type of response	Example
8	No	Only one number choice and/or would change number	There is only one number choice provided for the students to work with and it is in the high range of the 100s. Thus, I would provide additional number choices of one slightly above 100 like 112 and another number choice in the middle (e.g., 132) to provide access to a greater range of ability levels
14	Yes	Student develops/uses own strategy	There are multiple ways to draw 172 beans, but there really is not a clear "solution." They already know there are 172 beans and have to draw them. The only difference will be how they drew it
4	Yes	Number can be changed	For slower or higher learners, you could adjust the number of beans to an easier or more difficult number, and give more or less support to the students as needed
3	Yes	Students develops/uses own strategy and number can be changed	I think the lesson has multiple entry points. The lesson does not specifically give a way to illustrate the number. I think students could represent it in a lot of different ways. Because of the number, the lesson may be harder for lower-level students. I would adjust the number for a different range of learners
2	Yes	Count by ones	As long as students know the numbers above 100 and can count by ones, then yes it is accessible to wide-range learners

be adjusted by the teacher, as in the example response in Table 6. Twenty-three of the 34 PSTs thought that the lesson *did* have multiple entry points. For those 23 responses, a set of codes was developed to describe PSTs' reasoning. A response was coded as *student develops/uses own strategy* if the PSTs discussed that the students could develop or use their own strategy to represent 172; *number can be changed* if the PST thought the number could be changed to meet the range of learners in the classroom; and *count by ones* if the PSTs discussed the idea that if students could count by ones to represent 172, then the lesson had multiple entry points. Table 6 summarizes these responses.

All but three responses could be categorized by the codes in Table 6. Of those three responses, one talked about targeting multiple learning styles, another mentioned the teacher being able to ask students to count in a certain way, and the third suggested that the lesson did a good job of providing manipulatives. Another finding was that four PSTs suggested alternative number choices as in the first example

Table 7 Responses to Question 7 (what changes would you make?)

Number of PSTs	Type of response
11	Would make no changes
11	Would provide multiple number choices
7	Would change aspects of the lesson (e.g., add opening routine, count something more meaningful to students) that did not affect overall approach in the lesson
4	Would practice a model beforehand or give example

Table 8 Responses to Question 8 (questioning students about solution strategy)

Number of PSTs	Foci of question
8	Questioned if there was another way to group or represent the leftover 22
7	Questioned students as to why they used groups of 25
4	Questioned if there was another way to group or represent 172 due to having 22 left over
4	Asked another type of question about the strategy (e.g., any patterns)
3	Questioned students as to why they used groups of 25 and if there is another way to group or represent the leftover 22
2	Questioned if there was another way to group or represent 172
2	Questioned if there was a way to consolidate the groups of 25
3	Other

in Table 6. Many PSTs focused on the idea of offering multiple choices as a way to provide multiple entry points for students, or PSTs focused on the idea that students could develop their own strategies. Three PSTs thought a combination of those two ideas provided multiple entry points.

Adapting Curriculum Materials

Eleven PSTs would not make any changes to the given lesson. The remaining PST responses fell into three categories: *would provide multiple number choices, would change aspects of the lesson that did not affect the overall approach in lesson,* and *would practice a model beforehand or give example*. Table 7 summarizes responses to Question 7.

The most frequent adaptation ($n = 11$) was to provide multiple number choices. Seven other PSTs thought they would change aspects of the lesson that did not affect the overall approach in the lesson, and four wanted to provide a model or example before the students began to work.

Question 8 pertained to how PSTs might question students as they engaged with the 172 lesson: *Another pair of students represented 172 with 6 groups of 25 and had 22 left over. What would you say to or ask these students after they have shared their solution?* Responses were categorized according to which aspect of the solution PSTs questioned. In a few instances, a PST questioned multiple aspects of the solution. Table 8 summarizes the foci of PSTs' questions.

Table 9 Responses to Question 9 (use of extra time)

Number of PSTs	Suggested activity
15	Try multiple strategies with another number Either teacher or students choose new numbers Either for practice or trying other students' strategies
12	Lead a discussion on shared strategies Includes finding patterns or connections across strategies
11	Direct students to use a specific strategy Usually focused on grouping by 50 or 100
2	Lead a general discussion of place value
2	Have students play a math game

Most interesting to us was how PSTs reacted to the leftover 22. Eleven PSTs (first and fifth rows) asked if there was another way to group or represent the leftover 22 even with the other groups of 25 as in this response: "I would say, 'How can we group the leftover 22 in an organized way? How could we split those 22 beans into 5 groups?'" Another four PSTs wanted students to regroup the 172 entirely due to the leftover 22 as in the following response: "Could you have made less groups of more beans in order to not have so many left over?" Two others wanted students to consolidate their groups of 25.

Quite opposite from the disdain for the grouping-by-25 strategy was the connection one PST, who fell in the "other" category, made to money:

> That's another great idea as we know that just like in a dollar there are 4 quarters (25), right? (relating it to a real life situation) then break it down further like 2 quarters in 50 cents, so there would be 22 ones left over (if you are thinking in those terms).

The other responses seemed not to value or disvalue the grouping of 25 strategy, as PSTs just asked students to explain why they grouped by 25 or if there was another way to group or represent 172. The intent of these responses may have been to support or extend student thinking.

Finally, Question 9 asked PSTs how they would use the remaining 15 min of class after multiple student solutions had been shared. The question also asked PSTs to share a rationale for their use of this time. After reading through each of the 34 responses, we identified five distinct activities proposed by the PSTs, as listed in Table 9.

Some of the 34 responses included more than one of these activities. Of these, some suggested engaging students in more than one of the activities during the 15 min, while others provided two different possibilities along with (sometimes) the criteria they might use for choosing between these possibilities. If a PST's response included more than one activity, we coded for both activities, so that the total number of suggestions in the second column of Table 9 is 42, more than the number of PST responses ($n=34$).

The first two activities, identified by the majority of PSTs, seem most appropriate to us in terms of following the suggestion provided in the curriculum materials and building on children's mathematical thinking. The last two activities (general discussion and math game) do not contradict the goals of the lesson, but they are less closely related to those goals. Finally, the third activity, “Direct students to use a specific strategy,” seems to run counter to the overall approach indicated in the lesson, as well as contradicting, more generally, an approach to teaching that builds on children’s mathematical thinking.

Discussion and Implications

In thinking about a trajectory toward expert curriculum use, our results—combined with what we have learned from the research presented in the literature review—provide us with some preliminary trajectory levels. Schwarz et al. (2008) writes that learning progressions[4] specify “how knowledge and practices can be built over time, by articulating successively more sophisticated versions of the knowledge” (p. 2). In the remainder of this section, we will illustrate how our findings from the questions and responses presented above inform a trajectory of increasingly sophisticated instances of curriculum knowledge and use, particularly for PSTs.

Results from the goals question (Question 1) provide us with a clear initial trajectory. At the initial levels, PSTs stated procedural goals only. Goal statements were more sophisticated if they mentioned making connections across multiple strategies and/or representations, had a conceptual focus, or had both a procedural and conceptual focus. At the highest level, PSTs wrote goal statements that included a procedural and conceptual focus *and* mentioned making connections.

It is interesting to consider this progression of goal statements in light of Brown’s (2009) framework related to the ways in which teachers offload, adapt, or improvise when using curriculum materials. Our analysis process found that the goal statements provided in the materials were coded as procedures with connections and that was the code with the most frequency ($n=11$ or 33 %) in PSTs’ responses. In these instances, PSTs offloaded the goal statements. In the other cases, PSTs adapted the statements. Some of these adaptations lowered the level of sophistication, as when PSTs omitted the element of “connections” from their goals; other adaptations increased the sophistication beyond the provided goals by adding a focus on conceptual, as well as procedural, goals.

Not all curriculum materials, however, provide goal statements in the same manner as *Math Trailblazers*. For instance, consider the following goal statement from a third-grade lesson of *Investigations* (TERC 2008): “Solving addition problem with 2-digit numbers that involve more than 10 in the ones place and explaining the effect on the sum” (p. 44). Because this goal has a conceptual (explain the effect on the sum) and procedural focus (solving addition problems with two-digit numbers), offloading would be more appropriate than adapting unless that adaption included making connections. Our point is to indicate that knowing when to offload, adapt, and improvise is an important aspect of being an expert curriculum user, and that

[4] We are using the terms *progression* and *trajectory* as synonymous.

the choice of which strategy (offloading, adapting, or improvising) is most sophisticated will vary based on the particular lesson or set of materials teachers are using.

The results related to the student solution question (Question 2) also contribute to our development of the trajectory. Of the four given solutions, no PST mentioned the third solution of 72 more than 100, which led us to believe that PSTs did an analysis of the given solutions and thought that solution unlikely or uninteresting—or that they did not understand that strategy. Some PSTs applied solution strategies given in the materials to a new number, while some considered other likely strategies, and others did both. It is unclear if the PSTs who only considered other likely strategies did not attend to the ones given in the materials, or if they were expanding upon that solution set. Either way, being able to apply and generate solutions is the most sophisticated response, with not attending being the least.

Empson and Junk (2004) found that teachers' knowledge of children's solution strategies could be linked to their use of curriculum materials. Our results from Question 3 indicate that PSTs know something about the counting-by-ones strategy (either from previous knowledge or from reading the text): some students may use it, it is inefficient, and it works. PSTs' interpretations of the phrase, however, differed. While we want PSTs to learn something about children's strategies, accepting and building on those strategies as well as not trying to impose other strategies on students is also important (Jacobs and Ambrose 2008). Therefore, PSTs who did both—learn something about and honor students' strategies—are demonstrating knowledge at the most sophisticated level. Question 8 (responding to a student strategy) reiterates the honor and accepting aspect of this level, as most PSTs showed obvious disdain for or acceptance of the grouping by 25s solution.

Several prior studies have found that PSTs have varying levels of success in using constructs provided in class to evaluate curriculum materials (e.g., Beyer and Davis 2009; Lloyd and Behm 2005; Nicol and Crespo 2006). Each of these studies used different constructs with which PSTs were to evaluate curriculum materials. For the purpose of generating a trajectory, we use the term *evaluation constructs* to reflect any construct used in such a manner. In Question 6, the evaluation construct of multiple entry points was, in part, used by all PSTs. Some recognized the limitation of only one number choice, while others recognized that students could develop their own strategies. Three PSTs recognized both aspects. Further, some of those who recognized the number choice limitation stated that they would adapt the lesson by providing alternate number choices. Therefore, we assert that limited use to full use of an evaluation construct, as well as pairing an adaption within the construct, represents increasing sophistication.

Educative features in curriculum materials are meant for teacher learning (Davis and Krajcik 2005), but those features are supportive only if PSTs attend to and understand the intent of those features. "Counting One Hundred Seventy-two" provides educative features (e.g., student solution examples, lesson description, informal assessment strategies) within the lesson, which provides the opportunity for teacher learning. Many PSTs focused on student learning even when asked about strengths and weakness for the teacher. Other PSTs focused on features that attended to the supports for the procedural aspects of the lesson (e.g., preparation). At the most sophisticated level, PSTs focused on educative features (Davis and Krajcik 2005)—features meant for teacher learning. In these cases, PSTs noticed that teachers are provided with an occasion to observe students at work within the lesson and/or the lesson is adaptable to fit students' needs.

Finally, taken together, the results from Questions 7 and 9 provide us with some considerations about PST adaptations. In both questions, a few PSTs indicated that they would provide a model or direct students to a particular strategy either before or after the student exploration time, which we contend is the lowest level of sophistication. Using Seago's (2007) Categories of Adaptation—fatal, no impact, and productive—we classified this type of adaptation as fatal, because it is "contrary to the basic design or values of the materials" (p. 25). Adapting the lesson in ways that are not clearly related to the lesson goal, such as incorporating math games, reflects a middle level of sophistication, which is likely to have no impact. The remainder of the responses to these two questions cannot be sorted into varying levels of sophistication, as they all represent reasonable choices and adaptations in the enactment of this lesson, particularly given that the PSTs were planning this lesson for an abstract, generalized group of students. Using Seago's Categories of Adaptation, we would classify these types of adaptations as potentially productive.

Based on the above analyses, as well as the research provided in the literature review, we have generated a curriculum knowledge and use trajectory (Fig. 2). To be clear, the trajectory describes levels of knowledge and practices related to particular aspects of the use of curriculum materials. The trajectory cannot be used to describe individual PSTs, as it is possible that one PST might demonstrate knowledge and practices at multiple levels simultaneously. We present our trajectory as a construct map (Wilson 2005, 2009). "A construct map is a well thought out and researched ordering of qualitatively different levels of performance focusing on one characteristic" (Wilson 2009, p. 718). The figure should be interpreted as increasingly sophisticated knowledge about the domains, which are organized by bullet points. For instance, the first bullet point in each level is about the goal statements. Because the levels are organized around teaching domains, a PST could be in differing levels for each of the domains.

Conclusion

Our investigation into PSTs' curricular knowledge allowed us to begin to elaborate a trajectory of curriculum knowledge and use for PSTs (Fig. 2). In particular, we learned that PSTs could take supports given in the materials and use them in productive ways and, at times, even go beyond the materials. For instance, several PSTs provided goal statements that went beyond the sophistication of those given in the materials. At the same time, some PSTs interpreted supports in ways that did not necessarily align with the intent of the lesson. It is not surprising to find that PSTs' curricular knowledge and use are varied, but specifics about that variation will help mathematics teacher educators (MTEs) and researchers support PSTs' curricular knowledge development.

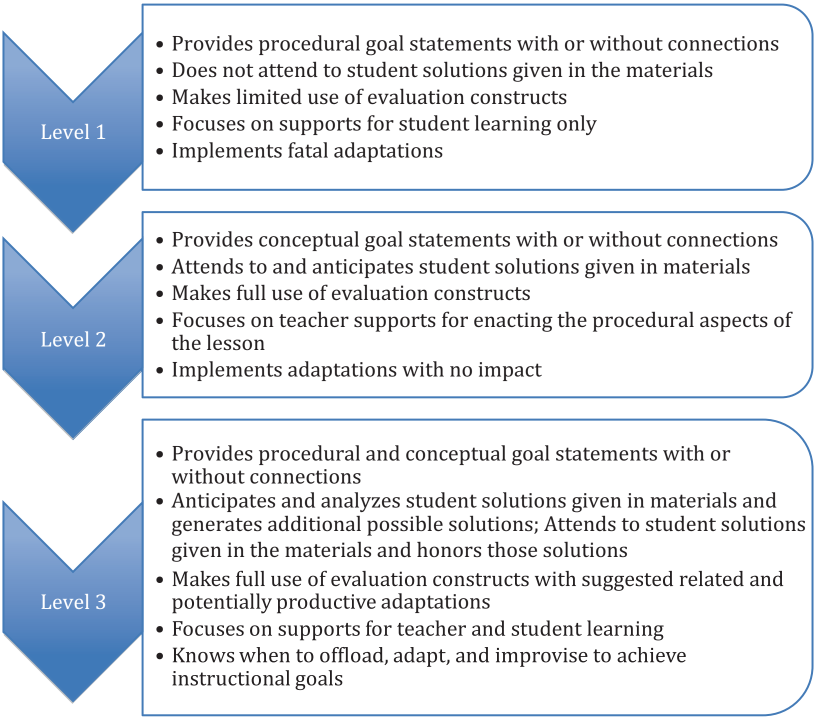

Fig. 2 Curriculum knowledge and use trajectory

The curriculum knowledge and use trajectory is an initial depiction of the development of expert curriculum use knowledge and practices. Further additions and refinement are needed. Specifically, we do not know how generalizable this trajectory is to other curriculum series or even to other lessons within the *Math Trailblazers* series. For instance, goal statements may be presented differently in any given set of materials (as we discussed earlier), making use of the various series different. Additionally, not all materials provide support in helping teachers anticipate student solutions, or the supports are designed differently, causing differences in how they are read, which would have implications for how they are applied (Tyminski et al. 2013). It seems likely that both the design of the materials and the capacity of the individual using the materials contribute to observed levels of curriculum use knowledge and practices. In other words, a PST (or practicing teacher) could exhibit very high levels of curriculum use when using one set of materials and lower levels when using another set of materials. The question of whether curriculum knowledge and expert curriculum use are generalized or curriculum-specific constructs remains an open question.

Nonetheless, this study represents a first step in the development of a measure of expert curriculum use. These survey questions and our findings about the range of PSTs' responses to the questions can help researchers to further develop the constructs of curriculum use and curricular knowledge, through an understanding of ways in which PSTs read, evaluate, and adapt curriculum materials. At the same time, these findings might support MTEs in designing learning experiences for PSTs that contribute to the development of PSTs' curriculum use practices. For instance, the survey questions and trajectory can help MTEs design course activities and anticipate PSTs' responses to those course activities. Additionally, the trajectory can help MTEs design interventions that will prompt them to respond to course activities in increasingly sophisticated ways. Ultimately, these questions, and others like it, can be used to understand PSTs' growth in knowledge and practices as they progress through teacher education courses and programs.

Appendix A: Survey Questions

1. Reading
   a. As a teacher, what would be your specific goal(s) for your students' learning with this lesson?
   b. Write a short (2–4 sentences) summary of the lesson.
   c. What is the teacher expected to do in this lesson?
   d. What are the students expected to do?
   e. What kinds of solutions do you think a teacher might see?
   f. How would you assess this lesson?
   g. On page 37 in the first bullet point under the assessment heading, the lesson plan states, "Even though counting by ones is an inefficient strategy, it works if done carefully." What does that mean?
2. Evaluating
   a. When thinking about student learning, what are the strengths and weaknesses of this lesson?
   b. From the perspective of the teacher who has to prepare for and enact this lesson, what are the strengths and weaknesses of this lesson plan?
   c. Does this lesson have multiple entry points? In other words, is the task accessible to a wide-range of learners? Explain your answer.
   d. Are strategies to find representations of 172 to be generated by the textbook/teacher or the student? Explain.
   e. What would be an ideal solution to this task? What would a child have to do or say exactly to convince you (the teacher) that he/she has mastered your learning goals?
3. Adapting and Children's Solutions
   a. Would you make any changes to the lesson before teaching it? If so, what would they be?
   b. A student starts making individual tallies, but stops at 45 and states that, "172 is too many." What would you say to or ask of this student?

c. A student has mistakenly made 163 individual tallies. What would you say to or ask of this student?
d. When asking students to share their solutions, the first pair of students represented 172 with 34 groups of 5 tally marks and had 2 left over. What would you say to or ask these students after they have shared their solution?
e. Another pair of students represented 172 with 6 groups of 25 and had 22 left over. What would you say to or ask these students after they have shared their solution?
f. During this lesson, these were the correct solutions that you saw and were shared:
   1. 172 individual tallies
   2. 34 groups of 5 with 2 left over
   3. 6 groups of 25 with 22 left over
   4. 17 groups of 10 with 2 left over

You still have 15 min of class after students share their representations of 172. What would you do with the remainder of class time? Why?

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Mathematical Knowledge for Teaching and its Specificity to High School Geometry Instruction

Patricio Herbst and Karl Kosko

In his description of paradigms for research on teaching, Shulman (1986a) called for a focus on teacher knowledge. With particular reference to mathematics, Ball et al. (2001) responded to Shulman's call by pointing out that research using traditional measures of teachers' content knowledge (e.g., degrees obtained or mathematics courses taken) "leaves obscured the nature of teachers' knowledge" (p. 443). Instead, they argued that research needed to focus on a particular kind of mathematical knowledge—*mathematical knowledge for teaching* (MKT). This MKT is knowledge of mathematics used in doing the work of teaching and it includes, but also goes beyond, the pedagogical content knowledge that Shulman (1986b) himself had proposed. The theoretical and empirical work on Ball et al.'s brand of MKT that followed has been vast, showing, among other things, that the possession of MKT can be measured, that MKT is held differently by teachers and non-teachers, that MKT is held differently by teachers of higher-grade level experience than those of lower grade-level experience, that it makes a difference in students' learning, and that scores on MKT correlate with scores on an observation measure of the mathematical quality of instruction (Hill et al. 2004, 2005, 2008b). The work of that group constructing measures of MKT has been concentrated mostly on the mathematical knowledge of elementary and middle school teachers (Hill 2007; Hill and Ball 2004); a more recent effort has developed MKT items in algebra (Mark Thames, personal communication, June 15, 2011).

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The purpose of this chapter is to report on a parallel effort to use the same approach to develop an instrument that measures *mathematical knowledge for teaching high school geometry* (MKT-G). While we recognize that other approaches to MKT at the secondary level have started from conceptualizing it independently (e.g., Adler and Davis 2006; Even 1990; McCrory et al. 2012; Saderholm et al. 2010; Silverman and Thompson 2008; see also Steele 2013), we have preferred to start from the conceptualization by Ball et al. (2008). While all of the other conceptualizations had compelling features, none of them had accumulated as critical a mass of empirical research as the MKT approach spearheaded by Ball and Hill. Furthermore, only the approach by McCrory et al. (2012) had been developed in a way that supported the creation of measurement instruments, and their approach did not seem to examine elements of pedagogical content knowledge that we saw as important to consider. Thus, our effort has attempted to follow the theoretical conceptualization of MKT and the item development procedures of Ball and Hill's group, even though, as we suggest below, we anticipated that some specification of MKT might need to be developed to respond to instructional demands specific to courses of study at the high school level. This chapter describes how we developed MKT-G items, provides pilot data that compares high school teachers with and without experience teaching geometry in terms of their possession of MKT-G as measured by selected items, and uses these results to raise some questions about the instruction specificity of the notion of MKT.

A crucial element in our development of items to measure MKT-G has been Ball et al.'s (2008) conceptualization of the different domains of MKT. According to Ball et al., the mathematical knowledge used in teaching can be conceptualized as the aggregation of knowledge from six domains. Our work focuses on four of these domains: *common content knowledge* (CCK), *specialized content knowledge* (SCK), *knowledge of content and teaching* (KCT), and *knowledge of content and students* (KCS). CCK is the mathematical knowledge also used in settings other than teaching, including, for example, knowledge of canonical methods for solving the problems teachers assign to students. SCK is knowledge of mathematics used particularly in doing the tasks of teaching, such as, for example, the knowledge teachers need to use in writing the problems they will assign to students or in figuring out "whether a nonstandard approach would work in general" (Ball et al. 2008, p. 400). KCT is defined as a combination of knowledge of teaching and knowledge of mathematics and includes the knowledge needed to decide on the best examples and representations to use for given instructional objectives. KCS includes a blend of knowledge of mathematics and of students' thinking, such as the capacity to predict what students might find confusing or what kind of errors students might make when attacking a given problem. In our effort to construct measures of MKT-G, we developed items that purport to measure each of those four domains—CCK, SCK, KCT, and KCS. Ball et al. also include horizon content knowledge (HCK) and knowledge of content and curriculum (KCC). At the time we started work on our MKT-G instrument, we felt that the KCC and HCK domains needed further conceptualization and that there were not sufficient examples of HCK and KCC items at other levels of schooling that could allow us to create analog items to measure teachers' knowledge of those domains in high school geometry.

Ball and Hill's Learning Mathematics for Teaching (LMT) project has developed items that measure the different domains of MKT and, over time, has paid attention to different content strands, including number and operation, patterns, functions and algebra, and geometry. These instruments have also included items that purport to measure mathematical knowledge for teaching middle school mathematics as well as for teaching elementary school mathematics. The extensive item development has yielded numbers of validated items that can be put together into forms that assess MKT for particular content strands. But there has not been, as of yet, a systematic development of items to measure MKT in different content strands or deliberate theoretical consideration about how content-strand differentiation might interface with the domains of MKT (Heather Hill, personal communication, February 8, 2012). As we think about conceptualizing and measuring mathematical knowledge for teaching high school mathematics, it is important to ponder whether and how the specific practice of teaching particular high school mathematics courses should be considered and featured in the process of designing measures of the mathematical knowledge for teaching those courses. In this chapter, we present our beginning attempts to conceptualize such instruction specificity, within the framework of MKT, by reporting on our development of an instrument to measure MKT-G. It is worth elaborating briefly on what we mean by instruction specificity here, since it is obvious that Ball et al.'s (2008) conceptualization of MKT is already rooted in a notion of instruction as the interactions among teacher, students, and content in environments (Cohen et al. 2003). As explicated in Herbst and Chazan (2012), we use *instruction* along similar lines, to allude to teacher–student transactions of knowledge. While *instruction* names that phenomenon in general, we expect that the nature of the knowledge at stake (as represented in syllabi and textbooks of a year- or semester-long course of studies) makes instruction specific (that is, shapes the nature of the transactions of knowledge that take place). Thus, a first approximation to the question of what an instruction-specific MKT looks like is to ask, what is the MKT that a teacher would need in order to teach a given course of studies? Thus, our purpose to design an instrument to assess MKT-G is concerned not with geometry as a mathematical domain but with high school geometry as a course of studies; we posit that similar approaches need to be taken regarding high school algebra and other such courses of study (further approximations might also consider specification to particular groups of students, particular environments, and particular "kinds" of teaching).

Our interest in MKT originated from our attempts to contribute to a theory of mathematics teaching that accounts for what teachers do in teaching in terms of a combination of, on the one hand, individual characteristics of practitioners and, on the other hand, practitioners' recognition of the norms of the instructional situations in which they participate and of the professional obligations they must respond to (Herbst and Chazan 2011, 2012). This effort contributes to a long-term agenda that seeks to understand the work of teaching in specific instructional systems, such as high school geometry. Our earlier work had been dedicated to conceptualizing and grounding (through examining records of intact classrooms and of instructional interventions) the didactical contract in geometry (Herbst 2002, 2003) and instructional situations in geometry (Herbst 2006; Herbst et al. 2009, 2010). In the context of that work, we developed a proposal for the description of teachers' actions and decisions as responses to norms of the role teachers play in activity systems (such as didactical contracts and their instructional situations) and obligations of the position of mathematics teacher in an institution (Herbst and Chazan 2012). In that context, we had asked the question of how individuals' MKT combines with their recognition of professional obligations and of norms of an instructional situation in accounting for the decisions they make in classroom events framed by those norms. The present chapter was part of a larger effort to develop measures of the constructs that we had contributed (particularly recognition of *norms* and *obligations*), as well as measures of other constructs that would allow us to estimate the resources possessed by individual teachers.

Development of the MKT-G Instrument

The conceptualization and disciplined approach to measuring MKT spearheaded by Ball and Hill (Ball et al. 2008; Hill and Ball 2004) provided us with important guidance for the development of MKT measures. This guidance included not only the conceptualization of the domains and some heuristics for how to create items, but also an awareness of the complexity of the task ahead. In the interest of proceeding toward our goal by taking manageable steps, we also followed the example from Ball and Hill's group in developing multiple-choice and multiple-response items as opposed to constructed-response or open-ended items. The following sections describe the work we did to develop an MKT-G instrument.

Constructing Items for MKT-G

Our item development process covered a relatively wide range of topics from the high school geometry course. We consulted curriculum guidelines in various states and on that basis sought to develop items dealing with definitions, properties, and constructions of plane figures, including triangles, quadrilaterals, and circles; parallelism and perpendicularity; transformations; area and perimeter; three-dimensional figures; surface area and volume; and coordinate geometry. Those topics by themselves provided sufficient guidelines to create items of CCK. But the definitions of the MKT domains, particularly the definition of SCK, call for items that measure knowledge of mathematics used in the tasks of teaching. To draft these items, we found it useful to create a list of tasks of teaching in which a teacher might be called to do mathematical work. The list included elements such as designing a problem or task to pose to students; evaluating students' constructed responses, particularly student-created definitions, explanations, arguments, and solutions to problems; creating an answer key or a rubric for a test; and translating students' mathematical statements into conventional vocabulary. Teachers of different courses of mathematical study might engage in those tasks of teaching, but as we sought to draft these items, we noted that those tasks of teaching could call for different kinds of mathematical work depending on specifics of the work of teaching geometry. For example, the task of designing a problem would involve a teacher in different mathematical work if the designed problem were a proof problem versus a geometric calculation. While the former might involve the teacher in figuring out what the givens should be to make sure the desired proof could be done, the latter might involve the teacher in posing and solving equations and checking that the solutions of those equations represented well the figures at hand. Thus, while a list of generic tasks of teaching was useful to start the drafting of items, this list appeared to grow more sophisticated with attention to tasks that are specific to different instructional situations in geometry teaching (Herbst et al. 2010). As we note below, this observation led to interest in comparing responses to items that were differently related to instructional situations in geometry, and to making some conjectures about the organization of the MKT domains.

The tasks of teaching were also useful in drafting items that measured KCT. To draft these items, we used as a heuristic the notion that the item should identify a well-defined instructional goal and the possible answers should name mathematical items that, while correct in general, would be better or worse choices to meet the specified goal. For example, teachers often need to choose examples for the concepts they teach. While different examples might be mathematically correct, they might not all be pedagogically appropriate to meet particular instructional goals. One example may be better than others if the instructional goal is to show a first or canonical example, while another example may be better if the goal is to illustrate an extreme case.

Finally, to create items that measured KCS, we were attentive to the definition provided by Ball et al. (2008) and especially sought to draft items that tested for knowledge of students' errors and misconceptions (Hill et al. 2008a). As in the case of other domains, there were specifics of the high school geometry class that shaped the items we developed. Thus, while we did create items that probed for teachers' knowledge of students' misconceptions about geometric concepts (e.g., angle bisector), we also created items that probed for their knowledge of students' misconceptions about processes or practices that are specific to geometry—such as the notion that empirical evidence is sufficient proof (Chazan 1993) or that definitions are exhaustive descriptions (Herbst et al. 2005).

Our research group drafted and revised an initial set of questions, including 13 CCK, 20 SCK, 26 KCT, and 16 KCS questions; this drafting and revision process relied, among other things, on general guidance and comments on specific items by Deborah Ball, Hyman Bass, Laurie Sleep, and Mark Thames.[1] The questions drafted took the form of multiple-choice items, as well as multiple-response items within a single question (e.g., a single stem with 3–4 yes/no questions following).

[1] Daniel Chazan, a co-PI of this project, was also involved in design discussions. Individuals involved in the drafting of items, in addition to the authors, included Michael Weiss, Wendy Aaron, Justin Dimmel, Ander Erickson, and Annick Rougee.

Table 1 Revisions to KCT item stem following cognitive pretesting

Original wording (before cognitive pretesting)	Revised wording (after cognitive pretesting)
While preparing to teach the theorem that says that base angles of an isosceles triangle are congruent, Ms. Gomez is pondering which among the following, valid mathematical proofs to use with her 9th grade geometry students. Given the following sketches of proofs, which ones would help students understand why the base angles theorem is true?	Ms. Gomez is preparing to teach the theorem that says that base angles of an isosceles triangle are congruent. She is pondering which among the following, valid mathematical proofs to use with her 9th grade geometry students. Of the valid proofs below, which one is the least appropriate for Ms. Gomez to use with her students in explaining why the theorem is true?

Teachers' Interpretations of MKT-G Items

Our items were submitted to a process of cognitive pretesting (Karabenick et al. 2007), which assesses the degree to which participants interpret items in the manner intended by those who wrote them. This process involved interviewing experienced geometry teachers ($n = 11$) as they completed the items. Teachers were asked to read the item prompt (or stem) and then tell us what they thought the item was asking them to do. The teachers would then be asked to select their response and tell us why they selected the option they did.

Responses to these prompts were used to examine the content validity of the items, as well as to improve such validity. We coded teachers' responses in regard to whether the stem was interpreted as intended, as well as whether the justification for their response was mathematical and/or pedagogical in nature. The vast majority of items were found to be interpreted as intended, with a subset of items being revised in accordance with evidence from the data. For example, one KCT item's stem was revised to make a provided theorem more explicit in the description of the task of teaching. The original and revised wordings of the stem are presented in Table 1.

In the original stem's wording, the base angles theorem is presented explicitly, but we found that some teachers' justifications for their responses did not focus as much on the theorem as anticipated. In reexamining the stem, we observed that the task of teaching was directed to choosing a proof for a theorem, but included some distracting wording (e.g., a special naming of that theorem, the description of the response options as sketches of proof). While seemingly minor, the revised wording centers the task of teaching (choosing a proof for the theorem) more clearly within the context of the base angles theorem. To further improve the way the item targeted the intended MKT domain of the stem, we also changed the question from a multiple-response format to a multiple-choice format and noted that the proofs were all valid.

All items were revised in accordance with evidence from the cognitive pretesting data to improve interpretability and validity. While we conducted cognitive pretesting as a means of identifying potential miscues in our items, the process also provided some preliminary evidence as to the construct validity of our items. Specifically, teachers were consistently observed to describe the mathematics of an item in selecting their response (approximately 95 % of the time). This was fairly consistent across all domains (CCK, SCK, KCS, KCT). Also, teachers were observed to refer to pedagogical issues much less frequently when focusing on CCK and SCK items (54 and 45 % of responses, respectively) than on KCS and KCT items (91 and 72 % of responses, respectively). This provided some evidence that items designed to measure KCT and KCS domains might indeed assess those aspects of pedagogical content knowledge, while items designed to measure the CCK and SCK domains might indeed assess content knowledge. Trends for particular items were used to inform revisions; for example, teachers did not discuss pedagogical issues at all for one KCT item, so this item was revised to appeal to such reasoning. The skewing of pedagogically related justifications in KCS and KCT items suggests a relatively good fit for most items per construct, even before these items were revised.

Piloting the Revised MKT-G Items

In July and August 2011, we piloted all MKT-G items revised from cognitive pretesting with a sample of 48 secondary mathematics teachers in a midwestern state. By design, each domain contained ten questions. These questions included multiple-choice and multiple-response formats. In examining the statistical reliability of our questions, multiple-response questions were treated as multiple items, with the number of items per a multiple-response item dependent on the number of accepted responses. Therefore, while each domain contained ten questions, different domains contained a different number of items for the purpose of statistical item analysis (CCK = 15; SCK = 29; KCT = 10; KCS = 10). In examining the fit of items for each separate domain, we used biserial correlations (Crocker and Algina 2006) to measure item discrimination or how well the items discriminated between higher- and lower-scoring test takers. Biserial correlations were examined in concert with item difficulty (percent of participants correctly answering the item) and Cronbach's alpha. This initial examination of the items allowed us to make some decisions regarding the removal of some items due to poor statistical fit. Particularly, responses for some items had near-zero or negative correlations with the overall response patterns of other items per domain. Based on this preliminary examination and pursuant to our goal to discard items that did not perform well, we removed 46.9 % of all items from the MKT-G item pool and retained 34 items, with the intent to collect additional data to confirm the reliability of the retained items.

Items were removed for various reasons. Some had low discrimination power; they lacked an ability to distinguish between lower- and higher-scoring participants per domain. Other items that performed poorly had apparent liabilities, such as being too long or having response options that were too similar to each other; these liabilities had been somewhat anticipated in the cognitive pretesting process. For example, one KCT item prompted teachers for the most appropriate feedback to provide a student in regard to an answer the student had presumably provided on a test. The correct response was answered by 34 % of participants, but one distracter was chosen by 40 % of participants. While scores of the 34 % of participants were marginally higher than the 40 % who answered incorrectly on the popular distractor, this difference was not great enough to help the item discriminate between lower- and higher-scoring participants. In principle, revisions might be made to improve the quality of those items. However, we were ever mindful of the time commitment necessary to complete our test, and chose to put revising those items aside for the time being. The retained 34 items comprised what we refer to in the remainder of this chapter as the MKT-G instrument. The items per domain ranged from 7 to 11 (CCK = 9; SCK = 11; KCT = 7; KCS = 7). See the Appendix for publicly released items.

Piloting the MKT-G Instrument

In May 2012, the MKT-G instrument was piloted with 35 secondary mathematics teachers in the mid-Atlantic region. For the purposes of the remainder of this chapter, all data reported are pooled from both the Midwest and mid-Atlantic samples ($n = 83$). Questions from each domain were uploaded into the Lesson*Sketch* online platform (www.lessonsketch.org) and completed by participants, who took them either by coming in person to a computer lab (73 participants) or by responding to the items online from their homes or workplace (10 participants). We note that neither the midwestern nor the mid-Atlantic sample had participated in any special professional development related to geometry prior to completing the MKT-G instrument. Participants were predominantly females (67.5 %) and Caucasian (67 %). Other reported ethnicities included African American (13 %), Asian American (5 %), and other (16 %). Participants varied in the amount of mathematics teaching experience ($M = 14.55$, $SD = 9.75$) and mathematics content courses taken in college ($M = 11.27$, $SD = 6.08$). Regarding geometry-specific experience, 60.2 % of participants were classified as experienced geometry teachers because they had taught geometry for 3 years or more ($M = 4.57$, $SD = 4.55$), and teachers had taken an average of 1.70 college-level geometry content courses ($SD = 1.41$).

While we later discuss MKT geometry in terms of a single score that we have found to have sufficient reliability, we briefly discuss item analysis and preliminary comparisons by domain (CCK, SCK, KCS, KCT). Such a description is not generalizable in nature, but provides interesting points of reference for later discussion and investigation.

Analyzing the MKT-G Instrument by Domain

We reexamined item fit on the smaller subset of items using biserial correlations (with a threshold of 0.30) as one indicator for reliability in each domain. Item difficulty in CCK ranged from 20.5 to 92.3 %. Item difficulty in SCK ranged from 21.7 to 88.0 %; in KCT it ranged from 18.1 to 48.3 %, and in KCS it ranged from 15.7 to 73.5 %. We also calculated Cronbach's alpha to provide preliminary data regarding

Table 2 Descriptive statistics by MKT domain

Domain	*M*	*SD*	*N*	*α*
[a]CCK—Geometry	0.66	0.20	83	0.58
[b]SCK—Geometry	0.59	0.21	83	0.65
[c]KCT—Geometry	0.39	0.21	83	0.50
[d]KCS—Geometry	0.38	0.25	83	0.49

[a] Common content knowledge
[b] Specialized content knowledge
[c] Knowledge of content and teaching
[d] Knowledge of content and students

Table 3 Correlations between MKT-G domain scores

	CCK	SCK	KCT	KCS
CCK	–	–	–	–
SCK	0.40***	–	–	–
KCT	0.36**	0.54***	–	–
KCS	0.69***	0.48***	0.43***	–

$*p<0.05$, $**p<0.01$, $***p<0.001$

reliability of each domain on its own (Table 2). While the alpha coefficients are not as high as desired (they are below the typically accepted threshold of 0.70), these statistics provided us with valuable information regarding future steps, should we choose to focus specifically on assessing particular domains in the future. One explanation for lower reliabilities in some domains is related to both the range in difficulty of items and the number of items per domain. For example, SCK, which has acceptable reliability ($\alpha = 0.65$), had a large range in item difficulty, but also had the most items (11); KCS had a much lower alpha coefficient, mainly due to having fewer items. Therefore, a more detailed study of KCS would require us to construct additional items of varying difficulty levels.

Correlations between the domain scores are presented in Table 3 and suggest moderate to strong relationships between the different domains. These results show similar trends to those found by Hill et al. (2004) for CCK and KCS, which suggests that the different domains are, to a degree, interrelated. Thus, these correlations encourage thinking of a single MKT-G measure made of aggregating the scores per domain. The use of items from different domains to make that single score helps argue that the MKT-G instrument assesses the MKT construct as conceptualized by Ball et al. (2008).

We also examined correlations, by MKT domain, with teaching experience in general and with experience teaching high school geometry in particular, as well as with college coursework in mathematics (we did not collect specific information about education classes taken or geometry topics covered in education classes because we did not anticipate enough variability in this number to warrant examining correlations). The correlations with experience and mathematics coursework (shown in Table 4) are preliminary. However, they provide an intriguing picture of how teachers with different experiences hold knowledge for teaching geometry and suggest that MKT-G may be learned from experience teaching geometry. These

Table 4 Correlations between domain scores and experience and coursework

	Years experience		Content coursework	
	Years teaching mathematics	Years teaching geometry	Total math courses	Total geometry courses
CCK-G	−0.06	0.38***	−0.05	−0.02
SCK-G	0.09	0.35**	0.05	0.04
KCT-G	0.04	0.19[a]	−0.00	−0.05
KCS-G	−0.02	0.25*	0.10	0.08

[a]$p<0.10$, *$p<0.05$, **$p<0.01$, ***$p<0.001$

results show that neither the total number of mathematics courses nor geometry courses taken in college correlate with MKT-G scores. Likewise, years of experience teaching mathematics in general also show near-zero correlations. However, for each domain in MKT-G, there are indications of a relationship with experience teaching the geometry course. Therefore, it seems that the MKT-G instrument may measure a domain of knowledge specifically related to the teaching of high school geometry.

Preliminary Evidence for Trends in Teachers' MKT-G

The previous section discussed preliminary findings for each domain (CCK, SCK, KCT, KCS) using raw scores. In order to better understand how teachers' demonstrated understandings of MKT-G as a single construct were affected by varying background factors, we constructed scores using Item Response Theory (IRT). Therefore, discussion of scores and item statistics in this section examine MKT-G as a single construct (i.e., CCK, SCK, KCT, and KCS items are pooled together). While it may also be useful to construct IRT scores for specific domains of CCK, SCK, KCT, and KCS, more items would need to be included to better assess the possible variance in each specific domain. Since more items mean a longer, lengthier test, we elected not to take such an approach at this time.

As Crocker and Algina (2006) describe:

> With item response theory the test developer assumes that the responses to the items on a test can be accounted for by latent traits…[that] accounts for the responses to items on a test. At the "heart" of the theory is a mathematical model of how examinees at different ability levels for the trait should respond to an item. (p. 339)

This latent trait is represented by the statistic theta (θ), which is calculated both for particular items and as a test score for particular individuals. Where classical test theory approaches take the percent correct on a test as the indicator for the latent trait being measured, IRT takes into account that some items are more difficult than others, and thus a raw score is not the most accurate measure of the latent trait (see Chap. 15 in Crocker and Algina 2006 or Wilson 2005, for introductions to IRT). We borrowed from approaches used by the Learning Mathematics for Teaching (LMT) project group led by Deborah Ball (Hill et al. 2004, 2008a) and used a 1-parameter IRT model, which is mathematically equivalent to a basic Rasch model. According to Linacre (1994), a minimum sample of 50 examinees is sufficient for obtaining estimates for a basic Rasch model, which our sample (n = 83) exceeds.

Results from our IRT modeling show sufficient item reliability (0.96) and person reliability (0.82). Item reliability indicates how well our survey distinguishes between easier and more difficult items, and is generally considered acceptable if above 0.90. Person reliability indicates how well our survey distinguishes between groups of people (e.g., lower and higher scorers), and is generally considered acceptable if above 0.80. Possible θ scores for individuals on the MKT-G ranged between −5.50 (low demonstrated MKT-G) and 5.45 (high demonstrated MKT-G). A person who scores zero demonstrates an average level of MKT-G. Such an individual would have a 50 % chance of correctly answering an item with θ value zero (that is, an item of average difficulty), but would have a greater chance of correctly answering items of lower θ values (and smaller chance of correctly answering items of higher θ). For our sample, participants' overall scores ranged from −2.27 to 3.43 (M = 0.19, SD = 1.03). This suggests that our sample did not contain individuals who had extremely high (−5.50) or low (5.45) θ scores. However, the mean is near zero and the standard deviation approximately 1.00, which are fairly good indicators of an assessment providing good data.

Building on our preliminary evidence from trends in the individual domains, and incorporating our IRT scores to ensure a more accurate depiction of demonstrated MKT-G, we conducted a multiple regression to examine the effect of being an "experienced" geometry teacher (3 or more years of experience) on MKT-G scores. To account for other potential factors, we included a dummy-coded variable indicating whether participants came from the mid-Atlantic or midwestern regions (course background and general experience were not considered given their near-zero correlations in the previous section). This regression equation is shown below. As modeled, the intercept, β_0 represents the average score of non-experienced geometry teachers (less than 3 years) who are from the Midwest.

$$MKTG = \beta_0 + \beta_1(ExperiencedGeometry) + \beta_2(dMidAtlantic) + e$$

Analysis showed that the model is statistically significant ($F_{(df=2)} = 9.10, p < 0.001$) with an r^2 of 0.19. Accounting for all independent variables, teachers who lack experience in teaching geometry and are from the Midwest had slightly lower than average MKT-G scores ($\beta_0 = -0.10$, $p = 0.58$), but this was not found to be significantly different from an average score of zero. Being from the mid-Atlantic, regardless of experience, was found to have a negative effect on MKT-G scores ($\beta_2 = -0.39$, $p < 0.10$). Accounting for the effect of geographic location, having at least 3 years of experience teaching high school geometry was found to have a statistically significant and positive effect on teachers' MKT-G scores ($\beta_1 = 0.78$, $p < 0.001$). Further, as this effect is approximately three quarters of a standard deviation unit, it is quite large. These results suggest that participants who have 3 or more years experience teaching high school geometry are more likely to have higher θ scores on the MKT-G instrument. The large size of the effect is double that of the negative effect associated with our mid-Atlantic sample; thus, experienced geometry teachers from the mid-Atlantic sample still appeared to have significantly higher MKT-G scores than non-experienced geometry teachers from either region.

As with the examination of the domain-specific correlations in the previous section, this analysis builds on our emerging understanding of teachers' MKT-G. First, there is evidence of potential differences in MKT-G performance depending on geographic location and, as noted before, none of these groups had participated in any special professional development that might account for the differences found. While it may be tempting to make conjectures regarding this finding (e.g., to suspect that demographic factors might account for those differences), it seems more reasonable for a larger sample to be collected in the future so that school- and district-level factors can be accounted for more accurately. What we can say about such findings is that where a teacher teaches may affect his or her demonstrated MKT-G. If, as the data suggest, experience teaching geometry matters in teachers' MKT-G scores, it is reasonable to expect that institutional differences may account for some of the variance, just as demographics may. Second, while a sizeable effect attributable to geographic location was found, the effect associated with being an experienced geometry teacher was a much stronger predictor of a teacher's MKT-G score. We believe this to be an important finding that could be influential for teacher education practice (e.g., how to organize courses that teach mathematical knowledge for teaching to prospective high school teachers) and departmental practices in schools (e.g., how to staff different mathematics courses in a high school). Obviously, such possible practical implications hinge not only on replication of this finding but also on institutional, moral, and political considerations that only policymakers and practitioners of particular locales can make. In what follows, we explore some of the theoretical consequences of this finding.

Relationships Between MKT-G Scores and Teaching Experience

Our interest in MKT contributes to a larger project that investigates the influence that individual factors (such as MKT) and socialization to the work demands of teaching a particular high school course (in this case, high school geometry), as indicated by teachers' recognition of instructional norms and professional obligations, have on the decisions that teachers make (see Herbst et al. 2013a, 2013b, for accounts of our attempts to measure teachers' recognition of norms). A question we posed to the pilot data discussed here is: What is the relationship between MKT-G and experience teaching high school geometry? To answer it, we correlated scores for each domain with teachers' general and geometry-specific background, including years of experience teaching high school and college mathematics courses completed. Noticing the consistent relationship between teachers' geometry-specific teaching experience and their MKT-G scores in each domain, we created and analyzed IRT scores to confirm this relationship.

Our findings indicated that having 3 or more years of experience teaching geometry had a statistically significant and positive effect on teachers' MKT-G scores. This effect was sizeable at approximately three-quarters of a standard deviation. To a certain degree, our findings support those by Hill (2010), who observed a near-zero correlation between elementary teachers' MKT scores and their general years of teaching experience. Our findings also suggest a similar near-zero correlation for general teaching experience. Yet, regarding general years of teaching experience, neither our findings nor those by Hill (2010) match those of Hill's (2007) examination of middle school mathematics teachers where (among other things) she found that middle school teachers with more years of experience teaching mathematics tended to possess higher levels of teaching-specific mathematical knowledge. What is intriguing is that in both of Hill's studies, specialized forms of experience (i.e., more experience teaching mathematics at higher grade levels) were found to have highly significant correlations to MKT scores. What Hill's findings have in common with ours is that when teachers have more experience at teaching the content in depth, they demonstrate higher MKT scores. Therefore, these results suggest that while teaching experience may affect MKT-G scores, it is the particular experience of teaching the geometry course that matters. To the extent that MKT is the knowledge of mathematics used in the work of teaching, the results lead us to ask how the specifics of the instructional work a teacher does in a course matter in the MKT the teacher has: How do the specific demands of the work of geometry instruction create opportunities for teachers to learn this mathematical knowledge?

MKT and Instructional Situations

In our earlier and parallel work (Herbst and Chazan 2012), we have argued that the particular nature of the *didactical contract* (Brousseau 1997) for a course of studies creates conditions of work that make the teaching of a specific course (e.g., high school geometry) different than the teaching of other mathematics courses (e.g., Algebra I). Taken together, the findings presented throughout this chapter seem to suggest that teachers of high school geometry have more MKT-G than other secondary mathematics teachers, with the difference not seemingly accountable to general experience teaching secondary mathematics or to the mathematics courses teachers took in their college education. While at one level one might not find these results surprising, the fact that three of the four domains of MKT we tested for (SCK, KCT, and KCS) are defined as mathematical knowledge used in the work of teaching raises questions for future inquiry.

One interesting finding from cognitive pretesting may suggest future directions for exploring the nature of how MKT-G is developed among teachers. Recall that teachers participating in cognitive pretesting were experienced geometry teachers. In providing justifications for their responses, it was anticipated that the domains of KCT and KCS would elicit a sizeable portion of pedagogically related comments, which they did. What is interesting is that approximately half of the justifications in CCK and SCK also elicited pedagogically related comments. While such evidence is limited in its breadth, when we take such evidence in context with the other data presented in this chapter—in particular that there was no correlation between the number of geometry courses taken in college and participants' scores on CCK and SCK—it seems that the specific experience of *teaching* geometry is important for all domains. That is, the nature of the work of teaching geometry may be more influential than academic knowledge of the geometry domain. Thus, there is reason to ask the question of how the specific instructional work a teacher does in the high school geometry course might matter in the MKT the teacher has.

As we noted above, the current conceptualization of MKT has not addressed theoretical differentiation within MKT domains according to the content of courses of studies: How might MKT domains be organized? An easy way of thinking about that differentiation could be by deferring to the way the discipline of mathematics organizes the topical content of items—items involving mathematical topics of the same branch of mathematics might be thought of as indicators of knowledge of a well-defined region of an MKT domain. But that approach seems to apply consistently with the way MKT is defined only to the domain of CCK. To the extent that the other domains are defined in relation to the work of teaching, it is plausible that specification within each domain will also require considerations of the specifics of the teaching involved and not only of the mathematical topics referred to. Indeed, it is not immediately obvious that combining general notions in the definitions of the domains (e.g., that KCS includes knowledge of misconceptions) and mathematical domain-specific topics (e.g., pentominoes, a topic that can be filed under geometry within the discipline of mathematics; see Tracy and Eckart 1990) would produce items that validly represent mathematical knowledge for teaching high school geometry. For example, it might be possible to create an item that asked about students' errors working with pentominoes, but one would have to question the extent to which such knowledge really plays any role in the actual work of teaching high school geometry. We posit instead that elements of MKT need to be identified from the specific work of mathematics instruction in given courses of studies, rather than by the pairing of generic features of instruction with lists of mathematical topics from mathematical domains.

The results from this study suggest that the teaching of high school geometry may entail specific mathematical knowledge demands. We would like to argue that those specific knowledge demands are not solely dictated by the mathematical topics that feature in the curriculum, but rather by the specifics of the actual work of teaching a course of studies (i.e., course-specific instruction). In what follows, we elaborate the argument using some item-to-item comparisons from our data.

Course Specificity in SCK

SCK is defined as the knowledge of mathematics used in doing the tasks of teaching. This definition seems operational to us, while an alternative one floating in the literature (that SCK consists of that knowledge of mathematics that nobody else but teachers use) is not so operational. The definition we chose is not completely without problems, but it permits creating items without having to establish the truth of an empirical negative proposition—that no one other than teachers has the same knowledge, which seems quite hard to verify. Our chosen definition is problematic in that it generates some overlap with CCK. For example, one task of teaching is to create answer keys for problems given to students, and such mathematical work involves (among other things) knowledge of canonical concepts and procedures, which is, by definition, part of CCK. In developing our SCK items, we have tried to stay away from that overlapping area or to emphasize not so much what the answer is but actually how the answer should be represented (e.g., writing an answer key that students would read, which would entail mathematical work less common outside of teaching if one thinks about the teacher's work of deciding what steps must be displayed vs. what steps can be taken without displaying them). Considering our definition of SCK as the mathematical knowledge used in doing the tasks of teaching, we speculate on whether and how a course-specific version of SCK is warranted.

We grant that it is possible that not all SCK may be course specific, in that some tasks of mathematics teaching may be generic, even if they involve doing some mathematics. The task of creating a grading system, to take an extreme example, involves a teacher making a mathematical model that feeds from grades in individual assignments; but there is no reason for this mathematical work to be different for teachers of different high school mathematics courses, and quite often high school mathematics teachers use the same system across all mathematics courses they teach. There may be other such tasks of teaching that involve similar mathematical work, no matter in what mathematics course they are engaged. We are not so interested in those tasks of teaching.

Instructional Situations We are interested in other tasks of teaching where course specificity is likely to shape the meaning of "task of teaching." Some tasks of teaching are amenable to generic statement (e.g., choosing the givens of a problem for students, checking on the correctness of what students wrote as they showed their work on a problem), but they may involve practitioners in different mathematical work depending on the specifics of the task (e.g., choosing the numbers for a word problem to be given in Algebra I involves the teacher in different mathematical work than the work he or she might be involved in when constructing a geometric diagram to include among the givens in a geometry worksheet). Are those differences merely differences in mathematical domain (algebra vs. geometry), or do they also reflect differences in the activity systems to which those tasks of teaching contribute? We argue for the latter and bring in for such purpose the notion of *instructional situation* (Herbst 2006).

Instructional situations are frames for the exchange of specific work done in a mathematical task for a claim on an item of knowledge at stake in the course of studies. For example, in Algebra I, when a student responds to $2x+1=x+5$ by subtracting x and subtracting 1 from both sides of the equality, the teacher has some evidence to claim that the student knows a method for the solution of equations in one variable. Instructional situations try to capture the folk notion that every item of knowledge to be learned has one or more problem types or canonical tasks used to teach it and assess it.

Instructional situations frame such exchanges by establishing norms for the elements involved—norms that specify what those mathematical tasks look like, what doing those tasks means, who has to do each component of the task, etc., as well as norms that specify what knowledge is at stake. In earlier work (e.g., Herbst and Chazan 2012), we have proposed that while some of those exchanges of work for knowledge may be done ad hoc (e.g., negotiated), many exchanges go without saying because they are framed by instructional situations (i.e., participants abide by the norms of a situation). These instructional situations are course specific, which means more than mathematics specific. The roles of teacher and student in regard to exchanges of work for knowledge vary depending on whether the exchanges happen in one or another course, though the work done or the knowledge at stake may exist in both courses. For example, properties of geometric figures are the subject of studies in elementary and middle school grades as well as in high school geometry, but these properties are studied differently in high school geometry—that is, through engaging in different mathematical work. To assert that students (have had the opportunity to) know those properties, a high school geometry teacher needs to see students at work on mathematical tasks that are specific to the high school geometry course (e.g., constructing quadrilaterals, calculating measures of quadrilaterals, doing proofs about quadrilaterals). Reciprocally, some tasks like answering the question "What is a rhombus?" might appear not only in high school geometry but also in an earlier grade, but the knowledge at stake in that question for each course of studies is likely to be different. While a high school geometry teacher is likely to put a premium on students' knowledge of a set of necessary and sufficient conditions (e.g., the definition of rhombus), an elementary teacher might take students' capacity to list true properties as evidence of knowledge and may not make much out of redundancy or conciseness.

We propose that the management of course-specific instructional situations involves teachers in singular mathematical work. The teacher's management of instructional situations includes, in particular, the choosing of the various mathematical tasks that students are to do, the observation of the proceeds (what students actually do), and the effecting of exchanges between such observed actions and the knowledge at stake (identifying at least for himself/herself, but possibly also publicly to the class, how what students have done indicates that they know the ideas at stake). While the definition of these tasks of teaching is general, the mathematical knowledge called forth in doing them would be different across different courses, as long as the specific exchanges were different. We expect that these differences between exchanges may ensue from different kinds of knowledge at stake, different kinds of students' mathematical work being transacted for such knowledge, or even different ways of effecting those transactions (for example, in attending to precision, a teacher might implement different mathematical sensibilities when appraising students' mathematical work depending on whether that work is done in second grade or in Algebra I).

A case in point that helps argue that instructional situations matter comes from one SCK item in the MKT-G instrument. This was a multiple-response question with two testlets; the stem spoke of a teacher needing to choose algebraic expressions for the sides of an isosceles triangle where the students would be expected to find the lengths of the sides of the triangle after solving an equation. Each item provided algebraic expressions for the three sides and asked whether they were an appropriate set of expressions. A quick examination of the responses to the item indicated that teachers with more or fewer years of experience teaching geometry (≥ 3 years and < 3 years, respectively) did not respond much differently for the item where the equation could not be solved. However, the two types of teachers' responses did show differences for the item where the equation could be solved: The less experienced geometry teachers tended to answer that the set of expressions was appropriate, while the experienced geometry teachers tended to respond that the expressions were not appropriate. In fact, the numbers obtained after solving the equations of that set of expressions would not work well to represent the sides of a triangle in that the triangle inequality[2] would not hold for those numbers. We conjecture that the experienced geometry teachers' familiarity with the instructional situation of "calculating a measure" (Herbst et al. 2010) mattered in their decision to check that the expressions would yield sides with positive lengths and that they would satisfy the triangle inequality. Our conjecture is not that the non-experienced geometry teachers did not know the triangle inequality. They may or may not know it and, in any case, that would be an example of CCK. We argue that the less experienced geometry teachers were less likely to know that the triangle inequality had to be checked in doing the task of teaching of choosing algebraic expressions for the sides of the triangle, possibly because they saw the problem only as an exercise in algebra rather than also as an exercise on the geometry of triangles. More generally, we propose that an element of SCK specific to the instructional situation of "calculating a measure" is the knowledge that, when assigning numerical (or algebraic) values to some dimensions of a geometric figure (a task that teachers have to do every time they create such problems), one has to check that none of the geometric properties of that figure are refuted by the set of choices made. Even more generally, we conjecture that tasks of teaching that are subservient to instructional situations that are specific to a given course of studies (e.g., high school geometry) might involve teachers in mathematical work that teachers who are experienced in managing those situations would better know how to do than teachers who are not so experienced. Thus, we propose that the instructional situations of a course of studies are natural containers of elements of SCK.

Novel Tasks There is reason to suspect that SCK might include items of knowledge other than those needed to manage the instructional situations of a course. In particular, a given course of studies is likely to include not only recurrent instructional situations framing familiar tasks but also novel tasks—tasks that may deviate enough from what is customary that teacher and students need to negotiate what it means to work together to complete them (see Herbst 2003). We argue that those tasks might also call for special mathematical work on the part of the teacher (choosing the givens for a problem, interpreting what students do and say, etc.), and along the lines of the definition of SCK, that work could also imply the existence of specialized knowledge. Is that SCK expected to be general to all mathematics teachers or expected to be special to teachers of high school geometry?

The MKT-G instrument also included SCK items that referred to mathematical tasks that depart from ordinary instructional situations, but that address objects of

[2] The triangle inequality states that if *a, b,* and *c* are the lengths of the sides of a triangle, then $a - c < b < a + c$.

study that are part of the geometry course. In one of those items, a multiple response question including two testlets, the participants were confronted with the scenario that a teacher had asked his geometry students to provide a definition of a figure that is ordinarily studied in the course. Participants were asked to consider two definitions purportedly proposed by students and decide whether they were correct, applied only to particular cases, or could only be consequences of a definition. We contend that this task is novel in that students are rarely asked to create definitions in high school geometry. Teachers of high school geometry, however, often consider definitions and distinguish them from other statements. Geometry teachers are confronted with the task of distinguishing between definitions and their consequences, for example, when they assess student proofs. Additionally, teachers do ask students to recall definitions for geometric concepts in tests and quizzes, and in such contexts they are likely to discriminate between definitions and other true statements about a concept. In contrast, in other courses the word *definition* is often used in much more relaxed way (i.e., defining is often used as synonymous with stating, describing, or clarifying; see Sinclair and Moss 2012; Zaslavsky and Shir 2005). Thus, we expected that teachers experienced in the teaching of the geometry course would do better on this item. Indeed, respondents who had experience teaching geometry were much less likely to respond to this item incorrectly than respondents who did not have experience teaching geometry. In particular, one of those testlets included a purported definition that, insofar as it expressed an element of the concept image of the figure under consideration, could have been considered correct in earlier grades or by people whose notion of definition was more relaxed. More experienced geometry teachers were far more likely than less experienced geometry teachers to single that definition out as not being correct. Thus, while having students create their own definitions may be a novel task in most geometry classrooms, experienced geometry teachers seemed better able to use their mathematical knowledge to handle student work. In contrast, in an item that asked teachers to assess purported student work in an area of geometry not commonly emphasized in high school geometry (transformations) and on an item where they had to assess purportedly student-generated construction procedures for a figure whose construction is not often studied in the course, the differences between more and less experienced geometry teachers were not as marked (both groups were similarly likely to answer the item incorrectly).

Course Specificity in KCS and KCT

In regard to the other MKT domains of interest, one could ask the same theoretical question as above: How are those domains of knowledge organized? In particular, are they specific to the instructional practices of particular courses of study? We suggest that considerations of the nature of the instructional situations in the high school geometry course could lead to analogous differentiation within the domains of KCS and KCT as we argued for SCK. As in the case of SCK, we concede the possibility that some KCS and some KCT might be general to mathematics teaching, but we propose that instructional situations within a course of studies can help organize elements of

KCS and KCT that are specific to the teaching of that course of studies. In particular, it is reasonable to expect that experienced geometry teachers might have more KCS if this domain is assessed with items that probe for their knowledge of students' conceptions and errors in tasks that are framed within the instructional situations of the high school geometry course, while they might have less KCS if the items concern students' conceptions or errors regarding geometric ideas that are not so central to the course. Among our KCS items, one item could be described as probing for knowledge of student errors in a task that is quite familiar to experienced geometry teachers (identifying the height corresponding to a given base of a given triangle), while another item could be described as not so familiar (determining the number of diagonals of a polygon of a large number of sides without access to the formula—a task that we would classify as novel because it is common for geometry students to calculate the number of diagonals of a polygon using the formula[3]). A quick glance at the responses to those items shows that the difference between the percentage of more and less experienced geometry teachers who answered the first item correctly is about three times the difference for the second item. That is, more experienced geometry teachers were a lot more aware than less experienced geometry teachers of the errors students could make in the first (familiar) task (about height of a triangle) than in the second (novel) task (about diagonals in a polygon)—despite the fact that both mathematical topics are equally common in the high school geometry course.

In inspecting the responses to KCT items, we found similar patterns. One item asked participants to choose a figure that would help a teacher show his or her students that the converse of a theorem is not true, an activity that we judge to be quite common when teachers are installing theorems in the geometry class (Herbst et al. 2011). Another item asked participants to choose among exploratory tasks the one that would best impress upon students that two geometric concepts (angle bisectors and diagonals) are different. The difference between the percentage of more and less experienced geometry teachers that answered the first item correctly was more than twice that difference for the second item.

Implications for Future Research

Clearly, the item-by-item commentary provided in the prior two sections cannot be taken as confirmation of any general trend, but it permits formulating conjectures that could be tested with more systematically constructed test forms. In particular, we conjecture that experienced geometry teachers would be more likely to answer correctly SCK items that involve a participant in doing mathematical work similar to the work they do in familiar instructional situations or that is related to an object of study that is a piece of the geometry course, than SCK items that concern geometric objects less common in the high school geometry course or tasks that are not usually used in teaching or assessing an item of the knowledge at stake. We speculate that the same conjecture could be used to study how course specificity plays

[3] The number of diagonals in a polygon of n sides is $n(n-3)/2$.

out in the case of MKT related to other high school courses of study, thus helping to think about how the domains of MKT could be organized.

The effort to discover an internal organization for each of the instruction-specific MKT domains is compelling theoretically. It could help us better understand the relationships between individual resources and situated activity—how different individuals may be differentially prepared for taking the teacher role in different instructional systems. The notion of instructional situation as an organizing element for the knowledge in each domain could also be used in the construction of instruments that measure the extent to which individuals are prepared to enact instruction that departs from customary instruction—by creating items that require the practitioner to engage in the kind of mathematical thinking needed to sustain tasks that more or less deviate from the thinking they need to do in customary instructional situations. From a more practical standpoint, the work of finding an internal organization to the MKT domains could inform the development of coursework in mathematics or mathematics education for future teachers, though such development also needs to attend to many other considerations of an institutional nature that policymakers and practitioners in particular locales are expected to make.

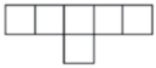
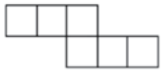
Conclusion

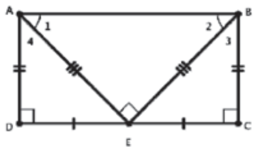
The present chapter describes efforts to construct and validate an instrument that measures mathematical knowledge for teaching high school geometry. Our construction of the items in the MKT-G instrument paid attention to generic tasks of teaching and combined those with geometric content that is or could be taught at the high school level. The instrument worked relatively well and pilot data show correlations between scores in each of the domains KCT, KCS, CCK, and SCK, and the number of years of experience teaching geometry. The aggregate MKT-G scores are significantly accounted for by years of experience teaching geometry, even when taking into consideration the significant effect of geographic region in which respondents taught. In contrast, neither years of experience teaching mathematics in general nor number of mathematics courses or geometry courses taken in college correlated with MKT-G scores. Our discussion of item-level differences in the sample, however, shows that experienced geometry teachers did not perform homogeneously—being much better than non-experienced geometry teachers in some items and not so much better in others. Further, item development and testing of experienced geometry teachers can help us understand individual differences and the differential difficulty individual teachers have with elements of MKT. Our discussion of item-level differences in the sample supports a conjecture about how the three teaching-specific MKT domains organize. We conjecture that experience doing the work called for by customary instructional situations in a course of studies (such as geometry) organizes practitioners' MKT more than the mere mathematical content of the items. This knowledge organization hypothesis can assist in item development that would allow for the study of individual differences among experienced geometry teachers and of the nature of expertise in mathematical knowledge for teaching geometry.

Appendix: Released Items

Appendix: Released Items

<table>
<tr><td>CCK released item</td><td>Students in Mr. Wingate's class have been creating nets that they want to be able to fold to create a cube. For each of the student-created nets shown below, identify whether it can be successfully folded into a cube?

<table>
<tr><th></th><th></th><th>Yes</th><th>No</th></tr>
<tr><td>i</td><td></td><td></td><td></td></tr>
<tr><td>ii</td><td></td><td></td><td></td></tr>
</table>
</td></tr>
<tr><td>SCK released item</td><td>While proving a claim on the board about the figure below, Joe wrote "1 + 2 = 90." Ms. Staples ponders how to correct that statement. Of the following, what is the best alternative?



A Do nothing. The statement is correct as is.
B Add the degree symbol so that the statement reads 1 + 2 = 90°.
C Replace what Joe wrote; write instead that "m∠A + m∠B = 90°".
D Replace what Joe wrote; write instead that "m∠EAB + m∠EBA = 90°".</td></tr>
<tr><td>KCS released item</td><td>Ms. Jamison asked students to figure out the number of diagonals in an 11-sided polygon. Of the following erroneous solutions, which one might have resulted from using the number of diagonals through one vertex?

A 11
B 72
C 88
D 99
E 121</td></tr>
<tr><td>KCT released item</td><td>After teaching her students the base angles theorem (which states that base angles of an isosceles triangle are congruent), Ms. Wellington is pondering what example she could give to illustrate how this theorem can help find new information. Which of the following would be the best example?

A If you have a triangle that you know is isosceles and you know two of the sides are 8 and 10, then the third side will be 8 or 10.
B If you have a triangle that you know the sides have length 8, 10, and 12 you know it does not have two angles that are congruent.
C If you have an isosceles triangle and two of its angles are 30 you can calculate the third angle by subtracting 60 from 180.
D If you have a kite and draw its minor diagonal, the kite is divided into two isosceles triangles and those have two congruent angles each.</td></tr>
</table>

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Using Coordination Classes to Analyze Preservice Middle-Grades Teachers' Difficulties in Determining Direct Proportion Relationships

Erik Jacobson and Andrew Izsák

Attention to the misapplication of proportional reasoning dates back to antiquity: Meno's slave famously told Socrates that doubling the side lengths would double the area of a square. In recent decades, studies from many countries and with participants ranging in age have documented the widespread, inappropriate application of direct proportion reasoning strategies in covariance situations (for a review, see Van Dooren et al. 2008). According to researchers involved in a recent special issue of *Mathematical Thinking and Learning* (Van Dooren and Greer 2010), however, the root psychological sources of this tendency remain unclear and pedagogical remedies to date have been unsatisfactory.

The large literature on proportional reasoning has concentrated primarily on students (e.g., Lamon 2007), but the handful of studies on teachers' capacities to reason about proportional relationships suggests that, in many cases, teachers' and students' difficulties are similar. Post et al. (1991) administered a test to upper elementary teachers consisting of items that, ideally, students should be able to solve. They were surprised by the low level of performance—for instance, less than half of the over 200 teachers could solve the following problem correctly: "Given 15 squares and [the] fact [that the] ratio of blue to red is 2:3, how many blue [squares]?" (p. 189). In other studies, teachers have (a) had difficulty distinguishing missing-value problems that describe proportional relationships from ones that do not (e.g., Cramer et al. 1993; Fisher 1988; Lim 2009; Riley 2010); (b) had trouble coordinating two quantities in a proportional relationship (e.g., Orrill and Brown 2012); (c) made inappropriate additive comparisons (e.g., Canada et al. 2008; Lim 2009; Son 2010); and (d) had trouble conceiving of a ratio as a measure of a physical attribute, such as steepness or speed (Akar 2010; Simon and Blume 1994; Thompson and Thompson 1994). With respect to problem-solving strategies, teachers have relied

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heavily on cross multiplication or other formal methods (e.g., Fisher 1988; Harel and Behr 1995; Orrill and Brown 2012; Riley 2010), guessed at operations (Harel and Behr 1995), and searched for key words (Harel and Behr 1995).

The present study builds most directly on results reported by Cramer et al. (1993), Fisher (1988), and Riley (2010). Cramer et al. (1993) reported that 32 out of 33 preservice elementary teachers solved the following task using the proportion $9/3=x/15$: "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?" The relationship between the runners' laps is not proportional. Rather, they remain an equal distance apart: $9-3=x-15$. Fisher (1988) reported that 12 out of 20 inservice secondary mathematics teachers did not solve the following problem correctly: "If it takes 9 workers 5 hours to mow a certain lawn, how long would it take 6 workers to mow the same lawn?" The common error was to set up a direct proportion such as $9/5=6/x$. Instead, the relationship between men and minutes is inversely proportional, because the amount of work is constant: $9(5)=6x$. More recently, Riley (2010) found that in a sample of 80 preservice elementary-grades teachers, less than 50% solved constant difference and inverse proportion problems correctly. In the present study, we investigated preservice middle-grades teachers' reasoning on tasks that also described constant difference, inverse proportion, and other relationships.

The present study contributes to research on teachers' reasoning about proportional relationships by not only documenting that preservice middle-grades teachers can have many of the same problems as students, but also providing new insight into why these problems can be persistent. In particular, we found several cases in which preservice teachers who correctly explained relationships between two quantities that were not direct proportions still tried to solve problems involving those quantities using direct proportions. To understand how this could occur, we used the coordination class construct (diSessa and Sherin 1998) developed as part of the knowledge-in-pieces epistemological perspective (e.g., diSessa 1993) to analyze knowledge resources that the preservice teachers used to judge whether the described relationship between two covarying quantities in a word problem was or was not a direct proportion. Our analysis suggested that novices decided to use the direct proportion equation based on different features of problem situations than experts would use.

The study contributes to research on proportional relationships more broadly by elaborating on a phenomenon that has been characterized as intuitive or impulsive responses to familiar missing-value problem presentations. For example, Van Dooren et al. (2010) examined the performance of Belgian sixth-grade students on tasks that described relationships that were and were not direct proportions. The authors attributed the pervasive misapplication of proportional reasoning strategies to "pupils' superficial approach of word problems—jumping too quickly to the calculating work and immediately reporting the outcome—rather than to being really unable to distinguish proportional from non-proportional word problems" (p. 34). Our findings suggest that recognizing direct proportions is more complex than reading problem statements carefully.

Theoretical Perspective

We employed the coordination class construct (diSessa and Sherin 1998) from the knowledge-in-pieces (e.g., diSessa 1993) epistemological perspective to gain insight into the performance of the preservice teachers with whom we worked. diSessa developed the knowledge-in-pieces perspective to explain emerging expertise in Newtonian mechanics and emphasized multiple related processes including not only the construction of new knowledge elements but also the coordination of diverse knowledge elements and the extension or restriction of conditions under which particular elements may be applied productively. Conceptual change from this perspective is characterized as the piecemeal construction and reorganization of knowledge elements (e.g., diSessa and Sherin 1998; diSessa and Wagner 2005; Smith et al. 1993) as learners gradually navigate the continuum from novice to expert. One strength of the knowledge-in-pieces perspective is that it supports analysis when novices seem to reason incoherently or inconsistently within and across problem situations. A handful of studies have applied this perspective to students' reasoning about mathematical topics, including functions (e.g., Monk and Nemirovsky 1994; Moschovich 1998), fractions (Smith 1995), probability (Wagner 2006), and rectangular area (Izsák 2005). A second strength is that the perspective explicitly asks about the forms that cognitive structures can take. Coordination classes are one such form.

diSessa and Sherin (1998) proposed *coordination classes* as empirically verifiable cognitive structures useful for describing certain kinds of concepts, such as force. Coordination classes are "systematically collected ways of getting information from the world" (p. 1171), and they allow a person to perceive instances of the same class among the diversity of features present within and across situations. Thus, coordination classes are specified in terms of performance and, among other things, provide tools for analyzing the Piagetian notion of assimilation as a complex, unfolding process.

Coordination classes are comprised of two components, *readout strategies* and the *causal net*. *Readout strategies* allow one to "see" information in situations and involve coordination in two senses. First, a person might need to select and combine multiple aspects within a given situation in order to perceive an instance of the class. This performance can be thought of as *integration* of information within the situation (p. 1176). Second, a person might deploy different combinations of knowledge resources to perceive instances of the same class across situations. This performance can be thought of as *invariance* (p. 1176), or transfer. Consistent with the general principles of the knowledge-in-pieces perspective described in the previous paragraph, one way that a person's coordination class could evolve is through refinements in the contexts where readout strategies are deployed and the perceived reliability of those strategies (see Wagner 2006, for a detailed and vivid example of such refinement in the context of expected value and the law of large numbers). The *causal net* is made up of syllogism-like ways of inferring new information not directly available from readout. For example, someone might use the equation

$F = ma$ to obtain information about acceleration from a situation that specified only force and mass (diSessa and Sherin 1998). Thus, a second way that evolution in a person's coordination class could occur is through refinement in understanding what things cause or determine other things. diSessa and Sherin (1998) noted that determining whether a particular candidate is a coordination class—that is, whether the way people reason about a concept can be specified in terms of particular sets of readout strategies and causal nets—requires extended empirical work (p. 1186). To be clear, the particulars of readout strategies and causal nets will likely vary from one person to the next.

In the present study, we provide initial data that suggest the coordination class construct may be useful for understanding when and how people discriminate relationships that are and are not direct proportions. Furthermore, we follow Wagner (2006) in specifying terms for more precisely discussing the contextual differences that problem solvers encounter in activity. The term *context* refers to the cover story for the problem. The *type* of problem is defined by appealing to normative or expert judgment. The *aspects* of a problem are the features or details perceived as relevant by the problem solver. *Contextual differences* are the amalgam of context, type, and aspect. These terms are analytically useful because they help distinguish the perspective of the problem solver (which may not be normative) from that of the expert. For example, the problems reported by Cramer et al. (1993) and Fisher (1988) are normatively of different *types* (one describes a constant difference and the other describes an inverse proportion), but a novice problem solver might read out the same *aspect* (the women run and the workers mow at equal rates) and solve both problems in the same way.

Methods

The Numbers and Operations Course and Participants

This study took place in an 18-week content and methods course on number and operations for preservice middle-grades mathematics teachers. The authors were the instructors for the course, which took place in Fall 2011 and met for two 75-min sessions each week. The course focused on multiplicative relationships and drawn models of quantities (e.g., number line and area models), and a main goal was for the preservice teachers to develop an understanding of how problem-solving strategies that make use of drawn models for quantities can provide the basis for developing general computation methods. This approach to number and operations in the middle grades is consistent with current curriculum standards (e.g., National Council of Teachers of Mathematics 2000; National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA and CCSSO] 2010).

The preservice teachers ($N = 28$) were being prepared to teach grades 4 through 8. They were in their third year of college and prior to the study had taken at least the first semester calculus course required by their program. The preservice teachers reported that using problems and drawn models to develop general computation methods was new to them: They had been taught only rules and algorithms when they were in middle school. Most class sessions began with small group work for approximately 40 min on a set of problems designed to target a particular mathematical idea. The preservice teachers then shared strategies and asked questions about each others' methods for solving the problems during whole-class discussion.

The course began with an examination of factors and multiples and the following explicit meaning for multiplication: $A \times B$ means the amount in A copies of B groups, all of which are the same size. The course continued with activities intended to focus the preservice teachers on ways that factors and multiples and the repeated groups meaning for multiplication can serve as resources for partitioning number lines and area models. Whole number multiplication and connections to partitioning served as the basis for a unit that developed meanings for fractions and fraction arithmetic. The unit began with the definition for fractions specified in the Common Core State Standards (NGA and CCSSO 2010). This is a two-part definition that first defines a unit fraction $1/b$ and then defines a/b as a copies of $1/b$. A main point made in the course is that this definition emphasizes repeated groups of size $1/b$ and thus underscores that fractions are fundamentally about multiplicative relationships. With the Common Core definition in hand, the preservice teachers next learned how to use area models to develop general numeric methods for multiplying proper and improper fractions and how to use double number lines (e.g., Orrill and Brown 2012) for partitive and measurement fraction division. Special attention was given to strategies for partitioning developed earlier in the course, referent units, and how solutions using drawn models could be used to deduce the general computation method of invert and multiply for fraction division.

We began the proportion unit by giving preservice teachers a set of word problems and asking whether relationships between quantities described therein were or were not direct proportions. The purposes were to (a) access preservice teachers' initial understandings of proportions, and (b) have the preservice teachers explicitly compare context[1] and mathematical structure as recommended by Van Dooren et al. (2010). Subsequently, we defined direct proportions as two quantities covarying in a fixed ratio and developed problem-solving strategies that generalized those used earlier in the course for solving partitive division problems (emphasizing that partitive division is a special case of reasoning about direct proportions). These strategies made use of composed units (e.g., Lamon 1994; Lobato and Ellis 2010) and double number lines. More details of the proportion unit are provided in the results section.

[1] *Context* as used in Van Dooren et al. (2010) has the same meaning as in Wagner (2006): the cover story of a word problem. In the rest of the chapter, we italicize other terms from Wagner that have a narrower definition than may be typical.

Data Collection

The data we used for this analysis were collected as part of a larger study. On the first and last day of the numbers and operations course, we administered a pretest and posttest consisting of items developed and validated by the Diagnosing Teachers' Multiplicative Reasoning (DTMR) project (see Bradshaw et al. 2014, and Izsák et al. 2010, for details). The items focused on several core kinds of reasoning about multiplicative relationships with drawn models of quantities, especially multiplication and division of fractions and proportional relationships. The kinds of reasoning about proportions most relevant for the present study were covariation and invariance (reasoning about two quantities covarying in a fixed ratio by partitioning and iterating composed units and by forming multiplicative comparisons within and between measure spaces) and appropriateness (judging whether a given relationship is or is not directly proportional).

We gave the preservice teachers problem situations during the unit and on assessments describing covarying quantities that were direct proportions or had other relationships. The present study focuses on the four problems shown in Table 1. The Running Problem and Combine Problem describe situations of constant difference, the Work Problem describes a constant product (inverse proportion) relationship, and the Interest Problem describes a piecewise linear approximation to exponential growth. Data for these four problems best illustrate our comprehensive findings concerning all problems used in the study and afford comparisons with results from prior studies that used similar problems (e.g., Cramer et al. 1993; Fisher 1988).

We video recorded each class session using two cameras. One camera captured activity in the whole classroom. The second camera recorded close-ups of written work. A research assistant shadowed the primary instructor (second author) with this camera and recorded written work of the preservice teachers with whom he worked. After each class session, the two videos were combined to create a restored view (Hall 2000). The second author reviewed the resulting video and wrote a summary that captured main points in whole-class discussion, strategies preservice teachers discussed and questions they had during group work and whole-class discussion, and screen shots of written work. We used these summaries to identify mathematical issues to pursue further during the interviews described next.

During the first week of class, we asked for volunteers to participate in a series of interviews. The only incentive offered was the benefit of additional time studying the course content with a partner. Eight of the 28 preservice teachers volunteered, and we used their overall performance on the pretest to form four pairs—one lower, two medium, and one higher performing. We interviewed each pair four times, before and after the unit on fraction arithmetic and before and after the unit on proportions described above. The interviews were semistructured (e.g., Bernard 1994) and lasted 60–90 min. During the interviews, we presented tasks similar to those used on the pre- and posttests and during the course. We asked the preservice teachers to solve the tasks together while reasoning aloud. The interviewer encouraged the preservice teachers to talk freely and occasionally asked clarifying questions. The present study focuses on the interviews before and after the four-session unit on

Table 1 Sample of covariation problems used during the proportion unit and on assessments

Problem	Type	Use
Running Problem. Determine whether the following problem is a *mathematically valid* illustration of the proportion A/B = C/D: Bob and Marty run laps together because they run at the same pace. Today, Marty started running before Bob came out of the locker room. Marty had run *A* laps by the time Bob had run *B* laps. How many laps *C* had Marty run by the time that Bob had run *D* laps?	Constant difference	Pretest, posttest, and interviews
Combine Problem. Two combines harvest grain at the same rate. The first combine starts harvesting 10 min before the second combine. After 20 min of operation, the second combine harvests 400 lbs of grain and the first harvests 600 lbs of grain. How many pounds will the second combine harvest by the time the first has harvested 1,000 lbs of grain?	Constant difference	Proportion unit
Work Problem. Determine whether the following problem is a *mathematically valid* illustration of the proportion A/B = C/D: If *A* men paint the outside of a house in *B* minutes, then how many minutes *D* would it take *C* men to paint the same house, if all the men work at the same rate?	Constant product	Pretest, posttest, and interviews
Interest Problem. Karl has a savings account that pays interest monthly at a rate of 5 %. Three months ago, there was US$ 300 in his account. If he did not withdraw any money from the account, how much is there now?	Exponential growth (a piecewise linear approximation)	Proportion unit

proportions (these interviews were the third and fourth interviews we conducted in the larger project, but we will call them interview 1 and 2 for the purposes of reporting the present study). Figure 1 shows the timeline for these data-collection activities. Interview 1 took place immediately before the proportion unit, and interview 2 took place after it.

Analysis

A third party transcribed the interviews in their entirety. We reviewed all of the interview data, refined the accuracy of the transcripts as needed, and wrote summaries comparing each pair's pre- and postunit interviews. Then we reviewed the classroom video data for episodes or written work that might inform our understandings of what took place during the interviews. Finally, we reviewed preservice teachers' written work on the pretest and posttest, on the course midterm and final exam, and on the homework assignments about proportions. For the present study, we focus on results from the pre- and posttests as a measure of the effectiveness of the unit on proportions and from the interviews. Data from the midterm, final exams, and homework provided corroborating evidence for the results reported in the next section.

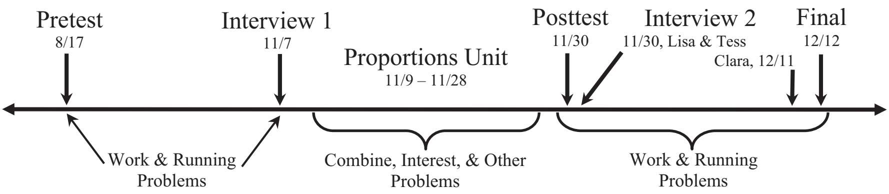

Fig. 1 Timeline of the data-collection activities

Results

Performance on the Pretest and Posttest

Analysis of the pretest and posttest suggested that the unit on proportions had little effect on preservice teachers' tendency to inappropriately apply methods for solving direct proportions. This was especially apparent in the stability of their responses to the Work and Running problems. Recall that the Work Problem described an inverse proportion, the Running Problem described a constant difference, and these specific problems were on both the pretest and the posttest. The preservice teachers worked on and discussed different problems of the same *type* (Wagner 2006) during the proportion unit.

Table 2 shows how many preservice teachers maintained or changed their responses for the Work and Running problems from pretest to posttest ($n=27$; one preservice teacher was absent for the posttest) and contextualizes the classroom and interview data described in the subsequent sections. The cells on the diagonal for each problem in Table 2 show preservice teachers who did not change their response between the pretest and posttest; overall, very few changed their responses. Preservice teachers were almost entirely agreed on the Work Problem, but the modal response was incorrect. By contrast, the responses on the Running Problem were split almost evenly between those who believed the problem was a direct proportion and those who did not. We used McNemar's (1947) test for matched data and found no statistically significant change in preservice teachers' responses on these items ($p_{Work}=1.00$, $p_{Running}>0.450$). This finding provided evidence that the proportion unit was not successful in fully developing the preservice teachers' capacities to accurately discriminate among relationships that were and were not direct proportions, even though the task *types* on the pre- and posttest aligned with the task *types* discussed during the unit.

At the same time that these preservice teachers experienced significant difficulties applying the direct proportion equation appropriately, we knew that they had been engaged in extensive discussions in class comparing the context and mathematical structure of similar tasks and had completed related homework solving similar word problems and classifying problems as directly or nonproportional. It was during retrospective analyses of the classroom and interview data that we

Table 2 Preservice teachers' pretest and posttest responses on two missing-value problems

	Posttest response			
	Work problem		Running problem	
Pretest response	Direct proportion	Not direct proportion[a]	Direct proportion	Not direct proportion[a]
Direct proportion	25	1	9	5
Not direct proportion[a]	1	0	2	11

[a] The correct response

recognized that the coordination class construct could help provide insight into their reasoning about proportional relationships. In particular, we noticed that preservice teachers' readout strategies focused on *aspects* (Wagner 2006) that did not support normative judgments across different problem *types* and *contexts* (Wagner 2006) and thus did not lead to *invariance* (diSessa and Sherin 1998).

Performance During Class Sessions

To gain insight into the stability of the preservice teachers' performance from pretest to posttest, we reviewed the video and summaries for the four class sessions that comprised the proportion unit. Recall that the pretest was administered in August at the beginning of the semester and the proportion unit took place in November, so there was a significant delay. The comparison of pre- and posttest performance suggested that any shift in preservice teachers' understandings of proportions—for instance, during the proportion unit or earlier in the semester when studying the special case of partitive division—was negligible.

As explained above, we began the unit by providing a set of missing-value word problems and asking whether the relationships described therein were direct proportions. Our intent when collecting these data was to uncover the understandings of proportions that the preservice teachers brought to the course. The set of problems included the Interest Problem (Table 1), as well as one problem that described a direct proportion relationship and one problem that described a linear relationship that was not proportional.

For the purpose of this present study, we conducted a retrospective analysis of the classroom data to understand preservice teachers' difficulties determining which *aspects* of the presented problem situations could reliably distinguish relationships that were and were not direct proportions. During the first class session, two main difficulties in the preservice teachers' readout emerged that could have contributed to the high rate of incorrect responses on the pretest. First, we observed that many, if not all, of them relied on rote procedures for missing-value problems, such as cross multiplication. For example, one preservice teacher said that she had been taught the following rule for transferring information from word problem texts to equations: "Is over of equals proportion over 100." She did not give a specific example to illustrate what she meant, but her comment clearly referred to a rote method based on key words and three numbers. Consistent with findings in past research reviewed above, these teachers seemed to read out a direct proportion based on the presence of three numbers and a request to determine the fourth.

A second main difficulty was distinguishing between solution methods and relationships between two covarying quantities. During group work, several of the preservice teachers (including three of the four we discuss in more detail below) focused on solution methods. As an example, for the word problem that described a relationship that was linear but not a direct proportion, these preservice teachers reported that they could use a proportion to solve $y=mx+b$ if they first subtracted b. As another example, one preservice teacher discussed how she could solve the Interest Problem by solving a separate proportion for each month. We highlighted the difference between solution methods and relationships between quantities during the whole-class discussion and emphasized that the relationships between quantities determine proportionality.

The second and third class sessions were intended to focus preservice teachers' attention on covariation in a fixed ratio. The tasks provided opportunities to partition and iterate composed units using tables, double number lines, and rectangles as supports. Main points that came out of whole-class discussions included connections between proportions and partitive division and the constant multiplicative relationships within and between measure spaces (e.g., Vergnaud 1983, 1988).

For the fourth and final class session, we returned to missing-value word problems and asked once again whether the relationships described therein were direct proportions. We were interested in seeing if experiences reasoning about covariation between quantities in a fixed ratio relationship helped the preservice teachers discriminate between relationships that were and were not direct proportions. The set of problems included the Combine Problem (Table 1), one problem that described a direct proportion, one problem that described a linear relationship that was not proportional, and one problem that described an inverse proportion similar to the Work Problem (Table 1). In contrast to the instructions for the first class session, the instructions for the fourth class session included making a graph.

Unfortunately, many of the preservice teachers misread the problem that described a linear relationship that was not a direct proportion. In their interpretation of the problem, the described relationship was, in fact, proportional. So we did not get data that we could compare to those from the first class session. Two main difficulties in the preservice teachers' readout did emerge around other problem *types*. First, several of the preservice teachers thought that the Combine Problem described a direct proportion. In some cases, they focused on the phrase "same rate" in the problem statement as their justification. Thus, they based their readout on key words or phrases. In other cases, the preservice teachers said that any situation that is linear has a constant rate and thus is a direct proportion. This occurred even as some preservice teachers recognized that the Combine Problem describes a constant difference. Second, some preservice teachers attempted to set up and solve a direct proportion to solve the problem that described an inverse proportion and then realized that their answer did not make sense, but did not understand why. In these cases, readout was based on understanding that two quantities were covarying but not on distinctions among different *types* of relationships. We pursued issues related to these during the interviews that took place after this lesson. We turn to those data next and make comparisons with data from the interviews conducted before and after the proportion unit.

Analysis of Interview Data

The interviews revealed that preservice teachers who correctly explained relationships between quantities that were not direct proportions in the Running Problem and Work Problem (see Table 1) still tried to set up and use the direct proportion equation in explanations of how they would solve the problems. To understand how this could take place, we examined the *aspects* of the problem situations that the preservice teachers read out and the inferences that they made when reasoning about the problems. Some *aspects* that the preservice teachers read out included the presence of covariation between quantities, references to "same rate," and whether they could describe the relationship between two quantities with a single equation. Here, we focus on the two pairs that most clearly demonstrated how appropriate understandings of covariation between quantities were not sufficient for judging whether relationships were direct proportions. Alice and Clara were the higher performing pair, and Lisa and Tess were the lower performing pair. We organize the analysis by tracing the reasoning of each pair within problem *type*. (*Note on transcription:* There are no deletions in the transcript provided. Pauses are indicated by ellipses and action is described within square brackets. Round brackets indicate our best guess about what was said when the audio trailed off).

Constant Differences We found that the preservice teachers used several *aspects* of the problems, like the phrase "same rate," to read out a direct proportion. At the same time, they also attended to the constant difference in the numbers of laps that Marty and Bob ran. That is, their causal net was sufficient to see that running at the same rate implied a constant difference in the numbers of laps but was insufficient to realize that a constant difference precluded a fixed ratio relationship. This allowed the preservice teachers to hold simultaneously two perspectives on the Running Problem that a more expert person would understand as mutually exclusive. The following examples illustrate these findings.

During the first interview, Alice[2] and Clara introduced two possible ways to think about the Running Problem. After reading the problem, Clara generated the equation $a/b = c/d$, keeping terms referring to the initial state on one side of the equal sign and terms referring to the later state on the other side. Alice tried to substitute specific values for numbers of laps: She supposed that Marty had run 2 laps by the time Bob had run 1 lap and concluded that Marty had run 3 laps by the time that Bob had run 2 laps. Clara agreed. Thus, when working with specific values,

[2] All names are pseudonyms.

Alice and Clara attended appropriately to the constant difference in numbers of laps. A moment later, Clara set up and solved a direct proportion and determined that Marty had run 4 laps by the time Bob had run 2 laps. Faced with the discrepant answers, the preservice teachers questioned whether the relationship between Marty and Bob's laps was, in fact, a direct proportion. Clara gave greater weight to the constant difference perspective when she concluded:

Clara If they're running the same pace … if Bob had run 1 more lap, then Marty should have run 1 more lap. He just started earlier but they're running at the same pace, so the same speed of 1 lap should just be in Marty's 1 more lap. So if Marty went and started and he ran 1 lap, and then Bob came and started and ran another lap, Marty is still running, so Marty would have run 2 laps by the time Bob ran 1. And then when they run another lap, Marty would have run 3 laps by the time Bob ran 2. But with this proportion, it's saying that Marty would have run 4 laps by the time Bob ran 2 because it's doubling.

Alice So basically Marty is always just going to be A laps ahead of Bob.

The resolution that Alice and Clara reached on the Running Problem appeared to stick at least with Clara. When reviewing the video recordings of the class lessons, we found one place where Clara focused on the constant difference in the Combine Problem and stated that the problem was not a direct proportion. When she read the Running Problem again during the second interview, Clara stated immediately that the difference between Marty's and Bob's laps would be constant, and therefore the relationship between the numbers of laps would not be proportional.[3] To support her explanation, she graphed two parallel lines, one for each runner, similar to graphs for the Combine Problem discussed during the class. Thus, by the end of the unit on proportions, Clara recognized that the constant difference and constant ratio perspectives were incompatible, but it remained unclear whether her causal net was sufficient for her to understand why.

During their first interview, Lisa and Tess introduced two possible ways to think about the Running Problem that were similar to the two ways introduced by Alice and Clara. In contrast, however, we found no evidence that Lisa or Tess ever perceived an incompatibility between the two perspectives. After Tess read the problem aloud, there were 30 seconds of silence. Then Lisa pointed out that there would be a constant difference *and* suggested that the relationship between the numbers of laps would be proportional.

Lisa I mean if they keep on at same pace,

Tess Right.

Lisa isn't that going to be the same difference between the two?

Tess Right.

Lisa So it would be an equal proportionality I would think…kind of like equal fractions. What are those called? Equivalent fractions?

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[3] Alice did not participate during the second interview due to a family obligation.

The preservice teachers then substituted specific numbers into a direct proportion but did not seem to notice that their result was inconsistent with the constant difference Lisa mentioned above. In particular, they decided that Marty ran 4 laps by the time Bob had run 2 laps, a difference of 2, and computed that Marty would run 16 laps by the time Bob had run 8 laps, a difference of 8.

During the proportion unit, both Tess and Lisa perceived the relationship described in the Combine Problem to be a direct proportion. Despite class discussions emphasizing that problems of this *type* do not describe direct proportions, they continued to think that the Running Problem was a direct proportion when they encountered it again during their second interview. In fact, they maintained the same two perspectives on the Running Problem that they demonstrated during their first interview. On the one hand, they continued to read out the phrase "same pace" as an indictor of a direct proportion. On the other hand, their causal net was sufficient to see the constant difference, but not the incompatibility of a constant difference and a constant ratio:

Tess If they run at the same pace, this says they run laps together because they run at the same pace. Even if Marty starts before Bob, however many … they're going to run at the same pace. So it's going to in … like the amount difference is going to stay the same the whole time because they're running at the same pace.
Lisa Mm-hmm. It's a constant increase.
Tess Right, if Marty starts, he runs two laps, when Bob starts. So by the time that he runs 4, by the time Marty runs 4 laps, Bob will have run 2 laps … then 6, 4 … 8, 6 … so the same amount of increase every time.
Lisa Yeah. And if you know the lap difference between the two, then you can give me any value of laps and I can figure out where they are.
Tess Right.

Lisa went on to explain that in problems like the Interest Problem, she knew the relationship between time and money was not a direct proportion because one used each monthly total to compute the next base amount, and thus how the two quantities related depended on the month.

Tess and Lisa's judgments about constant difference relationships remained non-normative. That limitations in Tess and Lisa's causal nets, and to a lesser extent Clara's causal net, allowed constant differences and constant products to coexist with direct proportions helped explain the stability in their performance on the pre- and posttests.

Constant Products We found that the preservice teachers routinely read out a direct proportion in the Work Problem, consistent with performance on the pre- and posttests (Table 2). The main *aspect* that they read out was that changes in the number of men were associated with changes in the time to paint a house. Furthermore, across all pairs, the preservice teachers read out correspondences between quantities and positions of variables in the presumptive proportion equation. When they succeeded in establishing these correspondences, they judged the proportion equation to be applicable. In contrast to their work on the Running Problem, the preservice teachers' causal nets were less able to construct appropriate relationships between covarying quantities that maintained a constant product. The following examples illustrate these results.

During their first interview, Alice and Clara decided very quickly that the Work Problem described a direct proportion. As they read the word problem, they set up the equation $a/b = c/d$ and stated that $d = bc/a$. Clara added that she would do the exact same thing if she were working with numbers, and Alice agreed. The preservice teachers then moved on to other tasks.

During the second interview, Clara began the Work Problem with confidence and set out to justify her use of the proportion equation by using a numerical example: 2/15 = 4/? In contrast to her work during the first interview, Clara read out the relationship between numbers of men and minutes more carefully, in that she recognized that as the number of men increased, the number minutes would decrease. She then interpreted the relationship as a negative proportion:

Clara It is proportional, but it's going to be a proportion going down this way if the men were increasing [draws a line starting at the origin with negative slope]. Because if more men are working on the house, then it's going to take fewer minutes. But if it takes 2 men 15 min, then it's going to take one man 7.5 min. So it is going to be proportional. I mean … not one man, 7.5, that doesn't make sense. It would take one man 30 min. It's proportional, it's just a decreasing proportion.

As Clara continued to think, she questioned her graph but continued to assert that the relationship between men and time was a direct proportion. She explained:

Clara The ratio of man to the amount of time he works would, like it's going to be the same sort of chart to determine if more men or less men work and how long it's going to take. They start at the same time; they work the same pace. It's just the amount of men that could change or the amount of work that could change. Then you'd have to determine who, how many people were working or how long they were working for based on the first proportion.

Although Clara read out several *aspects* accurately, she did not coordinate them appropriately to see that the correct relationship between men and time was not linear, a limitation in her causal net.

Similar to Alice and Clara, Lisa and Tess immediately agreed that the Work Problem was a direct proportion during their first interview. Lisa read out the *aspect* of "same rate" explicitly:

Tess I think [the proportion is] accurate because you have the men, the number of men on top over number of minutes
Lisa and they're going at the same rate
Tess and then same rate, so you have your second number of men over your second number of minutes.

These comments suggested that Tess and Lisa were translating information presented in the word problem to the direct proportion equation in a rote fashion.

During their second interview, Tess and Lisa continued to judge the relationship between men and time to be a direct proportion, even though they elaborated relationships among quantities to a much greater extent. After reading the Work Problem, Tess stated:

Tess It's proportional as well. If A men paint the outside of a house in B minutes, how many minutes D would it take C men to paint the house if they all worked at the same rate? 'Cause then we're having a, you know, steady difference or increase or whatever. If you had more men, it would take less minutes. If you have less men, it would take more minutes. (It would) be the same amount of increase or decrease no matter how many, like if you went up or down.

Tess and Lisa went on to explain that situations in which two quantities changed at the same rate were always a direct proportion. Lisa illustrated this point using numbers of men and minutes:

Lisa If I knew how many men worked in how many minutes, then I could divide the number of men by the number of minutes, other way, number of minutes by the number of men to find out the rate at which each man worked. And then if I knew that they were working at the constant, or the same rate, then regardless of the number of men, then I could still figure out the time (they each take).

Tess agreed that you could divide to find the time that each man worked, and she added that each additional man would increase the time by the same amount.

Results from interviews with Alice and Clara and with Tess and Lisa were consistent with our observations during classroom instruction that the preservice teachers' causal nets were not sufficiently developed to construct work as a product of men and time. Persistent reliance on correspondences between quantities and positions in the presumptive direct proportion equation and *aspects* like "same rate" for reading out a direct proportion, combined with an insufficiently developed causal net, could explain the stable performance of the preservice teachers from pre- to posttest. By contrast, the alternative explanation that impulsive responses explain the nonnormative application of direct proportion reasoning strategies does not adequately explain the careful and often partially accurate explanations that these preservice teachers provided.

Conclusion and Implications

Our quantitative results provide a partial replication with preservice middle-grades teachers of studies with elementary and secondary teachers (e.g., Crammer et al. 1993; Fisher 1988; Riley 2010) showing that, like children, teachers face challenges when discriminating among relationships that are and are not direct proportions. Recent research (e.g., Van Dooren et al. 2010) has suggested that thinking about problems rather than answering reflexively is necessary if students are to avoid applying the proportion equation to relationships that are not direct proportions. Our results suggest that simply reading problem statements carefully is necessary but insufficient for accurate identification of proportional relationships. In particular, an accurate understanding of the relationship between two covarying quantities may not be sufficient for the normative determination of direct proportions. In the case of the Running Problem, Tess gave an appropriate sequence of values for consecutive pairs of laps run: "Marty runs 4 laps, Bob will have run 2 laps … then 6, 4 … 8, 6." In case of the Work Problem, Clara said, "If it takes 2 men 15 min …[it] takes one man 30 min." In both cases, the preservice teacher identified accurate relationships between quantities yet endorsed the proportion equation.

We used the coordination class construct to obtain insight into how the preservice teachers in our study reasoned about relationships we presented through missing-value word problems and why their performance from pretest to posttest remained relatively stable. A strength of applying the coordination class construct to our data was the facility with which we were able to account for participants' seemingly inconsistent or contradictory perspectives on a given relationship between two covarying quantities—for instance, recognizing constant differences or constant products in problem situations yet still endorsing the direct proportion equation. In general, we found that incorrect performances were often the result of accurate but misleading readout combined with causal nets that were insufficiently developed to compensate for initial, incorrect judgments. Additional evidence provided by further studies will be needed in order to evaluate more fully the generality of findings we report here and the usefulness of coordination classes for examining how people reason about proportional relationships.

We conclude with a discussion of implications for teacher education. It is plausible that some of preservice teachers' (and students') difficulties stem from inadequate instruction, such as drill in solving missing-value problems that always involve a direct proportion. Avoiding these instructional methods is not difficult, but our experience teaching the content and methods course suggests teacher educators face significant pedagogical challenges when helping preservice teachers become more proficient at reasoning about proportions. We followed Van Dooren et al.'s (2010) recommendation and engaged preservice teachers in explicit discussion of context and mathematical structure of many different problems, and we found this approach insufficient to achieve the kind of understanding we hoped they would develop. We also engaged the preservice teachers in partitioning and iterating composed units during the proportion unit, with the notion that teachers would be better able to recognize proportional relationships if they had experience reasoning with two quantities covarying in a fixed ratio. This, too, proved insufficient to achieve the kind of understanding we hoped they would develop.

Our results suggest that preservice teachers would benefit from instruction designed to develop and refine their readout strategies and causal nets. With respect to readout strategies, one thing that might be helpful would be experiences in which preservice teachers saw that particular *aspects* (Wagner 2006) are not unique to proportional relationships and therefore are unreliable indicators of such relationships. Examples presented above illustrate that quantities that simply covary or that change at a fixed rate do not necessarily form a proportional relationship. With respect to causal nets, one thing that might be helpful is to study the behavior of relationships like constant difference and constant product (inverse proportion) relationships to see why they are not proportional. Preservice teachers could learn to distinguish among several well-understood models of covariation (such as indirect proportions and linear, quadratic, and exponential function models). Such experiences might help preservice teachers develop appreciation for the care with which functional relationships need to be analyzed. Finally, we suspect that preservice middle-grades mathematics teachers need to develop better understanding of the role that definitions play in mathematical reasoning. In the present study, we did not find many students who used the definition of proportion developed during the course when solving problems during the interviews. This calls into question not only the accessibility of the definition, but also whether the preservice teachers understood that definitions can be used to sort relationships into those that are proportional and those that are not. In related work with another data set, we are investigating whether preservice secondary teachers with more mathematical training have fewer difficulties distinguishing proportional and nonproportional problem situations because they are better able to apply mathematical definitions.

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A Processes Lens on a Beginning Teacher's Personal and Classroom Mathematics

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Policy documents and position papers have consistently pointed out the importance of engaging students in doing mathematics in their classrooms (National Council of Teachers of Mathematics (NCTM 2000); National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA and CCSSO 2010)). It seems reasonable to assume that teachers' mathematical knowledge is a key component in whether teachers are able to provide such opportunities for engagement to their students, yet studies related to this assumption have yielded

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mixed results (e.g., Rowland et al. 2000; Tchoshanov 2011; Wilkins 2008). Mathematical knowledge, as conceptualized in these studies, does not sufficiently explain classroom instruction in mathematics. Mathematical knowledge in these studies is often seen as knowledge of specific content or ability to generate answers to mathematical questions rather than engagement in doing mathematics. It is our suspicion that these ways of conceptualizing mathematical knowledge are limited in that they do not capture the ways that teachers do mathematics or the ways they engage their students in doing mathematics. We conceptualize mathematical knowledge as evidenced by how individuals engage in doing mathematics. In this study, we have attempted to focus on how teachers engage in mathematics, and how that engagement is related to the mathematics to which they expose students. By understanding that relationship, we hope to understand better the ways in which teachers' engagement in doing mathematics may facilitate or hinder their engaging their students in doing mathematics.

Research interest has burgeoned regarding the relationship between teachers' mathematical knowledge and the ways in which that knowledge impacts what happens in the classroom. Of particular interest is how teachers' knowledge affects both what teachers do and what students learn. Over the years, researchers (e.g., Eisenberg 1977; Hill et al. 2005; Monk 1994) have investigated the relationship between teacher knowledge and student achievement. These studies have found that teacher knowledge is related to student achievement, but they have shed little light on the question of how teacher knowledge affects what is happening in classrooms (Ball et al. 2001). This question has received much less attention from researchers, and when it has been studied, many researchers have not separated content knowledge and pedagogical knowledge when describing the relationship (e.g., Hill et al. 2007; Lehrer and Franke 1992; Swafford et al.1997), leaving one to wonder about the effects of content knowledge itself on instructional practice.

Studies that have examined the relationship between content knowledge and classroom practice have found relationships, although there is not always a consensus about what these relationships are. For example, Rowland et al. (2000) found that preservice elementary teachers who scored higher on an inventory of mathematical content knowledge received higher grades on their practice teaching lessons, and Tchoshanov (2011) found that teachers with a type of mathematics content knowledge that was rich and connected, as measured by a research-created survey instrument, presented higher quality lessons as measured by a classroom observation protocol. In contrast, Wilkins (2008) reported a negative correlation between elementary teachers' content knowledge, as measured by an inventory of mathematical content knowledge, and their self-reported use of reform teaching practices.

The aforementioned studies (i.e., Rowland et al. 2000; Tchoshanov 2011; Wilkins 2008) used written tests of predetermined categories of teacher knowledge (e.g., high and low content knowledge, cognitive type of content knowledge) and focused on limited aspects of classroom practice (e.g., cognitive demand of tests and homework, students' opinions about instruction, and teachers' self-reports of reform practices) rather than on the mathematics in which the teacher and students engaged. Although these studies identified relationships, they did little to explain the nature of these relationships. The study reported here used extensive sets of interviews and classroom observations focused on teachers' mathematical knowledge and its use in the classroom to characterize and explain the relationship between a teacher's mathematics and the mathematics presented in his or her classroom.

Perspective

Both *Principles and Standards for School Mathematics* (NCTM 2000) and the *Common Core State Standards for Mathematics* (NGA and CCSSO 2010) recognize the importance of mathematical processes to mathematics education.[1] Processes are constructive acts that operate on mathematical entities (e.g., concepts, procedures, principles) and result in the production of new mathematical entities. For the purposes of this study, we describe mathematical engagement in terms of four mathematical processes and actions on their respective products: the processes of representing, justifying, generalizing, and defining, and actions on representations, justifications, generalizations, and definitions (Zbiek et al. 2008, 2012). Our larger research group defines these processes as follows:

Representing The constructive act of creating external inscriptions, physical objects, verbal expressions, or movements intended to capture properties of a mathematical entity (e.g., concept, procedure, or principle).

Justifying The constructive act of explaining how one knows a mathematical claim is true or producing a rationale for belief of a mathematical claim.

Generalizing The constructive act of extending the domain to which a set of properties applies from multiple instances of a class or from a subclass to a larger class of mathematical entities.

Defining The constructive act of identifying and articulating, for a given mathematical entity, a set of mathematical properties and the relationship(s) among these properties in such a way that the combination can be used to determine whether an object, action, or idea belongs to a class of objects, actions, or ideas.

This study addressed the question of what characterizes a beginning secondary mathematics teacher's engagement with mathematical processes in personal

[1] The importance of mathematical processes such as representing and justifying is explicitly discussed in the process standards of the Principles and Standards for School Mathematics, whereas other processes such as generalizing and defining are referred to implicitly. The mathematical practices such as "Reason abstractly and quantitatively" and "Look for and express regularity in repeated reasoning" from the Common Core State Standards raise the importance of these same mathematical processes. We do not infer that mathematical practices and mathematical processes are interchangeable, but we do maintain that the two are strongly related, since engagement in mathematical practices often involves engagement in mathematical processes, and that attention to the former raises the importance of the latter.

mathematics and classroom mathematics and the relationship between the teacher's personal and classroom mathematics.

The processes on which we focus are clearly a subset of all possible mathematical processes, but we chose these particular processes because they were among the ones most commonly researched in the context of collegiate mathematics and advanced mathematical thinking. Because these processes are often evidenced in observable acts, we could gather data on our participants' engagement in processes and their uses of the products of those processes. This allowed us to use the processes and products to characterize a teacher's mathematical knowledge in several settings. We could examine the mathematics demonstrated in a teacher's nonroutine problem solving (*personal mathematics*), the opportunities the teacher provided students to engage in mathematics (*classroom mathematics*), and the relationship between the teacher's personal mathematics and his or her classroom mathematics. A second affordance of describing mathematical engagement in terms of mathematical processes and actions on products is that it transcends both mathematical content areas and grade levels because engagement in these processes and actions on products are not confined to particular topics. Capturing mathematical engagement in terms of mathematical processes allowed us to examine mathematical activity over a period of 3.5 years across three different content areas.

Methods

Our data for this chapter consist of verbatim records of 5 task-based interviews and 16 teaching observation cycles[2] of a beginning secondary mathematics teacher, Fiona (a pseudonym). Fiona was 1 of 10 students (out of 23 in her cohort) in a preparation program for secondary mathematics teachers who volunteered to commit to the 2–3-year study. We chose beginning mathematics teachers as participants in this study because their formal mathematics experiences would be fresh and conceivably more likely to influence the mathematics in which they would engage students. Fiona's case was of interest because her success in mathematics coursework seemed inconsistent with the ways in which she engaged students in doing mathematics. At the beginning of data collection, Fiona was enrolled in a 4-year undergraduate secondary mathematics certification program at a large US university. During this program, Fiona earned 34 credits in college mathematics and 12 credits in statistics. In addition, she earned 15 credits of mathematics education courses focused on content-specific pedagogy. Fiona was successful in her coursework, as indicated by above average course grades. This study followed Fiona through the last three semesters of her teacher preparation program and the first 2 years of her full-time teaching. As a full-time teacher, Fiona taught mathematics at a small public school

[2] The 16 teaching observation cycles included six cycles during Fiona's student teaching, four cycles during her first year of teaching, and six cycles during her second year of teaching. A teaching observation cycle consisted of a preinterview, observation of one class period, and postinterview.

serving several small communities in the suburbs of a large city in the eastern US. During the 2 years that we observed her teaching, the percentage of students at her high school scoring at or above the proficient level on state tests was above the state average.

The five task-based interviews (*area, count, cube, wrap,* and *defining*) were conducted during Fiona's teacher preparation program for the purpose of understanding her use of mathematical processes and products in her personal mathematics. To understand Fiona's use of processes and products in her classroom teaching, teaching observation cycles were conducted during her student teaching of precalculus, her first-year teaching of algebra, and her second-year teaching of geometry. An observation cycle consists of a preobservation interview, an observation of one class period, and a postobservation interview. Some cycles overlapped in that the postobservation interview for one observation occurred concurrently with the preobservation interview for the next observation.

Each of the task-based interviews was videorecorded and audiorecorded, transcribed, and annotated. Teaching observation cycles were audiorecorded, transcribed, and annotated. Still photos (e.g., the writing on the board, the manipulatives the teacher used, and posters on the classroom walls) from the teaching observation cycles were also collected. The photos were used to assist in the annotation of transcripts. The data were collected and prepared by members of the larger research team.

The six authors coded task-based interviews to identify and describe instances of Fiona's use of processes and/or products. For the most part, this coding occurred at whole-group meetings of the team. In some instances, several members of the team worked together to generate initial codes, and the group reviewed and came to an agreement on those codes. Any disagreements on coding were resolved by review and agreement of the data by the entire team. The coded instances were elaborated, categorized into the four process/product categories, and analyzed for emerging themes. This procedure was repeated for the teaching transcripts, but our procedure also included memoing about episodes that seemed to capture the mathematics to which Fiona exposed her students. For example, we memoed, "It is possible that Fiona cedes mathematical authority to an outside source, generally the textbook." This led us to examine how closely Fiona's teaching followed the textbook to gain insight as to whether her following the textbook could be a possible reason for not providing her students with opportunities to engage in mathematical processes. After the initial coding and analyses, the team then compared Fiona's use of processes and/or products in her personal mathematics with her use of processes and/or products in her classroom mathematics.

The data we use to describe Fiona's personal mathematics focus mainly on Fiona's work on two tasks from the task-based interviews. Those tasks were chosen because they were representative of the tasks used in the interviews and because Fiona's work on the tasks exemplified her work with the processes. Figure 1 illustrates tasks from the cube and area interviews. In the cube interview, Fiona was asked to describe the pattern and determine the surface area and volume of an n-layer model of a stack of cubes like the four layer one shown in the left panel of

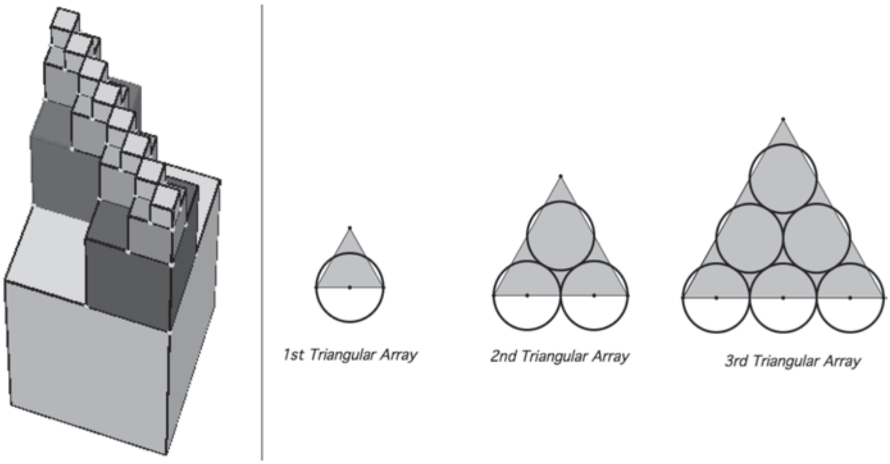

Fig. 1 Some of the illustrations accompanying the cube and area tasks

Fig. 1. She was provided with the paper figure, a physical model, and an interactive three-dimensional computer model of the arrangement of cubes. In the area interview, Fiona was given a diagram like the one illustrated in the right panel of Fig. 1. She was asked to describe the mathematical relationship between the sum of the area of the circles and the area of the equilateral triangle for the *n*th figure in the sequence as the number of circles on the base increased. We introduce a third task, from the defining interview, in the *defining and definitions* section.

Results

Our analysis of the interview and observation data provided us with insight into Fiona's engagement in the four processes, both in the context of her personal mathematics and in the context of her classroom mathematics. We saw consistencies between how she engaged in processes in her own mathematics and how she engaged her students in mathematics, and we also saw differences. The following sections describe Fiona's personal mathematics, her classroom mathematics, and the relationships between them, with an eye to identifying themes that characterized her engagement with processes. These themes transcended mathematical content areas (both in the task-based interviews and in her teaching) and allowed us to compare these themes across both Fiona's personal problem solving and the mathematics we observed in her classrooms. Although we focus each section on only one process at a time, many of the examples could illustrate several processes.

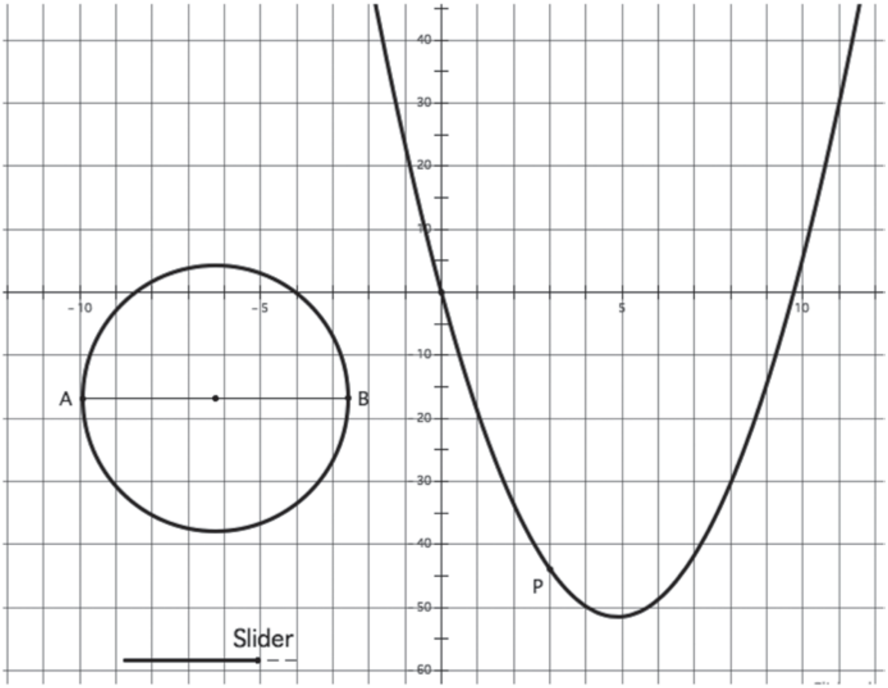

Fig. 2 Graph of the function defined as the difference between the area of the triangle and the sum of the areas of the circles with slider that controls the diameter of the circles. The configuration of triangle and circles is shown in the right panel of Fig. 1

Representing and Representation

Representing is one of the processes in which Fiona frequently engaged without prompting during her personal mathematics. Fiona seemed to use representing to help her in problem solving, often augmenting or modifying one representation to create other representations and connecting representations to provide justifications. Even though doing mathematics requires representing, the representing and interpretation of representations in which Fiona engaged did not go beyond what was necessary to get to an answer. Fiona tended to notice and pay attention to only selected features of a representation rather than accounting for the representation's more complete set of relevant characteristics. In the area interview, when Fiona was given the graph of the function defined as the difference between the area of the triangle and the sum of the areas of the circles (Fig. 2), she focused on the x-value of the minimum point of the function as invariant and interpreted it as the point at which the area of the triangle exceeds the sum of the area of the circles. Her error may have been related to her loosely constructed attention to negative-to-positive change. She seemed to recognize that the graph of the differences required a change from negative to positive, without recognizing that the negative-to-positive change captured by the minimum was a change in slope rather than a change in the output value of the function. She did not attend to the invariant *x*-intercepts, the feature of the graph most relevant to the question of whether the area of the triangle would exceed the sum of the areas of the circles, until she was specifically asked about them. At that point, Fiona did recognize a conflict between her interpretation of the *x*-value at the minimum point and the *x*-intercept—at different times she had identified each of the *x*-value of the minimum point and the *x*-intercept as the value of *x* for which the area of the triangle exceeds the sum of the areas of the circles. However, Fiona did not reconcile this contradiction because she seemed to be focusing on only one of these points at a time, and thereby not connecting her conflicting interpretations of these points.

In her classroom mathematics, in her student teaching as well as in her first and second year of teaching, Fiona occasionally directed students to use different representations in problem solving (e.g., directing students to draw a graph if they were struggling with writing an equation of a line, or having them use geometric figures to generate tables of values to look for a generalization). However, we observed a number of occasions on which it seemed to us that Fiona might have engaged students in linking multiple representations but did not capitalize on opportunities to do so. For example, in her student teaching, she purposefully did not link for students a graphical representation of derivative that appeared on an activity sheet. During the postobservation interview, Fiona showed her understanding of the representational link when she ably linked this graphical representation to a symbolic representation of derivative. During that interview, she explained the limit definition of a derivative by connecting the symbolic expressions in the numerator and denominator to the vertical and horizontal change of the slope of a line tangent to a point on a graph. However, Fiona gave "they [the students] don't know that, and if I would explain it to them I would have confused them–I think–endless amounts" as the reason for not discussing the graphical representation. She might have chosen not to explain the graph because her goal was only for students to be able to apply the limit definition to complete exercises, and she might have thought that the potential confusion that could arise from trying to develop further understanding of the limit definition might interfere with her goal. Not capitalizing on opportunities to link representations illustrates Fiona's tendency to focus on a particular representation, and not to incorporate other representations that may have helped in describing a mathematical entity. In the instance just discussed, Fiona chose to focus solely on the symbolic representation rather than link the symbolic representation to the graphical representation.

Another instance that illustrates this tendency to focus on representations in a limited way occured during her second year of teaching. While introducing polyhedra in a geometry class, Fiona used an unfolded cereal box to represent the net of a rectangular prism. However, Fiona did not mention to the students that because of the overlap for adhesive, the unfolded box is not exactly a net for a rectangular right prism. Fiona focused on parts of the unfolded box (the faces), but did not mention other aspects of the box (flaps for adhesive) that made the box an inadequate representation of a net. This omission was highlighted later in the same class session when the students, who were tasked with constructing different nets of a cube, generated "nets" that included overlapping faces. At this point, Fiona stated to the students that the nets cannot have overlapping faces but did not address the fact that the physical representation that she had shown them did have overlaps. This instance illustrates Fiona's tendency to focus on particular aspects of a representation in her teaching, but overlook other key aspects of a representation, the neglect of which could lead to an incorrect understanding (in this case, an incorrect understanding of net).

In both her personal and classroom mathematics, Fiona often focused on local features of representations and seemed not to account for other features of the representation relevant for the task at hand. This tendency for localization and inattention to links is frequently observed in her personal and classroom mathematics and suggests that mathematics as an integrated system is not central to her view of mathematics. Fiona also seemed to miss opportunities in her personal and classroom mathematics to incorporate multiple representations or to link different aspects of representations. Thus Fiona does not seem to fully exploit representations in her problem solving or teaching of mathematics.

Justifying and Justification

In her personal mathematics, Fiona seldom offered mathematical justification unless she was prompted by the interviewer with questions such as, "How might you convince someone of your claim?" However, she often offered complete and valid justification when she was involved in correcting an error. For example, during the area interview, Fiona had incorrectly labeled the sidelengths of a $30°-60°-90°$ triangle as 1, 2, and $\sqrt{3}$, with the hypotenuse labeled $\sqrt{3}$. When the interviewer prompted Fiona with, "I want you to convince me that those [sidelengths] make sense," Fiona immediately corrected her error and justified that correction by noting that the longest side of a triangle was opposite the largest angle. Despite her ability to engage in justifying, as exemplified by this instance, Fiona seldom engaged in unprompted justifying, even when it seemed reasonable to justify a claim or result.

When Fiona did engage in justifying, she did not always produce correct justifications. In particular, when Fiona justified by referencing properties of mathematical objects, she tended to attend to one property of the mathematical object while not attending to other relevant and necessary properties. This is similar to Fiona's tendency to attend to a limited set of features of a representation. In the area interview (see right panel of Fig. 1), for example, Fiona engaged in justifying that the sum of the areas of the circles in an array is larger than the area of the triangle in the same array. Having generated symbolic representations for the two areas, $(x+(x-1)+(x-2)+\ldots+0)\,\pi r^2$ for the sum of the areas of the circles and $x^2r^2/\sqrt{3}$ for the area of the triangle, she based this argument solely on one difference (one area formula involved multiplying by π and the other involved dividing by the square root of 3)

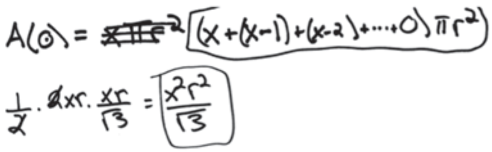

Fig. 3 Fiona's representations of the area of the triangle and the sum of the areas of the circles

without accounting for another essential factor $(x+(x-1)+(x-2)+\ldots+0)$ versus x^2 in the formulas (Fig. 3).

In the context of Fiona's classroom mathematics, Fiona seldom engaged in mathematical justification or asked students to justify even when she had opportunities to do so. One such opportunity occurred during student teaching as Fiona taught rules for producing derivatives. Having presented the usual rule for calculating the derivative of a product, Fiona introduced students to a generalized power rule, and continued the next day by demonstrating the quotient rule. She did not mention any connection between the two rules; even when a student pointed out the similarity between the product rule and the quotient rule, Fiona did not capitalize on the comment as a segue to recognizing and justifying that the quotient rule can be viewed as an instance of the product rule. Fiona was questioned about the similarity in a follow-up interview and her justification, "Because, well, division is multiplication by a reciprocal," suggested that she was aware of a justification, but chose not to use it with her students.

There were times in Fiona's classroom when students' opportunities to engage in justifying were limited because she did not pay attention to essential relevant features of a mathematical representation. An instance discussed in the previous section involved Fiona's introduction of nets of rectangular prisms. In the activity that followed the introduction, Fiona limited her students' justifying by insisting that they consider only two properties in deciding on whether a configuration was a net: (a) when folded, there should be no overlap; and (b) the net of a rectangular prism consists of six rectangles. Fiona did not ask students to pay attention to essential properties such as whether the proposed net folds into the desired shape. Not paying attention to essential relevant features of the net was further illustrated when Fiona showed the students an unfolded cereal box as an example of a net of a rectangular prism—a nonexample because the unfolded box overlapped when folded.

Although Fiona demonstrated her ability to justify in her personal mathematics, she rarely justified mathematically without prompting. Even when she did justify, she often attended to only some of the essential relevant features or properties of a mathematical object. Her tendency not to engage in processes and not to pay attention to essential properties of a mathematical object corresponds to very little justifying in Fiona's classroom by her or her students. The lack of justifying in Fiona's personal mathematics and in her classroom seems to indicate that Fiona does not see the role of justifying as a critical process in her or her students' mathematics.

Fig. 4 The third member of the sequence of pyramids in the count interview. (Published with permission of the Penn State Mid-Atlantic Center)

Generalizing and Generalization

In her personal mathematics, Fiona tended not to generalize without prompting even when it seemed reasonable to do so. For example, in the cube interview, Fiona was asked to find the volume and surface area of a stack of cubes configured in similarly structured layers of cubes (see the left panel of Fig. 1). Fiona recognized that the volume of a cube in one particular layer was one-eighth the volume of a cube in the previous layer, but was hesitant to conclude that this was true for all layers and did not offer this claim until she was prompted to state a conclusion.

Fiona's generalizations, when she did make them, were sometimes flawed because she focused on a limited set of properties and did not account for relevant possibilities. For example, in the count interview, when asked about a three-dimensional analogue (composed of layers of spheres as suggested in Fig. 4) of the circles situation shown in the right panel of Fig. 1, Fiona generalized that no matter the size of a pyramid constructed of spheres, there are no interior spheres. She based this observation on the fact that the three-layer pyramid of spheres had no interior spheres and that the spheres added to the three-tiered model to form the four-layer pyramid model were all exterior spheres. In this case, she failed to account for the fact that some of the spheres in previous layers become interior spheres when another layer is added. It is important to note, however, that when Fiona checked her assertion by physically deconstructing a three-dimensional representation of the five-layer pyramid, she saw that there was an interior sphere in the four-layer pyramid and was able to correct her error, justify by construction why interior spheres existed, and create a correct generalization about the number of interior spheres in any given figure.

Similar to the lack of unprompted generalizing in her personal mathematics, Fiona did not engage her students in generalizing in her classroom mathematics when it seemed appropriate to do so. For example, in an exercise involving Hooke's law, rather than giving students one general equation that could be used in three types of exercises, Fiona directed students to use three different equations for three different, but clearly related, cases: the equation $y/x = k$ to find the value of k, the equation $y = kx$ to find the value of x, and the equation $d = kt$ to find the value of k.

In her classroom mathematics, Fiona chose and implemented activities that seemed to have considerable potential for engaging students in generalizing. However, when she implemented activities aimed at generalizing, she usually led students to reach a generalization that she had predetermined, rather than allowing students to construct generalizations that she had not anticipated. For example, Fiona chose an activity in which students used a computer applet to count faces, edges, and vertices of polyhedra. The activity had the potential to engage students in generalizing about the relationship among these quantities. Fiona was focused on ensuring that all students arrived at the formula vertices − edges + faces = 2, but she did not seem concerned about whether students arrived at the formula by generalizing, by using the Internet, or by getting it from another student.

Fiona sometimes stated generalizations that were false, often seemingly basing them on an overly limited domain or set of examples. For example, when introducing a lesson on graphing lines, Fiona stated the incorrect generalization, "There is a y-intercept and an x-intercept for every single line." In this case, Fiona seemed to have focused only on the set of slanted lines, not accounting for the possibility of lines that were vertical or horizontal. Fiona's personal mathematics and her classroom mathematics had two main commonalities. First, Fiona often did not generalize when it seemed appropriate to do so. Second, Fiona often incorrectly generalized or stated incorrect generalizations, possibly because she focused on a limited domain or on a limited set of properties rather than accounting for all possibilities and relevant properties.

Defining and Definition

As with other processes, the process of defining did not seem to play a central role in either Fiona's personal mathematics or her classroom mathematics. Despite several opportunities in her task-based interviews and in her classroom mathematics, we rarely observed Fiona engaging in defining, although we saw her engage with the products of defining, namely, definitions.

Fiona tended to focus on elements of definitions rather than on the definition as a whole. She seemed to compartmentalize definitions and not to coordinate them in her teaching or in her personal mathematics. In the defining interview, she was presented with six ways in which people may talk about a parallelepiped (Fig. 5). She was asked which of the six descriptions are most similar to each other. We expected her to consider the mathematical entity defined by each statement and to compare those entities. However, she chose to examine parts of each statement and to compare them with parts of other statements. For example, she stated that descriptions B and F are similar to each other because they both describe a six-sided polyhedron. However, she never endeavored to examine each description as a whole.

a. A parallelepiped is a box with a lid.

b. A parallelepiped is a polyhedron with six faces that are parallelograms, which could be rectangles, squares or rhombuses but not trapezoids.

c. A parallelepiped is a cube whose sides are rectangles.

d. A parallelepiped is a prism the base of which is a parallelogram.

e. A parallelepiped is a hexahedron each face of which is a parallelogram.

f. A parallelepiped is a hexahedron with three pairs of parallel faces.

Fig. 5 Six ways people may talk about a parallelepiped

This trend of focusing on parts of a definition, rather than the whole definition as defining a mathematical entity completely, is also reflected in Fiona's classroom mathematics. For instance, Fiona presented the textbook's definition of a vertex as "a point at which three or more faces meet." However, later on during the same lesson, Fiona referred to a vertex of a cone without offering a definition for it. One of her students pointed out that the original definition of a vertex does not apply to the vertex of a cone, and Fiona merely agreed with the student without offering or seeking an explanation for why this is the case. Fiona seemed unperturbed by the potential conflict among these definitions. This incident involving vertex definitions highlights how Fiona did not insist on coherence across definitions of a mathematical term.

Discussion

Although her work was not always perfect, Fiona, during her task-based interviews, demonstrated an ability to engage in mathematical processes. From our characterizations of her personal and classroom mathematics, as well as the relationship between them, we noted two emerging themes regarding Fiona's use of her mathematical knowledge: (a) minimal use of mathematical processes and (b) overlooking of relevant features of mathematical objects. These two themes characterize Fiona's engagement with multiple processes and actions on the products of those processes, and were evidenced both in her personal mathematics and in her classroom mathematics. First, despite her ability to successfully engage in the processes, Fiona did not tend to use processes in her problem solving. Although Fiona demonstrated competence in engaging in the mathematical processes studied, the processes were not central to how she engaged in mathematics or how she engaged students in mathematics. In her task-based interviews, Fiona generally engaged in these processes only when prompted. Similarly, while teaching, she seldom engaged in processes or required that students do so, even though we observed several occasions (e.g., students asking Fiona for justification or Fiona providing students with activities designed to lead to generalizing) in which it would have seemed reasonable to engage in processes. The possible exception to Fiona's tendency not to engage in processes is the process of representing. In her personal mathematics, Fiona seemed to use representing to help her in problem solving, often augmenting a representation or manipulating one type of representation to create another and linking representations to provide justifications. In her classroom, Fiona occasionally directed students to use different types of representations in problem solving (e.g., directing students to draw a graph if they were struggling with writing an equation of a line or to use geometric figures to generate tables of values to look for a generalization). However, she seldom seized opportunities to have students interpret and link multiple representations even when it would seem to make sense to do so (e.g., not linking the limit definition of a derivative of a function to its graphical representation). Although Fiona showed that she was capable of engaging in each of the four identified processes, she tended not to engage in those processes in her own problem solving and did not engage her students in those processes.

Second, we observed a characteristic of Fiona's engagement in the processes: She tended not to pay attention to a sufficient set of relevant features of mathematical objects. Fiona had a tendency, when working with processes and products of those processes, to focus on some features of a product or mathematical object and not attend to other relevant features. In her personal mathematics, many of Fiona's justifications were incorrect because she had not attended to a sufficient set of the relevant characteristics of the object in question. In the defining interview, Fiona defined a particular set of polyhedra as having exactly one pair of parallel faces without recognizing that some of the polyhedra in the set have more than one pair of parallel faces. In her classroom mathematics, she used the term *vertex* as having universal applicability and failed to distinguish between definitions of a vertex of a polyhedron and a vertex of a cone. When working with processes and actions on the products of those processes, Fiona regularly focused on an insufficient subset of the features of a product or mathematical object and did not attend to other relevant features.

Although Fiona did not speak directly about it, it is conceivable that Fiona's view of mathematics could explain her approach to processes and actions on the products of those processes in both her personal mathematics and her classroom mathematics. Engagement in mathematical processes such as representing, justifying, generalizing, and defining are essential for making connections within mathematics. Fiona's tendency not to engage in mathematical processes without prompting and not to engage her students in processes seems to indicate that making connections within mathematics is not central to Fiona's view of what it means to do mathematics. Similarly, attending to only some of the relevant features within a problem situation was evidence of Fiona not treating mathematics as a connected system. Attending to all of the essential and relevant features of a problem situation would involve seeking and insisting on consistency in a mathematical system. Fiona's personal mathematics and her classroom mathematics suggest that the connectedness of mathematics is not central to her view of mathematics or of her role as a teacher.

One possible explanation for Fiona's tendency not to engage in mathematics as a connected body of knowledge is that she may have had limited experience with doing so. It is possible that Fiona may be accustomed to problem situations in which the mathematical tasks can be completed by attending to only a limited set of features without engaging in the processes. Even if this was not Fiona's personal experience, she orchestrated this kind of experience for her students. Many of the activities Fiona chose were ripe for involving students in mathematical processes, but Fiona's implementation favored getting to the intended end result instead of engaging in mathematical processes. In classrooms like Fiona's, doing math does not require accounting for a full range of problem features. Single cues suggest the procedure to follow. We are not suggesting that Fiona made conscious decisions about focusing on a single aspect to simplify mathematics for students; she also focused on limited aspects in her own mathematics. Nevertheless, she crafted for students environments in which doing mathematics meant being cued to do specific procedures based on a single aspect of the problem and in which doing mathematics did not require engaging in processes or attention to mathematics as a connected body of knowledge.

Implications

Our data show that Fiona, although capable of engaging in mathematical processes, tended to treat mathematics—whether personal or classroom—as if it was disjoint and disconnected. Doing so adversely affected her own problem solving and limited her students' mathematical opportunities. She often did not account for a sufficient set of relevant features of the mathematical entities with which she was dealing, nor did she account for needed domains. She did not tend to engage in processes without prompting or to engage her students in processes. Although more study is needed to determine why prospective secondary mathematics teachers with significant formal mathematics background may act on mathematics in this way, this behavior may be a reflection of their mathematical experiences. Teachers who have not engaged in mathematics as a connected, coherent whole themselves may have difficulty seeing the discipline that way. It might be productive to provide prospective secondary mathematics teachers who exhibit problem solving similar to Fiona's with experiences that not only require linking representations from different registers but that, through reflection, make those links overt. The key to doing so might be engaging them in reflection on connections in the context of their problem solving. They need not only to understand mathematics as a connected body of knowledge but also to feel empowered to engage in finding and using those connections in mathematics. Engaging prospective secondary mathematics teachers in mathematical processes, as a natural part of the way they do mathematics, will develop their ability to engage their students in mathematical processes and to place a value on doing so.

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Commentary on Section 1: Mounting Progress on Understanding Mathematics Teacher Content Knowledge

Mark Hoover

In his presidential address at the annual meeting of the American Educational Research Association, Lee Shulman argued that the absence of focus on subject matter was a "missing paradigm" in approaches to the study of teaching, with serious consequences for policy and research (Shulman 1986). He characterized research at that time as overlooking the essential role of teacher content knowledge in research on teaching.

> What we miss are questions about the content of the lessons taught, the questions asked, and the explanations offered. From the perspectives of teacher development and teacher education, a host of questions arise. Where do teacher explanations come from? How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems of misunderstanding? (p. 8)

In acknowledging that "mere content knowledge is likely to be as useless pedagogically as content-free skill" (p. 8), Shulman called for blending content and pedagogy. He introduced the term *pedagogical content knowledge* (PCK) to express a domain of content knowledge "that goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching…that embodies the aspects of content most germane to its teachability" (p. 9). Having argued that teaching requires content knowledge that sets a teacher apart from a content expert, he then suggested that a conceptual analysis of teacher content knowledge would need to combine a framework of domains and categories of teacher content knowledge with forms for representing that knowledge. He proposed three distinct forms: propositional knowledge, case knowledge, and strategic knowledge. As he explained, the latter two forms provide resources for linking propositional knowledge to relevant use of that knowledge in teaching: Case knowledge helps to make propositional knowledge applicable in practice, with strategic knowledge providing resources for managing situations when competing principles and precedents collide.

The notion of a distinctive content knowledge for teaching that is somehow different from knowledge of the subject being taught has appealed to scholars of teaching in all subject areas—from science, to physical education, to music, to language

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arts, and on. The idea of PCK seems to get at a basic issue that content knowledge matters for teaching, but that it is an understanding of subject matter distinct from that of the content expert—in particular, that it is useful for teaching and is understood in ways that enable it to be used in teaching. Unfortunately, progress on these ideas has been slow, especially empirical work grounded in classrooms, such as the development of observation instruments linked to the concept of PCK.[1] Nearly 30 years later, the basic issues raised by Shulman, about ways in which content knowledge for teaching might be different, persist as pressing questions. How is it different? What knowledge is most useful? Why do teachers often seem to not use what they seem to know? Shulman's suggested focus on case knowledge and strategic knowledge is compelling, but has not satisfied the field. Teacher content knowledge has garnered increased attention since Shulman's presidential address and several studies have offered important results, but consensus on what it is and how best to conceptualize it is still lacking. I suggest that the reason for the popularity of ideas about a distinctive content knowledge for teaching (PCK and others) is twofold. First, it turns attention to this set of important issues and questions about the relationship between content knowledge and teaching. Second, clarity about the dynamics of this basic relationship is key to improving teaching and learning.

Each of the four reports I discuss here takes a focused look at some constrained aspect of teacher content knowledge. At the same time, each simultaneously and consciously takes up the thorny set of issues laid out by Shulman. Each set of authors is concerned with identifying specific teacher content knowledge and with how teacher content knowledge is conceptualized in the field. Each is attentive to the consequences of choosing particular conceptualizations (of the problem and of teacher content knowledge) over others—consequences for research on teaching, for the improvement of teaching and learning, and for teacher education and professional development. Each is aware of the history and the debates surrounding teacher content knowledge. And each has written a chapter that speaks to these issues, directly and indirectly, as they report on their specific study.

Land and Drake report on an analysis of responses of 34 elementary preservice teachers to nine open-ended questions designed to gain insight into novices' practices of reading, evaluating, and adapting curriculum materials. They situate reading, evaluating, and adapting mathematics curriculum materials within a larger picture of curriculum use, and they use their focus on novices to propose an initial framework for a trajectory from novice to expert use. They propose five central skills: (a) writing goal statements, (b) learning about and honoring student strategies, (c) using explicit constructs to evaluate materials, (d) learning from educative materials, and (e) making adaptations to the materials. Further, they argue that these skills provide a basis for locating teachers at qualitatively different levels of curriculum-use performance. Their study contributes to an important larger effort to understand teaching practice related to curriculum use and to the development of such practice. Although it is unclear whether their five skills are the critical ones on which to fo-

[1] Patricio Herbst suggested this particular formulation of the problem to me—a dearth of observation instruments.

cus or whether their criteria for competent performance by beginning teachers are robust, their exploration draws attention to an important line of inquiry and provides a useful set of initial ideas about relevant practice and its knowledge demands.

Herbst and Kosko report on an analysis of the process of developing and piloting items their research group designed to measure mathematical knowledge for teaching (MKT) in the context of high school geometry. Building on the approach of Deborah Ball and her colleagues, both in conceptualizing MKT and in item development, they provide a detailed picture of the process for constructing and validating MKT items. Based on pilot results from experienced secondary teachers—specifically correlations between years of experience teaching geometry and assessed mathematical knowledge for teaching geometry—they argue that MKT is best organized by a simultaneous consideration of content and tasks of teaching as these are constituted within a larger instructional context, such as a high school geometry course. The detail in their report is likely to be helpful to those interested in developing MKT items. In addition, their proposals for distinguishing MKT by the context of instruction provide an important basis for future work in understanding the structure and organization of MKT.

Both of these reports conceptualize teacher content knowledge as fundamentally bound to teaching. Land and Drake manage the relationship between curricular knowledge and use of curriculum in teaching by blurring the distinction. They study knowledge by studying tasks of teaching. This approach permits them to finesse certain problems, but raises questions about what to make of the work and how to use it. For example, by focusing on the practice of curriculum use in their analytic framing without explicitly attending to the nature or role of mathematical knowledge in that practice, important mathematical issues get buried inside the details of their analysis. For instance, in their examination of goal statements for a lesson, they decide that a goal identified by a participant is more sophisticated than the explicit goals stated in the curriculum. Improving on the goal stated in the materials requires important mathematical knowledge. However, the analytic tools used in the study do not provide disciplined structure for identifying, categorizing, or elaborating the nature or role of this important MKT. Instead, it is the researchers' knowledge and insight that spot, highlight, and remark on the mathematical sophistication of the participant's goal. Throughout, Land and Drake describe levels of sophistication, while acknowledging the importance of exceptions. So, in their discussion about goal statements, they point out that knowing when to offload, adapt, and improvise depends on the specific lesson and set of materials teachers are using. However, if deciding when to override the goals given in curriculum materials is an important aspect of expert curriculum use, we are left to ponder the nature of the mathematical knowledge that informs the use of curriculum. The analytic framing features the work of using curriculum in teaching, so assures the relevance to teaching, but does not feature mathematics. Hence, provides no systematic and structured ways of attending to mathematics. Instead, readers need to rely on the mathematical judgment and insight of the researchers, which fortunately is good. My point here is not criticism of the study but the highlighting of a fundamental challenge for research on mathematics important for teaching—simultaneous attention both to mathematics and to the context of teaching in which it is to be used.

Herbst and Kosko use the conceptual framing of MKT, which is defined to be mathematical knowledge specifically referenced to carrying out the work of teaching mathematics (Ball et al. 2008). As with the concept of PCK, which Shulman described as "amalgam knowledge," the distinction between mathematical knowledge and teaching can be confusing. For instance, Herbst and Kosko characterize knowledge of content and teaching (KCT) items as identifying well-defined instructional goals, with possible answers naming mathematical options that, "while correct in general, would be better or worse choices to meet the specified goal." One might ask, if correct in general, on what basis are they better or worse? It is subtle to suggest that they are correct in the sense that they do not contain anything that is mathematically incorrect, but that they vary in the extent to which they meet a pedagogical goal and that deciding the extent to which they meet such a goal requires mathematical knowledge and judgment. It is this last point that qualifies these items as measuring a form of mathematical knowledge, or at least amalgam knowledge, and not just pedagogical knowledge. For instance, the item on choosing a proof for explaining the base angle theorem has teachers choose from mathematically valid proofs, with a focus on the extent to which the proofs support explanation of why the theorem is true for students with expected background knowledge. It seems reasonable to ask, if the options are all mathematically valid, what is the mathematical knowledge being measured? However, clearly this judgment draws heavily on mathematical knowledge and skill. I doubt this item could be answered simply by attending to nonmathematical features of the proofs. Certainly, it measures some form of mathematical knowledge, but characterizing such situated mathematical knowledge is not straightforward. By naming "tasks of teaching" and distinguishing items by the subdomains of MKT that Ball, Thames, and Phelps defined in terms of the relationship to the work of teaching, Herbst and Kosko provide analytic tools for making distinctions. While subtle, these tools allow them to tease apart interactions between tasks of teaching and mathematical demands associated with those tasks in systematic ways in cognitive interviews. Their use of "instructional situation" allows them to consider ways in which MKT might be organized based on particular ways that mathematical topics and pedagogical practices come together in instructional contexts.

In contrast, the other two reports maintain greater independence between mathematical knowledge and teaching. Jacobson and Izsák report on analyses of 28 middle-grades preservice teachers' responses to prompts about whether problem situations illustrate the proportion A/B = C/D and four, talk-aloud, pre- and post-interviews with pairs of teachers using a set of similar prompts. Beyond simply identifying shortcomings in teachers' knowledge in an area well documented to be difficult for school children, they examine why many teachers correctly identify relationships as not being direct proportions, but then solve them as if they were. The report helps to explain the apparent anomaly. Students can correctly identify relationships among quantities, yet may inappropriately apply methods for solving direct proportions because they (a) associate the solution technique for direct proportion problems with perceived features of direct proportion problems; and (b) do not have experiences that have alerted them to those features being present in the problems that are not directly proportional, nor experiences that help them see that other relationships do not have the properties of direct relationships. Jacobson and Izsák argue that, instead of only instructing students to think more carefully about the situations posed in word problems, successful intervention requires (a) more explicit attention to perceived features of such problems and to recognition that these features are not unique to direct proportion situations, and (b) instruction on the behavior of relationships that are not direct proportions (such as constant difference and inverse proportions) to see why these are not directly proportional.

Heid et al. report on an analysis of a beginning secondary mathematics teacher whose success in mathematics coursework was seen as inconsistent with the ways in which she engaged students in doing mathematics. They focus on four mathematical processes: representing, justifying, generalizing, and defining. They examine the teacher's engagement in these processes, both as she worked on mathematics problems that she was asked to solve and in her classroom instruction. They found that she rarely engaged in these mathematical processes unless prompted, and even then with limited success. Instead, her focus was on solving mathematics problems by cuing to do a specific procedure based on a single aspect of the problem. They conjecture that this may be a result of her experiences as a mathematics student, where such a focus was adequate for success and that it led her to views of mathematics in which doing mathematics does not require engaging in mathematical processes or attending to mathematical connections. The report draws attention to the potentially important role of a teacher's proficiency with mathematical processes in shaping the educational opportunities provided to students.

These two latter reports conceptualize teacher content knowledge as quite independent of teaching. The mathematical knowledge they consider is the same as that which might be considered when studying children's mathematical proficiency. These researchers do not focus on the relationship of the knowledge to teaching, but instead identify potential weaknesses in teachers' mathematical knowledge that may be crucial to their success in teaching mathematics. However, they do so as part of an effort to identify important mathematical knowledge for successful teaching. To get at the essential role of teacher content knowledge in teaching, Heid et al. examine knowledge as evidenced in how individuals engage in doing mathematics. Instead of examining how teacher content knowledge may be different from the knowledge that content experts have or the knowledge that students are expected to learn, these researchers argue that the way to identify content knowledge for teaching is by identifying aspects of the knowledge base that really matter for teaching. In this argument it is clear what mathematics is being studied, but it is less clear what the relationship of the mathematics is to teaching and whether and how it matters for teaching. The argument for the connection of teacher content knowledge to teaching is indirect. To establish the importance of teachers' knowledge of mathematical processes, researchers would need to develop measures of such knowledge and demonstrate effects of this knowledge on teaching and learning.

Similarly, Jacobson and Izsák do not focus on how teacher content knowledge is distinct from experts' content knowledge, but on a basic first step of assuring that teachers have a solid understanding of the content they are to teach. The mathematical preparation of teachers as envisioned in their study is about addressing immediate roadblocks to establishing that solid understanding for some of the most important and challenging topics in school mathematics. Certainly, such knowledge matters for successful teaching. Figuring out ways in which it matters and the extent to which it matters will be an important step in helping to establish its priority in an array of competing priorities. Having good measures for this knowledge (Bradshaw et al. 2014) will help to facilitate this ongoing work.

Having situated these studies in a broader picture of teacher content knowledge, I end with some comments and suggestions for future work. First, the mathematical terrain of K–12 education is extensive, as is the scope of varied tasks of teaching. Each topical area in mathematics and each task of teaching have significant implications for the mathematics that teachers need to know. From a research perspective, a developed understanding of the content knowledge required for teaching would need to include approaches for managing this sprawling landscape. Given the development of current work, the field is in need of studies, like the ones reported here, that continue to examine specific pieces of this landscape. Increasing numbers of dissertations and other small projects have begun this work. In addition, the more this stream of research can share conceptual tools and language, the more it will add up to something. This point is implicit in my above discussion of the individual reports and in my efforts to highlight the importance of considering how the relationship of mathematical knowledge to teaching is being conceptualized in any specific study and how mathematical knowledge is being conceptualized in any analysis of teaching. Finally, studies that examine how teacher content knowledge might best be organized, such as the one reported here by Herbst and Kosko, will be important in helping to structure and make sense of the overall landscape of teacher content knowledge.

An important, yet rarely considered, dimension of this landscape is the trajectory of different content knowledge needs and priorities along a spectrum of professional development, from the mathematical requirements for admission into a teacher education program, through preservice education, student teaching, entry into the profession, standards of established mid-career teaching, and mastery. Addressing such a trajectory is an asset of Land and Drake's study. More work along these lines is needed. At the same time, there is a distinct tendency in the field for researchers to study preservice teachers, generally students in their own institutions and classes. This is true for three of the four reports here. While understandable, this practice would be strengthened by explicitly situating such research in a conceptualization of developing knowledge, from novice to expert, and by expanding the design of studies, even in small ways, to include subjects that would afford data and contrast along such a trajectory. For instance, in hindsight, Land and Drake might have recruited even a small number of experienced teachers to complete their items. The additional contrasts between responses of preservice and practicing teachers, even small numbers, would have added to their study and might have led to stronger results about the interpretation of preservice teacher responses and the levels of performance. With a modest investment of time and thought, similar extensions to research designs could be generated for most studies that, for practical reasons, focus on a limited collection of readily available subjects.

Third, in order to move beyond a preponderance of studies that suggest novel ideas about mathematical knowledge that might be important for teaching, we need to continue to build reliable and valid measures of the most compelling ideas that have been proposed and to share them across studies. The development of robust measures requires extensive resources, both for construction and for validation. In addition, the necessary focusing of resources would require the building of consensus about underlying ideas and about confidence in the validity of the measures. Progress on this front is essential for building coherence across research on mathematics teacher content knowledge and for ultimately improving mathematics teaching and learning.

Last, researchers' clarity, in a particular study, about how teaching is being conceptualized, and about how the relationship between mathematical knowledge and teaching is being conceptualized, will help with communication among researchers about the assumptions, interpretations, and implications of studies. As Herbst and Kosko point out, the field is replete with different ways of conceptualizing teacher content knowledge and its relationship to teaching. Although much has been published, it is not clear that scholars working in this area understand one another's conceptions well or have conceptual framings that are readily understood by others in the field. University promotion based on distinguishing one's intellectual contribution can work against the need for in-depth understanding of one another's work and investment in collective endeavors. The inclination to focus on one's own novel ideas might, at times, be productively replaced with thoughts of joining existing efforts, forging extended collaborations, and pooling resources. Doing so might help in building the "accumulated critical mass of empirical research" referred to by Herbst and Kosko. Of course, at times, there are important differences of opinion and perspective. These, too, are essential. Nevertheless, efforts to contribute to an understanding of content knowledge for teaching might be more productively organized into distinct, collective lines of work that can operate at an adequate scale for addressing fundamental needs of research in this domain.

The suggestions for future work that I have laid out here are meant to focus our attention on issues that need to be addressed if we are to make progress that has real impact on teaching and learning. The reports in this section contribute to a surveying of the wide-ranging landscape of mathematics teacher content knowledge. Also, lying just beneath the surface in these reports are a set of foundational issues for work in this arena. Shulman's pedagogical address drew attention to the crucial, yet perplexing, relationship between content knowledge and teaching. I have argued here that explicit attention to how teaching is being conceptualized and explicit study of the nature of the relationship between content knowledge and teaching would strengthen our currently diverse efforts in the study of teacher content knowledge.

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Part II
Beliefs and Identities in Mathematics Teacher Education

Photo-Elicitation/Photovoice Interviews to Study Mathematics Teacher Identity

Theodore Chao

Over the last several decades, mathematics educational researchers have built a substantial body of work exploring how to support mathematics teachers. Spurred by Shulman (1987), researchers have focused on teachers' professional knowledge of mathematics (Hill et al. 2007), professional beliefs about how mathematics is learned (Philipp 2007), and professional dispositions toward mathematics (National Research Council 2001). Shulman, however, also warned about taking the professionalization of teaching too far, writing, "We must achieve standards without standardization. We must be careful that the knowledge-base approach does not produce an overly technical image of teaching, a scientific enterprise that has lost its soul" (Shulman 1987, p. 20).

What Shulman labeled as "soul" other scholars might refer to as an identity—the unique aspects that make each teacher human and inform his or her professional practice (de Freitas 2008; Drake 2006; Enyedy et al. 2006; Palmer 2007). Researchers often frame teacher identity as the interplay of a teacher's professional and personal lives (Gee 2000; Holland et al. 1998; Van Zoest and Bohl 2005; Wenger 1998). Yet, the research connecting this view of identity with actual mathematics teaching practice has, so far, been elusive (de Freitas 2008; Foote et al. 2011; Van Zoest and Bohl 2005).

In this study, I used alternatives to some of the traditional research methods used to explore mathematics teacher identity, something that the identity researchers have long called for (Nolan and de Freitas 2008; Philipp 2007; Van Zoest and Bohl 2005). I used photographs, allowing teachers to share how they envision themselves both personally and professionally. The photo-elicitation/photovoice interview

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(PEPI) is a relatively new education research method incorporating photographs of a mathematics teacher's life, selected and ordered by the teacher, into the interview context (Clark-Ibáñez 2004; Gauntlett and Holzwarth 2006; Harper 2002). By adding a photographic visual anchor, the PEPI helps structure teacher narratives into a series of stories teachers tell about each photograph in order to explore their mathematics teacher identity.

Explorations into the identity of mathematics teachers matter because the field of mathematics education is becoming increasingly sociopolitical (Gutiérrez 2013) as mathematics teachers are becoming more depersonalized (de Freitas 2004; Weisberg et al. 2009). Exploring the human element of why someone chooses to teach mathematics—their soul/identity—helps us as educational researchers and teacher educators to better support mathematics teachers as people, thereby supporting their students' learning.

In this chapter, I start by reviewing some of the ways mathematics teacher identity has been studied, specifically when using narrative methods (Bruner 1996; de Freitas 2008; Doyle 1997) and when trying to understand Latino/a mathematics teacher identity. Then, I show how a change in methods, specifically in using the PEPI, can help situate mathematics teacher identity as the stories mathematics teachers tell themselves but rarely share. Finally, I analyze the mathematics teacher identities of two Latino high school Algebra I teachers, framing the results through answering the research question: What identities do mathematics teachers present about themselves through photographs and stories within a PEPI?

Theoretical Perspective on Mathematics Teacher Identity

Drake et al. (2003, 2001) defined mathematics teacher identity as the stories that reveal how a teacher knows himself or herself and his or her life. Similarly, de Freitas (2008, 2010) defined mathematics teacher identity as the result of negotiating one's provisional self with regard to the various sociopolitical positions taken when teaching mathematics (e.g., sexuality, ethnicity, and economic status). Van Zoest and Bohl (2005) defined mathematics teacher identity as a socially constructed and stable attribute teachers develop through the various communities of practice they participate in throughout their career. This framing of identity as based upon community participation aligns well with Burell and Goldberg (2010), who defined mathematics teacher identity as a construct shaped and negotiated through everyday activities, goals, beliefs, knowledge, personal history, cultural socialization, experiences, and community memberships. Walshaw (2004, 2010) viewed mathematics teacher identity as formed through social negotiation and explored through narratives of pedagogical practice. Foote et al. (2011) found that, specifically for urban mathematics teachers, mathematics teacher development involves multiple strands of knowledge: mathematics content, mathematics teaching pedagogy, and one's students. Battey and Franke (2008) conceptualized mathematics teacher professional development through storied identities involving a teacher's relationships to teaching, mathematics content, students, and his or her community. Finally, Brown and McNamara (2005, 2011), in their exploration of primary mathematics teachers in the UK, found that identity is constantly negotiated, not just revealed through narrative, but created through it.

While these definitions and other frameworks of teacher identity vary on the surface, they share key attributes. First, mathematics teachers reveal their identities though *narrative and stories* (Bishop 2012; Brown and McNamara 2005; Clandinin and Connelly 1996; de Freitas 2010; Drake et al. 2001, 2003; Holland et al. 1998). Second, mathematics teacher identity involves *negotiating one's personal history with one's professional responsibilities* (Burrell and Goldberg 2010; de Freitas 2008; Drake et al. 2003; Enyedy et al. 2006). Third, mathematics teachers identities involve a teacher's *knowledge of or relationship to mathematics* (Battey and Franke 2008; Foote et al. 2011; Walshaw 2010), the various *communities* that a mathematics teacher lives within (Battey and Franke 2008; Van Zoest and Bohl 2005), and *sociopolitical identities,* such as gender, ethnicity, and sexuality (Battey and Franke 2008; Brown and McNamara 2011; de Freitas 2008, 2010).

Identity Revealed Through Narratives

Hiebert et al. (2002) wrote about teacher narratives as a way to understand teachers, echoing Bruner's (1986, 1996) call for a narrative construal for studying teacher identity. Doyle (1997) justified narrative-based teaching research to be as valid as any other form of research, particularly quantitative studies. I explore teachers' identity through narratives because narratives do what quantitative and traditional research methods cannot—narratives allow teachers to tell personal practitioner stories (de Freitas 2008; Doyle 1997). These personal practitioner stories elicited through narrative are important because teachers often learn to build their practice as individuals; they place value on the experiences they hear in colleagues' stories over the professional development and research-based practices they encounter in their careers (Britzman 1991; Hiebert et al. 2002; Lortie 1975). To uncover these stories, interviews must take place in "safe spaces, generally free from scrutiny, where teachers are free to live stories of practice" (Clandinin and Connelly 1996, p. 25).

Anchoring Structures to Focus Narratives

Narrative research, however, can get very messy; stories tend to overlap or tangent into other stories. Studies successfully exploring identity through a narrative lens often rely on specific structures to anchor these narratives into analyzable units. Vygotsky (1978) wrote about the need for physical anchors, images, or artifacts that helped him situate his exploration of a child's development from an inner speech to a social speech. Holland et al. (1998) found that the ability to self-reflect upon one's own "inner-speech/discourse" was difficult for all their subjects without some sort of anchor. Freire (1970) anchored his interviews with Brazilian literacy teachers using drawings and photographs. Structures, therefore, are helpful in separating a teacher's narrative into individual stories for analysis (Bruner 1996; Doyle 1997; Hiebert et al. 2002; Sfard and Prusak 2005). *Anchoring structures,* then, are specific tools used to ground a narrative so they can be divided into specific story units. Anchoring structures situate and direct an interview while still encouraging participant voice.

In this study, I used teacher-selected photographs of their "world as a mathematics teacher" as anchoring structures. This process is relatively new in educational research, but referred to as *photovoice* in nursing and health care research (Guillemin and Drew 2010; Hansen-Ketchum and Myrick 2008; Wang and Burris 1997) and *photo-elicitation* in anthropology and sociology research (Clark-Ibáñez 2004; Gauntlett and Holzwarth 2006; Harper 2002). These are research techniques that use a visual basis for self-authoring narratives, allowing the participants to tap into their "visual" voices to access memories and thoughts they might not recall in other ways (Hansen-Ketchum and Myrick 2008; Wang and Burris 1997). This visual self-reflection often elicits powerful and emotionally charged stories (Clark-Ibáñez 2004; Wang and Burris 1997).

Photographs Elicit Multiple Types of Stories

Anchoring structures preserve and encourage participants' voices and break narratives into analyzable *story units,* finite sequences of thoughts and reflections centered on a particular topic or idea. Teachers often tell two types of stories about their photographs: *professional* or *personal*. *Professional stories* are the stories, a teacher tells about the professional aspects of being a mathematics teacher. For instance, a teacher might tell a professional story about students working in groups, commenting about how group work is emphasized in his or her classroom. *Personal stories* are the stories connecting a professional story to a teacher's personal life. Personal stories explore who teachers see themselves to be: their prior experiences, their schooling, their family, the way they feel positioned, their dispositions, and other personal aspects that make them feel human. For instance, a teacher might tell a personal story about her family and how she grew up in a household that valued mathematics.

Although many studies situate teacher identity only within these personal and professional stories (de Freitas 2008; Drake et al. 2003; Enyedy et al. 2006; Palmer 2007), I found that identity actually involves a third story type that is touched upon in the literature (Clandinin and Connelly 1996; de Freitas 2008; Sfard and Prusak 2005). *Touchstone stories,* elicited through either professional or personal stories, are the stories teachers tell themselves but rarely share publically. The term is based on touchstones used to determine the quality and genuineness of precious metal alloys in ancient alchemy and modern chemistry. Teachers use touchstone stories as internal mechanisms of critique and decision making, holding these as standards upon which all their other experiences are compared.

Touchstone stories are the hardest to elicit because they sit prominently in a teacher's mind, where teachers constantly reference them but rarely share publically unless they are in a "safe" space (Clandinin and Connelly 1996; Sfard and Prusak 2005). Sfard and Prusak (2005) found that a depressing emotional response was sometimes generated when teachers confronted these stories. de Freitas (2010) echoed this sentiment, finding that the best way for mathematics teachers to confront their own identity was to "'zoom in' on highly emotional moments which had left a strong imprint on their memory" (p. 51). Triggering these emotional moments allows mathematics teachers to reveal and reflect upon their touchstone stories. Therefore, touchstone stories are revealed when a teacher confronts a professional or personal story in a way that generates an emotional reaction. A teacher might cry, get angry, look embarrassed, or laugh—all clues that a touchstone story is being shared.

Latino/a Mathematics Teacher Identity

The research into the specific sociopolitical ethnic identity of Latino/a mathematics teachers is sparse, yet emerging (Civil 2009; Gutiérrez 2013). Much of the work thus far has focused on debunking certain myths about Latino/a mathematics teachers. Achinstein and Aguirre (2008) found that teachers of color do not automatically connect to students from the same culture. Furthermore, Vomvoridi-Ivanović (2009) found it was very difficult for preservice Mexican-American teachers to bring their own culture into their mathematics teaching, even when working with children of Mexican heritage. These preservice teachers were able to bring their Mexican-American culture into their teaching only within open-ended projects, but not when using scripted curriculum activities and lessons. Finally, Gutiérrez (2013) used Latino/a critical race theory to show that assumptions cannot be made about cultural connections between Latino/a math teachers and their students. What this means for mathematics education research is that we need specific research studying Latino/a mathematics teacher identity. We need to understand how to support Latino/a mathematics teachers in drawing from their own culture and connecting to their students, thereby fostering the teacher–student relationships and increasing their students' mathematical learning (Achinstein and Aguirre 2008; Tellez 2004; Vomvoridi-Ivanović 2009).

In summary, the research on mathematics teacher identity agrees that identity is best studied through teacher narratives. But in order to organize these narratives into analyzable units, anchoring structures are needed. Because of the emerging work using photographs in the fields of health care and anthropology, I chose to use photographs as anchoring structures to study mathematics teacher identity. These photographs helped teachers frame their narrative into three specific story types: professional, personal, and touchstone stories. The touchstone story illuminated aspects of one's identity as a mathematics teacher since it was a story a teacher told to himself or herself, yet rarely shared publically. And specifically for sociocultural ethnic identities, such as a Latino/a mathematics identity, these stories might reveal that a shared cultural background between teacher and student is not enough for teachers to feel culturally connected to their students. In the next sections, I detail exactly how I elicited and analyzed these stories, as told by two Latino mathematics teachers, and the aspects of mathematics teacher identity these stories revealed.

Methods

In order to explore mathematics teacher identity through teacher-selected and teacher-captured photographs, I used a combination of the PEPI methods. In the fields of nursing and anthropology, visual and creative methods like the PEPI have proven to be especially useful for studying identities for a number of reasons (Brown et al. 2009; Clark-Ibáñez 2004; Gauntlett and Holzwarth 2006). First, Harper (2002) showed how the PEPI method was particularly suited for studying identity because photographs can retrieve events from a teacher's past and bring them freshly into the discussion; the teacher photographs become glimpses into various aspects of his or her life while the teacher creates commentary about how the specific moments in the photograph connect to his or her teaching. Second, the PEPI excels in generating a narrative told in the participants' "voice;" the stories they tell are unfiltered and raw (Brown et al. 2009; Clark-Ibáñez 2004; Gauntlett and Holzwarth 2006). Third, Gauntlett and Holzwarth (2006) found the PEPI forced participants to spend time thinking about each photograph during the selection process. This means that participants put a substantial amount of thought and reflection into what each photograph means before presenting them, rather than the instant responses typical of traditional interviews. Finally, the PEPI honors teachers' lives and busy schedules because it is noninvasive and requires a relatively short time commitment from each participant (Clark-Ibáñez 2004; Gauntlett and Holzwarth 2006; Harper 2002).

The Process

In this chapter, I focus on two self-identified male teachers of the six teachers who participated in my original dissertation study (Chao 2012). They taught Algebra I at two different high schools, located in a large city in a large Southwestern state. I had previously interacted with both teachers through interviews, professional development workshops, and classroom observations for another research project. I specifically chose the six teachers for the larger study, because, in our prior conversations, they expressed interests in exploring and talking about their identities in connection to their mathematics teaching. Through our prior interactions, I also knew they would tell rich and detailed stories.

I first visited each teacher to give him a digital camera and a loose prompt to "capture your world as a mathematics teacher" in at least 20 photographs. We then set up a time and date to sit individually for a formal PEPI 2 weeks later. During these 2 weeks, I observed at least one Algebra I class period for each teacher to get a feel for his teaching practice and style, to get to know his classroom and school culture, and to be available to answer any questions he might have about the study. A day or two before the scheduled interview, I sent an e-mail to both teachers reminding them about the interview, and I prompted them to choose their ten most important photographs. This forced shrinking of the photograph pool right before the interview was an attempt to make the teachers reflect deeper and engage directly with the photographs they chose.

I then sat with each teacher for the actual PEPI, each one taking place after school in the teacher's classroom and lasting at least 90 min. During the interview, each teacher shared one photograph at a time in order of self-selected importance. I used a minimal interview structure, using nonjudgmental and nonevaluative language such as "Tell me more about that," or "How does that connect to you as a mathematics teacher?" (Johnston 2004). I also used clinical interview strategies to get the teachers to elaborate more on how each image connected to their identity (Ginsburg 1997).

Data and Analysis

Audio data from each interview were captured with a digital recorder and then transcribed for analysis using a grounded theory coding structure. I coded for emerging themes of mathematics teacher identity centering on teachers' professional and personal lives (Corbin and Strauss 2008; Merriam 2009). The main data source was the interview of each teacher, which I transcribed in InqScribe[1] to build an emerging coding scheme. While the actual photographs each teacher shared and the notes I took during classroom observations added tremendous depth to the interview, I did not consider them as primary data and therefore did not analyze them.

I analyzed each teachers interview using two separate approaches: one based on qualitative education research practices and one based on the photo-elicitation/photovoice literature. The first approach involved "living" in the data—a straightforward grounded-theory approach based upon creating emergent codes on individual themes through successive analytical memos (Corbin and Strauss 2008; Merriam 2009). I used this approach to build trustworthiness in my interpretations of the data—what Erlandson et al. (1993) call *credibility, transferability,* and *dependability* when doing naturalistic inquiry. The second approach involved looking for stories and was appropriated from the research on photo-elicitation/photovoice techniques (Brown et al. 2009; Clark-Ibáñez 2004; Guillemin and Drew 2010; Hansen-Ketchum and Myrick 2008; Harper 2002; Oware et al. 2007; Wang and Burris 1997) and on studying identity through stories (de Freitas 2008; Drake 2006; Drake et al. 2003; Sfard and Prusak 2005).

The first part of analysis began when I transcribed each interview. I used InqScribe to tag thematic codes while I transcribed in order to build a general grounded theory (Corbin and Strauss 2008). These first-pass codes formed the initial stages of a categorical scheme to organize the data (Corbin and Strauss 2008).

[1] A video and audio transcribing tool built for educational researchers. http://inqscribe.com/.

After importing all the interviews into Transana,[2] I built a theoretical sample by analyzing the two longest interviews from the original dataset of six teachers. I listened to the audio and carefully read the transcripts in order to designate individual audio clips based upon emerging thematic codes (Corbin and Strauss 2008). These clips allowed me to create a more robust two-tiered analysis mechanism; each clip contained one primary keyword category and multiple thematic codes. For the first interview, I created 103 clips. For the second interview, I created 69 clips. Together, these interviews generated 446 thematic codes within 35 keyword categories.

At this point, I truncated the thematic code set with a specific focus on teachers' professional and personal lives to collapse the codes. I used this second-pass thematic code set to analyze the interviews, adding thematic codes and keyword categories only as necessary. Refining the codes and keyword categories with the full dataset, I focused the analysis on the construct of mathematics teacher identity without being too specific to any particular teacher. This third-pass (and final) code set contained 206 thematic codes within 14 keyword categories.

The second approach to the analysis involved breaking apart each interview into individual stories based upon the photographs each teacher shared. I started by dividing up each interview into individual chapters, based upon the photograph on the screen at that particular moment during the interview (Hansen-Ketchum and Myrick 2008; Harper 2002). I then divided up each chapter into individual professional and personal stories.

Then, I went back to the transcripts and my analytical memos and found that many of the thematic codes and categories were about increased emotion, such as "anger," "tears," or "raised voice." I found at least one instance of increased emotion in each of the original six teachers' interviews (Chao 2012). These instances of increased emotion always led to a revelatory type of story. So, based on other research into emotional teacher stories (Clandinin and Connelly 1996; de Freitas 2008; Sfard and Prusak 2005), I classified these emotional stories as touchstone stories. The touchstone stories were the stories a teacher told in ways that hinted they were revealing an internal dialogue they normally hid, usually preceded by a phrase like, "I didn't think about this until just now…" or "I never talk about this, but…" I then used thick description to write up each teacher's touchstone story and the professional or personal stories that led to the sharing of that touchstone story.

Finally, I wrote a summary of the teacher's stories, main keyword categories, thematic codes, and analytical memos for each teacher, along with my preliminary analysis. I e-mailed the teachers their summary, asking them to add to, edit, or clarify anything in the summary in order to create a form of member check validity (Corbin and Strauss 2008). I asked the teachers to help me keep what I was writing true to their "voice" and to ensure I was being "real" in an attempt to build credibility through member checks, since "no data obtained through the study should be included in it if they cannot be verified through member checks" (Erlandson et al. 1993, p. 31). I also asked each teacher to approve a pseudonym I chose from his or her interview that was both personal and unique.

[2] A qualitative analysis tool developed by the Wisconsin Center for Education Research. http://www.transana.com/.

These stories, along with the rigorous and multiple passes of the data through analytical memos, thematic codes, and keyword categories, allowed me to generate thick description of my findings (Erlandson et al. 1993). Because the interviews are specific to only the mathematics teachers in the study, I cannot generalize the findings to the general population of mathematics teachers (Corbin and Strauss 2008; Erlandson et al. 1993). However, the use of thick description and purposive sampling make these findings transferable to other contexts (Erlandson et al. 1993).

Results

In this results section, I focus specifically on a touchstone story each teacher shared, along with the connected professional and personal stories that elicited this touchstone story. Of the original six teachers in my dataset, I chose to focus on these two teachers because they both started the interview describing how their Latino heritage connected to their mathematics teaching, yet both teachers ended up telling very different touchstone stories (Chao 2012).

Mr. Leche

Mr. Leche was in his fifth year teaching at his second high school that served an urban, low-socioeconomic, almost entirely Latino/a student population. He self-identified as a Mexican male. Mr. Leche presented photographs that highlighted his childhood friends, his passion for ending school bullying, his experiences watching his Mexican hometown turn into a violent warzone, and his yoga practice. Mr. Leche shared that he immigrated to the USA from Ciudad Juárez when he was in high school. He saw his role as a mathematics teacher as supporting other immigrant students making the transition to life in the USA. For example, Mr. Leche made sure his Spanish-speaking students were grouped together so they could engage in mathematical dialogue together in Spanish. Mr. Leche shared a touchstone story about not being able to go home again. This story came about through a personal story about growing up in Juárez and a professional story about his struggles in teaching mathematics in a way that was culturally relevant to his students.

Personal Story Mr. Leche shared a photograph depicting the mass graves of over 370 murdered women discovered in his hometown of Ciudad Juárez in Chihuahua, Mexico (Fig. 1). This gravesite was very close to where Mr. Leche grew up. Mr. Leche remembered the discovery of this mass grave during his childhood, an incident he describes as “burned” into his psyche, so traumatic he had trouble saying exactly what happened.

> Back in the ’90s…Juárez was like, it’s a border town…they’re still finding a lot of like, missing girls and stuff. And this is actually, maybe 2 miles from my house, from my parents. And they found like a big, big (pause)…mass grave…. I was moving…to the States,

Fig. 1 Mr. Leche's photograph of a memorial in remembrance of the over 370 violently murdered women discovered in a mass grave near his childhood home in Ciudad Juárez, Chihuahua in Mexico

> and Juárez was getting a really bad reputation…. A lot of the girls were like 16 years, like 14 years, 14, 15-year-olds.

Mr. Leche shared that this incident was such a disturbing part of his childhood, he felt the need to honor these lost lives through presenting this photograph. "I just have to include this picture just to, just to honor them…. How does it relate this to my teaching? I don't know. Just, as a human being…I think it's part of me."

Mr. Leche's personal story was a reminder of the violence that engulfed his hometown. While looking at this photograph, he reminisced on the memories from his childhood of a safer Juárez that no longer exists.

> I remember…we were, like, alone in the streets in Juárez. And back then, it was not that dangerous…. On Sundays, we went to church and then we went to downtown to watch movies…. So it's kind of weird, like going back to my, like things that I remember…Juárez right now is kind of bad, it's like really, really bad.

In this personal story, Mr. Leche presented a traumatic event that engulfed his hometown, his family, and all of his memories of childhood. He was not sure how this story connected to his teaching, but he knew it was important to share it in this interview.

Professional Story Mr. Leche continued to reveal a professional story about the difficulty he had as a teacher living between the cultures of Mexico and the USA. He shared he felt like a perpetual foreigner, even though he had lived in the USA for over 15 years. He realized this feeling of being "othered" eroded his own tolerance for diversity.

> When I came to the States and I was really open and I wanted to know about different ethnicities…but now that I'm older and I have been more than 15 years in the States, not a

> lot of people feel comfortable with…like, maybe the community will not accept me…like, you're an outsider and you don't belong here.

He shared how he had felt "othered" on a recent professional development trip with a fellow teacher.

> Once I start speaking, they know, like, that's the first thing, like…you're not from here…. I went to this trip, I went with [a fellow teacher] and she's white. And it's kind of weird, like how…people treat you…if you go to a restaurant… and everyone is white…I just felt that presence…like, you're not welcome.

Mr. Leche saw this "othering" mirrored in his students. He said, "Here in the States, it's more racial. And it's also more, like, like, like sexual orientation and if you're a female. And what I have noticed is like, within the kids, within my students, yes."

Furthermore, Mr. Leche explained that he felt a high amount of *cultural congruence* (Ladson-Billings 1994) with his Latino/a students, yet he felt guilty invoking it or using this shared identity for fear of "othering" his non-Latino/a students. For instance, he noticed he was more macabre with his Latino/a students because they might understand his dark Mexican humor.[3]

> I'm more sarcastic with my Hispanic kids because I know the humor within the Mexican culture. And I know, like, it's like dark. Like Mexicans have like really dark sense of humors…. But I cannot be the same, I cannot say the same to a Black kid or to any, any, a White kid. Because they, they, they just, they will feel that it's like offensive. They wouldn't understand.

Mr. Leche shared he was growing in his awareness of how ethnicity, sexual orientation, and cultural congruence affected his mathematics teaching. But he readily admitted he was unsure how to address these issues. He was aware of the culture he shared with his Latino/a students. More than just a shared language, it involved shared social norms and registers.

> I consider myself Hispanic. And even the kids…like, to be honest, like, a lot of my Hispanic kids, they're more accepting because I'm Hispanic…most of the kids here are Hispanic and I think they relate to you…they say like Uncle, in Spanish…or they say, like, Dad…I think it has to do with culture.

Mr. Leche shared how his conceptions of discrimination by race, gender, sexual orientation, and culture had evolved through his experiences as an immigrant dealing with assimilation into American culture. He used these lenses to view his students and shared he was still trying to figure out how to teach mathematics with this awareness. He felt uncomfortable invoking the mutual Latino/a background he shared with many of his students. He disclosed that he rarely had an opportunity to talk about these issues, especially about sexual orientation and culture. He felt they were hidden topics in his professional development, spoken about only in shallow ways or glossed over completely. He knew these issues should be a primary part of his mathematics teaching practice, but he was unsure of how to incorporate them into his teaching.

[3] I observed Mr. Leche opening one of his lessons with the joke: "I have a few requests. The first one is that no one dies on me."

Touchstone Story There was one moment in the interview when Mr. Leche calmed down from his nervous disposition, his speech slowed, he made direct eye contact, his eyes watered, and he referred to me by name. This felt like a rare moment of clarity in Mr. Leche's narrative, in which I noted he seemed reflective and vulnerable. I took this reflective and emotional moment as a sign Mr. Leche was telling a touchstone story.

Speaking of his childhood in Juárez, Mr. Leche stated, "I think it marked, it marked, there's, kind of like what I say, there was a lot of things that I, living in Juárez that marked, that marked, kind of like a burn into your brain." Revealed through Mr. Leche's stories as an outsider trying to make sense of the segregated nature of American culture is a touchstone story about the lasting trauma from the destruction of his hometown. Mr. Leche indicated that Juárez was a city so devastated by violence he had trouble finding images tame enough for this interview; many of his photographs were too gruesome. He shared the pain of seeing memories from his childhood engulfed in violence. Yet, he disclosed that he felt no one in his current world understood what he was going through. He shared that he felt ostracized from his teaching community, from American culture, and even from other Latino/as. He found that his connection to Juárez, while fascinating to his teacher colleagues, also tokenized him as "the teacher from Juárez," which colleagues used to characterize him rather than to understand him.

> I remember, I was in [fellow teacher's] house, and he had like a lot of friends and all these people over, like from [an elite, private, Southern university]. Like, and we were talking and they knew that I was from Juárez and they started asking questions…there was a picture that actually was published in the newspapers, they were like beheadings, so they were like four bodies, four bodies…. And they, they cut the heads and they place them. And when there's things like, when you're driving, you see dead bodies, like on the bridges. So I think like people here in the States really take the things for granted. And they just don't understand how in other parts of the world, it can be so bad.

Mr. Leche's touchstone story involved his feelings of isolation in knowing firsthand the violence of his hometown, of feeling "othered" from being from Juárez, and knowing he could not go home again. He felt neither his colleagues nor his students understood this feeling of displacement. Mr. Leche remembered, "And even though it was simple, I felt like at home…even though, like, I was in Juárez, and Juárez is not a really safe place." And as much as living in the USA troubled him, he could not return to Juarez. Mr. Leche's hometown was now a place people routinely got killed, even when demonstrating for an end to the violence. Mr. Leche shared, "We went to, went to do this walk, a walk for the peace. Now you cannot do that, anymore. You get killed, if you do that. Yes."

Mr. Leche's touchstone story was an extension of his personal story involving his feelings as a refugee, and the intolerance and trauma that haunted his day-to-day existence. He shared that he felt the school system he worked in was unable to support him talking about these experiences, especially since he knew that many of his students were also immigrants from violent, drug-war regions of Mexico. He felt he did not know how to talk about the issues that troubled him in a meaningful, professional way with his colleagues or his students. He felt he had to keep these ideas hidden. Mr. Leche's touchstone story was about not being able to go home again, as his hometown had been destroyed by drug-war violence.

Listening to Mr. Leche's stories, I wondered where the professional development opportunities were for Mr. Leche to learn how to navigate the tension he felt in his students' lives and in his own life. Mr. Leche, who shared an ethnic and cultural background with most of his students, felt ostracized and guilty about capitalizing on that cultural congruence to improve his teaching. He also directly identified the kind of support mechanisms he needs: a way for him to work through the trauma of growing up in Juárez, particularly since his story parallels those of some of his students. Through his stories, Mr. Leche revealed a mathematics teacher identity focusing on the lack of professional development and support he needs when it comes to understanding how to talk about ethnicity, culture, and geopolitical identity within his mathematics teaching.

Mr. Ginobili

Mr. Ginobili was also in his fifth year teaching, all in the same middle/upper-socioeconomic status suburban high school serving a primarily White and Latino/a student body. He taught in a dual-language immersion bilingual program and also coached the girls' soccer team. He self-identified as Mexican and male. Mr. Ginobili shared photographs about his family, his house, his love of sports, the effort he put into engaging his students, his coaching, and his pride in being from Mexico. Mr. Ginobili immigrated to the USA from Mexico City when he was 6 years old and he saw his role as a bilingual mathematics teacher as bridging the mathematics learning cultures of Mexico and the USA. For instance, he always had his newly immigrated students share with the class how they had learned and discussed Algebra in their former countries. This student sharing was Mr. Ginobili's attempt to foster student agency and help him understand his students' prior mathematics learning experiences. Mr. Ginobili shared a touchstone story about how he had grown so frustrated with the culture of mathematics teaching he had taken on a new identity of soccer coach rather than mathematics teacher. This touchstone story came through a professional story about how being a coach allowed him to teach and inspire his players in ways he never could in the mathematics classroom.

Professional Story Mr. Ginobili shared a photograph of himself, in a suit, ready to coach a girls' soccer game (Fig. 2). He shared that he brought many of the things he did as a soccer coach into his mathematics classroom. For instance, he noticed his players paid attention to bright colors and flashy objects on the soccer field. In order to hold his players' attention on the field, he wore lime green shoes on the soccer field. So, in his mathematics classroom he wore bright orange shirts. He also shared he had a lot more fun coaching soccer than teaching mathematics.

Fig. 2 Mr. Ginobili just before he coaches a soccer game. Note the lime green shoes

> I love being on the soccer field. That's, that's, I mean, if I could coach year-round, I would. It's a—I have fun with it. I think the kids have fun when I'm coaching, so definitely a lot of joy. You know, a couple stressful moments. But not as much as when I'm on the, in the classroom. (laughs)

Mr. Ginobili enjoyed coaching so much that he would never consider taking a future teaching position at another school unless it also came with a head coaching position.

> I like where I'm at in my coaching. If you want to offer me a head coaching job there, then I'll go there for that. But if it's just for teaching, I don't see myself going there. Because I see it as something that's become a part of me. It's become a part of who I am, and I wouldn't want to leave it. I wouldn't want to leave it.

Mr. Ginobili's professional story was about how he saw himself as a coach, just as much, if not more so, than how he saw himself as a mathematics teacher. He shared that the reason he loved coaching was that it allowed him to teach and inspire his soccer players, which he found harder and harder to do in his mathematics classroom. As a mathematics teacher, he spent most of his time trying to get students interested in the mathematics content, whereas on the soccer field the students came already wanting to play, wanting to get better. Mr. Ginobili described coaching as feeling more natural, more "free flowing." He felt mathematics teaching, however, required a lot more effort and meticulous planning.

> It's a lot different. Having kids that you're forcing to get this information, and kids that are, you know, thirsty for it and are like sponges, wanting to get your coaching. So it's a lot different…you get the bright side of it out there because you get these kids that are

> passionate about something. While, in here [the mathematics classroom]…sometimes, I'm pulling teeth! (laughs)

Mr. Ginobili continued to share about how student passion looked very different on the soccer field as compared to his mathematics classroom. He became frustrated with the culture of mathematics education in how it dissuaded students from building passion for mathematics.

> Now that I've become a coach…it's really become an integral part of what I am, and what really motivates me to keep coming back…like, sometimes you'll have a gloomy day here in the classroom and you're like, "Oh my god. These kids are killing me!" And then you'll go out onto the soccer field and you have girls who are, you know, busting their little butts off for you and trying really hard and running and doing everything for you. And you're like, you know, you grow to appreciate their effort.

Mr. Ginobili saw the difference in passion for sports and mathematics through students that were both on the soccer team and in his mathematics class.

> I coached some of my students before and you see such a difference in them in the classroom and on the soccer field, or on any field, really…they'll be rowdy little kids and they'll be talking to their friends and kind of, you know, messing around in the classroom. But… you see that focus and that drive in them on the field, that they're like, "Oh my god, I want to go to practice because I want to get better." And, "I want to make sure that I'm on the team. And I want." And it drives them and it's like, "Wow, okay, I don't remember seeing this drive in you in the classroom."…you see another side of them. You definitively see another side of them.

Touchstone Story Mr. Ginobili maintained a laid-back, jovial tone throughout most of the interview. But he raised his voice, expressing feelings of exasperation, when he talked about the state of modern mathematics teaching. In his first few years as a teacher, Mr. Ginobili was part of a grant emphasizing small group teaching and project-based instruction. But now that the grant was over, he was stuck with increased class sizes and felt that he had to resort to a more "traditional" style of classroom teaching. This frustrated him.

> I started teaching here in a grant position where we had max ten kids in a room…just all small group, just…intense, focus on the kid. And I did that for…three years. And...it worked. It did work. But I mean, the grant went away…I tried to do that my first year that I had a large classroom with 30-some-odd kids…I had them in groups, in small groups… it didn't quite work…the dynamics of the classroom and the dynamics of the kids. It was so different.

Mr. Ginobili lamented he ended up creating a classroom culture in which he supported only students who were motivated to learn. He felt forced to ignore students who had already decided to drop out of high school.

> The kids that realized that they had made a mistake their freshman year and were really trying to regain their credit for Algebra sat in the front. The kids that were just like, "You know what, I'm going to drop out and no matter what you do, I'm going to drop out by next month." They sat in the back and, you know, I didn't bother them.

Mr. Ginobili felt burnt out on mathematics teaching because he was unable to use methods he knew would lead to more student engagement. "Real" teaching, inspiring students to become the best they could be, existed only in his coaching. He shared that he felt the support structures he needed to teach mathematics in ways he knew were best for his students had been systemically removed. Mr. Ginobili's touchstone story was about his disillusionment with mathematics teaching. He saw himself as a motivator, someone who inspired his students to be their very best. But he did not have the time and resources to inspire all of his students in his mathematics classroom. If his students came to his classroom wanting to learn, he could teach them. But if they wanted to drop out, he did not have the resources to engage them. So he ended up ignoring them. Letting students slip through his mathematics classroom without learning or becoming inspired to learn frustrated him. Yet, on the soccer field, he felt he had all the resources necessary to inspire every single one of his players to always try their hardest.

Mr. Ginobili saw vast differences between coaching and teaching culture. He felt passionate about inspiring and working with his soccer players to get better and work as a team. But with his mathematics students, he felt like a glorified babysitter who could reach only a certain number of students at a time. I wonder what kind of supports it would take for Mr. Ginobili, a fifth-year veteran bilingual mathematics teacher, to embrace his mathematics teacher identity again. And while he initially mentioned his Latino/a cultural congruence to his students as being an important part of his teaching identity, he still spoke with frustration about not being able to inspire and engage his mathematics students. He saw himself as a soccer coach first, and yet he never mentioned his Latino heritage as being a part of his coaching identity. The culture of sports might be less entrenched with issues of social class, immigration, or colonialism than the culture of being Latino/a (de Freitas 2008; Duncan-Andrade 2010). The professional identity switch is a much easier path than delving into the politics of inter-group Latino/a sociocultural conflict (Valenzuela 1999). Perhaps the difference between Mr. Ginobili, who emigrated from Mexico City as a young child, and his students, many of whom recently emigrated from less cosmopolitan areas as teenagers, are accentuated when Mr. Ginobili invokes his Latino/a identity in the classroom, but not on the soccer field.

Additionally, by embracing an alternative identity, Mr. Ginobili entered a world with more community capital and prestige. He mentioned that becoming a coach was a difficult career ladder to climb; he had to work his way up. But the community prestige of being a coach was rewarding. People in the community knew him and respected him much more as the soccer coach than as a mathematics teacher. And even within his own school, he felt he held more power and proximity with his administrators as a soccer coach than he ever had as a mathematics teacher.

Discussion

I originally had a few hypotheses in this study, based on the existing literature on mathematics teacher identity. First, mathematics teachers would be able to talk about their identity through stories (Brown and McNamara 2005; de Freitas 2008;

Drake et al. 2003). Second, mathematics teacher identity would center on the intersection of personal and professional stories (de Freitas 2008; Drake et al. 2003; Enyedy et al. 2006). Third, mathematics teachers' identities would involve a teacher's relationship to or knowledge of mathematics (Battey and Franke 2008; Foote et al. 2011; Walshaw 2010); sociopolitical identities, such as gender, ethnicity, and sexuality (Battey and Franke 2008; de Freitas 2008); or the various communities a mathematics teacher lives within (Battey and Franke 2008; Van Zoest and Bohl 2005). Finally, a shared cultural connection between teacher and student would not be enough for teachers to feel they were invoking their Latino/a mathematics teacher identity (Achinstein and Aguirre 2008; Tellez 2004; Vomvoridi-Ivanović 2009).

Most of the findings confirmed these hypotheses. First, teachers were able to share stories focused on specific aspects of their identity through the PEPI. Second, however, beyond professional or personal stories, mathematics teacher identity actually involved a third type of story, the touchstone story. I did not plan to elicit this type of story originally. But the PEPI revealed these touchstone stories teachers told themselves but rarely shared (Clandinin and Connelly 1996; Sfard and Prusak 2005). Third, the aspects of mathematics teacher identity that both Mr. Leche and Mr. Ginobili shared did not involve their knowledge of or relationship to mathematics at all. Instead, as expected, they talked about their sociopolitical identities and the communities they lived in. Finally, as expected, both teachers expressed difficulty in connecting with their students through a Latino/a identity they assumed they shared with their students.

The two teachers in this study communicated very different aspects of their mathematics teacher identities, which related to the nature of the photographs. First, through a photograph of a mass grave, Mr. Leche talked about his personal trauma of not being able to return to his hometown, which connected to struggles in his mathematics teacher identity with connecting culturally to his students. Like Mr. Leche, other mathematics teachers might pull on their own geopolitical and cultural experiences to define themselves and their connection with their students, but become paralyzed in their teaching if they do not have the support or professional development to authentically use these experiences in their teaching. Second, through a photograph of himself before coaching a soccer game, Mr. Ginobili talked about how he saw himself more as a soccer coach than as a mathematics teacher, which connected to his frustration in his mathematics teacher identity about the support he had to teach mathematics in ways that inspire his students. Like Mr. Ginobili, other mathematics teachers might adopt different teacher identities because these new identities allow more power and opportunities to actually teach.

The PEPI successfully elicited stories from these two mathematics teachers for a number of reasons. First, a visual basis for self-authoring allowed each teacher to tap into his "visual" voice, accessing memories and thoughts that might not be recalled in other ways (Hansen-Ketchum and Myrick 2008; Wang and Burris 1997). For instance, Mr. Leche related that he often had trouble talking about Juárez and that the photographs helped him immensely. Mr. Ginobili echoed this sentiment, explaining that the photographs helped him think about parts of his teaching he never thought about. Mr. Leche even shared that revealing the touchstone story through photographs was reflective and illuminative:

> When I was like talking about the…picture…you can be biased and you don't realize…it's reflective. Reflective. It's good. So it's good…like, sometimes, like something happens and it affects. And you don't realize that that's like the root of the problem.

Second, the visual nature of the PEPI helped teachers keep focused because both interviewer and participant had a visual reference to guide the interview. Mr. Leche shared that he often had trouble keeping focused within other professional development settings. Third, the PEPI method encapsulated how mathematics teachers saw themselves in a quick and efficient manner. The entire process took 2 weeks, with the majority of the effort concentrated on the 90-min interview. Teachers indicated they spent about an hour or two during these 2 weeks taking and assembling the photographs.

Regarding the limitations to this study, one issue stood out specifically. Neither teacher talked much about or connected his stories to specific students. This absence of students was an artifact of the school district's institutional review board permissions for this study, which did not allow teachers to use photographs that identified students. Consequently, the stories each teacher shared were noticeably devoid of any mention of students. I suspect that students are integral to the way mathematics teachers identify themselves, so future studies using photography-based elicitation must include a way for teachers to tell stories about their students.

The identities this study illuminated show how mathematics teachers can feel hindered from actually teaching mathematics. Mr. Leche shows how paralyzed he is by his sociocultural identity as a refugee—it stops him from using his cultural congruence with his students in his teaching. And Mr. Ginobili started to move away from actually teaching mathematics by taking advantage of an opportunity to pursue a coaching career, an expression of his frustration with the culture of modern mathematics education. These conclusions lead to some implications for us as a field of mathematics educators. First, we must be better at supporting mathematics teachers through all the various aspects of their identity, beyond just knowledge of mathematics, and to include specifically sociocultural and community-based identities. Second, we need better practices for mathematics teachers so that teachers will not grow frustrated and leave the profession, especially teachers who value the actual aspect of teaching and inspiring students. Third, we need new ways for mathematics teachers to be heard, to share their touchstone stories so they can be seen as individuals with rich histories. Finally, we need more support mechanisms within our field for teachers to talk about their own traumatic stories and to explore their own sociopolitical nuances of identity.

This study fills a current gap in the research on understanding mathematics teachers through the lens of identity. By incorporating the PEPI as a change in method, I explored a research tool that captured teachers' voices without hovering or disrupting their practice. This experience allowed teachers to tell stories revealing how they really saw themselves. Perhaps if there were more developed constructs to understand these identities, we as a research community could better show the value in knowing what makes each mathematics teacher human, unique, and special.

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Teachers, Attributions, and Students' Mathematical Work

P. Holt Wilson, Cyndi Edgington, Paola Sztajn and Jessica DeCuir-Gunby

An understanding of the relation between teachers' knowledge of students' mathematics and their beliefs about teaching and learning is inherent in the work of mathematics teacher educators seeking to support teachers in learning to make instructional decisions based on students' mathematical thinking. Schoenfeld (2011) proposed that individuals' decision making in well-practiced, knowledge-intensive domains can be fully characterized as a function of three factors: orientations, resources, and goals. Schoenfeld broadly defined orientations to include a myriad of concepts such as dispositions, beliefs, values, tastes, and preferences. He explained that people's orientations shape what they perceive, the meanings they make of these perceptions, the goals they establish for the situation, and the resources they put to use to achieve the established goals. Most importantly, Schoenfeld discussed decision making in relation to teaching and stated that in mathematics classrooms, teachers' orientation toward mathematics, students, learning, and teaching shapes their instruction.

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Thompson et al. (1994) used the concept of orientation to describe different approaches to teaching mathematics and included teachers' knowledge, beliefs, and values within this concept. They proposed that orientations molded teachers' images, views, intentions, and goals for mathematics instruction. Similarly, Magnusson et al. (1999) considered that teachers' orientations influenced instructional practice by shaping teachers' knowledge and beliefs about curriculum, students, teaching, and assessment. Philipp (2007) suggested that teachers' orientations were operationalized through language and action.

From our perspective, we consider teachers' discourse about students' mathematical work, in particular the *attributions* that teachers use as they discuss students' mathematics, as one aspect of teachers' orientations toward students. Attributions are perceptions of causality or judgments regarding the occurrence of an incident (Weiner 1972). In the classroom, teachers' attributions refer to the judgments or causal explanations teachers construct to explain students' successes and failures. Teachers' attributions influence their expectations regarding student ability and subsequently impact student performance (Graham 1991). This process of connecting teachers' attributions to students' performance has been documented in various disciplines, including different areas of mathematics (Dobbs and Arnold 2009; Middleton and Spanias 1999). Thus, teachers' attributions in mathematics classrooms are an important aspect of instruction and of concern for mathematics teacher educators working to support teachers in student-centered instruction.

Because our work is in professional development, we extended the role of teachers' orientations and attributions in instruction to professional development settings. Similar to Philipp (2007), we considered that teachers' attributions were operationalized through their discourse. We believe that teachers' attributions play a fundamental role in the conversations teachers have as they engage in professional learning tasks focused on students' mathematical thinking. Therefore, our work examined teachers' attributions by investigating their discourse about students' mathematical successes and failures. We were particularly interested in teachers' discourse within a professional development setting as teachers analyzed students' mathematical work. We explored the following research question: *To what do elementary teachers attribute students' mathematical successes and failures when they consider research results about students' mathematical thinking and learning*?

The results we report are part of a larger design experiment (Cobb et al. 2003) that involves a professional development setting purposefully planned to teach teachers about students' mathematical thinking and learning. Guiding the design and implementation of the professional development was our initial conjecture that learning about students' mathematical thinking would change teachers' discourse by adding new explanations for students' mathematical work to teachers' existing repertoires. Through ongoing analysis, we identified the attributions teachers used throughout the professional development and developed a codebook (DeCuir-Gunby et al. 2011) to investigate teachers' uses of these attributions. The initial phase of the retrospective analysis examined the nature of the attributions and will ultimately characterize changes in teachers' uses of the attributions over time.

In this chapter, we share the attributions that emerged from the ongoing analysis as teachers learned about student mathematical thinking, and we report on the initial phase of the retrospective analysis. We begin by briefly reviewing the literature on attribution theory, focusing particularly on teachers' attributions for students' work. Then, we introduce our research methodology, describing the professional development setting in which we worked, as well as the data collection and analysis processes. We define the attributions identified in our professional development, share examples of how these attributions were present in our work with elementary teachers, and offer findings related to teachers' uses of the identified attributions during the professional development. We conclude with a set of next steps for our research, including a shift in framework to use positioning theory to conceptualize teachers' uses of these attributions as acts of stereotyping.

Teachers' Attributions

Bar-Tal (1978) defined attributions as the inferences made about the causes of one's own or someone else's behaviors. Attribution theory allows for individuals to gain a better understanding of their environments and the determinants of individual behavior (Schunk et al. 2013). The general attribution model (see Weiner 1986, 1992, 2010) consists of the creation of attributions through the *attribution process* and the use of those attributions through the *attributional process*. The following sections describe the various components of the model.

Attribution Process

The attribution process involves understanding the development of attributions. Specifically, it concerns the exploration of antecedent conditions or the causal determinants of behavior. The process of creating attributions considers both environmental factors and personal factors. Environmental factors in the case of academic achievement include issues such as the type of school, the testing environment, and teacher quality, among many others. Personal factors, on the other hand, consist of a variety of features including beliefs about causality, rules used to make attributions, prior knowledge, and individual differences (Schunk et al. 2013). These environmental and personal factors influence the creation of perceived causes to explain behavior.

Antecedent conditions serve as a foundation for understanding the perceived causes of behavior. Early research indicated that ability, effort, task difficulty, and luck were seen as the most common perceived causes of the outcomes of events (Cooper and Burger 1980). Weiner (1986) elaborated this list to delineate explanations for academic success or failure to include ability, skill, stable effort, unstable effort, task difficulty, luck, interest, mood, fatigue, health, and help from others. Once a perceived cause is established, it then impacts the attributional process.

Attributional Process

Attributions may be classified along three dimensions according to their causal structure. The first dimension, locus, establishes whether the source of the outcome is internal or external to the individual. Internal causes are aspects that individuals can control, such as effort spent studying for a test; external causes are beyond the control of individuals, such as luck in answering correctly on the test. The second dimension, stability, explores how consistent the cause is over a period of time. Stability involves understanding a cause as fixed and stable or variable and unstable. For example, intelligence is commonly viewed as a fixed trait and thus stable. The last dimension, controllability, addresses the amount of control a person has over a cause. For instance, effort can be considered controllable because one may put forth more or less effort, whereas ability is often viewed as uncontrollable because one cannot change his or her inherent ability. These attributional dimensions have both psychological and behavioral consequences. They impact expectations for success and emotions associated with achievement. The dimensions also impact specific behaviors including future choices, persistence at engaging in tasks, level of effort placed to complete tasks, and achievement (Schunk et al. 2013). When attempting to infer the causes of another's behavior, the *fundamental attribution error* may result from attributing another's behavior to a personal trait without attending to situational factors (Schunk et al. 2013).

In our work, we take a view of designing professional development to support teachers in learning a framework for students' mathematical thinking as contributing to teachers' attribution and attributional processes. Through learning research results about students' mathematical thinking, teachers engage in the attribution process by considering the antecedent conditions that affect learning and that serve as a foundation for the perceived causes of students' mathematical successes or failures, including environmental factors such as opportunities to learn and personal factors such as previous experiences and current understandings. Our investigation of teachers' attributional processes concerns their perceptions of the causes of students' successes and failures, in particular the fundamental attribution error, which we see impacting teachers' expectations and efficacy. By attributing student failure to an internal, fixed, and/or uncontrollable cause, a teacher may perceive no recourse for teaching, whereas attributing a students' failure to an external, variable, and/or controllable factor suggests that learning may be affected by instruction.

Attributions and Mathematics Education

Attribution theory has been applied to a variety of contexts and tasks within mathematics education. Although research regarding students' attributions of their success and failure exists (e.g., Seegers et al. 2004), a substantial amount of research in this area has focused on the attributions that teachers make regarding students' mathematics learning. For example, Middleton and Spanias (1999) noted that teachers' attributions for their students' successes and failures were reflected in the ways teachers interacted with their students during mathematics instruction. In examining preschool settings, Dobbs and Arnold (2009) claimed that teachers' attributions of students' behavior shaped their behavior toward the child, which in turn often elicited the expected behavior from the child, having a self-fulfilling prophecy effect.

Within mathematics education, studies have indicated that stereotypes related to gender, race, and socioeconomic status can influence teachers' attributions for student success and failure. For instance, Fennema et al. (1990) studied 38 first-grade teachers' attributions for boys' and girls' successes in mathematics. They found that teachers tended to attribute boys' successes and failures to ability while attributing girls' successes and failures to effort. Reyna (2000) found that stereotypes could serve as the foundation for the attributions made regarding the mathematics achievement of students of color. For example, she discussed that whereas some people believe African Americans or Latino/as as a group are lazy, others believe they are underprivileged. As a result, teachers may attribute a students' ability to internal factors, such as effort, or external factors, such as opportunities related to the beliefs they hold about groups of students. Similarly, Reyes and Stanic (1988) examined how students' socioeconomic status impacted teachers' perceptions regarding achievement. They found that teachers' attitudes about students' achievement, as measured through classroom processes, varied based on students' sex, socioeconomic status, and race. However, the causality of these connections had yet to be established. Together, these studies suggest teachers' attributions, such as ability or effort regarding mathematics achievement, may be an extension of social stereotypes.

Methods

The overarching purpose of our research was to understand the ways in which teachers come to learn about students' mathematical thinking in the context of a professional development setting. We used a design experiment methodology within a school-based professional development setting to work toward this purpose. Design experiments are "iterative, situated, and theory-based attempts simultaneously to understand and improve education processes" (diSessa and Cobb 2004, p. 80). They are used to develop "a class of theories about both the process of learning and the means that are designed to support that learning" and they "entail both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb et al. 2003, p. 9).

In line with this methodology, we examined both teacher learning and the set of professional learning tasks that supported their learning experiences. Although we expected teachers' orientations toward students to shape the ways in which they engaged with the professional learning tasks we designed for the study, it was the ways that teachers talked about students' successes and failures that emerged as a major component in their discourse, playing a fundamental role in teachers' engagement with the professional learning tasks and shaping professional conversations around students' mathematics. As the professional development unfolded, we designed tasks that further brought forth various attributions and focused our analysis on exploring these attributions.

Context

Learning Trajectory Based Instruction (LTBI) is a multiyear NSF-funded research project with a strong mathematics professional development component for elementary teachers based on the concept of learning trajectories (LTs). When Simon (1995) coined the expression "hypothetical learning trajectory," he indicated that teachers create representations of the "paths by which learning might proceed" (p. 135) when students progress from their own starting points toward an intended learning goal. He named these trajectories hypothetical because each student's individual learning path was not knowable in advance. However, he suggested that these learning paths represented expected tendencies and that commonalities across students allowed teachers to develop expectations about the progression of learning.

Over time, the concept of LTs developed beyond the notion that teachers have expectations about how learning might proceed to include an empirical search for the highly probable sets of levels through which students progress as their learning of specific mathematics topics evolves. Current work on LTs uses research on student learning from clinical interviews and large-scale assessment trials to seek clarification of the intermediate steps students take as learning proceeds from informal conjectures into sophisticated mathematics. Recently, research on LTs has progressed from an agenda for studying student learning to include an agenda for research on teaching. Daro et al. (2011) called for the translation of LTs into "usable tools for teachers" (p. 57) and indicated the need to make these trajectories available to teachers so that they can guide classroom instruction.

Content Over the course of 1 year, teachers in the LTBI project learned about students' early rational number reasoning through study of the equipartitioning learning trajectory (EPLT). Confrey et al. (2009) defined equipartitioning as the cognitive behaviors that have the goal of producing equal-sized groups or parts as typically encountered by children in constructing "fair shares." The EPLT empirically describes how children begin with informal knowledge of fair sharing, and through instruction, build an understanding of partitive division that unifies ratio reasoning and fractions—see Confrey (2012) for a more detailed description of the EPLT.

The LTBI professional development included both a summer institute and academic-year monthly meetings. These two components of the intervention were designed with different goals in mind. The summer institute offered teachers opportunities to learn about the EPLT and develop an appreciation for the role of the trajectory in understanding student mathematics. In contrast, the academic-year monthly meetings focused on establishing connections between the trajectory and instructional practices. The two components of the professional development totalled 60 h of face-to-face, whole group interactions.

Professional Learning Tasks All tasks developed for use in the professional development were guided by a set of design principles stating that professional learning tasks for LTs: (a) attend mostly to issues of pedagogy, (b) embed opportunities for teachers to examine all facets of their knowledge for teaching, (c) use instructional sequences that begin with practice-based activities that challenge teachers' views of students' mathematics and mathematics learning, and (d) use artifacts similar to the ones researchers used in developing the LT (Wilson et al. 2013). As such, the professional learning tasks incorporated videos of clinical interviews, samples of students' written work, and examinations of teachers' curricular materials. During the summer institute, these artifacts consisted of "anonymous" students' work, whereas during the school year, teachers discussed and examined their own students' work. When possible, teachers brought in their own curricular materials supplied by the school district in which they worked.

One example of a task used in the summer institute was to engage teachers in watching and discussing videos of clinical interviews with students from different grade levels solving similar mathematical problems. Teachers were asked to describe the ways in which each child solved the problem, conjecture about each student reasoning for that particular solution, consider the sophistication of the various strategies, and examine what surprised them about each student's work. In the discussion of the task, despite the facilitator's effort to focus the discussion on what each child did and why, teachers' discourse focused mostly on alignment or deviations from their expectations based on the child's grade level. That is, teachers attributed what the children did to their grade level, and the information about each child's grade level that was offered to teachers as part of the context for the clinical interviews became the center of teachers' subsequent discussion.

Participants

The professional development was offered in partnership with one K–5 elementary school in a mid-size urban area in the southeast USA. The school had approximately 600 students: 35 % Caucasian, 29 % Hispanic, 25 % African American, 7 % Asian, and 4 % other; 54 % of the children qualified for free or reduced-price lunch. Teachers at the school volunteered to participate, and all professional development meetings were conducted at the school at times convenient to the teachers. Of the 24 teachers who started the professional development in July 2010, 22 completed the program 1 year later in June 2011. The initial group of teachers included six kindergarten teachers and three Grade 1, five Grade 2, three Grade 3, two Grade 4, and one Grade 5 teacher. Four teachers taught multiple grade levels.

Data Sources and Analysis

All data were collected by a research team comprising two principal investigators and one graduate student (the first, third, and second authors, respectively). Data sources included the researchers' field notes, 69 video files from the professional development meetings, and 90 transcripts of audio recordings of teachers' small group discussions during the 60 h of summer institute and monthly meetings. Following Cobb (2000), our analysis included both an ongoing and retrospective phase where we used a grounded-theory approach (Strauss and Corbin 1990) to code the data. Open coding was utilized to create concepts from the raw data. These data-drive codes were supplemented with additional codes derived from attribution theory and our research goals. We used the constant comparative method in that we compared various project data sources, including field notes and transcripts as well as research literature (Glaser and Strauss 1967).

In what follows, we describe our data analysis process, focusing on the first two of three stages of the work. The ongoing analysis occurred as the professional development was unfolding. During this time, we generated field notes and used these notes to revise and refine the professional learning tasks for the professional development. The retrospective analysis began after the conclusion of the professional development and was divided into two phases. The first phase included the development of a codebook, data reduction, and analysis of the frequencies of codes by each participant. The second phase consisted of more in-depth analysis of a subset of data to further understand changes in the teachers' discourse over time. In this chapter, we report on the ongoing and first phase of the retrospective analyses.

Ongoing Analysis Teachers' attributions for students' mathematical successes and failures emerged early in our analysis as a fundamental aspect of teachers' discourse, shaping their engagement with the designed professional learning tasks. For example, teachers talked about students not completing a task because they were "low students" or because of the way the task was presented to them. Thus, throughout the first year of the project, we conducted an ongoing analysis to examine the ways in which teachers talked about students' successes and failures in mathematics. We noted emerging attributions in our field notes, and the research team discussed them in regular meetings throughout the implementation of the professional development. From a design perspective, we continued to create and refine professional learning tasks in order to provide new opportunities for teachers to examine the ways in which they talked about their students as mathematics learners as well as their students' successes and failures. For instance, following the example above of teachers talking about students having low ability, we created teaching scenarios in which such vocabulary was used and then asked teachers to discuss these scenarios. We posed teachers' own attribution language back to them for explicit discussion, enabling us to use the design experiment setting to better understand the various attributions that emerged throughout the yearlong professional development. This process assisted us in further eliciting and understanding the teachers' uses of various attributions and the role the learning trajectory played in their discourse.

Retrospective Analysis After the completion of the first implementation of the LTBI professional development, we began a retrospective analysis to understand the attribution process of generating a new explanation for students' mathematical successes and failures based on the EPLT, as well as teachers' uses of the various attributions in their discourse about students' mathematical work. We initially engaged in a grounded theory approach to data analysis (Strauss and Corbin 1990) and coded our field notes generated during the ongoing analysis using open coding. This process enabled us to create concepts from our raw data, offering a first set of attributions that we used as codes for the subsequent analysis. These attributions varied in locus, stability, and controllability and will be elaborated on in the results section.

Building on the concepts identified in the ongoing analysis and refined through our examination of the complete data corpus, we created a codebook through an iterative process between our data and theory (DeCuir-Gunby et al. 2011). In line with the grounded theory approach to data analysis, we used constant comparison methods to compare various project data sources, including field notes and transcripts of small group discussions, and the research literature (Glaser and Strauss 1967) in order to identify and refine our codes. The codebook included definitions and examples of each code, and we revisited the data frequently as we refined our definitions until the codebook was finalized.

Four independent coders were trained to use the codebook to code the transcripts of audio data and the video data with 85% interrater reliability. Because we began with a large data set, we used this process to reduce the data to turns where teachers were explicitly or implicitly talking about students. We defined a turn as one person's statement in a conversation that is not interrupted by another's idea. Coders used the definitions from the codebook to code every turn teachers made in all whole group and small group discussions that were related to students' work, using one or more of the codes identified. In all, 2,868 turns were identified and coded, with 123 turns marked with multiple codes. Each coder coded approximately 40 files and 10% of the files were double coded to maintain reliability and prevent drifting.

Because we were interested in teachers' uses of the language from the EPLT to explain students' mathematical thinking, all turns were then examined a second time for evidence of language from the EPLT and given an additional code of "LT" when evidence of such language was found. For example, when teachers were reviewing students' written work, one teacher commented, "The way the child divided the pizzas was he did benchmarking, so they did the halving first and then they did the radial cuts." Because this teacher is describing what she perceived the student to do mathematically, this turn was coded as "Math." Because she is referencing specific strategies described by the EPLT, in particular benchmarking and radial cuts on a circle, the turn was later also given a code of LT. Because the data were first reduced to turns that concerned students' mathematical work and then were coded based on evidence of the LT, turns coded as LT referenced students' mathematical work in relation to one of the eight identified attributions. Two research team members carried out the coding for LT, discussing any unclear turn until agreement was reached.

Results

Our research sought to identify to what elementary teachers attributed students' mathematical successes and failures when working on professional learning tasks designed to share with them research results about students' mathematical thinking and learning. First, we present the attributions that emerged during our ongoing analysis that informed the development of the codebook. Then, we examine teachers' uses of these attributions during the 60-h professional development from the first phase of our retrospective analysis.

Attributions

We identified eight different attributions that teachers used when explaining students' mathematical work. Common attributions noted by Cooper and Burger (1980) were present in our data, specifically, ability, effort, luck, and difficulty of task. We identified additional attributions of age or grade level, out of school context, teaching, and previous math knowledge. Moreover, these attributions varied in terms of locus of causality, stability, and controllability. In our interpretation, we considered these dimensions from the teachers' perspective, viewing locus of causality and stability in relation to the student and examining whether the teacher has control over a particular attribution. In this section, we present each attribution, beginning with the attributions previously identified in the literature, and then describing additional attributions we identified, along with selections from the data that exemplify how each was presented in teachers' discourse.

Ability Ability was internal to the student, fixed, and an uncontrollable attribution that included the personal traits of students and characteristics that defined fixed qualities related to students' aptitude in mathematics. Often times, teachers used achievement as a proxy to consider students' abilities and attributed their performance to an innate capacity. One example involved a teacher describing her work with a previous student in a discussion during the summer institute. In her comment, she indicated that ability was a fixed characteristic of the student. She stated, "We had evaluated this student, and we were convinced there was a learning disability. The work was really low." Another teacher, also describing past students, expressed a similar explanation related to students' innate abilities. She said, "I had a lot of math geniuses and they can figure things out when they are so young."

Effort Effort was internal to the student, variable, and a controllable attribution that referred to the level of students' attention and engagement with a particular task at a particular moment. This attribution indicated that performance did not always represent a fixed characteristic of the student but depended on how carefully or how speedily that particular student progressed through the work at a particular moment and was thus subject to change. When examining her own students' written work on an assessment during a monthly meeting, one teacher explained a student's incorrect solution by saying, "Well, he just zipped through all this, so, no wonder." In another instance during the same activity, a teacher commented, "He worked on this so carefully." Other teachers explained students' work by speculating about the student's attention during instruction, such as "In my mind, this kid just wasn't paying attention to me while I was teaching and he played connect the dots."

Luck Luck was external to the student, variable, and an uncontrollable attribution that included the idea that what students did had no intentionality behind it. Teachers who attributed students' success or failure to luck implied that students had no real explanation for what they did or why they did it, or questioned whether students knew what they were doing. For instance, during the summer institute, in response to viewing a clinical interview with a child who was equipartitioning a collection of 24 coins among four friends, one teacher remarked, "When questioned how did you know, that is when I realized she really randomly chose to give each one, two pieces. It was not that she had the number fact or she understood." During the same discussion, another teacher commented, "I thought she was just guessing and she was just lucky."

Difficulty of Task Difficulty of task was external to the student, variable, and a controllable attribution for students' work that expressed the notion that students' difficulty was determined by the clarity (or lack thereof) of the question posed to them. It had the embedded idea that there was a perfect way to ask a question so that students would not make a mistake. For example, in the summer institute, when analyzing two video recordings of students sharing a collection of coins among various numbers of people, one teacher said, "The proctor asked her to put things together and then divide them, so, she shared differently [than the first student] because the proctor asked a different question." In another case, teachers were examining written work on equipartitioning assessment items and one teacher commented:

> When we teach a group of students and over half of them make the same mistake, then we have to go back and look at the way we presented it and ask ourselves…"Is it some fault in the way the question was presented?"

Age or Grade Level Age or grade level was internal to the student, fixed, and an uncontrollable attribution that described the expectations teachers had for students' performance given normalized definitions for what a generic student should be able to do at certain points in his or her development. Teachers used grade level to create groups of students at similar developmental levels who should perform in certain expected ways, assessing the quality of a students' work or their mathematical reasoning based on whether it conformed to what is expected of children at that age or grade level. For instance, during the summer institute, one teacher commented on a video recording of a student sharing 24 coins among three friends: "I had expected the third grader to not share dealing it one by one." In response to a similar video of another student, a teacher stated, "I taught Kindergarten, and I would have guessed she would share using one for you, one for you, one for you; what she did was more advanced because she counted two plus two plus two."

Out of School Context Out of school context was external to the student, variable, and an uncontrollable attribution that included out of school understandings and explanations that teachers expected students to generalize to the academic context. This attribute indicated that teachers took into account the experiences students bring with them from their own lives. For instance, when conjecturing in the summer institute about a video of one student's work sharing a collection of coins among two people, a teacher said, "She just shared and she thought, 'It is fair because we each got some,' and that is because of how we use the word 'share' in the real world. She thought, 'We both have some, so we have shared.'" Another teacher remarked about a written task where students were asked to equipartition a rectangular birthday cake among six friends:

> I think that was a problem for a lot of these kids, dishing out the whole birthday cake [to fair share it]. I just wonder if you called it something else besides a birthday cake if they would have seen the whole differently.

These comments indicate that students' out of school experiences are uncontrollable and not necessarily places from which to build instruction, but as justifications for students' mistakes.

Teaching Teaching was external to the student, variable, and a controllable attribution that indicated that students' mathematical work depends on what teachers present to them. Teachers expected students to know or not know a topic depending on whether or not a teacher had already taught the topic. The attribution also indicated that teachers suggested that students had no way of knowing a topic that they were not yet taught. For example, during the summer institute, one teacher stated,

> Sometimes students can say something even when we had not taught it, like, this is 1/2 of 10 so that part has to be 5 as well. It seems simplistic, but I don't know how they would have known that already.

In another case, after examining two tasks related to identifying "one-sixth," one with a circle already partitioned into six equal sized parts and one that asked students to equipartition a circle for six, a teacher remarked, "Don't you think that's kind of hard too? Because like you said, this one's already done for them, and kids have a lot of trouble until you teach them on how to actually divide it I thought." Both of these comments indicate that students' successes can be attributed only to what has been taught to them.

Math Math was internal to the student, variable, and an uncontrollable attribution that described the idea that students' mathematical work can be attributed to their cognitive development based on previous mathematical experiences. It included descriptions of students' mathematical thinking and used mathematical language to talk about students' successes and failures. Given that the nature of the professional development focused on students' mathematical thinking, we expected teachers to use specific language from the EPLT to describe students' mathematical work as they learned about students' mathematics through the EPLT. For example, during the summer institute, teachers were asked to anticipate the way that a child would equipartition a circle into six equal-sized parts. After viewing a clinical interview

Table 1 Frequency of attributions by teacher

	Attribution								
Teacher	Ability	Effort	Luck	Task	Grade	Context	Teaching	Math	Total
A	8	6	3	23	22	5	28	93	188
B	6	4	4	28	6	4	22	100	174
C	12	2	1	11	22	6	20	62	136
D	3	0	2	8	4	2	16	102	137
E	3	1	1	9	7	0	12	22	55
F	8	0	2	15	6	2	32	97	162
G	3	2	5	8	7	4	29	47	105
H	0	3	1	12	2	2	12	48	80
I	3	3	1	15	8	1	25	70	126
J	10	0	0	1	2	2	10	43	68
K	15	4	8	25	18	3	44	135	252
L	4	0	1	19	7	3	24	89	147
M	7	1	5	19	10	0	19	121	182
N	13	1	0	4	12	2	16	68	116
O	6	2	7	24	12	3	52	178	284
P	4	0	0	22	13	9	28	65	141
Q	15	4	5	13	6	3	21	89	156
R	3	3	5	13	1	7	9	52	93
S	4	1	1	11	6	4	20	71	118
T	3	1	0	3	4	2	6	20	39
U	6	1	3	10	4	0	22	67	113
V	8	1	2	12	7	3	19	67	119
Total	144	40	57	305	186	67	486	1706	2991

of the student, one teacher commented, "I thought she would benchmark the half, as she did…. She knew she had to go to sixths, so she drew the diagonal." Because the teacher described what the student did mathematically rather than focusing on other nonmathematical factors, this statement was coded as math. More specifically, the teacher used language from the EPLT to describe what the student did and attributed the student's work on the task to what she knew about common strategies for equipartitioning.

Teachers' Use of the Various Attributions

During the initial phase of the retrospective analysis, we examined the total number of turns coded for the 22 teachers who completed the professional development using the eight attributions identified in our data. As shown in Table 1, most teachers used all of the attributions during the professional development. Column totals showed there was considerable variability in the number of turns coded for each of the attributions. Teachers did not use the attributions of luck and effort as much as ability and age/grade, whereas math was the most used attribution, followed by teaching and task. An examination of the columns also shows that most attributions

Table 2 Frequency and percentage of LT attribution

	Turns coded LT	
Teacher	Frequency	Percentage of teacher turns
A	64	34
B	43	25
C	30	22
D	46	34
E	9	16
F	36	22
G	32	30
H	28	35
I	41	33
J	12	18
K	47	19
L	31	21
M	51	28
N	39	34
O	101	36
P	45	32
Q	37	24
R	11	12
S	29	25
T	17	44
U	48	42
V	22	18
Total	819	27

were used by all teachers, with few zeroes in each column. For instance, only five teachers did not use effort as a way to explain students' mathematical successes and failures, four did not use luck, and three did not use context. In addition, we take the prevalence of turns coded as "Math" to be an indication that the design of the professional development supported teachers in using research-based knowledge to understand students' mathematical thinking. The next phase of the retrospective analysis will address questions of changes in the attributional process over time during the professional development.

During this phase of the retrospective analysis, we also examined teachers' attributions that included the EPLT. Table 2 depicts the number of instances where each teacher made reference to the EPLT within the set of coded turns, that is, the number of turns that were coded as one of the eight attributions in the first round of coding and then later also received an LT code. The table shows that 819 turns referenced the EPLT, indicating that teachers used their learning from the professional development to explain students' mathematical work. This number represents 27 % of the total previously coded turns. The percentage of each attribution that later was double coded as LT ranged from 12 to 44 %.

We conjectured that the LT code would emerge solely within the math attribution, and we considered the math code as a way to capture teachers' emerging use of the LT language. However, as we coded our data, we found that references to the

LT emerged in all of the eight codes, not only mathematics. For example, during one of the last monthly meetings, one teacher discussed her instruction, saying: "We've done things like reallocation with the kids. We present a story and then ask, 'If this kid leaves, how many would each get?'" This turn was coded as teaching because the teacher was describing how she would teach (present a story) in order for students to learn. However, because the teacher is also referencing ideas from the LT (reallocation), this turn was also coded as LT. This example demonstrates how teachers used the LT to talk about students as well as their teaching. Yet, from our field notes and ongoing analysis, our data suggested that teachers also maintained the attributions they had been using to judge or provide causal explanations for students' mathematical work. From this perspective, the LT language neither eliminated nor added to previous attributions. Instead, the various attributions became more complex as teachers used LT language in conjunction with previous language related to other attributions. The next phase of the retrospective analysis will seek to understand the emergence and prominence of the LT attribution over time during the professional development.

In summary, the analysis of teachers' attributional processes showed that the majority of the teachers used all of the eight as causal explanations for student work during the professional development. More specifically, all 22 teachers used at least six of the attributions at least once, and 20 teachers used at least seven of them. Likewise, all teachers made reference to the LT when using these attributions to varying degrees. Together, these observations suggest that teachers do not hold one attribution for students' mathematical work but rather employ a variety when conceptualizing students as mathematics learners. Further, teachers can learn to use students' mathematical thinking as represented in LTs to explain students' successes and failures, adding a new attribution to their repertoire.

Discussion and Next Steps

We sought to identify the attributions that elementary teachers use to discuss students' mathematical successes and failures when working on professional learning tasks designed to share with them research results about students' mathematical thinking and learning. We started with the initial conjecture that as teachers learned about a mathematics LT, they would change their attributional discourse and add new explanations for students' mathematical work to their repertoire. Here, we have reported on the initial phases of our analysis; our ultimate analysis aims to understand the changes in teachers' discourse about students as mathematics learners over the span of the professional development.

Our study documented eight different attributions brought forth in the context of our professional development that teachers used to explain students' mathematics successes or failures when examining student work. These attributions went beyond the traditional attributions of ability, effort, luck, and difficulty of tasks to also include age or grade level, out of school context, teaching, and previous mathematical knowledge. Further, we did not find explicit attributions of gender, race, or socioeconomic status for students' mathematical successes and failures, as previously reported in the literature.

In addition, our work with the teachers around LTs led teachers to include references to students' specific mathematical thinking when discussing their work. As a result of professional development focused on an LT, teachers began to use this research-based knowledge to explain students' mathematical work in their discussions. However, the use of the LT did not substitute or displace the existing attributions teachers used; rather, it added to and was included as part of teachers' previous attributions. Instead of holding one or two particular attributions, teachers used a variety of attributions throughout the professional development to talk about students as mathematics learners. For mathematics teacher educators working to support teachers in learning to make instructional decisions based on students' mathematical thinking, our research suggests that although teachers my acquire such expertise in professional development, they may persist in attributing students' mathematical successes and failures to nonmathematical factors.

Our larger investigation of teacher learning of students' mathematical thinking includes an examination of the relation between teachers' discourse in the professional development and their learning of mathematics LTs. Framing this investigation with positioning theory (van Langenhove and Harré 1999), we are currently examining the ways that teachers position themselves in the discourse of the professional development. Yet, the analysis presented in this chapter highlights that not only do teachers position themselves in discussions in professional development, they also position students through attributional processes. Thus, we are currently reconceptualizing the attributional processes identified in the data as acts of stereotyping students as mathematics learners.

van Langenhove and Harré (1999) questioned the notion of locating stereotypes within the individual, with words and actions being the expression of personally held beliefs. Rather, they considered that stereotypes reside as positions in public collective discourse and that individuals appropriate them in conversation. They defined an act of personal stereotyping as a speech-act that is part of a conversation's storyline and is used to position both the speaker and the object of the stereotyping. Stereotyping draws upon social representations of the stereotyped objects that are available in certain communities. For example, rather than considering one teacher's statement, "I had expected the third grader to not share dealing it one by one" as attributing the student's work to grade level, our reconceptualization suggests that the teacher was calling upon a representation of what a third-grade student should do that is available and accepted in her community.

In continuing our research, we conjecture that the array of personal stereotyping available to teachers within their professional discourse communities is influenced, as teachers learn a framework for students' mathematical thinking. In future analysis, we will seek to understand the ways these stereotypes were called upon over time, the changes in stereotyping as teachers learned about the LT, and the relation between their use and the professional learning tasks designed to support their learning.

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Teacher Identity and Tensions of Teaching Mathematics in High-Stakes Accountability Contexts

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Both the National Council of Teachers of Mathematics *Standards* (NCTM 2000) and the *Common Core State Standards for Mathematical Practice* (CCSS-MP; NGA Center and CCSSO 2010) encourage mathematics instruction to focus on problem solving, reasoning, and mathematical communication, and to include all students in those activities. These instructional practices are further encouraged in mathematics teacher professional development and preservice teacher education. School- and district-based accountability pressures, such as curriculum standards and large-scale testing, however, push teachers toward instructional practices that are less focused on rigorous mathematics and more focused on skill-based teaching and coverage of content (e.g., Ellis 2008; Horn 2007; Valli et al. 2008).

There are thus potential tensions between the ambitious instruction practices suggested by mathematics teacher professional development, methods courses for preservice teachers, or even the *CCSS-MP* and the pressures teacher candidates (TCs) feel as they begin teaching in a high-stakes accountability school context, contexts characterized by public pressure to improve school performance, particularly on standardized exams (e.g., Rinke and Valli 2010). High-stakes accountability policies have been shown to give rise to test-driven school cultures, including instructional methods to support test achievement and a narrowed curriculum focused on the test (Valli et al. 2008). In a test-driven school culture, where teaching quality and students' learning are equated with high-stakes test scores (Cochran-Smith 2005), teachers, administrators, and students feel the pressure of the importance of high student performance on these assessments. Research shows that these schools often enact practices that favor test preparation in lieu of authentic mathematics instruction (Ellis 2008; Lattimore 2005). The influences of accountability systems also reach deeper than teaching practices or teachers' roles in the classroom. State-mandated reforms involving imposed standards and curricula and reductive

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testing lead to ability tracking, lowered expectations of students, and reduced opportunities to learn, often for the very children the policies are designed to support (e.g., Apple 2004; Valenzuela 1999). In addition, the ways that students are labeled and grouped by ability, as well as how mathematics achievement is characterized as related primarily to speed, motivation, and accuracy, influence the teachers' identities as teachers of mathematics (Brown and McNamara 2005; Horn 2007).

It is important to support elementary TCs in enacting ambitious pedagogical practices in their mathematics classrooms in high-stakes accountability contexts. Research on mathematics teacher identity suggests potential relations between teacher identity and teacher practice (Peressini et al. 2004), as well as the ways in which accountability pressures influence teacher identity and self (Brown and McNamara 2005; de Freitas 2008; Horn 2007). Furthermore, other elements of TCs' identities, such as their existing beliefs about mathematics (Gellert 2000), their personal, perhaps damaging, mathematical experiences (Drake 2006), and their understandings of students (Sleeter 2008), have been shown to influence their interpretations of curricula, reform, and best instructional practices.

Because of potential relations to teaching practice and the many influences on identity, understanding more about elementary TCs' identities and the pressures that they feel in these contexts when teaching mathematics, and specifically how TCs position themselves in relation to these tensions and pressures, may support mathematics teacher education in preparing elementary TCs for teaching mathematics. This study focused on how elementary TCs position themselves in relation to tensions of teaching mathematics in high-stakes accountability contexts.

Theoretical Framework

Becoming a teacher is not developing an identity; it is developing *identity* as a continuous process of constructing and deconstructing understandings within the complexities of social practice, beliefs, and experiences. In this research, I conceptualized teacher identities as ways of seeing and understanding oneself and one's positionality that are constructed within discourses. Specifically, TCs' identities as well as their visions of appropriate and possible mathematics teaching practice are shaped by the political, social, and institutional forces that structure the mathematics teaching they have experienced and continue to experience—in other words, their identities are shaped by the *discourses* that provide systems of categories, terms, and beliefs that organize ways of thinking and acting in relation to mathematics, mathematics teaching, and learning (Davies 2000; St. Pierre 2000). Discourses are political, historical, and social forces that are outside an individual and "imply forms of social organization and social practices that both structure institutions and constitute individuals as thinking, feeling, and acting subjects" (Walshaw 2001, p. 481). Foucault (1980) is credited with this conceptualization of discourses, wherein he emphasizes how knowledge, truth, and subjects are produced in language and cultural practice and how discourse works in very material ways through social institutions and socially regulated rules to construct realities that position individuals and control their actions.

Discourses, as theorized in poststructural feminist discussions of education, constrain and enable what teachers do, say, and even conceive of as appropriate. For a teacher, "constituted by her relationship, among others, with her students, their parents, her school, and the wider community, discourses provide taken-for-granted ideas and ways of practice that come before any views she might have about herself as a teacher" (Walshaw 1999, p. 100). Prevailing discourses about mathematics teaching and learning include institutional discourses around curriculum and testing (Brown and McNamara 2005); social discourses around race, class, and abilities (de Freitas and Zolkower 2009); and discourses about mathematics as skills or as practices of "making sense" (Fuson et al. 2005). In addition to the institutional and social dynamics specific to elementary classrooms (Soreide 2006), elementary teachers' and TCs' beliefs about or negative experiences with mathematics complicate relations between teacher identity and practice (Drake 2006). Test-driven school cultures in the USA, for example, create institutional discourses about teaching and students, including notions of fixed student abilities and the equivalence of learning and test performance (Apple 2004). These discourses thus interfere with teachers' teaching and learning relationships with children in their schools (Olson and Craig 2009).

Consistent with the above framing of discourses, the conceptualization of identity in this research reflects how *identities* are simultaneously personal and social, thriving within the complexities of social practice, personal beliefs and experiences, and the constraints of social constructions and social norms. This perspective is different from conceptualizations of identity that focus on identity as related to an individual's role that he or she takes in various situations (e.g., Stets and Burke 2000) or teacher actions in line with these roles (e.g., Ronfeldt and Grossman 2008), or that conceptualize identity as a constellation of beliefs and knowledge (e.g., Collopy 2003). A focus on role does not attend to the dynamic nature of identity or the ways in which individuals negotiate their identity in context (Britzman 1993). Research that sees identity as personal or individual may not incorporate society's influence on identity construction or attend to the social norms about mathematics or teaching that influence or construct teachers' experiences, beliefs, or identities. That is, conceptualizing identity as individualistic may not be productive for addressing TCs' identities in the complexity of their school contexts.

In response to calls to work at the intersection of identity and context (Beijaard et al. 2004; Thomas and Beauchamp 2011), research in mathematics education from a poststructuralist perspective on identity has addressed this concern. In a poststructuralist perspective on identity, elementary mathematics teacher identity is understood as transitory and influenced by systems of power (Brown and McNamara 2011; Walshaw 2004, 2010). Walshaw (2004), for example, emphasized the ways in which TCs' identities are related to both coursework and internship experiences, that is, both "formal and informal educational discourse and practice" (p. 70), and are continuously shifting:

> Pre-service teachers are not only redefining their teaching identities in relation to the available discourse in the classroom and to the complex selves of others, they are also learning what is defined as “normal” practice through the school’s organizational procedures…. The school, construed as a regime of power, constructs specific regulatory practices for the normalization of and ultimately the production of the self-governing individual teacher. (p. 72)

This perspective on identity emphasizes the role of power and knowledge in shaping identities and in TCs’ understandings of the availability of certain responses to these influences on their identities. In their school contexts, TCs

> set about to accentuate the identities of their teaching selves in contexts that are already overpopulated with the identities and discursive practices of others…. Within such contexts, where desires are assigned and fashioned, student teachers strive to make sense and act as agents in the world. Indeed much of their time is taken up with negotiating, constructing, and consenting to their identity as a teacher. (Britzman 2003, p. 221)

As TCs are constantly negotiating their identity, embracing research perspectives that focus on positioning and negotiation attends to teaching context and to the multiple influences on TC identity.

Because the taking up of an identity is in constant social negotiation, it is not synonymous with role or function (Britzman 1993). Moving away from identity as operationalized through the idea of “role,” the concept of “positioning…focus[es] attention on the dynamic aspects of encounters in contrast to the way in which the use of ‘role’ serves to highlight static, formal and ritualistic aspects” (Davies and Harre 1990, p. 43). Through focusing on the discursive practices and how they constitute the speakers, this perspective can embrace, if not explain, discontinuities in productions of self in conversations with others: “An individual emerges through the processes of social interaction, not as a relatively fixed end product but as one who is constituted and reconstituted through the various discursive practices in which they participated” (Davies and Harre 1990, p. 45).

Butler (1999) used the theoretical premise of *performativity* to connect identities and discourses and to explain how identities are shaped by contextual and historical elements and discourses. Butler suggested that the process of identity construction is regulated and constructed within discourse, and thus is a constrained choice. *Performativity* is a continuous process of being naturalized by these outside forces and suggests that identities are under constant influence, not present from the beginning, but instituted in specific contexts. For example, a TC’s teacher identity involves negotiation of discourses and experiences within power relations and discursive practices that are already present in the situation, as well as TCs’ previous experiences (Walshaw 2004). Performativity is used to highlight prevailing discourses and the subsequent process of *construction* of identities to open opportunities for individuals to engage in *deconstruction* of these identities (Butler 1999). Using performativity within mathematics teacher education suggests exposing the prevailing or restricted discourses that shape some teacher identities and troubling these discourses in order to support TCs in understanding themselves as mathematics teachers and (re)authoring their mathematics teacher identities.

Using Butler’s (1999) perspective on identity recognizes that there are many social, political, and institutional dynamics in schools and in mathematics classrooms that complicate and regulate teacher practice and identity. These dynamics, such as the institutional framing of student learning, prevailing discourses about students' abilities and mathematics as a set of skills, and discourses about race, class, and gender, create a complex teaching environment for novice teachers. Teaching mathematics is fundamentally about teaching within and navigating this complexity. The consequences of leaving teachers unprepared for these dynamics are damaging, not only for those teachers, but also for students, as teachers who are unprepared for the realities of schooling may be unable to enact ambitious mathematics teaching practices. As a theoretical research tool, embracing a perspective of performativity means not categorizing or focusing on solely individual elements of identity, but investigating how these identities are shaped and highlighting their contextual and historical elements as a means toward preparing teachers for negotiating their identities within the complex dynamics of school contexts.

Modes of Inquiry

From this theoretical perspective on identity and the need to support TCs in their high-stakes accountability contexts, the following research question guided this study: For elementary TCs participating in a seminar on critical self-examination and mathematics teaching in context, in what ways do they position themselves in relation to tensions of teaching mathematics in high-stakes accountability contexts?

Study Context

Certification Program The participating TCs were students in an intensive 15-month master's degree program, which led to elementary certification, at a large university in the mid-Atlantic region. This certification program was well suited for the study context because of its focus on elementary teacher preparation and the emphases on teaching for understanding, inquiry, and reflection in the coursework and during the internship.

The program began in June, and the full cohort of 32 students took three courses together during the summer term. In the fall, the students took four courses: mathematics methods, science methods, reading methods, and a diversity course. Two sections of each course were offered, and I taught one of the mathematics methods courses in Fall 2009. The coursework was offered 2 days a week, and students were immersed in a yearlong internship in an elementary school 3 days a week. In the spring semester, students were immersed in their internship for 5 days a week, took two evening courses, and met monthly with their advisor to address their action research projects. TCs took a Capstone course in the final summer of the program. Upon completion of the program, students earned a Master's of Education and were eligible for state certification to teach Grades K–6.

Noncredit Seminar I designed and then facilitated a noncredit seminar that met biweekly during the spring and final summer of the program, for a total of eight sessions. All students in the cohort were invited to participate. In the seminar design and facilitation, I operationalized critical pedagogy (e.g., Kumashiro 2000) and feminist poststructuralist notions of identity (Butler 1999; St. Pierre 2000) in mathematics teacher education. The seminar goal was to encourage TCs to question prevailing discourses of students, mathematics, and teaching and "bring this knowledge to bear on his or her own sense of self" (Kumashiro 2000, p. 45). This stance resonates with "giving reason" to prospective teachers (D'Ambrosio and Kastberg 2012), in the exploration of TCs' existing knowledge, not to "ignore it or criticize it, but to explore with" (p. 26) TCs along their journeys of becoming mathematics teachers. The seminar focused on specific objectives: identifying and examining the many prevailing social and political discourses that shape mathematics, mathematics teaching, and their positioning; analyzing the implications; rethinking these discourses and their implications in relation to self; and problematizing teaching in relation to these discourses. Seminar activities designed to work toward these objectives included case analysis, group discussions, and reflective writing about positioning and teaching experiences. The primary data for this study were collected during this seminar.

Participants

I solicited participation for the seminar from all 32 students in the program cohort. Ten female TCs between the ages of 25 and 35 volunteered to participate in the eight seminar sessions as well as the study. Seven of the TCs had been students in my mathematics methods course in Fall 2009; three were in the other course section. Eight of the participating TCs self-identified as white, one as an immigrant from Argentina, and one as African American. Three TCs, Brooke,[1] Candice, and Laura, were chosen for case analysis because of their consistent participation in the seminar (e.g., actively participated in discussions, completed coursework, and discussion board reflections) and because across their participation, institutional discourses related to their test-driven school contexts and the accountability pressures emerged as salient and could be conceptualized as central to each TC's identity. There were, however, emergent differences in the ways in which they positioned themselves in relation to these discourses of test-based accountability pressures, how they understood themselves as mathematics teachers and understood teaching in their school contexts, and their differing interactions with the seminar activities. Analysis, as detailed below, sought to understand more about the differences in these three TCs' self-understandings, specifically, their identities and tensions

[1] All names of participating teacher candidates, students, schools, and counties are pseudonyms.

related to teaching mathematics in high-accountability contexts. Each TC is introduced briefly below.

Brooke was a white female in her mid-twenties. Before beginning the program, she worked as a consultant in the private sector. She said that a speech by former DC School Chancellor, Michelle Rhee, inspired her to teach in an underserved school and to begin a certification program. Brooke never had liked mathematics in elementary school, but described her mathematics learning experiences as "not overly positive or negative" (Mathematics Autobiography, September 2009). Brooke interned in a second-grade classroom that she described as having "a lot" of inclusion students. When the students were regrouped for mathematics instruction, her mathematics class was "above grade level," meaning that she taught and tested this group of second-grade students on third-grade mathematics objectives. In her methods course, Brooke presented herself as confident about being an elementary mathematics teacher.

Candice was in her late twenties, and she self-identified as an African–American female. She had moved from Los Angeles specifically to attend this elementary master's certification program because of its geographic proximity to areas that she felt were rich with African–American history. Candice had previous experience working with students in alternative schools and with programs for students who are at risk for educational failure. Candice described her experiences as a mathematics learner and how they related to her mathematics teachers: "Math is not hard. Many people, like myself, have not had many great teachers of mathematics, because many math teachers did not have great teachers and the cycle continues" (Mathematics Autobiography, December 2009). Unlike other TCs in the program, Candice rotated through three different classrooms in her placement school and had four different mentor teachers and experiences in first-, third-, and fourth-grade classrooms during her internship.

Laura was a white female in her mid-twenties. Before beginning this program and her internship in a third-grade classroom, she worked with students in after-school settings and taught religion classes on the weekends. Laura described positive mathematics learning experiences and herself as a "straight A student": "Math came to me somewhat naturally in elementary school, and as the material became more difficult in middle and high school, I worked harder to fully understand the material and to earn good grades" (Mathematics Autobiography, August 28, 2009).

Although these three TCs interned in three different elementary schools (and in three different counties), teachers and the participating TCs reported that educational reforms and the resultant accountability systems created a high-stakes accountability context at each of these elementary schools. Administrators and the school culture at all three elementary schools emphasized high-stakes testing and curriculum pacing, and each school used student labeling and grouping by ability in organizing teachers' classrooms and students.

Data and Analysis

The primary data included video and transcripts of the eight seminar sessions; TCs' written work produced during the seminar, including seminar homework and discussion board reflections; and a final interview with each participating TC on her experience in the seminar and in the program in general. Additional data included TCs' written mathematical autobiographies completed during their mathematics methods courses. Although TCs discussed many issues related to their identity as elementary teachers and TCs, the analysis for this study focused specifically on relations between teachers' conceptions of their identities and discourses of test-based accountability, which was an emergent theme across participants' talk.

In analysis, I conceptualized TCs' understandings of being a mathematics teacher as positioning (Davies and Harre 1990), which entails how discursive practices constitute speakers and hearers in certain ways and also serve as resources through which speakers and hearers can negotiate new positions. I focused analysis on how TCs were being reflexive of their positioning in social interaction, that is, recognizing one's positioning and the circular nature of positioning in social structure (Mauthner and Doucet 2003). I attended to the complexity of TCs' identity in analysis by focusing on each participant's discursive practices in context; then, each participant served as a case, and case analysis (Yin 2009) focused on an individual's identity.

Based on close reading of the empirical literature on being reflexive about positioning (Ellsworth and Miller 1996; Williams 1991) together with emergent categories from my own analysis, I identified four discursive practices that constitute aspects of being reflexive of one's positioning, specifically focusing on how TCs were identifying, specifying, and responding to *positioning as contextualized, positioning as relational, positioning in performance,* and *repositioning* (Table 1). In my analysis of TCs, the first three discursive practices suggested how TCs were reflexive about their positioning—how they were understanding themselves as mathematics teachers.

Attending to *repositioning,* including the *possibilities* for repositioning, suggests repositioning or shifting understandings of self. Butler (1999) emphasized both acts of subverting positioning and "thinking through the *possibility* of subverting and displacing" (p. 46) for "intervention and resignification" (p. 45) of identities or, in this analysis, shifting understandings of self as mathematics teacher. Repositioning as theorized for my analytical purposes is not directional or analyzed as movement in a particular direction, either toward or away from a particular discourse; analytically, repositioning is creating a more articulated position and there is not a normative response to the direction. For example, repositioning in relation to a discourse of test-based accountability does not include being in favor of or not in favor of testing; rather, it includes describing personal actions one can take in relation to the discourse, perhaps identifying subversive behaviors to contest the way one is positioned.

My analysis followed how each TC engaged in these discursive practices specifically in relation to issues of test-based accountability. Across the data corpus, I

Table 1 Discursive practices of being reflexive about positioning and repositioning

Discursive practice	Description
Positioning as contextualized	Identifying, specifying, or attending to social, political, historical, or institutional contexts and how they relate to positioning by a particular discourse or in a particular manner
Positioning as relational	Identifying, specifying, or explaining positioning as in relation to others or gives explicit attention to the multiplicity of positioning and the constant process of and opportunities for negotiating positioning and membership in different groups
Positioning in performance	Identifying, specifying, attending, or responding to one's own or other's actions as suggesting positioning by a particular discourse
Repositioning	Identifying, specifying, or explaining the actions that one has to take in order to subvert or contest a particular discourse or the possibilities for repositioning

analyzed how TCs engaged in identifying, specifying, and responding to positioning as contextualized, as relational, and in performance and to repositioning around this theme to explore TCs' developing identities. Data were organized into case studies of each TC because of my attention to the participating TCs as individuals. I chose a multiple-case design in order to present the different and various ways TCs positioned themselves in relation to tensions of teaching mathematics in high-stakes accountability contexts.

Results

Across seminar sessions, TCs articulated tensions related to teaching in high-stakes accountability contexts. I first present an analytic episode, which describes the three TCs' participation in discussions about mathematics teaching in high-stakes accountability context during the seminar, and illustrates the different ways in which they positioned themselves in relation to institutional discourses of accountability and related pressures. Then I present the cases of each TC (Brooke, Candice, and Laura) and how each TC's positioning suggests a different emergent tension.

Analytic Episode

During Session 1, after writing a vision of herself teaching mathematics in 5 years, Brooke looked up from her writing and asked me a question about teaching volume to her current second-grade students:

> Can I ask you something? I'm, I need some help with a math lesson. So, tomorrow is Day 5. I'm trying to teach volume. They get, like, the point. They understand, like, the concept of volume. They don't get how to count on flat figures. So, when you look at a 2D object on a piece of paper, and it's an irregular [figure]—if it's regular they can do it, but as soon as it's

> like stairs and you can't see the behind ones, then they can't do it.… We built them for three days and they're just not seeing the connection at all. (Group discussion, March 23, 2010)

Brooke suggested that her students "understood the *concept* of volume," but were not demonstrating a particular understanding or skill, that is, counting hidden cubes when presented as flat figures. Differentiating between concepts and skills, Brooke emphasized that she wanted to focus her teaching on this test objective because it was on the assessment, even as she questioned the "practical [nature] of this" or how this test objective related to mathematics outside of the test.

Although the initial question was directed at me, other TCs were interested and joined in the discussion. Candice and Laura (among others) asked follow-up questions:

Candice	What grade is it?
Brooke	Second graders doing third-grade math.
Laura	I was going to say, "Why are you talking about my objectives?" [Recall that Laura is teaching 3rd grade.]
Jill	So, what's the goal?
Candice	Is it a 3rd grade concept? Are they supposed to master that at 3rd grade?
Jill	I don't, I don't know. So, that's the idea, is like—
Laura	It's like one objective.
Brooke	It's definitely on their, I hate to say, it's on their assessment. I don't know a lot about the practical of this.
Laura	I mean, they need to count. There are two pictures, and I think that they are odd-shaped. They have to count the cubes.
Jill	They did build their own?
Brooke	So, today I had them build, I did it on the screen and they did it at their desk. So, we all built the same thing, but they just. They literally can't see what's behind it. It's like, you know when you're working with an infant and you have a ball and you put it behind your back and they just think the ball is gone, like magically into thin air. They just think the blocks are gone. (Group discussion, March 23, 2010)

Brooke identified that her focus was on the assessment and indicated that her mathematics teaching was contextualized by these assessment pressures (positioning as contextualized). She discussed her students' difficulty with seeing the composition of the figure and made an important step to try to understand why her students were miscounting the blocks. However, Brooke focused on evaluating the correctness of students' responses, and there is no evidence of identifying how she was positioning her students or how her students' positioning influenced her own mathematics teacher identity. Brooke positioned her students as infants unable to understand object permanence. Brooke rooted this deficit perspective of her students in how they were not successful with a particular test objective; together with her reference to the assessment, her participation suggests how she felt positioned by the associated test-based accountability pressures and understood student understanding as related to test objectives.

In response to Brooke's description of her students, Paige suggested students could trace the figures, and Michelle suggested having the students use Jell-O boxes because the unit cubes fit perfectly. Brooke replied that these ideas would not work because her students were "not seeing the connection" (Group discussion, March 23, 2010). The "connection" that Brooke was concerned about is how the understandings that students could build through engaging in activities such as building with blocks or tracing figures related to the particular test objective. She wanted her students to do well on this objective and was searching for a solution to this concern; she was not focused on building student understanding of volume more generally. Brooke's emphasis on this lesson and assessment suggested that the high-stakes accountability pressures influenced her teaching (e.g., Rinke and Valli 2010).

As the conversation about Brooke's lesson ended, Candice then began sharing her own teaching experiences in her elementary school, particularly her responses to test-based accountability systems:

> The [state assessment], standardized tests, they influence just how I teach because there are certain things. It's well, the connections, like legislation, I guess, more so, or so, because legislation is like. It's saying basically, like, you are not a good teacher if your students don't do good on the [state assessment] or whatever test it is. Like, with the No Child Left Behind thing, um, it might influence where I teach because I might choose not to teach at a school where I won't, where I already know the students won't make AYP [Adequate Yearly Progress] or, you know, reach those benchmarks, whereas, that's, I'm all about them right now. And I see the struggles that my teachers are going through because they have made the sacrifice to teach at a school where the students were not high-achieving. (Group discussion, March 23, 2010)

Candice identified how education policy, the resultant accountability pressures, and the related tests positioned her as a teacher and had implications for her teaching and how she understood her options of where to teach. Analytically, she identified the contextual nature of her positioning and the related actions that this positioning led to—her positioning in performance. She felt pressure to ensure that her students met particular benchmarks and felt that her reputation as a teacher was related and dependent on her students' achievement on high-stakes tests; in this manner, she also highlighted the relational elements of her positioning. Although Candice consistently identified her desires to support all students to learn in her participation across the seminar, she also explained that there are personal sacrifices that she would have to make in order to reach all students if she were to teach in certain schools. In this comment, Candice identified that in order to reposition herself and teach mathematics in struggling schools, teachers need to make sacrifices or recognize that job security is tied to their students' performance. This suggests repositioning because of the ways in which Candice identified actions that need to be taken in order to contest discourses of test-based accountability, or specifically, to contest the ways in which those discourses limit particular students' opportunities to learn.

In response to Candice, Laura described a similar tension. She emphasized how her teaching context would likely limit the type of instructional practices she could enact:

> I imagine myself as a certain, teaching a certain way but recognizing that in certain places I can't do that. I can't have those freedoms. But then I recognize that I am really limiting myself to places, where I can, you know, if I, if it's a priority to me to teach in a way that I want to, I'm really limited to places that are going to give you that freedom. (Group discussion, March 23, 2010)

Laura shared how she felt that in certain schools she would have less freedom in how she could teach, specifying how school context matters for how she is positioned (positioning as contextualized) and how she is able to position herself or engage in certain practices. By "certain places" she was referring to schools where she did not think her teaching practices would be limited by the administration and its emphasis on standardized testing, which she also felt was most prevalent at underserved schools. Laura also identified that particular teaching practices suggested her positioning (positioning as performance) and how she was not sure if she would be able to reposition herself in some teaching environments.

Following Laura's comment, Brooke responded to the tensions that Candice and Laura raised and suggested that these same accountability pressures, or the standardized assessments in particular, did not influence her in the same ways that Candice and Laura mentioned:

> But see, I don't feel like that. I don't feel like that influences me all that much because like I'm spending 5 days on this stupid volume lesson, which really doesn't have the huge practicality of counting these imperfect like block shapes, but I'm doing it because it is going to be on their unit assessment. But the majority of the stuff I teach is all beyond what is being assessed on those tests. So I feel like the test doesn't necessarily, like I make sure that I hit those things but it is only like, you know, a couple of things that I need to do and I have a lot more time that I get to do a lot more enriching things with my kids. (Brooke, March 23, 2010)

Brooke discussed her practice differently from Laura. First, she noted that she was indeed spending 5 days on a test objective and then asserted that she did not feel these same institutional pressures as Laura; she felt that she had autonomy over what she taught. While Laura described how the accountability pressures in her school contexts influenced how she taught and what she taught, in her discursive participation, Brooke seemingly contested that her positioning was in context or related to her teaching context.

It is significant that this discussion took place less than 3 min after Brooke asked for assistance with her volume lesson and emphasized a focus on test objectives. I sought to encourage Brooke to identify how some elements of her context, in particular her students' status as "above-grade-level," influenced how she was positioned as a teacher and the options she had to teach:

Jill	Now, how is that, how is that in your situation because, there is this prevailing way of thinking that your kids are above grade level, right?… What are they? In second grade doing third grade math?
Brooke	Right.
Jill	How does that allow you then to not have these same struggles?
Brooke	Well—

Laura I'm teaching the same curriculum but to the lowest, [laugh] I hate to say it, third graders, and so we probably see it, very, this exact same curriculum in very different ways. (Group discussion, March 23, 2010)

Both Laura and Brooke administered the third-grade mathematics assessments to their students, but Laura identified that because she taught the lower-ability third graders, she saw the curriculum differently. In this manner, Laura recognized that how her students were positioned in mathematics influenced her practices, her options for her practice, and her conception of the material she taught.

Three Distinct Cases

Across the three cases, issues of high-stakes accountability emerged in the TCs' identities, and different tensions emerged related to the TCs' discursive practices of being reflexive about their positioning. Specifically, these three TCs' tensions with mathematics teaching in test-based accountability can be understood as three iterations of the question, "What can I do?"—questions first referenced by Pollock et al. (2010) in discussions of teaching teachers about race. The different iterations of this question emphasize TCs' questioning of individual agency when teaching mathematics in high-stakes accountability contexts: What *can* I do? (questions of personal agency in responding to issues of teaching in these contexts); What can I *do*? (questions in search for actionable steps in their mathematics classroom); and What can *I* do? (questions of personal readiness to become the elementary teacher of mathematics that he or she would like to be). The analysis that follows describes how each TC's positioning suggests one of these tensions.

Brooke: What *can* I do? Across the data corpus and as evidenced in the analytical episode above, Brooke did not consistently identify the contextual, relational, or performance-based elements of her positioning or engage in repositioning. Brooke seemed comfortable identifying the accountability pressures as related to her specific lesson, but less comfortable, even resistant, to framing her understanding of herself as a mathematics teacher as related to the institutional discourses of accountability and their influences on her practice. For example, she identified that the unit test and its objectives drove her instructional choices but did not generalize to the influence that the prevailing pressures of accountability have on her practice, and instead suggested that these pressures did not influence her practice or identity, describing her teaching as including "more enriching things." In this manner, Brooke seemingly struggled to examine the contextual or relational elements of her positioning, how her positioning, by test-based accountability, surfaced in her teaching (positioning as performance), or to consider repositioning *in response* to these discourses.

Earlier in Session 1 (prior to the analytic episode), when discussing a vignette of teachers discussing teaching a high school algebra course to all eighth grade students, Brooke identified how there are expectations for students and teachers and that teachers may feel frustrated when their students do not meet these expectations. In the analytic episode, however, she did not identify how she felt frustrated by her administration's expectations for teachers or students, or the role of institutional pressures of accountability systems in teaching; rather, she seemed to resist identifying the way she was positioned. Her discussion of her teaching seemingly aligned with these pressures, but she asserted that the political and institutional pressures of accountability systems did not influence her or how she or her colleagues taught. In this way, Brooke was not reflexive about *her* positioning as contextualized, as relational, or in performance, not only because she did not identify these particular influences but because she did not identify other elements either, maintaining that her teaching context did not influence how she was positioned or how she saw her practices.

Analysis suggests that, when reflecting on teaching in high-stakes accountability contexts, Brooke struggled with the question, "What *can* I do?" Brooke questioned her agency in her teaching in these contexts, suggesting that student performance, such as on the assessment items on volume, and student positioning were problems of *students* needing to understand the mathematics and take responsibility for their performance and positioning. Brooke's questioning of her agency means not only that she felt her practice was confined by particular contexts, but also that she did not feel that the problems and tensions were hers to contest. There is no evidence that Brooke engaged with the interaction between herself and her context when reflecting on her teaching practice or how her positioning by test-based accountability surfaced in her teaching. Even as her talk of her teaching seemingly aligned with these pressures, Brooke asserted that the political and institutional pressures of accountability systems did not influence her self-understanding or how she or her colleagues taught. Brooke was uncomfortable, even resistant, to framing her understanding of herself as a mathematics teacher as related to the institutional discourses of accountability or generalizing about the influences of these discourses on her practice.

This analysis is not meant to be an indictment against Brooke or a criticism of the lack of evidence of her discursive practices of being reflexive about her positioning. Across the seminar sessions, how Brooke was positioned by these pressures of high-stakes accountability contexts, in particular how she understood math abilities and students' responsibilities, framed her discussions of students, teaching, and herself, and she struggled to interrogate her positioning towards mathematics, the framing of abilities in mathematics, and how her school framed learning and progress. Brooke's participation highlights the challenges TCs have with working at the intersections of these dynamics and their self-understanding.

Candice: What can I *do*? Across Candice's participation, when discussing herself and her instruction, she identified how she and her students were positioned by discourses of test-based accountability and tensions of teaching in her school, where standardized assessments mattered more than what was learned from students' work. In her discursive participation she identified, specified, and responded to positioning as contextualized, positioning as relational, positioning in performance, and repositioning. For example, when discussing the standardized assessments and pressures of AYP, Candice identified her positioning as well as possibilities for repositioning. In positioning herself and her students in relation to discourses of test-based accountability, Candice asked, "What can I *do*?," seeking actionable steps to address her positioning.

This question is evident in her discursive participation in the above analytic episode and in her participation across sessions when she sought for actionable steps to address tensions of teaching in test-based accountability contexts. In her participation, Candice identified the contextual and relational elements of her positioning, as detailed in the analytic episode above, and she sought for how to engage in repositioning herself and her students. Across seminar sessions, Candice asked questions about supporting students' abilities, creating opportunities to learn, and contesting discourses of test-based accountability, her positioning, and her students' positioning. When Candice discussed grading students' mathematics homework and working with parents and her mentor teachers, for example, she attended to both the test-based accountability context and her goals of repositioning herself and her students with an eye for equity. Candice also actively listened to others during seminar sessions, consistently asking other TCs to share their teaching practice. Candice would ask clarifying questions about TCs' practices, such as, "How do you grade that?", "Do you do the same for students with IEPs?", and "What grade [level] do you start that?" In this manner, Candice sought to make connections between herself and what she could do in teaching in a test-driven context.

While Candice identified the ways in which there were challenges and constraints to teaching high-stakes accountability contexts, she did not cite the contextual challenges as excuses for teachers' lack of attention to rigorous mathematics teaching or suggest that the contextual challenges should be seen as limits to her agency, even as those challenges impeded her current teaching. For example, she emphasized in Session 2 that all teachers, specifically the teachers in the vignettes that were in high-stakes accountability contexts, should emphasize teaching higher-order thinking skills and include explorations of geometric shapes:

> I think teaching could be like that [including explorations of geometric shapes], especially if we allow them to explore, like show them different buildings and like and different things like that. Or give them those experiences like at museums or zoos or whatever. But I think a lot of times, teaching is just like the definition in isolation or the skill in isolation and not. Because we're talking about rigor at our school and how to allow them to create and synthesize and analyze, all those higher level thinking and like a majority of the things that we do are lower level thinking skills. Even though in the younger grades, it seems like we should be able to still allow the students to practice those, those higher order thinking skills. (Group discussion, May 11, 2010)

Candice emphasized that the way teachers teach, in addition to the content they include, is important for all students, and she critiqued the mathematics teaching of the teachers in the vignette who lowered the cognitive demand and rigor. She identified how she and the teachers had options in how they positioned their students in relation to the mathematics and emphasized providing students access to opportunities by including rigorous mathematical thinking. These comments emphasize how Candice felt that she and other teachers did have agency in repositioning themselves and engaging in ambitious mathematics teaching practices in high-stakes accountability contexts, even as she asked questions about specifically what those practices would look like in her mathematics instruction.

Laura: What can *I* do? Across the seminar, analysis of Laura's participation suggests that she also identified, specified, and responded to positioning as contextualized, positioning as relational, positioning in performance, and repositioning. While these elements surfaced in her participation, similar to Candice, analysis of Laura's participation and her positioning suggests that she struggled with understanding how to be the teacher she wanted to be within her teaching context. In this manner, while she engaged in similar discursive practices of positioning herself in relation to discourses of test-based accountability, Laura asked questions about her own readiness to engage in repositioning or to enact her teacher identity in her high-stakes accountability context. In this manner, Laura asked, "What can *I* do?"

In the analytic episode above, Laura's questions of personal readiness emerged when she identified that she did not feel that she would be able to engage in the same teaching practices in test-driven school contexts. In this manner, Laura understood the potential steps to take to teach across school contexts, but she did not see herself as being able to take those steps or specifically to *engage* in the practices of repositioning herself. A similar tension emerged in Session 2, when Laura discussed compromises she would make if she were to teach in different school environments.

> I think that, maybe we brought this up last time or maybe a little bit today, you have to make compromises because like, when we were walking around the school and then we came back like and everybody was kind of like, "Wow, look at how nice it is," and "They have these freedoms." They have this amazing arts program here, and like so many freedoms in what they teach, so much creativity I guess I'd say, but you know it's also a very privileged population here, and so we have to make decisions where maybe you'd want to work with a different population or where the school is much more regimented in what you do and so we have to make compromises. (Group discussion, April 27, 2010)

Laura's comments are consistent with her earlier description of the choice she felt she had to make. We held Session 2 in a newly renovated arts-based school, and Laura recognized how aesthetically pleasing the school is and suggested that the teachers at this school did not respond to the pressures of test-based accountability through the instructional practices that are prevalent in test-driven schools. In her comment, Laura associated schools where teachers have autonomy over what and how they teach with schools that serve students of privilege. She contrasted this environment with schools that have what she deemed characteristics of test-driven contexts—schools that serve a less-privileged population and that are racially and socioeconomically diverse. Tying teachers' autonomy over their practice to the teaching context and student population, Laura's self-understandings as a mathematics teacher were tightly anchored to teaching context.

Laura set up a contrast between (a) schools where teachers have more flexibility and "freedom" about what they teach and schools that serve students of privilege, and (b) schools that are test-driven, serve a less privileged population, and are also inferred to be racially and socioeconomically diverse. By making this link between practices and school population, she felt limited to teaching in schools that served students from higher socioeconomic backgrounds because these schools would also support progressive pedagogical practices and not be test-driven. Laura wanted autonomy over her teaching and opportunities to engage in practices that she deemed progressive, but she felt that these pedagogical practices were accepted only in schools that served students from higher socioeconomic backgrounds, not in schools that served students from lower socioeconomic backgrounds or underserved schools where she felt teachers were more needed. She was conflicted, however, about teaching in the more regimented school context where she would have a particular school population that she would like to serve or in the school context where she would have more freedoms. Across these data, Laura evidenced reflection on and awareness of the many elements of her positioning, specifically her context and relationships, but questioned her personal readiness in contesting it.

As this study and seminar were framed in relation to sociopolitical discourses of teaching, Laura's discursive participation suggests that through this attention to what is framing her positioning, more tensions about enacting this work in context emerged for her. The self-understanding and awareness that she evidenced may have helped her better understand how teaching context was a critical element of her practice and her positioning, but nevertheless, she remained uncertain about how to engage in the instructional practices that were important to her and were encouraged by her methods courses when she was teaching in her school context. Her responses are consistent with other research on teacher identity and self-understanding that suggests that social, political, and institutional discourses constrain and enable what teachers do, say, and even conceive of as appropriate (Britzman 1993; Walshaw 1999). Understanding herself as constrained by the discourses present in her context, Laura identified many tensions about teaching mathematics in high-stakes accountability contexts as related to her personal readiness.

Discussion and Conclusions

The TCs in this study experienced various tensions as related to their identities and self-understandings as teachers of mathematics in high-stakes accountability contexts. As related to teaching mathematics in high-stakes accountability contexts, TCs identified differing tensions about understandings of self, their identity, and their practice and were asking different questions about themselves, their agency, and instruction. These findings contribute to research on teacher identity, specifically on understandings of TC identity in high-stakes accountability contexts. This section details implications for mathematics teacher education and suggestions on how to support TCs in understanding themselves, their positioning, and their practice.

Research suggests that teacher education settings may be advantageous for supporting TCs' identity work and specifically their reflections on themselves and their practice as mathematics teachers (e.g., Ponte and Chapman 2008). Furthermore, "a teacher education program seems to be the ideal starting point for instilling not only the awareness of the need to develop an identity, but also a strong sense of the shifts that will occur in that identity" (Beauchamp and Thomas 2009, p. 186). In a review of mathematics education research on teacher identity, Ponte and Chapman (2008) asserted that teacher identity is seen as "a by-product of teacher education programs rather than as a targeted outcome" (p. 246), and therefore, more focused research on *how* identity develops and how teacher education can foster this is needed. Furthermore, most research on teacher identity development in teacher education programs was theoretical in nature, leaving the field without many details on how to foster teacher identity (Rodgers and Scott 2008).

The results of this study contribute to research on teacher identity by suggesting increased attention to teacher positioning and to TCs' understanding of their positioning when seeking to support TCs' conceptions of their identity and self-understanding. Specifically, TCs need to understand, on one hand, that how they position themselves as mathematics teachers and are positioned in their school contexts is shaped by social and political discourses about mathematics and teaching, and, on the other hand, that they can use this understanding to (re)author their positions toward mathematics teaching and learning and engage in ambitious practices. Research suggests that teachers are influenced by how students are labeled and understood by other teachers and administrators in their environment and also by the ways in which mathematics achievement is understood as related to correctness and motivation (Ellis 2008; Horn 2007). Teachers' conceptions of students are also reinforced through schools' curriculum pacing and ability grouping systems (Horn 2007). Mathematics teacher education can have a role in supporting TCs in understanding their positioning and navigating the sociopolitical discourses that interfere with their teaching, their relationships with students, and their engagement in ambitious practices.

In mathematics teacher education coursework, for example, discussing the different iterations of "What can I do?" may support TCs in making connections between self, students, and teaching contexts, particularly understanding that there are multiple dynamics and discourses in classrooms that influence their identity and multiple tensions that may be influencing their questions. Brooke, Candice, and Laura struggled with different questions and tensions in relation to their positioning as mathematics teachers in high-stakes accountability contexts. Distinguishing between these questions about agency and responsibility (What *can* I do?), about the steps one can take to enact ambitious pedagogical practices in high-stakes accountability contexts (What can I *do*?), and about feelings of personal readiness to teach mathematics in these contexts (What can *I* do?) may help TCs discuss, for example, their potential responsibilities for student learning or conflicts between their visions of themselves teaching and what they understand as the realities of their contexts. The process of distinguishing between TCs' different questions and concerns may also create opportunities for TCs to problematize practice and address others' concerns, understanding that they may be asking different questions (Pollock et al. 2010). It is also possible that prompting TCs to articulate the tensions they feel in their teaching contexts may highlight the ways in which their teaching practice both is and is not constrained by their context, as well as other reasons behind a TC's instructional practice. For example, a TC's own capacity for teaching in a certain manner, rather than the context of his or her teaching, may be limiting his or her practice or student engagement. Furthermore, research on teacher practice and teacher identity may benefit from understanding how sharing these iterations of these questions in mathematics methods course supports TC self-understanding and feelings of agency.

Given the tensions and dilemmas that TCs feel about their positioning, TCs may benefit from focused discussions of mathematics teaching as situated in teaching contexts and classrooms and within the multiple dynamics that constrict and construct elementary mathematics teaching, specifically in test-driven contexts. Opportunities in mathematics methods coursework for problematizing pressures specific to high-stakes accountability contexts, such as narrowed curricular content and multiple choice testing, may support TCs in discussing teaching practices and assessments in relation to ambitious teaching. Vignettes and case studies on complex issues in mathematics teaching could also support discussion in mathematics methods courses, specifically about high-stakes assessment pressures, category systems, and ability groupings that are prevalent in mathematics classrooms. To prepare TCs to enact ambitious mathematics teaching practices *while* navigating the many social, political, and institutional dynamics in mathematics classrooms and schools, TCs may benefit from both self-understanding and understanding of ambitious practices in context.

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Teachers' Learning Journeys Toward Reasoning and Sense Making

Lindsay M. Keazer

Recommendations for improving the nature of teaching and learning mathematics have been ongoing over the past 30 years. In the USA, recommendations can be traced back to 1980 with the National Council of Teachers of Mathematics (NCTM) publication of *An Agenda for Action,* followed by a series of standards documents (NCTM 1989, 1991, 1995, 2000) to clarify new goals and curricular recommendations. A recent iteration of recommendations for improving mathematics teaching focused specifically on the improvements needed in high school mathematics classrooms. *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009) (hereafter referred to as *Reasoning and Sense Making*) proposed that engaging high school students in reasoning and sense making about mathematics is critical for them to develop conceptual understanding and to recognize connections between concepts, rather than simply memorize procedures.

In the past, some have defined reasoning narrowly as an informal precursor to mathematical proof (e.g., Ellis 2007; Stylianides 2009, 2010), as in the "Reasoning and Proof" standard (NCTM 2000). In *Reasoning and Sense Making,* however, NCTM shifted its conceptualization of reasoning to a form of mathematical logic or thinking that is necessary in all mathematical activities (e.g., Bergqvist et al. 2008; Lithner 2008; NCTM 2009; Sternberg 1999). Sense making is defined as "developing understanding of a situation, context, or concept by connecting it with existing knowledge" (NCTM 2009, p. 4). The authors of *Reasoning and Sense Making* ar-

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gued that these practices are a significant part of what is missing from high school mathematics classrooms, as currently too many students associate mathematics learning with carrying out meaningless procedures.

While the focus of national recommendations for mathematics teaching has been clarified over time through standards and policy documents published by NCTM, evolving recommendations have maintained a common thread of the importance of student thinking and engagement in authentic mathematical activities. Despite a longstanding presence of policy recommendations suggesting ways to improve mathematics teaching and learning (e.g., NCTM 1980, 1989, 1991, 1995, 2000; NRC 1989), the lack of impact of these recommendations on mathematics teachers' practice is a common topic of discussion. Evaluations of mathematics teaching across the USA have identified a persistent deficit, or gap between recommendations for mathematics teaching and actual classroom practice (Hiebert 2013). Hiebert (1999), for example, declared that "the same method of teaching persists, even in the face of pressures to change" (p. 11). Similarly, the Conference Board of the Mathematical Sciences asserted back in 1975 that "teachers are essentially teaching the same way they were taught in school" (p. 77), referencing the lack of impact of the earlier "new math" movement of the 1960s.

Discussions of the deficit between recommendations and actual practice often use a framing that suggests that the blame is on teachers. Researchers have addressed how framing student learning using a deficit perspective is unproductive to improve students' learning (e.g., Brown et al. 2011; Gutiérrez 2008). Similarly, framing mathematics teaching in this way, while it helps to identify areas for improvement, needs to move beyond blaming teachers to focus on interrogating the factors associated with the deficit. Hiebert (2013) investigated reasons why mathematics teaching has historically been resistant to change. He argued that "believing that teaching can be learned and that such learning takes time and practice would dramatically alter the nature of schools" (p. 53). He concluded that efforts to improve teaching should focus on teacher learning, and the responsibility for teacher learning does not lie solely with teachers. In order to address the gap between recommendations for mathematics teaching and actual classroom practice, research is needed that investigates how to support teachers as learners in taking up and adopting new practices in line with recommendations for mathematics teaching.

One contributor to the deficit between recommendations and actual classroom practice is the complex transitional process involved for teachers learning to teach mathematics through pedagogical methods distinct from how they learned mathematics. Mathematics education researchers have described challenges they encountered when they studied their own teaching as they instigated changes (e.g., Ball 2000; Cady 2006; Chazan 2000; Heaton 2000). These struggles, expressed by researchers familiar with underlying theories of student learning, indicate the potential for practicing teachers to struggle even more with enacting recommendations. Changes proposed by reformers and policy recommendations have the underlying assumption "that teachers will change their world view of mathematics, mathematics teaching, and mathematics learning" (Shaw and Jakubowski 1991, p. 13). This shift in worldview is no simple transition. Different knowledge is needed in order to teach mathematics in pedagogically different ways. In short, the literature suggests that aligning one's practice with recommendations for mathematics teaching is a complex process.

In order to move beyond the deficit perspective, research is needed that frames mathematics teachers as learners and develops an understanding of their varied experiences attempting to change their teaching. The study shared in this chapter was designed for the purpose of understanding the experiences of mathematics teachers testing and enacting recommendations published through *Reasoning and Sense Making*. NCTM proposed these recommendations as timely and relevant to high school mathematics teachers across the country. The findings of this study capture both the complexity and variation of seven teachers' experiences, as well as the salient common features in teachers' learning journeys. An understanding of teachers' experiences is useful to mathematics teacher educators considering ways of supporting the learning of other teachers in enacting recommendations to change practice.

Theoretical Framework

Two constructs were important to the theoretical framework of this study: *experience* and *teacher change*. Dewey (1938) suggested that the study of experience should be at the forefront of educational research. Building off the ideas of Dewey, Clandinin and Connelly (2000) developed an approach to study experience, known as narrative inquiry. Clandinin and Connelly conceptualized experience as a collection of lived moments of struggle that, when pieced together, make up a phenomenon. It is this definition of the *experience* construct that underpins the study discussed in this chapter. Narrative inquiry is well suited for the study of experience, as experiences are shared through narratives that convey the complexity from the perspective of the participant. Narratives highlight teachers' voices, support them in sharing their experiences (Barone 2010) and allow an outsider to vicariously experience the phenomenon. Thus, narratives of experience are uniquely positioned to illuminate experiential factors from the teacher's perspective that contribute to our understanding of how to reduce the deficit between teaching recommendations and teachers' classroom practice.

The construct of *teacher change* is conceptualized here as changes in instructional moves that were mutually recognized by teacher and researcher. Measures of teacher change in mathematics education are often used to assess the impact of professional development experiences on teachers' instruction. According to Shaw and Jakubowski (1991), genuine change can come only from within each individual teacher. As such, I attempted to understand teachers' changes from their perspective. Teachers' own conceptualizations of *Reasoning and Sense Making* undoubtedly impact what changes they choose to make. I relied on teachers to describe their actions taken to align their practice with *Reasoning and Sense Making* and to identify the changes they made. I used this information, along with my own developing understanding of their experience obtained through collaboration and observations of their teaching, to co-construct an understanding of their changes.

I drew on Shaw and Jakubowski's (1991) theory of six phases of teacher's change as a lens through which to make sense of teachers' experiences making changes. Past research on mathematics teacher change has measured changes in practice along continuums or stages that gauge the degree to which teachers' instructional practices adhere to preconceived change objectives (e.g., Fennema et al. 1996). Shaw and Jakubowski's phases, however, are not stages of change in a particular direction, such as states of instructional practice that align with particular teaching strategies. Rather, these six phases describe the process a teacher moves through while navigating autonomous change, driven by the teacher's own goals. The first phase of change requires teachers to be mentally provoked to realize a reason for change, or experience a perturbation. Second, teachers make a commitment to the change. Third, teachers reenvision their practice to include the change. Fourth, teachers project themselves into the image of the changed classroom, envisioning themselves enacting their vision. Fifth, teachers decide to take action to begin to enact the change in their practice. Finally, in the sixth phase, teachers continuously reflect on their practice, comparing it to their vision. A teacher's progress in making changes is contingent upon successfully navigating each phase of this complex change process. Collaboration with other teachers is supportive of teachers' progress through the six phases (Shaw and Jakubowski 1991). Later in this chapter, I discuss what these phases looked like within the varied journeys of teachers in this study.

The constructs of experience and teacher change were integral to the framework of this study, as the purpose was to understand change in ways that highlighted teachers' experience, or daily moments of struggle as they attempted to test and align their practice with the recommendations of *Reasoning and Sense Making*. The ideas behind the national recommendations were presented to teachers through reading and discussing the document itself. The purpose of this study revolved around understanding teachers' changes to align their practice with *Reasoning and Sense Making,* through the lens of the individual teacher. Thus, teachers' changes were each directed by their individual conceptualizations of reasoning and sense making. Their conceptualizations were articulated through their own definitions and illuminated by their actions and reflections on their teaching. In the following section I examine past research related to mathematics teachers' reading and responding to NCTM recommendations. I examine this literature through the lens of my theoretical framework and build a case for the importance of this study.

What Is Currently Known of Teachers Engaging with NCTM Recommendations

The inherent challenges of supporting teachers to understand and take up recommendations from initiatives to improve mathematics teaching has been recognized, and these "problems of dissemination and implementation…loom large" (Howson et al. 1981, p. 9). The challenges involve both disseminating information and supporting teachers in reenvisioning their practice. *Reasoning and Sense Making* represents a recent iteration of reform recommendations disseminated through NCTM, and the first that has narrowed its focus to specifically target high school mathematics. Although research has not yet explored teachers' experiences interpreting these particular recommendations or testing them in their teaching, studies have examined teachers' interactions with other NCTM policy documents. Previous iterations of NCTM recommendations (1989, 1991, 1995, 2000) were considered to encompass the *NCTM Standards*. *Reasoning and Sense Making* stems from a similar perspective on mathematics teaching and learning as the *NCTM Standards,* and thus a review of this research informs this study.

After the release of the Curriculum and Evaluation Standards for School Mathematics (NCTM 1989) and the Professional Standards for Teaching Mathematics (NCTM 1991), the 2-year project, Recognizing and Recording Reform in Mathematics Education (R^3M) was formed to monitor and describe the national impact of these standards (Ferrini-Mundy and Johnson 1994). Their investigation into various schools attempting to put these standards into practice found that the context of the district, school, and classroom strongly impacted interpretation of the standards and the emphasis placed on their implementation (Ferrini-Mundy and Johnson 1997). Teachers' perceptions of the recommendations, the collaborations available, and the support in their schools all impacted the versions of the recommendations they implemented. The study described the process of change as "slow and arduous" (p. 111) and found that teachers' confidence as well as their dispositions toward change were connected to their tendency to make changes to their practice.

Other studies have taken a closer look at the impact of reform in smaller groups of teachers studied in specific contexts. Brosnan, Edwards, and Erickson (1996) examined four sixth-grade teachers' changes in beliefs and practice as they attempted to align their teaching with *NCTM Standards* (in this case, NCTM 1989, 1991). After 2 years of collaboration and support, researchers suggested that changes had occurred in teachers' beliefs about the use of classtime and in the ways they focused their lessons.

Examples of change in secondary classrooms are much less prevalent than those in elementary classrooms. Frykholm (1999) investigated how secondary mathematics student teachers interpreted and took up recommendations from *NCTM Standards* (1989, 1991, 1995) into their teaching practice during their student-teaching internship. Student teachers in Frykholm's study were completing a teacher education program that was designed around the goals and recommendations of these documents. Findings showed that student teachers developed an understanding of the ideas expressed in the standards and became fluent in talking about them to colleagues and administrators. They conceptualized the recommendations as a fixed set of rules to be rigidly followed, yet observations of their teaching revealed little evidence of these objectives in their teaching. Only 11 % of their lessons showed any deviation from a traditional direct instruction format. These findings further underscore the challenge of supporting teachers to enact recommendations in a sustainable way. Using the framework of Shaw and Jakubowski's phases of change as a lens through which to view Frykholm's findings, it appears that while student teachers developed an understanding of the recommendations, they did not experience a perturbation to see a need for change: The necessary first phase of sustainable change.

Berk (2004) investigated 14 middle school mathematics teachers' impressions and interpretations of the *Principles and Standards for School Mathematics* (NCTM 2000) through a study group. Over half of the teachers had taught for 8 or more years. Teachers understood the document as "living" and flexible, rather than containing fixed knowledge, and conceived it could lead different people to envision their curriculum in different ways. Berk found that teachers engaged actively in extensive discussions about the document in study group meetings and continued their enthusiasm in thinking about their work outside of meetings. Teachers expressed that this document was a useful tool for examining their curriculum and spurring their learning. The scope of Berk's study did not include following teachers into their classrooms to assess impact on teachers' classroom practice. As a result, we do not know whether they moved through any phases of the change process (Shaw and Jakubowski 1991). The fact that teachers identified the document as a tool to foster their learning, however, suggests that reading the recommendations may have mentally provoked them to recognize areas in which they wanted to change, thus fostering their entrance into the first phase of change. Teachers' conceptualizations of the recommendations as flexible, rather than fixed, also suggests the potential for any changes fostered to be autonomous and teacher-directed.

These studies offer different pictures of the ways teachers conceptualized the knowledge embodied by the recommendations. While student teachers (Frykholm 1999) conceptualized the recommendations as a fixed set of rules, teachers in Berk's (2004) study viewed them as fluid and flexible. It is possible that these differences were a product of the structure of participants' learning environments, with teacher education programs likely involving more structure and authority than a teacher study group. Differences also could be influenced by the ways in which facilitators presented the recommendations to the teachers. Stenhouse (1975) suggested that reform proposals be presented to teachers such that "the crucial point is that the proposal is not to be regarded as an unqualified recommendation but rather as a provisional specification claiming no more than to be worth putting to the test of practice" (p. 142). Teachers in the studies conducted by Berk (2004) and Brosnan et al. (1996) took a flexible orientation toward implementing the recommendations, while student teachers did not come to realize this freedom of knowledge (Frykholm 1999). This flexibility in orientation may have contributed to an increased impact on teachers' practice.

One limitation of the study conducted by Berk (2004) is that it did not explore the impact on teachers' practices. While teachers talked enthusiastically about integrating the recommendations into their practice, we cannot be sure that the reality of their teaching practice would not reflect the same lack of impact that Frykholm (1999) observed in student teachers' classroom practice. However, Berk's indications of the ways that teachers had begun to reenvision their practice suggested entrance into the early phases of change (Shaw and Jakubowski 1991) in ways that the student teachers had not. Although these studies highlight teachers' perceptions of reform recommendations and outcomes, an aspect that remains untold in these studies is that of the *experiences,* the ongoing lived struggles and successes that occur within the work of these teachers as they attempt to make changes in their practice to align with recommendations. The findings of this study are timely and relevant to the current recommendations for high school mathematics and illustrate teachers moving through various stages of the change process.

The purpose of this study was to develop an understanding of the experiences of mathematics teachers investigating the recommendations of *Reasoning and Sense Making* (NCTM 2009) and attempting to incorporate them into their practice. The overarching research question guiding this study was: What are the experiences of mathematics teachers as they examine recommendations from *Reasoning and Sense Making* and test them in their teaching?

Research Methods

Narrative epistemologies have emerged in response to positivistic paradigms for doing research in the social sciences, embracing the view that humans lead storied lives and that knowledge is contained and shared through narratives (Clandinin and Rosiek 2007). The methods of narrative inquiry are closely linked to the epistemology, with the aim to understand and illustrate a phenomenon under study through the gathering of stories and the construction of narratives of experience. While holding similarities to other forms of qualitative research, such as ethnography and phenomenology, the key differences are rooted in the epistemology, where a significance is afforded to stories as a form of knowledge, and the potential for narratives to convey a deep understanding of experience. The narrative researcher participates in close collaboration with participants, attempting to gather their stories and construct a narrative that retains their voice and gives power to their story through the creation of a research text. The narratives produced are believed to hold the potential to convey the meaning of the experience such that a reader can experience it vicariously (Clandinin and Connelly 2000). I explored the experiences of teachers through collaborating with them as a participant–observer, using various sources of data to construct narratives of their experience, and testing and revising narratives with teachers' feedback.

Study Context

A group of seven high school mathematics teachers was obtained by recruiting teachers who expressed an interest in reading recommendations for mathematics teaching and putting aspects of those recommendations to the test of their practice (Stenhouse 1975). The teachers worked in six different schools and had 0–11 (mean 3.5) years of teaching experience. Three teachers were men and four were women.

Six teachers were Caucasian and one was of an Indian descent. Teachers were given the option of receiving graduate credits in compensation for their participation, which six of the seven teachers chose to do because they were working on their master's degrees or getting credits for continuing teacher licensure.

Teacher action research (AR) was used to support teachers' learning through the change process. AR is conceptualized as a self-critical inquiry into one's practice in order to improve it (Carr and Kemmis 1986). AR aligns with recommendations for sustainable professional development (e.g., Clarke 1994; Darling-Hammond and McLaughlin 1995; Elmore 2002) and facilitates teacher-driven changes. The process of AR is depicted as a spiral or series of cycles, with phases of investigation, planning, taking action, monitoring, and evaluation (Kemmis 1988). The process of AR also aligns well with the phases of teacher change (Shaw and Jakubowski 1991). Although the five phases of AR do not hold a one-to-one correspondence with the phases of teacher change, both describe an ongoing process of autonomous action and reflection. For example, phase three of the AR cycle and phase five of the theory of teacher change both encapsulate a teacher adopting new actions in their practice, followed by phases for reflection and evaluation, which often lead to additional actions.

I attempted an equitable collaboration (Feldman 1993) as I served in the roles of both researcher and facilitator. I attended to issues of power inherent in my relationship with the teacher participants, issues characterized by a researcher–teacher dichotomy as well as by the university-based versus K–12 settings of our work. I asked teachers to share their desires regarding the meeting schedule and agenda and to negotiate them with me based on their needs and interests. We met every 3 weeks from October to May during the 2010–2011 school year, a total of nine times. In initial meetings, teachers discussed *Reasoning and Sense Making* and considered areas of their teaching in which they felt they should incorporate these recommendations. Based on their self-recognized teaching weaknesses and the aspects of *Reasoning and Sense Making* that held meaning for them, they each selected actions to take in their practice. The elements they chose to take up and test in their teaching included improving the structure of the questions they asked, selecting activities that would engage students in more reasoning, changing their role in class discussion, prompting students to justify their mathematical ideas, and incorporating writing into assessments.

Teachers were introduced informally to the methods of AR through PowerPoint presentations, excerpts from methods handbooks, and narrative examples of AR (e.g., Gronewold 2009; Robinson 2006). I created a library of practitioner readings following the method Herbel-Eisenmann and Cirillo (2009) used in their teacher collaboration. Because readings that discussed the intertwined ideas of "reasoning and sense making" as conceptualized by NCTM (2009) were limited due to the newness of the document, I used the teachers' interpretations of these ideas that they vocalized at group meetings to select related readings for this library (e.g., Eggleton and Moldavan 2001; Thompson et al. 1994; Umbeck 2011). I continuously added to this library over the course of the collaboration as I clarified my understanding of teachers' conceptualizations of reasoning and sense making. Teachers' collective choices of readings evolved over time as they progressed through the phases of their AR and developed the strategies they were enacting to foster students' reasoning and sense making in their teaching. Meetings served as a time for teachers to process their understandings of readings and to share their goals, challenges, and successes.

Data Collection

I collected a variety of data sources, known as *field texts* (Clandinin and Connelly 2000), to generate a holistic understanding of teachers' experiences. The field texts that informed this analysis were transcripts of teachers' discussions, teacher interviews, teachers' written reflections, observations of classroom teaching, curricular documents provided by teachers, and my own research journal.

I audio-recorded teacher discussions to capture conversations about readings, goals, and classroom concerns. I conducted semistructured interviews at the beginning of the study to learn about teachers' past experiences as a teacher of mathematics. Interviews lasted 30–60 min, depending on how freely the teachers responded to the open-ended questions. I also interviewed them at the end of the study to ask each participant to reflect on the experience of attempting change. Teachers wrote reflections prior to each meeting, which were an important data source to capture their experience as they progressed through the journey of making changes. For each reflection, I provided open-ended journal prompts to facilitate them in reflecting on their current phase of their AR. For example, when teachers were at the stage of planning the actions they would take in their classroom, they received the following prompts:

- Describe your action strategy, or what you will do and why. What challenges do you anticipate?
- How will you assess if your actions have any impact on your students?
- What sort of data will you collect in order to determine the impact?

In addition to the prompts, teachers were encouraged to journal about anything they were thinking about or noticing that seemed relevant to their AR.

Observations of classroom teaching were conducted 3–4 times for each teacher, at times when each teacher invited me to visit. I asked them to select times for me to observe their teaching that would be illustrative of the actions they were describing in their reflections. Some teachers targeted a particular course for their AR, while some focused more generally on all their courses. In addition to being observed, teachers provided copies of relevant curricular documents such as lesson plans and worksheets they felt illustrated changes they were implementing. Additionally, my own research journal was a source of data, as this was where I recorded my field notes and reflections, monitored my role as a researcher and facilitator, and journaled about my "Subjective I's" (Peshkin 1988).

Data Analysis

The individual field texts consisted of snapshots of each teacher's experience at particular times throughout his or her experience. I continuously reviewed the field texts as I collected them. This continual review informed my decision making as facilitator. For instance, prior to entering a teacher's classroom, I reviewed all field texts pertaining to that teacher. This review developed my understanding of the teacher's goals for enacting reasoning and sense making in his or her teaching, and impacted the aspects of the teaching on which I focused my observations. I attempted to observe the teaching through the lens of what he or she valued and was trying to improve, in addition to viewing it through my own theoretical lens.

After all data were collected, all excerpts of field texts pertaining to a particular teacher were organized chronologically into a spreadsheet. I reviewed each spreadsheet and divided the data into six categorical bins that originated from initial research subquestions: contextual information, conceptions of reasoning and sense making, changes made in their teaching, challenges, opportunities, and the teachers' interpretations of their AR. This categorical scheme served the purpose of organizing the large quantity of data into manageable sections with a common focus, so that I could review it and look for recurring and connecting ideas.

The method of analysis in narrative analysis is termed *emplotment and narrative configuration* and involved synthesizing the field texts together to "develop or discover a plot" of the experience (Polkinghorne 1995, p. 15). Through this process, data were tested against each other among the multiple sources, similar to the traditional idea of triangulation. Narrative analysis differs from traditional qualitative research, however, in that it requires synthesizing the data rather than separating them into distinct parts. The data in each category were continuously reviewed until recurring ideas and connections emerged to synthesize the information into the plot of their experience. The process of writing involved repeatedly experimenting with writing *interim texts* (Clandinin and Connelly 2000) or smaller drafts of each teacher's experiences, with the goal to combine together individual field texts revealing different aspects of the experience, to create a holistic narrative. I relied heavily on teachers' own quotes and descriptions, to keep the teachers' own perspectives as central in the narratives. At three different stages, I shared interim texts with the teachers, presenting them as my understanding, and asking them for revisions to improve my understanding. Teachers appreciated these opportunities to review what they had said and done and saw them as an opportunity for feedback on their AR. They provided minimal revisions to align these interim texts with their understandings. Through repeated experimentation with the writing process while writing interim texts, comparing them to the data, and getting teacher feedback, I eventually produced the final research texts. The teachers' narratives told a unique story with respect to their past experiences, the actions they selected to take in their practice, and their struggles and reflections as they began to engage in the change process.

These narratives convey an understanding of the process of change in practice that occurred as a result of teachers' efforts to test aspects of *Reasoning and Sense Making* in their practice.

Through this analysis, I became aware of the complexity of each teacher's journey of experience. I use the term *journey* to capture the ongoing, teacher-directed nature of their experiences enacting changes across the time period of the study. At various times the teachers described the potholes, wrong turns, and rerouting their journeys involved. The uniqueness of each teacher's past experiences led to variation in the changes each one made in response to the recommendations. As I engaged in the process of analysis, subtle similarities became evident among the teachers' complex, unique journeys. To clarify my understanding of these similarities, I continuously compared plotlines of the developing narratives, examining the differences across them. As I read and reread my data, I tested different grouping schemes in my own sense-making process of understanding the connections between teachers' narratives.

As I made sense of the similarities and differences in teachers' journeys, I saw connections between teachers' experiences. Looking for ways to group teachers according to their experiences led me to make three pairs of teachers, grouped according to their similarities with one additional teacher whose experience did not fit into any of the groups. By comparing and contrasting these groups I generated four analogies that conveyed the similarities and differences among their journeys: a linear function, a piecewise function, a step function, and a scatterplot. I chose to use mathematical analogies to convey the variation among teachers' journeys because mathematics is a language that I use to make sense of the world. I do not use mathematics in its traditional form, to claim precise mathematical precision of measurement. Neither are they an attempt to measure teachers' alignment with recommendations, but to convey the meaning of teachers' challenges and successes from their perspective as they tested and accommodated recommendations in their practice.

The independent variable in these analogies of a mathematical relationship represents time in the collaboration; the dependent variable represents degree of alignment with the recommendations for *Reasoning and Sense Making*. Evolutions, or changes in any direction, that occurred in teachers' strategies are recognized through teachers' interpretations of their progress to align with recommendations, in conjunction with my observations of their teaching. These mathematical relationships illustrate teachers' experiences over the 7 months of this study, but are neither representative of teachers' previous journeys nor predictive of their future journeys. Experiences, as defined in this study, are best expressed through narratives and are impossible to truly mathematize. The strength of these analogies, rather, lies not in their capability to cleanly model an individual's journey of experience, but to convey the variation and complexity of teachers' experiences through the phases of change. These four analogies are most effective when considered together as representing distinct journeys resulting from teachers' common attempts to align their practice with *Reasoning and Sense Making*.

Findings and Discussion

Here I present condensed descriptions of teachers' journeys, summarized by connecting them to an analogy of a mathematical relationship. The narratives and accompanying analogies frame mathematics teachers as learners and illuminate the complexity of the experiences of teachers testing reform recommendations in their teaching. The four analogies introduced here are organized from the least to the most complex, beginning with teachers who were able to successfully navigate the phases of change (Shaw and Jakubowski 1991) and inherent challenges with little obstacles to their progress, to teachers whose progress involved more complexity in the form of impediments, false starts, and progressive development. An understanding of these teachers' experiences provides a valuable background for considering future ways of supporting teacher learning.

A Linear Journey

Two teachers, Peter and Alexis, entered the collaboration having already problematized many aspects of mathematics teaching that *Reasoning and Sense Making* sought to change. Both talked openly in our initial meetings about the problematic consequences of teaching mathematics through providing a list of procedures, consequences they had observed firsthand in their classrooms. Peter used humor to relate the negative effects of students' reliance on procedures or the teacher's authority, such as the following example he offered of students' lack of reasoning and sense making.

> I really want my students to start critically thinking. I swear that I could say, "Your lesson today is to learn that $5+8=22$" and they will just write $5+8=22$, and not even think a thing about what they're actually writing, whether it even makes sense at all.

Peter felt that his students had learned to rely on procedures and the teacher's authority, and that they were not accustomed to engaging in reasoning and sense making. He talked often about how "we're fighting a decade's worth of ingrained math," after seeing indications that his students were well practiced with years of experience learning mathematics *without* reasoning. As Peter and Alexis each read *Reasoning and Sense Making,* they agreed wholeheartedly with the proposition of the document that teaching mathematics through steps and procedures did not produce positive student learning outcomes.

Peter (in his eleventh year of teaching) and Alexis (in her third) both began the collaboration having already experienced a perturbation that made them realize the need to shift their practice toward reasoning and sense making. As a result of reflections on their past experiences, each had realized that allowing students to do math by following steps or procedures was not good for students and not an effective instructional strategy. They talked about specific ways that their role as teacher impacted students' opportunities to engage in reasoning and sense making. Both agreed with the philosophy of *Reasoning and Sense Making* and could provide

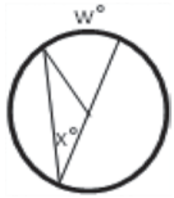

Fig. 1 A figure Peter posed to foster students' reasoning about angle relationships

examples of ways they had already had success teaching certain lessons by engaging students in reasoning and sense making. Despite these examples, upon reading *Reasoning and Sense Making* they both seemed convinced that more of their instructional practices needed revision in order to successfully achieve the vision the recommendations offered.

Their goals for change varied slightly, but both focused on teaching through discussion. Peter's goal was to improve the teaching practice of posing tasks and facilitating class discussion about those tasks. He wanted students to reason and make sense rather than rely on his hints or his evaluation of their ideas. He focused on changing his role during discussions, monitoring his words and actions in response to students' ideas. He developed new habits he described as "keeping silent"—not answering questions but instead "firing them back at the class" and "going along with wrong ideas." Alexis, meanwhile, wanted to redesign lessons that she had previously taught through direct instruction. She summarized her AR focus as follows:

> My big goal was trying to replace lecture with discussion, with activity, with the students being more active participants in class rather than just listeners. So I did things like spent more time preparing warm-ups and activities for class, that had them work in groups and had them discussing with people around them. I had them discovering ideas on their own, rather than me giving them procedures, trying to get them to make connections between concepts.

Peter and Alexis talked about the internal struggle they felt, at times, trying not to revert back to their traditional roles in the classroom. They each discussed instances of frustration with students' resistance to their efforts to foster reasoning. Alexis shared, "I can't tell you how many times I'd say, 'Do you have a question?' and somebody would raise their hand and say (*changing to a more assertive tone*) 'Yea, how do you do it?'" Through these struggles, however, they were confident of the importance of their efforts and resisted the temptation to revert back to their prior ways. They seemed surprisingly comfortable with the uncertainty of changing their practice and seemed to find the challenge of looking for new ways of fostering reasoning to be exciting and enjoyable.

Peter talked about the enjoyment of the mathematical challenge of selecting a geometric figure, such as the one pictured in Fig. 1, and posing questions that would support the class in making and justifying conjectures to reach a specific mathematical objective. In this instance, he drew the figure on the board and asked students to comment on what they knew to be true. He found it intellectually stimulating to solicit all students' mathematical ideas, being challenged to think about how to respond with questions that did not give hints, but helped students refine or justify

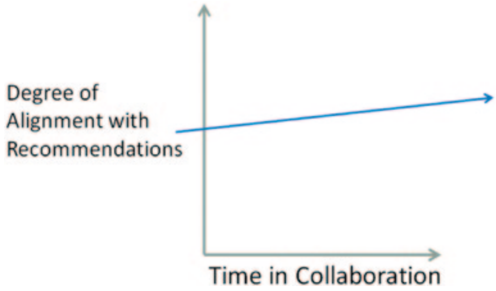

Fig. 2 The analogy of the linear function journey

their conjectures. He particularly monitored his ways of giving feedback and tried not to give any indications of the correct strategy, which he felt would take away some of the mathematical reasoning from the students.

Alexis taught a semester-long remedial algebra course and shared her enthusiasm about her second semester students' growing capabilities to reason, which she saw as a result of applying her new strategies from day one during the second semester.

> When I started that way at the beginning of the semester, it was so much easier…. I don't have to push as hard. Like with the slope [lesson], I didn't have to say much at all. I put the two ordered pairs on the board and said, "All right, so how could I use these ordered pairs to find a slope?" And when I had two or three students that were able to put that formula together by themselves, I was a little shocked. I thought I was going to have to guide them a little more.

Peter and Alexis each recognized progress and positive results from their persistence. Peter enthusiastically shared, "It's to the point now where they will explore an idea for 10–15 min. It's really neat!" Although Peter and Alexis each discussed challenges, their process of changing their practice seemed relatively smooth compared to the other teachers. As they tested out changes in their practice, their reflections on their changes continued to move them forward in a productive way. Thus, to illustrate the similarities among their journeys, and the ways their journeys were distinct from others, I use the analogy of a linear function (Fig. 2).

Shaw and Jakubowski's (1991) theory of teacher change can be used as a lens through which to examine Peter and Alexis's experience. It appears that prior to entering our collaboration, both had experienced some sort of perturbation (phase 1), as they both saw a strong need to change their practice to engage students in reasoning and sense making. They both began their AR with a strong commitment to the change (phase 2). They were eager to engage in reenvisioning their practice, projecting themselves into their vision, and taking action to enact their vision (phases three, four, and five) shortly after they read *Reasoning and Sense Making* and began discussing their ideas. Throughout the remainder of the collaboration, Peter and Alexis seemed to be continuously engaged in cycles of action and reflection to compare their vision to their practice (phase 6). The analogy of a linear function is similar to the journey of their experiences, navigating through the challenging transition of change with confidence and persistence, yielding smooth progress toward their vision for enacting *Reasoning and Sense Making.*

A Piecewise Journey

Teachers Logan and Melinda joined the collaboration hoping it would broaden their awareness of ways to improve their teaching. Unlike Peter and Alexis, they had not conceptualized specific ways they had already worked on changing their teaching to increase students' engagement in reasoning and sense making, and they were initially unsure how to begin. Logan's 3 years of teaching experience had been primarily lecture-based, and he wanted to incorporate more discussion into his lectures. Melinda had 7 years of teaching experience and had already incorporated many student-centered activities into her teaching over the years, for the purpose of making her lessons "less boring" and keeping her freshman students moving around. She had not necessarily attended to the quality of students' reasoning or sense making when selecting activities. Both teachers clarified their goals for change over the course of several weeks, while monitoring their practice and considering options. Logan eventually settled on the goal of taking "baby steps" to incorporate discussion into his existing lectures, and Melinda worked on finding and incorporating more student-centered learning activities, with an eye on fostering students' reasoning and sense making.

Both Logan and Melinda shared at group meetings that they were seeing some initial success and were satisfied with the changes they were making. Later, however, at different times of the year, they each experienced periods of frustration where they felt students were responding to their questions with silence. When their best efforts were met with resistance, they became discouraged and wondered if some of their students were capable of engaging in reasoning and sense making. This challenge was experienced differently than those encountered by Peter and Alexis, as it resulted in Logan and Melinda each questioning their goals and reducing their efforts to enact their action strategies. Rather than making more pointed attempts to counter students' resistance, their reflections revealed a growing tone of frustration and negativity, and they considered giving up because the obstacle seemed insurmountable. Melinda described her students' dependency on her to do the reasoning and sense making for them:

> I think now that I'm concentrating on trying to get my kids to think for themselves, I'm realizing more and more how they can't do that. Like it used to be, they would come up and be like, "Oh, I don't know how to do this problem," and I'd be like, "Oh! Let me show you." And I would grab my pencil and I would work it out for them and they would watch. And now they're like, "I don't know how to do it." So I'm like, "Well, what do you think you should do?" and then…oh, my god, they just don't know anything.

Melinda was frustrated with students' desire for her to do all of the work for them, although she recognized her own compliance in fostering this dependency through her former methods of "helping" them. Now, as she attempted to foster their independence in mathematical reasoning and sense making by turning their questions back at them, students responded with a helplessness that left her irritated and at a loss for how to respond. Similarly, Logan described his challenge of not knowing how to spark students' independence in reasoning and sense making:

> I think I have sort of hit a rough spot in the process. I am not sure if it is my teaching or if it is the students…. Since most of my focus has been on class discussions and questioning, it is pretty easy for me to monitor how things are going. I still find that I am forced to lead the students way too much. If I don't lead them, then they will literally become silent multiple times during my lectures. I have tried a few different things, but it has been tough.

Both Logan and Melinda reached a point where they felt like their efforts to engage students in reasoning and sense making had limited impact. This frustration and discouragement caused them to second-guess their goals and diminished their motivation.

During this time, I observed Melinda teach a lesson on dimensional analysis to her Algebra I students, where she asked many questions in an attempt to facilitate a discussion, but students primarily responded with silence and nonengagement. I contemplated how I might help her recognize some of the ways that her questioning could be more open-ended to allow students more entry points into the discussion. After teaching the first lesson, she mentioned that she would be teaching the same lesson five more times and asked if I would be willing to teach the lesson to the next class walking in. She said that she grew tired of teaching the same lesson all day long, and suggested that may be I would like the opportunity to be a classroom teacher again. This proved to be a valuable opportunity to help her realize the impact of different questioning patterns. After watching me teach her same lesson and noticing the ways her students responded to my questions, she said, "I thought the problem was that my students couldn't reason. But now I see that I was just asking the wrong questions." After that episode, when I visited her classroom on subsequent observations, I was surprised by the dramatic differences in the questioning that Melinda used. Rather than questioning patterns that resembled those described as "funneling" (Wood 1998), her questioning changed to resemble more closely the pattern described as "focusing" (Wood 1998). For example, previous questions had directed students toward a particular procedure she had in mind, such as "Which fraction should we use? What if we use this one? Can we cross anything out?" Her new questioning patterns were more open to allow students to determine their own solution paths, such as "How can you find the side length of a square with an area of five?" and, when solving equations, "Steve subtracted and then divided. Do we have to do it in that order? Why or why not?" The following school year after our collaboration ended, Melinda continued to email me to share ongoing successes she saw as a result of long-term use of her new questioning strategies.

A similar pattern of struggle and resulting reenvisioned practice happened in Logan's AR journey. He became discouraged for several months during the spring semester and began to wonder if the juniors and seniors in his lower-level Algebra II courses were capable of reasoning. As I observed this happening and saw Logan starting to give up, I wondered how I could help him see that his students *could* reason, but perhaps needed to be asked different questions and held accountable for engaging in reasoning. Remembering the impact of Melinda observing me teach a lesson with her students, I asked Logan if he would allow me to teach a lesson in his classroom. He agreed, and I began to plan a lesson about the graphs and equations of ellipses (Kysh et al. 2009) that I hoped would serve as an "existence proof" that these students *could* reason mathematically. The following is an excerpt of his reflection on observing me teach that lesson:

> When watching Lindsay teach my class, I noticed how she was able to get everyone involved. She was calling on students who had not volunteered to share an idea in months. I have made a point to call on each and every student in my class since then. I also do not let students get away with just saying, "I don't know." They were actually saying, "I don't want to think right now," so I have to make them tell me something that they do know.

For both Logan and Melinda, observing someone else teach a lesson to their students served as one form of perturbation to jolt them into a new awareness of ways their questioning strategies impacted student engagement in reasoning and sense making. They not only saw proof that their students were willing and able to engage in reasoning and sense making, but they each identified particular strategies they wanted to adopt in order to address the challenges they had been experiencing.

Analyzing Logan and Melinda's experiences through the lens of Shaw and Jakubowski's phases of change, it seemed that although reading and discussing *Reasoning and Sense Making* piqued their interest, that alone was not enough to support them in navigating through all of the phases of change. It appeared that they had initially made a commitment to making a change (phase 2), reenvisioned their practice (phase 3), projected themselves into the image of their changed classroom (phase 4), and taken action (phase 5). They began to stall during phase 6, however, the phase in which teachers continuously compare their vision to their practice in order to further develop their actions to align their practice with their vision. One of Logan's reflections captured his discouragement over the gap he saw between his vision and actual classroom practice:

> I have really found all of this to be quite difficult lately. Through all of our readings, reflections, meetings, feedback, etc., I really know what I want to see happening in my classroom. Unfortunately my Algebra II Intro class is not where I would like them to be. There are several kids who have made great strides, but it is always a struggle trying to get all of the students to reason. The process has been great in reminding me that I am not where I want to be. It has been a humbling experience that is hopefully making me a better teacher.

At this point, Logan was discouraged because he lacked the knowledge or awareness of what strategies he could use to address the gap between his vision and his practice. When Logan and Melinda each reached this point after having made efforts at change, they felt disappointment in their perceived lack of success. The opportunity to observe me teach a lesson, however, raised their awareness of additional strategies to foster students' reasoning and sense making and spurred new motivation and persistence after seeing students engage in ways they hadnot previously seen. This experience served as a form of perturbation that prompted them to renew their commitment to change (phase 2), reenvision their practice given their new knowledge (phase 3), project themselves into a new classroom vision (phase 4), and take actions that were a revision of their previous actions (phase 5).

Logan and Melinda were engaged in phase 6, continuously comparing their vision with their practice, when the study ended. Each had overcome their obstacles to varying degrees. After Melinda realized weaknesses in her questioning strategies, there were several months left in the school year in which she invited me back to

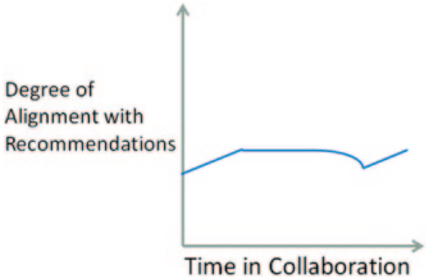

Fig. 3 The analogy of the piecewise function journey

observe her new strategies. I observed her successfully engaging many different students in reasoning through either small group or large group work. Logan, on the other hand, talked openly and critically about how the experience had left him dissatisfied with his practice:

> I think I changed more than the students did during the school year through my action research. More than anything, I became more aware about myself and how I was teaching. I found myself not giving enough wait time, not making everyone get involved, and not making all of the students reason in my math class.

Logan's realization happened near the end of the collaboration, and there were no more opportunities for observation to see how his practice was impacted by this realization. However, he described new action strategies of providing opportunities for students to discuss ideas with their peers, moving around the room more, calling on every student, and refusing to accept "I don't know" as an answer. He also shared a picture of his newly arranged classroom, with desks organized in pairs of rows to facilitate easier student-to-student discussion.

To illustrate the similarities between Logan's and Melinda's journeys, I draw on the analogy of a piecewise function (Fig. 3). Although Logan and Melinda initially saw short-term improvement in their students' engagement in reasoning and sense making, both later experienced a plateau, followed by a decrease in their efforts to align with recommendations of *Reasoning and Sense Making*. They overcame the obstacle to varying degrees when they became aware of ways they could improve their questioning strategies and subsequently increase students' engagement in reasoning. A new awareness of their teaching prompted them to plan further actions to support student's reasoning and sense making.

While Logan's and Melinda's journeys were piecewise in nature, one could argue that a more precise attempt to mathematize their journeys would yield two different graphs. Logan's plateau seemed to last longer than Melinda's, and his downward curve seemed to dip lower as he backtracked in his attempts at changing strategies. Additionally, the linear pieces could be drawn with different slopes—Melinda's steeper and longer in the positive direction at the end. The purpose of this study, however, was not to claim precise measures of teachers' changes made, but to convey the experience of undergoing the process of making changes. This piecewise function is productive in illustrating the pattern of their journeys, and in setting their journeys apart from those of the other teachers.

A Step Function Journey

Sarah, a fourth-year teacher of high school geometry and algebra, shared that she had not previously considered the importance of fostering reasoning and sense-making opportunities until joining our collaboration. She conceptualized the document as an authority on good teaching, and its argument convinced her of the importance of developing such practices in students to prepare them for their future. It seemed initially she was not quite sure what to change in her teaching to get it in line with the recommendations. She started by making a list of instructional strategies mentioned in *Reasoning and Sense Making*. Then through a process of trying to envision what they would look like in her own classroom and reading other practitioner articles, she narrowed her focus to asking more questions and requiring students to justify all ideas. Upon initial visits to her classroom, it was difficult for me to recognize the changes that she had described, as classroom discourse seemed to follow a structured initiation-response-feedback pattern. In conversations with Sarah, I ask her to describe what she had changed in her teaching. These conversations helped me to understand that she had increased the quantity of her questions and required each student to give a procedural justification of his or her answer, which resulted in increased student talk.

At times, the increased student talk created opportunities for students to "surprise" Sarah with their mathematical ideas. Sometimes a student's suggestion or justification was not what Sarah had anticipated. Through studying her teaching, she reflected and journaled about these moments of surprise, and each of them became individual opportunities for her own learning, raising her awareness of new ways she could impact students' engagement in reasoning and sense making. Sarah described one such moment of surprise:

> You remember the Algebra class where they wanted to use synthetic division? (*laughing*) I was so caught off guard because I've never thought of using that method [in that context] before in my *life*. I was like, "Okay let's go with it." But I was really surprised. And I should've been more calm about it…because then they wanted to know what "my way" was. But [the idea of using synthetic division] totally caught me off guard.

I was visiting Sarah's classroom when this instance occurred, and her surprise was very evident when a student suggested simplifying the expression using the method of synthetic division. In previous instances, when a student had suggested a strategy that was not what Sarah thought was best, she would either continue soliciting other students' suggestions, or suggest her preferred strategy. In this instance, however, Sarah's response was "Oh!" and she appeared deep in thought as she processed this student's suggestion. During her pause, the student asked, "Would that work?" "Yeah, that would work. Let's try that," Sarah responded. Then she solicited students' input to complete the procedure of synthetic division while she stood at the board recording the process. When they arrived at the answer of 3, the student who had suggested this approach asked, "Is that right?" Sarah responded, "Yeah, that's the answer." This ninth grader's pride was evident in her response of, "Wow!" Several students began asking Sarah to share her anticipated strategy for simplifying this expression. One student bargained, "We gave you *our* way, you

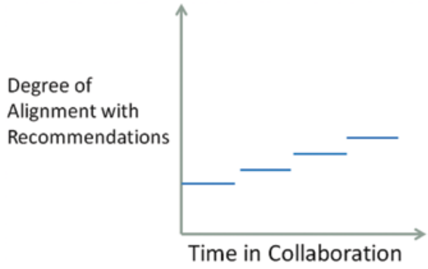

Fig. 4 The analogy of the step function journey

give us *your* way," as if the whole class was somehow united with their classmate and her strategy. Rather than suggest her own strategy as I had seen her do previously, Sarah responded with, "Do you want to see another way? Does anyone see a different way?" One student suggested factoring, and together they factored and got the same answer. Then a buzz of excitement around the room could be heard, and some students discussed how they did *not* think this method was easier than synthetic division, evidencing their continued pride and ownership over their peer's idea. Sarah responded with, "Well, something that is easier to you might not be easier to someone else. So it's good to have more than one way." Some students opted to continue to use the method of synthetic division in later examples, comparing their answers to the representations generated by factoring.

In Sarah's process of changing her teaching, unexpected surprises fueled new learning and furthered the changes she was making:

> Before, I wouldn't let them [solve problems] the way that they wanted to…. I think a lot of times I would just be like, "Well, didn't you see this method?" instead of just letting them do it their way. I think it's okay now just to let them do it a different way, even if it's the hard route. Just let them be, because that's the way they understand. Giving them that freedom.

In the synthetic division instance, the students' visible pride and developing sense of community that resulted from their realization that they had contributed an important mathematical idea was powerful. This surprise caused Sarah to become aware of the value of allowing students to generate their own solution paths.

I use the analogy of a step function to illustrate the journey of Sarah's experience attempting to align her practice with *Reasoning and Sense Making* (Fig. 4). Each new action strategy she adopted was subtle and sometimes hard to distinguish initially in observations. In addition, initial periods of testing out the change in her teaching were reminiscent of a plateau. Each plateau was disrupted when Sarah experienced learning as a result of her AR. Students surprised her with their mathematical ideas when she monitored their comments and discussions in search of examples of reasoning and sense making. For example, Sarah described a moment of surprise when a student challenged a textbook problem with a prescribed solution path:

> As the students became more familiar with justifying and explaining their thoughts, they began asking questions themselves. I found a journal entry where this was illustrated: "I posed a question in Geometry that gave four vertices of a quadrilateral. You had to determine whether or not you had a parallelogram. The directions stated that you had to use the

> specified formula to come to a conclusion. After the problem was complete, a student made the following comment: 'It would be much easier to use the midpoint formula. Why can't we just use that one?'" This student went on to explain why we could use the midpoint formula to show that a quadrilateral is or is not a parallelogram.

This student's bold example of reasoning about a more efficient strategy caused Sarah to reconsider her use of that textbook problem. She reflected on what she learned from this instance:

> I have noticed that textbook problems can hinder the critical thinking process because they tell you which method to use. This does not allow the students to think for themselves. As a result, in my next two Geometry classes I decided to let them choose which method to use.

In instances such as these, Sarah was impressed by individual examples of students' reasoning that occurred in her classroom, and these incidents made her aware of additional opportunities where she could engage students in reasoning and sense making. This new learning, and the subsequent changes Sarah made in her teaching, resulted in new steps in the function. While each step in the function would not be the same length, or always be exactly horizontal, I was not present in Sarah's classroom on a daily basis in order to make claims about a precise model. Compared to other teachers attempting to align their practice with *Reasoning and Sense Making,* Sarah's experience was unique by the pattern of repeated instances where students' ideas triggered a new awareness that prompted her to revise and develop her action strategies.

Using Shaw and Jakubowski's (1991) theory of change as a lens through which to analyze Sarah's experience, it is difficult to identify a single event that served as a key perturbation in her process of change. Sarah's experience, rather, was defined by multiple mini-cycles through this change process, as multiple events provided enough dissonance to propel her to commit to a subtle but new change in her teaching (phase 2), reenvision her practice in a new way (phase 3), project herself into her new vision (phase 4), and take new actions (phase 5). Through repeated instances of a experiencing a perturbation, committing, reenvisioning her practice, and enacting a change, the changes Sarah made in her teaching evolved and developed in response to her learning.

A Scatterplot Journey

Claudia and James were in their first year of teaching, and both juggled many new responsibilities. Claudia reflected on the initial months of our collaboration and discussed the challenge of trying to focus her actions:

> With it being my first year and everything, I didn't know what my teaching style was and how I wanted to change or improve it…. I kept kind of trying the different things I heard people talking about, thinking, "Is this what I need to work on? Is this something that interests me?"

It took both Claudia and James much more time than the other participants to determine the focus of their changes. Their goals changed frequently and they experimented with a variety of different strategies as they learned different ideas from other teachers and from the readings. Both teachers eventually narrowed their efforts to posing open-response writing prompts, one in the context of summative assessments and the other in the context of daily formative quizzes. These approaches to incorporating reasoning were more like an add-on to their teaching than a part of their everyday routine. James explained in a written reflection why he picked a "subtle" approach:

> I would love to hold classroom discussions and ask questions where students learn from their mistakes, discuss problems with one another, and problem solve when they do not get the correct answer (Eggleton et al. 2001). That type of classroom environment is one that I envision for the future, but I do not believe my classes are ready for such radical changes all at once. To me, writing seems like a natural and subtle way for students to convey their reasoning and sense making.

Both teachers were at schools that had been targeted by state policy as needing improvement, and both teachers felt pressures, in particular, to raise students' scores on the statewide algebra exam. With the many other things vying for their attention, Claudia and James at times would forget the changes they were trying to make. In the final reflection, they each talked about recognizing the need to incorporate reasoning and sense making into their teaching beyond assessments. They had each attempted an occasional student-centered activity in order to foster reasoning through lessons. Limitations in time to explore better instructional strategies and lack of readily available quality instructional resources, very real challenges for many teachers, hindered them from incorporating activities on a more frequent basis. By the end of the collaboration, Claudia and James had narrowed and clarified the focus of the new strategies they were trying out in their practice to a focus on writing prompts in their assessments, but these practices had not transformed into being an integral part of their teaching. Claudia and James each described room for improvement to change their teaching further and discussed plans to continue their changes in subsequent small steps in the future. I illustrate Claudia's and James's journeys using the analogy of a scatterplot with a positive correlation that became slightly stronger over time (Fig. 5). This analogy is distinct from the others as it illustrates the variety of disconnected changes, such as their brief efforts to test out each of the strategies discussed by other teachers within our collaboration. Claudia and James made isolated attempts to incorporate student-centered tasks, manipulatives, and discussion, but their strategies were not connected in a meaningful way. After they narrowed their focus to implementing open-ended writing prompts, their strategies became slightly more focused and consistent. They still struggled, however, to give their AR the attention they needed as they were distracted by the many other demands on their attention.

Analyzing Claudia's and James's experiences through the lens of Shaw and Jakubowski's (1991) phases of change, it is not evident that either of them ever experienced a perturbation in their teaching to make them realize a need for the changes outlined by *Reasoning and Sense Making*. This finding is not surprising considering these are first-year teachers who do not yet have established norms for their teaching practice. As a result, they are not yet in a position to evaluate their practice and recog-

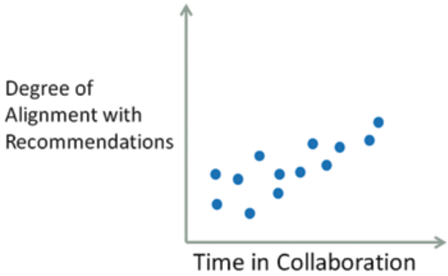

Fig. 5 The analogy of the scatterplot function journey

nize a need for change. As Claudia's quote indicated, she did not know in what ways she should change her teaching, and her AR was characterized by "trying out" many different strategies that she read about or learned from other teachers. While James suggested that his students were not "ready for such radical changes all at once," the data suggest that perhaps he wasnot ready for such radical changes because he had not yet recognized the need for such changes. None of the actions that Claudia or James tested in their practice seemed like consistent changes that would be sustained, which aligns with Shaw and Jakubowski's theory that sustainable change cannot occur without first the teacher realizing a need for such a change.

Conclusion

The findings presented through four analogies portray teachers' experiences as they attempted to align their practice with recommendations for *Reasoning and Sense Making.* In the case of the linear journey, Peter and Alexis were able to navigate the phases of change and respond productively when faced with challenges, with little hindrance to their efforts. In the case of the piecewise journey, Melinda and Logan reached a point where they struggled to know how to resolve the gap between their vision and their actual practice. In their case, the opportunity to observe me teach their students provided a perturbation that renewed their confidence by helping them identify additional action strategies. In the case of the step function journey, Sarah's initial changes had relatively little impact on aligning her practice with recommendations, but each new change opened the opportunity for her to listen to students' mathematical thinking, triggering an awareness of new strategies. Finally, in the case of the scatterplot journey, first-year teachers Claudia and James implemented a variety of strategies, but the absence of a perturbation hindered them from making a strong commitment to specific and sustainable changes. The varied journeys, when considered together, illustrate both the importance of a perturbation to foster awareness of the need for a change, and also the variety of ways that teachers interpreted their students' responses to their instructional changes.

It is widely acknowledged that, despite recommendations, the status quo of mathematics teaching across the US has historically experienced little change (e.g., Hiebert 2013). Examinations of underlying hindrances to enacting recommendations are productive to move our understanding forward. Anyone who has spent time in schools recognizes underlying issues such as access to appropriate curriculum materials, student motivation, increasing demands on teachers, or standardized tests and accountability that may promote different goals. These factors each have the potential to confound teachers' efforts to change. The findings from this study, however, illuminate a different underlying issue: the sheer complexity of the experience of navigating through the stages of change. Navigating change is challenging. Although the complexity of this challenge has been established by individuals who have written about their personal experiences attempting change (e.g., Cady 2006; Chazan 2000; Heaton 2000; Umbeck 2011), as well as the national R^3M study (Ferrini-Mundy and Johnson 1997), this study offers a snapshot of the change efforts of a group of teachers, positioned in conjunction with one another. The multiplicity of experiences captured in this study illustrates complexity through the variety of experiences, as well as the differences in the ways that teachers responded within their experiences while undergoing change. These teachers had a desire to test recommendations; they had guidelines in the form of *Reasoning and Sense Making*; they had the support of a collaborative AR group. Nonetheless, their needs for support differed and their outcomes varied.

The responsibility for teacher learning does not lie solely with teachers. Hiebert (2013) suggested that efforts to improve teaching should focus on teacher learning and called for mathematics teacher educators to acknowledge the extended time and practice involved in improving one's teaching. In order to address the discrepancy between recommendations for mathematics teaching and actual classroom practice, mathematics teacher educators need a common understanding of teachers' experiences enacting recommendations. These seven teachers' journeys, depicted through four analogies, convey the meaning of the experience and illuminate the complex and varied journeys through change. The analogies and the corresponding descriptions offer a glimpse at teachers' narratives, providing an avenue for the reader to vicariously experience teachers' efforts at change. These findings provide mathematics teacher educators with an understanding of teachers' experiences in the midst of making changes. This understanding is productive and timely for informing the consideration of ways to foster perturbations and support the learning of teachers through their diverse journeys attempting changes in practice to align with *Reasoning and Sense Making*, in particular, and reform recommendations in general.

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Commentary on Section 2: Attending to Teachers in Mathematics Teacher Education Research

Denise A. Spangler

Attending to Teachers in Mathematics Teacher Education Research

It is widely accepted that the teacher is the critical classroom factor that determines students' opportunities to learn mathematics (cf. Sanders and Horn 1998; Sanders and Rivers 1996). Thus, research that seeks to understand how teachers, both pre-service and in-service, view themselves and the ways in which teachers develop the knowledge, skills, and dispositions to enact ambitious teaching (Franke et al. 2007; Kelly-Peterson 2010; Lampert et al. 2010; Lampert and Graziani 2009; Newmann and Associates 1996) is timely and critically important to the field of mathematics education.

The four chapters in this section frame research on teachers and their learning in different ways, but all are consistent with the notion of helping teachers develop ambitious teaching by attending to various aspects of their beliefs or identities. Although each is different, the four chapters have much in common. Two chapters (Chao, DePiper) deal explicitly with teachers' identities and how they position themselves in various contexts; the other two chapters (Keazer, Wilson et al.) deal explicitly with supporting teachers as they attempt to change their practice. Two of the chapters (DePiper, Wilson) also look at how teachers position students with respect to mathematics learning. I first provide a brief overview of each chapter and then discuss implications for teacher education and future directions for this type of research.

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The Chapters

Chao illustrates a way researchers can adapt research methods used in other fields, in this case social work and nursing, and apply them to teaching. Using the method of photo elicitation, in which teachers bring photographs that they have identified as significant to them in some way and connected to their teaching of mathematics to an interview, Chao uncovers teachers' identities as people outside of the classroom and ways in which their identities outside of the classroom intersected with the work of mathematics teaching. In particular, he highlights how two Latino secondary mathematics teachers reflect on their experiences as ethnic minorities and the ways this status influenced their thinking about teaching.

Wilson et al. set out to study the ways elementary school teachers used their knowledge of student learning trajectories (LTs) to shape instruction, but what arose from the professional development sessions was a focus on the ways that teachers talked about their students and their abilities to learn mathematics. The authors use attribution theory to describe the sources to which teachers attributed student success and failure. As a result of opportunities to learn about LTs, teachers began to include this language in their discourse about student success and failure in mathematics.

DePiper documents the ways preservice teachers struggled with positioning students and themselves in the context of the sociopolitical demands of public school classrooms and the ways this context clashed with their desires to engage in ambitious mathematics teaching. She also highlights the ways that teachers' identities were constructed by the discourses of teacher education and of the schools in which they completed their field experiences.

Keazer shares the journeys of four teachers attempting to implement practices related to reasoning and sense making (National Council of Teachers of Mathematics 2009) into their secondary mathematics classrooms. She presents the teachers' journeys using mathematical functions as analogies for their paths, using the teachers' points of view to help us understand teachers' perspectives on professional development and instructional change.

These four chapters highlight how much can be learned from small-scale, qualitative studies that shed a fairly small circle of light on the large field of mathematics teacher education. Gaining an in-depth look at a particular set of teachers in specific circumstances raises questions for the reader to consider in one's own teaching, professional development work, and research. Playing off of the work of Pollock et al. (2010), DePiper poses the question "What can I do?" with three different emphases: What *can* I do? What can *I* do?, and What can I *do*? As mathematics teacher educators and researchers, it behooves us to take up the third question and consider what we can *do* as individuals and as a field to help both preservice and in-service teachers enhance their ability to enact ambitious instruction. Thus, in the remainder of this chapter, I offer some possible answers to this question that were prompted by my reading of these chapters.

Implications for Teacher Education

Embracing Reality

Three of the four chapters speak to the idea of change—changing instructional practices, changing individuals' beliefs and practices, and changing identities. Keazer's chapter, in particular, offers us glimpses of teachers' perceptions of their efforts to change their instructional practices. As I think about how to support teachers as they seek to enact ambitious instruction, I am reminded of an idea posed by Tsamir and Tirosh (2000), who said that our task as teacher educators is to prepare teachers to be bicultural—to exist in schools as they are today and to be agents of change. The preservice teachers and the in-service teachers with whom we work must be able to succeed in the current educational system, regardless of whether we or they agree with every aspect of it. If they do not succeed in the system as it is currently constituted, they will have little credibility when they try to implement change, in their own classrooms and in wider venues.

This tension underlies much of the research that has been done on teachers' beliefs to date. Many studies show that some teachers who hold more progressive beliefs enact classroom practices that are more traditional than their espoused beliefs would suggest. Both the DePiper and Keazer chapters give us some insight into how preservice and in-service teachers feel constrained by this tension of trying to succeed in the existing system while also being exposed to ideas about changing the system (or at least their practice within the system).

As teacher educators and researchers, sometimes we fail to acknowledge the enormous impact of "the system" on teachers' lives, even the lives of preservice teachers. We sometimes present ideas in teacher education as though they should be implemented immediately and that implementing them is simply a matter of will. We assume that teachers have seen enough of the status quo on a daily basis in schools, so we must present them with ideas from the opposite end of the spectrum in hopes that their practice will somehow become a reasonable melding of the two. I suggest that we do our cause and our teachers a disservice when we take this approach of extremes. I suspect that we would get far more buy-in from teachers and that teachers would be far more successful if we admitted up front that teaching mathematics, at least in this day and age, is a balancing act between the progressive or reform-oriented ideas espoused in teacher education and the more conservative/traditional ideas that are often the norm in schools. DePiper argues for helping teachers "trouble" the discourses that exist in schools, such as discourses about ability grouping/tracking and mathematics as being about speed and accuracy. To trouble these discourses, we must admit that they exist and that there are rationales behind them.

I have heard former students say, "I feel so horrible when I give my students a worksheet," which suggests to me that I have painted teaching as entirely too black and white (worksheets = bad, group work = good) and have failed to acknowledge and help them appreciate the competing masters that teachers must serve. Particular instructional practices are neither inherently good nor bad; context matters. If I stood outside a classroom and peeked in through the narrow glass window in the door without being able to hear what is happening in a classroom, I would be likely to conclude that rows of students seated quietly in desks is bad instruction, whereas groups of lively energetic students engaged with one another is good instruction. The problem with these assumptions is that I cannot really tell what the students are doing. The students sitting in rows could be engaged in the "think" part of "think–pair–share" with an enriching mathematical task, and the students sitting in groups could be off task or working collectively on lower level recall tasks or "fun" activities with little mathematical substance. Perhaps, too often we give our students extreme definitions of what constitutes good and bad mathematics instruction, which may drive them to the oft-cited practices of relying primarily on survival advice from their mentor or peer teachers and of seeing university-based mathematics teacher educators as living in ivory towers and lacking understanding of what happens in "real" classrooms. The teachers in DePiper's study provide authentic examples of the challenges many teachers face as they try to enact ambitious practices in their classrooms.

Facilitating Discourse

The Wilson et al. and DePiper chapters suggest that teacher educators can help preservice and in-service teachers acquire language to talk about students, learning, curriculum, assessment, and other contemporary issues in mathematics education and can provide spaces in which they can try out discourses on such topics. The teachers in these studies were struggling to make sense of new ideas and about their places and the places of their students in an ever-changing system. The Wilson et al. chapter offers an example of a professional development project that provided teachers with both knowledge of and language about children's learning trajectories in early rational number reasoning. The authors found that teachers used both the ideas and the language from the learning trajectory when describing students' successes and failures with mathematical tasks. It is also very encouraging that teachers did not attribute student success or failure to gender, race, or socioeconomic status.

DePiper's chapter provides an example of a teacher educator engaging preservice teachers in discourses around students, testing, accountability, and instruction. In this case, the teachers were enrolled in a voluntary seminar outside of mathematics education instruction, but the ideas could be incorporated into a student teaching seminar or as part of a course that runs parallel to an early field experience. In order to foster such discourse, however, it is imperative that teacher educators first seek to understand what is happening in schools and not simply degrade the experiences of preservice teachers and suggest alternatives. As DePiper notes, "troubling" these ideas is not easy ground to tread, and resolutions will not occur in a single discussion.

Abandoning Deficit Models of Teachers

Keazer's study raises the notion that, in the same way we avoid using a deficit model when talking about students, we need to examine our discourse to ensure that we are not employing a deficit model of teachers. Certainly there is a lot of deficit discourse about teachers in the press, but I hear it from teacher educators, too, although not usually in print. For instance, I hear that preservice teachers are interested only in grades and not learning, that classroom teachers are taking professional development workshops just for the stipend, or that we will never make a dent in the local school district because there are so many teachers and administrators who "don't get it." With students, we are asked to consider what they *do* know and to think about how we can leverage existing knowledge in service of new learning. If we take this same approach with teachers, then we seek to meet them where they are and to provide learning experiences within their zone of proximal development (Vygotsky 1978). If we assume that teachers have come to their views for rational reasons and seek to understand them, then we will have a much better basis on which to build future instruction.

For instance, a common deficit view of preservice elementary teachers laments that they often expect their methods courses to provide them with a "bag of tricks," a "recipe book," or a collection of "cute activities" they can use in their classrooms. I find that preservice teachers often come to these views through one of three paths. Many of them have had negative experiences as learners and are therefore looking for ways to make mathematics "fun" and less painful for their students; thus, they are looking for cute activities. Others have been very successful as mathematics learners because they are good memorizers and are good at executing procedures, so they believe that teaching mathematics is all about explaining things clearly and sometimes cleverly; thus, they are looking for a recipe book that tells them the correct order in which to teach things for the greatest success. Other preservice teachers' experiences with mathematics have been neither overwhelmingly positive nor negative, but they have developed an instrumentalist view of mathematics (Ernest 1989) due to their experiences as learners, and thus they seek a recipe book and tricks to make learning easier. It is easy to take a deficit view of these teachers, but if we accept that they have arrived at these conclusions logically through their own experiences, then we frame our task in teacher education as showing them a different view of mathematics as opposed to correcting the error of their ways. This perhaps seems like a subtle shift of language, but it implies substantive differences in our approaches to instruction. For me, showing them a different view of mathematics entails, in part, engaging them in mathematics learning experiences that mirror those we want them to provide for children, and then debriefing those experiences by discussing the nature of the task I posed; how I responded to their questions, requests for help, and errors; how concrete or visual materials were used; the ways in which the experience was intellectually and socially enjoyable (a reengineered definition of "fun"); and many other topics. This type of discussion can lead to building a bridge between where they have been as mathematics learners to where we want them to go as mathematics teachers.

Recognizing and Embracing Multiplicity

The Chao, DePiper, and Keazer chapters all remind us that teachers are complex individuals, shaped by multiple personal and professional forces in their lives. As teacher educators, we would do well to seek to understand teachers as people first and then as mathematics teachers. For example, many of us have our preservice teachers write mathematics autobiographies the first week of classes to draw out beliefs about mathematics teaching and learning. Perhaps we should ask students to write autobiographies of themselves as learners and/or ask them to illustrate their autobiographies with photos, similar to Chao's use of photo elicitation. We might then learn who has an affinity for languages, for taking things apart, for poetry, for playing piano, or for running. We might learn something about their families and how they valued schooling. We might learn something about how the teachers view teaching and learning mathematics in contrast to other content areas. As Chao illustrates, we might learn something about the teachers' cultural identities that is profoundly influencing the ways they learn about the teaching and learning of mathematics. We may be able to leverage what we learn to connect mathematics teaching and learning to other aspects of teachers' lives, or we may simply be able to connect with them on a personal level in a different way, which may lead to them viewing our instruction differently.

Another common task in a methods course is to have preservice teachers write lesson plans, teach them, and write reflections on them. Asking preservice teachers to provide a bit of narrative about how the topic of the lesson was chosen; how it fits into a larger instructional sequence; and what expectations were provided by the mentor teacher with respect to standards to be covered, materials and tasks to be used, and methods of instruction would help us see how the lesson is shaped by the school context (as noted by Keazer). Preservice teachers sometimes tell me, for instance, that their mentor teacher has said that his/her students cannot work in groups because they will not behave, which constrains what the preservice teacher can do. I have also seen teachers hand preservice teachers complete lesson plans and tell them to follow them to the letter. We might have preservice teachers write elaborated lesson reflections in which they describe changes they would make to the lesson if they were to teach it in the same circumstances again, as well as what circumstances they would change along with why and how those changes would affect instruction.

Implications for Future Research

Some might argue that research on beliefs and identity is past its prime, but these four chapters make a convincing argument that it is important to continue to look in depth at small numbers of teachers to better understand how they view themselves and the enterprise of mathematics teaching and learning. The chapters also spur some thoughts about future research on beliefs and identity.

Chao's study reminds me that education is a field made of many disciplines and that most of our research methods are borrowed and adapted from other disciplines. Chao used the method of photo elicitation, borrowed from social work and nursing, to gain deeper insights into teachers' lives than one typically uncovers in a standard question-and-answer interview. Many of the methods of studying beliefs have well-known limitations, and methodological advances have been few. If this line of inquiry is to continue in fruitful directions, it will be necessary for researchers to borrow, develop, or adapt new methods that allow for scalability and/or that have greater validity than those now in use (such as Likert scale questionnaires).

It may be beneficial for researchers to back up a bit, giving teachers a chance to tell us about the variety of influences in their lives, rather than immediately honing in on beliefs and identities related to teaching mathematics. Chao introduces us to one method, photo elicitation, for taking a wider lens on teachers' experiences, but existing and popularly used methods could be retooled to start at a different grain size. In a related vein, DePiper's study reminds us that teacher education programs are not the only influences on preservice teachers; they are shaped by the experiences they have in schools. Much research on preservice teachers seeks to document the "impact" of the teacher education program on teachers' beliefs and practices, and much of that research shows little evidence of significant impact, at least in the short term. Studies that seek to make sense of the ways in which teachers process and prioritize the many competing messages they hear could be useful to the field in designing teacher education and professional development programs.

I mentioned above that teacher educators would do well to examine their discourse for evidence of a deficit model of teachers and of teacher learning. A similar admonition applies to research on teachers. I urge us to examine our stance toward teachers by looking at the way we frame studies in grant proposals, the interview protocols we use, the analytical tools we use, and the ways we write about teachers to become aware of when and how we are explicitly or implicitly taking a deficit view of teachers in our research. One way in which we implicitly take a deficit view of teachers that has received some attention in the literature is the focus on gaps between teachers' beliefs and practices. Leatham (2006) has offered the field another way to look at teachers' beliefs and actions as a sensible system that gets us out of the deficit approach.

The Wilson et al. study shows how existing research can be used to leverage new research. Wilson et al. designed a professional development program around existing research findings on learning trajectories and sought to understand teachers' uptake of these ideas in instructional decision making. This layering of research programs is one way that we can help shape the body of research in our field from a collection of stories (Cooney 1994) to a coherent thread of research that builds over time into a solid theoretical frame. The work on SimCalc (http://www.kaputcenter.umassd.edu/projects/simcalc/) provides a nice example of a body of work that has been built up deliberately over time. The work began with research on students' learning about change and variation and proceeded to the development of software to illustrate these ideas, then to the development of curriculum materials to teach these ideas, then to pilot studies, and on to scale-up studies. What would research on beliefs or identity look like if we tried to plot a similar trajectory for a systematic research program? I will not pretend to have the answer to this question, but I submit that it is worth the collective time and attention of those who are passionate about research on beliefs and identity.

Conclusion

The chapters in this section offer much food for thought about our work as mathematics teacher educators and researchers, both as individuals and as a collective. From the practical to the theoretical, these chapters have both immediate and long-term implications for our work as we seek to support teachers as they engage in ambitious instruction and to understand what it means for teachers to do so.

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Part III
Tools and Techniques for Supporting Mathematics Teacher Learning

Preservice Elementary Mathematics Teachers' Emerging Ability to Write Problems to Build on Children's Mathematics

Andrew M. Tyminski, Tonia J. Land, Corey Drake, V. Serbay Zambak and Amber Simpson

Effective mathematics teaching requires teachers to employ a variety of knowledge, skills, and dispositions. Sowder (2007) suggested one goal of teacher development should be to develop teachers' understanding of how students think about mathematics." Research suggests that teachers who understand how students think about particular mathematical ideas will be better positioned to recognize, interpret, and support these ideas in their instruction (Brown and Borko 1992; Fennema and Franke 1992). Research on Cognitively Guided Instruction (CGI) has demonstrated that teacher knowledge of student thinking, reasoning, and strategies can lead to gains in student achievement (Carpenter and Fennema 1992; Carpenter et al. 2000). Ball and colleagues' work on mathematical knowledge for teaching has identified knowledge of content and students and knowledge of content and teaching as the crucial facets of pedagogical content knowledge necessary for teaching mathematics effectively (e.g., Ball et al. 2005; Hill et al. 2008).

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As the research on student learning of mathematics has produced a substantial collection of findings, it has become increasingly important for mathematics teacher educators to assist teachers in developing an understanding of this knowledge base as well as the means to leverage this knowledge during instruction. Recently, Jacobs et al. (2010) introduced professional noticing of children's mathematics as a framework to understand the ways in which teachers attend to, interpret, and respond to students' mathematical thinking. These three aspects of professional noticing are intertwined and used in concert as teachers interact with children and their mathematics. Jacobs and colleagues stated that the professional noticing of students' thinking is a difficult practice and not one that comes naturally to adults. However, they also provided evidence that this practice is something that can be learned through targeted professional development. In this study, although preservice teachers (PSTs) were asked to attend to, interpret, and respond to children's mathematical thinking, we focused our analysis on responding, as responding is the most difficult of the three skills to acquire (Jacobs et al. 2010). Additionally, Vacc and Bright (1999) found that PSTs were able to acknowledge the tenets of CGI, but had difficulty using children's mathematical thinking in instruction. Therefore, we wanted to design course activities that provided PSTs the opportunity to develop responses to children's mathematical thinking, in the form of contextualized problems and number choices. We enacted those activities and analyzed PSTs' responses as a way to understand how to develop and support the ability to respond to children's mathematics in PSTs.

In this chapter, we present our analyses of PSTs' responses to a sequence of three activities that we designed and implemented in our methods course, in order to scaffold and support PSTs' development of responding to children's mathematical thinking. These activities presented PSTs with authentic classroom situations and required them to engage in increasingly complex tasks involving professional noticing. The sequence of tasks progressed from noticing an expert teacher's task design, to designing a task to address a single mathematical concept, to designing a task that addressed a wide range of student's needs. In each case, PSTs interpreted or designed a problem whose intent was to build on the children's mathematical thinking as evidenced in a prior task. Our research question for this study was, "How can our series of professional noticing activities support PSTs' abilities to pose problems that build on students' mathematical thinking?"

Theoretical Frame

We view PST learning from a social constructivist viewpoint and employ an inquiry approach to our instruction, providing PSTs with structured opportunities to explore content and resources designed to support PSTs in learning to teach mathematics. The design of our elementary mathematics methods course was guided by three main ideas: (a) children's mathematical ideas and understandings emerge from solving problems; (b) teachers can use questioning to scaffold the development of

Table 1 Trajectories of scaffolding PSTs' professional noticing within activities

Aspect of scaffold	Trajectory
Observing to doing	From observing an expert teacher's subsequent task to having PSTs design subsequent tasks themselves
Number of concepts	From designing a subsequent task to address a single concept to a task that addressed a wide range of student understandings
Number choices	From analyzing an expert teacher's number choice, to making number choices for a prewritten task, to writing an entirely new task, complete with number choices

children's mathematical understanding and sense making; and (c) *Standards*-based[1] curriculum materials can be useful learning tools for teachers and students (Drake et al. in press). Our course goals for PSTs were teaching mathematics for understanding through problem solving, identifying and evaluating worthwhile mathematical tasks, and attending to children's mathematical thinking. PSTs' ability to leverage children's mathematical thinking in posing a subsequent problem would involve successful integration of these three goals. Further, the activities would result in records of PSTs' "practice," affording the instructors the opportunity to evaluate their progress.

In order to design our sequence of activities and frame our analysis, we drew on the work of Jacobs et al. (2010) related to professional noticing of children's mathematical thinking. Three interrelated skills comprise the construct of professional noticing of children's mathematical thinking: attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings. Because these three skills are interrelated, we believed it was unwise to scaffold PSTs' development by practicing the skills individually through a decomposition of the practice (Grossman et al. 2009). Our approach was to develop a series of activities for PSTs that would gradually increase the complexity of the situation in which PSTs would engage in the professional noticing of children's mathematics. Our activities included three different trajectories of scaffolding for PSTs: observing to doing, number of concepts, and number choices. These dimensions are outlined in Table 1. We viewed this approach as a potentially effective means of supporting the development of PSTs' professional noticing.

The processes of attending, interpreting, and responding require teachers to utilize a variety of knowledge bases simultaneously. Shulman (1986) suggested three types of knowledge important for teaching—subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Ball and colleagues (Ball et al. 2005; Hill et al. 2007) built on Shulman's work and provided the Mathematical Knowledge for Teaching (MKT) framework, further defining subject matter knowledge (SMK), and pedagogical content knowledge (PCK), and identifying subsets of these

[1] In using the term standards-based curriculum, we are referring to the curriculum materials funded by the National Science Foundation and aligned with the NCTM Standards (1989, 2000), including: Investigations in Data, Number, and Space (TERC 2008); Everyday Mathematics (UCSMP2007); and Math Trailblazers (UIC 2008).

knowledge bases. The common content knowledge and specialized content knowledge subsets of SMK are germane to the work of professional noticing. Common content knowledge is defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball et al. 2008, p. 400) and includes "the mathematical knowledge teachers are responsible for developing in students" (Hill et al. 2007, p. 132). Specialized content knowledge is "the mathematical knowledge and skill unique to teaching" (Ball et al. 2008, p. 401) and includes "the mathematical knowledge that is used in teaching but not directly taught to students" (Hill et al. 2007, p. 132). Error analysis, analyzing and validating a child's nonstandard approach to computation, and asking productive mathematical questions are all examples of specialized content knowledge (Ball et al. 2008). We also ground this study in two subsets of PCK: knowledge of content and students, which includes knowledge of common student conceptions and misconceptions, the ability to anticipate possible student solutions paths to given tasks, and the ability to hear and comprehend the often primitive mathematical explanations of students; and knowledge of content and teaching—"mathematical knowledge of the design of instruction, includes how to choose examples and representations, and how to guide student discussions toward accurate mathematical ideas" (Hill et al. 2007, p. 133).

While Ball and colleagues have identified and differentiated these subsets of knowledge for the purpose of assessing teacher knowledge (Hill et al. 2007), we recognize that engaging in the professional noticing of children's mathematics requires that these knowledge bases are accessed and utilized in a coordinated and integrated manner as teachers attend to, interpret, and respond to children's mathematics. We utilize combinations of these subsets of SMK and PCK in our interpretations and explanations of PSTs' ability to engage in the professional noticing of children's mathematics. As an example, we posit specialized content knowledge and knowledge of content and students as integral to attending to and interpreting student thinking. PSTs use their specialized content knowledge in order to decipher children's mathematics, while simultaneously drawing on their knowledge of children's common strategies. We suggest, PSTs who are more accurate and effective in interpreting children's mathematical thinking are likely to have more developed knowledge of these subdomains. Due to the integrated nature of these knowledge bases, however, we will not attempt to attribute ability to specific knowledge bases. Rather, we will explain our results using combinations of the domains of the MKT framework.

Methodology

Context of the Study

Data presented in this chapter were collected from 72 PSTs from three sections of elementary mathematics methods courses at two university sites during the 2011–2012 academic year. Our decision to examine the work of PSTs from different universities was a purposeful aspect of our research design, as it allowed the authors not only to learn from one another, but also to potentially disentangle the contributions of context and instructor in advancing PSTs' ability to pose subsequent problems. Although the activities reported were completed within the first 9 weeks of each methods course, PSTs' educational and mathematical experiences prior to the methods course were highly varied. PSTs at one institution took mathematics methods in the first semester of their senior year. In their program, they were required to take two to three mathematics content courses specifically designed for elementary school mathematics teachers. PSTs at the other institution took mathematics methods at varying points within their program, but most often during their junior year. Their program required mathematics courses, but not necessarily elementary-specific content courses. As such, some PSTs had completed none, one, or two elementary-specific mathematics content courses. While we did not perform analysis on the individual university sites, the differences in the educational experiences of our participants are a reflection of the reality of diverse math content requirement in US teacher education programs.

The three activities we report on were posed within the first 9 weeks of the course, during which PSTs also had other instructional experiences involving the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA Center, CCSSO) 2010), the Thinking Through a Lesson Protocol (Smith et al. 2008), the Levels of Cognitive Demand Framework (Stein and Smith 1998), and CGI problem types and solution strategies (Carpenter et al. 1999). The three activities were built from records of practice of our collaborating classroom teachers—Natalie Franke, Molly Sweeney, and Jenny Johnson—each of whom had taken professional development in CGI and taught in a district that had adopted Investigations in Number, Data, and Space (TERC 2008).

Data Collection

Our activities were designed and sequenced to scaffold and support PSTs, as they developed the capacity to make sense of student strategies and to use their understanding of student thinking, to write appropriate subsequent tasks for students. The three trajectories we followed within the design of the three activities are listed above in Table 1. We next describe each of the activities and the data sources we collected from each.

Natalie's Class the Next Day The first activity, Natalie's Class the Next Day, given on the first day of the semester, was designed to give PSTs the opportunity to notice and analyze how an experienced teacher used her students' current knowledge of division with fractional remainders to design a subsequent story problem and number choices. After a brief overview of the learning goals for the course, we showed PSTs "Sharing Cookies," a video with transcript of Natalie and her second-grade class as they solve and discuss two partitive division story problems (Fig. 1). The video shows Natalie introducing the two problems to the children. At this point, we

Problem #1
Trisha and Allie are sharing _____ chocolate chip cookies. If they are shared equally, how many will each of them get?

2	4	5	8	9	12	13
30	31	50	51	66	67	83

Problem #2
Trisha, Allie, Lance, and Kathy are sharing brownies. If they are sharing _____ brownie equally, how many will each person get?

4	5	8	9	16	17	20
32	33	44	45	48	49	50

Fig. 1 Partitive division problems presented in the Sharing Cookies video

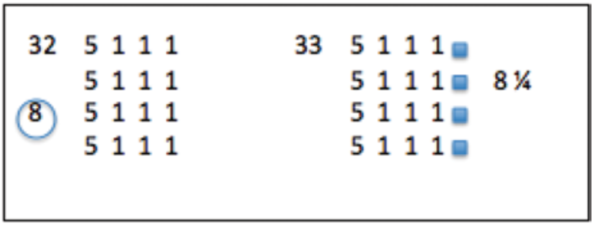

Fig. 2 Carter's solutions from the Sharing Cookies video

paused the recording to make sure PSTs also understood the format of the problems, which contain one blank and multiple number choices for the same story problem. The multiple rows of number choices are given to provide differentiation, with each child selecting the row of number choices "just right" for them. Natalie instructs the children to work on the first number choice in their chosen row and to show her their work when they are finished. After she has seen their work, they then work on the remaining number choices in the row.

As the video continued, PSTs had opportunities to see several examples of children's work and some brief interactions between children and the videographer, as well as between children and Natalie. They next viewed Natalie's whole-class discussion of the two problems, which begins with a child, Carter, (all student names are pseudonyms) sharing his solution to 32 brownies shared fairly among four people (displayed in Fig. 2). As a class, we discussed evidence of children's understanding and the specific teacher moves made by Natalie in helping her class understand Carter's solution to 32 brownies, how it connects to his solution to 33 brownies, and how sharing a single object equally among two people relates to sharing a single object equally among four people.

The Sharing Cookies video, its transcript, and our class discussion provided opportunity for PSTs to attend to and interpret children's mathematical thinking. The Natalie's class the next day assignment was given after our viewing and discus-

The next day, Natalie posed the following problem. Solve the problem for a few of the number choices. Then, answer the questions below.

There are ___ miniature candy bars. Dustin, Jose, Sam, and Joe are going to share the candy bars. If they split up the candy bars equally, how many will each of them get?

11	17	22	35	48
65	83	75	99	104

1. Why do you think Natalie posed this particular problem next?
2. What do you notice about the number choices in this problem compared to the number choices given the day before?

Fig. 3 Natalie's Class the Next Day assignment

Fig. 4 Counting Sequences

30, 40, 50, _____, _____, _____
44, 54, 64, _____, _____, _____
57, 67, 77, _____, _____, _____
157, 167, 177, _____, _____, _____

sion of the video, collected at the beginning of our next class meeting, and used as a source of data. The assignment asks PSTs to notice and interpret how Natalie attended to, interpreted, and responded to her students' mathematical thinking in her problem and number choices for the next day (Fig. 3). After grading and returning the assignment, we discussed PSTs' responses as a class, focusing on the facets of Natalie's number choices.

Counting Sequences The second activity, Counting Sequences, required PSTs to write an opening number routine (ONR) and problem, including number choices to address a single mathematical concept—counting by tens. The Counting Sequences Assignment is designed around a video with transcript of Jenny's first- grade class. Jenny begins with an ONR that poses a series of Counting Sequences to her class that focus on base-ten concepts. She presents a sequence of numbers and asks students to supply the next three numbers in the sequence. Jenny begins by posing the sequences in Fig. 4. The students are able to complete the next three terms for each of these with little difficulty. Jenny's ONR concludes as she poses an open number sentence for her class, 30 + ___ = 70, with which, again, the children have little trouble. The video ends with Jenny posing the main problem for the day, a join-change unknown (JCU) story problem about a paleontologist (Fig. 5). Like Natalie, Jenny often posed one story problem with multiple number choices, allowing for differentiation for the children in her class.

As in the first assignment, the Counting Sequences video, its transcript, and our class discussion provided opportunity for PSTs to attend to and interpret children's

Paleontologist Problem
A paleontologist had ____ dinosaur bones. He found some more. Now, the paleontologist has ____ dinosaur bones. How many bones did he find?

10, 70 20, 84 26, 126 15, 65 60, 150 42, 53

Fig. 5 Paleontologist problem

Now that you have seen the Counting Sequences video (and its transcript), consider these questions related to students' solutions to the Paleontologist Problem.

1. What is the disconnect between how students counted in the opening routine and the counting strategies they used when solving the problem?
2. Why do you think the disconnect exists?
3. Considering this disconnect, generate two artifacts for the next day's lesson: and opening number routine and number choices for the Paleontologist Problem given below. Briefly justify your choices.

"Today, the paleontologist is looking for fossils. He already had ___ fossils in his collection. He found some more. Now, the paleontologist has ___ fossils. How many fossils did he find?"

Fig. 6 Counting Sequences assignment

mathematical thinking. At its conclusion, we introduced the Counting Sequences assignment, which PSTs would complete for our next class meeting. The assignment begins with Jenny's description of the children's work from the paleontologist problem.

> Most of the children solved the paleontologist problem by using a hundreds chart, but many counted by ones when counting up to the second number instead of counting by tens. Some children did count by tens. For 20 and 84, the children who were counting by tens either counted by ones from 20 to 84, or counted by tens to 80, then counted 4 more. Nobody solved for 42 and 53 (Drake et al. in press).

Jenny's description indicates there are children in her class who had earlier showed evidence that they could count by tens, but did not invoke their counting by tens schema when solving the JCU problem (with numbers 20 and 84). The Counting Sequences assignment is built around this inconsistency. Considering the above information, PSTs were asked to complete the assignment in Fig. 6. After grading and returning the assignment, we discussed sample PST responses as a class. We discussed the inconsistency the authors saw in the situation, presented PSTs' responses echoing some of the possible reasons for it, and emphasized how prescribing "more of the same" (as many PSTs had done) is not an effective approach to addressing students' mathematical misconceptions.

Sam had ___ fish bowls. He had ____ fish in each bowl. How many fish did Sam have?

A	B	C	D
(2, 10)	(4, 20)	(3, 11)	(4, 12)
(5, 10)	(8, 20)	(6, 11)	(8, 12)

Fig. 7 Fishbowl problem

Write a problem for the next day along with a rationale. What do you think will be an appropriate problem that will meet the range of needs in Molly's classroom? Reference at least four students or group of students specifically in your rationale.

Fig. 8 Fishbowl problem assignment

Fishbowl Problem The third activity, fishbowl problem, asked PSTs to analyze children's multiplication strategies and write a subsequent problem with number choices to address the wide range of solution strategies and learning goals. The fishbowl problem is set in the context of Molly's second/third-grade mixed-age classroom. This task was built around PSTs' examination of examples of student work from 14 children in Molly's class in response to the multiplication problem in Fig. 7. Molly's students were instructed to pick a pair of number choices to solve, show their work, and move on to the next pair of number choices.

In the methods class, we first asked PSTs to predict what Molly's learning goals for the lesson could be, based on what they saw in the problem she had posed. Molly had shared her goals for this lesson with the authors and we shared her response with the PSTs:

> I had a couple of different goals for the lesson with the goldfish bowl problem. For some of my students I wanted to see if they were able to skip count by multiples of ten. For others, that I knew could, I wanted to see if they could see any relationships between the numbers I had chosen for them to solve. I had asked them to pick a pair of numbers to solve, hoping they would see this. Also, when choosing the numbers 11 and 12, I was looking to see if any of the students used the distributive property and their knowledge of tens to help them solve the problem (Personal communication, 2010).

With these goals in mind, PSTs were to examine the samples of student work in order to (a) attend to the details in each child's strategies and (b) interpret the strategies in relation to Molly's learning goals. Then, using this information, PSTs were to complete the assignment in Fig. 8 for the next class.

Data Analysis

Within our study, there were two main stages of analysis—an analysis of the tasks and an analysis of PSTs' responses to the tasks. In the first stage, the first three authors performed an analysis of each of the three tasks in order to establish a series of codes and operational definitions. This process involved the authors completing each task individually, discussing our responses collaboratively, and when necessary, included examination and discussion of PSTs' responses for the same assignments from prior implementations. The authors' process for analyzing each task is described below.

Natalie's Class the Next Day For the Natalie's Class the Next Day assignment, we report results from Question 2 (Fig. 3), which addressed Natalie's number choices. This question is most germane to our research question and our trajectories, as it provided PSTs an opportunity to interpret how an experienced teacher can build on children's mathematical understanding through appropriate number choices. Our analysis of Natalie's number choices revealed three aspects of those numbers we anticipated PSTs might notice, and we coded PSTs' responses for the number of aspects they identified.

Counting Sequences The Counting Sequences assignment is comprised of three questions (Fig. 6). The first question asked PSTs to "identify the disconnect evident in Jenny's classroom." The authors used the term disconnect to refer to what we identified as the inconsistency between Jenny's students using skip counting by 10 in completing the Counting Sequences within the opening routine but not applying the skip counting strategy when solving the JCU story problem, resorting instead to the less efficient strategy of counting by ones. Responses that identified the disconnect were coded as 1; those that did not were coded as 0.

We used a process of open and emergent coding (Strauss and Corbin 1998) in order to develop our codes for the reasons PSTs gave in Question 2 for the identified disconnect. This process resulted in three primary codes for the reason for the disconnect, as well as subcategories of more specific reasons within categories.

PSTs' responses to the third question were analyzed in order to determine if their proposed opening routine for the next day addressed the primary reason for the disconnect they identified. PSTs' responses were coded as 0 if the opening routine did not address their primary reason and as 1 if it did. We elected to use a binary coding, as at this point in the course we were most interested in understanding if PSTs were attempting to draw on their interpretation of the children's mathematical thinking as they responded with the next task.

Fishbowl Problem The fishbowl problem assignment asked PSTs to respond to the 14 samples of children's work from Molly's classroom by writing a problem for the next day including number choices. The authors first analyzed the 14 samples of children's work, resulting in a classification of children's strategies with four categories. Table 2 displays the four categories, their descriptions, and the names of Molly's students who used those strategies. We collaboratively examined several

Table 2 Children's strategy, classification, and description

Strategy	Description	Students using strategy
Direct modeling	Students either could not solve any of the multiplication tasks, or did so by directly modeling the solution with drawings	Dante, Whitney, Alex, and Kris
Skip counting	Students skip counted by tens and/or multiples of 10 to solve 4 × 20 = 80 20, 40, 60, 80	Tom, Olivia, Sarah, and Amber
Repeated addition and break apart by place	Students solved by writing the multiplication problems as repeated addition and then broke the two-digit numbers like 11 and 12 apart by place value and added the tens and ones separately 3 × 11 = 11 + 11 + 11 10 + 10 + 10 = 30, 1 + 1 + 1 + = 3 30 + 3 = 33	Matt, Wes, Gwen, and Max
Doubling	Students used repeated addition to solve the first number choice in the pair, but were able to solve the second number choice using the doubling relationship between doubling the number of by simply doubling the product	Seth and Hank

responses to this activity from a previous course to establish our codes and operational definitions.

We examined PSTs' entire response to the fishbowl assignment in order to classify their ability to attend to, interpret, and respond to the children's work. Within their response, PSTs were asked to address specific children or groups of children. In order for a response to be coded for evidence of understanding children's thinking, PSTs needed to correctly identify or describe the work of each child or groups of children they included within their justification.

We next coded each response for story problem appropriateness. We considered an appropriate story problem for Molly's students to be a multiplication story problem. We also expected PSTs to write a problem similar in structure to the one Molly's students had already solved, with the first blank indicating the number of groups and the second blank indicating the number in each group. Molly's students had not yet met her learning goals and therefore the authors did not see a need to stray from this problem type or structure.

PSTs' number choices and justifications were coded for evidence of addressing current children's understanding and evidence of addressing Molly's learning goals. As we coded the responses in terms of addressing children's current understanding, we first looked for explicit evidence in the rationale that PSTs were attempting to choose numbers for a strategy used by a specific child or groups of children. If we found evidence, we then examined the number choices they selected in order to determine if they had successfully done so. In order to be coded as a successful attempt, the number choice had to address the children's current understanding.

We coded PSTs' number choices in terms of addressing Molly's learning goals in a similar manner. If PSTs explicitly mentioned a learning goal in their rationale, we coded it as an attempt. If an attempt was made, we then determined if the number choices were appropriate. If so, we coded it as a success. In order to be coded as a successful attempt, the number choice had to support the learning goal it was intended to address.

Coding PSTs' Responses In the second stage of coding, we used the established codes in order to analyze PSTs' responses to the tasks. Two researchers carried out the coding of each assignment independently and a reliability/agreement measure was calculated for each set of coding. The agreement percentages across all coding sets fell between 75 and 96%. The 75% agreement was from the coding of PSTs' given reasons for our identified disconnect in the Counting Sequences assignment. This set of data involved certain subtleties that were not uncovered until discussion of disagreements. One such example was the difference in the two codes: "more comfort counting by ones" and "less comfort counting by tens." While these two codes essentially suggest the same thing, we coded them as different depending on the strategy explicitly identified by PSTs. The 96% agreement was in our coding of PSTs' number choices within the paleontologist problem. We believe this high percentage was due largely to the well-defined nature of the codes (e.g., *counting by tens from a decade number* (20, 60)).

Results

We begin by presenting results from each of the three activities and briefly describing the strengths and weaknesses of PSTs' responses within each particular context. Following this presentation, we conclude this section by discussing patterns of PSTs' responses across the three assignments and characterizing PSTs' progress in attending to, interpreting, and responding to student thinking. We include examples from PSTs' responses to serve as evidence for our claims.

Natalie's Class the Next Day

The Natalie's Class the Next Day assignment supplied evidence of PSTs' ability to interpret Natalie's problem and number choices in designing a problem for the next day. The authors had identified three facets of Natalie's number choices: (a) the numbers are larger than the day before; (b) the numbers are more complex in that students had to think not only about sharing remainders of 0 and 1, but also 2 and 3 as well; and (c) the next consecutive number scaffold that had been used the day before has now been removed. We examined the 72 PSTs' responses to determine how many of the facets of Natalie's number choices were identified by each PST and the number of PSTs that identified each facet. The results are presented in Tables 3 and 4.

We present examples of PSTs' responses in order to illuminate our results. Julia's response (all PST names are pseudonyms) did not identify any of the facets we had identified in Natalie's number choices:

Table 3 Number of facets identified

# Correctly identified	# PSTs	Percent PSTs
0	8	11.1
1	31	43.1
2	27	37.5
3	6	8.3

PSTs preservice teacher

Table 4 Percentage of PSTs identifying each facet

Facet	# PSTs	Percent PSTs
Larger numbers	31	43.1
More complex numbers	37	51.4
No scaffolds	36	50.0

PSTs preservice teacher

> I noticed that this problem has 10 choices versus the 14 number choices from the problems of the day before. I think there are less options because the numbers are more difficult, or will take more time for the children to solve the problems with these choices.

Although Julia noticed the numbers were "more difficult," she was not able to specify what made them so. Tally's response was an example that identified all three facets:

> Like the day before, there are two sets of numbers and the students choose what "best fits them." The first row of numbers does a couple of things. It reinforces the idea of halves as well as adding the concept of fourths, using 1/4. Also to note, the largest number in the first row (48) is a number that is divisible by 4. These numbers are also larger than the first row numbers the day before and there is a little less scaffolding from number to number. The second row of numbers does not review the concept of halves, but introduce fourths by using 1/4 as well as 3/4 (helping these students understand how to add fractions together properly). Like the last number in the above row, the largest number (104) is divisible by 4.

We interpreted these data through the lens of MKT. Natalie was building on her students' understanding through the use of specific number choices, and we wanted to see how well PSTs were able to identify the specific changes Natalie made in her number choices as a result of children's work the day prior. The ability to identify facets of Natalie's number choices serves as an indication of PSTs' knowledge of content and teaching, which is knowledge of how to choose examples and design instruction. From the data, one can see the majority of PSTs (43.1 %) identified one facet of the number choices, with slightly less (37.5 %) identifying two facets. Smaller percentages of PSTs were at the extremes; 11.1 % did not identify any of the three facets and 8.3 % identified all three. The average coding score for PSTs is 1.43, demonstrating they identified roughly half of the facets of Natalie's number choices. Based on prior implementations of this assignment, we anticipated it might be more likely for PSTs to notice the larger numbers and the lack of scaffolds than recognize the complexity of the numbers. Identifying the complexity of the numbers would require PSTs to recognize that there are only four possible remainders when dividing a whole number by four, and that all of them were accounted for within Natalie's number choices. Identifying the first facet required less-developed knowledge of content and teaching, only to recognize that the numbers were larger than the day prior, and the second facet was explicitly discussed in the video of Natalie's strategy sharing session. In examining the rate of individual facets, however, we see the likelihood of PSTs' identification of each as relatively the same (approximately 50%), with larger numbers being identified by slightly less than half of the group. Taken together, these results suggest at the beginning of the methods course, after completing their respective university's mathematics content requirements, PSTs had an emerging ability to make sense of the number choices and to consider their role in posing problems based on students' thinking. In the next task, we examine PSTs' ability to professionally notice children's mathematics.

Counting Sequences

The questions for this assignment require PSTs to attend to and interpret both the children's work in the ONR and Jenny's description of their work on the first paleontologist problem. Then they must respond by creating an ONR, theoretically to be used at the beginning of Jenny's next lesson. In this activity, we were interested to see if PSTs could (a) identify "the disconnect," the inconsistency in children not using the count by tens strategy from the sequence activities when solving the JCU story problem; (b) posit reason(s) for the disconnect; and (c) design an ONR to address the reason(s) stated in "b." We believe this task was an appropriate progression from the previous task, as it provided PSTs an opportunity to engage in all facets of professional noticing, but required them only to attend, interpret, and respond to a single mathematical concept (counting by tens) within a class of students.

We examined the data on identifying the disconnect in order to investigate PSTs' ability to attend to children's thinking. Of the 72 PSTs, 51 (70.8%) were able to successfully identify the disconnect between children solving the Counting Sequences tasks by counting by tens and the counting by ones strategy many used in solving the JCU problem. Alice's response is indicative of PSTs who were able to identify the disconnect:

> When the students were counting in the opening routine they were able to count by 10's. Although when they were solving the problem, they were unable to see the connection between the opening routine and the current problem and didn't use the strategy of counting by 10's to solve the problem. The teacher hoped the students would count by 10's to solve the problem, but they used other strategies they knew instead.

The data concerning PSTs' reasons for the disconnect provided evidence about their ability to interpret student thinking. We employed three codes as main reasons for the disconnect: (a) children did not see the counting by tens strategy as applicable in the JCU problem, (b) children's comfort with one strategy as opposed to another, and (c) number choices. We identified subcategories of more specific reasons within the applicability and comfort categories and identified the number of PST responses in each subcategory (Table 5). To illustrate further, we present PST responses representative of each of the main categories:

Table 5 Number of PSTs identifying each reason for Counting Sequences disconnect

Code	# PSTs identified
Applicability of counting by tens	
No scaffolds included (no pattern progression, no teacher scaffolds)	14
No further reason given	12
Problem type in word problem (JRU, JCU, etc.)	6
Context/no context	5
Not paying attention	1
Applicability totals	38
Comfort with strategies	
Students more comfortable, familiar, experienced counting by ones	9
Comfort in using 100 chart to count by ones	9
Students less comfortable, familiar, experienced counting by tens	8
Students need more practice with counting by tens	5
Comfort with strategies totals	31
Number choices totals	15
Overall totals	84

PSTs preservice teacher, *JRU* join-result unknown, *JCU* join-change unknown

> I think that the scaffolding was missing. I think that the students did not see that they could count by 10s and it would solve the problem faster than counting by ones and would also be more accurate. (Pamela) I believe this disconnect happened for maybe different reasons for different students: some students may have simply felt more comfortable with counting by 1s, and they may not have wanted to take a chance in making a mistake with the new concept of counting by 10s. (Abbie) In the counting sequences Jenny did not introduce numbers that you couldn't count up by ten's and this may have confused the students. Possibly if Jenny had introduced them to one of these, the students would have been more successful. (Laura)

Pamela's response is an example from the applicability category; she interpreted the disconnect as resulting from a lack of scaffolds. Abbie's response is from the comfort category and was coded both as "more comfort with counting by ones" as well as "less comfort counting by tens." Laura's response attributes the issue to the difference in number choices within the ONR and the JCU problem.

As is evident in Abbie's response, some of the 51 PSTs who correctly identified the disconnect provided more than one possible reason. Some PSTs' (21) reasons were coded in more than one of the main categories, and other PSTs (8) provided reasons coded for multiple subcategories within a given reason. When we adjusted the numbers to account for various forms of double counting, we found 37 of the 51 (72.5 %) PSTs who correctly identified the disconnect posited it was a result of students not being able to see the strategy as applicable in the JCU problem, 22 of 51 (43.1 %) attributed the disconnect to students' comfort with certain strategies, and 15 of 51 (29.4 %) attributed the disconnect to number choices.

We agreed that children not recognizing the count by tens strategy as applicable was the most likely reason for the disconnect in children's strategies. From this viewpoint, 37 of 72 PSTs (51.4 %) both accurately identified and interpreted the thinking of Jenny's students. The ability to identify and interpret the disconnect is a reflection of PSTs' use of specialized content knowledge and knowledge of content and students. These two knowledge bases would be used concurrently in PSTs' understanding of the ways in which children solved these particular tasks, other possible solution paths, and an understanding of how children learn to develop and use strategies. We also viewed PSTs' responses that were coded "comfort with strategies" as reasonable. These responses could be interpreted as evidence of knowledge of content and students, as PSTs recognize children might be comfortable with a new strategy when doing it as a group, but then be more comfortable with a more familiar strategy (i.e., counting by ones) when they are solving a problem on their own. However, the responses coded as "number choices" seemed less reasonable, as the number choices were essentially of the same types in the counting sequences as they were for the JCU problem.

The final part of the assignment asked PSTs to respond to Jenny's students by creating an opening number routine Jenny would present the next day. From our viewpoint, PSTs who intended to respond based on their current understanding of the children's thinking would address the disconnect within their opening routine. As such, we examined PSTs' opening routines in order to determine if they had addressed their main reason for the disconnect. Although we did not necessarily agree with all PSTs' interpretations of the disconnect, all responses providing evidence of PSTs' addressing their interpretation in their opening routine were coded as having done so. We present opening routines from two PSTs, Pamela (applicability of counting by tens) and Allison (comfort with using the 100s chart), as examples to further illustrate our process.

> I am going to give you some number sentences. Solve for the missing numbers.
> 23 + 50 = ____
> 46 + 70 = ____
> 30 + ____ = 50
> 42 + ____ = 92
> ____ + 60 = 77
> ____ + 30 = 52
> By starting with solving a problem where you add the tens, then moving to see that you can count by tens, I think students will make the jump between counting by tens and then counting by tens to solve. (Pamela)

Pamela's response was coded 1 as she addressed her identified scaffolding issue by employing only open number sentences in her ONR. She included result unknown, change unknown, and start unknown tasks. Below, we can see Allison's response does not address her interpretation of the disconnect (comfort in using the 100s chart). Her response was coded 0. If Allison had explicitly mentioned introducing children's use of the 100s chart to solve her true/false number sentence, her response would have been coded 1.

> True/False Number Sentences
> 10 + 40 = 50
> 20 + 30 = 50
> 22 + 44 = 60
> Equations: Show Work
> 10 + ____ = 30
> 20 + ____ = 50

Table 6 Number of PSTs who identified and addressed their reason for the disconnect

Code	# PSTs identified	# PSTs addressed
Applicability of counting by tens	37	13
Comfort with strategies	22	13
Number choices	15	8

PSTs preservice teacher

> My rationale for starting with true and false statements as my opening routine is to make sure that students are on the right track with realizing how to count by tens and place value. I realize there was some disconnect the day before so I want to see if they can determine if the sentences are true or false given the whole problem. I then give students some equations and ask them to show work because I want to see the strategies they use to solve the equations so I can address any mistakes or disconnects before we move on in the lesson. (Allison)

Table 6 displays the number of PSTs who identified each code as their reason and the number of those PSTs who addressed their reason within their opening routine. Both columns of data have accounted for the double counting described earlier. In terms of each of the three main codes, 8 of 15 (53.3 %) PSTs addressed the number choices, 13 of 22 (59.1 %) PSTs addressed students' comfort with strategies, and 13 of 37 (35.1 %) PSTs addressed the applicability of counting by tens.

We present a final summary of PSTs' responses to the Counting Sequences activity in order to frame their emerging ability to engage in the professional noticing of students' thinking. While a large proportion of PSTs (70 %) at this stage of the course were able to identify the disconnect (attending), fewer (51.4 %) could also provide a reason based upon aspects of their MKT (interpreting). In terms of being able to also respond appropriately (as defined in our coding), and thus being successful in all three facets of noticing, 18.1 % (13/72) of PSTs demonstrated an ability to do so. This result lends credence to the complexity of utilizing student thinking in instructional decisions.

Fishbowl Problem

We believed the fishbowl activity was an appropriate next task in the trajectory to support PSTs' noticing, as it required them to attend to and interpret several examples of students' thinking and to respond by writing a story problem appropriate for the entire class, while simultaneously attending to specific strategies and learning goals when writing number choices. This activity required PSTs to use a variety of knowledge bases in concert to make sense of, interpret, and respond to the authentic samples of students' work. Additionally, PSTs had the added benefits of reading and interpreting the students' work without the normal time constraints of face-to-face interactions, as well as the opportunity to discuss the students' work with their peers and instructor.

To begin, we examined PSTs' ability to attend to and interpret the student work samples. As described previously, we coded PSTs' responses to the assignment for evidence of understanding students' multiplication strategies. In order for a response to be coded for evidence of understanding students' thinking, PSTs needed to correctly identify or describe the work of each student or groups of students they included within their justification. For example, Elise's response was coded 0 (not demonstrating evidence) because she incorrectly interpreted Dante's work; Dante's work showed he simply added the two numbers provided in the fishbowl problem. An excerpt from Elise's response is below.

> Problem A and B are for students like Dante. He worked the problem from left to right to see the relationship. So, I made a relationship from left to right by doubling both numbers. It was easy for him to get 2, 10 and 5, 10. Since there is a relationship now, maybe it will be easier for him to see the pattern and get the correct answer. Here, I would like for him to use doubling to answer the problem correct. (Elise)

Amy's response was coded as demonstrating evidence of knowledge of students' thinking as she correctly identified the strategies of each student she alludes to in her rationale.

> Since Olivia is partitioning, I would give some higher set numbers so that she can start to explore other ways to solve this problem that may be more efficient than partitioning. For Tom and Matt I would take out the even counting by 10 numbers from the harder set so they can start to look at partitioning or other options for multiplying numbers. But for Amber, who is using addition I would like to make an easier set so she can start to grasp the concept of multiplying the numbers instead of adding. (Amy)

In this particular context, 75 % of the PSTs (54 of 72) demonstrated evidence of attending to and interpreting student thinking, a slight increase from the Counting Sequences results. PSTs employed specialized content knowledge in concert with their knowledge of content and students in order to be able to correctly interpret and identify the samples of students' work. Specialized content knowledge is employed when PSTs use their own knowledge to make sense of the students' work and to describe mathematically their process. PSTs then used their knowledge of content and students in order to connect their description with common student strategies for multiplication, such as direct modeling, skip counting, and repeated addition.

We examined PSTs' proposed story problems and number choices in order to evaluate their ability to respond to the wide range of student strategies and understandings in Molly's class. We believed the most appropriate story problem for the next day would be a multiplication problem with the result unknown. Further, we decided in order to support the development of Molly's learning goals, the story problem structure should mirror the structure of the fishbowl problem, in that the first blank would represent the number of groups of objects and the second blank would represent the number of objects in each group. Below are three sample problems:

> Sam had ________ fish. Sam then gave ________fish to his friend Henry. How many fish does Sam have? (Angie)
> Pencils are packaged in boxes of ________. There are ________ boxes. How many pencils are there? (Becca)
> Sarah had _______ flower vases. There were _______ flowers in each vase. How many flowers does Sarah have? (Cassie)

Angie's problem was coded as inappropriate, as it was a subtraction problem. Becca's result unknown multiplication problem was also coded inappropriate, as it did not adhere to the expected format by first listing the number of objects in each group. We viewed the order in the format as important, as it would support Molly's learning goal of students noticing the relationship between doubling the number of groups and the final product. Although our coding rubric did not reflect a distinction, we viewed Becca's multiplication problem as "closer" to the original multiplication problem than Angie's subtraction problem. Cassie's problem was coded as appropriate, as it was a result of unknown multiplication problem in the expected format. Employing these criteria, 60 of 72 PSTs (83.3 %) wrote an appropriate story problem for the students in Molly's class.

We then examined the number choices within the 60 appropriate problems. We wanted to determine if PSTs addressed their interpretations of the students' thinking as well as Molly's learning goals within their number choices and how successful they were in doing so. We provide and interpret examples from Alice and Laney to clarify our analysis. For each PST we present her story problem, number choices, and a part of her justification. Alice's response is below.

> Sam had ________ fish bowls. He wanted ________ rocks in each fish bowl. How many rocks did Sam have?

A	B	C	D
(4, 10)	(3, 20)	(5, 20)	(6, 12)
(6, 10)	(6, 20)	(10, 20)	(12, 12)

> For column A, my focus was on students that weren't able to skip count by tens to solve the problem. This includes students like Alex, Whitney, and Gwen. Since I didn't think that these students would benefit from having one number that was exactly half of another, I chose two numbers that were lower, even numbers. It should be easiest for them to use tens to skip count. Therefore I chose ten for the second number in each number choice in column A. (Alice)

In Alice's response, we saw evidence of attending and responding to the direct modeling group (Alex and Whitney) as well as to Molly's learning goal of having students use skip counting by tens and groups of tens to solve the problem. We coded Alice's number choices for Column A as successful attempts, as the choices (4, 10) and (6, 10) could potentially scaffold both student thinking as well as progress toward the learning goal. Laney's work provides an example of a coded attempt, but with number choices coded as unsuccessful.

> Sam had _________ cookie jars. He had ________ cookies in each jar. How many cookies did Sam have?

A	B	C	D
(2, 5)	(5, 20)	(3, 14)	(4, 15)
(5, 5)	(10, 20)	(6, 14)	(8, 15)

> The next [second] set of numbers I chose were (5, 20) and (10, 20). I chose these because I think they are numbers that students can use the skip counting method with. Several students were able to skip count well (Max, Matt, and Wes) which seemed to help them. For

Table 7 Fishbowl number choices—number of PSTs addressing groups of student thinking ($N=60$)

Number of strategies addressed	# PSTs	Percent PSTs
4	5	8.3
3	17	28.3
2	19	31.7
1	10	16.7
0	9	15.0

PSTs preservice teacher

Table 8 Fishbowl number choices—number of PSTs addressing specific strategies ($N=60$)

Strategy category	# PSTs addressed (%)	# PSTs successful (%)
Direct modeling	31 (51.7)	21 (67.7)
Skip counting	42 (70.0)	18 (42.9)
Repeated addition and break apart by place	24 (40.0)	12 (50.0)
Doubling	22 (36.7)	15 (68.2)

PSTs preservice teacher

> those students who are ready to move onto skip counting (Sarah, Gwen), this problem is a great starter. Since all of the numbers are multiples of 5 it helps students to think of the relationships of skip counting by 5's. (Laney)

Laney's second pair of number choices is attempting to address students who were not able to solve using skip counting (Sarah and Gwen), as well as the learning goal of skip counting in general. Her number choices, however, were coded as unsuccessful for both attempts, as they did not scaffold students who were not yet using skip counting by tens. Further, Laney's justification does not distinguish between identifying the number of groups and the number in each group.

We examined number choices addressing children's mathematical thinking from two perspectives: How many of the four different categories of children's strategies did PSTs address, and which specific strategies were addressed most often. The results displayed in Table 7 reveal a vast majority of PSTs (~85%) attempted to address some form of student thinking in their number choices, an increase from the Counting Sequences assignment. Approximately, 48% of PSTs attempted to address one or two of the groups of children's strategies, and 37% attempted to address the thinking of at least three of the four groups (Table 8). On an average, each PST who wrote a successful multiplication story problem addressed roughly two (1.98) of the four groups of strategies. PSTs were most likely to address children who used the skip counting strategy than those who used direct modeling, repeated addition, or a doubling strategy.

Unlike the Counting Sequences assignment, for this assignment we also coded the success of PSTs' attempts to address student thinking. In order to be coded as a successful attempt, the number choice had to address the student's current misconception with the use of the strategy, or "encourage" students to adopt a more efficient strategy. We organized our examination of these data according to the specific

Table 9 Fishbowl number choices—number of PSTs addressing learning goals (N=60)

Number of goals addressed	# PSTs	Percent PSTs
3	11	18.3
2	16	26.7
1	24	40.0
0	9	15.0

PSTs preservice teacher

Table 10 Fishbowl number choices—number of PSTs addressing specific learning goals (N=60)

Learning goal	# PSTs addressed (%)	# PSTs successful (%)
Skip count by 10s	31 (51.7)	14 (45.2)
Doubling	34 (56.7)	28 (82.4)
Distributive property	19 (31.7)	17 (89.5)

PSTs preservice teacher

strategy categories. Using the results from Table 9, we determined success rates for the 60 PSTs who wrote appropriate multiplication problems ranged between 43 and 68 % in addressing student thinking within their number choices. It is interesting to note that, although PSTs were most likely to address students who used a skip counting strategy, their number choices were least likely to be appropriate. In many cases, PSTs wrote skip counting number choices that were less challenging than those Molly posed in the fishbowl problem.

In addition, we also examined PSTs' number choices within the 60 appropriate multiplication problems to determine if they explicitly addressed Molly's learning goals. We were interested in how many goals PSTs addressed and which specific goals were addressed most often. Although less than 20 % of PSTs addressed all three learning goals (Table 9), the vast majority explicitly addressed at least one. The doubling and skip counting goals were addressed by over 50 % of the PSTs. We also examined PSTs' number choices in terms of their support of the learning goals. In order to be coded as successfully addressing a goal, the number choice had to support student development for the learning goal it was intended to address. Success rates for the skip count by tens, doubling, and distributive property goals were 45.2, 82.4, and 89.5 %, respectively (Table 10). As in the thinking strategies data, the learning goal addressed least frequently, the distributive property goal had the highest success rate among the 60 PSTs who wrote appropriate multiplication problems.

We conclude with an examination of the overall success of PSTs to successfully engage in the professional noticing of students' thinking within the fishbowl problem assignment in relation to all 72 PST responses. Our results indicate PSTs were relatively successful in attending to and interpreting students' thinking (75 % of 72 PSTs). PSTs were also successful in responding to students in the context of writing an appropriate story problem for the next day (83.3 % of 72 PSTs). However, when asked to create and justify specific number choices to address or further student thinking, PSTs were much less successful. Approximately, 29 % of 72 PSTs provided appropriate number choices for the direct modeling group, 25 % provided appropriate number choices for the skip counting group, 21 % provided appropriate number choices for the doubling group, and 17 % provided appropriate number choices for the repeated addition group—an average of 23 % for each category of student strategies. Similar results are evident when we examine PSTs' ability to consider specific learning goals while responding to student thinking. Less than 20 % of the 72 PSTs selected appropriate numbers to address the skip count by tens learning goal, 24 % selected appropriate numbers to address the distributive property learning goal, and 39 % selected appropriate numbers to address the doubling learning goal.

The fishbowl problem data support our results from the Counting Sequences activity. We can see by this stage in our instructional sequence a majority of the PSTs attended to and interpreted the student work provided and were able to write an appropriate story problem type. However, the complexity of this task made it very difficult for PSTs to write appropriate number choices for the next story problem. Our results demonstrate (a) PSTs had difficulty in addressing multiple groups of student thinking simultaneously; (b) when PSTs did attempt to write specific number choices to address or further student thinking, they were not often successful in doing so; and (c) PSTs had difficulty writing number choices that attended to both student thinking and learning goals.

Discussion

We set out to examine the question, "How can our series of professional noticing activities support PSTs' abilities to pose problems that build on students' mathematical thinking?" We intended to demonstrate how a sequence of activities, designed to scaffold PSTs' development along three dimensions, could be useful in helping PSTs learn to engage in the professional noticing of children's mathematics. Our results indicated PSTs' experiences across the three activities and assignments resulted in relatively successful engagement in the three facets of professional noticing, as measured within the Counting Sequences and fishbowl assignments. The average percentage from these two activities revealed 73 % of PSTs demonstrated evidence of attending to student strategies, 63 % demonstrated evidence of interpreting student thinking, and approximately 20 % demonstrated evidence of utilizing student thinking in posing their next problem.

As a frame of reference for these percentages, we cite Jacobs and colleagues' (2010) examination of teachers' professional noticing of children's mathematics. In the study, teachers were given two mathematical tasks, complete with examples of children's responses to interpret and respond to. Teachers' responses were analyzed and rated for each of the three skills involved in professional noticing. The study also included a group of PSTs enrolled in a first mathematics for teachers content course. The PSTs' results served as baseline data and helped demonstrate that as teachers gain experience and training, they are more successful in engaging in the professional noticing of children's mathematics (Jacobs et al. 2010). PSTs in their study demonstrated evidence of attending to student strategies 46 % of the time. In terms of interpreting student strategies, 47 % of PSTs demonstrated limited evidence, while 53 % provided no evidence. Results for responding based on student understanding were even less encouraging, as 14 % provided limited evidence and 86 % provided a lack of evidence (Jacobs et al. 2010).

We recognize that the differences in tasks, scales, and measures used in the two studies make it harder to justify direct comparisons between the groups of PSTs. Further, the PSTs in our study were provided explicit instruction in the aspects of professional noticing, while the PSTs in the study Jacobs et al. (2010) had conducted had not received any instruction prior to their participation in the study. Given these differences, it is not surprising that different percentages of PSTs in each study were successful at attending, interpreting, and responding. More interesting are the similarities across our findings, despite the differences in the studies. As we examine the relative success rate across the three skills of professional noticing, we see PSTs in each study are far less successful in responding to student thinking. Though both groups of PSTs demonstrated emerging abilities to interpret student thinking and "diagnose" mathematical inconsistencies, they have either not yet developed the appropriate knowledge bases, or are unaware how to leverage them, in order to respond effectively in "prescribing" the next treatment. This result supports the importance of teaching experience on the development of this practice (Jacobs et al. 2010).

In other ways, our results differ from those of Jacobs and her colleagues. Examining the success rates within each skill, our participants appear to be more advanced in their professional noticing abilities. PSTs' educational experience can account for some of these differences. These differences suggest that the continued development of PSTs' mathematical knowledge for teaching, either through mathematics courses, elementary content specific courses, or methods course experiences, can improve PSTs' ability to successfully engage in professional noticing. One implication for the field of mathematics education is to ensure elementary mathematics content courses provide opportunities for PSTs to engage in the analysis of children's work, as well as to develop, understand, and practice commonly developed student strategies for computation.

Beyond the differences explained by educational background, we posit our sequence of activities also influenced PSTs' ability to develop their skills in the professional noticing of students' thinking within the context of the methods course. As evidenced by the percentage comparisons between the Counting Sequences and fishbowl assignments, our PSTs increased their abilities in attending (71–75 % of PSTs successful), interpreting (51–75 %), and responding to student thinking (18–23 %). The increase in attending and interpreting percentages suggests PSTs improved their ability to make sense of and evaluate students' thinking strategies in a variety of mathematical contexts. This improvement may reflect a development in or employment of specialized content knowledge and knowledge of content and students through repeated opportunities to examine and discuss authentic student work (both video and written). PSTs' progress within responding is still lagging much further behind. We believe there must also be ways to address these concerns within the methods course, an issue to which we now turn our attention.

We recognize the skills of professional noticing as interrelated, and as teachers interact with children's mathematics, they shift between and within attending, interpreting, and responding in a fluid manner. In terms of attending to and interpreting children's mathematics, we believe our PSTs' prior educational experiences, supported by our series of activities, provided them with a strong foundation for success in these aspects of noticing. In comparison, however, these experiences did not result in comparable gains in responding to children's mathematical thinking. Our findings have two main implications for teacher education in terms of the overall development of PSTs' professional noticing of students' thinking: (a) methods course activities require significant scaffolds to support PSTs' incremental development; and (b) beyond approximations of practice (Grossman et al. 2009), PSTs require opportunities to engage directly with students and their thinking.

Scaffolds and educative supports that address PSTs' ability to respond appropriately to student thinking are vital in methods course activities that target professional noticing. The professional noticing of children's mathematics is a difficult practice, even for experienced teachers who have received professional development in this area. While we believed our trajectories of development gradually increased the complexity of situations in which PSTs were engaged, we recognize that still more supports are necessary. For example, we see a need for another activity and assignment prior to the fishbowl problem, as there is a relatively large leap in complexity from asking PSTs to respond to a single general misconception, to asking them to respond to a large number of students' work, while also considering a variety of learning goals. Including an activity that expands PSTs' responsibility from addressing a single concept to a task situated in small group instruction addressing a single learning goal might be an appropriate next step in supporting PSTs' development.

A second support necessary in our case, and worthwhile as a suggestion for all methods courses, would explicitly address the question, "What makes a number choice appropriate or inappropriate to support/extend a student's current way of thinking?" While many PSTs in the fishbowl activity wrote a story problem with appropriate structure, their number choices did not often reflect a successful response to students' thinking. An activity that presents an example of student thinking and requires PSTs to select and justify an appropriate number choice from a list of possibilities would support PSTs' developing ability to interpret, evaluate, and write appropriate number choices in consideration of children's current and future strategies.

We believe our sequence of activities, as approximations of practice, can provide PSTs with opportunity to develop facility with the skills of professional noticing of students' thinking. However, as Jacobs et al. (2010) demonstrated, teaching experience also plays a large role in developing expertise in this area. While sustained teaching experience is not a part of the methods course, we believe affording PSTs opportunity to enact this practice in real time with actual students within the structure of a methods course would be extremely beneficial. To begin with, it is crucial we help PSTs to problematize the work of teaching. In our experience, this rarely happens for PSTs outside of face-to-face interactions with children. We suggest that engaging PSTs in a short one-on-one problem-posing experience with a student prior to the sequence of methods course activities can help develop PSTs' understanding of the complexity of teaching. After PSTs are faced with this struggle, they may be more able to relate to the construct of professional noticing. Then, following our sequence of experiences, a final opportunity for PSTs to interact one-on-one with a student and engage in the practice of professional noticing of student thinking would then take place.

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Examining the Relationship Between Preservice Elementary Teachers' Attitudes Toward Mathematics and Professional Noticing Capacities

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The past 2 decades have seen a growing interest in research on teachers' attitudes about mathematics due to the potential influence they have on the way teachers enact classroom practices and understand mathematics (Grootenboer 2006; McLeod 1992, 1994; Pajares 1992; Philipp 2007; Wilkins 2008). Research on attitudes is critical because students and teachers often develop negative attitudes toward mathematics, which can

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later result in anxiety (Quinn 1997). Preservice elementary teachers (PSETs) have tended to view mathematics as a system of rules and procedures that must be transferred to students (Ball 1990; Foss and Kleinsasser 1996). Moreover, studies have reported that PSETs tend to view mathematics negatively or neutrally, but rarely positively (Ambrose 2004; Ball 1990; Bekdemir 2010; Quinn 1997). It is important to understand the attitudes PSETs hold because, these attitudes have the potential to influence the instructional practices they adopt. Ambrose (2004) suggested that mathematics educators focus on the range of strengths PSETs bring, such as the PSETs' view of teachers as nurturers of children. Jong and Hodges (2011) have also argued that mathematics educators take an asset-based approach to build on PSETs' prior knowledge and experiences. They found that it was possible for PSETs to experience positive changes in attitudes as a result of completing a mathematics methods course that connected past experiences with reform-oriented methods. We believe that our module on professional noticing within the mathematics methods courses could also positively affect PSETs' attitudes toward mathematics, because the module supports PSETs' development of strategies that might improve their own view of mathematics and teaching mathematics. Thus, our work focuses on PSETs and their potential changes in attitudes toward mathematics by engaging them in professional noticing of children's mathematical thinking in the context of an early numeracy progression.

This research is informed by the literature on teachers' attitudes toward mathematics (Grootenboer 2006; McLeod 1992; Philipp 2007), professional noticing (Jacobs et al. 2010), and the progression of early numeracy (Clements and Sarama 2009; Steffe 1992; Steffe et al. 1988; Steffe et al. 1983). Our previous analyses of the data from this project focused specifically on a learning experience using video vignettes to teach PSETs about professional noticing within the context of early numeracy. That analysis revealed significant gains in the professional noticing skills of the PSETs involved in this instructional module (Schack et al. 2013). Research has shown that positive correlations exist between PSETs' attitudes toward mathematics and their content knowledge (Matthews and Seaman 2007; Quinn 1997). Based on the literature and our previous analyses, we aimed to expand our analyses to examine whether similar relationships exist between our PSETs' attitudes toward mathematics and their professional noticing skills. Specifically, we investigated the following research questions: *Did PSETs who engaged in a professional noticing learning experience exhibit improvements in their attitudes toward mathematics*? *If so, what correlations exist between aspects of PSETs' professional noticing performance and attitudes toward mathematics*?

Theoretical Framework and Literature Review

Preservice Elementary Teachers' Attitudes Toward Mathematics

Research on attitudes toward mathematics has become an increasingly prominent area of study. Attitudes are often defined as a component of *affect* (Philipp 2007), which has various meanings in the field of psychology (Chamberlin 2010). Mathematics educators have attempted to differentiate attitudes from beliefs by characterizing beliefs as having true or false orientations (Philipp 2007), but the field still varies on its use of definitions of beliefs (Beswick 2005; Pajares 1992). We focus on attitudes because mathematics educators are more consistent in their definition of attitudes in contrast to beliefs. For the purpose of our study, we draw upon Philipp's (2007) definition of attitudes as "manners of acting, feeling, or thinking that show one's disposition or opinion…. Attitudes, like emotions, may involve positive or negative feelings" (p. 259).

In the context of teacher education, researchers have studied PSETs' attitudes toward mathematics through coursework and field experiences (Jong and Hodges 2011; Quinn 1997; Wilkins 2008). Several studies have shown positive changes and significant relationships between PSETs' attitudes toward mathematics and their content knowledge (Matthews and Seaman 2007; Quinn 1997; Young-Loveridge et al. 2012). It is not a surprise that PSETs with stronger content knowledge also develop a more positive attitude toward mathematics because they are more likely to show favor toward a subject in which they are more confident. Quinn (1997) used Aiken's Revised Mathematics Attitudes Scale to investigate attitudes and found that there were statistically significant positive changes in PSETs' attitudes toward mathematics, but no significant changes in attitudes toward mathematics of preservice secondary mathematics teachers, who entered with higher attitude measures. He argued, and we agree, that the methods course could play a critical role in the improvement of attitudes toward mathematics, especially for PSETs who have negative attitudes toward mathematics. Matthews and Seaman (2007) also used Aiken's Revised Mathematics Attitudes Scale to examine the mathematics attitudes of PSETs enrolled in a course that emphasized a conceptual understanding of mathematics. Their results indicated that the course had a significant positive influence on PSETs' attitudes toward mathematics, along with their content knowledge. Young-Loveridge et al. (2012) found that if PSETs were "good at mathematics [it] did not automatically mean that [they] liked mathematics" (p. 38). Their study challenged the notion that those who have strong mathematical content knowledge also have a positive attitude toward mathematics. While studies have examined the relationship between PSETs' attitudes and their mathematics content knowledge, there is an absence of studies that examine PSETs' attitudes and their professional noticing skills.

We also know that stronger content knowledge alone does not lead to effective teaching (Hill and Ball 2004). In fact, Wilkins (2008) found that strong content knowledge was negatively related to beliefs about effective mathematics instruction that aligns with practices recommended by the National Council of Teachers of Mathematics (2000). That is to say, teachers who had stronger content knowledge did not necessarily agree with more reform-oriented practices in mathematics. Wilkins (2008) also found that beliefs about mathematics have a strong influence on teachers' practices, although teachers with more positive attitudes toward mathematics were more likely to have an orientation toward inquiry-based practices. For example, teachers who believe that mathematics is primarily a system of rules and procedures that must be transferred to students will instruct in a manner that reflects this belief whether or not they have a positive attitude. However, those with a positive attitude are more likely to hold beliefs that reflect inquiry-based practices. This suggests that building positive attitudes toward mathematics in PSETs could lead to their adoption of more inquiry-based practices. Unlike the aforementioned studies, Wilkins used a combination of items to measure attitudes to include items about the enjoyment of teaching mathematics along with liking mathematics as a subject. Schackow (2005) used the Attitudes Toward Mathematics Inventory (ATMI) (Tapia and Marsh 2005), which was not originally designed for use with preservice teachers; however, by changing two questions to make it more suitable for preservice teachers, she found significant gains in PSETs' attitudes toward mathematics. In our study, we used Schackow's revised version of the ATMI, since it was edited for use with preservice teachers, and the ATMI (Tapia and Marsh 2005) also measures multiple factors of attitudes rather than a single factor. While many mathematics education researchers have attempted to examine changes in PSETs' attitudes, there is still more work needed in the field on identifying conditions to support PSETs in developing positive attitudes (Grootenboer 2006; Jong and Hodges 2011; Quinn 1997; Wilkins 2008). Thus, we have established an instructional module to aid in increasing PSETs' professional noticing skills, and subsequently, supporting PSETs' attitudinal improvement.

Our instructional module situates the professional noticing of children's mathematics in the context of mathematics progressions, specifically an early numeracy progression, illustrated through video representations of children's work. The focus on children's mathematical work capitalizes on PSETs' nurturing attitudes about teaching (Ambrose 2004) and also reveals to them the complexities of the mathematics content. The content of early numeracy, on the surface, seems simple for PSETs to understand, because it encompasses such skills as forward and backward counting, skip counting, and addition and subtraction of numbers within 100. As PSETs view video vignettes of children engaged in mathematical thinking along the early numeracy progression, they are exposed to the idea that counting, for example, is not an "all or nothing" skill. The children in the videos used during our instructional module display nuanced understandings and skills that demonstrate the incremental but important steps through which children progress. Providing meaningful and focused experiences in both coursework and clinical or field activities, as the aforementioned studies show, can potentially relate to a positive shift in PSETs' attitudes toward mathematics. Our intent is to investigate whether engaging PSETs in a focused experience that integrates representations of practice with coursework and is specifically aimed to develop professional noticing of children's mathematical thinking results in a positive shift in PSETs' attitudes toward mathematics. If PSETs' attitudes shift positively in relation to developing professional noticing skills, the improved attitudes could result in PSET adoption of more reform-oriented instructional practices. And, incorporation of professional noticing skill development in preservice coursework would be one tool for improving PSETs' attitudes toward mathematics.

Research on attitudes toward mathematics and noticing skills of teachers, individually, is not new; however, there is no previous research comparing the professional noticing capacities with attitudes toward mathematics, within the same study. Due to the importance of both of those constructs, and supported by the evidence by Wilkins (2008) in which attitudes were found to have an influence on teachers' practices, we think a better understanding of this relationship seems warranted.

Professional Noticing

The construct of professional noticing, as defined by Jacobs et al. (2010), is "a set of three interrelated skills: attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings" (p. 172). The first skill, attending, contains physical evidences observed from the student and teacher, such as eye movements, finger counting, and touching objects to count. The second skill, interpreting, is determining how those observations in the attending category can inform the observer on the mathematical abilities of the students. Finally, deciding involves the next steps in the process, which can include diagnostic or instructional decision making. Jacobs et al. (2010) found that teaching experience alone does not contribute to an increase in professional noticing skills; professional development in the area of professional noticing is needed to adequately develop these skills, especially in the deciding component. Several studies have found that closer attention to children's mathematical thinking can significantly impact student learning (Carpenter et al. 1999; Kersting et al. 2010); however, attention to the three interrelated components of professional noticing is missing from much of the previous research.

Numeracy and Counting

The term *numeracy,* a portmanteau of "numerical literacy," is typically invoked to describe an understanding of number and arithmetic operations. This area of mathematical learning has been the subject of considerable study over the past four decades (Gelman and Gallistel 1978; Siegler and Robinson 1982; von Glasersfeld 1982; Steffe et al. 1983; Fuson 1988; Steffe 1992; Wright et al. 2006; Clements and Sarama 2009; Thomas and Harkness 2013). In the USA, young children's initial numeracy experiences typically involve counting (Steffe et al. 1983, 1988; Wright 1994). Depending upon the context, counting can describe several different activities. For example, the term might be used to describe the production of a specified verbal sequence (e.g., "four, five, six, seven"). Similarly, counting might also refer to verbal utterances when presented with a sequence of numerals. These descriptions of counting, however, fail to capture the potentially quantitative aspect of the activity. To capture this aspect, Steffe et al. (1983) investigated children's counting activity in the face of problematic situations dealing explicitly with quantities. Wright and his colleagues describe this type of *quantitative counting* as "the coordination of each uttered number word with the conceptual production of a unit item" (Wright et al. 2006, p. 52).

To further define early numeracy progression, Steffe and his colleagues (Olive 2001; Steffe et al. 1983, 1988; Steffe 1992) put forward a framework, ultimately termed the *Stages of Early Arithmetic Learning* (SEAL), that is an extremely descriptive progression constructed from highly authentic research methods. This progression was used in our previous research (Schack et al. 2013) to further investigate the interpretations of mathematical thinking within the professional noticing context. We hypothesized that combining the early numeracy progression of SEAL into a learning experience focused on professional noticing and then using representations of practice in an authentic diagnostic setting would improve PSETs' attitudes toward mathematics.

Methodology

Instructional Module Description

The instructional module was a component of the methods or blended course at each institution. The module consisted of multiple in-class sessions in which professional noticing was developed in the context of early numeracy, specifically, SEAL, and organized around multiple video cases of children engaged in early numeracy experiences. The decomposition of professional noticing into three interrelated skills allowed for the skills to be progressively nested (Boerst et al. 2011) throughout the module sessions. The instructional module embedded video vignettes throughout the lessons to provide examples of one-on-one interviews with children for PSETs to practice their newfound professional noticing skills. The first two sessions focused solely on the development of attending. Subsequent sessions further developed attending with interpreting and deciding. SEAL was nested within the development of professional noticing and integrated through the video cases as representations of practice. Additional assignments throughout the instructional module included assigned readings on SEAL and video-based assessments and other activities, such as role-playing, small group discussions, whole-class discussions, and another activity entitled "World Café," where PSETs rotate around the room discussing their observations from a selected video clip and creating Venn diagrams of their findings. The culminating experience was an assignment that required the PSETs to conduct and record (on video) at least one diagnostic interview with a child during their field experience—an approximation of practice, one of the three pedagogies of practice proposed by Grossman et al. (2009), where they addressed the three stages of professional noticing within their written analyses of their interview. Schack et al. (2013) provides further information about the individual modules, rationale for the selection of early numeracy as the mathematics focus, and measures taken to ensure fidelity across multiple sites.

Participants

All 270 PSETs enrolled in the participating researchers' 11 mathematics education course sections over a two-year period participated in the module. They were predominantly women, enrolled in an elementary mathematics method or a blended content and methods course at one of the five participating public universities in a south central state. The five institutions' populations, taken as a whole, represent varying regions of the state because of their locations in central, northern, western, and eastern parts of the state. The populations also represent rural, suburban, and urban areas. Notably, two of the regional universities draw much of their population from central Appalachia, a traditionally underrepresented population in science, technology, engineering, and mathematics (STEM) fields. The majority of the participants were traditional university students, while some were second-career students.

Data Sources and Analyses

Data analyzed for this study include that which was collected from the 123 PSETs who provided consent and responded to all questions on both the pre- and post-assessments of professional noticing and the pre- and post-assessment of the ATMI (Tapia and Marsh 2005). The number of PSETs who did not complete all pre- and post-assessments and those who completed them yet left items blank, particularly on the ATMI, affected the response rate. One can only speculate as to the reasons for leaving items blank.

Professional Noticing Assessment A pre- and post-assessment was used to measure the changes in professional noticing at the beginning of the semester and again at the end of the semester. A video of a diagnostic interview with a child completing a comparison, difference unknown task (Carpenter et al. 1999) was used, and both pre- and post-assessments were identical in prompts and video. The brief 25-sec video shows an interviewer presenting a first-grade student with a partially screened task that extends beyond finger range. The screened component consists of 11 seashells hidden by the interviewer's hand and the visible component is seven red counting bears in a row. The student is asked to determine how many more shells there are than bears. Counting the bears from one and continuing the count on his fingers until he reaches 11, the student then glances at his raised fingers and correctly responds, "I'm gonna have four left over."

After the PSETs watched the video, they were asked to respond to the following prompts and questions: (1) *Please describe in detail what this child did in response to this problem,* (2) *Please explain what you learned about this child's understanding of mathematics, and* (3) *Pretend that you are the teacher of this child. What problems or questions might you pose next*? *Provide a rationale for your answer.* These prompts were drawn from the work of Jacobs et al. (2010) and each prompt addresses one of the components of professional noticing.

The researchers, individually and as a group, examined the clip to reach a consensus on the key response details for each professional noticing prompt to develop scoring benchmarks along the continuum for each of the prompts. A sample set of data was examined to extract qualitatively different response types for each prompt and the response types were subsequently examined for emergent themes (Glaser and Strauss 1967). The emergent themes were coupled with the researchers' key response details to define the benchmarks or rankings. This process was used for each prompt and resulted in four potential rankings for attending, three for interpreting, and three for deciding. The high rank of four for attending represents an emergent theme from PSET responses that represented an elaboration beyond the salient attending features. A similar elaborating theme did not emerge for the remaining two professional noticing components.

Teams of two scorers ranked the data, and ranks by different scorers were compared. Discrepancies in ranks were resolved through discussion and/or a third scorer if a consensus was not met. The desire to better standardize rankings by multiple scorers and to make the ranking process more efficient led to the development of a decision tree-scoring device with multiple levels of questions to guide the scorers' rankings. Interrater reliability across six scorers using the decision trees averaged 83 % (Schack et al. 2013).

Attitudes Toward Mathematics Inventory The ATMI was administered as a pre- and post-assessment at approximately the same time as the professional noticing assessment. The ATMI is an instrument consisting of 40 Likert-scale items with five response choices ranging from *strongly disagree* to *strongly agree*. Eleven items are reverse scored. We selected the ATMI over Aiken's (1963) Revised Mathematics Attitudes Scale, because the ATMI included more factors about mathematics. Factor analysis on the ATMI resulted in four factors associated with attitudes toward mathematics: value, enjoyment, self-confidence, and motivation (Tapia and Marsh 2005), whereas Aiken's instrument considered only enjoyment and value. Table 1 illustrates sample items by factor with reverse-scored sample items noted. The total number of items per factor is also included. Scores are determined by summing all items and items within factors. Maximum possible scores for each factor vary because of the differing number of items per factor.

The ATMI was determined by Tapia and Marsh (2004) to be a reliable instrument with a Cronbach alpha coefficient of 0.97. Cronbach alpha coefficients for each of the four factors range from 0.88 (motivation) to 0.95 (self-confidence). Test–retest reliability was established with a Pearson correlation coefficient of 0.89 for the total scale. Coefficients of the subscales ranged from 0.70 (value) to 0.88 (self-confidence) (Tapia and Marsh 2004). The ATMI was initially used with secondary students but has since been used with postsecondary students (Schackow 2005; Tapia 2012). Schackow (2005) modified two items for use with a sample of PSETs. She determined a Cronbach alpha coefficient of 0.98 for this sample, indicating internal consistency of the modified instrument. Our study used the instrument modified by Schackow for use with preservice teachers.

Table 1 Sample Attitudes Toward Mathematics Inventory items by factor

Factor	Sample attitudes
Value (10 items)	Mathematics is a very worthwhile and necessary subject. Mathematics courses would be very helpful no matter what grade level I teach.
Enjoyment (10 items)	I really like mathematics. I have usually enjoyed studying mathematics in school.
Self-confidence (15 items)	Mathematics does not scare me at all. Studying mathematics makes me feel nervous. (reverse scored)
Motivation (5 items)	The challenge of mathematics appeals to me. I would like to avoid teaching mathematics. (reverse scored)

Table 2 Pre- and post-assessment results of paired *t*-tests

	N	*t*	*p*-value
Value	123	−1.543	0.125
Enjoyment	123	−3.070	0.003
Self-confidence	123	−5.057	<0.001
Motivation	123	−2.733	0.007

Analyses Paired *t*-tests were used to examine the relationships between the pre- and post-assessment scores for the ATMI. However, due to the ordinal nature of the data for the professional noticing scores, similar parametric tests could not be used to test for significance for professional noticing measures or the correlation between attitudes and professional noticing. A statistical test using Spearman's rho was used to determine, if a correlation exists between the change scores of the two assessments, and further investigation by quartile on the ATMI was also conducted.

Results

Attitudes Toward Mathematics Inventory

Paired *t*-tests were applied to ATMI data to examine PSET change in attitudes toward mathematics from pre- to post-assessment. There were statistically significant increases in the enjoyment, self-confidence, and motivation factors. Table 2 summarizes the paired *t*-test results for the 123 cases. There was not a statistically significant change in the fourth factor, value, when all 123 cases were included. However, when the 15 cases that achieved the maximum possible score on the value factor on the pre-assessment were removed, there was a statistically significant increase in this factor from pre- to post-assessment ($t = 2.181$, $p = 0.031$).

The significance of the change in the value factor, after removing the data for maximum possible scores on the pre-assessment, indicated a need to further examine the change scores of all factors more closely. For example, if a PSET started at

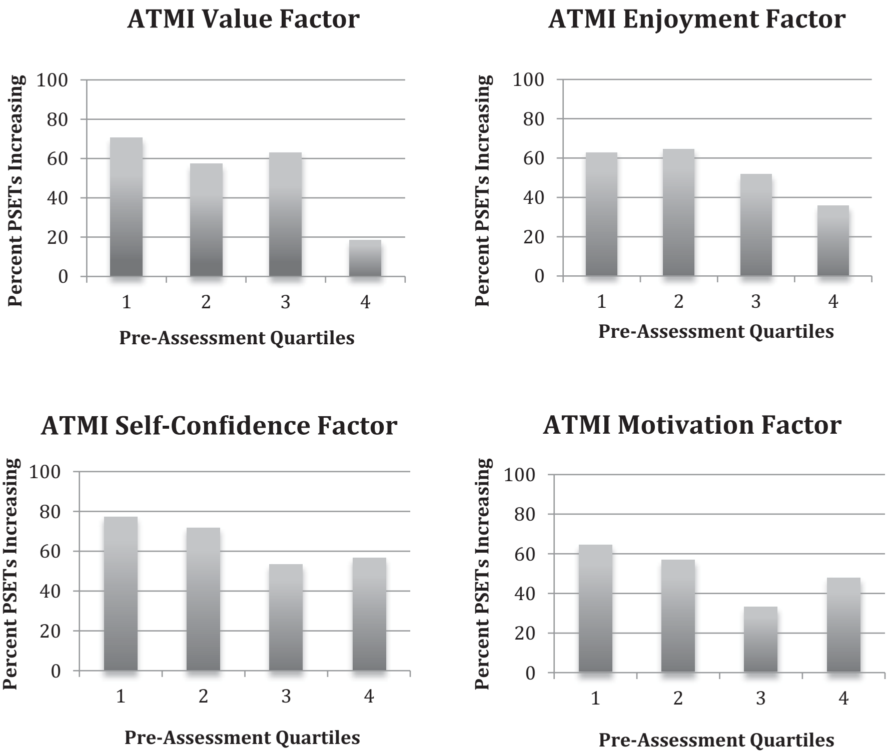

Fig. 1 Percent PSET increase within factors by pre-assessment quartiles

a lower score for a factor, the potential range of increase was much greater than, for example, a PSET with a pre-assessment score at or near the maximum. Hence, the ATMI data were examined within pre-assessment quartiles. Figure 1 shows the percent of PSETs increasing for each factor by pre-assessment quartile. The percent of PSETs with a pre-assessment score in the lowest quartile showing an increase on the post-assessment ranged from 63 to 77 % for the four factors. Remarkably, even those PSETs in the highest pre-assessment quartile for motivation and self-confidence showed that 48 and 57 % of the PSETs increased their scores, respectively.

Overall, more than 50 % of the PSETs increased from pre- to post-assessment for each of the four factors. The self-confidence factor had the greatest percent of PSETs increasing with 65 %. The 95 % confidence interval for the "improvement" rate for this factor was 55.9–73.4 %. The value that scored the lowest percentage of increase was motivation, with 51 % of PSETs demonstrating an increase. Value and enjoyment revealed increases at 54 and 55 %, respectively. Regrettably, approximately 30 % of PSETs decreased in each category except self-confidence, where the percentage of decrease was 21 %; thus, the remaining PSETs remained unchanged in their scores on those factors.

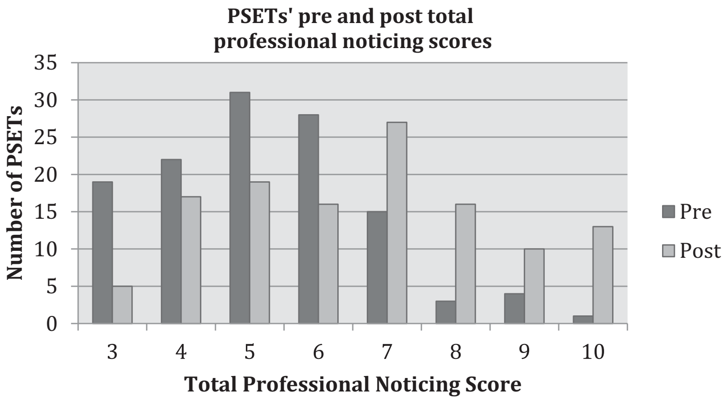

Fig. 2 Professional noticing total score frequencies

Professional Noticing Assessment

Our research questions sought to determine the correlation of changes between PSETs' professional noticing and attitudes toward mathematics. Figure 2 shows the overall trend of improved PSET professional noticing total scores. Ten is the maximum possible score, while a score of three is the minimum.

Professional noticing is a complex construct (Jacobs et al. 2010) with the potential for the three components—attending, interpreting, and deciding—to be interdependent. Spearman's rho correlations were performed to determine correlations between the professional noticing components on the pre- and post-assessments as well as on the interrelationships of the changes in the components. Interestingly, the results indicated no significant correlation between any two components on the pre-assessment and no significant correlations between changes in components from pre- to post-assessment, yet there were significant positive correlations between components on the post-assessment. Tables 3 and 4 show the results of the Spearman's rho correlations.

The professional noticing data were analyzed using nonparametric statistics because of the ordinal nature of the rankings. Wilcoxon signed-rank tests were employed to determine if there was growth in each component of PSETs' professional noticing. The results, indicating statistically significant increases in all three components, are displayed in Table 5. The larger *z*-score in deciding can be attributed to that component receiving the largest overall growth, relative to attending and interpreting. This growth is illustrated by comparing the top section of bars from pre to post for each component in Fig. 3.

At first examination, the lack of correlations among components on the pre-assessment and on changes from pre- to post-assessment, along with the significant correlations among the professional noticing components on the post-assessment, is a bit confounding. However, consideration of the trends we observed in the data

Table 3 Spearman's rho correlations between professional noticing components on pre- and post-assessments

			Pre-assessment		Post-assessment	
Variable 1	Variable 2	*N*	Spearman	*p*-value	Spearman	*p*-value
Attending	Deciding	123	0.117	0.196	0.270*	0.003
Attending	Interpreting	123	0.108	0.234	0.339*	<0.001
Deciding	Interpreting	123	0.169	0.061	0.235*	0.009

* Significant at $p = .01$.

Table 4 Spearman's rho correlations between professional noticing components on pre/post changes

			Pre/Post change	
Variable 1	Variable 2	*N*	Spearman	*p*-value
Change in attending	Change in deciding	123	0.134	0.139
Change in attending	Change in interpreting	123	0.146	0.108
Change in deciding	Change in interpreting	123	0.124	0.173

Table 5 Results of Wilcoxon signed-rank tests on professional noticing components from pre- to post-assessment

	Scale	*N*	*z*	*p*-value
Attending	1–4	123	−3.466	<0.001
Interpreting	1–3	123	−3.841	<0.001
Deciding	1–3	123	−5.378	<0.001

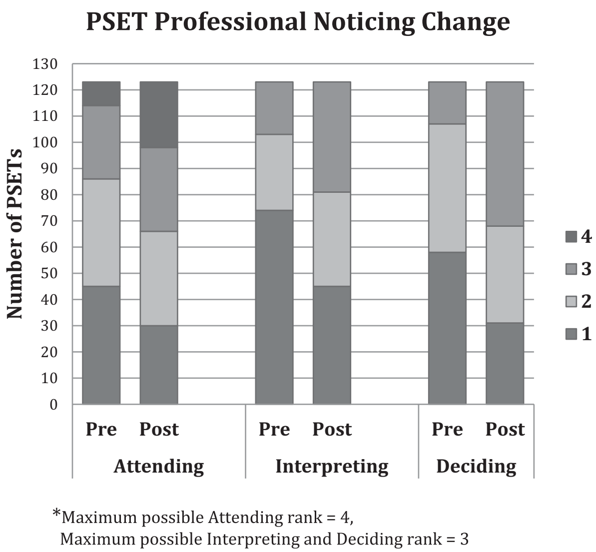

Fig. 3 Number of PSETs scoring each rank (by component) at pre- and post-assessment

enlightens this result. Figure 3 illustrates the relative changes in the number of PSETs scoring each rank on pre- and post-assessment for each professional noticing component. On the pre-assessment, PSETs' attending scores were spread fairly evenly across ranks of 1, 2, and 3 or greater, while their pre-assessment scores in interpreting and deciding tended to cluster in the lowest rank of 1, resulting in a lack of correlations among components but greater room for growth in interpreting and deciding. Growth in these two components, particularly deciding, was greater than attending, resulting in a lack of correlation among components on growth. The encouraging outcome is on the post-assessment, where the spread across ranks was more similar on all three components (Fig. 3), especially when comparing attending and deciding, resulting in significant correlations among components. One might interpret this to mean that many PSETs enter methods with an ability to observe the details of children's mathematics, but with less skill in interpreting and deciding. This is consistent with Jacobs et al.'s (2010) conclusion that learning to professionally notice requires deliberate professional development in each of the components of professional noticing. Furthermore, the ability to attend to children's mathematics that many PSETs bring to a methods course lends support to the perspective of Ambrose (2004) and Jong and Hodges (2011) to take an asset-based approach to PSET education, building upon their view of teachers as nurturers of children.

Correlation Between Professional Noticing Assessment and Attitudes Toward Mathematics Inventory (ATMI)

Spearman's rho correlations were employed to examine the correlation between the change in professional noticing and the change in attitudes toward mathematics. For reasons described earlier, the individual components of professional noticing and the factors of ATMI were analyzed for significant growth. While significant PSET growth was found in all three professional noticing components and in three of four factors of the ATMI, there was no statistically significant correlation between changes in professional noticing and changes in attitudes toward mathematics at the component/factor level or when overall scores of both professional noticing and ATMI were analyzed ($r_s = -0.020$, $p = 0.828$). The results of the correlation of changes in each professional noticing component and ATMI factor are found in Table 6.

Discussion and Implications

In this study, we hypothesized that developing PSETs' professional noticing skills in the context of children's early numeracy would relate to the improvement of PSETs' attitudes toward mathematics. Our hypothesis included the expectation that viewing children doing mathematics might capitalize on an oft-reported reason PSETs give for entering the teaching profession—their view of teachers as nurturers of children

Table 6 Spearman's rho correlations between changes in professional noticing components and attitudes factors

		Change in value	Change in enjoyment	Change in self-confidence	Change in motivation
Change in attending	Correlation coefficient	0.127	0.045	0.136	−0.009
	p-value	0.163	0.621	0.134	0.924
Change in interpreting	Correlation coefficient	−0.151	−0.081	−0.127	−0.052
	p-value	0.097	0.376	0.161	0.566
Change in deciding	Correlation coefficient	0.023	0.102	0.033	0.091
	p-value	0.799	0.261	0.715	0.316

(Ambrose 2004)—by motivating them to think mathematically themselves in order to better assist children in learning mathematics. Furthermore, the context of SEAL provides details of the common progression of early numeracy and children's mathematics (Steffe 2013), encouraging PSETs to see mathematics through the lens of a child, focusing on what children can do conceptually rather than on the procedures of mathematics that children cannot yet do, procedures that characterize what many PSETs believe mathematics to be. Our intention of this research is to push PSETs beyond their adult-level "first-order mathematical knowledge," as described by Steffe (2013), and to better understand the "second-order" knowledge, also known as the "mathematics of children." It is "frequently necessary to construct new ways of thinking mathematically to make adequate interpretations" (Steffe 2013, p. 368). By enhancing PSETs' understanding of children's mathematical thinking, they can, in turn, increase their own understanding of mathematics by constructing new ways of thinking, both pedagogically and mathematically, leading to greater self-confidence and, in general, more positive attitudes toward mathematics, and can contribute to closing the gap between research and practice.

The significant increase by PSETs in this study on three of the four factors of the ATMI (enjoyment, self-confidence, and motivation), and in the fourth factor (value) when maximum possible pre-scores are removed, is encouraging. This finding revealed the possibility that components of PSETs' attitudes can improve when experiencing a course where professional noticing skills are explicitly taught, modeled, and reinforced. Building on Wilkins' (2008) finding that positive attitudes toward mathematics resulted in beliefs aligned with reform-oriented practices, the positive shift of PSETs' attitudes in this study has the potential to result in the PSETs' adopting more reform-oriented instructional practices in mathematics. We are cautious with this claim as attitudes have many influences, including parents, peers, and past teachers (Tapia and Marsh 2004).

While we are encouraged by the growth in both professional noticing skills and in attitudes toward mathematics, the lack of correlation between the two presents the question as to why. There are several possible reasons why the lack of correlation occurs in this study. We will address two of these possible reasons. One possible reason for this lack of correlation could be due to the unmatched ordinal and interval scales of the professional noticing and ATMI assessments. The strength to draw more conclusions from parametric tests was not possible, with the professional noticing data scored as ordinal data and the ATMI measure calculated as interval data. Thus, the correlations had to be computed using a more simplistic nonparametric measure. At this time, other research studies that attempt to correlate attitudes about mathematics with professional noticing skills do not exist; thus, there is no prior literature with which to compare these results.

Another possible reason for a lack of correlation relates to the nature of the ATMI instrument. The ATMI primarily focuses on attitudes toward mathematics as a discipline, rather than attitudes toward teaching and learning mathematics, which could very well be more closely connected to gaining pedagogical skills, such as professional noticing. Further investigation on the lack of correlation between changes in attitudes and changes in professional noticing will be conducted in future studies to better understand this result, and we hypothesize that future use of the Mathematics Experiences and Conceptions Surveys (MECS), developed by Hodges and Jong (2012), to measure attitudes, beliefs, and dispositions will more closely correlate with attitudes toward teaching and learning of mathematics and not primarily on attitudes toward the discipline of mathematics.

Next Steps

PSETs experience a plethora of mathematical experiences throughout the duration of a methods semester, including field placements, in-class experiences, and other educational activities. The variety of influences makes it difficult to directly attribute PSETs' attitude change solely to their participation in the professional noticing instructional module. Nonetheless, PSET attitudes did improve, and a look at the results of the factors of the ATMI in this context is encouraging. Most notable is that more than 50 % of the PSETs increased in all four factors from pre- to post-assessment. The largest increase, 65 %, was exhibited in the self-confidence factor. We are hopeful that the increase in self-confidence is built upon an increase in Mathematical Knowledge for Teaching (MKT) (Hill and Ball 2004). The researchers have collected data on MKT (via the Learning Mathematics for Teaching Assessment) that will be examined independently and in relation to PSET changes in attitudes toward mathematics, as well as in relation to changes in professional noticing. More broadly, inquiries into the relationships among teaching practices (e.g., professional noticing), knowledge (e.g., MKT), and attitudes have the potential to shed increasing light on the mechanisms of change in each of these areas.

It is also interesting to note that 15 of the 123 PSETs entered the study with the highest possible rating on value of mathematics, indicating that at least some PSETs come to their teacher education courses valuing mathematics, despite the frequently reported fragile mathematical understanding harbored by PSETs (Ball

1990; Foss and Kleinsasser 1996; Quinn 1997). For those PSETs who already value mathematics, additional investigation of the processes by which these individuals developed such attitudes could aid in the refinement of the professional noticing module. Indeed, if one of the aims of the module is to positively affect PSETs' attitudes toward mathematics, a greater understanding of such change *with respect to our local contexts,* such as prerequisite courses, field placement requirements, and previous student experiences in education, holds the potential for increasing the effectiveness of the PSET learning experience.

Finally, the examination by quartile indicated that a large percentage of the PSETs scoring within the lowest quartile increased in all four attitudes toward mathematics factors. There seems to be a positive influence on these PSETs' attitudes toward mathematics in the methods and blended methods courses of this project, despite the lack of significant correlation to change in PSETs' professional noticing capacities. It is likely that incorporating a mid-semester measure of attitudes immediately following the professional noticing module would further illuminate the impact of the module on PSET attitudes. The data indicate that attitudinal growth is somewhat decoupled from growth in professional noticing capacities. However, a mid-semester measure may indicate a significant attitudinal growth linked to experiences within the professional noticing module.

Our primary research focus of PSET professional noticing has yielded positive results (Fisher et al. 2012; Schack et al. 2013). Collecting comparison data with similar groups of PSETs in their mathematics methods courses, where the professional noticing instructional module was not implemented, would help provide a picture of the effectiveness of the module (with respect to attitudinal change) in comparison with other teacher education activities. This could aid in determining whether other influences contributed to the growth in both professional noticing skills and attitudes toward mathematics.

Conclusion

We remain optimistic that professional noticing (i.e., attending, interpreting, deciding) is a *teachable* skill (Schack et al. 2013) and that research refinements can lead to more robust conclusions regarding the relationship between professional noticing and attitudes toward mathematics. Given the highly complex nature of both professional noticing and attitudes toward mathematics, it is, perhaps, expected that initial attempts to determine a relationship have proven elusive; however, there is much cause to continue such lines of inquiry. Returning to the notion that productive attitudes toward mathematics are crucial for the development of effective teaching practice, a firm understanding of mechanisms that promote such attitudinal change in conjunction with responsive teaching practices is essential.

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Transitions in Prospective Mathematics Teacher Noticing

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Student-centered instruction (Ball and Cohen 1999; Feiman-Nemser 2001; National Council of Teachers of Mathematics (NCTM) 2000; Sherin 2002) requires that teachers carefully attend to, assess the potential of, and respond to student ideas during instruction. To support such instruction, one important transition that prospective teachers (PTs) need to make is in the way classroom instruction is viewed—transitioning from viewing it as a student concerned with his or her own learning, to viewing it as a teacher attentive to student learning.

One venue where this transition might occur is in school-based field experiences, which have long been advocated as an important component of teacher education programs (Ishler and Kay 1981; Myers 1996). Such experiences have been criticized, however, for failing to provide a structure that allows PTs to learn about teaching in significant ways (Feiman-Nemser 2001; Leatham and Peterson 2010a; Philipp et al. 2007). It has been found, for example, that many PTs lack the skills to meaningfully observe and make sense of classroom interactions during field experiences (Masingila and Doerr 2002; Orland-Barak and Leshem 2009). Furthermore, the goals of field experiences are often not well articulated, resulting in the experience having little focus on mathematical content or students' understanding of it (Leatham and Peterson 2010a, 2010b). Thus, it has been suggested that substantial teacher educator involvement, including collaborative viewing and discussion of instances of practice, might be critical to supporting meaningful learning from field experiences (Masingila and Doerr 2002; Oliveira and Hannula 2008).

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Central to the transition from viewing the classroom as a student to viewing it as a teacher is the construct of *teacher noticing* (Sherin et al. 2011; Sherin and van Es 2005). Underlying this construct is the notion that some events that occur during instruction are more productive for a teacher to attend to because of their potential to be used to support student learning. Although experienced teachers might intuitively recognize and respond to productive moments that arise during a lesson, novices often fail to notice or respond to these same instances (Peterson and Leatham 2010; Stockero and Van Zoest 2013). In fact, a major difference between expert and novice teachers' practices has been found to be their ability to attend to and capitalize on important instructional events (Berliner 2001; Mason 1998). Although targeted noticing is not something that adults automatically know how to do (Jacobs et al. 2010), the results of numerous studies suggest that it is a skill that can be learned (Jacobs et al. 2010; Santagata et al. 2007; Sherin et al. 2011).

Accumulating evidence suggests that structured viewing of video recordings of classroom lessons is one effective method of supporting both prospective (Santagata et al. 2007; Star et al. 2011; Stockero 2008a, 2008b) and practicing (Jacobs et al. 2011; Sherin and van Es 2009; van Es and Sherin 2008) teachers' ability to notice important events that occur during instruction. By slowing down practice and eliminating a need to immediately react to classroom situations (Sherin 2004), the structured analysis of video has been found to help teachers develop a stronger focus on students and their learning of mathematics (Sherin and van Es 2005; Stockero 2008a, 2008b) and increasingly consider the impact of teacher decisions on student learning (Santagata and Guarino 2011). Teachers have also been found to become more specific in their description of classroom events (Santagata and Guarino 2011; van Es and Sherin 2008) and to increasingly use evidence to support analyses of teaching and learning (e.g., Sherin and van Es 2005; Stockero 2008a). Importantly, Sherin and van Es (2009) have also found that noticing skills developed via video analysis can transfer to classroom teaching situations.

This study builds on prior work to support prospective teacher noticing in two important ways. First, much of the work to promote prospective mathematics teachers' noticing currently discussed in the literature has taken place in mathematics methods courses, typically later in a teacher education program (e.g., Santagata et al. 2007; Star et al. 2011; Stockero 2008a, 2008b). Although this work has been found to be effective, this study examines whether noticing can be taught at the start of a teacher education program in order to provide a strong focus on students and their mathematics upon which PTs might build in later courses.

Second, many video-based teacher learning interventions discussed in the literature have used video clips that were purposely selected by experienced teacher educators (e.g., Borko et al. 2008; Seago 2004), eliminating the opportunity for teachers to determine which instances that arise during instruction are worthy of analysis. This may be problematic, since to productively notice and use student thinking during instruction, teachers need to learn to sift through the complexity of classroom interactions to recognize instances that can be capitalized on to support mathematical learning. Thus, teacher educators need to develop not only PTs' analytic abilities, but also their abilities to notice which instructional events should be analyzed because of their potential to support student learning. This study uses unedited classroom video that requires PTs to "sift through" the complexity of the classroom to identify important events.

This chapter reports on findings from the first two iterations of an ongoing design experiment in which prospective mathematics teachers were engaged in research-like analysis of unedited videos of mathematics instruction during an early school-based field experience. Key features of the learning intervention included individual analysis of teacher-perspective classroom videos and group discussion of the analysis supported by a mathematics teacher educator. The goal of the work was to cultivate prospective mathematics teachers' ability to notice, analyze, and consider how to capitalize upon important mathematical moments that occur during instruction. The research questions for the study included: (a) To what extent were project activities effective in helping PTs learn to notice important instances of students' mathematical thinking? and (b) What particular aspects of the activities supported the PTs' learning? The results reported here primarily address the first research question by reporting on transitions in the participants' noticing that resulted from this work. Preliminary insights into the second research question, as well as implications for future work, are discussed.

Theoretical Perspectives

The work is grounded in a particular vision of teaching—one in which teachers continuously build on student mathematical thinking in ways that are responsive to students' current understanding (Ball and Cohen 1999; Feiman-Nemser 2001; NCTM 2000). Teaching in this way involves the teacher carefully listening to students' ideas, analyzing the mathematics underlying them, and then making in-the-moment decisions about whether and how these ideas might be used to develop students' understanding of important mathematics ideas.

Teacher noticing is defined in a variety of ways in the literature (e.g., see chapters in Sherin et al. 2011). Sherin and van Es (2005) defined teacher noticing to include three components—identifying what is important in a situation, making connections between a particular classroom situation and broader principles of teaching and learning, and reasoning about the situation. Jacobs et al. (2010) added a fourth component of noticing to this definition—deciding how to respond. This project adopts a definition of noticing that includes all four components, with a particular focus on noticing important mathematics.

Although instances of student thinking that occur during instruction are central to teacher noticing, this work is also grounded in the perspective that not all instances of student thinking have the same potential to help achieve the goal of supporting students' mathematical learning. Thus, this research project aims to promote *mathematical noticing*—noticing of students' mathematical ideas that surface during instruction and have the potential to be built upon to support students' understanding of important mathematics. This is not to say that there are not other things that are valid to notice, such as instances that provide opportunities to develop student confidence or to cultivate norms for working in small groups. The work of the project, however, deliberately narrows the focus to student mathematics in order to help PTs consider what student mathematical thinking might be most productive to focus on during instruction in order to help students develop a deeper understanding of the mathematics in a lesson. Thus, this project adopts a definition of mathematical noticing that includes the following four components: noticing important student ideas, analyzing the mathematics within them, making connections to a particular framework, and deciding how to respond. The analysis reported here focuses on the first two components.

This focus on mathematical noticing is informed by two related research projects. In a study of novice mathematics teachers' instruction, Stockero and Van Zoest (2013) characterized pivotal teaching moments (PTMs)—defined as an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding. In this study, five types of PTMs were identified that had significant or moderate potential to support students' learning of mathematics: extending, incorrect mathematics, sense making, contradiction, and mathematical confusion. The finding that mathematical moments that are important to notice during a lesson might fall into a small number of categories informed decisions about where to focus participants' attention during the project activities.

An ongoing research project is working to characterize "teachable moments" in a mathematics classroom—referred to as Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOSTs) (e.g., Van Zoest et al. 2013). MOSTs are seen as occurring at the intersection of student mathematical thinking, significant mathematics, and pedagogical opportunity. In other words, high-leverage instances of student thinking that occur during a lesson must be student generated, involve mathematics that is related to a learning goal for students in the class, and provide the teacher an opportunity to build on student thinking to develop understanding of important mathematics. This framework provides a strong mathematical focus to teacher noticing and was used in this study to focus participants on moments that occur during a lesson that have the potential to be used by the teacher to enhance students' learning of mathematics.

Methodology

Context

The participants in the study were seven secondary mathematics prospective teachers (PTs); four were members of the first cohort that participated in the project in the Fall 2011 semester, and three were members of the second cohort that participated in Fall 2012. The teacher education program in which the participants were enrolled is very small, so the participants included all students with a mathematics major and some students with a mathematics minor who were enrolled in an early field experience course during each semester. Each participant was paired with a different experienced mathematics teacher at a local middle or high school to complete a 14-week early field experience. The field experience course is one of the first education courses that students in the teacher education program enroll in and is usually taken either late in students' sophomore year or early in their junior year of study.[1] Typically, this field experience consists of observing classroom instruction and assisting the teacher as needed, usually by working with individual students. Additional activities were added to the course as part of the project.

The Classroom Video

Each week, one PT in the cohort was assigned to video record a mathematics lesson taught by his or her cooperating teacher. The recording took place on a rotating basis, resulting in each PT recording two or three classroom lessons during the semester. Due to school scheduling and technical problems, there were two weeks where cohort 2 PTs did not collect the video, so the video from cohort 1 was used instead. The video was recorded from a front-of-classroom perspective to allow the participants to view the classroom as a teacher, rather than as a student. The instructional portions of each video were left unedited for the PTs to analyze; portions of the video where the teacher was tending to administrative tasks or where students were working quietly at their seats were cut from the video and excluded from the analysis since students' mathematical ideas could not be heard during these activities.

Members of each PT cohort analyzed the same set of eight (cohort 1) or nine (cohort 2) different videos during the semester. Note that because of the way the video was collected, the two PT cohorts did not generally analyze the same video, except for the two videos from the first cohort that were also used with the second cohort. Although an attempt was made to place the members of each cohort in a range of mathematics classrooms, the specific mathematics content of each video was not controlled because the focus was on noticing students' mathematics regardless of content. In short, the instruction that was recorded was mostly teacher led with various degrees of student interaction; the courses ranged primarily from middle school to geometry classes, with one calculus lesson analyzed by cohort 1. Table 1 summarizes the videos by PT cohort; note that videos C and F were coded by both cohorts.

Project Learning Activities

Each week, the PTs used the Studiocode video analysis software (SportsTec 2011) to individually code one classroom video, guided by an evolving and increasingly explicit coding framework focused on noticing instances of students' mathematical thinking. Early analysis activities were intended to focus the PTs on the students in

[1] Students do not apply to the teacher education program until they have established a GPA at the university; most students apply during their second year and start taking education courses during their third year of study.

Table 1 Summary of the video coded by each cohort

Video number	Cohort 1		Cohort 2	
	Video	Description	Video	Description
1	A	8th grade—patterns and variables	I	6th grade—absolute values; equations and expressions
2	B	8th grade—decimals and scientific notation	J	Algebra—exponents
3	C	Geometry—special triangles and quadrilaterals	K	Geometry—introduction to proof, grounded in segment congruence
4	D	8th grade—Pythagorean theorem and Pythagorean triples	C	Geometry—special triangles and quadrilaterals
5	E	Calculus—product and quotient rules	L	6th grade—factoring and exponents
6	F	8th grade—probability	M	Algebra—parallel and perpendicular lines
7	G	Geometry—inscribed and circumscribed circles	N	6th grade—percents and decimals
8	H	8th grade—test review; lines and scatterplots	O	Algebra—unit conversion
9			F	8th grade—probability

the classroom video and on their mathematics; later activities also incorporated the idea that some instances of student mathematics provide better opportunities for supporting student learning than others. The PTs' coding frameworks and interactions with the researcher are described in the following.

Early in the semester, PTs were asked to code the classroom video for "mathematically important moments (MIM) that a teacher needs to notice during a lesson" and write a brief explanation of their reasoning. The definition of a MIM was intentionally left ill-defined to allow the researcher to understand what the participants viewed as mathematically important during a lesson. The researcher and a graduate research assistant independently coded the same video as the members of the PT cohort and then met to discuss their own and the PTs' coding and decide which instances to discuss in a weekly meeting with the PTs. When coding, the Studiocode software produces a "timeline" of coded instances that can be merged to compare the coding of multiple coders. The merged timeline that showed all of the PTs' coding was used during the weekly meeting so the PTs could visually compare their own coding to that of the other members of their cohort. The researchers' coding was not shared with the PTs.

During the early group meetings, the researcher pushed the PTs to consider what was important mathematically about a particular instance and what the teacher had to notice; this pushing was intended to focus participants on students and their mathematics, rather than on teaching. Approximately six to eight instances were discussed at each weekly meeting; they included instances identified as a MIM by one or more participants (including both instances that were and were not identified as MIMs by the researchers), as well as some instances that were not identified by any PT but were deemed important by the researchers. The latter instances were primarily used to push PTs to attend to more subtle mathematical issues underlying student ideas—for example, ideas that were incomplete or that included imprecise language.

During the weekly meetings, the PTs were also pushed to consider whether instances that they collectively agreed were MIMs might fall into categories. These categories—informed by the researcher's prior work on PTMs (Stockero and Van Zoest 2013), but not made explicit to the PTs—were codeveloped by the PTs over the course of several meetings. As the categories were developed, the PTs were given the additional task of labeling each MIM that they identified in a video according to the category to which they felt it belonged. Subsequent meetings included refining and adding to the categories to more accurately describe the types of moments that were deemed important to attend to during a lesson. Although the names of the categories differed slightly, both cohorts had final categories related to student questions, wrong answers, incomplete answers, and unexpected (alternate) correct answers. Cohort 1 also developed categories of generalizing and multiple answers, while cohort 2 had a common misconceptions category.

Approximately midway through the semester—following the fourth video (video D) for cohort 1 and the fifth video (video L) for cohort 2—the framework for coding became more explicit when the PTs were asked to read an excerpt from a paper about MOSTs (e.g., Van Zoest et al. 2013)[2] and to recode two formerly analyzed videos to identify instances that met three defined criteria—student thinking, mathematically significant, and pedagogical opportunity. This reading was meant to provide language to discuss the student thinking and mathematics of an instance, as well as to introduce the idea that some moments that occur during a lesson provide the teacher a pedagogical opportunity to build on students' thinking to support their understanding of mathematics. After this framework was introduced, the PTs were asked to code subsequent videos for MOSTs instead of MIMs, and to discuss all three components of a MOST in their explanation of each selected instance in addition to labeling each instance by type.

Table 2 chronologically summarizes the project activities and shows the alignment between the activities and the videos described previously for each cohort.

Data Collection and Analysis

Data for the study included the PTs' individually coded Studiocode video timelines and video recordings of the weekly group meetings. The researchers used Studiocode to analyze all of the video timelines. The analysis process included adding an additional level of research coding to the timelines previously coded by the PTs; the meeting videos were used to help with the coding if a PT's written explanation of an

[2] The paper that the participants read during the project was a precursor to the current work on MOSTs. In it, the construct was referred to as a Mathematically Important Pedagogical Opportunity to Build on Student Thinking (MIPO) (Leatham et al. 2011).

Table 2 Summary of project activities, aligned with videos

Activity	Cohort 1	Cohort 2
PTs individually code for MIMs and give general explanation of importance of each moment	Videos 1–4 (A, B, C, D)	Videos 1–5 (I, J, K, C, L)
PTs codevelop categories of MIM types at weekly meetings	Primarily videos 2–4; modified as needed in later videos	Primarily videos 2–5; modified as needed in later videos
PTs read MOST paper and recode two prior videos using new framework	After video 4; recoded videos B and C	After video 5; recoded videos C and L
PTs individually code for MOSTs and give explanation of three components	Videos 5–8 (E, F, G, H)	Videos 6–9 (M, N, O, F)

instance was unclear. The unit of analysis for the research coding was any instance (MIM or MOST) that was coded by a PT. Thus, the length of an instance was determined by the PTs' coding, not by the researchers.

Building from coding frameworks used in previous studies (van Es and Sherin 2008; Stockero 2008a, 2008b), each important instance in a video that was identified by a PT was coded by the researcher and a graduate research assistant for agent, topic, and mathematical specificity (see Table 3 for codes). As defined in these prior works, agent refers to *who* the participants focused on in their description of an instance, and topic refers to *what* was focused on. For the agent code, student–teacher interactions were coded as either student–teacher, or teacher–student, depending on who was focused on first in a PT's comment. Some instances received two codes for agent or topic if the participant discussed two separate ideas in his or her explanation of the instance; in general, however, double coding was kept to a minimum. Because the focus of this work was on mathematical noticing, van Es and Sherin's (2008) specificity code—defined as how the teachers in their study discussed the events they noticed—was modified to focus on the specificity of the mathematics that the PTs discussed. Thus, in addition to using the subcodes of *general* and *specific,* a third code, *nonmathematical* was added to code instances of noticing that were focused on issues or events that were not mathematical in nature. During the coding process, the researchers met regularly to discuss, refine, and verify the coding; each instance was discussed until agreement on the coding was reached.

To give the reader a sense of the coding, consider the following PT explanation of an instance: "*The student asks a question about the placement of negative signs and the order [of the points] in finding slopes and the teacher uses this opportunity to go more in depth about finding slopes.*" For this instance, the agent was coded as *student–teacher* because the statement focuses first on the student's question, and then on the teacher's response. In this case, the student was considered the primary agent and the teacher was considered the secondary agent. The topic of this instance was double-coded as *question* and *explanation* because the PT noticed both the question that the student asked and the teacher's action—in this case, re-explaining

Table 3 Code definitions and sample codes

Coding category	Description	Codes[a]
Agent	Who the PT focused on in an instance	Teacher (T) Individual student (SI) Group of students (SG) Teacher, then students (T/S) Students, then teacher (S/T) Mathematics (no person)
Topic	What about the agent was focused on	Teacher explanation Student question Student thinking Student participation Understanding of the students as a group
Specificity	Whether and in what way the mathematics was discussed	Nonmathematical Specific mathematics General mathematics

[a] The agent and specificity code lists are complete, but the topic code list is a sample that includes the codes that are discussed in this chapter

how to find a slope. The specificity was coded as *specific mathematics* because it is clear that the PT is noticing mathematics related to calculating the slope of a line using two points.

After all of the PT-identified instances were coded by the researchers, the researchers' coding was analyzed to characterize shifts in the PTs' noticing in each of the three coding categories. Even though the two PT cohorts did not analyze the same set of videos, the goal for the two groups was the same: to scaffold their noticing to increasingly focus on students' mathematics in the detailed and nuanced way that is the foundation of student-centered instruction. Thus, the aim of the analysis was to understand the extent to which the PTs were focused on students, specific mathematics, and evidence of how students were thinking mathematically or mathematical issues they were encountering. The analysis involved examining the PTs' foci at different times during the learning experience, and how their foci changed during the experience. The findings are discussed in the following section.

Results and Discussion

The data revealed that the PTs' noticing shifted in all three of the coding categories. These shifts in agent, topic, and specificity, as well as initial conjectures about what caused these shifts, are discussed in the following. Because the number of instances coded by PTs varied by video, the results are reported as percentages. To help the reader make sense of the findings, the total number of instances coded by each PT cohort, the average number of instances per PT, and the range of the number of instances coded by each PT are given in Tables 4 and 5 for cohorts 1 and 2, respectively.

Table 4 Summary of instances coded by cohort 1

Video	Total coded instances	Instances per participant	Range of number of coded instances
A	25	6.25	4–12
B	16	4	3–5
C	25	6.25	5–9
D	14	3.5	3–4
E	12	3	2–4
F	21	5.25	3–7
G	13	3.25	2–5
H	14	3.5	3–4

Table 5 Summary of instances coded by cohort 2

Video	Total coded instances	Instances per participant	Range of number of coded instances
I	15	5	3–7
J	13	4.3	3–6
K	12	4	3–5
C	12	4	2–6
L	13	4.3	3–5
M	5	1.7	1–2
N	8	2.7	1–6
O	5	1.7	1–3
F	12	4	2–7

Agent

The analysis of the agent coding revealed three different shifts in the PTs' noticing: (a) from teacher as the primary agent to students as the primary agent; (b) from the teacher alone to interactions between the teacher and students; and (c) when the student was the primary agent, from focusing on groups of students to focusing on individual students.

Figure 1 shows the percent of PT-coded instances in each video with a primary student focus, depicted chronologically. That is, it shows the percent of PT-identified instances coded as focused on individual students, SI (e.g., "The student explained her thinking [about] how to multiply percentages. The opportunity arises to explain why that is." PT6); groups of students, SG (e.g., "The students are trying to come up with a correct answer to something they don't know. Even if they aren't sure, they're still trying and coming close to the right answer." PT7); or student–teacher interactions where the student was discussed first in the PT's explanation of the instance, S/T (e.g., "[The student] was able to put her understanding of the pattern into a basic mathematical model. This helps the teacher drive the lesson forward." PT1). As can be seen in the figure, early on, fewer of the PTs' coded instances had a primary focus on students. This pattern is particularly evident for cohort 2, where in

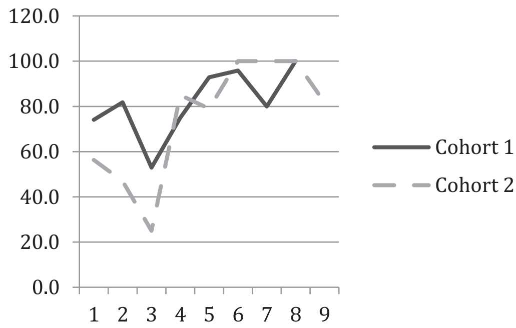


Fig. 1 Percent of participant-coded instances (by video) with a primary student focus

the second and third videos, less than half of the participant-identified instances had a primary student focus. After the third video, both groups exhibited a more consistent focus on students, with over 80% of coded instances for both groups focused primarily on students in videos 5 and later. This shift is significant because focusing on students is key to implementing mathematics instruction that is responsive to students and their ideas about mathematics.

Although there was some continued focus on the teacher throughout the experience, the PTs' focus shifted from noticing the teacher alone to noticing teacher–student interactions. The percentages of teacher-focused instances for each cohort that were focused on teacher-student interactions can be seen in Table 6. Early on, when the PTs focused on the teacher, they often discussed only what the teacher did (e.g., "The teacher explains the difference between expressions and equations." PT1) without considering how the teacher action they noticed was likely to support student learning. Beginning with the fifth video, however, a significant shift was evident for both groups, with 89% or more of all instances that considered the teacher (coded as T, T/S, or S/T) focusing on teacher–student interactions (coded as T/S or S/T). For example, in one instance, PT2 explained, "The student gives an equation for slope but she has the numerator and denominator switched. Instead of

Table 6 Summary of teacher-focused instances focusing on teacher–student interactions

	Cohort 1		Cohort 2	
Video	# Teacher-focused instances	Percent of teacher-focused instances focused on teacher-student interactions	# Teacher-focused instances	Percent of teacher-focused instances focused on teacher-student interactions
1	8	25.0	5	80.0
2	5	20.0	8	25.0
3	13	23.1	11	18.2
4	7	57.1	2	100.0
5	7	100.0	1	100.0
6	14	92.9	1	100.0
7	9	88.8	0	n/a
8	11	100.0	0	n/a
9	–	–	3	100.0

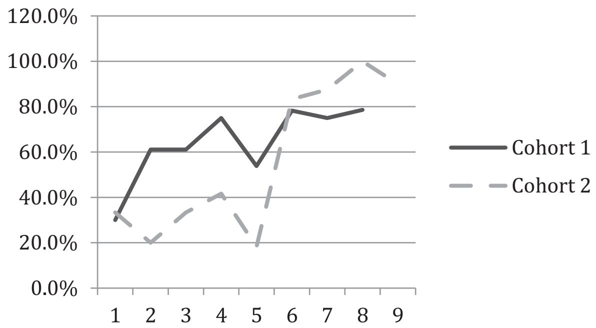


Fig. 2 Percent of primary student instances focused on individual students

just saying that the answer was wrong and giving the correct answer, [the teacher] connects the equation to rise/run. In doing this he gets the students to notice the problem with the equation." This shift was particularly strong for cohort 2, who collectively had no comments coded as focused on the teacher alone after the third video. This shift is significant in that it provided evidence that the PTs were beginning to think about how teacher actions support student learning, rather than on teacher moves independent of students.

A third significant shift occurred in how the PTs noticed students in the videos. Figure 2 shows the percent of coded instances with a primary student focus (coded as SI, SG, or S/T) for which the PTs were focused on individual students (SI). As can be seen in the figure, in the first video, when the primary focus of the PTs' noticing was students, the focus was on individual students a relatively small percentage of the time (30 % for cohort 1; 33 % for cohort 2). Their focus was more often on groups of students, with many PT comments offering assessments of the understanding of the group of students as a whole based on a single student comment. For example, the PT comment, "Students struggle to give a definition for [quadrilateral]" was coded as focused on a group of students, while the explanation, "[The student] was able to see the connection, which led to finding an equation" was coded as focused on an individual student.

The shift in focus from student groups to individual students began as early as the second and third videos for cohort 1, with about 60 % of student-focused instances centered on individual students in these videos. This same shift did not occur until later for cohort 2—in the sixth video—although after this time, their focus on individual students appeared stronger than cohort 1. The cause of this difference in timing warrants further investigation, but may be related to the specific classroom videos that each group analyzed. Overall, however, both PT cohorts demonstrated a significant shift in their focus on students during the experience, with the percent of student-centered comments focused on individual students ranging from 75 to 100 % in the last three videos.

Topic

Although the topic of the PTs' noticing was the most difficult to make sense of—mainly due to a larger number of codes spread out over a relatively small number of participants—the analysis revealed several shifts that are worth noting. In particular, decreases in the PTs' noticing of teacher explanations, claims about the understanding of an entire class, and student participation were documented, as were increases in PTs' noticing of student thinking and student questions and other evidence of mathematical confusion. These are discussed in the following.

Some topics of the PTs' noticing were coded less frequently as the learning experience progressed. Two of these shifts were closely related to the previously discussed shifts in agent. First, related to the decreased focus on the teacher alone (i.e., not in interaction with students), the PTs in cohort 2 became less focused on the teacher's explanation of the mathematics for its own sake—that is, explanations that were not in response to students' questions or comments. In the first three videos, 38 % of their instances were coded as teacher explanation, whereas only one instance of teacher explanation was coded after this time, and this instance was focused on how the teacher used a student idea to better explain a concept. Cohort 1 began with a low focus on teacher explanations (10 % of instances in videos 1–4) and increased it slightly (20 % in videos 5–8); it was encouraging, however, that all but 3 % of instances in these latter videos were in response to student questions or comments. Second, related to the previously discussed shift in agent from groups of students to individual students, there was a decrease in PT comments that made overgeneralized claims about the understanding or lack of understanding of the class as a whole. In the last four videos, the two cohorts combined had only 4 % of documented instances (3.3 % for cohort 1; 6 % for cohort 2) focused on making claims about group understanding, compared to about 21 % of instances in the first four videos (18 % for cohort 1; 26 % for cohort 2).

Another decrease in focus was on how students participated in the class or worked with one another (e.g., "There are students with their hands raised that give up on what they wanted to say, and the guy in the back doesn't try to participate the remainder of the class." PT3). This more affective noticing focus was relatively prevalent in the first few videos, particularly for cohort 1, which had 10 documented instances (15 % of total instances) in the first three videos. During the latter half of the experience, it was documented only once for each cohort (1.7–3.3 % of instances). Although important to learning in general, affective noticing such as this does not directly support the learning of mathematics content; thus, this seems to be a productive shift in attention as it indicates that the PTs were becoming more attentive to issues directly related to students' mathematical learning.

There were also some topics that were focused on more frequently by PTs as the experience progressed. One such topic was student thinking, which overall was stronger in the later videos (Table 7). This shift was not consistent, however, as cohort 1's focus on student thinking was strongest in the middle videos, while cohort 2's focus was strong in the first video, and then diminished before becoming

Table 7 Percent of instances coded with student thinking as a topic

	V1	V2	V3	V4	V5	V6	V7	V8	V9
Cohort 1	0.0	0.0	4.0	21.4	8.3	14.2	7.7	0.0	–
Cohort 2	25.0	0.0	0.0	8.3	7.7	40.0	62.5	80.0	25.0

strong again in the later videos. To illustrate a focus on student thinking, consider PT5's explanation of the importance of one instance: "This is a common mistake in probability problems where the items aren't replaced. The student mentally removes [student]'s name from the number of girls but forgets to remove her from the total number of students." As can be seen in this example, when focusing on student thinking, the PTs were making sense of how students seemed to be thinking mathematically, rather than just whether they were thinking correctly or incorrectly. This more nuanced focus on student thinking is essential in order for a teacher to use student thinking during a lesson. Because helping PTs learn to notice student ideas was a primary objective of the learning activities, the increasing and decreasing nature of the PTs' focus on this topic requires further investigation.

A second topic that received increased focus was student questions—particularly those that were conceptual in nature—and evidence of student mathematical confusion, which was implied from both their questions and mathematical comments (e.g., "A student is confused about concurrent and [the teacher] answers his question by explaining the difference between concurrent and the circumcenter. I think this is the best method to answer his question because you can see after that he understood after the explanation." PT2). This shift in focus was most significant for cohort 1, which had only 5 % of instances coded as questions or confusion in the first four videos, compared to 58 % in the last four videos. Although the shift was not as dramatic for cohort 2, they still had more than twice the number of instances coded in this category for the latter half of the experience compared to the first (8 % of instances early; 21 % of instances late). As data from future iterations of the work are analyzed, it is likely that this coding category will need to be refined to separate conceptual questions that introduce new mathematical ideas or provide an opportunity to go beyond the mathematics of the lesson, from questions that indicate confusion about the mathematical ideas at hand. This more fine-grained level of coding may provide insight into whether PTs were focused on instances that provided opportunities to clarify the mathematics in the lesson, or those that might be used to make connections among lessons. Both of these foci are important, but very different, ways that a teacher could use student thinking productively, so it would be helpful to understand in more detail the degree to which the PTs focused on each.

Specificity

The analysis of the specificity coding indicated that the PTs collectively transitioned to becoming both more focused on noticing instances that were mathematical in nature and more specific in their discussion of the mathematics of an instance. These shifts are discussed in the following.

Despite the fact that the PTs were given instructions to code "MIMs that a teacher needs to notice during a lesson," some of their early noticing was focused on nonmathematical instances. For example, PT2 identified a teacher incorporating exercise into her lesson as a MIM, even though this activity is clearly not mathematical by nature. In the first two videos, 15 % of instances identified by cohort 1 were not mathematical; this group identified no nonmathematical instances after this time, however. In their first three videos, 13 % of instances coded by cohort 2 were not mathematical in nature, with only one nonmathematical instance documented subsequently in video 5. The shift away from nonmathematical noticing seemed fairly easy to facilitate by pushing the PTs to discuss the mathematics in each coded instance, but it is an important shift nonetheless since focusing on nonmathematical issues necessarily shifts teachers' attention away from noticing students' mathematics.

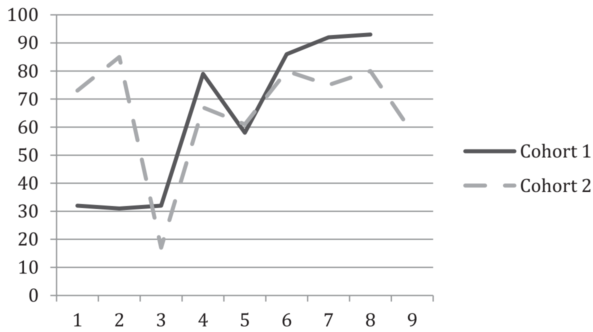
A second shift that was particularly strong for cohort 1 was from noticing general mathematical issues to those that were specific in nature. This shift is important because it indicates that the PTs were engaged in more detailed analysis of the mathematics in the videos and were able to attend to specific student ideas or teacher moves, rather than making general observations about the mathematics of the lesson. An example of general mathematical noticing is, "The student gave the wrong answer to the teacher's question, but the teacher went along with the reasoning and then went through the process of double checking," whereas an example of specific mathematical noticing is, "The student asks a question about the placement of negative signs and the order [of the points] in finding slopes and the teacher uses this opportunity to go more in depth about finding slopes." Note that in the first example, while the PT was focused on the students' mathematics, she did not discuss in her explanation a specific mathematical issue, just that the student gave a wrong answer. In fact, the comment itself gives no indication of the mathematics of the lesson. In the second example, it is clear that the PT was focused on calculations related to finding the slope of a line.

In the first three videos, cohort 1's explanations of their coding were more general in nature, with less than 35 % of instances coded as related to specific mathematics (Fig. 3). Cohort 2's noticing was fairly specific from the start (with the exception of video 3), but the data were somewhat skewed by one PT whose explanations were consistently specific throughout the experience. Relatively early in the experience, before they had engaged with the MOST framework, both groups of PTs' noticing became consistently more specific than general.

Explaining the Shifts

Although the analysis of the facilitation of the learning activities is ongoing, the timing of some of the documented shifts allows one to make informed conjectures about what might have prompted them. In general, there seems to be three factors that may account for the PTs' shifts in noticing: (a) targeted facilitation moves during the weekly meetings intended to focus the PTs' attention on specific aspects of

Fig. 3 Percent of PT-identified instances (by video) focused on specific mathematics



the video, (b) the reading of the MOST paper approximately midway through the experience, and (c) ongoing engagement in the project activities.

Targeted facilitation moves, particularly those employed early in the experience, appear to account for some of the documented shifts in the PTs' noticing. The relatively early shift toward a more consistent focus on students, for example, seems to be the result of the facilitator pushing participants to articulate what the teacher had to notice in each instance that they had documented (recall that the participants were prompted to identify MIMs that a *teacher needs to notice* during a lesson). Because a teacher does not need to notice his or her own actions, this move seems to have had its intended effect of focusing the PTs on the students in the video. Facilitator questioning also seems to account for the early shift away from nonmathematical noticing, as participants were consistently asked to discuss the mathematics in each instance during the weekly group meeting. In response to the facilitator questioning PTs about both of these foci, it was not uncommon for one PT to highlight that a moment identified by a peer either had nothing for the teacher to notice because it was just teacher talk, or that it was not mathematical in nature.

Other shifts, particularly those related to making sense of student contributions and the mathematics within them, may have been prompted by a different facilitation move—replaying portions of the video during the weekly meetings, sometimes several times, so that PTs could carefully listen to what was being said. This move seemed to help PTs learn to attend to the nuances of student comments, which may have helped them both to become better able to recognize when student contributions contained important mathematics and to shift toward becoming more specific in the way they discussed the mathematics of an instance. Shifts such as these are significant in that they support the focused noticing and ongoing analysis that is necessary to teach in a way that is responsive to student thinking.

After coding four or five videos, each cohort read an excerpt from a paper describing the MOST framework that highlighted key characteristics of "teachable moments" in mathematics: student thinking, significant mathematics, and pedagogical opportunity. After reading this paper, the PTs were asked to use the framework to inform their coding, which appears to have contributed to several shifts in their noticing. The timing of the PTs' increased focus on teacher–student interactions, for example, suggests that it was facilitated at least in part by introducing the MOST framework. In this case, it is conjectured that asking the PTs to think about the pedagogical opportunity that student thinking might provide helped them begin to consider ways in which a teacher might be responsive to student ideas.

Because the PTs engaged in the project activities over an entire semester, it is also possible that some of the documented shifts in noticing were the result of ongoing engagement over time. The focus on individual students, for example, seemed to continue to increase throughout the experience. It is also possible that some of the shifts were the result of a combination of activities, including the PTs' interactions with one another. Although the effects of sustained engagement and the interaction of learning activities are more difficult to document, understanding these effects will be the focus of ongoing data analysis. In any case, it is encouraging that significant changes in teacher noticing were documented over the course of the learning experience—changes that are likely to support their ability to engage in student-centered instruction.

Discussion and Conclusions

The results of this work indicate that it is possible to facilitate transitions in mathematics teacher noticing, even early in a teacher education program. While other studies have used videos to prompt similar changes in mathematics methods courses (e.g., Santagata and Guarino 2011; Sherin and van Es 2005; Stockero 2008a, 2008b), which typically take place near the end of teacher candidates' university coursework, this study attempted to do so at the start of a teacher education program, during a school-based early field experience. Focusing PTs on students and their learning of mathematics from the start of a teacher education experience offers the potential to build on this foundation during subsequent coursework, possibly resulting in an even stronger student-centered focus at the end of a teacher education program.

The data revealed that the PTs in this study were, in fact, able to increasingly notice important mathematical instances in unedited video recordings of classroom instruction. Their noticing became more focused on individual students and how teacher–student interactions affect learning. They also became better able to attend to the specific mathematical details of important instances that surfaced during a lesson. The coding of the topic of the PTs' noticing provided some indication that the participants became more focused on instances that were important mathematically rather than for social or affective reasons, less prone to make claims about groups of students, less focused on teacher explanations, and more attentive to issues directly related to student understanding—how they might be thinking mathematically, conceptual questions that they asked, and evidence of mathematical confusion. Together, these transitions are significant because noticing students and the details of their mathematical thinking is foundational to student-centered instruction; a teacher cannot build on ideas that they do not notice or of which they cannot make sense.

The results are also significant in that they provide evidence that PTs not only can learn to analyze short preselected video clips of instruction (e.g., Borko et al. 2008; Seago 2004; Stockero 2008b), but can also learn to identify mathematically important instances in unedited classroom videos. Although short preselected video clips offer some advantages—such as less time for analysis and possibly a more focused discussion about particular issues the clip is intended to raise—sifting through the complexity of classroom interactions to figure out which student ideas have potential to be used to develop students' understanding of the mathematics is exactly what teachers need to do in order to enact student-centered instruction. In this way, learning to analyze unedited classroom videos might be advantageous in terms of helping teachers transfer noticing skills developed through teacher learning experiences to their classroom instruction.

The intervention in this study included many elements that are often missing from school-based field experiences: a clear and consistent focus on mathematics, engagement in structured analysis, and collaborative learning about practice with substantial mathematics teacher educator support. Each of these elements is critical to helping PTs take the most away from the time they spend in schools (e.g., Leatham and Peterson 2010a; Masingila and Doerr 2002). First, the early noticing of the participants in this study suggests that, without a clear push to focus on content, it is likely that PTs will focus instead on elements of instruction that are less central to student learning of mathematics. Second, as has been found by others (e.g., Santagata et al. 2007), analysis frameworks seem essential to give both structure to what is observed in the complex environment of a classroom and a language to discuss it. The evolving analysis framework in this study, including the labeling of important instances and the eventual introduction of the MOST framework, provided this structure and language for the participants and seemed to prompt changes in how and what the PTs attended to in the classroom videos. Finally, it has been suggested that teacher educator support—something that is often lacking during school-based field experiences—is necessary to effectively facilitate prospective teacher learning from such experiences (e.g., Leatham and Peterson 2010a; Oliveira and Hannula 2008). In this study, discussing common classroom videos with a teacher educator maintained a clear mathematical focus during the field experience and provided a means to challenge PTs' emerging ideas about the teaching and learning of mathematics. Together, these elements appear to have been effective in supporting desired transitions in noticing in this intervention.

Although the results are promising, further work is needed to fully understand the transitions that have been documented, as well as what specific activities and structures supported them. One of the limitations of the results reported here is the small number of participants; in particular, this limitation makes it difficult to make sense of some of the differences in noticing between the two PT cohorts. Recall, for example, that cohort 1 shifted from noticing student groups to noticing individual students earlier than cohort 2; cohort 1 also developed a stronger focus on students' conceptual questions and evidence of confusion. Cohort 2, on the other hand, developed an overall stronger focus on students and had a stronger focus on student thinking at the end of the experience. They were also more specific in their discussion of the mathematics throughout. Some of these differences seem to be attributable to the individual students in each cohort. For example, cohort 2 included a PT who consistently discussed the specific mathematics in each instance right from the start. Other differences may be the result of the interactions of the PTs within the group. Cohort 1, for instance, included a participant who challenged others when they made claims about the understanding of whole groups of students based on a single student comment, possibly reducing the number of such claims. Other differences may be the result of the way the group discussion was facilitated each semester, or participants' understanding of the mathematics of each lesson that was analyzed. These factors are being investigated as part of the ongoing analysis of the data; additional data are also being collected to determine whether similar trends are seen with other PT cohorts.

In addition, the analysis of the topic of the PTs' noticing has raised issues related to video selection that require further investigation. One of the reasons that the coding of the topic of the PTs' noticing was more difficult to make sense of was that some of the noticing topics seemed to be video or context specific. That is, instances related to some noticing foci were not seen in all of the videos. In the study, classroom video that was recorded by the participants was used to maintain a strong connection to practice. Analyzing the video from classrooms in which the PTs worked was intended to give a sense of reality to the experience; that is, the video portrayed students who were real to the PTs, rather than "other" students from classrooms that were special in some way. This meant, however, that what each PT cohort had available to notice was not always the same. In particular, there were three topics that seemed strongly dependent on the classroom culture or nature of a lesson: multiple student solutions, unexpected correct answers, and students correcting one another's mathematical thinking. Instances that instantiate these topics are only likely to occur, and thus be available to be noticed, in classrooms where the teacher allows multiple ideas or nonstandard ways of thinking to be made public and where students are engaged in discussion about mathematical ideas. Thus, although the mathematical content of the videos did not seem to be a factor, the context of the classrooms and, in particular, the nature of student–teacher interactions seem to be an important consideration. This context dependency raises interesting questions related to video selection that need to be explored in future work. Specifically, it may be the case that in future iterations of the work, videos need to be deliberately "inserted" into the sequence of videos that are analyzed to ensure that participants have access to a wide range of mathematically important instances.

In summary, this study provides an initial understanding of the outcomes of a set of activities designed to facilitate PTs' mathematical noticing and gives some insight into elements of such activities that might be critical to helping PTs learn to attend to important mathematical instances that arise during a lesson. Understanding the details of transitions in noticing and how to best support them in this context has the potential to inform interventions to support mathematics teachers in a range of contexts to engage in productive mathematical noticing during instruction.

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Teachers' Uses of a Learning Trajectory as a Tool for Mathematics Lesson Planning

Cyndi Edgington

Attention to student thinking has been identified as a critical tool to initiate changes in teachers' knowledge for teaching and improvements in classroom instruction (Fennema et al. 1996; Franke et al. 2001; Kazemi and Franke 2004; Sherin and van Es 2009). Moreover, an emerging hypothesis in the field is that the construct of a *learning trajectory* (LT) has the potential to support teachers in making sense of and using student thinking to improve teaching and learning. The authors of the Common Core State Standards (National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA Center & CCSSO) 2010) emphasized the use of research-based LTs in the development of the new standards and committed to use research and evidence of student learning to inform future revisions. Daro et al. (2011) stated that LTs serve "as a basis for informing teachers about the (sometimes wide) range of student understanding they are likely to encounter, and the kinds of pedagogical responses that are likely to help students move along" (p. 12). However, little is known about how teachers come to learn about LTs and appropriate them into their instruction. In this study, I identify the ways in which five elementary teachers used an LT to support attention to students' mathematical thinking as they plan mathematics lessons. In particular, I address the following research question: *In what ways do teachers use LTs to choose instructional tasks and learning goals, and anticipate students' approaches to intended instructional tasks?*

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Background

LTs utilize research on student learning to describe probable pathways of learning over time. Researchers who have studied the implications of LTs for teachers have found that LTs provide a framework for making instructional decisions (Bardsley 2006; Sztajn et al. 2011a; Wilson 2009) and afford teachers with a means to focus on their students' mathematical thinking (Clements et al. 2011; Edgington et al. 2011; McKool 2009; Mojica 2010). While these studies addressed teachers' knowledge of student thinking, little is known about the ways in which LTs support specific teaching practices. As teachers increasingly attend to student thinking in lesson planning and instruction, researchers must consider the role of LTs in supporting teachers' complex work. Research has yet to address how teachers plan for instruction when they have information about the progression of more sophisticated levels of thinking inherent in LTs, or how teachers use evidence of student thinking to inform future instruction in light of LTs. This study contributes to the research on teachers' uses of LTs to support attention to student thinking in planning for mathematics instruction.

The research presented reports, five teachers' uses of an LT in lesson planning through two different processes: identification of instructional tasks and learning goals, and anticipation of students' work. Often considered a core routine of teaching, lesson planning refers to the time teachers spend preparing for instruction before students enter the classroom. Grossman et al. (2005) refer to this as the "preactive" aspect of practice, where teachers focus on lesson planning, unit planning, or even planning for classroom management.

Early studies on teachers' lesson planning attended to the resources teachers used to plan, indicating that teachers focused on ideas such as content, activities or tasks, materials, textbooks, routines, as well as students' needs and backgrounds (Brown 1988; Fernandez and Cannon 2005; McCutcheon 1980). In a study of 12 elementary school teachers, McCutcheon (1980) found that teachers used their textbook as a main source for activities and depended heavily upon suggestions from the teachers' guide. In a later study, Brown (1988) examined the lesson planning practices of 12 middle school teachers in various content areas. She found that teachers relied heavily on curriculum materials, building their lessons off of objectives expressly stated in the curriculum resources.

More recently, studies on mathematics lesson planning have sought to identify high-leverage planning practices. Superfine (2008) studied three teachers' lesson planning with respect to a specific mathematics curriculum. Her study revealed two planning difficulties: (a) anticipating student work, misconceptions, and potential errors for a given task; and (b) understanding the treatment of the content in the curriculum. She concluded that teachers' conceptions of teaching and learning mathematics, as well as years of experience, mediated their management of the planning problems.

In an attempt to articulate high-quality mathematics instruction, Corey et al. (2010) studied conversations between seven Japanese student teachers and three cooperating teachers from one school. These conversations occurred during planning meetings that took place between the student teachers and their cooperating teachers, where the pairs met on average three times prior to teaching a lesson. The findings described six principles of high-quality instruction, with three of the principles directly related to lesson planning: (a) an ideal lesson is guided by a set of explicit long- and short-term goals, (b) a particular lesson is created with clear connections to previous and future lessons, and (c) high-quality instruction requires anticipating student thinking in relation to the goals of the lesson. This study offers support for the importance of teachers' ability to choose learning goals and to consider students' approaches to intended instructional tasks in order to foster meaningful learning.

Conceptual Framework

In light of reform efforts to improve the teaching and learning of mathematics, one may question what *should* be the focus of planning when instruction attends to students' mathematical thinking. Teaching from a reform perspective requires consideration of how to construct lessons that address specific learning goals and allow teachers to gather evidence of their students' understanding toward the chosen goals. Moreover, as student learning progresses over time, teachers must be able to consider how to build on students' current conceptions to reach intended learning goals.

The conceptual framework for this study draws upon the work of Hiebert et al. (2007) as well as that of Smith and Stein (2011). These two frameworks were chosen because of their emphasis on student thinking as a central feature of lesson planning and instruction. During lesson planning, teachers not only *choose intended learning goals,* but they decompose learning goals into smaller subconcepts that comprise larger goals (Hiebert et al. 2007). Careful unpacking of larger mathematical goals as well as specifying necessary subconcepts can provide teachers with more detailed information with which to build a lesson. Hiebert et al. (2007) recognized the importance of teachers' subject matter knowledge with respect to this skill and that it is often a challenge for novice teachers.

As teachers *choose mathematical tasks*, it is important that the task aligns with the chosen learning goals (Smith and Stein 2011). Moreover, teachers must consider how different tasks elicit different opportunities for student thinking. Smith and Stein (2011) also noted that teachers should consider "the extent to which the task permits entry to students who bring with them a range of prior knowledge and experiences" (p. 18). Open tasks that can be solved in multiple ways are more likely to be accessible to a wide range of learners and provide opportunities for productive mathematical discussions.

In considering the mathematical task proposed in a lesson, teachers use their own content knowledge as well as their knowledge of how students are likely to approach the task to *anticipate students' responses* and common areas of difficulty (Smith and Stein 2011). In this way, teachers can consider how students' responses, both correct and incorrect, can lead to the intended learning goals. By comparing evidence of student learning to the intended learning goals, teachers can determine which aspects of their instruction helped or hindered their students' understandings (Hiebert et al. 2007). Once instruction has been evaluated, careful planning is important for considering new learning goals and instructional tasks that build on students' current conceptions and move students to more complex mathematical understanding.

The structure for the current study joins components of the Hiebert et al. (2007) and Smith and Stein (2011) frameworks in relation to findings from previous research on lesson planning to specify what it means to plan mathematics lessons when teachers attend to student thinking. Because identifying tasks and learning goals are closely related and often intertwined, I chose to examine these two areas together. Hence, this study examined two aspects of lesson planning: choosing tasks and learning goals, and anticipating students' approaches to intended tasks.

Attending to student thinking can support teachers as they engage in lesson planning. However, as noted earlier, specifying learning goals and anticipating student work in relation to chosen learning goals can be challenging (Hiebert et al. 2007; Superfine 2008). Careful attention to students' current conceptions can support teachers in designing lessons that build on prior knowledge. Furthermore, as teachers consider evidence of students' thinking, they can more explicitly connect students' conceptions to important mathematical ideas. As representations of student thinking, LTs can not only enhance teachers' subject matter knowledge (Mojica 2010; Stzajn et al. 2011b; Wilson 2009), but also potentially serve as tools to help advance teachers' abilities to make sense of student thinking and use it to develop instruction that addresses students' existing conceptions and moves learning forward.

Method

This study seeks to understand how teachers use the construct of an LT to support attention to students' mathematical thinking within two areas of lesson planning: (a) choosing mathematical tasks and learning goals, and (b) anticipating students' approaches to intended tasks. A qualitative case study approach was chosen in order to understand participants' created meaning of their use of one particular LT in mathematics instruction. Specifically, case studies allow the researcher to uncover and examine significant interactions that are characteristic of the phenomenon under study, as well as provide concrete and contextual knowledge as evident in the end product (Merriam 1998). Since a goal of the research is to propose a framework for teachers' uses of LTs, the emphasis is on the cross-case analysis to consider the variability of teachers' uses of LTs.

Context

Learning Trajectory-Based Instruction (LTBI) is a research project with a strong professional development component for elementary school teachers (Sztajn et al.

2011b). The project was motivated by the adoption of the Common Core State Standards (NGA Center & CCSSO 2010) and current research on LTs in mathematics education (Battista 2004; Clements and Sarama 2009; Confrey et al. 2009). The project's goals include examining the ways in which teachers learn about LTs and use them in their classrooms to define what it means to teach from an LT perspective.

The Equipartitioning LT (EPLT) In the first year of the project, one cohort of teachers learned about a specific LT: the EPLT. Based on a synthesis of research and clinical interviews, Confrey and her colleagues developed the EPLT to describe how children use their informal knowledge of fair sharing as a resource to build an understanding of partitive division that unifies ratio reasoning and fractions (Confrey 2012).

The EPLT begins with experiences of fairly sharing collections of items or single wholes. In equipartitioning, students must learn to coordinate three criteria: (a) create equal-sized groups or parts, (b) create the correct number of groups or parts, and (c) exhaust the entire collection or whole. As students enact strategies to complete these tasks, they gain proficiency in mathematical reasoning practices such as justification and naming (e.g., as a count, fraction, or ratio) and begin to develop understandings of fundamental mathematical properties that later influence the ways that they fairly share multiple wholes, for example, sharing three wholes among two people (Confrey et al. 2010). The trajectory describes how these strategies, practices, and properties ultimately unify as a generalization of partitive division that relates ratio reasoning and fractions. Important to the trajectory are not only the levels of sophistication of reasoning but parameters associated with the tasks, including the number of wholes and number of sharers. Beginning with equipartitioning collections, the next task parameters address equipartitioning single wholes (rectangles and circles), building on primitive splits such as halves and powers of two, to eventually include arbitrary integer splits. The upper levels of the trajectory address tasks that involve multiple wholes and multiple sharers when the number of wholes is both less than and greater than the number of sharers. The full trajectory can be found in Fig. 1.

To further illuminate the EPLT, I offer possible task sequences, a teacher might follow depending upon the overall mathematical goals for a group of students. For example, if the focus is on strategies for equipartitioning a single whole (Level 2), a possible task sequence may be to remain at that level but alter the task parameters over time, beginning with a two-split on a rectangle or circle, then progressing to splits for powers of two (4, 8, 16) where a repeated halving strategy might be used, then examining more difficult even and odd splits (6, 3, 5, etc.). In contrast, a teacher may choose to hold a task parameter constant (e.g., a two-split) and move vertically through the levels of strategies, justification, naming, and transitivity.

The LTBI Professional Development In the first year, the LTBI project partnered with one elementary school in an urban district in the Southeastern USA. The school had approximately 600 students, 35 % Caucasian, 29 % Hispanic, 25 % African American, 7 % Asian, and 4 % other; 54 % of the children qualified for free or reduced lunch. In all, 24 teachers at the school volunteered to participate, and 22 teachers completed the program one year later. All professional development meet-

Equipartitioning Learning Trajectory Matrix (grades K-8) Task Parameters→ Proficiency Levels		A Collections	B 2-split (Rect/Circle)	C 2^n split (Rect)	D 2^n split (Circle)	E Even split (Rect)	F Odd split (Rect)	G Even split (Circle)	H Odd split (Circle)	I Arbitrary integer	J $p = n + 1$; $p = n$-1	K p is odd, and $n = 2^i$	L $p >> n$, p close to n	M all p, all n (integers)
16	*Generalize*: a among $b = a/b$													
15	*Distributive property,* multiple wholes													
14	*Direct-, Inverse- and Co-variation*													
13	*Compositions* of splits, multiple wholes													
12	Equipartition *multiple wholes*													
11	Assert *Continuity principle*													
10	*Transitivity* arguments													
9	Redistribution of shares (*quantitative*)													
8	Factor-based changes (*quantitative*)													
7	*Compositions* of splits; *factor-pairs*													
6	*Qualitative compensation*													
5	*Re-assemble*: n times as much													
4	*Name* a share w.r.t. the referent unit													
3	*Justify* the results of equipartitioning													
2	*Equipartition single wholes*													
1	*Equipartition Collections*													

Fig. 1 The equipartitioning learning trajectory (adapted from Confrey 2012)

ings were conducted at the school, during times selected based on convenience to the teachers, and teachers received a stipend for their participation.

The professional development was designed over a 12-month period and began with a 30-h intensive summer institute in which participants engaged in professional learning tasks on equipartitioning and different aspects of the EPLT, including video analysis of students working through equipartitioning tasks, videos of classroom instruction, analysis of students' written work, and curricular connections. By the end of the summer institute, the various ideas participants experienced were formalized with an emphasis on the first 12 levels. Some of the professional learning tasks were designed to allow teachers to make connections to existing curricula and current practice.

Throughout the school year, teachers met with project leaders monthly for 2 h after school to continue to build their knowledge of the EPLT and to try out tasks that incorporated equipartitioning concepts in their classrooms. During these meetings, teachers engaged in activities such as analyzing student assessments and watching video clips of participants' equipartitioning lessons. The professional development concluded with a 2-day follow-up summer institute.

Participants

The sample for the current study was a subset of participants from the first year of the LTBI project. Project participants were offered the opportunity to continue working with the research team in some respect in the following school year. The second-grade team, consisting of five teachers, expressed an interest in developing a set of equipartitioning lessons based on the EPLT. These teachers volunteered and, as such, their participation indicated a willingness to explore the utility of the EPLT as a tool for lesson planning, justifying their selection as a purposeful sample for this study. All data for this study were collected after the conclusion of the LTBI professional development.

The five selected teachers were previously identified by the project research team as highly engaged with the LTBI professional development, which increased the possibility of observing the phenomenon of interest, that is, the use of the EPLT in the classroom. At the same time, these teachers varied in their mathematical knowledge for teaching (as measured by the University of Michigan's Learning Mathematics for Teaching content assessment for rational number reasoning grades 4–8; Hill and Ball 2004), years of experience, and beliefs about instruction (as measured by the *Teachers' Beliefs about Mathematics and Mathematics Teaching* instrument; Campbell et al. 2011), which created variation for the investigation. The fact that the teachers met regularly to plan and discuss their mathematics instruction in a professional learning community setting supports including all five teachers. The use of the same curriculum and the selection of similar tasks for implementation made the connection between teachers' uses of the EPLT and their curriculum (Wilson 2009) in certain ways uniform across the cases, further illuminating variations among the teachers' uses of the EPLT.

In summary, the selected sample had a high probability of producing cases that were both information-rich and varied, offering in-depth details on the teachers' uses of the EPLT for lesson planning. Each teacher was given a pseudonym used to report findings of the study in order to maintain confidentiality. The five participants are Bianca, Ellen, Emma, Lara, and Tracy. Table 1 provides a summary of the participants.

Data Sources and Analysis

The primary sources of data for this study are transcripts from three grade-level planning meetings, pre-lesson questionnaires, classroom observations of teachers' instruction, and transcripts of teacher interviews. There were three data cycles and

Table 1 Participant information

	Bianca	Ellen	Emma	Lara	Tracy
Race	Hispanic	White	White	White	White
Years of teaching experience	5	6	6	8	18
LMT (scaled score)[a]	0.63	−0.65	0.63	−0.51	0.47
Beliefs score[b]	135	137	146	116	146
Certification	Bachelor's degree, K–6	Bachelor's degree, K–6	Bachelor's degree, K–6	Bachelor's degree, K–6	Master's degree, K–6

[a] For the LMT, 22 teachers completed the assessment with a mean scaled score of 0.19, st. dev. = 0.92. A positive scale score is associated with more items answered correctly, whereas a negative scale score is associated with fewer items answered correctly

[b] A higher beliefs score indicates teachers' beliefs align more often with student-centered instruction, with a maximum score of 200

each cycle began with a grade-level lesson planning meeting, followed by individual classroom observations, and concluding with individual post-lesson interviews. The grade-level lesson planning meetings were audio recorded and took place approximately once a month over a 3-month period. Hence, while the three lessons were sequenced by topic, they were not taught on three consecutive days. During these meetings, my role was mainly as an observer, but I also served as a resource or knowledgeable other about the EPLT when my input was requested or when I considered my input helpful. Prior to each lesson, four of the five participants[1] completed a pre-lesson questionnaire to provide information about the teachers' learning goals and any adaptations they may have made to the lesson plan. The pre-lesson questionnaire consisted of open-ended questions in order to acquire specific information from each participant and was collected prior to the lesson observation, either through email or as a hard copy. Examples of questions asked are "what are the learning goals for this lesson?" and "Anticipate what you think will happen as you implement this lesson."

Observations took place in each participant's classroom during the regularly scheduled math instructional time and were video recorded. They were video recorded using a blue tooth microphone and one video camera that followed the teacher in order to capture dialogue between the teacher and students during whole-group and small-group classroom interactions. Following each lesson, a semistructured interview was conducted with the participant to discuss the teacher's perceptions of what learning took place, as well as evidence of that learning and how the teacher used that evidence to inform future learning goals. The post-lesson interviews took place within 1–3 days after a lesson, and on two occasions took place immediately following the lesson. The interviews were audio recorded and semistructured in

[1] For unknown reasons, Lara did not complete any of the pre-lesson questionnaires. Information about her learning goals and interpretations of the lessons were collected during the post-lesson interviews.

order to collect similar data from each participant, but they also allowed the researcher to respond to the individual participants' views that emerged through data collection (Merriam 1998). Interview questions focused on discussing important moments from the lesson, evidence of student learning, and decisions the teacher made during the lesson. At the end of each interview, participants were asked, "did you draw upon the EPLT as you implemented your lesson? If so, how?" in order to obtain specific information regarding the EPLT.

Data were analyzed using ATLAS.ti (2012), qualitative data analysis software. Evidence from the grade-level meetings and pre-lesson questionnaires were used to examine the ways in which teachers used the EPLT to select learning goals and tasks and anticipate students' responses. Observations were used to seek evidence of the teachers' use of the EPLT as they monitored their students' progress on tasks, and as they structured whole-class discussions. Evidences from post-lesson interviews and from grade-level meetings were considered to determine the ways in which the EPLT was used to reflect on the impact of instruction on student learning, to evaluate evidence of student learning, and to inform future instruction. The findings reported here focus on the use of the EPLT to choose tasks and identify learning goals, and to anticipate likely student responses for three lessons.

Coding of Lesson Planning Meetings First, data were organized into what Patton (1990) referred to as a case record; all data for one participant were gathered for analysis. Since all five teachers participated in the planning meetings, transcripts from these meetings were analyzed separately. That is, for each case, the participation of the particular teacher in the lesson planning meetings was analyzed with the other teachers serving as the context for that case. Analysis began by coding the transcripts of the grade-level planning meetings using codes identified a priori based on the conceptual framework for the study; these included *task and learning goal,* and *anticipating*. Open coding was used to capture ideas that emerged that were not included in the initial codes. For example, during the first planning meeting, issues of when to teach equipartitioning concepts surfaced in the teachers' discussions. I coded this as "curricular connections" and began to look for further instances when the teachers discussed the EPLT in relation to other concepts in their curriculum, or difficulties they perceived in fitting equipartitioning in with their existing curriculum. Codes were applied to chunks of the transcripts from the lesson planning meetings according to idea units that consisted of dialogue by one or more individuals about one particular idea.

Coding of Individual Cases After the initial coding of the lesson planning meetings, each individual case, consisting of the teacher's pre-lesson questionnaires, observations, and interviews, was coded in chronological order. The pre-lesson questionnaires were coded for *task and learning goal* and *anticipating*, as well as for emerging ideas that were identified through open coding. Then, the post-lesson interview transcripts were coded in the same way as the lesson planning meetings. When new codes emerged, I revisited previously coded data to check for further evidence of the new codes. For the purposes of lesson planning, the observations were used to triangulate teachers' statements in the pre-lesson questionnaires and post-lesson interviews. This process was repeated for the second and third lessons, then for each of the remaining cases. Memos were created throughout the coding process to develop emerging themes and categories.

Within- and Cross-Case Analyses Once all data were coded, I completed a within-case analysis by answering the research question using detailed descriptions for each case and its context. Once the within-case analyses were completed, a cross-case comparison (Merriam 1998) was conducted to facilitate clarification about the teachers' uses of the EPLT. I began to look across the cases to create categorical aggregations and establish patterns. For example, four of the cases provided evidence of coordination between proficiency levels and task parameters of the EPLT to calibrate tasks to fit the needs of their students and the mathematical goals. Because the teachers varied in how they used the EPLT in this way, I used this idea, along with others, to categorically describe the ways teachers used EPLTs to choose tasks and specify learning goals.

Various forms of data were collected in order to triangulate the participants' perceptions and interpretations of the use of the EPLT for lesson planning. In addition, because multiple observations were conducted over the course of one semester, repeated observations of the same phenomenon were conducted, increasing the validity of the findings (Merriam 1998). As each within-case analysis was completed, I shared with participants descriptions of their uses of the EPLT as a form of member checking to solicit their views of the credibility of my interpretations. Once the cross-case analysis was completed, I met with a colleague not associated with the project, who reviewed the findings as an external check.

Results

One goal of the research was to propose a framework for teachers' uses of LTs with respect to lesson planning. To this end, emphasis was on the cross-case analysis; cases are not presented individually, but instead information from the cases is distributed throughout each section (Yin 2003). I begin by providing an overall description of the grade-level lesson planning meetings in order to describe the instructional tasks the teachers chose to use and the issues the teachers discussed as a group. Next, I share themes that emerged related to two aspects of lesson planning: choosing tasks and specifying learning goals, and anticipating students' approaches.

Lesson Planning Meetings and Instructional Tasks

Second-grade teachers who participated in this research met weekly to discuss their instruction. Therefore, when it came to planning for the three EPLT lessons, details of the lesson planning took place in similar grade-level planning meetings scheduled just for the lessons they would use for the research. Although not part of the

Table 2 Tasks descriptions with potential proficiency levels and task parameters (see, Fig. 1)

Task description	Proficiency levels	Task parameters
Lesson #1: Fairly share a collection of 24 counters for 2, 4, then 3 friends. Explain, and name the resulting shares	Equipartitioning collections Justification Naming Reassembly Qualitative compensation Factor-based change Reallocation	Sharing collections for 2-splits 2^n-splits Odd splits
Lesson #2: Fairly share a rectangle and collections of 6, 8, and 10 for 2, name the resulting shares	Equipartitioning collections Equipartitioning wholes Justification Naming Reassembly	Sharing single wholes (rectangle) and collections for 2–splits
Lesson #3: Fairly share a rectangular piece of wrapping paper for different numbers of equal-sized gifts	Equipartitioning wholes Justification Naming Reassembly Qualitative compensation Composition of splits with multiple methods Transitivity	Sharing single wholes (rectangle) for 2-splits 2^n-splits (e.g., 2, 4, 8) Even splits (e.g., 6) Odd splits (e.g., 3)

individual cases, the planning meetings are described in what follows to provide context for the results.

In general, the teachers used the lesson planning meetings to choose instructional tasks. Although they mentioned specific goals at times, they did not use the meeting to discuss in detail or come to consensus on particular learning goals they wanted to address for each of the lessons. At the time of the study, it was unclear whether this was a normal practice of their common planning time. Table 2 presents a summary of the three tasks the teachers chose to use. Also included is an analysis of potential proficiency levels and task parameters that the tasks could address depending on implementation, which varied from teacher to teacher.

For the first lesson, it was important for the teachers that the task addressed multiple levels of the trajectory, so they could get a sense of what conceptions their students held at the beginning of the school year, even though they may not explicitly attend to each level during the lesson. Using the EPLT to consider connections between equipartitioning collections and the second-grade curriculum came up during this first meeting. Part of their curriculum during the first half of the school year was to develop students' facility with doubles facts as a strategy for addition and subtraction. A few of the teachers noticed and brought up the idea that students could potentially use doubles facts as a strategy to share collections between two people and as a way to generally strengthen students' number sense. The teachers decided to use a task they used the previous year during the LTBI professional development that engaged students in sharing 24 counters among two, four, and three friends. The task asked students to explain how they shared the counters, what they would name each person's share, and to predict what would happen to the size of the share when more or less friends shared the counters.

The second lesson planning meeting took place after school, once all of the teachers had taught the first equipartitioning lesson, and the teachers spent the first half of this meeting sharing their observations from this lesson. All of the teachers deemed their students successful on the task and agreed that, in general, their students struggled to consider mathematical names for the shares they created. The teachers discussed various routines they developed for their classes that incorporated equipartitioning concepts, including daily story problems that used equipartitioning contexts as well as estimation activities that utilized the idea of sharing a collection in half.

As the teachers considered possible follow-up activities for the second lesson, they struggled to articulate where on the trajectory they wanted to focus. Because they considered that equipartitioning was only explicitly addressed in their curriculum in relation to naming fractional parts, some of the teachers found it difficult to conceptualize how equipartitioning concepts related to the curriculum they were teaching at that time. Others suggested that since they were currently working on developing doubles facts—an idea that came up in everyone's classroom during the first equipartitioning lesson—perhaps this idea could be further developed in conjunction with equipartitioning concepts. In the end, they chose to focus on naming, but restrict the task parameters to sharing a whole and collection for two. The lesson they developed began by having students explore how to share two different sized rectangles for two and how each share can be named "half," but of a different sized rectangle. Then, students explored sharing small collections of 6, 8, and 10 counters that were arranged in rectangular arrays for two friends and named the resulting share. Through discussion, the goal was for students to begin to understand the importance of the referent unit and that each share can be named as "one-half" though the sizes of the collections varied.

Similar to the second lesson planning meeting, the teachers spent the first part of the third meeting reflecting and sharing observations of their instruction from the second lesson. Overall, the teachers believed their students were beginning to understand the concept of "one-half" as one of two equal shares, and as a way to name the resulting share when equipartitioning for two. When they shifted the conversation to what they might teach for their third lesson, several ideas surfaced.

First, one of the teachers suggested a possible follow-up lesson would be to do a similar activity as the second lesson with different numbers of counters and without using the array structure in order to further develop the idea of naming in relation to the whole collection. Another teacher also suggested doing a similar activity, but to move beyond sharing for two to include sharing for four and three. They discussed how to potentially connect this lesson with other topics that they were teaching at the time, such as congruent shapes and area. Because they were moving into geometry topics, they decided a lesson that focused on sharing wholes would be more in line with their curriculum and decided to use a task from the LTBI professional development that they called "the wrapping paper task." The task used the context of fairly sharing holiday wrapping paper and could be adapted to address a number of proficiency levels and task parameters. As indicated on their pre-lesson questionnaires, the teachers modified the task on their own, focusing on sharing a rectangle for various splits including 2, 3, 4, 6, and 8.

In what follows, I present themes that emerged from the analysis related to the teachers' uses of the EPLT for two areas of lesson planning: choosing tasks and learning goals, and anticipating students' responses.

Choosing Tasks and Learning Goals

Themes that emerged related to tasks and learning goals are as follows: (a) considering the purpose of the task, (b) attending to short- and long-term goals, (c) coordination of proficiency levels and task parameters, and (d) considering a range of instructional moves. The teachers utilized the EPLT to choose instructional tasks that spanned multiple proficiency levels, thereby providing entry points for students with a variety of levels of proficiency with equipartitioning concepts. The tasks supported multiple approaches and utilized various representations, and each teacher implemented the tasks with the expectation that students would communicate with each other about mathematical ideas. The tasks were considered high cognitive demand in that they were not routine and asked students to consider the mathematics beyond general procedures (Smith and Stein 2011), although not all teachers maintained the high demand of the tasks as they were implemented.

Considering the Purpose of the Task All five teachers changed the chosen tasks slightly, either in form or in presentation, to fit their individual teaching styles and students, and varied in their purposes for using the tasks. While they all saw the first task in part as an informal assessment to determine what their students knew about equipartitioning collections, Emma and Tracy both used the lesson as an opportunity to teach their students how to organize their work and explain their thinking beyond reporting a numeric answer. During the first lesson planning meeting, Tracy commented:

> I think also when we think about the beginning of the year and we're trying to teach the tools of being a mathematician and the organizational skills that are necessary, we need to emphasize that. And sort of just gather, what are the strategies that they're using and model, or have other kids model, in a more systematic way for divvying things up.

Based on the results of the first lesson, four of the participants chose to focus on the long-term goal of developing the concept of naming fractional parts, in line with the second-grade curriculum. In the second planning meeting, Bianca and Tracy agreed that naming was an important concept to address:

> Bianca: I feel as if the naming is the hardest part… I think it would be great to have some particular lessons to start really pulling that out of kids and then help scaffolding them with that. Because when we teach fractions explicitly, I feel like they get to the wholes and they get to the actual sharing of things. But I feel as if we'd be doing our kids a disservice if we didn't hit on what they are most needing. Which I, from my class, I definitely think the naming thing.

> Tracy: Yeah, well, I'm thinking—I'm always thinking of how difficult naming is. And to me the biggest hope of getting them to name is understanding half and if, you know, by the end of second grade they can get [what] "half" is with a collection, and maybe more than that later during the year, I'd be happy. Because it is such a difficult concept. Not just count them, to give the number a count—to refer to the whole.

However, one participant, Lara, saw all three of the tasks as various forms of formative assessment to find out what her students knew about equipartitioning. Lara did not identify specific learning goals for any of the lessons. During the second planning meeting, the teachers discussed how they structured the first lesson:

> Tracy: So are you saying, did you just give them this? And have them read, you didn't…
> Lara: Yes. I didn't stop, because I guess I thought the benefit was I wasn't polluting by interjecting too much. I didn't want to, well maybe that's not a good way to put it. If I say something, I could skew their thinking and I didn't want to give them any help. This is like raw, I just wanted to see what they can do on their own.
> Tracy: Yeah, I understand that for assessment purposes…
> Lara: Mm-hmm.
> Tracy: But I guess I'm seeing this as an instructional task.
> Lara: I guess I was looking at this as assessment like, what do they know if I don't say anything? What can they do on their own?

Overall, Lara struggled with making connections between equipartitioning and the second-grade curriculum and did not see the three lessons as instructional, as opportunities to strengthen her students' number sense, or as opportunities to build a foundation for fractional concepts. Her beliefs about instruction were something different from having students engage in a novel task and then discussing students' ideas to bring forth important mathematical concepts.

Attending to Short- and Long-Term Goals In contrast to Lara, the other four teachers specified goals related to equipartitioning for each of the three lessons. They were able to use the EPLT to consider short-term goals in relation to the long-term goal of naming fractional parts. For example, for the second lesson, Bianca hypothesized that naming would be easier with a whole, so she suggested starting with sharing a rectangle for two and naming the resulting share to scaffold students' ability to name the resulting share from equipartitioning a collection. She also recognized from the first lesson that students readily made connections to doubles facts, so that could potentially also scaffold students' ability to name two-splits. She suggested,

> What if we went, this is, I'm just throwing this out there, this could be, you know. But what if we went to wholes and just worked on halving to see if a name came out of that? And then we went back to doubles with collections and see, saw if the, you know if the vernacular, if the vocabulary came out with a whole, if they would transfer it then to collections.

In addition, the teachers decided to keep the size of the collections smaller in the second lesson in order to provide opportunities for students to use their knowledge of doubles facts to determine the size of the share. In this way, they were attempting to scaffold students to move from dealing strategies to the more sophisticated strategy of using number facts. During the second planning meeting, Tracy stated,

> Well, I do like how it sort of scaffolds their thinking in a visual sense. So it's connecting the collections with the whole. And where as we don't have to, and it is very complicated for second graders to be able to name collections, half is you know, you can get them to half.

Emma outlined specific goals for the second lesson and reported in her pre-lesson questionnaire that her goals were (a) recognizing that "half" is one of two equal-sized shares of a whole, (b) recognizing the importance of naming a half in relation to the whole, (c) seeing connections between sharing a whole and sharing a collection, and (d) seeing that there is more than one way to name a share. As a group, the teachers used the trajectory to identify naming as an important goal and to integrate equipartitioning concepts with their curriculum.

Coordinating Between Proficiency Levels and Task Parameters Ellen, Bianca, Tracy, and Emma coordinated the proficiency levels of the EPLT with the task parameters to calibrate tasks so they were appropriate for their students and the mathematical goals they had chosen. For example, Ellen decided for the third lesson to have her students explore various ways to share a rectangular whole for different task parameters. She knew from the trajectory that odd splits were more difficult for students than even splits, but she also considered repeated halving to be too easy for her students. Consequently, she chose to have students explore sharing for 2, 4, and 6, with an emphasis on justifying fair shares and naming the resulting shares using fractional names. Her specific goals were "To be able to understand why each piece of paper is the same size, to be able to name each piece of wrapping paper 1/2, 1/4, 1/6, possibly 1/3." When asked during the post-interview why she chose 2, 4, and 6, she stated,

> Because we had done it before and I felt comfortable with it. I actually thought about doing two, four, and eight, but I didn't want them to just halve, and halve, and halve. And I felt like they might halve, and halve, and halve. So I wanted to see how they would do the sixths. And I didn't want to get them into odd yet. I'd rather have them get real comfortable for halves, fourths, sixths and then we'll talk about third.

Similarly, for the third lesson, Bianca attended to the task parameters as a way to address naming with her students. Her specific learning goals were for students to "share a whole fairly for 2, 4, and 8 people. Students will focus on how they might name the share in relation to the whole, for example, each person got 'one of 8 pieces.'" She also considered that because her focus was on naming, which is higher in the trajectory, keeping the task parameters lower would allow her students to focus more easily on the name rather than on the strategy for equipartitioning, as evidenced in this interview excerpt:

> I: Why did you choose two, four, and eight?
> Bianca: Yeah, so I wanted to keep with repeated halving just knowing the trajectory. You know, I know that that's easier and since naming is a little bit harder, I didn't want the sharing to be too difficult for them.
> I: I see.
> Bianca: I wanted them to be able to feel successful sharing so that they could focus on what do we call what we've just shared.

In this way, Bianca and Ellen used the task parameters in relation to the proficiency levels to adjust their lessons to the appropriate level of difficulty for their students.

Considering a Range of Instructional Moves Some of the teachers also used the EPLT to consider a range of potential instructional moves in their planning based on their students' understandings. For example, after the second lesson, Bianca recognized that she could repeat the task either by changing the task parameter to share among four, or by changing the size of the collection but still focus on naming, as shown in this dialogue from the second post-lesson interview:

> Bianca: Because I feel like some kids, I would break them off and we would start talking about collection—like, naming of collections when it's not just half, maybe. You know, or maybe doing the same kind of lesson but with fourths.
> I: Okay.
> Bianca: Like, the exact same thing where we take [a rectangle] and we share for four people and then we take a collection and you share for four people and what can we call it. You know, maybe we could go that route. Or I would go a completely different route for the kids who are, like, I'm not really sure if they got halves. Let's work on sharing wholes and naming them and seeing, you know, and can we call the same—you know, if we had the same size. You know, exploring that more deeply. So, I feel at this point, it could go—there's, like, a couple different ways you could go. I could do something, like, really similar again, which is what I tried to do with the last lesson.
> I: Right.
> Bianca: But, like, let's—
> I: With different numbers.
> Bianca: —do it, like, increase the numbers. Now, seeing what I did last time, increasing the numbers, I wouldn't do it too much more, but I could—especially if it was structured this way—I wonder if we did [collections of] 12, 14, 16.

After the third lesson, both Bianca and Emma considered a possible follow-up activity would be to change from sharing a rectangle to sharing a circle. In her post-lesson interview, Emma stated:

> It would be good to do some more multiple methods. I mean, we could also try it with a circle. And see how that would vary—couldn't do wrapping paper, but, coming up with something else. Snowmen, or I don't know, cookies.

Similarly, Bianca commented during the post-lesson interview after the third lesson:

> Well, I think—I mean I would like to see—I would probably do something the same, maybe with circles. And still focus on naming because we've kind of gotten there. But I now we're taking it down a—I would still do two, four and eight, but let's do circles and see can we still name them, but are our shares—but, like, at the side be like, "okay, what's going to happen when we get a circle? Can we share it?" Because here, they were successful sharing it and so they could be successful naming it, all right so now we're just going to take a circle and we're going to try to share it fairly.

Both teachers knew from the trajectory that equipartitioning circles is more difficult than equipartitioning rectangles, and since their students were successful with strategies and naming using rectangles, a possible move would be to explore strategies and naming using circles while keeping the task parameters the same (2^n-splits). The trajectory supported the teachers in considering both horizontal and vertical movement along the trajectory to refine and push their students' understanding related to equipartitioning.

Anticipating Students' Approaches

The participants varied in their use of the EPLT to anticipate how students would approach the intended instructional tasks. Because Lara did not complete any of the pre-lesson questionnaires, there is no evidence about whether she used the EPLT to engage in this practice or anticipated at all how her students might approach the tasks she used. For the other four teachers, themes that emerged are (a) identifying specific strategies and misconceptions, (b) attending to levels of sophistication, and (c) going beyond the goals of the lesson.

Identifying Specific Strategies and Misconceptions The four teachers for whom there was evidence of anticipation used the EPLT in different ways. Ellen used the EPLT to anticipate some of the strategies her students would use to equipartition collections and wholes, as well as how they might name the resulting shares, but she often underestimated what her students could do. For example, for the third lesson, she anticipated that students would use vertical and horizontal cuts to create halves and fourths, and that they would unintentionally create eighths by repeated halving when trying to share for six. She did not anticipate that students would use diagonal cuts, which gave her pause during her lesson. After the lesson, she stated:

> I thought it was very interesting when they did the diagonal. And when—as I was teaching it and we were talking about it and I was asking them to explain it, [how] it was "equal pieces," I was in my head going, "how am I going to explain to them?" So, I'm just going to let them give their simple explanation and just go from there because I didn't want to take the time to cut and those kinds of things to do that.

Had she anticipated the variety of approaches students often use when equipartitioning a rectangle (e.g., using diagonal cuts), she might have been more prepared to discuss how to justify the equivalence of shares produced by diagonal cuts, including a common misconception that diagonal cuts can be used to create six equal-sized pieces. Perhaps she would have considered her own mathematical thinking about using diagonals to create six unequal-sized pieces and addressed this with her students when it came up in her lesson.

Tracy also used the EPLT to anticipate common strategies. In the first pre-lesson questionnaire, she anticipated specific strategies that she expected her students to use, such as "dealing systematically one at a time while others may give a few at a time to each person, then distribute the remaining chips one at a time." Her anticipations connected to how she planned to select student work during the whole group discussion. She expected to see a range of dealing strategies, which allowed her to consider how to highlight different strategies during the whole group discussion to make the mathematics available to all of her students and to encourage more efficient strategies. In her first post-lesson interview, she compared her knowledge of student thinking prior to and after the LTBI professional development:

> I [was] pretty good at noticing what kids do and how it's different from each other, but still not necessarily understanding that one represents a more sophisticated thought. For some things, maybe…but I just had never thought of that before. I think whenever we did fair shares, it was "did they get it, or didn't they get it?" You know, maybe observing how they did it, but not really reflecting and analyzing and giving much thought to how they do it.

Bianca used the EPLT in a general way to consider what might be difficult for her students by noting on her pre-lesson questionnaire, "I think one difficulty will definitely be naming the shares. I know that it is a more difficult task on the learning trajectory and they have not had many experiences doing so." She also used it to expect specific behaviors from her students, such as using dealing strategies along with number facts and doubles facts as strategies to determine fair shares of collections. For the third lesson, because she purposefully chose powers of 2, she predicted that her students would use a repeated halving strategy, saying, "I hope that a few of them notice the repeated halving and give their own language and explanation as we go from 2 to 4 to 8." Bianca also used the EPLT to expect different mathematical names such as "one out of four, or one part of the four whole parts, or one part out of eight parts, etc." Bianca used the trajectory as a tool to consider how her students would approach the tasks, thereby allowing her to be better prepared to connect their strategies with her goals for the lessons.

Attending to Levels of Sophistication Tracy, as an experienced teacher, recognized the different strategies students used to share collections, but learning the EPLT gave her more precise language with which to describe her students' anticipated behaviors. In her pre-lesson questionnaire, she used specific language from the EPLT to describe how she thought her students would approach the first task:

> I imagine most students will not name a share mathematically, though a few may do so with prompting. I do expect that the vast majority of my students will be able to create fair shares… I anticipate the greatest difference among students to be in how they go about equipartitioning. Some will be able to reallocate chips when going from sharing between two and sharing among four by simply halving each person's share to make two new shares. Others may need to reassemble the collection and begin dealing from one all over again.

The specificity that the trajectory provided supported Tracy in recognizing levels of sophistication among the strategies her students might use, such as using a composition of splits ("halving each person's share to create two new shares") or needing to reassemble and re-deal. She also used the trajectory to recognize difficulties her students might have, such as naming a share without using the referent unit and proving the equivalence of noncongruent shares on a rectangle. Similar to Tracy, Emma used the EPLT to anticipate levels of sophistication among the approaches she expected her students to use. For the second lesson, Emma anticipated that her students would be familiar with the word "half" and would be comfortable naming shares from collections using a count, but they may struggle to see the connections between naming one of two shares of a rectangle as "half" and naming one of two shares of a collection as "half." She used the EPLT to think of the levels of sophistication among the various names she expected her students to use (e.g., "3," "3 out of 6," "one-half") and this guided how she intended to share students' ideas during the whole group discussion.

Going Beyond the Goals of the Lesson For the third lesson, Emma anticipated that the task might bring up ideas related to the equivalence of noncongruent shares (transitivity) and was prepared to discuss these ideas when they did in fact arise during her lesson. In her pre-lesson questionnaire, she wrote, "I think they will easily generalize 'half' for familiar representations, but not necessarily for less common ones. There may be some discussions about transitivity. I think naming fourths and eighths will be more difficult." Her ability to anticipate ideas not directly related to the goals of her lesson supported her in listening to her students' thinking and using that to guide her instruction "in the moment." In fact, Emma found the EPLT most useful as a tool to anticipate how her students might approach a given task. In her third post-lesson interview, she stated, "I knew a lot of the time that most of my children were not necessarily going to already be *on* the level—but then I would see some things. And so [the trajectory] was good for helping me to review, okay, these are some things that I might expect."

Discussion

With a focus on theorizing, a goal of this multi-case study was to conduct cross-case analysis to consider the various ways in which teachers use LTs for lesson planning. Thus, I propose a framework for teachers' uses of LTs as a first step toward a theory of teaching based on LTs. Previous research recognized the importance of identifying specific learning goals and anticipating students' approaches to tasks during the lesson planning process (Corey et al. 2010; Superfine 2008). Identifying and unpacking learning goals to specify necessary subconcepts can provide teachers with more detailed information with which to build a lesson (Hiebert et al. 2007). In addition, the selection of open tasks that provide students with opportunities to engage with the mathematics and discuss their solutions forms the foundation for rich classroom discussions (Smith and Stein 2011). Teachers in this study used LTs to choose tasks and specify learning goals and to anticipate students' approaches in a variety of ways. Based on the observed similarities and differences, I use the terms *initial, intermediate*, and *proficient use* to describe various levels of teachers' uses of LTs for lesson planning.

Choosing Tasks and Learning Goals

Some teachers in this study coordinated both task parameters and proficiency levels of the EPLT to calibrate tasks to meet the needs of their students; others had difficulty relating equipartitioning to the mathematics they were teaching at the time. The teachers worked collaboratively to choose tasks that spanned multiple proficiency levels, and in doing so, supported the engagement of students with a variety of zones of proximal development (Sztajn et al. 2012).

Despite the fact that these teachers regularly planned together, participants interpreted the agreed-upon tasks in their own way, choosing a variety of learning goals. One teacher, Lara, maintained the purpose of each lesson as assessment. Some of the teachers used the EPLT to specify short-term goals of equipartitioning collections and single wholes in relation to the longer-term goal of naming resulting shares. Those teachers with a stronger content knowledge maintained a focus on specific learning goals, simultaneously aware of the potential to connect to other important ideas from the trajectory that might surface during their lessons.

When the goal of a task is for general purposes, initial use supports choosing tasks that are open ended and address multiple levels of the trajectory for the purpose of determining where students are in the progression of their learning. As such, teachers gather evidence of students' understandings in relation to general ideas associated with the trajectory. As teachers shift from initial to intermediate use, tasks are used for instruction, and the LT supports them in choosing short-term goals in relation to long-term goals (Heritage 2008). Intermediate use promotes the development of open-ended tasks that address multiple proficiency levels but also focus on specific mathematical ideas.

Proficient use of the LT supports the choice of open-ended tasks, the ability to specify particular learning goals, and also the ability to consider connections to other mathematical concepts that may emerge during a lesson. The coordination among proficiency levels supports teachers in listening to students' mathematical ideas and using these ideas during instruction to further enhance their students' learning toward long-term goals highlighted in the LT. Proficient use draws upon aspects of the trajectory (in the case of the EPLT, task parameters and proficiency levels) to adjust tasks to fit the needs of students, and considers a range of next instructional steps based on students' understanding.

Anticipating Students' Approaches

Smith and Stein (2011) claimed that anticipating students' approaches to a task prior to instruction permits teachers to begin to think about how students' work relates to the intended mathematical goals. The cases in the current study allowed further refinement of the specific ways in which the trajectory was useful for anticipating prior to whole-class instruction and how language supported this practice. Teachers ranged from not anticipating to using the LT to anticipate levels of sophistication among known strategies, highlighted in the LT. The language provided by the LT supported already knowledgeable teachers by providing specificity to their expectations of how students approach equipartitioning tasks.

Initial use primarily considers if a task is accessible to students, or whether it will be easy or difficult. When teachers begin to use the LT to consider how students might approach a task, they look for information about likely strategies and misconceptions (intermediate use). As teachers become proficient with LTs, they draw upon the LT not only to anticipate known strategies and misconceptions, but they

Table 3 Teachers' Uses of LTs for lesson planning

	Initial use	Intermediate use	Proficient use
Task and learning goals	Selects open-ended tasks Tasks are used for assessment purposes only	Selects open tasks Chooses short-term mathematical goals in relation to long-term goals detailed in the LT	Selects open tasks Chooses short-term goals in relation to long-term mathematical goals detailed in the LT Coordinates among proficiency levels in the LT to adjust tasks based on students' understanding
Anticipating	Anticipates holistically if tasks will be easy or difficult for students	Anticipates likely strategies and misconceptions detailed in the LT	Anticipates likely strategies and misconceptions from the LT Relates the anticipated strategies and misconceptions to learning goals detailed in the LT Anticipates levels of sophistication among students' approaches as highlighted in the LT Anticipates beyond the goals of the lesson

use these anticipations in relation to the mathematical goals of the lesson and the long-term goals detailed in the LT. The LT is used to consider a range of student approaches, and often language from the LT gives specificity to teachers' descriptions of student behaviors and related understandings. Proficient use effectively draws upon the LT to judge levels of sophistication among the strategies teachers expect students to use. This type of anticipating supports teachers in making sense of students' ideas that emerge as students engage with a task, preparing them to use and build off students' ideas to expand and extend students' mathematical knowledge.

Table 3 answers the research question by providing a summary of teachers' uses of LTs for lesson planning. The EPLT proved useful for considering instructional tasks that spanned multiple proficiency levels and provided teachers with information about short-term learning goals in relation to the broader mathematical ideas outlined in the trajectory. Specific information about common approaches and misconceptions supported the teachers in anticipating how students would approach intended tasks.

The findings suggest variation in the ways that teachers use LTs. Through an in-depth study of individual teachers' uses of one particular LT, differences emerged across the cases. Teachers ranged from not using the trajectory to focus on student thinking, to use it purposefully to calibrate tasks, attend to levels of sophistication among students' approaches, and to structure lessons that facilitated students' movement to more sophisticated ideas. Although it is beyond the scope of this study, conjectures can be made about factors that mediate or moderate teachers' uses of

LTs. For example, it would make sense that teachers' content knowledge would influence their ability to understand and make sense of the mathematics described in a LT. Moreover, proficient users would more likely hold beliefs that are compatible with student-centered instruction.

The differences among initial, intermediate, and proficient uses highlight key aspects for researchers and teacher educators to attend to as we continue to study the ways in which teachers use LTs to focus on students' mathematical thinking. The intent of the framework is not to "label" teachers, but as an aid to teacher educators in supporting teachers' movement from initial to proficient use of LTs. The results are an initial step toward a theory of teaching based on students' LTs. These findings were gained through the cross-case analysis of five teachers, and this should be taken into consideration. Further work is needed to refine the initial framework and to empirically test these levels of uses of LTs to test its applicability across teachers from multiple grade levels and with other LTs. Future research can consider the impact of teachers' participation in professional development on LTs for student learning. Is there a relationship between teachers' uses of LTs and student learning? Moreover, if trajectories support teachers to consider the range of ideas present in their classrooms, then a consequence would be to examine issues of equity in relation to teachers' uses of LTs. Do LTs support teachers in providing meaningful learning opportunities for all students? Researchers should continue to study teacher learning of LTs, their use of LTs in instruction, and necessarily the impact of teachers' uses of LTs on student learning.

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Commentary on Section 3: Research on Teachers' Focusing on Children's Thinking in Learning to Teach: Teacher Noticing and Learning Trajectories

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With recent movements toward more ambitiously defining *mathematical proficiency* (National Research Council (NRC), 2001)—including publication of the Common Core State Standards in Mathematics (CCSSM) which calls for all students to learn deeper mathematics while engaging in sophisticated mathematical practices (National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA and CCSSO) 2010)—we are poised for a new kind of ambitious mathematics teaching. At the core of *ambitious teaching* is instruction aimed at ambitious learning goals in which *all* students are supported in acquiring, understanding, and using the knowledge to engage with complex mathematics and to solve authentic problems (Franke et al. 2007; Lampert et al. 2010; Lampert and Graziani 2009; Newmann and Associates 1996).

Ambitious teaching, however, is complex because it lies at the intersection of mathematical content and students' mathematical reasoning. A teacher who understands a particular mathematical topic in a deep way sees interrelationships among the concepts and procedures, which may enable the teacher to construct a logical and rich presentation, but for that content to be meaningful to particular students, the instructor must also understand the students' understanding. The four chapters in this section focus on (a) the space lying at the intersection of mathematics and students' understanding, and (b) teachers' understanding of that space. In this book, Edgington used a mathematics-learning trajectory as a tool to support teachers' attention to students' mathematical thinking. Stockero studied prospective secondary school teachers enrolled in an early field experience and described their transitions while they learned to focus on students' mathematical thinking. Tyminski et al. and Fisher et al. studied prospective elementary school teachers (PSTs) while they were developing deeper understanding of children's mathematical thinking in the context of an elementary mathematics methods course. Authors of all four papers placed students' mathematical thinking at the core. Edgington approached her work through the lens of a learning trajectory, whereas the authors of the other three studies focused on teacher noticing. The latter are addressed first here.

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Teachers' Noticing

In a relatively new movement in teacher education that is gaining voice, researchers call for decomposing teaching into core activities that can be productively discussed and practiced (Ball and Cohen 1999; Ball et al. 2009; Grossman and McDonald 2008; Lampert 2001). By decomposing the complexity of teaching into specific activities, we educators can more feasibly and directly address key practices and develop a common language for discussing these practices. One construct that has emerged as a useful means of focusing on professional practice is teacher noticing.

Noticing is a term used in everyday language to indicate the act of observing or recognizing something, and people engage in this activity regularly as a means of helping to organize and make sense of a perceptually complex world. Different professions have strategic ways of noticing, and these ways of noticing are affected by the knowledge and expertise held by these professionals (Miller 2011). Two particularly influential conceptualizations that have served as the foundation of current work on mathematics teacher noticing are Goodwin's (1994) ideas about *professional vision* and Mason's (2002) *discipline of noticing*. Understanding and promoting productive noticing by mathematics teachers has become a growing area of study (Sherin et al. 2011; Sherin and van Es 2009), and two components, *attending* and *making sense,* have emerged as major constructs in the study of teacher noticing. Recently, one group of researchers expanded mathematics teacher noticing, which they called *professional noticing of children's mathematical thinking,* to include three components—*attending* to children's strategies, *interpreting* children's understandings, and *deciding how to respond* on the basis of children's understandings (Jacobs et al. 2010; Jacobs et al. 2011). These researchers included *deciding how to respond* in their conceptualization of professional noticing of children's mathematical thinking because teachers are responsible for doing more than understanding their students' reasoning—they must constantly grapple with how they might extend that reasoning; as such, that to which teachers attend and their interpretations are intertwined with teachers' in-the-moment instructional decision making. Clearly, the way that a teacher decides to respond to a student, in the moment, depends upon whether the teacher attended to that student's words or actions and how the teacher interpreted them. But the relationships among attending, interpreting, and deciding how to respond are complex, because often a teacher's sensing a need to respond can affect the manner in which the teacher attends to and interprets a student's thinking. In a sense, the three components of practice—attending to students' strategies, interpreting students' reasoning, and deciding how to respond in the moment to students' reasoning—are tightly bundled.

Authors of the three of the four chapters in this set applied the theoretical lens of noticing of children's mathematical thinking, with two, Tyminski et al. and Fisher et al., focusing on all three components. Although the third research study, by Stockero, is part of a project in which all three components of professional noticing are being addressed, the study reported here narrows the focus to prospective secondary school teachers' attending to students' ideas and interpreting the mathematics within those ideas. By investigating circumstances under which noticing might be enhanced, the authors of these studies extend our understanding of teacher noticing.

Tyminski et al. studied 72 students in 3 elementary mathematics methods courses over the first 9 weeks of the semester using three carefully crafted sets of activities designed to support PSTs while they learned to attend to and make sense of students' strategies and use their understanding to write subsequent tasks for children. In the first activity, the elementary mathematics methods students watched a video of an experienced teacher posing two equal-sharing problems to her second-grade students; the PSTs then reflected on why the teacher might have selected particular numbers to use the subsequent day and the ways that those number choices differed from numbers used initially. The intent of this activity was for the PSTs to focus on the nuanced number choices that might be made by a teacher. The second activity required the students to write an open-number routine and a problem, including selecting numbers for the tasks, to address counting by tens. In the third activity, the PSTs were asked to analyze children's multiplication strategies and write a subsequent problem, with number choices, to support students' expanding understandings. The tasks were designed to scaffold PSTs' learning: from observing an expert teacher select a subsequent task to the PSTs' designing subsequent tasks; from designing a subsequent task addressing a single concept to designing a subsequent task to address a wide range of student understandings; and from analyzing an expert teacher's number choices, to choosing numbers for a prewritten task, to choosing numbers for tasks created by the PSTs. The general result of the study was that whereas the PSTs, even at the beginning of the elementary mathematics methods course, possessed a strong foundation for attending to and interpreting children's mathematical thinking, the PSTs' success at responding on the basis of children's mathematical thinking was more mixed: The PSTs were relatively successful in responding to students in the context of writing a story problem to be used the next day but struggled to create and justify number choices to address or extend student thinking.

To highlight a contribution of this study to the work on teacher noticing, I describe and analyze Task 1, used the first day of the methods class. Although the PSTs had previously completed varying numbers of mathematics content courses, they had no prior instruction to support them in completing a noticing task. The PSTs observed a teacher using two partitive-division tasks with her second-grade students. In the first, two students shared two and four cookies, with no need to subdivide cookies; they then shared larger numbers of cookies, including odd numbers. In the second, four children shared brownies, with the number of brownies carefully sequenced—4, 5, 8, 9, 16, 17, etc. (each multiple of 4 was followed by a number 1 larger)—to scaffold the learning; for example, after solving the problem of eight brownies, the children would note that they had one more than eight, encouraging them to use their previous strategy to share eight before deciding how to partition the one leftover brownie into four equal-sized pieces. The methods instructors discussed these tasks, including looking at the evidence of children's understandings and focusing on the difference between sharing an object (a cookie or brownie) between two children versus among four children. At the beginning of the next class meeting, the methods instructors asked the PSTs to reflect in writing upon a third equal-sharing task given to the same students: Four children share miniature candy bars (11, 17, 22, 35, 48, 65, etc.). The authors highlighted three salient ways these numbers differed from the previous day's numbers (differences also used in analyzing the PSTs' reasoning): (a) the numbers were larger; (b) they were more complex in that the remainders were not just 0 and 1 but also included 2 and 3, thereby requiring children to deal with more complex sharing; and (c) the previous day's scaffolds had been removed, so that, for example, sharing 17 bars was not immediately preceded by sharing 16.

The tasks in this study provide fine examples of the kinds of tasks that might be used with PSTs, either for purposes of assessment or instruction, because they specifically focus on particular content, are subtle enough to tease out distinctions in PSTs' thinking, and provide opportunities for PSTs to learn how to notice children's thinking. Furthermore, these tasks could also be used with practicing teachers while they learn to focus more deeply on children's mathematical thinking. Finally, researchers interested in studying prospective or practicing teachers' noticing would do well to consider the tasks used in this study, because the three embedded salient aspects in these tasks are adaptable to a variety of assessment opportunities.

Fisher et al. conducted a study of PSTs' attitudes toward mathematics and their noticing in the context of a mathematics methods course. In particular, they studied the changes in PSTs' attitudes toward mathematics and the changes in their noticing as a result of their work in a methods course focused on professional noticing. Furthermore, they tested for correlations both among noticing components and between the professional-noticing change scores and attitudes-toward-mathematics change scores. The study was conducted with 123 PSTs in 11 elementary mathematics methods courses across five institutions; the PSTs completed pre- and post-assessments of professional noticing and of their attitudes toward mathematics. The Professional Noticing Assessment was based around PSTs' responses to three questions about a 25-second video of a first-grade child solving a comparison, difference-unknown task. The questions, drawn from work by Jacobs et al. (2010), addressed attending to the child's thinking, interpreting the child's thinking, and deciding how to respond on the basis of the child's thinking. The Attitudes Toward Mathematics Inventory (Tapia and Marsh 2005) consisted of 40 Likert-scale items. On the basis of a factor analysis, the authors determined that four factors associated with attitudes toward mathematics emerged from the assessment: value, enjoyment, self-confidence, and motivation. Results included statistically significant differences in the PSTs' attitudes from pretest to posttest on enjoyment, self-confidence, and motivation, but not on value. Furthermore, although the students showed significant growth for three of the four attitudes, the story is more complicated because, whereas at least half of the PSTs' scores increased from pre- to post-assessment for each of the four factors, attitude scores for each of the four components decreased for at least one fifth of the PSTs. For professional noticing of children's mathematical thinking, the authors found statistically significant differences between pretest and posttest on each of the three noticing constructs. The authors also tested for correlations between each pair of noticing constructs (attending vs. interpreting;

attending vs. deciding how to respond; interpreting vs. deciding how to respond) on the pretest, on the posttest, and on the pretest-to-posttest change score, and only one of the nine, attending and interpreting on the post-assessment, was statistically significant. Finally, the authors looked for correlations between each of the four attitude-change scores and each of the three noticing-change scores, and none of the 12 correlations was significant.

The contribution of this study to our understanding of noticing is twofold. First, the authors present additional data that PSTs' noticing can improve over the course of a semester. Second, the authors have opened the door to considering relationships between noticing and other constructs, in this case, attitudes, and even though no key correlations were found between noticing and attitudes in this study, the authors present some steps to be considered by others interested in studying noticing.

Stockero studied seven prospective secondary mathematics school teachers (PSMTs), in the context of an early field experience, to determine the extent to which project activities effectively supported their noticing important instances of students' mathematical thinking. She situated her study in the context of teacher noticing and focused her analyses on attending to and interpreting students' ideas. She drew upon three frameworks, including mathematically important moments and Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOSTs), enabling her to distinguish student thinking in general from those cases of high-leverage student thinking that stand out because they might potentially be built upon to support students' understanding of important mathematics. For Stockero, these high-leverage instances of student thinking must be student-generated, involve mathematics related to learning goals for the students, and provide an opportunity for the teacher to build on student thinking. The study took place over a 14-week early field experience, with four PSMTs enrolled in the course one semester and the other three PSMTs enrolled in a different semester. During the early field experience courses, the students took turns video recording a mathematics lesson taught by the teachers they observed, and the instructional portions of each video were left unedited for the PSMTs to analyze. Early in the semester the PSMTs were told to analyze the video for mathematically important moments and targeted facilitation moves were taken by the instructor to redirect the PSMTs toward specific aspects of the video. About halfway through the semester, the PSMTs read a paper entitled *Mathematically Significant Pedagogical Opportunities to Build on Student Thinking,* and this paper was used to further narrow the PSMTs' foci. The data were analyzed along three constructs: on whom the participants focused (agency), about what the participants focused (topic), and the mathematical details on which the participants focused (specificity). Results showed that the PSMTs' noticing shifted for all three constructs. Regarding agency, the PSMTs shifted from the teacher to the student as the primary agent, from the teacher alone to interaction between the teacher and students, and from focusing on groups of students to focusing on individual students. For topic, the PSMTs decreased in their noticing of teacher explanation and in claims about the entire class and increased in their noticing of student thinking, student questions, and evidence of mathematical confusion. Finally, for specificity, PSMTs shifted their noticing from the general to instances that were more specifically mathematical. In conclusion, Stockero suggested possible explanations for the shifts that took place.

Through this study, Stockero makes several contributions to the research on teacher noticing. First, by investigating noticing of prospective teachers that occurs in an early field experience, the author provides evidence that prospective teachers can develop noticing skills even earlier in their studies than in methods courses. Second, many of the studies of noticing, especially with prospective teachers, have used preselected videos. However, because the PSMTs in this study shifted in their noticing skills even when observing unedited classroom video, we see that a wide selection of video can be used in developing noticing skills. Third, the theoretical lens applied in this study, in particular, the MOSTs, appears to have supported the shifts in the PSMTs' noticing, showing the usefulness of this theoretical lens in particular and focusing upon mathematical moments that have the potential for making a difference in general.

Learning Trajectory

Edgington, like the other authors in this section, focused on teachers' attention to students' mathematical thinking, but instead of applying a noticing framework, she used a mathematics-learning trajectory as a tool to support her teachers. Edgington's theoretical framework was built upon the work highlighting that teachers not only choose intended learning goals but also unpack and decompose these learning goals and choose tasks that align with them. As developers of learning trajectories draw upon research on student learning to describe paths of student learning over time, the trajectories can be tools for supporting teachers while they develop effective goals and tasks. Edgington's purpose was to investigate how teachers plan for instruction when they have the kind of information about the learning progression provided in learning trajectories, and she drew upon transcripts from grade-level planning meetings, pre-lesson questionnaires, classroom observations, and transcripts of teacher interviews to answer her research question. She studied a team of five highly engaged and experienced second-grade teachers in their first year of professional development while they learned about a learning trajectory for equipartitioning, a subtopic of fractions.

The results of the study were presented as four themes that emerged from the analysis of the teachers' use of the learning trajectory. First, Edgington found that all five teachers altered the chosen tasks to fit their individual teaching styles and students, and they varied in their purposes for using the tasks. Second, four of the five teachers outlined specific goals related to equipartitioning for each of the three lessons. Third, some teachers used the learning trajectory to consider a range of potential instructional moves. Finally, the teachers used the learning trajectory to anticipate how students would approach tasks, but they varied in the ways they did this. Edgington proposed a framework for teachers' initial, intermediate, and proficient use of learning trajectories.

Learning to focus on students' mathematical thinking can be challenging, and Edgington contributes to our understanding by showing how teachers, when looking through the lens of a learning trajectory, can view their students' thinking with deeper understanding and more nuance and then use that understanding to develop long-term and short-term learning goals, together with tasks, that might support their students' learning. Furthermore, the framework presented by Edgington could be useful to other researchers seeking to study the use of learning trajectories and by teacher educators who are considering ways to support teachers.

Final Comments

Teachers engaged in ambitious teaching advance their students' mathematical thinking by teaching in ways that respond explicitly and in the moment to what students do (Kazemi et al. 2009). But focusing on students' mathematical thinking is particularly difficult, and teaching experience alone is generally insufficient to enable teachers to shift their instruction so as to focus on students' mathematical thinking (Jacobs et al. 2010). The authors of the four chapters in this section are committed to develop a deeper understanding of ways educators might support teachers in refocusing their attention on their students' thinking, and though not all apply the same theoretical lens, they are all committed to use their lenses to focus on the same outcome: teachers who understand and build on their students' thinking.

Teachers have always been confronted with a "blooming, buzzing confusion of sensory data" (Sherin and Star 2011, p. 69), so they need to find ways to make sense of the complexity of classrooms. Furthermore, in this confusion, focusing on the reasoning of students is often lost. These authors provide additional evidence for the role that decomposing practice (Grossman et al. 2009; Grossman and McDonald 2008) can play in our study of teaching, in general, and in ambitious teaching focused on students' mathematical thinking, in particular.

Two examples of core activities around which teacher educators and researchers may focus their attention when decomposing practice are using learning trajectories and focusing on the noticing of teachers. For the field to continue to grow in these areas, we will need to focus on specifics. This specificity might be around the subjects in a study, be they K–3 teachers who have engaged in differing numbers of years of professional development (Jacobs et al. 2010), or, as in these studies, highly engaged and experienced second-grade teachers in their first year of professional development, or prospective elementary or secondary school teachers at particular points of their education. The specificity also should focus on particular, well-defined content areas. For example, I provided details about the task used in one of the noticing studies because the authors carefully selected numbers for their equal-sharing tasks, and the relationships among those numbers provide specificity around which teachers may learn to notice and researchers may study teacher noticing. I encourage additional work in learning trajectories and teacher noticing in particular, and more generally around decomposing practice, focused with this level of specificity. For our field to encourage the kinds of changes in teaching that will lead to ambitious teaching, we must support prospective and practicing teachers in reflecting upon their knowledge, beliefs, and practices by focusing on the subtleties made possible by such detailed specifics.

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Overall Commentary: Understanding and Changing Mathematics Teachers

Olive Chapman

The main reason for researching mathematics teachers is to understand their mathematical knowledge, practice, and learning, and how to impact them. Despite significant research and progress in these areas over the past few decades, the slow pace of reform in mathematics education suggests that our understandings of teachers are still lacking. Ongoing efforts to reform the teaching of school mathematics suggest the need for continuing efforts to understand teachers and how to help them achieve change or growth in their knowledge, thinking, and practice. The studies reported in this book make a significant contribution to both our understandings of mathematics teachers and ways to support their learning. In particular, the book highlights contributions to three central areas of research in mathematics teacher education: mathematical knowledge for teaching, teacher identity, and tools to facilitate teachers' learning. This chapter discusses the nature of these three areas, highlights specific contributions of the studies in this book, and suggests implications for future research in this field.

Central Themes of the Book

The first section of the book highlights research on mathematical knowledge for teaching. There is a general consensus that teachers need to hold deep content knowledge, as their knowledge affects both what they teach and how they teach it. However, while teachers who do not have strong knowledge of mathematics are likely to be limited in their professional competence, having such knowledge does not guarantee that they will be effective mathematics teachers (e.g., Baumert et al. 2010; Ma 1999). It is not only important what mathematics teachers know but also how they know it and what they are able to mobilize for teaching. As Ball et al. (2008) pointed out, "general mathematical ability does not fully account for the

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knowledge and skills entailed in teaching mathematics" (p. 396). Thus, "a special type of knowledge is needed by teachers that is specifically mathematical, separate from pedagogy and knowledge of students, and not needed in other professional settings" (Chapman 2012, p. 107). This knowledge has become an important basis of the recent research on mathematics teachers' knowledge (Ponte and Chapman in press). However, there is neither a consensus nor a common perspective regarding the nature of this knowledge. For example, Ruthven's (2011) overview of chapters in the book *Mathematical Knowledge in Teaching* (Rowland and Ruthven 2011) distinguished four approaches to subject knowledge for mathematics teaching: subject knowledge differentiated—approaches that categorize knowledge; subject knowledge contextualized—approaches "strongly influenced by material and social contexts" (p. 87); subject knowledge mathematized—approaches concerned with "mathematical modes of enquiry" (p. 91); and subject knowledge interactivated—approaches concerned with "epistemic and interactional processes" (p. 89). While these approaches broaden our understanding of teaching-specific mathematics knowledge, they also illustrate the complex nature of this knowledge, which contributes to the challenges of educating the mathematics teacher.

Despite differences in conceptualization, knowledge specific to teaching is widely valued as important for teaching mathematics. Recent large-scale studies (e.g., Baumert et al. 2010; Hill et al. 2005) have reported positive correlations among this kind of knowledge, teaching quality, and student achievement. Given this situation, a trend in the current research on understanding the mathematics teacher is to investigate the nature of this knowledge that he or she possesses. A review of the recent studies on prospective mathematics teachers (Ponte and Chapman in press) suggested that Ball et al.'s (2008) categories of mathematics knowledge for teaching provided the theoretical perspective for most of these studies, which dealt with topics such as rational numbers (most dominant), functions, reasoning, representation, evaluating students' achievement, and providing explanations. This focus on Ball et al.'s categories is also evident in the studies in this section of the book. These studies add to our understanding of mathematical knowledge for teaching in a variety of ways. For example, they addressed this knowledge in relation to problem solving (Heid, Grady, Jairam, Lee, Freeburn, and Karunakaran), geometry (Herbst and Kosko), proportional relationships (Jacobson and Izsák), and curriculum knowledge (Land and Drake). Thus, they cover content, mathematical processes, and curriculum—all key areas in mathematics education.

The second section of the book focuses on teacher's professional beliefs/identity. There is more to professional practice than mathematics knowledge for teaching. Teachers are engaged in practice not just with their knowledge but also with their whole being. As Palmer (1998) argued, "good teaching cannot be reduced to technique" as it "comes from the identity and integrity of the teacher" (p. 149). In other words, "we teach who we are" (p. 2). The teacher's way of being, his or her identity, is important as a means of understanding the teacher, teaching, and teacher education. For example, identity, as a construct, can inform studies that consider not only what teachers know but also who they are and how they see themselves as teachers, relate to students, deal with problems, reflect on issues, and identify themselves with the profession. Thus, in recent years, there has been growing interest in identity in educational research (e.g., Beijaard et al. 2004; Gee 2000; Juzwik 2006; Sfard and Prusak 2005). While this interest is reflected in mathematics education (e.g., de Freitas 2008; Hodgen and Askew 2007), studies involving an explicit focus on identity are underrepresented in the literature (Ponte and Chapman 2008). However, given the complex nature and multiple perspectives of identity, the extent of the underrepresentation is not clear-cut because of the overlap between aspects of identity and other constructs (e.g., beliefs, attitude) that could lead to different classifications of related studies. For example, identity has been considered from a sociocultural perspective, as a person's sense of belonging to a group or achieving within the norms of the group or as a function of participation in different communities (e.g., Wenger 1998). Sfard and Prusak (2005) suggested that "identities may be defined as collections of stories about persons or, more specifically, as those narratives about individuals that are *reifying, endorsable,* and *significant*" (p. 16). Identity can also be considered as being made up of personal (psychological) features as well as social (contextual) features, which come together in a construct that encompasses factors such as knowledge, beliefs, image, values, emotions, relationships, contexts, and experiences. Specific to mathematics education, for example, Hodgen (2011) related identity to mathematics knowledge in teaching, while Bjuland et al. (2012) related it to a "teacher's engagement and critical alignment in the community of participants" (p. 405).

In their review of current studies on mathematics teachers, Ponte and Chapman (in press) identified studies that addressed specific aspects of identity associated mainly with a psychological perspective, for example, teachers' confidence, values, efficacy beliefs, views, motivation, and attitudes. The studies in this section of the book contribute to our understanding of the mathematics teacher's identity in a variety of ways. Chao focused on sociocultural aspects of mathematics teachers' identity. Keazer used a narrative perspective focused on teachers' experiences of the change process. Related to a psychological perspective, DePiper considered identity in terms of teachers' positioning in relation to high-stakes accountability teaching contexts, while Wilson et al. focused on teachers' attributions of students' mathematical successes or failures. Together, then, these studies highlight different aspects of teachers' identities in ways that broaden our understanding of mathematics teachers in terms of their personal and professional lives.

The third section of the book focuses on tools and techniques for supporting teachers' learning. The importance of tools in mathematics teacher education is a focus of the international handbook edited by Tirosh and Wood (2008), in which "a range of tools and processes, often used in mathematics teacher education to facilitate various proficiencies needed for teaching mathematics, are described and critically analyzed" (p. 1). Tasks as tools are also presented as significant in mathematics teacher education in edited books at the primary level by Clarke et al. (2009) and at the secondary level by Zaslavsky and Sullivan (2011). The studies in this section of the book contribute to this growing area of research in mathematics teacher education through innovative use of various tools/techniques. For example, to facilitate teachers' learning, Edgington used a mathematics learning trajectory, Fisher et al. used a framework called *Stages of Early Arithmetic Learning,* Stockero used unedited classroom video with research-like analysis, and Tyminski et al. used three different scaffolding trajectories. These tools/techniques involved meaningful use of research-based constructs as a basis for teacher learning.

The studies in this book, then, are situated in three sections that are representative of areas of research in mathematics teacher education that are both established and growing in importance, and they contribute to these areas in a variety of ways related to understanding and changing mathematics teachers. The next section further highlights some of these contributions.

Themes of Contributions from Across the Studies

Viewed across the three sections of the book, the studies contribute in specific ways to our understanding of mathematics teachers and of facilitating their learning or change. These contributions are considered next in terms of three themes: understanding the teacher, supporting teachers' learning, and research tools.

Understanding the Teacher

The first theme highlighted by the studies in this book involves contributions to understanding the teacher. Given the importance of the relationships between the teacher and teaching and the teacher and reform, understanding the teacher is a central factor in creating twenty-first century mathematics classrooms. The studies in this book provide current insights about the teachers' knowledge and identity that suggest ongoing trends and new considerations in the field of mathematics teacher education, classified here as (a) issues with mathematics knowledge for teaching, (b) sense making of mathematics knowledge for teaching, and (c) personal-professional self.

Issues with Mathematics Knowledge for Teaching Research has consistently raised concerns about teachers' knowledge of mathematics for teaching being problematic in relation to what is considered to be adequate to teach mathematics with depth (Llinares and Krainer 2006; Ponte and Chapman 2006, 2008). Such research findings have been useful to understand the teacher and to inform teacher education of possible issues that need attention. Some of the studies in this book, directly or indirectly, suggest ongoing issues with various aspects of teachers' mathematical knowledge for teaching, thus providing further insights to our understanding of the mathematics teacher. Following is a summary of these issues.

Jacobson and Izsák found that prospective teachers often struggled and misapplied methods when dealing with direct proportions. Herbst and Kosko, in assessing experienced teachers' mathematics knowledge for teaching geometry, found that experienced geometry teachers did much better than their non-experienced counterparts on items that were most directly connected to what is commonly taught in geometry classrooms. Heid et al. found that their participant seldom enacted mathematical processes like representing and justifying, although she was capable of doing so. This lack of engagement in these processes, along with a tendency to underrepresent the important features of mathematical objects, resulted in a somewhat disconnected treatment of mathematics. Tyminski et al. found that prospective teachers struggled to coordinate attending to student thinking while simultaneously attending to alternate thinking or learning goals. A number of prospective teachers in Land and Drake's study attended more to surface-level, procedural aspects of their curriculum materials than to aspects related to interpreting and assessing student thinking. Collectively, these studies address both content and pedagogical content knowledge and raise awareness of ongoing issues in teacher knowledge that have implications for mathematics teacher education.

Sense Making of Mathematics Knowledge for Teaching In addition to identifying issues, some of the studies draw attention to teachers' sense making (i.e., their meanings, interpretations, or capabilities). Addressing teachers' sense making is important to understand teachers in positive ways that can help to explain their classroom actions and provide a meaningful basis to attend to and build on in facilitating their learning. Examples of teachers' sense making of mathematical knowledge for teaching are reflected in the studies as follows. Land and Drake found that prospective teachers were capable of using curricular supports from within materials to extend beyond the scope of those materials. The prospective teachers in Jacobson and Izsák's study often did not make sense of the situations, but merely applied rote procedures and failed to attend to important mathematical relationships. Edgington's study showed teachers' sense making in using a learning trajectory to plan lessons, ranging from considering accessibility to anticipating student approaches and pitfalls. Finally, Tyminski et al. found that, through intervention, prospective teachers were able to improve their ability to make sense of and evaluate students' thinking strategies in a variety of mathematical contexts. A majority of them were able to attend to student strategies and interpret student thinking. Collectively, these studies provide examples of teacher's sense making for different aspects of mathematical knowledge for teaching that contribute to our understanding of what teachers are able to do with or without intervention.

Personal-Professional Self As discussed above regarding identity, understanding teachers in terms of their personal and professional selves is central to making sense of and reforming mathematics teaching. Some of the studies in this book provide insights into the aspects of teacher identity that show how self-knowledge (as opposed to content or pedagogical knowledge) can impact, for example, the teacher's classroom behavior, process of change, and knowing the students culturally and mathematically. A summary of these ways of understanding the teacher follows.

Chao's study allows us to understand two mathematics teachers through their personal and professional stories. One teacher's personal story involved feelings of isolation and was grounded in quite traumatic experiences. Because of the sensitive nature of this background (which he shared with many of his students), he was reticent to capitalize on it, even though he recognized that the cultural connection could be beneficial. The other teacher's identity was reflected through how he valued himself and felt valued by others more as a coach than as a mathematics teacher. He felt as if he was much more successful at motivating and inspiring his soccer players than his mathematics students, and he dealt with these feelings of impotence by teaching only those mathematics students who were self-motivated. He too felt ethno-cultural congruence with his students yet could not capitalize on it to his students' benefit. In both cases, the teachers experienced challenges linking an identity valued or meaningful to them with a professional identity as a mathematics teacher, even when they shared an ethno-cultural connection with their students.

Keazer's study provides insights into teachers' professional identity associated with professional change based on their personal experiences in adopting reform-oriented practice. Implied in the findings is how teachers' affective characteristics impacted whether or how they changed. Some teachers became excited and enthusiastic when confronted with the uncertainty of change, while others became frustrated and discouraged. Some grew in confidence and commitment, while others became disappointed and disenchanted. Collectively, the cases of these seven teachers illustrate the relationship between personal attributes and dealing with the complexity of change.

DePiper's study provides insights into prospective elementary teachers' identity in relation to how they viewed themselves teaching mathematics in high-stakes accountability contexts and how this positioning could influence how and what they taught. One teacher doubted her abilities to enact particular teaching practices because of the relationship she perceived between student achievement and her personal reputation, whereas another felt capable and at liberty to enact such practices, but nevertheless uncomfortable in doing so. Collectively, these prospective teachers' positioning also draws attention to how beliefs, implicit or explicit, are important to identity and to shape the teachers they become as opposed to the teachers they want to be.

Wilson et al.'s study allows us to understand the teacher in terms of attributions—"perceptions of causality or judgments regarding…students' successes and failures" (p. 116). The authors identified eight attributions teachers used to explain students' mathematics successes or failures when examining students' work during a professional development involving an equipartitioning learning trajectory: ability, effort, luck, task difficulty, grade/age, out of school context, teaching, and previous mathematics knowledge. Most of the teachers used all eight attributions at one time or another, with prior mathematics knowledge as the most frequently used attribute and luck and effort as the least. The professional development provided the teachers with useful research-based attributions, but they still persisted in employing nonmathematical attributions as well. Thus, the teachers' attributions, as part of their teacher identity (i.e., their ways of perceiving students), seemed well entrenched in their ways of being. Since "teachers' attributions influence their expectations regarding student ability and subsequently impact student performance" (p. 116), this aspect of teacher identity, without appropriate intervention, could negatively impact students' learning.

Finally, Fisher et al.'s study draws attention to teachers' attitudes toward mathematics. They found "significant increase" in pre- and post-assessment "on three of the four factors (enjoyment, self-confidence, and motivation), and in the fourth factor (value) when maximum possible pre-scores [were] removed" (p. 232), suggesting that initially these factors may be of concern for many. These affective factors are important to teachers' professional selves, and this study implies that, without intervention, they may be an issue in supporting meaningful mathematics teaching.

Collectively, these three categories of studies focused on *understanding the mathematics teacher* and allow us to understand the teacher from various perspectives of identity. Across these categories, the studies raised awareness of the ongoing issues in teachers' content and pedagogical content knowledge, provided examples of teacher's sense making for different aspects of mathematical knowledge for teaching, and highlighted the nature of and possible challenges associated with teachers' personal-professional selves**.** They drew attention to challenges both prospective and practicing teachers could experience as a result of their personal or professional identity and the need for providing meaningful support for further development or growth in their professional identity. Given the underrepresentation of published studies on identity in mathematics teacher education, this emphasis on identity is desirable in terms of providing insights to the field and drawing attention to the importance of future research on it, as is discussed later.

Supporting Teachers' Learning and Change

The second theme highlighted by the studies in this book involves contributions to ways of supporting mathematics teachers' learning and change. Given the importance of teachers in reforming the teaching and learning of school mathematics, ongoing efforts to understand learning opportunities that will help them to enhance their knowledge and develop new instructional practices are central to mathematics education. Some of the studies in this book show that a variety of approaches can be used to facilitate or support teachers' learning with positive outcomes. Four important areas in which they contribute insights in understanding teacher learning are learning of content, learning of pedagogy, learning to notice, and changing identity.

Learning of Content Current perspectives of prospective teacher learning of content include engaging them in learning or relearning the mathematics they will teach in ways consistent with current curriculum recommendations, revisiting familiar content to examine it in ways unfamiliar to them, and probing more deeply fundamental mathematical ideas from the school curriculum (Ponte and Chapman 2008). Jacobson and Izsák's study is an example of these views of teacher learning and provides insights into how a course focused on multiplicative relationships and "drawn models of quantities (e.g., number line and area models)" can support prospective teachers' development of an understanding of how such problem-solving strategies "can provide the basis for developing general computation methods" (p. 50).

Learning of Mathematical Pedagogy Current studies of practicing teachers' learning and change suggest a trend that includes teachers working together to improve their practice and embedding professional study within the everyday practice of teaching (e.g., Even and Ball 2009). Edgington's study adds to this view. It shows how a mathematics learning trajectory can be used as a tool to support primary teachers' planning of meaningful student-centered lessons by helping them to become aware of students' mathematical thinking. Participants studied the equipartitioning learning trajectory through a series of professional learning tasks, some of which "were designed to allow teachers to make connections to existing curricula and current practice" (p. 266). For example, teachers experimented with equipartitioning-related tasks in their classrooms then came together to reflect on and analyze their lessons. Teacher learning resulting from the experience included using the learning trajectory in their planning to "choose tasks…, specify learning goals…, anticipate students' approaches in a variety of ways" (p. 279), and "consider connections to other mathematical concepts that may emerge during a lesson" (p. 280).

Learning to Notice An emerging body of research related to teachers' noticing supports the importance of it in teaching (e.g., Ainley and Luntley 2007; Mason 2008; Scherrer and Stein 2012; Sherin et al. 2011; Star and Strickland 2008). Noticing involves not only the attention that teachers give to significant classroom actions and interactions, but also their reflections, reasoning, and decisions based on it, i.e., *attention* and *awareness* (Mason 2008). The extent to which a teacher can notice in this way impacts his or her teaching. Many events and interactions occur at once in the classroom (in student-centered classrooms in particular), and a teacher needs to be able to identify key moments that require attention, for example, moments of student thinking that can be used to advance instruction. Thus, helping teachers to enhance their ability to notice is an important goal of teacher education. Some of the studies in this book provide insights about tools and approaches that offer promising directions to accomplish this goal. The following summary highlights these approaches.

Edgington's study was discussed above as offering an intervention for pedagogy. However, its primary goal was to help teachers to learn to notice. Thus, it also shows that a mathematics learning trajectory and the process, as already noted, can be used to help teachers to notice students' mathematical thinking, in particular, conceptions (strategies) and misconceptions. While the focus is on equipartitioning concepts and an equipartitioning learning trajectory, the study illustrates the potential for using learning trajectories to develop a stance of noticing. In general, it suggests that using learning trajectories as representations of student thinking could help teachers to notice and attend to students' mathematical thinking in their planning of lessons.

Stockero showed how activities including "research-like analysis of unedited classroom video and group discussions" supported by a teacher educator early in a teacher education program led to several "transitions in the participants' noticing" (p. 241). The prospective teachers recorded several of their cooperating teachers' mathematics lessons, then analyzed those lessons using a framework that focused on "mathematically important moments that a teacher needs to notice during a lesson"

(p. 244). The approach helped the participants to learn to notice such moments, as well as what individual students were thinking and the effect of teacher–student interactions on learning. Their descriptions of the mathematics of an instance also became more detailed. Thus, the study illustrates how certain mathematics-focused activities can help prospective teachers "learn to attend to important mathematical instances that arise during a lesson" (p. 257).

Fisher et al. showed how an approach based on a framework called *Stages of Early Arithmetic Learning* (SEAL) led to statistically significant growth in professional noticing capabilities of prospective teachers, providing opportunities for prospective teachers to "see mathematics through the lens of a child" and focus "on what children can do conceptually rather than on the procedures of mathematics that children cannot yet do" (p. 232). The intervention used video cases and interviews with children as contexts for developing prospective teachers' attending, interpreting, and deciding skills. The approach helped participants to change in all three of these components of noticing, thus suggesting the importance of being explicit about these components in activities based on children's thinking to guide prospective teachers' noticing.

Tyminski et al. demonstrated the potential success of an approach they developed to engage prospective teachers in professional noticing. The approach was framed in three trajectories of scaffolding (observing to doing, number of concepts, and number choices) with an associated sequence of tasks that "progressed from noticing an expert teacher's task design, to designing a task to address a single mathematical concept, to designing a task that addressed a wide range of student needs" (p. 194). The study suggests, however, that such a scaffolding framework has promising potential to help prospective teachers develop skills of professional noticing of students' thinking.

Finally, Heid et al.'s study, while not intended to be about intervention or noticing, also implied the importance of noticing for oneself as one engages in mathematics and how a restrictive noticing ability could hinder how teachers engage students in doing mathematics. For example, their participant needed to be prompted to notice essential properties of a mathematical object other than local features of the representation relevant for the task at hand and opportunities to incorporate multiple representations. Without the prompting, her noticing ability adversely affected her problem solving and limited her students' mathematical opportunities. These findings suggest that intentional prompting could play a useful role in designing activities to support teachers' noticing in their learning and teaching.

Collectively, these studies add to the growing body of research that indicates the importance of helping teachers to notice students' mathematical thinking in order to teach flexibly and adapt lessons to accommodate students' ideas. They provide further evidence that noticing can be taught and learned and that a variety of approaches can lead to positive outcomes. They suggest the importance of incorporating a specific framework and a structured sequence of activities to help guide teachers' learning to notice and of developing teacher noticing in the context of a specific domain, rather than attending to student thinking in general, as doing so may better support noticing with depth. In particular, such activities are important to help focus prospective teachers' attention on more complex aspects of teaching and learning, since observing videos alone may not lead them to notice what is intended or to notice productively.

Changing Identity The final area of contribution to our understanding of teachers' learning and change highlighted involves changing identity. Given the perspectives of identity involved, changing a teacher's identity could be a challenging endeavor since it could involve trying to change who the person is. However, constructs such as beliefs and attitudes have been shown to change in response to intervention, and this possibility is supported by two of the studies in this book. Wilson et al. showed that although teachers were able to augment their attributional discourse related to students' mathematical successes and failures, the approach "did not substitute or displace the existing attributions teachers used; rather, it added to and was included as part of [their] previous attributions" (p. 130), suggesting it is easier to impact growth in teachers' identity than to change it. Fisher et al.'s study "revealed the possibility that components of [preservice elementary teachers'] attitudes can improve when experiencing a course where professional noticing skills are explicitly taught, modeled, and reinforced" (p. 232). Based on the attitude scale used, there was a significant increase in their enjoyment, self-confidence, and motivation, and in value when maximum possible prescores were removed. The fact that some prospective teachers had maximum possible prescores on value suggests possible issues related to improving value (a central aspect of identity).

Research Tools

The third theme highlighted by the studies in this book involves contributions to research tools. For research to provide meaningful ways for us to understand teachers and to support their learning, the tools and processes employed are of critical importance. Two important areas in which the studies contribute to this need are in researching mathematics knowledge for teaching and identity.

Mathematics Knowledge for Teaching Appropriate and productive tools are needed to understand teachers' mathematics knowledge for teaching and changes in that knowledge. Some of the studies provide insights about the nature of possible tools that could contribute to research on different components of mathematical content and pedagogical knowledge. Herbst and Kosko focused on developing an instrument to measure mathematical knowledge for teaching high school geometry. They include sample items and describe a process for developing and testing such items. Their study provides insights into the nature of the tool and how it can be used for research. Based on their studies, Land and Drake and Edgington developed trajectories of teachers' learning that provide examples of what such trajectories could look like and tools that can be used to frame further exploration. Land and Drake developed a trajectory of mathematics curriculum knowledge and use for prospective teachers that provides a depiction of the development of expert curriculum-use knowledge and practices, while Edgington developed a trajectory of teachers' movement from initial to proficient use of a student mathematics learning trajectory as a basis to support teachers' planning of meaningful student-centered lessons. As research tools, these trajectories can provide meaningful starting points to develop more robust trajectories and theories of curriculum use, curricular knowledge, and lesson planning to support teachers' learning.

Identity Researching identity is as challenging as defining it. Narratives/stories are considered most appropriate to study identity, and Chao's study draws attention to a unique and meaningful way of accessing teachers' stories via the use of photographs. The study shows that, as a research tool, teacher-selected photographs could be used as "anchoring structures" (p. 95) to study teacher identity by framing teachers' narratives into professional, personal, and touchstone stories. These photographs provide a visual representation of the narratives of teachers' experiences that unfold during a photo-elicitation/photovoice interview. Chao described how he elicited and analyzed these stories and the aspects of mathematics teacher identity they revealed. He demonstrated how this approach is effective in accessing stories focused on sociocultural aspects of teacher identity and, in particular, how teachers' internal stories can be surfaced through their personal stories and photographs. Thus, the study illustrates a tool with powerful potential to study identity.

Other studies imply that the use of group communication is a meaningful approach to researching aspects of a teachers' identity. For example, in their studies, DePiper used teachers' discussion of mathematics teaching and practices in high-stakes accountability teaching contexts to study teachers' positioning, and Wilson et al. used teachers' discourse about students' mathematical work to study teachers' attributions of students' successes and failures. Each demonstrated how these processes can provide ways of capturing aspects of teachers' identity in a context of actual experience that are less likely to be captured by individual interviews.

Themes for Future Research

We have gained significant insights about mathematics teachers and their learning from the large body of research in the field of mathematics teacher education. But despite the significant progress resulting from it, and given the slow pace of reform in classrooms, there is still much more we need to know to help teachers transform their practice and make a difference to mathematics education. In this section, I focus on some general implications for future research, organized around the three themes of the book: mathematics knowledge for teaching, identity, and noticing.

Mathematics Knowledge for Teaching

The importance of understanding mathematics knowledge for teaching cannot be overstated. While many studies are exploring it in different ways, the complexity of this knowledge (based on, for example, various perspectives for conceptualizing it and various classroom, institutional, and sociocultural contexts that impact aspects of it) makes ongoing research of it necessary. Studies in this book have indicated possible issues with teachers' knowledge that could impact practice, suggesting the need for future research to consider not only the nature of the knowledge but how it is used in actual practice and how it impacts students' learning in actual classrooms (as in the case of Heid et al.'s study). Similarly, as demonstrated in some of the studies, in order to understand the ways teachers hold their knowledge and make sense of content and pedagogy, we need more attention to understanding teacher knowledge from the teacher's perspective and in light of how it informs actual practice.

Some of the studies used or implied approaches that hold promise for producing positive outcomes for teachers' development of mathematics knowledge for teaching. In particular, learning trajectories of students' thinking were shown to be useful in supporting teachers' learning. Research could focus on developing such learning trajectories for different content areas that can be used in teacher education. Finally, measuring or assessing mathematics knowledge for teaching is also an area that deserves more attention. Tools such as those employed by Herbst and Kosko can inform future research in other topics and on exploring the relationship between knowledge in practice and mathematics knowledge for teaching. For example, such tools could inform research on how specific aspects of the actual work of teaching a subject (e.g., Algebra, Geometry) or topic are related to specific mathematics-knowledge-for-teaching demands of teaching that subject or topic.

Identity

As Bjuland et al. (2012) pointed out, "the notion of teacher identity is considered to be a key theme for future directions of research in a sociocultural perspective" (p. 406). However, it is still underrepresented in mathematics teacher education research as an explicit research focus. Identity, depending on how it is defined, can provide a way to connect cognitive, affective, social, and cultural dimensions in considering teachers' knowledge, practices, and development. The studies in this book provided examples of the aspects of these factors that draw attention to the importance of understanding the mathematics teacher from both sociocultural and psychological perspectives and the need for future research to address identity in ways that consider both perspectives. In particular, these studies imply the need for research not only about the nature of identity, but also about, for example, the relationship between identity and actual practice, identity and change in practice, identity and equitable mathematical instruction for diverse students, and identity and noticing. The studies also suggest ways of accessing identity and approaches that could lead to growth in specific aspects of teacher identity that could be further explored. But more generally, given that the development of a teacher's professional identity is shaped by multiple influences prior to, during, and after teacher education, it is important to understand these influences and ways to explicitly support the development of professional mathematical identity.

Noticing

As previously discussed, noticing is emerging as an important construct in mathematics teacher education research, with particular attention being paid to teachers' noticing of students' mathematical thinking. The studies in this book provided evidence of noticing being a teachable skill, thus suggesting the importance of research to further understand the nature of teachers' noticing and how to support its growth and development. They also imply that such research should investigate teachers' noticing of student thinking for specific mathematical domains to understand what the teachers pay attention to and how they use it to support student learning. In addition, such research should explore the nature of and how to incorporate structured frameworks to help guide teacher noticing and approaches to support and prompt prospective teachers, in focusing their attention on more complex aspects and significant moments of teaching and learning.

In general, these three themes and the studies in this book collectively suggest we need a better grasp regarding how personal, educational, professional, and institutional factors influence teachers' practices and to further explore ways of facilitating teachers' learning and change.

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