Chapter 11
Transporting Concentrates and Tailings

Abstract Ore, water and mineral pulps are transported among the different operational units of a mineral processing plant. Water is pumped through pipelines to the grinding plant to be mixed with the ore to form the pulp that constitutes the mill feed. The mill overflow is again mixed with water to adjust the solid content and is sent through pipes to be classified in hydrocyclones. Cyclone underflow with coarse material is sent back to the mill and the overflow goes to the flotation plant. Transport in the flotation plant and between flotation sections and solid–liquid separation units is through pipelines, and finally flotation tailings are transported to tailing ponds through pipelines or channels. This chapter of the book is related to the transport of pulps in mineral processing plants. Starting from the continuity equation and the equation of motion for a continuous medium, the expression for the pressure drop during fluid flow in a tube is obtained. Newtonian fluid behavior is used to treat cases of laminar and turbulent flows. The concepts of friction factor and Reynolds number are introduced and the distribution of velocity, flow rate and pressure drop in a tube are obtained. The transport of suspensions in pipelines is then treated, defining the different regimes separated by the limiting deposit velocity. First, the flow of heterogeneous suspensions is introduced and the form to calculate head loss is presented. Next, homogeneous suspensions modeled by different rheological approaches are discussed. Finally equations for the transport of suspensions in open channel are dealt with.

Ore, water and mineral pulps must be transported among the different operational units of a mineral processing plant. In the crushing plant, where the ore is essentially dry, it is transported efficiently by conveyor belts. Water is pumped through pipelines to the grinding plant to be mixed with the ore to form the pulp that constitutes the mill feed. The mill overflow is again mixed with water to adjust the solid content and is sent through pipes to be classified in hydrocyclones. Cyclone underflow with coarse material is send back to the mill and the overflow goes to the flotation plant. Transport in the flotation plant and between flotation sections and solid-liquid separation is through pipelines, and finally flotation tailings are transported to tailing ponds through pipelines or Channels.

F. Concha A, *Solid–Liquid Separation in the Mining Industry*,
Fluid Mechanics and Its Applications 105, DOI: 10.1007/978-3-319-02484-4_11,

Pipelines in mineral processing plants enable transporting maximum loads with a minimum of space using conventional centrifugal pumps and pipes that in most cases do not exceed 24 inches in diameter. Pipelines are extremely flexible and can be used for short distance tailing disposal and long distance concentrate transportation, No matter how complex the topography; a pipeline can always be laid out.

Slurries can be classified as *homogenous* and *heterogeneous* suspensions. Homogenous suspensions behave like fluids with increased density and particular rheology, while in heterogeneous suspensions, also called mixed slurries; solid particles settle and form a solid vertical concentration profile and some bed formation while being transported.

A suspension at low solid concentration with particle sizes of less than 270 mesh (50 μm) behaves heterogeneously and requires high transport velocity to prevent particles from settling. The same suspension for high concentrations behaves homogeneously at any transport velocity. The latter suspension behaves as a mono phase fluid with particular rheological behavior. Knowledge about the rheological properties of dense slurries is fundamental to design pipeline systems. The power consumption to pump 100 (tph) of homogeneous slurry horizontally is between 0.1 and 0.2 (kW/ton-km) (Condolios and Chapus 1967). Gravity transport of homogeneous slurries is possible if there is a gradient of at least 1.5 (m) per 100 (m).

Slurries with particles larger than 270 mesh (50 μm) form heterogeneous or mixed slurries that produce vertical concentration profiles and bed formations while being transported. Particles in these suspensions are transported by saltation, by bed movement or with concentration gradients that depend on the size of the particles and the flow velocity. Higher velocities must be used to prevent settling. Pulp with particle sizes under 9 mesh (2 mm) and at least 20 % of material under 270 mesh can be transported by centrifugal pumps with a power consumption of about 3–4 kW/ton-km for a capacity of 100 tph (Condolios and Chapus 1967). Materials with sizes over 9 mesh (2 mm) require more power, in the range of 6–12 kW/ton-km (Condolios and Chapus 1967), and subject pipes to severe wear.

11.1 Transporting Fluids in Pipelines

Incompressible stationary flow in a horizontal circular tube can be described by the following variables, the fluid (1) density $\rho(\boldsymbol{r}, t)$, (2) velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ and (3) stress tensor $\boldsymbol{T}(\boldsymbol{r}, t)$, where r and t are the position vector and time respectively. These three field variables must obey the mass and linear momentum field equations:

$$\nabla \cdot \boldsymbol{v} = 0 \tag{11.1}$$

$$\rho \nabla \boldsymbol{v} \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{T} + \rho \boldsymbol{g} \tag{11.2}$$

where $\boldsymbol{g}$ is the gravitational constant.

Since there are three field variables and only two field equations, a constitutive equation must be postulated for the stress tensor:

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{T}^E(\boldsymbol{r}) \tag{11.3}$$

where p is the pressure and $\boldsymbol{T}^E$ is the extra stress tensor.

Cylindrical tubes have axial-symmetry and cylindrical coordinates can be used. Thus for the horizontal tube shown in Fig. 11.1, the following equations are valid:

$$\text{Continuity} \quad \frac{\partial v_z(r,z)}{\partial z} = 0, \Rightarrow v_z = v_z(r) \tag{11.4}$$

Since the velocity varies in the r direction only, T^E_{rz} must be a function solely of r.

$$\text{Momentum component r:} \quad 0 = -\frac{\partial p}{\partial r} + \frac{\partial T^E_{rz}(r)}{\partial z} + \rho g_r \tag{11.5}$$

$$\text{Momentum component } \theta\text{:}\ 0 = -\frac{\partial p}{\partial \theta} + \rho g_\theta \tag{11.6}$$

$$\text{Momentum component z:}\ 0 = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r T^E_{rz}(r)\right) \tag{11.7}$$

Equation (11.7) can be written in the form:

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r T^E_{rz}(r)\right) = K \tag{11.8}$$

Integrating by parts and writing the pressure drop $\Delta p = p_0 - p_L > 0$, the left side of (11.8) yields:

$$\begin{aligned} &\int_{p_0}^{p_L} dp = \int_0^L K dz, \quad \Rightarrow p_L - p_0 \equiv -\Delta p = KL \\ &K = -\frac{\Delta p}{L} \end{aligned} \tag{11.9}$$

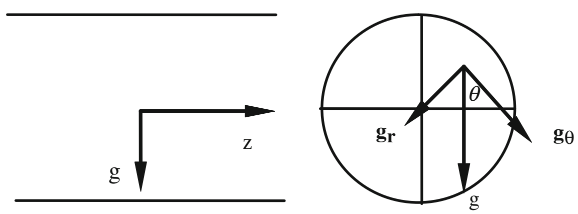


Fig. 11.1 Flow in a *horizontal* tube

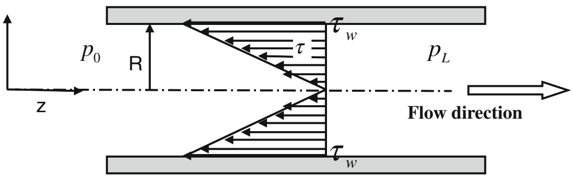


Fig. 11.2 Shear stress distribution for the flow in a *cylindrical* tube

For the right side of (11.8), integrating by parts yields:

$$\int d\left(rT_{rz}^{E}(r)\right) = \int Krdr$$

$$T_{rz}^{E}(r) = \frac{1}{2}Kr + \frac{C}{r}$$

Since the stress is finite at the tube axis, $T_{rz}(0) \neq \infty$, for $r = 0\, C = 0$, then:

$$T_{rz}^{E}(r) = \frac{1}{2}Kr$$

Substituting K from (11.9) yields the distribution of shear stress in a cylindrical tube:

$$T_{rz}^{E}(r) = -\frac{1}{2}\frac{\Delta p}{L}r \tag{11.10}$$

Designating $T_{rz}^{E}(r) \equiv \tau(r)$, Eq. (11.10) is usually written in the form (See Fig. 11.2):

$$\tau(r) = -\frac{1}{2}\frac{\Delta p}{L}r \quad \text{with}\, \Delta p = p_0 - p_L > 0 \tag{11.11}$$

If we call τ_w the shear stress at the wall, from Eq. (11.10) we can write:

$$\tau_w = -\frac{1}{2}\frac{\Delta p}{L}R \tag{11.12}$$

The ratio of shear stress at r and at the wall is:

$$\frac{\tau(r)}{\tau_w} = \frac{r}{R} \tag{11.13}$$

It is important to realize that Eqs. (11.10–11.13) are valid for all types of fluids, since we have not invoked any type of constitutive equation for $T_{rz}^{E}(r)$.

11.2 Newtonian Fluids

11.2.1 Laminar Flows

For a Newtonian fluid, the constitutive equation for the extra stress $T^E_{rz}(r)$ is:

$$T^E_{rz}(r) = \mu\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \tag{11.14}$$

Using the continuity Eq. (11.4) and substituting (11.10) gives:

$$-\frac{1}{2}\frac{\Delta p}{L}r = \mu\frac{\partial v_z}{\partial r} \tag{11.15}$$

Velocity distribution

Integrating (11.15) with boundary condition $v_z(R) = 0$ at the wall gives:

$$v_z(r) = \frac{1}{4}\frac{\Delta p R^2}{\mu L}\left(1 - \left(\frac{r}{R}\right)^2\right) \tag{11.16}$$

The velocity distribution is parabolic as shown in Fig. 11.3.

Volume flow rate

The volume flow rate is given by $Q = \int_0^R 2\pi v_z(r) r dr$, then:

$$\begin{aligned} Q &= \frac{1}{2}\frac{\pi \Delta p R^4}{\mu L}\int_0^1 \left(1 - \left(\frac{r}{R}\right)^2\right)\frac{r}{R}d\left(\frac{r}{R}\right) \\ Q &= \frac{1}{8}\frac{\Delta p \pi R^4}{\mu L} \end{aligned} \tag{11.17}$$

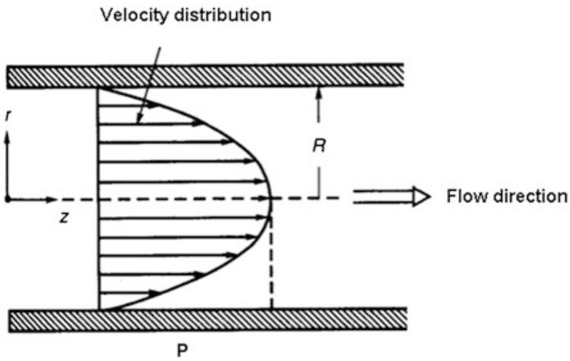


Fig. 11.3 Velocity distribution for the flow of a Newtonian fluid in a *circular* tube

Pressure gradient

$$\frac{\Delta p}{L} = \left(\frac{8\mu Q}{\pi R^4}\right) \tag{11.18}$$

Average velocity

The average velocity can be obtained from the volume flow rate by $\bar{v}_z = Q/A$, where $A = \pi R^2$ is the cross sectional area of the tube:

$$\bar{v}_z = \frac{1}{8}\frac{\Delta p R^2}{\mu L} \tag{11.19}$$

Shear rate at the wall

Defining the shear rate $\dot{\gamma}_w = \partial v_z/\partial r|_{r=R}$ at the wall as $\tau_w = \mu\dot{\gamma}_w$, from Eq. (11.12) for τ_w, we get:

$$\dot{\gamma}_w = \frac{1}{2}\frac{\Delta p R}{\mu L} \tag{11.20}$$

and using (11.19) we can write:

$$\dot{\gamma}_w = \frac{8\bar{v}_z}{D} \tag{11.21}$$

where $D = 2R$ is the tube diameter.

Maximum velocity

The maximum velocity is obtained from (11.16) for $r = 0$:

$$v_m = \frac{1}{4}\frac{\Delta p R^2}{\mu L} \tag{11.22}$$

Problem 11.1 Calculate the velocity distribution of three fluids with different viscosities, $\mu = 0.001,\ 0.002$ and 0.008 [Pa-s] in a tube 1 inch in diameter and 50 m in length, subjected to a pressure drop of 172 (Pa). As well, calculate the flow rate, average and maximum velocity, wall shear stress and shear rate and the Reynolds flow number.

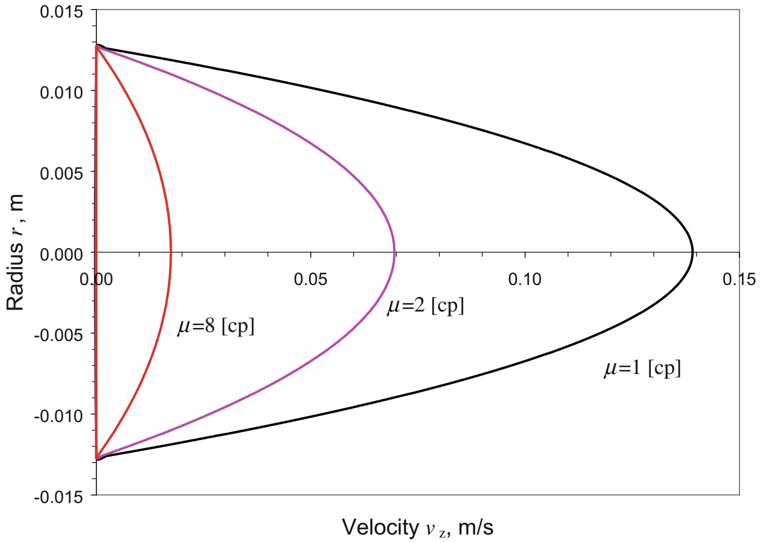


Fig. 11.4 Velocity distribution for the flow of three fluids; with viscosities 1, 2 and 8 cp, in a cylindrical tube 1 inch in diameter and 500 m in length

Data are: $R = 0.0127$ m, $L = 50$ m, $\Delta p = 172$ Pa, $\mu = 0.001$; 0.002, 0.008 Pa-s As an example, calculate with $\mu = 0.001$ Pa-s (see Fig. 11.4).

$$v_z(r) = \frac{1}{4}\frac{\Delta p R^2}{\mu L}\left(1 - \left(\frac{r}{R}\right)^2\right) = \frac{172 \times 0.0127^2}{4 \times 0.001 \times 50}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$v_z(r) = 0.14\left(1 - \left(\frac{r}{R}\right)^2\right) \text{ m/s}$$

Maximum velocity:

$$v_m = 0.14 \text{ m/s}$$

Volume flow rate:

$$Q = \frac{1}{8}\frac{\Delta p \pi R^4}{\mu L} = \frac{172 \times \pi \times 0.127^4}{8 \times 0.001 \times 50} = 3.52 \times 10^{-5} \text{ m}^3/\text{s}$$

Average velocity:

$$\bar{v}_z = \frac{Q}{\pi R^2} = \frac{3.52 \times 10^{-5}}{3.14 \times 0.0127^2} = 0.035 \text{ m/s}$$

Shear rate and shear stress at the wall

$$\dot{\gamma}_w = \frac{8\bar{v}_z}{D} = \frac{8 \times 0.035}{0.0127 \times 2} = 21.9\,\mathrm{s}^{-1}$$
$$\tau_w = \mu\dot{\gamma}_w = 0.001 \times 21.9 = 0.022\,\mathrm{Pa}$$

Reynolds number $\mathrm{Re} = \frac{\rho D \bar{v}}{\mu} = \frac{1000 \times 2 \times 0.0127 \times 0.035}{0.001} = 1.77 \times 10^3$

Summary:

Newtonian fluid			
μ (Pa-s)	0.001	0.002	0.008
L (m)	50	50	50
R (m)	0.0127	0.0127	0.0127
Δp (Pa)	172	172	172
Q (m^3/s)	3.52E−05	1.76E−05	4.40E−06
v_{av} (m)	0.070	0.035	0.009
v_m (m)	0.14	0.07	0.02
γ_w (s^{-1})	21.9	10.9	2.7
τ_w (Pa)	0.022	0.022	0.022
ρ (kg/m^3)	1.00E+03	1.00E+03	1.00E+03
Re	1.77E+03	4.41E+02	2.76E+01

Friction factor for Newtonian fluids

The dimensionless solid-fluid resistance coefficient, called the *Fanning friction factor,* is defined as the ratio of friction at the wall to the dynamic pressure:

$$f = \frac{-\tau_w}{1/2\rho\bar{v}_z^2} \tag{11.23}$$

From Eq. (11.12) $\tau_w = -\frac{1}{2}\frac{\Delta p}{L}R$, substituting (11.23) yields:

$$f = \frac{\Delta p}{L}\frac{D}{2\rho\bar{v}_z^2} \tag{11.24}$$

Equation (11.24) shows that the Fanning friction factor can also be interpreted as the ratio of the pressure gradient to halve the dynamic pressure. Substituting the value of $\Delta p/L$ from (11.19) with (11.24) results in:

$$f = \frac{16}{\rho D\bar{v}_z/\mu}$$

Using the definition of the Reynolds number $\mathrm{Re} = \rho D\bar{v}_z/\mu$, the Fanning friction factor for the laminar flow of a Newtonian fluid is:

$$f = \frac{16}{\mathrm{Re}} \tag{11.25}$$

Another definition of the friction factor is the ratio of head loss to velocity head: $\lambda = h_L / \left(\frac{\bar{v}_z^2}{2g}\right)\left(\frac{L}{D}\right)$ [m], and since $h_L = \frac{\Delta p}{\rho g}$, $\lambda = \frac{\Delta p}{L} \frac{D}{(1/2)\rho \bar{v}^2}$, then $\lambda = 4f$. This version is called the Darcy-Weisbach friction factor. In terms of λ, the friction factor for Newtonian fluids is:

$$\lambda = \frac{64}{\mathrm{Re}} \tag{11.26}$$

11.2.2 Turbulent Flows

The transport of suspensions occurs in laminar or turbulent regimes. The parameters defining the transition between laminar and turbulent flows are the *Fanning friction factor f* and the *Reynolds number* Re.

Due to the overriding effect of viscosity forces in the laminar flow of Newtonian fluids, even flows over asperous surfaces appear smooth. Therefore, the roughness of the walls, unless it is very significant, does not affect flow resistance. Under these flow conditions the friction coefficient is always a function of the Reynolds number alone.

As the Reynolds number increases, inertia forces, which are proportional to velocity squared, begin to dominate. The turbulent motion is characterized by the development of transverse component of the velocity, giving rise to agitation of the fluid throughout the stream and to momentum exchange between randomly moving masses of fluid. All this causes a significant increase in the resistance to the motion in turbulent flow compared to laminar flow.

When the surface of the wall is rough, separation occurs in the flow past the rough section and the resistance coefficient becomes a function of the Reynolds number and the relative roughness ε^*, defined as the ratio of the roughness height and the tube diameter:

$$\varepsilon^* = \frac{\varepsilon}{D} \tag{11.27}$$

where ε is the average height of the asperities and D is the tube diameter. While for low velocity flows in smooth tubes the friction factor decreases with higher Reynolds numbers, in rough tubes the friction factor increases with the Reynolds number and constant relative roughness. This is because at low flows the viscous sublayer δ is greater than the roughness protuberances $\delta > \varepsilon$ and the fluid moves smoothly past irregularities, while at higher velocities the sublayer becomes thinner than the roughness protuberances, $\delta < \varepsilon$, which enhances the formation of vortices and increases the friction factor and pressure drop. Tubes are considered

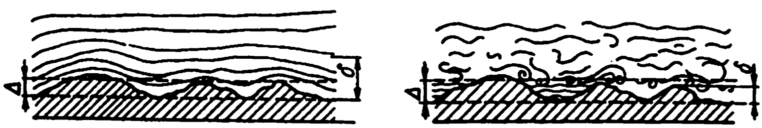


Fig. 11.5 Flow past rough tube walls for different ratios of viscous sublayer to roughness asperity

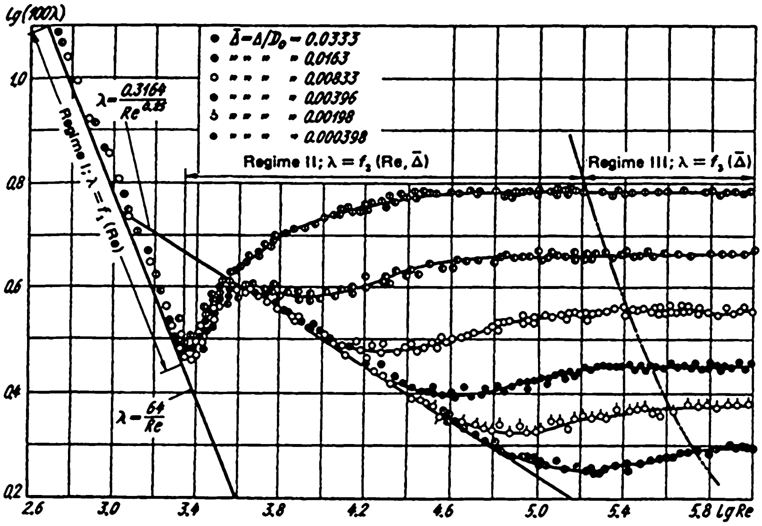


Fig. 11.6 Friction factor λ versus Reynolds number Re for tubes with uniform roughness, with $\bar{\Delta} = \varepsilon^*$, according to Nikuradse (1933), Idelchik et al. (1986)

smooth as long as the height of the asperity is less than the thickness of the laminar sublayer. See the following Fig. 11.5.

Nikuradse (1933) made flow experiments with tubes covered with different sizes of sand to simulate uniform roughness. His results are given in Fig. 11.6, which can be interpreted as consisting of three regimes (Tamburrino 2000): (1) laminar flow, (2) transition to turbulence and (3) rough walls regime.

First regime. In the first regime, with Reynolds numbers lower than 2,100, f is independent of the roughness of the tube and is given by:

$$\lambda = \frac{64}{\text{Re}} \tag{11.28}$$

Second regime. With $\text{Re} > 2,100$ and $\text{Re}_{\in} < 5$ the friction factor is given by the Blasius equation for all roughness (see Fig. 11.6):

$$\frac{1}{\sqrt{\lambda}} = -2\log\left(\frac{2.51}{\text{Re}\sqrt{\lambda}}\right) \tag{11.29}$$

For $\text{Re} > 2,100$ and $5 < \text{Re}_{\in} < 70$ the friction factor increases with the Reynolds number diverging to different lines for different degrees of constant relative roughness:

$$\frac{1}{\sqrt{\lambda}} = -2\log\left(\frac{2.5}{\text{Re}\sqrt{\lambda}} + \frac{\varepsilon^*}{3.7}\right) \quad \text{for} \quad \text{Re} > 2{,}100 \text{ and } 5 < \text{Re}_{\in} < 70 \tag{11.30}$$

where the Reynolds roughness number is $\text{Re}_{\in} = \varepsilon^* \sqrt{f/2}\text{Re}$.

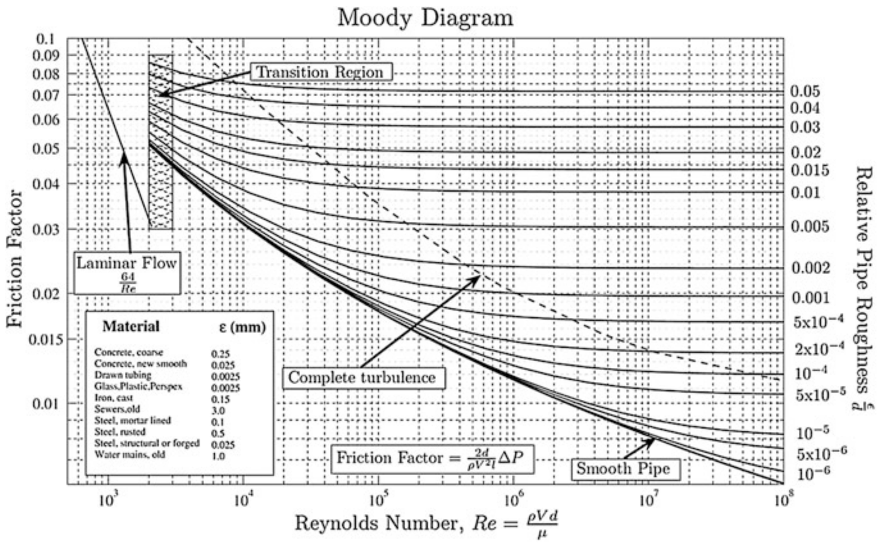


Fig. 11.7 Moody diagram

Third regime. In the third regime the friction factor becomes a different constant for each relative degree of roughness, independent of the Reynolds number:

$$\frac{1}{\sqrt{\lambda}} = -2\log\left(\frac{\varepsilon^*}{0.854}\right) \quad \text{for} \quad \mathrm{Re} > 2{,}100 \text{ and } \quad \mathrm{Re}_{\in} > 70 \qquad (11.31)$$

Moody diagram for commercial pipes is a version of Eq. (11.30). See Fig. 11.7.

Problem 11.2 If D and $\bar{v}_z$ are known, calculate the pressure gradient due to friction when water flows through a 4-inch diameter pipe at 1.5 m/s for pipe roughness $\varepsilon = 0\,(\text{smooth}),\ 0.1, 0.5$ and 1 mm. Applying Eqs. (11.29–11.31) yields:

ε (mm)	0	0.1	0.5	1
D (inch)	4	4	4	4
ρ (kg/m^3)	1,000	1,000	1,000	1,000
μ (Pa-s)	0.001	0.001	0.001	0.001
v_{av} (m/s)	1.5	1.5	1.5	1.5
D (m)	0.1016	0.1016	0.1016	0.1016
E	0	0.00098425	0.00492126	0.00984252
Re	152,400	152,400	152,400	152,400
f	0.00413	0.00534	0.00771	0.51021
$1/f^{0.5} - 1/f^{0.5}$	(0.00)	(0.00)	(0.00)	(0.00)
Re_E	–	8	47	758
	183	237	342	22,598

Explicit equations for the friction factor

When the value of the average velocity or of the volume flow rate is not known, the Reynolds number and the friction factor cannot be calculated directly. To avoid using iterative calculations, Concha (2008) observed that $\mathrm{Re}\sqrt{f}$ is a dimensionless number independent of *average velocity*. From Eq. (11.24) $f = \left(D/2\rho\bar{v}_z^2\right) \times \Delta p/L$, so that

$$\mathrm{Re}^2 f = \left(\frac{\rho D\bar{v}_z}{\mu}\right)^2 \left(\frac{D}{2\rho\bar{v}_z^2}\right)\left(\frac{\Delta p}{L}\right) = \left(\frac{\rho}{2\mu^2}\frac{\Delta p}{L}\right) \times D^3 \tag{11.32}$$

Since the left-hand side of this equation is dimensionless, the right-hand side should also be dimensionless and a parameter Ξ, with dimensions of size, which can be defined as:

$$\Xi^3 = \left(\frac{2\mu^2 L}{\rho\Delta p}\right) \tag{11.33}$$

so that (11.32) can be written in the form:

$$\mathrm{Re}^2 f = \left(\frac{D}{\Xi}\right)^3 = D^{*3}; \quad \text{and}\ \mathrm{Re}\sqrt{f} = \left(\frac{D}{\Xi}\right)^{2/3} = D^{*2/3} \tag{11.34}$$

Similarly, Re/f is a dimensionless number independent of the *pipe diameter*. Consider the function:

$$\frac{\mathrm{Re}}{f} = \frac{\rho D\bar{v}_z}{\mu}\frac{2\rho\bar{v}_z^2}{D}\left(\frac{\Delta p}{L}\right)^{-1} = \frac{2\rho^2}{\mu}\left(\frac{\Delta p}{L}\right)^{-1}\bar{v}_z^3 \tag{11.35}$$

Defining the parameter Z, with the dimension of velocity by:

$$Z^3 = \left(\frac{\mu}{2\rho^2}\frac{\Delta p}{L}\right) \tag{11.36}$$

then Eq. (11.35) can be written in the following form:

$$\frac{\mathrm{Re}}{f} = \left(\frac{\bar{v}_z}{Z}\right)^3 = \bar{v}_z^{*3}; \quad \text{and} \quad \frac{1}{\sqrt{f}} = \frac{\bar{v}_z^{*3/2}}{\mathrm{Re}^{1/2}} \tag{11.37}$$

Multiplying (11.32) and (11.37) yields:

$$\mathrm{Re} = D^*\bar{v}_z^* \tag{11.38}$$

From (11.34) $f = \dfrac{D^*}{\bar{v}_z^{*2}}$

and from (11.29) to (11.31) we get:

$$\frac{\bar{v}_z^*}{D^{*1/2}} = -4\log\left(\frac{1,26}{D^{*3/2}} + \frac{\in}{3.7}\right), \tag{11.39}$$

Since for Re < 2,100 $f = \frac{16}{\text{Re}}$ and $\frac{\text{Re}}{f} = \bar{v}_z^{*3}$, we finally obtain:and

$$\begin{aligned} \text{for Re2, 100} \quad \bar{v}_z^* &= \frac{1}{16} D^{*2} \\ \text{for Re2, 100} \quad \bar{v}_z^* &= -4 \log\left(\frac{A}{D^{*3/2}} + B\varepsilon^*\right) D^{*1/2} \\ \text{where} \quad \text{Re}_{\in} 5 : A &= 1.25; \quad B = 0 \\ 5 \le \text{Re}_{\in} \le 70 : A &= 1.25; \quad B = 0.270 \\ \text{Re}_{\in} 70 : A &= 0; \quad B = 0.171 \end{aligned} \tag{11.40}$$

$\text{Re}_{\in} = \varepsilon^* \sqrt{f/2}\text{Re}$.

Problem 11.3 (If D and $\Delta p/L$ are known) Calculate the flow rate that will be achieved when water is forced through a pipe 8 inches in diameter under a pressure gradient of 200 Pa/m, if the pipe roughness is $\varepsilon = 0.25$ mm.

ε (mm)	0.25	0.25	0.25	0.25
D (inch)	8	8	8	8
ρ (kg/m^3)	1,000	1,000	1,000	1,000
μ (Pa-s)	0.001	0.001	0.001	0.001
	200	200	200	200
D (m)	0.2032	0.2032	0.2032	0.2032
$\varepsilon^* = Z$	0.0000508	0.00123031	0.00123031	0.001230315
Ξ	0.00021544	0.00021544	0.00021544	0.000215443
Z	0.00464159	0.00464159	0.00464159	0.00464159
D*	943.17	943.17	943.17	943.17
v^*_{av}	7.6778	11.4969	420.7164	349.0550
v_{av}	0.036	0.053	1.953	1.620
Re	7,241	10,844	396,807	329,219
f	16.00000	7.13560	0.00533	0.00774
Re_E	1	25	25	25
Q (m^3/s)	0.00115568	0.00173055	0.06332765	0.052540943

The 4th column with $\text{Re}_{\in} = 25$ gives the correct result.

Problem 11.4 (If D and $\Delta p/L$ are known) Calculate the flow rate that will be achieved when water is forced through a 4-inch diameter pipe under a pressure gradient of 180 Pa/m, if the pipe roughness is $\varepsilon = 0$ (smooth), 0.1, 0.5 and 1 mm.

ε (mm)	0	0.1	0.5	1
D (inch)	4	4	4	4
ρ (kg/m^3)	1,000	1,000	1,000	1,000
μ(Pa – s)	0.001	0.001	0.001	0.001
$\Delta p/L$(Pa/m)	180	180	180	180
D (m)	0.1016	0.1016	0.1016	0.1016
$\varepsilon^* = Z$	0	0.00098425	0.00492126	0.00984252
Ξ	0.000223144	0.00022314	0.00022314	0.00022314
Z	0.00448140	0.00448140	0.00448140	0.00448140
D*	455.31	455.31	455.31	455.31
v^*_{av}	331.7712	290.4213	242.0344	218.0251
v_{av}	1.5	1.3	1.1	1.0
Re	151,059	132,232	110,201	99,269
f	0.00414	0.00540	0.00777	0.00958
Re_E	–	6.76166	33.80828	67.61657
Q (m^3/s)	0.012053971	0.01055164	0.00879364	0.00792133

11.3 Mechanical Energy Balance

In an open flow the mechanical energy balance reads:

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2$$

Table 11.1 Friction head losses

Fitting	X
45° elbow	0.3
90° elbow	0.7
90° square elbow	1.2
Exit from leg of T-piece	1.2
Entry into leg of T-piece	1.8
Unions and couplings	Small
Globe valve fully open	1.2–6.0
Gate valve fully open	0.15
Gate valve 3/4 open	1.0
Globe valve 1/2 open	4.0
Globe valve 1/4 open	16
Sudden expansion	$\left(1-(D_1/D_2)^2\right)^2$
Discharge into a large tank	1
Sudden contraction	$X = 0.7867(D_2/D_1)^6 - 1.3322(D_2/D_1)^4 + 0.1816(D_2/D_1)^2 + 0.363$
Outlet of a large tank	0.5

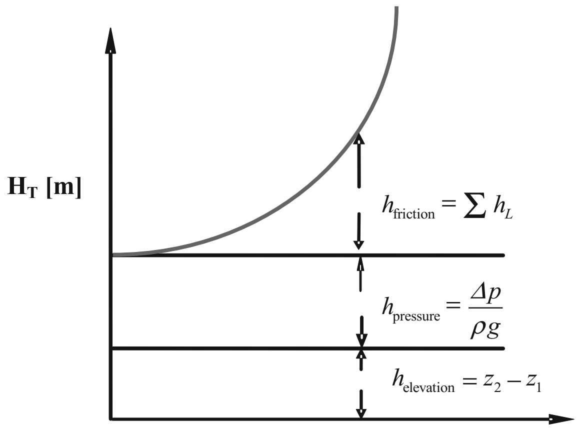


Fig. 11.8 Total head versus flow demand

that is:

$$\frac{\Delta p}{\rho g} = \frac{1}{2g}\left(\bar{v}_{z2}^2 - \bar{v}_{z1}^2\right) + (z_2 - z_1) \tag{11.41}$$

where the first term is the *pressure head* with $\Delta p = p_2 - p_1 > 0$, the second term is the *velocity head* and the third term is the *head*.

The basis to calculate flow in conduits is the mechanical energy balance in open flows to which two additional terms are added, one for the positive head H_T imposed by the pump and one for the loss $\sum h_L$ due to the friction within the fluid, on the pipe walls and on the fittings.

$$\begin{aligned} p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 + H_T - \sum h_L &= p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2 \\ H_T &= \frac{\Delta p}{\rho g} + \frac{1}{2g}\left(\bar{v}_{z2}^2 - \bar{v}_{z1}^2\right) + (z_2 - z_1) + \sum h_L \end{aligned} \tag{11.42}$$

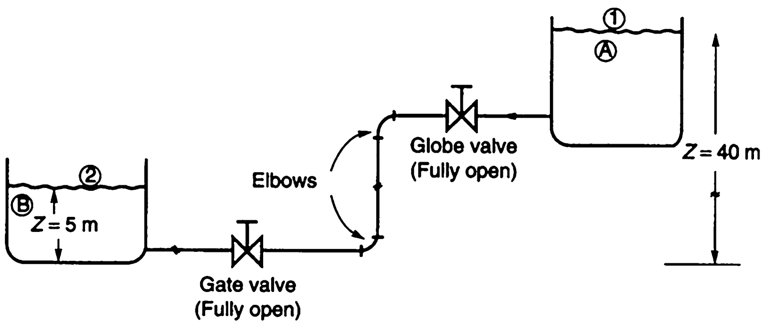


Fig. 11.9 Figure for problem 11.5

In Eq. (11.42) $H_T = \frac{P_o}{\rho g Q_f}$ and $\sum h_L = \frac{\dot{E}_v}{\rho g Q}$, where P_o and Q_f are the power and the flow rate delivered by the pump and $\dot{E}_v$ is the speed of energy dissipation by friction and $\sum h_L$ is the sum of the head loss in the pipe line $h_{pipe} = \left(L\bar{v}_z^2/gD\right) \times f$ and pipe line fittings given as the numbers X of velocity heads, $X \times \bar{v}_z^2/2g$. Table 11.1 gives the head loss for different fittings.

Figure 11.8 is a graphic description of the total head that a pump must deliver to a given flow rate of a Newtonian fluid.

Problem 11.5 Water flows under gravity from reservoir A to reservoir B, both of which are of large diameters. Estimate the flow rate through a 6-inch diameter pipe, with a roughness $\varepsilon = 0.4$ mm, and 75 m length. See Fig. 11.9.

Apply Eq. (11.42):

$$0 = 0 + 0 + (z_2 - z_1) + \sum h_L$$
$$\sum h_L = z_1 - z_2 = 40 - 5 = 35\,[\text{m}]$$

Head loss:

$$\sum h_L = h_{friction}(\text{pipe}) + h_{fitting}(1\,\text{gate valve}) + h_{fitting}(1\,\text{globe valve}) + h_{fitting}(2\text{elbows}) + h_{entrance} + h_{\text{outlet}}$$
$$\frac{1}{\sqrt{\lambda}} = -2\log\left(\tfrac{\varepsilon^*}{0.458}\right)\text{for}\text{Re} > 2{,}100 \text{ and } \text{Re}_\in > 70$$
$$\lambda = (-2\log(\varepsilon^*/0.854))^{-2}$$
$$h_{friction} = \lambda\left(\frac{\bar{v}_z^2}{2g}\right)\frac{\text{L}}{\text{D}}$$
$$\text{Re}_\in = \varepsilon^*\sqrt{\lambda/8}\text{Re}$$

Data		X_{outlet}	0.5
D (in]	6	X_{inlet}	1
L (m)	75	X_{elbow}	0.7
ε (mm)	0.4	$X_{globe\ valve}$	6
z_1 (m)	40	$X_{gate\ valve}$	0.15
Z_2 (m)	5	λ	0.039606674
P_f (kg/m^3)	1.000	assume v [m/s]	5.11
μ (Pa-s)	0.001	$h_{friction}$	25.94999706
Results		$h_{fittings}$	9.05
D (m)	0.1524	h_L [m]∼	34.99999706
ε (m)	0.0004	h_L–h_L∼0	2.94393E−06
ε* [-]	0.002624672	Re∼	778,898
hL = $Z_1 - Z_2$	35	Re_ε>70	143.85

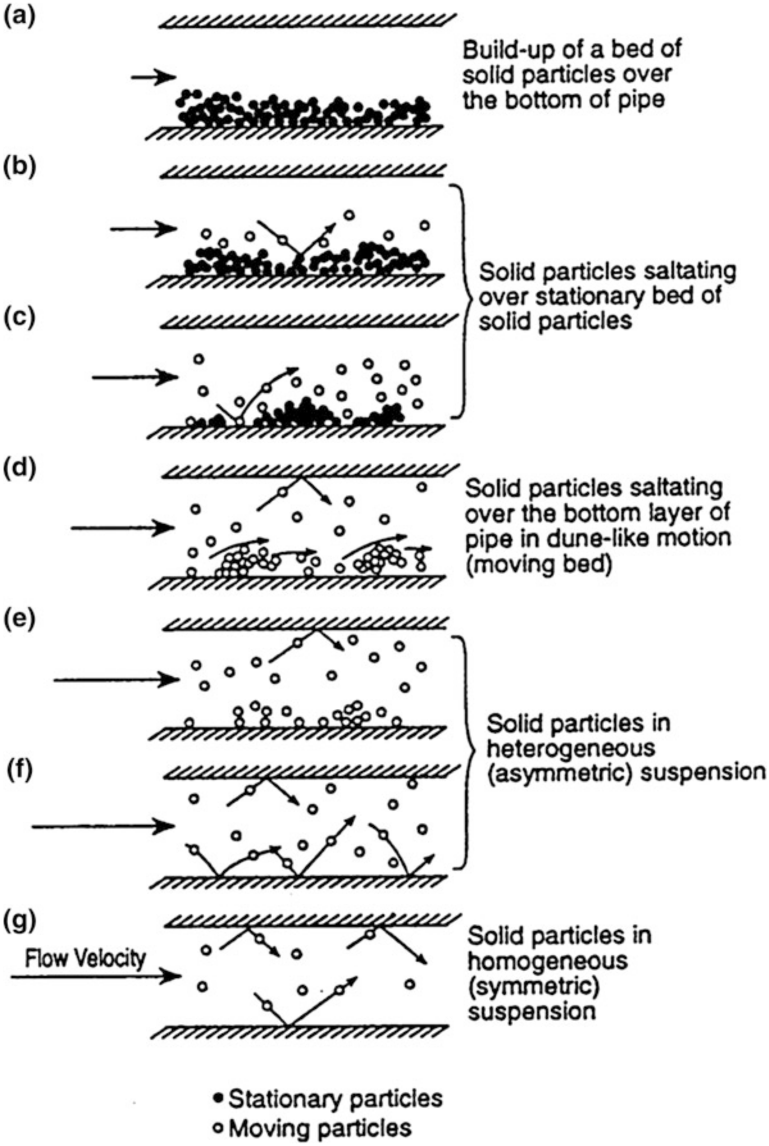


Fig. 11.10 Particle behavior for the flow of a suspension through a tube according to Chien (1994)

Table 11.2 Relationship between flow patterns and solid concentrations

Mixture velocity v_{Mi}	Flow pattern	Fraction of concentration φ/φ_F
v_{M1}	Homogeneous suspension	1.0
v_{M2}	Asymmetric suspension	0.7–1.0
v_{M3}	Living bed with asymmetric suspension	0.2–0.7
v_{M4}	Stationary bed with some particles in suspension	0–0.2

11.4 Transporting Suspensions in Pipelines

The flow patterns of suspensions in tubes depend on the transport velocity. See Fig. 11.10. At low velocities, the particles form a bed at the bottom of the tube and are not transported by the fluid. As the velocity increases, particles at the surface of the bed start moving. At higher velocities, the sediment moves as a cloud in saltatory motion, and some particles are suspended and carried away with the fluid. If the velocity increases, most particles are suspended but some settle. Under this condition the suspension is termed a *settling suspension* and the flow regime is *heterogeneous*. Increasing the velocity further, all particles are suspended and particles and fluid behave as a homogeneous mixture, the suspension is non-settling and the flow regime is *homogeneous*. Each of these behaviors corresponds to a pressure drop and the type of motion can be controlled by the pressure gradient.

The flow pattern for transporting suspensions in a tube is closely related to the suspension concentration. When particles begin to move above a stationary bed, the fractions of the feed concentration φ_F in suspension is in the fraction range of 0.7–12.0. Motions of the bed yield fractions of the feed concentration between $0.2 < \varphi/\varphi_F < 0.7$. Partial suspension gives fractions of feed concentration of $0.7 < \varphi/\varphi_F < 1$ and a complete suspension of particles gives $\varphi/\varphi_F = 1$ (Table 11.2).

Settling velocity

Since particles will settle from a flowing suspension, it is important to be able to calculate the settling velocity of the particles at several concentrations. This can be obtained from laboratory experiments or by calculations from sedimentation models. A useful model was proposed by Concha and Almendra (1979a), which was discussed in Sect. 4.1.6.

For a suspension of spherical particles, Concha and Almendra (1979b) proposed using the same equation as for single particles, but with the P and Q parameters depending on the particle concentration. See Sect. 4.1.7.

11.4.1 Flow of Heterogeneous Suspensions

The flow of a suspension is heterogeneous if some particles segregate and settle. This happens when the average flow velocity is not fast enough to maintain the

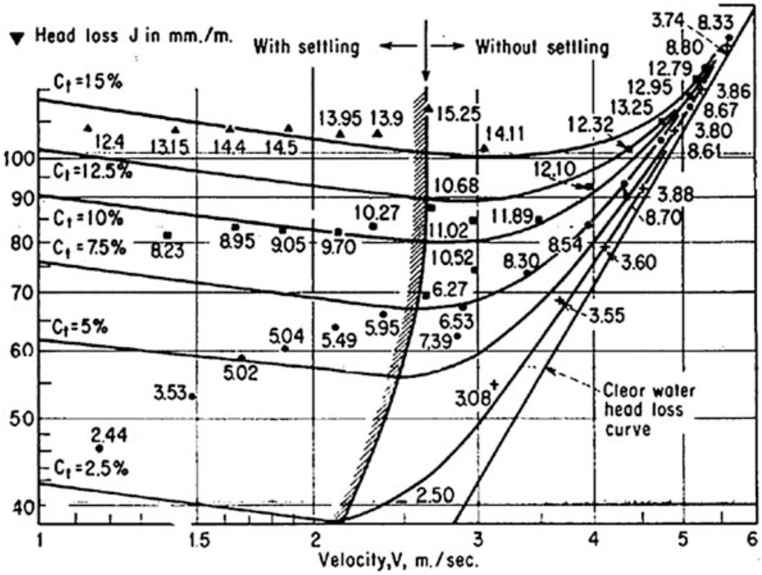


Fig. 11.11 Head loss versus average transport velocity for 0.44 mm fine sand at several concentrations (Condolios and Chapus 1963)

largest particles in suspension. In a heterogeneous regime, particles will form a concentration gradient without sediment at the bottom of the tube.

Figure 11.11 shows the pressure drop for the flow of a suspension as a function of the average flow velocity, with concentration as a parameter. For each suspension concentration there is a minimum flow velocity for which the pressure drop has a minimum. This is the lowest velocity that avoids sediment formation in the tube. Joining the minimum points for each concentration, a curve representing the *limiting settling velocity* is obtained.

Limiting deposit velocities

As discussed in the previous section, between the initiation of particle motion and the complete suspension at several concentrations, there is a small range of velocities at which the pressure drop is minimal. This range of velocities $v_L(\varphi)$ is termed *limiting deposit velocities*. The limiting deposit velocity is the lowest velocity at which no particles are deposited. The optimal transport velocity for a suspension produces the lowest pressures drop without depositing particles, that is, slightly to the right of the limiting deposit velocity in a pressure drop versus flow velocity graph.

In practice, the most common regime is with heterogeneous suspensions where the largest particles in the suspension settle, but this does not change the rheological characteristics of the pulp, although a concentration gradient is present in the pipe.

Correlations for the limiting deposit velocities

The simplest way that particles do not settle in a heterogeneous regime is to ensure that the regime is turbulent and that the Reynolds number for the largest particles is in Newton's regime (Faddick 1986):

$$\mathrm{Re} = \frac{D\bar{v}_z}{v} > 4{,}000 \text{ and } \mathrm{Re}_p = \frac{du}{v} > 1{,}000 \tag{11.43}$$

where D and $\bar{v}_z$ are the diameter of the pipe and the average velocity of the flow, d and u are the diameter of the largest particle in the suspension and its settling velocity and v is the kinematic viscosity of the fluid.

Problem 11.6 Design a pipe for the flow of 600 tph of magnetite mineral slurry that behaves as a Newtonian fluid with density 1,667 kg/m^3 and viscosity 5 mPa-s. The magnetite density is 5,000 kg/m^3 and its maximum particle size is 5 mm. Make sure that the flow regime is heterogeneous with an average velocity of 2.00 m/s.

Pulp volume flow is $Q = \frac{F}{\rho} = \frac{600}{1{,}667/1{,}000} = 0.100\ \mathrm{m^3/s}$
Particle size: $d = 0.005\,\mathrm{m}$
Magnetite density

$$\rho_s = 5{,}000\,\mathrm{kg\ /m^3}$$

Water density

$$\rho_f = 1{,}000\ \mathrm{kg/m^3}$$

% of solids:

$$w = \frac{100 \times \rho_s \times (\rho - \rho_f)}{\rho \times (\rho_s - \rho_f)} = \frac{100 \times 5{,}000 \times (1{,}667 - 1{,}000)}{1{,}667 \times (5{,}000 - 1{,}000)}$$
$$= 50\,\%\ \text{solid by weight}$$

Volume fraction:

$$\varphi = \frac{\rho_f \times w}{\rho_s \times (100 - w) + \rho_f w} = \frac{1{,}000 \times 50}{5{,}000 \times (100 - 50) + 1{,}000 \times 50} = 0.167$$

From Eq. (4.45):

$$P = \left(\frac{3}{4}\frac{\mu_f^2}{\Delta\rho \times \rho_f \times g}\right)^{1/3} \left(\frac{3}{4}\frac{0.001^2}{(5{,}000 - 1{,}000) \times 1{,}000 \times 9.81}\right)^{1/3} = 2.674 \times 10^{-5}\ \mathrm{m}^{1/3}$$

$$Q = \left(\frac{4}{3}\frac{\Delta\rho \times \mu_f \times g}{\rho_f^2}\right)^{1/3} = \left(\frac{4}{3}\frac{(5{,}000 - 1{,}000) \times 0.001 \times 9.81}{1{,}000^2}\right)^{(1/3)} = 0.0374\,(\mathrm{m/s})^{1/3}$$

$$d^* = \frac{d}{P} = \frac{0.005}{2.674 \times 10^{-5}} = 187.01$$

From Eq. (4.51):

$$u^* = \frac{20.52}{d^*}\left(\left(1 + 0.0921 \times d^{*3/2}\right)^{1/2} - 1\right)^2$$

$$= \frac{20.52}{187}\left(\left(1 + 0.0921 \times 187^{*3/2}\right)^{1/2} - 1\right)^2 = 22.698$$

$$u = u^* \times Q = 22.689 \times 0.03740 = 0.849 \text{ m/s.}$$

$$\mathrm{Re}_p \frac{\rho_f u d}{\mu} = 4243.0 > 1,000$$

Select the average transport velocity: $\bar{v}_z = 2.0$ m/s

$$D = \left(\frac{4Q}{\pi \bar{v}_z}\right)^{1/2} = \left(\frac{4 \times 0.100}{3.14 \times 0.304}\right) = 0.2524 \text{ m} = 10.0 \text{ in}$$

$$\mathrm{Re} = \frac{\rho_f \bar{v}_z D}{\mu} = \frac{1,667 \times 2.00 \times 0.2524}{0.005} = 1.6827 \times 10^5 > 4,000$$

Reynolds number fulfills the conditions for a heterogeneous flow. Summary:

F (tph)	600	P	2.673688E−05
d (m)	0.005	Q	3.740152E−02
p_s (kg/m^3)	5,000	d*	1.870076E+02
p_f (kg/m^3)	1,000	u*	22.689
P (kg/m^3)	1,667	u (m/s)	0.849
η_f (Pa-s)	0.005	Rep	4243.0
μ_f (Pa-s)	0.001	v_z (m/s)	2.00
Q_F (m^3/s)	0.09998	D (m)	0.2524
w (% by weight)	50.0	D (inch)	9.94
ϕ	0.167	Re	1.6827E+05

It has not been possible to establish the limiting deposition velocity from fundamentals, but many correlations have been proposed in the range of particle sizes from 50 (mμ])to 5 (mm) and pipes from 50 mm (2 in) to 300 mm (12 in).

Chien (1994) reviewed the work of many researchers, among them Durand (1953), Durand (1953), Spells (1955), Newitt et al. (1955), Cairns et al. (1960), Govier and Aziz (1961), Schulz (1962), Sinclair (1962), Condolios and Chapus (1963), Yufin and Lopasin (1966), Zandi and Govatos (1967), Babcock (1968), Shook (1969), Bain and Bonnington (1970), Charles (1970), Wilson (1979), Thomas (1979), Oroskar and Turian (1980) and Gillies and Shook (1991).

Unfortunately these equations are valid for different particle sizes, densities and pipe diameters, and therefore give different values of limiting velocities that range from 0.5 to 7. Figure 11.12 shows the application of these equations to suspensions of particles 150 microns in size and 2,650 kg/m^3 in density, in an 8-inch pipe and volume fractions from 0 to 0.30.

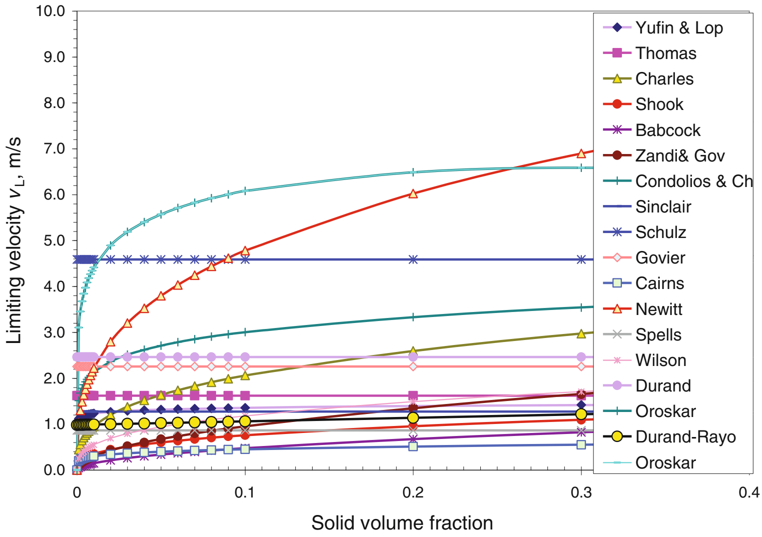


Fig. 11.12 Several correlations for the limiting velocities versus suspension concentration

In what follows, we use Durand's equation (1953) with parameters by McElvian and Rayo (1993):

$$v_L(\text{cm/s}) = F_L(\varphi)\sqrt{2\,gD\Delta\rho/\rho_f}\,; \quad \text{for } \varphi < 0.20 \tag{11.44}$$

$$F_L(\varphi) = \begin{cases} 1.1\left(\Delta\rho/\rho_f\right)^{1/5} F_{LM}(\varphi) & \text{for} \quad \text{small } d_{50} \text{ and small } D \\ 1.25\left(2\,gD\Delta\rho/\rho_f\right)^{-1/4} F_{LM}(\varphi) & \text{for} \quad \text{small } d_{50} \text{ and big } D \\ \left(d_{80}/d_{50}\right)^{1/10} F_{LM}(\varphi) & \text{for} \quad \text{big } d_{50} \text{ and extended distribution and small } D \end{cases} \tag{11.45}$$

$$\begin{aligned} F_{LM} = {} & (0.1248\varphi + 0.165)\ln(d_{50}) \\ & + (0.6458\varphi + 1.224) \text{ for } 0.005 < d_{50}(\text{mm}) < 0.5 \end{aligned} \tag{11.46}$$

In these equations φ is the volume fraction of solids in the suspension, F_{LM} is given by, with the particle diameter in mm and $F_L(\varphi)$ by Rayo (1993), where the units of the variables are v_L m/s, D m, d m; ρ kg/m^3 and $g = 9.81$ m/s^2. Rayo's equation is based on numerous years of experience designing pipelines for the copper mining plants in Chile.

Problem 11.7 Determine the limiting sedimentation velocity of quartz suspensions flowing in pipes 200 m long and 2 and 8 inches in diameter. The particle diameters are $d_{50} = 50$ mμ, $d_{80} = 374.5$ mμ and $d_{50} = 1.5$ mm and concentration 20 % solid by weight. The solid density is $\rho_s = 2{,}650$ kg/m^3; water density $\rho_f = 1{,}000$ kg/m^{3}and suspension viscosity $\mu = 5$ cp. Use Durand's equation

(1953), with parameters by Rayo (1993). For the same data draw a figure of the limiting velocity versus concentration.

Utilizing Eqs. (11.44–11.46) and Fig. 11.13 yields:

Durand and Rayo	d_{small};D_{big}	d_{small}; D_{small}	d_{big}; D_{small}
ρ_s(kg/m^3)	2,650	2,650	2,650
ρ_f (kg/m$_3$)	1,000	1,000	1,000
L (m)	200	200	200
D (m)	0.2032	0.0508	0.0508
d_{50} (m)	1.500E−04	1.500E−04	1.500E−03
d_{50} (mm)	1.500E−01	1.500E−01	1.500E+00
d_{80} (m)	3.7450E−04	3.7450E−04	3.7450E−04
g (m/s^2)	9.81	9.81	9.81
X (%sol)	20	20	20
μ (Pa-s)	5.000E−03	5.000E−03	5.000E−03
φ	0.2284	0.2284	0.2284
P	1.050E−04	1.050E−04	1.050E−04
Q	4.761E−02	4.761E−02	4.761E−02
d*	1.428E+00	1.428E+00	1.428E+01
u*	0.082	0.082	2.994
u	3.923E−03	3.923E−03	1.426E−01
$\Delta\rho/\rho_f$	1.650	1.650	1.650
ρ (kg/m^3)	1377	1377	1377
v_L (m/s)	1.161	1.340	1.706
Re_p	1.177E−01	1.177E−01	4.277E+01
Re	4.717E+04	1.362E+04	1.733E+04
C_D	2.112E+02	2.112E+02	1.597E+00

Pressure drop in a heterogeneous regime

In a heterogeneous regime the head loss J_m has two contributions: J_L to maintain the turbulent fluid flow in a Newtonian fluid, and, J_S to maintain the particles in suspension in the fluid. Both values are measured in columns of water per meter of pipe length ($J = h/L = \Delta p/\rho g L$), evaluated at the average mixture velocity:

$$J_m = J_L + J_S \tag{11.47}$$

There is no generally accepted equation for the head loss for the flow of suspensions. In a form similar to the limiting velocity, there are several empirical equations that give results with great scatter. We will use the Durand and Condolios equation (1953).

$$J_m = J_L\left(1 + 81\varphi A^{-3/2}\right); \quad \text{where } A = \frac{\bar{v}_z^2\sqrt{C_D}}{gD\,\Delta\rho/\rho_f} \tag{11.48}$$

If particle size is widely distributed, (Wasp et al. 1977) recommended calculating total head loss by weighing the individual head loss by its volume fraction;

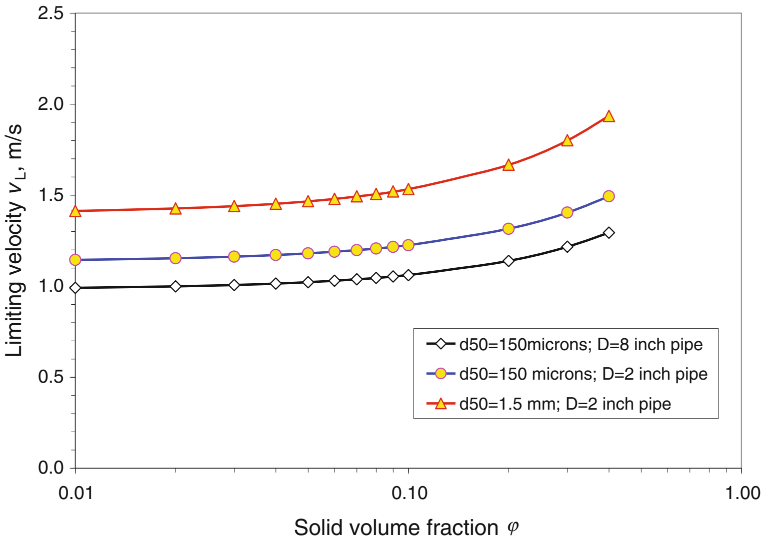


Fig. 11.13 Limiting sedimentation velocity for quartz particles of two different sizes and three different diameters, according to Durand's equation with Rayo's parameters

$$J_m = \sum_i J_i \varphi_i \tag{11.49}$$

where J_i and φ_i are the head loss associated with particle size x_i in a suspension with particle size distribution.

Problem 11.8 Calculate the pressure gradient due to friction when slurry, composed of 1 mm silica particles with a density of 2,700 kg/m^3, is pumped through a 5 cm diameter and 75 m pipeline at velocities of 3.5 m/s. The slurry contains 30 % silica by volume and the water has a density and viscosity of 1,000 kg/m^3 and 0.001 Pa-s.

$$\sum h_L = 23.96\left(\frac{\bar{v}_z^2}{2\,g}\right) \qquad J_L = 23.96/75\left(\frac{\bar{v}_z^2}{2\,g}\right) = 0.3195\left(\frac{\bar{v}_z^2}{2\,g}\right)$$

$$J_m = J_L\left(1 + 81\,\varphi A^{-3/2}\right); \quad \text{where } A = \frac{\bar{v}_z^2\sqrt{C_D}}{gD\Delta\rho/\rho_f}$$

$$\varphi = \frac{\rho_f w}{\rho_s(100 - w) + \rho_f w} = \frac{1{,}000 \times 30}{2{,}650(100 - 30) + 1{,}000 \times 30} = 0.139$$

$$P = \left(\frac{3}{4}\frac{\mu_f^2}{\Delta\rho\rho_f g}\right)^{1/3} = 3.59 \times 10^{-5} \text{ and } Q = \left(\frac{4}{3}\frac{\Delta\rho\mu_f g}{\rho_f^2}\right)^{1/3} = 2.7842 \times 10^{-2}$$

$$d_{50} = 1 \times 10^{-4}\ \mathrm{m} d^* = d_{50}/P = 2.78 f_p(\varphi) = (1-\varphi)^{-2.033} = 1.3563\,,\ f_q(\varphi)$$
$$= (1-\varphi)^{-0.167} = 1.0254 u^*$$
$$= \frac{20.52}{d^*} f_p(\varphi) f_q(\varphi)\left(\left(1 + 0.0921 f_p^{-3/2} d^{*3/2}\right)^{1/2} - 1\right)^2 = 0.7521 u = u^* \times Q$$
$$= 2.094 \times 10^{-2}\ (\mathrm{m/s})\mathrm{Re}_p = \frac{d_{50}\rho_f u}{\mu} = 2.094 C_D$$
$$= 0.28(1-\varphi)^{-2.01}\left(1 + \frac{9.08(1-\varphi)^{-1.83}}{\mathrm{Re}^{1/2}}\right)^2 = 32 A = \frac{\bar{v}_z^2\sqrt{C_D}}{gD\,\Delta\rho/\rho_f} A/\left(\bar{v}_z^2/2g\right)$$
$$= \frac{2\sqrt{C_D}}{D\,\Delta\rho/\rho_f} = 2.31\,,\ A = 2.31\bar{v}_z^2 J_m = J_L\left(1 + 81\varphi A^{-3/2}\right)$$
$$= 0.3195\left(\frac{\bar{v}_z^2}{2g}\right)\left(1 + 81 \times 0.138 \times \left(10 v_z^2\right)^{-3/2}\right)$$

Assuming that $v_z = 4.41\ \mathrm{m/s}$, with $J_m = 35/35 = 1$

$$1 - 0.9183\left(\frac{\bar{v}_z^2}{2g}\right)(1 + 81 \times 0.138 * (10 v_z^2)^{-3/2}) = 0.0858$$

Using the solver results in:

$$1 - 0.9183\left(\frac{\bar{v}_z^2}{2g}\right)\left(1 + 81 \times 0.138 \times \left(10 v_z^2\right)^{-3/2}\right) = 0.0858$$
$$v_z = 4.61\ \mathrm{m/s}\,,\ Q = \left(\pi D^2/4\right) v_z = 0.0842\ \mathrm{m^3/s}.$$

11.4.2 Flow of Homogeneous Suspensions

A homogeneous flow regime is such that suspensions are non-settling. According to Faddick (1985), if particles are small enough to be in a Stokes regime, their settling velocity will be low in relation to their transport velocity, and the suspension can be considered homogeneous.

Depending on the constitutive equation of the extra stress tensor, homogeneous suspensions can behave as Newtonian or no-Newtonian. If a suspension behaves Newtonian, the discussion and design criteria of Sect. 11.2 are valid. For non-Newtonian suspensions, we will consider flows of Bingham, power law and Herschel-Bulckley fluids in a tube.

(a) **Bingham Fluids**

Bingham fluids have the following constitutive equation for the shear stress in cylindrical coordinates:

$$T_{rz}^E(r) = \tau_y + K\frac{\partial v_z}{\partial r} \tag{11.50}$$

where K is a constant called *plastic viscosity*. From (11.10), $T_{rz}^E(r)$ is given by:

$$T_{rz}^E(r) = -\frac{1}{2}\frac{\Delta p}{L}r \tag{11.51}$$

Calling R_y the radius for which the stress is $T_{rz}^E = \tau_y$, we have:

$$T_{rz}^E(R_y) = \tau_y = -\frac{1}{2}\frac{\Delta p}{L}R_y, \tag{11.52}$$

Since the stress at the wall is given by (11.12), $\tau_w = -\frac{1}{2}\frac{\Delta p}{L}R$, the relationship between the yield stress τ_y and the wall shear stress τ_w is:

$$\frac{\tau_y}{\tau_w} = \frac{R_y}{R} \tag{11.53}$$

Velocity distribution

Substituting (11.50) with (11.51) yields:

$$\frac{\partial v_z}{\partial r} = -\left(\frac{1}{2}\frac{\Delta p}{KL}r + \frac{\tau_y}{K}\right)$$

Using (11.53) for τ_y results in: For

$$T_{rz}^E(r) > \tau_y \quad \frac{\partial v_z}{\partial r} = \frac{1}{2}\frac{\Delta p}{KL}\left(R_y - r\right) \tag{11.54}$$

Integrating this expression yields:

$$v_z = \frac{1}{2}\frac{\Delta p}{KL}\left(R_y r - \frac{1}{2}r^2\right) + C$$

For $r = R,\ v_z(R) = 0$, therefore:

$$C = -\frac{1}{2}\frac{\Delta p}{KL}\left(R_y R - \frac{1}{2}R^2\right)$$

and

$$v_z = \frac{1}{2}\frac{\Delta p}{KL}\left(R_y r - \frac{1}{2}r^2\right) - \frac{1}{2}\frac{\Delta p}{KL}\left(R_y R - \frac{1}{2}R^2\right)$$

For

$$T^E_{rz}(r) > \tau_y \qquad v_z(r) = -\frac{1}{2}\frac{\Delta pR^2}{KL}\left(\frac{R_y}{R}\left(1-\frac{r}{R}\right)-\frac{1}{2}\left(1-\left(\frac{r}{R}\right)^2\right)\right) \tag{11.55}$$

For

$$T^E_{rz}(r) \le \tau_y \qquad v_z(r) = -\frac{1}{2}\frac{\Delta pR^2}{KL}\left(\frac{R_y}{R}\left(1-\frac{R_y}{R}\right)-\frac{1}{2}\left(1-\left(\frac{R_y}{R}\right)^2\right)\right) \tag{11.56}$$

Using (11.53) we obtain the alternative expressions:

$$\begin{aligned} v_z(r) &= -\frac{1}{2}\frac{\Delta pR^2}{KL}\left(\frac{\tau_y}{\tau_w}\left(1-\frac{r}{R}\right)-\frac{1}{2}\left(1-\left(\frac{r}{R}\right)^2\right)\right); \quad \text{for} \quad \tau > \tau_y \\ v_z(r) &= -\frac{1}{2}\frac{\Delta pR^2}{KL}\left(\frac{\tau_y}{\tau_w}\left(1-\frac{\tau_y}{\tau_w}\right)-\frac{1}{2}\left(1-\left(\frac{\tau_y}{\tau_w}\right)^2\right)\right); \quad \text{for} \quad \tau \le \tau_y \end{aligned} \tag{11.57}$$

Volume flow rate

The volume flow rate is given by $Q = \int_0^R 2\pi v_z(r) r dr$, then:

$$Q = \int_{R_y}^{R} 2\pi v_z(r) r dr + \int_0^{R_y} 2\pi v_z(r) r dr$$

$$Q = \int_{R_y}^{R} 2\pi\left(-\frac{1}{2}\frac{\Delta pR^2}{KL}\left(\frac{\tau_y}{\tau_w}\left(1-\frac{r}{R}\right)-\frac{1}{2}\left(1-\left(\frac{r}{R}\right)^2\right)\right)\right) r dr$$

$$+ \int_0^{R_y} 2\pi\left(-\frac{1}{2}\frac{\Delta pR^2}{KL}\left(\frac{\tau_y}{\tau_w}\left(1-\frac{\tau_y}{\tau_w}\right)-\frac{1}{2}\left(1-\left(\frac{\tau_y}{\tau_w}\right)^2\right)\right)\right) r dr$$

$$Q = -\frac{\pi \Delta pR^4}{KL}\left(\int_{R_y/R}^{1} \frac{\tau_y}{\tau_w}\left((1-\xi)-\frac{1}{2}\left(1-(\xi)^2\right)\right)\xi d\xi\right.$$

$$\left. + \int_0^{R_y/R} \left(\frac{\tau_y}{\tau_w}\left(1-\frac{\tau_y}{\tau_w}\right)-\frac{1}{2}\left(1-\left(\frac{\tau_y}{\tau_w}\right)^2\right)\right)\right)\xi d\xi$$

Integrating this expression we obtain:

$$Q = \frac{\pi \Delta pR^4}{8KL}\left(1-\frac{4}{3}\left(\frac{\tau_y}{\tau_w}\right)+\frac{1}{3}\left(\frac{\tau_y}{\tau_w}\right)^4\right) \tag{11.58}$$

Average velocity

The average velocity is given by $\bar{v}_z = Q/\pi R^2$, then:

$$\bar{v}_z = \frac{\Delta p R^2}{8KL}\left(1 - \frac{4}{3}\left(\frac{\tau_y}{\tau_w}\right) + \frac{1}{3}\left(\frac{\tau_y}{\tau_w}\right)^4\right) \quad (11.59)$$

Shear rate at the wall

Using a similar procedure as in the case of Newtonian fluids, we have:

$$\dot{\gamma}_w = \frac{8\bar{v}_z}{D} = \frac{1}{4}\frac{\Delta p D}{KL}\left(1 - \frac{4}{3}\left(\frac{\tau_y}{\tau_w}\right) + \frac{1}{3}\left(\frac{\tau_y}{\tau_w}\right)^4\right) \quad (11.60)$$

Maximum velocity

From (11.57) the maximum velocity is that for $0 \leq r \leq R_y \quad (\tau < \tau_y)$

$$v_m = \frac{1}{4}\frac{\Delta p R^2}{KL}\left(1 - \frac{\tau_y}{\tau_w}\right)^2 \quad (11.61)$$

Friction factor

Defining the Reynolds numbers $\mathrm{Re}_B = \rho \bar{v}_z D/K$ and $\mathrm{Re}_\in = \in/\sqrt{f}\,\mathrm{Re}_B$, the friction factor according to Eq. (11.24) can be written as $f = \Delta p D/2\rho L \bar{v}_z^2$:

$$f = \frac{16}{\mathrm{Re}_B}\left(1 - \frac{4}{3}\left(\frac{\tau_y}{\tau_w}\right) + \frac{1}{3}\left(\frac{\tau_y}{\tau_w}\right)^4\right) \quad \text{for} \quad \mathrm{Re}_B < 2,100 \quad (11.62)$$

$$f_{smooth} = \left(4.53\log\left(\mathrm{Re}_B\sqrt{f}\right) - 2.3 + 4.5\log\left(1 - \tau_y/\tau_w\right)\right)^{-2} \text{ for } \mathrm{Re}_B > 4,000;\ \mathrm{Re}_\in < 5 \quad (11.63)$$

$$f_{rough} = f_{mooth} \times \left(\frac{f_{water;\,rough}}{f_{water;\,smooth}}\right) \quad \text{for} \quad \mathrm{Re}_B > 4,000;\ 5 < \mathrm{Re}_\in < 70 \quad (11.64)$$

$$f = \left(4.07\log\left(\frac{D}{2\in}\right) + 3.36\right)^{-2} \quad \text{for } \mathrm{Re}_B > 4,000\ \mathrm{Re}_\in > 70 \quad (11.65)$$

Problem 11.9 For three suspensions of clay with density $\rho = 1,275\ \mathrm{kg/m^3}$ that can be represented by the Bingham model in the range $10 < \dot{\gamma} < 500\,[\mathrm{s}^{-1}]$ with

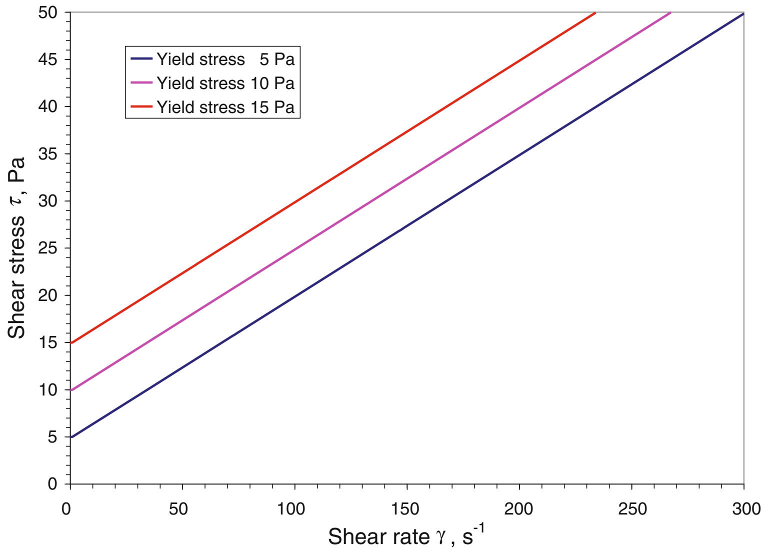


Fig. 11.14 Shear stress versus shear rate for a Bingham model of a material with plastic viscosity of 150 (mPa-s) and yield stresses of 5, 10 y 15 (Pa)

$\tau_y = 5,\ 10$ y 15Pa respectively and a plastic viscosity of $K = 150\,\text{mPa-s}$, flowing in a cylindrical tube 1 inch in diameter and 200 m in length. See Fig. 11.14. Calculate the pressure drop, the shear stress at the wall and the velocity distribution necessary to transport 100 l of suspension per minute.

Pressure drop:

$$Q_1 = 100\,\ell/\text{min} = 100/(60 * 1{,}000) = 1.6667 \times 10^{-3}\ \text{m}^3/\text{s}$$

$$Q_2 = \frac{\pi}{8}\frac{\Delta p R^4}{KL}\left(1 - \frac{4}{3}\left(\frac{\tau_y}{\tau_w}\right) + \frac{1}{3}\left(\frac{\tau_y}{\tau_w}\right)^4\right),\ \bar{v}_z = \frac{Q_f}{\pi R^2} = 0.923\ (\text{m/s});\ \tau_w = \frac{1}{2}\frac{\Delta p R}{L}$$

$$Q_2 = \frac{\pi}{8}\frac{\Delta p \times R^4}{K \times L}\left(1 - \frac{4}{3}\left(\frac{\tau_y}{0.5 \times \Delta p \times R/L}\right) + \frac{1}{3}\left(\frac{\tau_y}{0.5 \times \Delta p \times R/L}\right)^4\right)$$

Using solver to minimize the error $\Delta Q = Q_1 - Q_2$, by changing Δp leads to:

$$\Delta Q = 1.6667 \times 10^{-3} - \frac{\pi}{8}\frac{\Delta p \times R^4}{K \times L} \times \left(1 - \frac{4}{3}\left(\frac{5}{0.5 \times \Delta p \times R/L}\right) + \frac{1}{3}\left(\frac{5}{0.5 \times \Delta p \times R/L}\right)^4\right)$$
$$= 6.517 \times 10^{-9}$$

$$\Delta Q = 1.6667 \times 10^{-3} - \frac{\pi}{8}\frac{\Delta p \times R^4}{K \times L} \times \left(1 - \frac{4}{3}\left(\frac{5}{0.5 \times \Delta p/L}\right) + \frac{1}{3}\left(\frac{5}{0.5 \times \Delta p \times R/L}\right)^4\right)$$

$$= 6.517 \times 10^{-9}$$

$$\Delta p = 4.10 \times 10^5 \text{ Pa}$$

$$\Delta p = 4.10 \times 10^5 \times 1.45 \times 10^{-4} = 60 \text{ psi}$$

$$\tau_w = \frac{1}{2}\frac{\Delta p \times R}{L} = 26.09 \text{ Pa};\ R_y = R \times \frac{\tau_y}{\tau_w} = 0.00487 \text{ m};\ \text{Re} = \frac{\rho \times R \times \bar{v}_z}{\eta} = 1.40 \times 10^4$$

The velocity distribution:

$$v_z(r) = -\frac{1}{2}\frac{\Delta p R^2}{\eta L}\left(\frac{\tau_y}{\tau_w}\left(1 - \frac{r}{R}\right) - \frac{1}{2}\left(1 - \left(\frac{r}{R}\right)^2\right)\right)$$

For the three cases, calculations are in this excel sheet. See Fig. 11.15:

τ_y (Pa)	5	10	15
K (Pa-s)	0.150	0.150	0.150
Q (l/min)	100	100	100
R (inch)	1	1	1
L (m)	200	200	200
ρ (kg/m^3)	1,275	1,275	1,275
Q (m^3/s)	1.667E−03	1.667E−03	1.667E−03
R (m)	0.0254	0.0254	0.0254
τ_w (Pa)	26.09	32.76	39.42
τ_y/τ_w	0.19	0.31	0.38
γ_w (s^{-1})	173.94	218.39	262.83
V_{Zav} (m/S)	0.823	0.823	0.823
Δp (Pa)	4.109E+05	5.159E+05	6.209E+05
Δp (psi)	59.58	74.80	90.02
R_y (m)	0.00487	0.00775	0.00966
R_y (inch)	0.1916	0.3053	0.3805
Re	1.40E+04	1.40E+04	1.40E+04
ΔQ (m^3/s)	6.517E−09	9.775E−09	9.492E−09
v_m (m/s)	1.44	1.34	1.28

Problem 11.10 For three suspensions of materials with densities $\rho = 1,275$ kg/m^3 that can be represented by the Bingham model in the range $10 < \dot{\gamma} < 500$ s^{-1} with $\tau_y = 15$ Pa respectively and a plastic viscosity of $K = 150, 300$ and 500 mPa-s, see Fig. 11.16, flowing in a cylindrical tube 1-inch in diameter and 200 m in length, Calculate the pressure drop and velocity distribution necessary to transport 100 l of the suspension per minute.

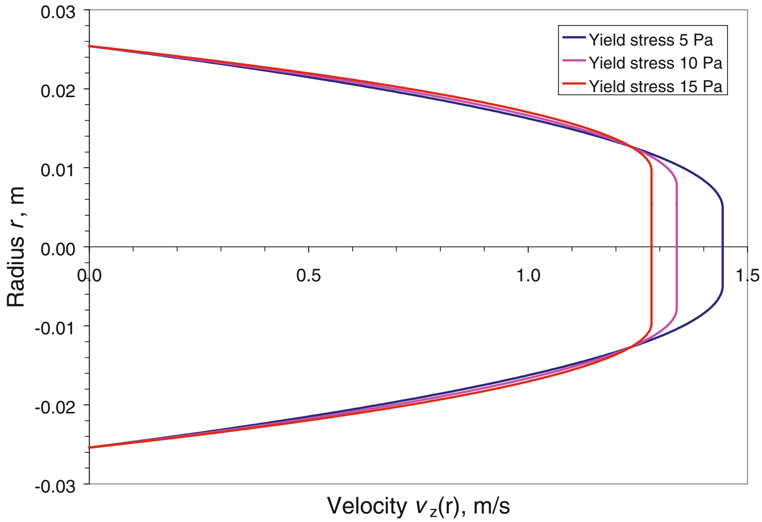


Fig. 11.15 Velocity distributions for a Bingham model of clays

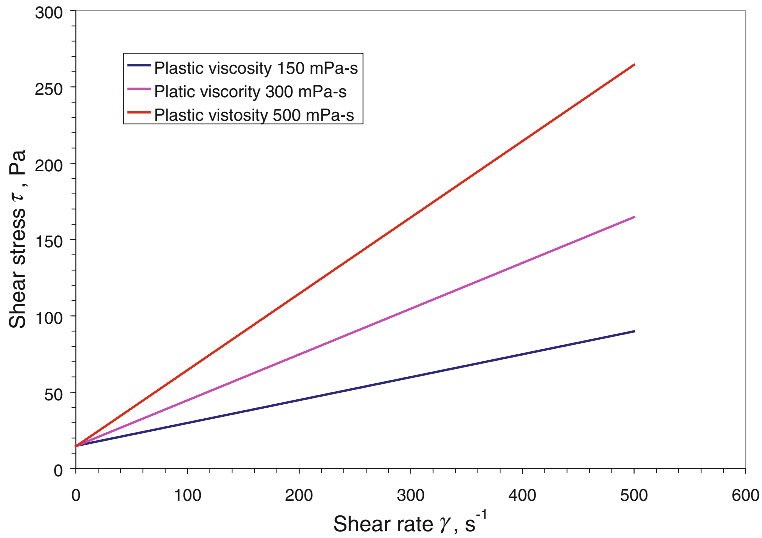


Fig. 11.16 Shear stress versus shear rate for a Bingham model of a material with plastic viscosity of 150, 300 and 500 (mPa-s) and yield stress of 15 (Pa)

$$Q = 100[\ell/\mathrm{m}] = 100/(60 \times 1{,}000) = 1.6667 \times 10^{-3}\,\mathrm{m^3/s}$$

$$Q = \frac{\pi}{8}\frac{\Delta p R^4}{KL}\left(1 - \frac{4}{3}\left(\frac{\tau_y}{\tau_w}\right) + \frac{1}{3}\left(\frac{\tau_y}{\tau_w}\right)^4\right)$$

$$\tau_w = \frac{1}{2}\frac{\Delta pR}{L} = \frac{\Delta p \times 0.2}{2 \times 200} = 5 \times 10^{-4}\Delta p$$

$$Q = \frac{\pi}{8}\frac{\Delta p \times 0.2^4}{0.2 \times 200}\left(1 - \frac{4}{3}\left(\frac{15}{5 \times 10^{-4}\Delta p}\right) + \frac{1}{3}\left(\frac{15}{5 \times 10^{-4}\Delta p}\right)^4\right)$$

$$\text{Error} = 1.6667 \times 10^{-3} - \frac{\pi}{8}\frac{\Delta p \times 0.2^4}{0.2 \times 200}\left(1 - \frac{4}{3}\left(\frac{15}{5 \times 10^{-4}\Delta p}\right) + \frac{1}{3}\left(\frac{15}{5 \times 10^{-4}\Delta p}\right)^4\right)$$

$$= 1.000 \times 10^{-3}$$

Using solver minimizing the Error by changing Δp leads to:

$$\Delta p = 1.462 \times 10^6 \text{ Pa}$$

$$\tau_w = 5 \times 10^{-4}\Delta p = 5 \times 10^{-4} \times 1.462 \times 10^6 = 73.11 \text{ Pa}$$

The velocity distribution is given by:

$$v_z(r) = -\frac{1}{2}\frac{\Delta pR^2}{KL}\left(\frac{\tau_y}{\tau_w}\left(1 - \frac{r}{R}\right) - \frac{1}{2}\left(1 - \left(\frac{r}{R}\right)^2\right)\right)$$

$$= \frac{1}{2}\frac{0.2^2}{0.15 \times 200}\left(\frac{15}{73.11}\left(1 - \frac{r}{0.2}\right) - \frac{1}{2}\left(1 - \left(\frac{r}{0.2}\right)^2\right)\right)$$

For the three cases, see Fig. 11.17.

τ_y (Pa)	15	15	15
K (Pa-s)	0.150	0.300	0.500
Q (l/min)	100	100	100
R (inch)	1	1	1
L (m)	200	200	200
ρ (kg/m^3)	1,275	1,275	1,275
Q (m^3/s)	1.667E−03	1.667E−03	1.667E−03
R (m)	0.0254	0.0254	0.0254
τ_w (Pa)	39.14	58.87	84.79
τ_y/τ_w	0.38	0.25	0.18
γ_w (s^{-1})	260.95	196.24	169.57
v_{zav} (m/s)	0.823	0.823	0.823
Δp (Pa)	6.210E+05	9.271E+05	1.335E+06
Δp (psi)	90.05	134.43	193.61
R_y (m)	0.00973	0.00647	0.00449
R_y (inch)	0.3832	0.2548	0.1769
Re	1.40E+04	6.99E+03	4.20E+03
ΔQ (m^3/s)	9.995E−07	1.000E−06	1.000E−06
v_m (m/s)	1.27	1.38	1.46

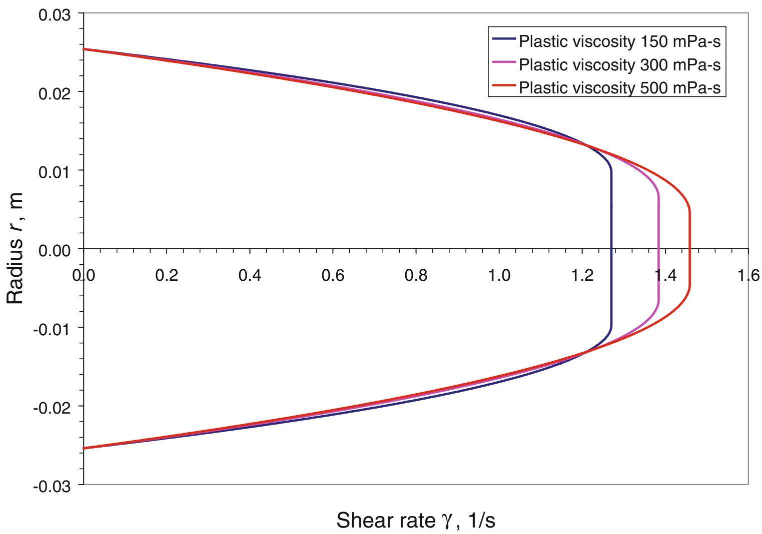


Fig. 11.17 Velocity distributions for a Bingham model of a material with plastic viscosity of 20, 50 and 100 mPa-s and yield stresses of 15 Pa

(b) **Power-Law Fluids**

The constitutive equation for the shear stress of power law fluids flowing in a circular tube is:

$$T^E_{rz}(r) = m\left(\frac{\partial v_z}{\partial r}\right)^n \tag{11.66}$$

where m is the consistency index and n is the power index.

Velocity distribution

Replacing (11.66) with (11.51) we have:

$$m\left(\frac{\partial v_z}{\partial r}\right)^n = -\frac{1}{2}\frac{\Delta p}{L}r$$

Integrating yields

$$\frac{\partial v_z}{\partial r} = \left(\frac{-\Delta p}{2\,mL}r\right)^{1/n} \tag{11.67}$$

$$v_z(r) = \left(\frac{\Delta p}{2\,mL}\right)^{1/n} \int (-r)^{1/n} dr + C$$
$$= \left(\frac{\Delta p}{2\,mL}\right)^{1/n} \frac{n}{n+1} \left(-r^{n+1/n}\right) + C$$

Using boundary condition $v_z(R) = 0$, results in:

$$v_z(r) = \frac{nR}{n+1} \left(\frac{\Delta pR}{2\,mL}\right)^{1/n} \left(1 - \left(\frac{r}{R}\right)^{(n+1)/n}\right) \quad (11.68)$$

Figure 11.18 of Problem 11.11 shows the velocity distribution for the flow of a power law fluid in a tube for different values of the power index.

Volume flow rate

The volume flow rate is given by $Q_f = \int_0^R 2\pi v_z r dr$, then substituting (11.68) and integrating:

$$Q_f = \int\limits_0^R 2\pi \frac{nR}{n+1} \left(\frac{\Delta pR}{2\,mL}\right)^{1/n} \left(1 - \left(\frac{r}{R}\right)^{(n+1)/n}\right) r dr$$

$$Q = 2\pi \frac{nR^3}{n+1} \left(\frac{\Delta pR}{2\,mL}\right)^{1/n} \left(\int\limits_0^1 \frac{r}{R} d\left(\frac{r}{R}\right) - \int\limits_0^1 \left(\frac{r}{R}\right)^{(2n+1/n)} d\left(\frac{r}{R}\right)\right)$$

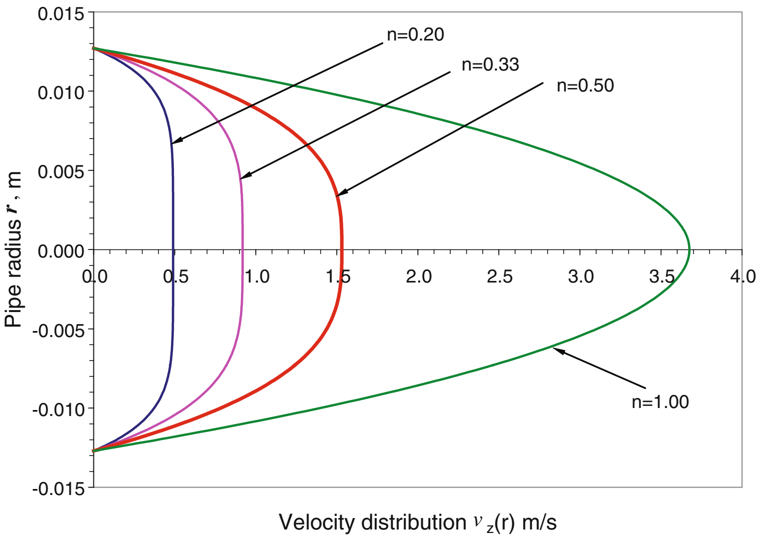


Fig. 11.18 Velocity distribution of power law fluids with consistency index $m = 3\text{Pa - s}^n$ and power law indices 0.20; 0.33; 0.50; 1 and 3

$$Q_f = 2\pi R^3 \frac{n}{n+1}\left(\frac{\Delta pR}{2\,mL}\right)^{1/n}\left(\frac{1}{2}\left(\frac{r}{R}\right)^2\Big|_0^1 - \left(\frac{1}{((2n+1)/n)+1}\right)\left(\frac{r}{R}\right)^{((2n+1/n+1))}\right)$$

$$Q_f = \pi R^2 \frac{n}{(3n+1)}\left(\frac{\Delta pR^{n+1}}{2\,mL}\right)^{1/n}$$

Average velocity

The average velocity is given by $\bar{v}_z = Q_f/\pi R^2$, then:

$$\bar{v}_z = \frac{n}{(3n+1)}\left(\frac{\Delta pR^{n+1}}{2\,mL}\right)^{1/n} \tag{11.70}$$

$$\text{and} \qquad \frac{8\bar{v}_z}{D} = \frac{4n}{(3n+1)}\left(\frac{\Delta pR}{2\,mL}\right)^{1/n} \tag{11.71}$$

Shear stress and shear rate at the wall and maximum velocity

Since the shear rate and shear stress at the wall are given by:

$$\dot{\gamma}_w = \frac{\partial v_z}{\partial r}\Big|_{r=R} = \left(-\frac{\Delta pR}{2\,mL}\right)^{1/n} \qquad \tau_w = m\dot{\gamma}_w^n$$

from (11.71) we finally have:

$$\dot{\gamma}_w = \frac{(3n+1)}{4n}\frac{8\bar{v}_z}{D} \qquad \tau_w = m\left(\frac{(3n+1)}{4n}\frac{8\bar{v}_z}{D}\right)^n \tag{11.72}$$

The maximum velocity is obtained from (11.68) for $r = 0$, then:

$$v_m = \frac{nR}{n+1}\left(\frac{\Delta pR}{2\,mL}\right)^{1/n} \tag{11.73}$$

$$v_z(r) = v_m\left(1 - \left(\frac{r}{R}\right)^{(n+1)/n}\right) \tag{11.74}$$

Pressure drop

From (11.69)

$$\boxed{\Delta p = \frac{2\,mL}{R^{n+1}}\left(\frac{(3n+1)}{n}\frac{Q}{\pi R^2}\right)^n} \tag{11.75}$$

Problem 11.11 For a mass flow of $F = 1{,}000$ kg/h of a non-Newtonian fluid of the potential type with a density of $\rho = 1{,}074$ kg/m^3, consistency index of $m = 3$

and power law indices of $n = 1/5,\ 1/3,\ 1/2$ and $1,\ 3$, calculate the pressure drop Δp and draw a figure for the velocity distribution. See Fig. 11.18.

n	0.20	0.33333333	0.50	1.00
ρ (kg/m^3)	1,074	1,074	1,074	1,074
m (Pa-s^n)	3	3	3	3
d (inch)	1.00	1.00	1.00	1.00
L (m)	10	10	10	10
F (kg/h)	1,000	1000	1,000	1,000
Q (m^3/s)	9.311E−04	9.311E−04	9.311E−04	9.311E−04
R (m)	0.0127	0.0127	0.0127	0.0127
v_{zav} (m/s)	1.838	1.838	1.838	1.838
v_m (m/s)	0.49	0.92	1.53	3.68
Δp (Pa)	1.40E+04	3.12E+04	8.99E+04	2.73E+06

Wall shear stress and Reynolds number

Defining the friction coefficient in the same way as for Newtonian fluids in laminar flow, $f = 16/\mathrm{Re}$, we can define *a Reynolds number* Re_M for a power law fluid as the ratio of the wall shear stress τ_w to the dynamic pressure Metzner and Reed (1959). Then we have:

$$f = \frac{-\tau_w}{\frac{1}{2}\rho \bar{v}_z^2} = \frac{16}{\mathrm{Re}} \tag{11.76}$$

From (11.72) $\tau_w = m\left(\frac{(3n+1)}{4n}\frac{8\bar{v}_z}{D}\right)^n$

So that Re_M is:

$$\boxed{\mathrm{Re}_M = \frac{\rho \bar{v}_z^{2-n} D^n}{8^{n-1} m \left(\frac{3n+1}{4n}\right)^n}} \tag{11.77}$$

Transition to a turbulent regime

As in the case of fluids with Newtonian behavior, the friction factor gives the transition from a laminar to turbulent flow.

With the Reynolds number defined by (11.77), the roughness Reynolds number by $\mathrm{Re}_\in = \in \sqrt{1/f}\mathrm{Re}$ and the friction factor $f = -\Delta p D/2\rho L \bar{v}_z^2$, we have:

$$f = \frac{16}{\mathrm{Re}} \quad \text{for} \quad \mathrm{Re} < 2,100 \tag{11.78}$$

$$f_{rough} = \frac{f_{water-rough}}{f_{water-smooth}} \frac{16}{\mathrm{Re}} \quad \text{for} \quad \mathrm{Re} < 2,100\,,\ 5 < \mathrm{Re}_\in < 70 \tag{11.79}$$

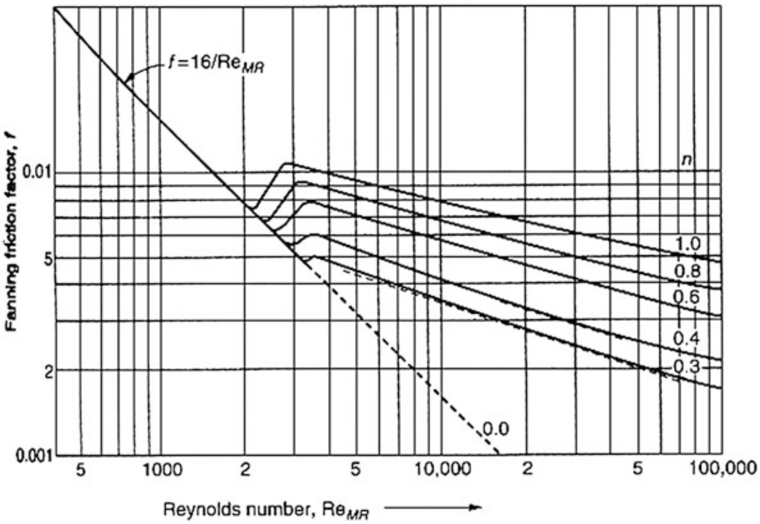


Fig. 11.19 Friction factor as a function of Metzner's Reynolds number for different values of the power function n (Chhabra and Richardson 1999)

$$\left(f = \frac{4.53}{n}\log\left(\mathrm{Re}\sqrt{f^{2-n}}\right) + \frac{2.69}{n} - 2.95 + 0.68\,\frac{5n-8}{n}\right)^{-2}, \quad \text{for} \quad \mathrm{Re} > 4,000\,,\ \mathrm{Re}_{\in} < 5 \tag{11.80}$$

$$f_{rough} = \left(4.07\log\left(\frac{1}{2\in}\right) + 6 - \frac{2.65}{n}\right)^{-2} \quad \text{for} \quad \mathrm{Re} > 4,000\,,\ \mathrm{Re}_{\in} > 70 \tag{11.81}$$

Figure 11.19 shows the friction factor as a function of Metzner's Reynolds number for different values of the power function n for smooth walls according to Chhabra and Richardson (1999).

Problem 11.12 A polyacrilamide solution of $\rho = 1,074\ \mathrm{kg/m^3}$ in density is to be pumped through a tube one inch in diameter and 10 m in length at a rate of 2.500 kg/h. Measurement in the laboratory showed that the fluid can be represented by the power law model with $m = 3\mathrm{Pa\text{-}s}^n$ and $n = 0.5$. Calculate the necessary pressure to maintain the flow and calculate the velocity distribution, average and maximum velocity.

Volume flow

$$Q = \frac{F}{\rho \times 3,600} = 6.466 \times 10^{-4}\ \mathrm{m^3/s}$$

Pressure drop:

$$\Delta p = \frac{2\,mL}{R^{n+1}}\left(\frac{3n+1}{n}\frac{Q}{\pi R^2}\right)^n = 1.059 \times 10^5 \text{ Pa}$$

Average velocity

$$\bar{v}_z = \frac{Q}{\pi R^2} = 1.128\,\text{m/s}$$

Velocity distribution:

$$v_z(r) = \frac{nR}{n+1}\left(\frac{\Delta pR}{RmL}\right)^{1/n}\left(1-\left(\frac{r}{R}\right)^{(n+1)/n}\right) = 2.128 \times \left(1-\left(\frac{r}{R}\right)^{(n+1)/n}\right)$$

Maximum velocity:

$$v_z(r) = \frac{nR}{n+1}\left(\frac{\Delta pR}{RmL}\right)^{1/n} = 2.128$$

$$\text{Velocity distribution} \quad v_z = 2.128 \times \left(1-\left(\frac{r}{R}\right)^{(n+1)/n}\right)$$

Problem 11.13 A non-Newtonian fluid with density equal to that of water flows in a tube 300 mm in diameter and 50 m long at a rate of 300 kg/s. Rheological measurements yield the following power law parameters: $m = 2.74\,\text{Pa-s}^{0.3}$ and $n = 0.30$. Determine the necessary power of a pump and the wall shear stress. See Fig. 11.20.

ρ (kg/m^3)	1,074
m (Pa-s^n)	3
n	0.5
D (inch)	1
L (m)	10
F (kg/h)	2,500
Q (m^3/s)	6.466E−04
R (m)	1.270E−02
v_{zav} (m/s)	1.277
Δ (Pa)	1.059E+05
v_m (m/s)	2.12787

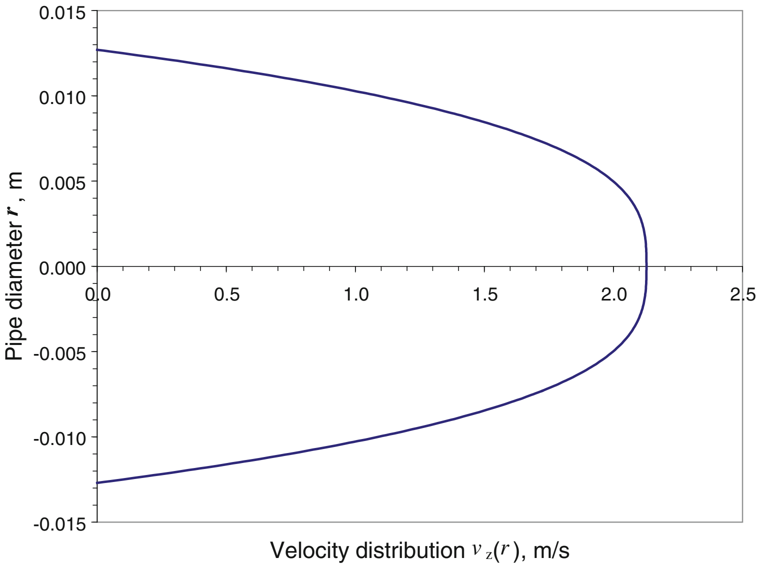


Fig. 11.20 Velocity distribution for a polyacrilamide solution with a power law model: $m = 3[\text{Pa-s}^{0.5}]$ y $n = 0.5$

$$\text{Average velocity}: \quad \bar{v}_z = \frac{Q_f}{\pi R^2} = \frac{300/1,000}{3.14 \times (0.15)^2} = 4.24\,(\text{m/s})$$

The transitional or critical Reynolds number is $\text{Re}_c \approx 2,100$.

$$\text{Re}_{MRc} = \frac{\rho v_c^{2-n} D^n}{8^{n-1} m \left(\frac{3n+1}{4n}\right)^n} = 2,100$$

then the critical velocity, that is, the velocity at which the flow changes from laminar to turbulent, is:

$$v_c = \left(\frac{8^{n-1} m \left(\frac{3n+1}{4n}\right)^n 2,100}{\rho D^n}\right)^{n-2} = 2.91\,(\text{m/s})$$

Since the average velocity 4.24 (m/s) is greater than the critical velocity 2.91 (m/s), the regime is turbulent. The actual Reynolds number is:

$$\text{Re}_{MR} = \frac{\rho \bar{v}_z^{2-n} D^n}{8^{n-1} m \left(\frac{3n+1}{4n}\right)^n} = 11090$$

With $\text{Re}_{\text{MR}} = 11,090$ and $n = 0.3$, from Fig. 11.16, we get a friction factor $f = 0.0033$.

The value of the pressure drop necessary to produce the flow is obtained from the friction factor definition: $f=-\tau_{\mathrm{w}}/1/2\rho\bar{v}_z^2$; $\tau_w=-\frac{1}{2}\frac{\Delta p}{L}R\rightarrow f=\frac{\Delta p}{L}\frac{R}{\rho\bar{v}_z^2}$ therefore:

$$\Delta p=\frac{2\rho\bar{v}_z^2L}{D}f=1.9814\times 10^4\ (\mathrm{Pa})$$

and, the pump power P*o* is:

$$Po=Q_f\times\Delta P=6.0\,\mathrm{kW}$$

The wall shear stress is

$$\tau_w=-\frac{1}{2}\frac{\Delta p}{L}R=29.7\,\mathrm{Pa}$$

D (m)	0.3
L (m)	50
ρ (kg/m^3)	1,000
F (kg/s)	300
m (Pa-s$^{0.3}$)	2.74
n	0.3
Re_{MRc}	2,100
R (m)	0.15
Q (m^3/s)	0.300
v_{zav} (m/s)	4.24
v_c (m/s)	2.921
Re_{MR}	11,090
f (11090; 0.30)	0.0033
Δp (Pa)	19,814
Po (W)	5,944
τ_w (Pa)	29.7

(c) **Herschel-Bulkley Fluid**

The constitutive equation for the stress tensor for Herschel-Bulkley fluids in a pipe has the form:

$$T_{rz}^E(r)\begin{cases}<\tau_y; & \frac{\partial v_z}{\partial r}=0\\ \geq\tau_y; & T_{rz}^E(r)=\tau_y+m\left(\frac{\partial v_z}{\partial r}\right)^n\end{cases}\tag{11.82}$$

Since for any fluid $T_{rz}^E(r)=-\frac{1}{2}\frac{\Delta p}{L}r$, for Herschel-Bulkley fluids we have:

$$T_{rz}^E(R_y)=\tau_y=-\frac{1}{2}\frac{\Delta p}{L}R_y$$

Then:

$$T_{rz}^{E}(r) - \tau_y = -\frac{1}{2}\frac{\Delta p}{L}\left(r - R_y\right)$$

Velocity distribution

For $T_{rz}^{E}(r) \geq \tau_y\,;\;\; R_y \leq r \leq R$:

$$\frac{\partial v_z}{\partial r} = \left(\frac{T_{rz}^{E}(r) - \tau_y}{m}\right)^{1/n} = \left(-\frac{\Delta p\left(r - R_y\right)}{2\,mL}\right)^{1/n} \tag{11.83}$$

Integrating with boundary condition $v_z(R) = 0$

$$\begin{aligned}
v_z(r) &= \left(-\frac{\Delta p}{2\,mL}\right)^{1/n} \int \left(r - R_y\right)^{1/n} dr \\
v_z(r) &= \left(-\frac{\Delta p}{2\,mL}\right)^{1/n} \times \frac{n}{n+1}\left(r - R_y\right)^{(n+1)/n} + C_1 \\
v_z(R) &= \left(-\frac{\Delta p}{2\,mL}\right)^{1/n} \times \frac{n}{n+1}\left(R - R_y\right)^{(n+1)/n} + C_1 = 0 \\
\mathrm{C}_1 &= -\left(-\frac{\Delta p}{2\,mL}\right)^{1/n} \times \frac{n}{n+1}\left(R - R_y\right)^{(n+1)/n} \\
v_z(r) &= \left(-\frac{\Delta pR}{2\,mL}\right)^{1/n} \times \frac{nR}{n+1}\left(\left(\frac{r}{R} - \frac{R_y}{R}\right)^{(n+1)/n} - \left(1 - \frac{R_y}{R}\right)^{(n+1)/n}\right)
\end{aligned} \tag{11.84}$$

Using Eqs. (11.12) and (11.53) for the shear stress at the wall and the yield stress, Eq. (11.84) can be written in the form:

$$\boxed{v_z(r) = \left(\frac{\tau_w}{m}\right)^{1/n} \times \frac{nR}{n+1}\left(\left(\frac{r}{R} - \frac{\tau_y}{\tau_w}\right)^{(n+1)/n} - \left(1 - \frac{\tau_y}{\tau_w}\right)^{(n+1)/n}\right)};\quad \text{for} \quad R_y \leq r \leq R \tag{11.85}$$

For $T_{rz}^{E}(r) < \tau_y\,;\quad 0 < r < R_y \quad$ where $\quad \tau_y = T_{rz}^{E}\left(R_y\right)$:

$$\frac{\partial v_z}{\partial r} = 0 \qquad \rightarrow \qquad v_z(r) = v_z\left(R_y\right) \tag{11.86}$$

From (11.84),

$$v_z(r) = -\left(-\frac{\Delta pR}{2\,mL}\right)^{1/n} \times \frac{nR}{n+1}\left(1 - \frac{R_y}{R}\right)^{(n+1)/n} \quad \text{for} \quad 0 \leq r \leq R_y \tag{11.87}$$

$$\boxed{v_z(r) = -\left(\frac{\tau_w}{m}\right)^{1/n} \times \frac{nR}{n+1}\left(1-\frac{\tau_y}{\tau_w}\right)^{(n+1)/n}} \quad \text{for} \quad 0 \le r \le R_y \tag{11.88}$$

Volume flow rate

The volume flow rate is given by $Q_f = \int_0^R 2\pi v_z r dr$, then substituting (11.84) and (11.87) into this equation and integrating yields:

$$Q_f = 2\pi\left(\frac{\tau_w}{m}\right)^{1/n}\frac{nR}{n+1}\left(\begin{array}{l}\int\limits_0^{R_y} -\left(1-\frac{\tau_y}{\tau_w}\right)^{(n+1)/n} rdr \\ +\int\limits_{R_y}^{R}\left(\left(\frac{r}{R}-\frac{\tau_y}{\tau_w}\right)^{(n+1)/n} - \left(1-\frac{\tau_y}{\tau_w}\right)^{(n+1)/n} rdr\right)\end{array}\right)$$

$$= 2\pi\left(\frac{\tau_w}{m}\right)^{1/n}\frac{nR^3}{n+1}\left(\begin{array}{l}\int\limits_0^{R_y/R} -\left(1-\frac{\tau_y}{\tau_w}\right)^{(n+1)/n} \xi d\xi \\ +\int\limits_{R_y/R}^{1}\left(\xi\left(\xi-\frac{\tau_y}{\tau_w}\right)^{(n+1)/n} - \left(1-\frac{\tau_y}{\tau_w}\right)^{(n+1)/n} \xi\right)d\xi\end{array}\right)$$

Integrating yields (Bird et al. 1987):

$$Q_f = \pi n R^3\left(\frac{\tau_w}{m}\right)^{1/n}\left(\frac{1}{3n+1}\left(1-\frac{\tau_y}{\tau_w}\right)^2 + \frac{2}{2n+1}\frac{\tau_y}{\tau_w}\left(1-\frac{\tau_y}{\tau_w}\right) + \frac{1}{n+1}\left(\frac{\tau_y}{\tau_w}\right)^2\right) \tag{11.89}$$

Average velocity

The average velocity is given by $\bar{v}_z = Q/\pi R^2$, then:

$$\bar{v}_z = nR\left(\frac{\tau_w}{m}\right)^{1/n}\left(\frac{1}{3n+1}\left(1-\frac{\tau_y}{\tau_w}\right)^{(n+1)/n} + \frac{2}{2n+1}\frac{\tau_y}{\tau_w}\left(1-\frac{\tau_y}{\tau_w}\right) + \frac{1}{n+1}\left(\frac{\tau_y}{\tau_w}\right)^2\right) \tag{11.90}$$

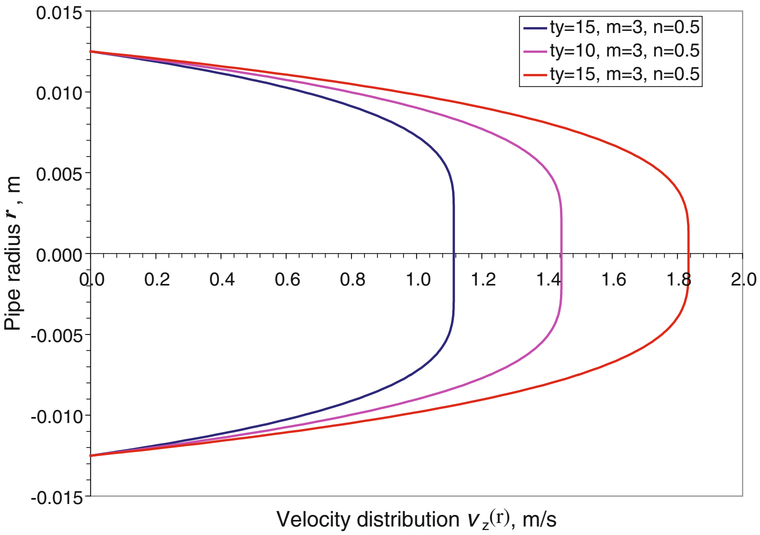


Fig. 11.21 Velocity distribution of Herschel-Bulkley fluids with yield stresses τ_y=5; 10 and 15; Pa, consistency index m = 3 Pa-s^n and power law index of 0.50

Maximum velocity

The maximum velocity is obtained from (11.88), then:

$$v_m = -\left(\frac{\tau_y}{\tau_w}\right)^{1/n} \times \frac{nR}{n+1}\left(1 - \frac{\tau_y}{\tau_w}\right)^{(n+1)/n} \tag{11.91}$$

Problem 11.14 Figure 11.21 shows the velocity distribution for Herschel-Bulkley fluids with yield stress $\tau_y = 5,\ 10$ and15Paconsistency index $m = 3$ Pa-s^n and power law indices of 0.50. See Fig. 11.21.

τ_y (Pa)	15	10	5
n	0.5	0.5	0.5
m (Pa-s^n)	3	3	3
ρ (kg/m^3)	1,074	1,074	1,074
d (cm)	2.5	2.5	2.5
L (m)	10	10	10
F (kg/h)	2,500	2,500	2,500
Q (m^3/s)	6.466E−04	6.466E−04	6.466E−04
R (m)	1.250E−02	1.250E−02	1.250E−02
v_m (m/s)	1.318E+00	1.318E+00	1.318E+00
Δp (Pa)	1.125E+05	1.125E+05	1.125E+05
R_y	2.667E−03	1.778E−03	8.890E−04

Transition to turbulent regime

As in the case of Newtonian fluids, the friction factor gives the transition from laminar to turbulent flow. The Reynolds number is the same as that of pseudo plastic fluids. See Eq. (11.77):

$$\mathrm{Re}_{HB} = \frac{\rho \bar{v}_z^{2-n} D^n}{8^{n-1} m \left(\frac{3n+1}{4n}\right)^n} \; ; \; \mathrm{Re}_{\in} = \in \sqrt{1/f}\,\mathrm{Re}_{HB} \tag{11.92}$$

Now the friction factor $f = -\Delta p D / 2 \rho L \bar{v}_z^2$ is given by:

$$f_{smooth} = \frac{1}{2^{2n-4}} \left(\frac{\pi R^3}{Q}\right) \left(\frac{m}{\tau_w}\right)^{1/n} \times \frac{1}{\mathrm{Re}_{HB}} \; ; \quad \text{for} \quad \mathrm{Re}_{\mathrm{HB}} < 2,100 \tag{11.93}$$

$$f_{rough} = \frac{f_{water-rough}}{f_{water-smooth}} \frac{1}{2^{2n-4}} \left(\frac{\pi R^3}{Q}\right) \left(\frac{m}{\tau_w}\right)^{1/n} \times \frac{1}{\mathrm{Re}_{HB}} \; ; \quad \text{for} \quad \mathrm{Re}_{PL} < 2,100\,,\ 5 < \mathrm{Re}_{\in} < 70 \tag{11.94}$$

$$\left(f_{smooth} = \frac{4.53}{n} \log\left(\mathrm{Re}_{PL} \sqrt{f^{2-n}}\right) + \frac{2.69}{n} - 2.95 + 0.68 \frac{5n-8}{n}\right)^{-2}, \quad \text{for} \quad \mathrm{Re}_{HB} > 4,000\,,\ \mathrm{Re}_{\in} < 5 \tag{11.95}$$

$$f_{rough} = \left(4.07 \log\left(\frac{1}{2 \in}\right) + 6 - \frac{2.65}{n}\right)^{-2} \quad \text{for} \quad \mathrm{Re}_{PL} > 4,000,\ \mathrm{Re}_{\in} > 70 \tag{11.96}$$

Problem 11.15 The rheology of copper tailings is described by the values in the following table:

%	ty (Pa)	η (mPa-s)			
		10	100	150	200
55	0.678	166	32	25	21
60	1.035	230	44	34	29
65	1.579	318	61	46	39
70	2.409	441	83	63	53

Calculate the pressure drop necessary to transport 5,000 l per minute of a copper tailing with a density of 2,650 kg/m^3, at 55, 60, 65 and 70 % of solid by weight in a pipe 4 inches in diameter and 200 m long, if the rheological parameters of the pulp are those given in the table. Model the rheology of the tailing and draw the Rheological curves.

Pressure drop:

$$Q_f = 5{,}000\,\ell/\text{m} = 5{,}000/(60 \times 1{,}000) = 8.333 \times 10^{-2}\ \text{m}^3/\text{s}$$

$$Q_f = \frac{\pi}{8}\frac{\Delta p R^4}{KL}\left(1 - \frac{4}{3}\left(\frac{\tau_y}{\tau_w}\right) + \frac{1}{3}\left(\frac{\tau_y}{\tau_w}\right)^4\right)$$

$$\dot{\gamma}_w = \frac{8\bar{v}_z}{D} = \frac{32Q}{\pi D^3} = \frac{32 \times 1.6667 \times 10^{-3}}{3.14 \times 0.02^3} = 26.5\ 1/s$$

$$\tau_w = \frac{1}{2}\frac{\Delta p R}{L} = \frac{\Delta p \times 0.1016}{2 \times 200} = 2.54 \times 10^{-4}\Delta p$$

$$\tau_y = 0.678\ \text{Pa}$$

$$Q_f = \frac{\pi}{8}\frac{\Delta p \times 0.1016^4}{0.818 \times 200}\left(1 - \frac{4}{3}\left(\frac{0.678}{2.54 \times 10^{-4}\Delta p}\right) + \frac{1}{3}\left(\frac{0.678}{2.54 \times 10^{-4}\Delta p}\right)^4\right)$$

$$\text{Error} = 8.333 \times 10^{-2} - \frac{\pi}{8}\frac{\Delta p \times 0.1016^4}{0.818 \times 200}\left(1 - \frac{4}{3}\left(\frac{0.678}{2.54 \times 10^{-4}\Delta p}\right) + \frac{1}{3}\left(\frac{0.678}{2.54 \times 10^{-4}\Delta p}\right)^4\right)$$

$$= 6.998 \times 10^{-7}\Delta p =$$

The following excel sheet permits the calculation of all solid percentages:

% solids	55	60	65	70
ρ_s (kg/m^3)	2,650	2,650	2,650	2,650
ρ_f (kg/m^3)	1,000	1,000	1,000	1,000
τ_y (Pa)	0.678	1.035	1.579	2.409
η (Pa-s)	0.818	1.142	1.612	2.266
Q (l/min)	5000.00	5000.00	5000.00	5000.00
R (inchs)	4.00	4.00	4.00	4.00
L (m)	200	200	200	200
ρ (kg/m^3)	1520.80	1596.39	1679.87	1772.58
Q (m^3/s)	8.333E−02	8.333E−02	8.333E−02	8.333E−02
R (m)	0.1016	0.1016	0.1016	0.1016
τ_w (Pa)	83.66	116.92	165.19	232.46
$\tau_y/\ \tau_w$	0.01	0.01	0.01	0.01
γ_w (s^{-1})	102.28	102.38	102.48	102.59
v_{zav} (m/s)	2.571	2.571	2.571	2.571
Δp (Pa)	3.294E+05	4.603E+05	6.504E+05	9.152E+05
Δp (psi)	47.76	66.74	94.30	132.71
R_y (m)	0.00082	0.00090	0.00097	0.00105
R_y (inch)	0.0324	0.0354	0.0382	0.0415
Re	3.82E+04	2.88E+04	2.14E+04	1.61E+04
ΔQ (m^3/s)	9.998E−07	9.999E−07	1.000E−06	1.000E−06
v_m (m/s)	5.11	5.11	5.11	5.10

% solid	τ_y Pa	m	n − 1	n
55	0.678	818.26	−0.6963	0.3037
60	1.035	1141.8	−0.6996	0.3004
65	1.579	1611.7	−0.7072	0.2928
70	2.409	2266.1	−0.7134	0.2866
Average			−0.70	0.30

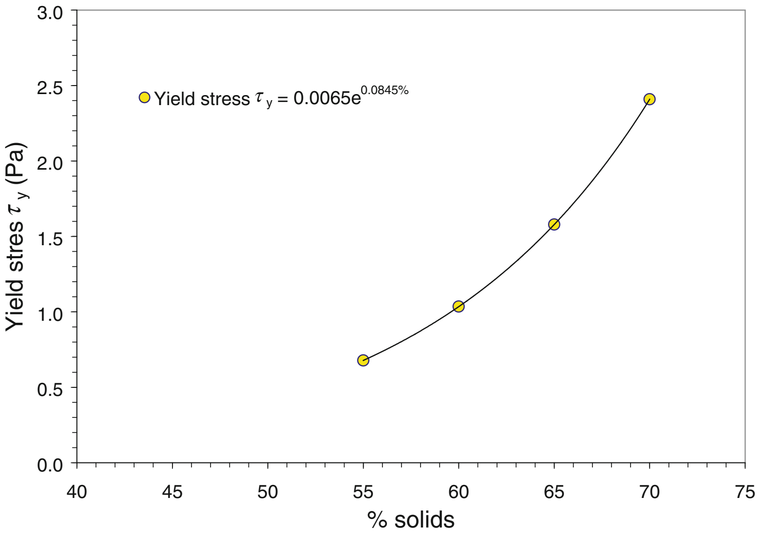


Fig. 11.22 Yield stress versus % solid by weight for a copper tailing

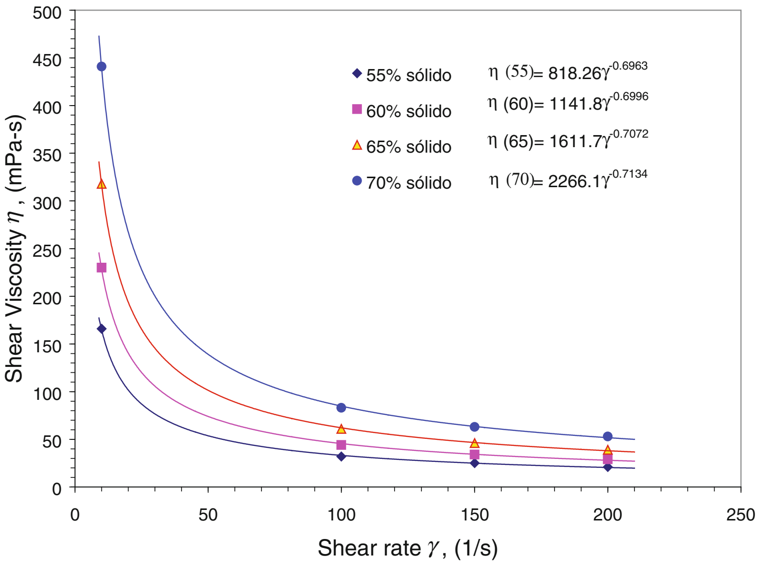


Fig. 11.23 Shear viscosity versus shear rate for a copper tailing with % solid by weight as parameter

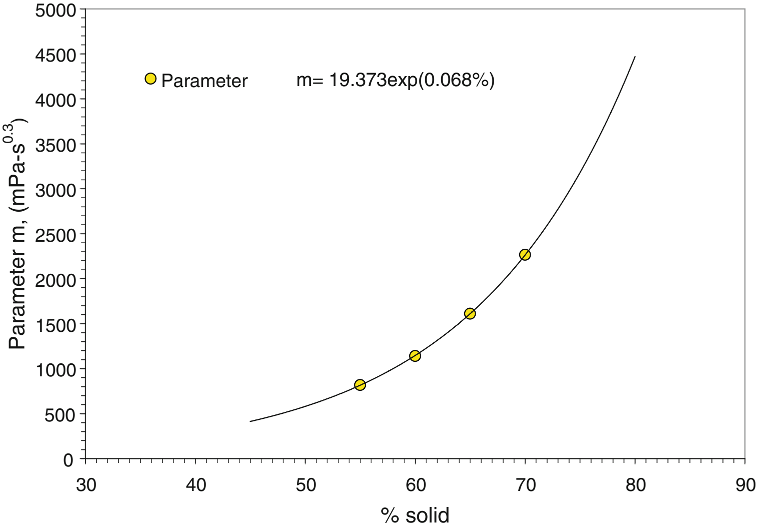


Fig. 11.24 Parameter m for a copper concentrate versus % solid

$$\eta(\text{mPa-s}) = 19.3673 \exp(0.067\,\%)\dot{\gamma}^{-0.7}$$
$$\tau_y\,(\text{Pa}) = 0.0035 \exp(0.0845\,\%)$$
$$\tau(Pa) = 0.0035 \exp(0.0845\,\%) + 19.3673 \exp(0.067\%)\dot{\gamma}^{0.3}$$

Figures 11.22, 11.23, 11.24, and 11.25 show the results graphically.

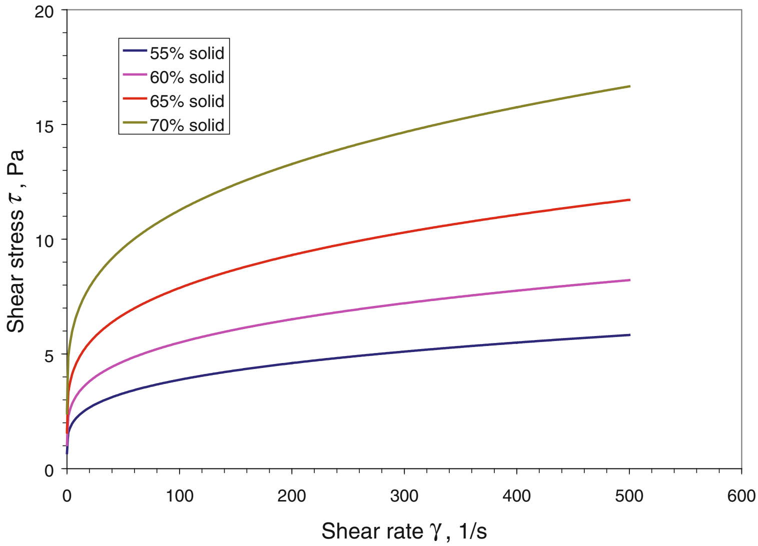
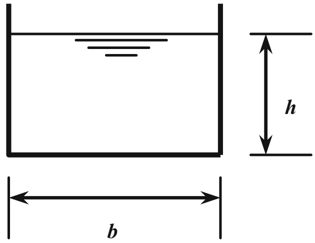
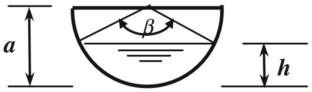
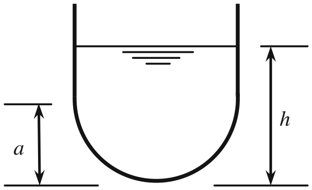


Fig. 11.25 Shear stress versus shear rate for a copper tailing

Table 11.3 Geometrical parameters of typical channels used for slurry conveyance

Geometry	Area	Wetted perimeter
	$A = bh$	$P = b + 2h$
 $h = a\left(1 - \cos\left(\frac{\beta}{2}\right)\right)$	$A = \frac{a^2}{8}(\beta - sen(\beta))$	$P = \frac{a}{2}\beta$
	$A = 2a(h - a) + \frac{\pi}{2}a^2$	$P = \pi a + 2(h - a)$

11.5 Transporting Suspensions in Open Channels

Due to the natural slopes of the land around mines, it is often convenient to use Channels instead of pipelines to transport tailings. From a fundamental point of view, the problem of Channel flow is more complex than tube flow because the flow area is not known in advance and it can change while the flow is developing. The case is simpler if the flow is uniform.

Although several Channel geometries are used in open Channel flows, the most commonly used in slurry transport are the rectangular section, semicircular, and a composed semicircular-rectangular section. The relevant geometrical parameters are the cross-section flow area, A, and the wetted perimeter, P, as given in Table 11.3.

Steady open Channel flow is classified as uniform or non-uniform. A uniform flow is one for which the fluid depth h above the Channel bed is constant. Non-uniform flows are further classified into gradually varied flows, where the curvature of the free surface is small compared to the depth of the fluid, and rapidly varied flows, where the curvature is comparable to the fluid depth. Analysis of gradually varied flow is simpler because a hydrostatic pressure distribution can be assumed. Curvature in rapidly varied flows adds a radial acceleration to the fluid particles that must be added to the gravity effect to compute the pressure.

In studying steady state gradually varied flows in open Channels we want to determine the flow depth h as a function of the distance x for a given flow rate Q_f.

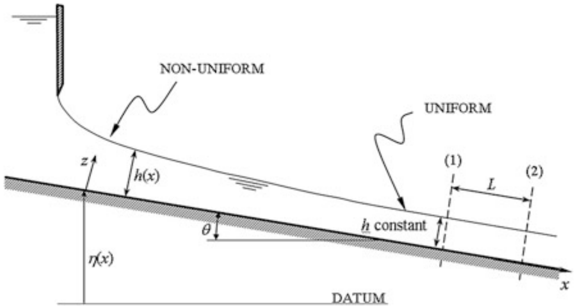


Fig. 11.26 Uniform and non-uniform flow in a channel

In this case the flow depth depends on the Channel characteristics (geometry, slope, wall roughness), and fluid properties (density and viscosity). See Fig. 11.26.

Flow of copper tailings in Channels has more favorable conditions than flow in pipelines. For example, the concentration of the pulp in Channels has no influence on the speed of the flow and whether or not the flow is turbulent, (velocities above 0.8 m/s will yield turbulent flow). The viscosity has no influence on the transport velocity but influences in limiting deposit velocity. On the other hand, the head loss can be calculated using the methods for water with similar wall conditions. These simplifications are not valid for pipe flow.

The slope of the Channel is important. As a rule of thumb, slight slopes, such as 0.3 %, need transport velocities greater than 1.2 m/s to avoid embankment. Velocities of 1.5 m/s are recommended for copper tailings (Kleiman 1960). If water is added to a developed flow with high solid content, such as 45 % by weight in a Channel with a slight slope and slow transport velocity, particles will settle. This is because the water dilutes the pulp and larger particles can segregate. Once a bed forms under this condition, it cannot be eliminated by washing with water. A flow with high concentration at velocities higher than 1.0 (m/s) will eventually removes the bed. Channels with slopes greater than 0.6 % and flows with high solid concentration will not segregate particles if water is added, and Channels with slopes greater than 0.9 % will not embank even with low flows.

11.5.1 Sub-Critical and Super-Critical Flow

Open Channel flows can be classified (Tamburrino 2000) in several ways, depending on the aspects we are interested in. We already distinguished between gradual and rapid flow, depending on the flow curvature. Other aspects we can consider are also found in pipe flows, such as the flow variation over time (steady or unsteady flow), and the importance of viscous effects with respect to inertia (viscous or turbulent regime). For non-homogeneous suspensions, a turbulent flow is required to avoid particle settling.

Another important classification arises from comparing mean flow velocity $\bar{v}_x$ to the speed of the small surface wave c. Assuming low wave amplitude and negligible surface tension effects, the speed of a surface wave is given from potential flow theory by:

$$\frac{c^2}{gh} = \frac{\lambda}{2\pi h} \mathrm{tgh}\left(\frac{2\pi h}{\lambda}\right) \tag{11.97}$$

where λ is the wave length and g is acceleration due to gravity. When $\lambda \gg h$, $\mathrm{tgh}(2\pi h/\lambda) \approx 2\pi h/\lambda$, and the speed of a small perturbation on the free surface of a flow having a finite depth is given by:

$$c = \sqrt{gh} \tag{11.98}$$

A further classification arises when $\bar{v}_x$ is compared to c, it being customary to work with the ratio $\bar{v}_x/c = \bar{v}_x/\sqrt{gh} \equiv Fr$, which is termed the Froude number. Thus, the following classification arises in open Channel flows:

$Fr < 1$, the flow is called sub-critical (or tranquil flow).
$Fr = 1$, the flow is called critical.
$Fr > 1$, the flow is called supercritical (or rapid flow).

When flow conditions are such that the Froude number moves in the range 0.8–1.2, the flow is called trans-critical. Design of open Channels usually avoids trans-critical flows due to the presence of water surface oscillations and flow depth variations. A supercritical regime is recommended for slurry transport in Channels.

The speed given in (0.91) is valid for two-dimensional flows. A general definition, valid for a Channel with any shape is as follows:

$$c = \sqrt{g\frac{A}{b_s}} \tag{11.99}$$

where A is the flow cross sectional area and b_s is the free surface width.

11.5.2 Steady Uniform Flow

In steady uniform flow there is equilibrium between the force generating the motion (gravity) and the resistance force opposing the flow. Theoretically, a gradually varied sub-critical flow will became uniform at an infinite distance upstream, and a super-critical flow reaches the condition of uniform flow at an infinite downstream. In practice however, as shown in Fig. 11.26, we can consider a finite distance for the uniform flow to develop.

In the figure, the sluice gate provides a control section that imposes a boundary condition for the flow downstream the gate. The flow that develops close to the gate is non-uniform, with h as a function of x. After a certain distance, variations of h with x are very small and of the order of the natural water surface fluctuations. Here we can consider that the flow has reached the uniform condition. The uniform flow is also called *normal flow* and its depth *normal depth*.

Mass balance

The macroscopic mass balance indicates that the volume flow rate Q is constant, then:

$$\begin{aligned} Q_2 &= Q_1 \\ A_2\bar{v}_2 &= A_1\bar{v}_1 \end{aligned} \tag{11.100}$$

where A is the wetted area and $\bar{v}_i$ is the average velocity. Since $A_2 = A_1$, we have:

$$\bar{v}_2 = \bar{v}_1 \tag{11.101}$$

Momentum balance

The macroscopic momentum balance at steady state applied to the control volume defined between sections (1) and (2) in Fig. 11.26 results in:

$$\oint_S \rho \boldsymbol{v}\boldsymbol{v} \cdot \boldsymbol{n} dA = \int_V \rho \boldsymbol{g} dV + \oint_S \boldsymbol{T} \cdot \boldsymbol{n} dA \tag{11.102}$$

The wall shear stress τ_w is defined by:

$$\tau_w = -\frac{1}{S}\int_S \left(\boldsymbol{T}^E \cdot \boldsymbol{n} dA\right) \cdot \boldsymbol{i}$$

where ρ is the pulp density and $\boldsymbol{i}$ is the unit vector in the direction of the flow. Since the velocity and the areas are constant, the first term of Eq. (11.102) vanishes and the other two terms become:

$$0 = \rho\,(\boldsymbol{g} \cdot \boldsymbol{i})\,AL - \tau_w S \tag{11.103}$$

$(\boldsymbol{g} \cdot \boldsymbol{i}) = g\sin\theta$ is the slope of the Channel, $S = LP$ is the wetted surface, that is, the Channel surface in which the shear stress is acting. Then,

$$\tau_w = \rho g \sin\theta \frac{A}{P} \tag{11.104}$$

Thus, Eq. (11.97) provides an expression for the average wall shear stress in terms of the Channel characteristics (A, P and θ), the density of the substance being conveyed and the acceleration of gravity. Note that the equilibrium does not discriminate between liquid and mixture, so that Eq. (11.104) is as valid for water as for slurries.

The ratio between the cross sectional flow area and the wetted perimeter is called *hydraulic radius* R_h, which is an important geometric parameter of the flow representing the ratio between the slurry (or liquid) volume, where gravity is acting, and the Channel surface where there is shear stress between the liquid (or slurry) and the wall.

$$R_h = \frac{A}{P} = \frac{bh}{P} \frac{\langle \text{cross sectional area} \rangle}{\langle \text{wetted perimeter} \rangle}$$

Note that Eq. (11.104) is valid for steady uniform flow in any geometry. The only restriction is that the Channel must be prismatic, that is, its shape must not change with distance in the flow direction:

$$\tau_w = \rho g \sin\theta R_h \tag{11.105}$$

For a rectangular Channel $A = bh$ and $P = b + 2h$

$$\tau_w = \rho g \sin\theta \frac{bh}{b + 2h} \tag{11.106}$$

Flow velocity

In terms of the dimensionless wall shear stress, known as Fannig friction factor, defined by $f = 4\tau_w/(1/2)\bar{v}_x^2$, we have:

$$f = 8\rho g \sin\theta \frac{bh}{b + 2h} \frac{1}{\bar{v}_x^2} \tag{11.107}$$

The most popular expression for the Fanning friction f factor is:

$$f = 116 \frac{\chi^2}{R_h^{1/3}} \tag{11.108}$$

so that the average flow velocity is:

$$\bar{v}_x = \sqrt{\frac{8\rho g \sin\theta}{116\chi^2} \left(\frac{bh}{b + 2h} \right)^{4/3}} \tag{11.109}$$

where χ is the roughness coefficient. Table 11.4 shows friction factors f for several Channels of uniform cross sections.

Volume flow rate

From Eq. (11.109), the flow rate for rectangular Channels is:

$$Q_f = bh \sqrt{\frac{8\rho g \sin\theta}{116\chi^2} \left(\frac{bh}{b + 2h} \right)^{4/3}} \tag{11.110}$$

Table 11.4 Friction factors for channels

Type of channel of uniform cross section	χ, ft$^{1/6}$	χ, m$^{1/6}$
Sides and bottom lined with wood	0.009	0.0074
Neat cement plaster; smoothest pipes	0.010	0.0082
Cement plaster; smooth iron pipes	0.011	0.0090
Unplanned timber evenly laid; ordinary iron pipes	0.012	0.098
Best brick work; well-laid sewer pipes	0.013	0.0170
Average brick work; foul iron pipes	0.015	0.0123
Good rubble masonry; concrete laid in rough form	0.017	0.0139

Fluid depth

Calculating h from (11.110) yields:

$$h = \frac{1}{b}\left(\frac{116(b+2\,h)^{4/3}\chi^2 Q_f^2}{8\,g\mathrm{sen}\theta}\right)^{3/10} \tag{11.111}$$

The height h is calculated from the implicit Eq. (11.111) by iteration. Another version of this equation is:

$$h = \frac{1}{b}\left((b+2\,h)\left(\frac{116\chi^2 Q^2}{8\,g\mathrm{sen}\theta}\right)^{\frac{3}{4}}\right)^{\frac{2}{5}} \tag{11.112}$$

Limiting velocity

Domínguez and Harambour (1989) proposed the following limiting deposit velocity to ensure that particles do not settle:

$$v_L = 0.6505\left(8g\left(\frac{\rho_s}{\rho}-1\right)d_{85}\right)^{0.5}\left(\frac{d_{85}}{4R_h}\right)^{0.342}\left(\frac{d_{99}}{d_{85}}\right)^{0.386} \tag{11.113}$$

where ρ_s and ρ are the solid particle and pulp densities, d_i are the sizes where i % of the material passes and R_h is the hydraulic radius.

Mechanical energy balance

The mechanical energy balance is:

$$\oint_S \left(1/2\rho\bar{v}_x^2 \boldsymbol{v}\cdot\boldsymbol{n}\right)dA = \oint_S \boldsymbol{v}\cdot\boldsymbol{T}\cdot\boldsymbol{n}dA - \oint_S \rho\phi\boldsymbol{v}\cdot\boldsymbol{n}dA - \dot{E}_v$$

$$\oint_S (1/2)\rho\bar{v}_x^2 \boldsymbol{v}\cdot\boldsymbol{n}dA = -\oint_S p\boldsymbol{v}\cdot\boldsymbol{n}dA - \oint_S \rho\phi\boldsymbol{v}\cdot\boldsymbol{n}dA - \dot{E}_v$$

Since

$$\phi = g(\eta(x) + z\cos\theta)$$
$$p = \rho g[(h - z)\cos\theta]$$

$$(1/2)\rho\bar{v}_2^2 A_2 - (1/2)\rho\bar{v}_1^2 A_1 = -\rho g\big((\eta_2 + h\cos\theta)\bar{v}_2^2 A_2 - (\eta_1 + h\cos\theta)\bar{v}_1^2 A_1\big) - \dot{E}_v$$

but $\bar{v}_2 A_2 = \bar{v}_1 A_1, \bar{v}_2^2 A_2 = \bar{v}_1^2 A_1, p_2\bar{v}_2 A_2 = p_1\bar{v}_1 A_1, z_2\bar{v}_2 A_2 = z_1\bar{v}_1 A_1$, so that this equation reduces to:

$$\eta_1 - \eta_2 = -\big(\dot{E}_v/\rho g Q\big)$$

Since:

$$\eta_1 - \eta_2 = L\text{sen}\,\theta \text{ and } h_f = \big(\dot{E}_v/\rho g Q\big)$$

$$h_f = L\text{sen}\,\theta \tag{11.114}$$

Thus, the *viscous dissipation*, or *head loss*, is just equal to the decrease in potential energy for uniform flow.

Problem 11.16 A uniform flow of copper flotation tailings takes place in a rectangular Channel constructed of concrete. If the angle between the Channel and the horizontal is 1.0° and the Channel is 0.9 m wide and water is 0.50 m deep, calculate the velocity and the volume flow rate.

$$\chi = 0.0139\ \text{m}^{1/6}$$

$$Q_f = bh\sqrt{\frac{8\,g\text{sen}\,\theta(bh)^{4/3}}{116\chi^2(b+2\,h)^{4/3}}} = 0.9\times 0.5 \times \sqrt{\frac{8\times 9.8\times \text{sen}(1\times\pi/180)\times(0.9\times 0.5)^{4/3}}{116\times 0.0139^2(0.9+2\times 0.5)^{4/3}}} = 1.347\ \text{m}^3/\text{s}$$

$$\bar{v} = \frac{Q}{b\times h} = \frac{1.346}{0.9\times 0.5} = 2.99\ \text{m/s}$$

b (m)	1.00
θ (°)	1
Q (m^3/s)	1.346
assume h* (m)	0.45
h*-h = 0	9.18E−06
g (m/s^2)	9.81
χ ($m^{1/6}$)	0.0139
v ((m/s)	146683.42

Problem 11.17 A Channel 0.9 m wide and 1 m high with a slope of 1.0° carries 1.347 m^3/s of copper flotation tailings. Calculate the height of the water in the Channel.

Using solver from Excel by assuming $h = 1$ m in Eq. (11.112) results in:

Problem 11.18 Design a Channel of rectangular cross section to transport a volume flow rate of 0.3 (m^3/s) of copper tailings. The Channel should have a slope of $\tan\theta = 0.0157$ and a height to width ratio of $h/d = 0.5$.

Assume $h = 1$ (m) in Eq. (11.112) and search for objective, with the result

tanθ (°)	0.0157
Q (m^3/s)	0.300
h/b	0.5
θ (°)	0.01569871
assume h* (m)	0.27
b (m)	0.54
h*-h = 0	2.83E−04
g (m/s^2)	9.81
χ ($m^{1/6}$)	0.0139
v (m/s)	2.05

Problem 11.19 For a smooth concrete Channel 2 m wide with a slope of 0.001 has a volume flow rate of 1.0 m^3/s, determine the wall shear stress per unit length.

Using the solver of Excel and assuming $h^* = 1$ in Eq. (11.112) and calculating τ_w from (11.106) gives:

ρ (kg/m3)	1,000
tanθ (°)	0.001
Q (m^3/s)	1.000
b (m)	2.00
g (m/s^2)	9.81
θ (°)	0.001
senθ	0.001
assume h* (m)	0.21
h*-h = 0	2.62E−06
χ ($m^{1/6}$)	0.0139
v (m/s)	2.41
τ_w (Pa/m)	1.68

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