Chapter 6
Discussions of Part I Chapters

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6.1 Introduction

This chapter could be taken as a breath in this book, giving a space for a reader's reflective point of view on the previous chapters. It is one of the moments where the three authors have decided to question each other, from their own perspective (mathematical, cultural, historical, epistemological…). In this chapter, I have the initiative of this questioning. Here is the way I have proceeded: in the following sections, I ask my questions either to John or Jon, and each of them has a special place for developing his point of view, feeling free also to answer to a question that is not addressed to him! After receiving their answers, I do not propose to 'answer the answers', or provide a conclusion but provide some reflections arising from my colleagues' answers, wishing to empower the reader to enter the debate.

This chapter is organised in three sections. In the first one, I continue previous interactions I had with Jon and John during the process of writing the chapters. In the second one, I ask a set of new questions arising from re-reading the chapters once when the book was in a state of near completion. Then comes my conclusion.

6.2 Interactions with John and Jon Follow-Up

During the design of the book, two discussions were particularly interesting. The first one with John, about history, the second one with Jon, about proving 'graphical and numerical evidence'.

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6.2.1 Writing and Mathematics, a Dual Invention?

I had a discussion with John about the historical chapter: was it desirable to embrace the whole history of tools in a single chapter, or was it better to focus on specific periods? Finally, we decided to choose… both, with Chap. 4 dedicated to *Tool, human development and mathematics*, and Chap. 5 dedicated to the *Mesopotamian scribal schools*. Reading now these two chapters, I wonder if, with this organisation, we did not miss something. Actually, Chap. 4 skips from 'Tools use and phylogenesis' (Sect. 4.2) to 'Ancient Greece' (Sect. 4.3), i.e. from Prehistory to History. The essential element, distinguishing these two periods, is the invention of writing (that could not be reduced to some inscriptions on a bone). To what extent this invention is linked to mathematics? It is certainly a complex question:

- On the one hand, it seems to be clear that writing and information processing… including mathematics, came together, as stated by Nissen, Damerow, and Englund (1993): 'This innovation [the writing] was quite certainly more than a simple change in the means of storing information, or in the representation of language. Observing that at the end of the third millennium BC, during the so-called Ur III period, the human labour force was subjected to complete administrative control made possible through the developed techniques of writing (see Chap. 11), we must realise that this level of centralisation would have been impossible without the methods of information processing developed more than 1000 years earlier'.
- On the other hand, the most ancient texts that are known (called 'proto-cuneiforms') are written in an unknown language. The only part of them (constituting actually the main part of the corpus) that has been understood is composed of computations and have been deciphered by historian of mathematics: 'It may surprise some that the most important recent advances in the decipherment of the proto-cuneiform documents have been made by and in collaboration with mathematicians with no formal training in Assyriology, J. Friberg and P. Damerow. But remembering that the great majority of archaic texts are administrative records of the collection and distribution of grain, inventories of dairy fats stored in jars of specific sizes, and so on, that is, documents above all made to record in time quantifiable objects, it is reasonable to expect that such documents would contain, no less than the accounts of current institutions, evidence of mathematical procedures used in the archaic period and that they would thus contain the seeds of the mathematical thinking which developed during the third millennium'. (Englund, 1998, p. 111)[1]

So, the most ancient translated written texts seem to be mathematical texts because there only understandable part was about mathematics. Anyway, there seems to be a very strong relationship between the emergence of writing and the emergence of mathematics. Could you develop on that, John?

[1] Thanks to Christine Proust, historian of mathematics, who gave me the two references evoked in this section.

Emergence of Writing and Mathematics

John: As a mathematics educator my expertise is in the didactics of mathematics. Because mathematics comes with a long tradition (culture) and doing mathematics is a social and a cognitive activity my research is informed by work in the history of mathematics, philosophy, psychology and sociology but I am not an expert in these areas. I thus shy away from any pretence to have a definitive answer to the question on the co-emergence of writing and mathematics. But my reading in the history of mathematics provides support for this co-emergence.

In Sect. 4.3.2, informed by Netz (1999), I note that the development of ancient Greek mathematics was co-temporal with the development of an alphabet, lettered scripts and media approximating to pen and paper. And Netz (1999, p. 86) hypothesises that a common origin to mathematical propositions was 'to draw a diagram, to letter it, accompanied by an oral dress rehearsal—an internal monologue perhaps—corresponding to the main argument; and then proceed to write down the proposition'. Further to this, and with reference to your statement, Luc, that 'most ancient translated written texts seem to be mathematical texts because there only understandable part was about mathematics', Singh (2000), adds support to this statement. In a discussion about a twentieth century attempt to decipher an ancient Cretian text from 1450 to 1375 BCE, he writes, 'Many of the tablets seemed to contain inventories. With so many columns of numerical characters it was relatively easy to work out the counting systems, but the phonetic characters were far more puzzling'. (Singh 2000, p. 220)

6.2.2 *Proving (What Appears As) Numerical and Graphical Evidences*

In 2014, I had a very challenging discussion with Jon about a mysterious (to me) property. 20 years ago, I was studying with my students a Riemann sum (see T_n right side). This sum converges towards I, and the proof is quite easy to establish. Seeing the successive values of the sum (0.500000, 0.650000, 0.697436, 0.720294, 0.733732...), and supported by a strong visual support (the more numerous are the rectangle under the curve, the better they fit this curve), it seems quite obvious that this sum is increasing. Quite confident with that, I ask a student to prove it... and it appeared quickly that it was not so obvious! I try to prove it myself... and did not succeed... I ask a colleague, who also could not prove it. Finally, after some new attempts, I renounced, but I kept always in mind this open ended question... until I read the first version of Jon Chapter (Example 5.3), where this example was presented, from the point of view of the convergence, without questioning the monotonic point of view. I questioned Jon at once: what about the increasing nature of this sequence? There ensued an email interaction, see below.	$I = \int_0^1 \frac{1}{1+x^2} = \frac{\pi}{4}$ $S_n = \sum_{i=0}^{n-1} \frac{n}{i^2+n^2}$ $T_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2}$

Luc to Jon: June 14th, 4:21

[. . .] to illustrate how a « simple » mathematical question could resist to a « naive » thinking. I have « in me » the following question, since I was a teacher in an university (see the formulas below). It is « visually obvious » that $T_n < I < S_n$, and that both S_n and T_n converge towards I. And the students can prove this. If we consider the table of values of the two sequences, it is again quite obvious that S_n is decreasing, and that T_n is increasing. *I* asked once a student: can you prove it? He was quite confident (at the beginning), me too. . . But he cannot do that. . . neither me. What should be the answer of an experimental mathematician facing this question?

Jon to Luc: June 14th, 5:08

I do not need to since I know it is a Riemann sum, but I could mention it.

Luc to Jon: June 14th, 6:30

Yes we do know that this sequence is a convergent one, and we do not need to know any thing regarding its variation but. . . My question is a kind of aesthetic one, the sequence appears to be decreasing, do I have some means to prove it or to explore further this 'potential' property? I have just read your Example 5.3, but you do not evoke the decreasing aspect of the sequence?

Jon to Luc: June 14th, 5:45 a.m

I will think on this....

Luc to Jon: Jun 14th, 8:24 p.m

Yes, it seemed to me very easy to prove that the Riemann sum was decreasing, as the function $1/(1+x^2)$ is decreasing. . . Of course no direct relationship between the behaviour of the function on [0, 1], and the behaviour of the corresponding Riemann sum. Actually it is easy to prove that the sequence $S(2^n)$ is decreasing, not at all the same easiness for $S(n)$. . .

Jon to Luc: June 14th, 8:36

Yes the powers of two are easy, but I think the monotonicity of the whole sequence is subtle. . .

Jon to Luc: June 19th, 13:37

Dear Luc, did this paper trigger your example? Szilard Andras (2012). Monotonicity of Certain Riemann-Type Sums. The teaching of mathematics, 15(2), 113–120.

Sadly, the argument at the top of page 16 has an error and the proof does not work, although it does work for the general results on convex and concave functions. But $1/(1+x^2)$ is neither and the sufficient condition fails despite the claimed proof to the contrary.

Interesting. I checked in Maple and got a different condition to check.... I found the article typing 'monotonicity of Riemann sums' into Google. Digital assistance in action!

Luc to Jon: Jun 19th, 15:31
Thank you Jon (and digital assistance!). Interesting for convex and concave function, but, if I have well understood, the proof doesn't work (top of the page 116) and it remains to be done? Will try to find extra digital assistance!

Jon to Luc: June 19th, 20:05
Yes it is unproven. Did you know this paper?

Luc to Jon: Jun 20th, 1:49
No, I didn't know. The paper is interesting, and the way of access too! Thank you.

As the reader could check, the current—and last—version of Jon's Chapter contains a complete proof of this property.

Then, my questions to Jon: could you tell the 'end of the story'? To what extent looking for proving a property is interesting for you, as the main result (the sequence at stake is convergent) is already achieved? Could you develop on your way of reflecting to a new problem?

What Is to Be Proven, and Why?

Jon: We eventually found a clean proof of much but not all that we wanted. In particular (https://www.carma.newcastle.edu.au/jon/riemann.pdf) we proved that[2]: if the function f is decreasing on the interval [0, 1] and its symmetrisation $f(x)+f(1-x)$ is concave, then T_n increases with n.

How much I care about a proof—once I know the result is true—is context dependent; if it is just a step on a route, then I have little interest unless the destination is interesting. This issue bedevils teaching proofs in classes as often the thing the student is asked to generalise and prove is intrinsically uninteresting. Why bother?

When I am given a new problem what do I do?

If it is a conjecture, I usually try to falsify it numerically, graphically or symbolically. Even if the question is not directly computational, I look for a consequence or a similar question which is. I find that the act of quantifying the problem sufficiently to play with it in Maple or Mathematica is enormously helpful. It forces a deeper understanding of the question, of unintended ambiguity and much else.

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[2] Interestingly we cannot prove the decreasing behaviour by our methods for the increasing case.

If the computations lead to a counter-example, then great. If not, and they add reassurance that the conjecture is probably true, I then let things slosh around in my head for a few days. I hunt for similar things I have seen or topics it reminds me of. I probably also then ask one of my network of collaborators if they know more.

If after that no progress comes, I try to judge if more effort is needed. After over 40 years, I trust my gut feelings. If my mind wants to keep worrying about the problem, I let it. And so it goes. This can be a long process. One of my post doctoral students and I are currently making sense and making an article out of a set of overheads from a talk I gave in 1983 but never turned into a paper.

6.3 And Some Fundamental Questions

Re-reading the chapters in this part of the book, I realised how complex were the questions arising from the consideration of tools, mathematics and learning. My reflections developed in two directions: What is mathematics? In the digital area, what links can be established between mathematics and computer science?

6.3.1 Mathematics, What Else?

Chapter one opens with these two essential questions addressed by John and me: what is a tool? What is mathematics? We can read in the first page of this chapter '[...] the view that mathematics is just a tool-box is a pretty impoverished view of mathematics'. Re-reading that, I realised that this sentence (written by John and me) witnessed perhaps an impoverished view on tools, and on tool-boxes, and perhaps a view that is not in line with the purpose of the book, this purpose giving tools a great potential for doing, evolving, thinking... What should be your comment, John, on that?

Mathematics, As a Tool-Box

John: The statement 'mathematics is just a tool-box' appears to ignore essential dimensions of what mathematics is, for example semiotic and social dimension, and what is involved in doing mathematics. Doing mathematics is usually directed at an outcome (an answer, a proof, a construction, etc.) in which tool use is essential but there is 'doing mathematics' prior to this outcome. For example, in extending and validating the pattern $1+2=3$, 4

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$+5+6=7+8$, $9+10+11+12=13+14+15$ there is 'recognition': the numbers are consecutive; the terms on either side of the equal sign have patterns (the number of terms on the left hand side is 1 more than the right hand side, the number of numbers increases by 1 each time); the leftmost number of the nth equation is n^2. At this point we can bring in a tool-box of sorts and formulate a conjecture in algebraic terms.

Further to this, 'mathematics is a tool-box' is a metaphor. Metaphors are important. They can be things of beauty in the literary arts and it is very difficult to communicate (especially in instruction) without using metaphors but 'scientific inquiry' should, I hold, try to eliminate metaphors whenever possible. The term 'tool' is often used metaphorically. Vygotsky (1978, p. 53) wrote of this, 'Expressions such as "the tongue is the tool of thought" . . . are usually bereft of any definite content and hardly mean more than what they really are: simple metaphors . . .' I've met 'mathematical tool-box' used as a metaphor many times over the decades. The main occurrence is in undergraduate mathematicians' accounts of solving a problem—when their accounts refer to looking into their mathematical tool-box and choosing the appropriate algorithm or theorem or whatever to solve the problem. I think/hope I'm very tolerant of students' metaphorical reference to a tool-box and, like many metaphors, there is a sense in which it 'rings true' but I think these students have overlooked such things as 'recognise' and 'formulate' actions which preface their use of a tool to solve their problem. Further to this, I think much of school/institutional mathematics encourages students to view mathematics as a toolbox. For example, a question like 'Factorise x^2+3x+2' calls for the 'factorise tool' a bit like 'Dig that garden' calls for a digging tool but both directives will be (one hopes) embedded in a wider activity, otherwise the actions following the directives are what Chevallard (2005) calls 'monumental' acts.

With regard to our general purpose, we are writing this book because we think tool use is an essential part of mathematics (it is impossible to do mathematics without tools) and there isn't a book devoted to the place of tools in mathematics but mathematics is more than just tool use.

Otherwise, Jon, you described yourself (Chap. 3, Sect. 1.2) as 'a computer-assisted quasi empiricist'. Reading this, I remembered the famous address of the Russian mathematician Arnold at the discussion on teaching of mathematics in Palais de la Découverte of Paris on 7 March 1997. His first sentence was: 'Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap'.[3] Jon, as an experimental mathematician, what could you say about that?

[3] See the whole address at http://pauli.uni-muenster.de/~munsteg/arnold.html (retrieved in 12th February 2015).

Mathematics, Experience, Experiment and Cheapness

Jon: This is a great aphorism. Yet it becomes less true as scientific experiments have moved from in vivo to in silico. As I have discovered over the years, mathematical problems can be as computationally demanding as any. That said, governments and agencies are more likely to pay for global warming models than for an attack on the Riemann Hypothesis. Moreover, in particle physics, it is hard to see any experiments to validate string theory. Some see this as an approaching crisis in Physics. At any rate, as I hope I illustrated, there are indeed many cheap and insight-laden computer experiments available to the working mathematician.

6.3.2 Mathematics and Computer Sciences

The French mathematician Jean-Pierre Kahane, who was always interested in the questions of teaching,[4] chaired the French CREM (Commission of Reflection on Mathematics Teaching) from 1998 to 2002. This commission wrote a report (Kahane, 2002), trying to define mathematics (our translation):

Mathematics is the oldest of the sciences and of whose values are more permanent. However, the approach and means of study varied according to civilisations and eras. Printing, navigation and astronomy contribute to shape the usual functions and calculus.

Today, computer science creates both new ways and new areas of study, all the sciences improve using mathematical tools and help to forge new, the link to the physical strengthens, and mathematical research benefits from the intuition of physicists.

The vision of mathematics has changed considerably over the last 50 years. Mathematical then seemed to have regained its unity on the basis of a solid construction of its foundations and structures. But she was impoverished. Then applied mathematics have made a breakthrough. Currently, the movement of math reveals a multitude of sources and impacts, together with a considerable work in constituted mathematics.

Mathematics enriches themselves with problems, methods and concepts from other sciences and practices, creating new concepts and theories, and provides material to sometimes unforeseen applications. The mathematical models, allowing simulations are everywhere, and mathematics develop through interactions with other disciplines together by interactions within them. Thus mathematics is far from being the affair of the only mathematicians.

[4] He chaired the ICMI—International Commission on Mathematical Instruction—from 1983 to 1990.

In the pumping process, distillation and irrigation they represent today, we must take into account the mathematical activity engineers, physicists, computer scientists, engineers, biologists, economists, chemists at the same time as that of mathematicians in the strict sense. It is good to no longer think only in terms of 'mathematic', 'pure mathematics and applied mathematics', but to consider all 'Mathematics' in the variety of their actors and their users.

In this view of the 'mathematics sciences', it could be relevant, in the frame of this book, to question the particular relationships between mathematics 'themselves' and computer sciences. These two fields develop complex interactions. A number of curricula (for example: France, see Sect. 12.4.3) now integrate from the primary and middle school elements of teaching on programing. There is then an emerging debate (see Sect. 12.4.3) on the relationship between mathematics and computer science teaching. Who should teach programing and algorithmic? The mathematics teacher? And why, on the contrary, computer science teachers should not teach mathematics? Or could we conceive mathematics and computer sciences as a new scientific field giving matter to a new teaching in secondary mathematics? What could you say on that, John and Jon?

Teaching Mathematics vs. Teaching Computer Science

John: I am more interested in Jon's response to this than mine as he is a research mathematician and the use of computers (and ideas/techniques from computer science) is essential in his research but my experience in computer science (other than simply using computers) is limited to teaching processor architecture a long time ago. Further to this, computer experiments in school mathematics, though 'well intentioned', are very different to the computer experiments Jon reports on in Chap. 3. They are often of the form 'use software X to generate a pattern and use mathematics (possibly computer-aided mathematics) to find mathematical relationships in the pattern generated'. A 'didactical transposition' (cf. Sect. 10.3) of sorts, from research mathematics to school mathematics, has taken place.

But I see no reason why school mathematics could not evolve to include aspects of computer science. There is an historical precedence for such an evolution. School mathematics (at the senior level) in England (and in some other countries influenced by British culture) includes Newtonian mechanics as a part of mathematics. The reason for this appears to be simple, Newton was an English mathematician and his mechanics was an important part of his oeuvre.

Jon: There are various nascent curricular coding proposals in different countries. At the school level, I share John's concern that they are largely ill considered. But my nine years old grandson is keen to learn to build his own

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video games; any tools that ensure he learns some of the fundamentals of programing while 'designing' his games could only do good.

At the university level, I would love to see computation deeply embedded in the curriculum but to the degree that this is happening it is slow and haphazard. Some years ago Penn State mandated the use of Maple in all entry-level classes. Six weeks later the hottest selling item on campus was a 'F**K Maple' T-shirt. The administrators had forgotten to tell the teachers!

6.3.3 Mathematics and Tools, Serendipity, vs. Intentionality?

John (in Chap. 1) used the beautiful word of *serendipity*[5] for describing the history of tools. Thinking of serendipity, I have always in mind the history of the steam engine:

> A tool does not have a pre-assigned function. The 'logical of use' can deviate at any time its trajectory. The twists of this sartorial give the history of technology, material and intellectual, a baroque and poetic charm that closer, for our greatest benefit and pleasure, its polar opposite: an anthology of wonderful. The first steam engine was not designed by Savery in 1698 to drive a vehicle, but to draw water from the bottom of a well (Debray, 2001, p. 106, our translation).

In the same time, a tool is also intentionally developed to meet a given need, to support a given activity. Actually the design of tools cannot be thought independently of their usages. Designing trajectories and usages trajectories appear completely interrelated.[6] Intentionality aims to meet a given necessity. We could understand with this perspective the so-called Mohr–Mascheroni theorem (Maschieroni, 1797), stating that each construction with a ruler and a compass can be realised with a single compass: a compass is a more accurate tool than a ruler. And that, perhaps, is the reason why this theorem had a practical interest.

I wonder, Jon, if, and how, you could identify, and feel, serendipity, intentionality, and necessity in your own activity of mathematician, as you state in your chapter that '[mathematicians] produce so many unneeded results'. . .

[5] Following Wikipedia (2015, February 12th), serendipity means a 'fortunate happenstance' or 'pleasant surprise'. It was coined by Horace Walpole in 1754. In a letter he wrote to a friend Walpole explained an unexpected discovery he had made by reference to a Persian fairy tale, *The Three Princes of Serendip*. The princes, he told his correspondent, were 'always making discoveries, by accidents and sagacity, of things which they were not in quest of'.

[6] I described this dialectic (Chap. 10) as an interplay between instrumentation and instrumentalisation processes.

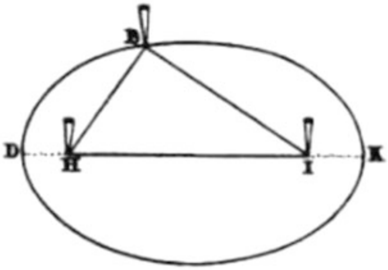

Fig. 6.1 The gardener's ellipse

Serendipity, Intentionality, Necessity and Mathematical Activity

Jon: I can do no better than quote Pasteur who said correctly that 'fate favours the prepared mind'. One attractive role for an intelligent agent in the future is to be aware of things that have been of interest in the past to a given researcher. Then metaphorically, the computer could tap the researcher on the shoulder when a potentially interesting phenomenon arose. It has often seemed to me that one difference between good and excellent researchers is in the ability to recall things in context.

I have also in mind the Descartes' description of the drawing of an ellipse (Fig. 6.1), linking one of its essential property to a practical way of drawing it:

> L'ellipse ou l'ovale est une ligne courbe que les mathématiciens ont accoutumé de nous exposer en coupant de travers un cône ou un cylindre, et que j'ai vu aussi quelquefois employer par des jardiniers dans les compartiments de leurs parterres, où ils la décrivent d'une façon qui est véritablement fort grossière et peu exacte, mais qui fait, ce me semble, mieux comprendre sa nature que la section du cylindre ni du cône. (Descartes, 1637)

The development of tools (here a nail and a string), using—implicitly—a mathematical property answers here to a practical (or aesthetic) necessity: drawing a given shape.

I wonder, John, how you feel about serendipity, intentionality and necessity? Do new tools really arise from serendipity? In Chap. 2, John, you describe the different achievements of one given task (bisection of one angle) by four tools. Could you describe this in terms of necessity? For what reason should we bisect an angle? Who needed to do this? A gardener? A painter? Which tools have been designed, and by whom, for achieving such a task?

Serendipity, Necessity and Tools

John: I said ‘new tools often arise from serendipity’. Yes, I stick by what I said, though the word ‘often’ might be replaced by ‘sometimes’. I suspect I was thinking of two things. The first is Wertsch’s (1998) 9th thesis on mediational means: *Mediational means are often produced for reasons other than to facilitate mediated action*. I will cite myself from Sect. 13.2:

> *Sometimes they* [*mediational means*] *are produced for the purpose for which they are used but this is not always the case*, *sometimes they are a spin-off. Wertsch cites fibreglass pole-vaulting poles. Fibreglass was developed by the military for reasons that had nothing to do with pole-vaulting. But once the material was produced it was available to be made into poles for pole-vaulting*.

We see spin-offs in mathematics classes at the moment in the form of calculators and spreadsheets. Small(ish) electronic calculators, as far as I have been able to ascertain, came about because advances in electronics enabled the production of such devices (cf. Sect. 7.1), not because of a need for such devices (though once they appeared they were put to use). Spreadsheets (cf. Sect. 7.1) were developed for financial use; their use in school mathematics could be said to be serendipitous (or not, depending on the value one attaches to the use of spreadsheets in school mathematics, the attribution of serendipity to something is a value judgement).

The second thing I was thinking was of the future, indeed, what I said in context was:

> *This book will also tentatively consider the future of mathematics and the role of new tools and new ways of using tools in this future. We say* ‘*tentatively*’ *because new tools often arise from serendipity and it is somewhat foolish to say that things will develop in this way*.

I am wary about predicting the future and one reason for this is that new artefacts will appear and people will appropriate (some of) these and this appropriation of new artefacts will impinge on future practices in ways I, at least, cannot imagine prior to the new artefacts and practices appearing.

But the other side of spin-offs is ‘need’ (or, at least, the perception of need). In Sect. 4.4 I write of the need to calculate accurately in astronomy in the sixteenth century and the development of multi-digit trigonometrical tables was an artefactual design to satisfy this need. Needs arise in mathematics—to understand why something is so, to solve a problem, to prove a theorem, etc.—and new mathematical algorithms are developed specifically to satisfy these needs as Jon’s ‘top 10 twentieth century algorithms’ (cf. Sect. 3.5) aptly show. But with regard to your question about who first needed to

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bisect an angle, I do not know the answer. The ancient Greeks could certainly do this using a compass and straight edge but whether this came from a need to understand or do something or whether it came about from serendipitous doodling with a compass, I do not know if this is in historical records.

Tools as well as mathematics arrive as answers to humans needs or questions. It has a practical consequence for teaching: as stated by Chevallard (2005), mathematics has to be taught and learnt as such, and not as monuments left by the great elders to the admiration of future generations.

6.3.4 Words, Images, Gestures and Proving

In Chap. 3, Jon underlines the power of visualisation for discovering new properties and proving them. We know well the power of images. Debray (2013) remarks that: 'magie et image ont même lettres'.[7] I wonder if this centration on *images* was not a reduced view of *experiencing/feeling* things. In French the deep dialectic between experiencing and proving appears more clearly through the duo éprouver/prouver.

I have in mind two examples for illustrating this idea:

- First example, about the question 'Is it possible to tile the plane with any quadrilateral?' The spontaneous students' answers are 'no!', and the using of a Dynamic Geometry Software appears not so easy for exploring such a question. But, if they use cardboard templates, and if they move them on a table (Fig. 6.2), they arrive, quite rapidly, to a solution, and this set of gestures convince them that the answer is 'yes'. In other words, the conviction is shaped by the activity, not the final image.
- Second example, about what is finally, for me, as my best memory of my years of teaching in a secondary school. My intention was to prove that the orthogonal projection of a right angle on a plane is a right angle if and only if one of the sides of the right angle is parallel to the plane. For this purpose (it was a quite long time ago), without announcing the targeted knowledge, I ask my students to come to school with four potatoes and a knife (I do not know if, today, it would be possible to enter a classroom with a such dangerous tool...). Then I ask them to cut their potatoes in order to have some cubes. And then comes the question: is it possible for a plane section of a cube to be a right triangle? The students try and try, and they felt the impossibility: 'to obtain such a triangle, I am obliged to curve the knife when cutting the potatoes'...

[7] Such a playing on words cannot happen in English with the corresponding words "magic" and "image".

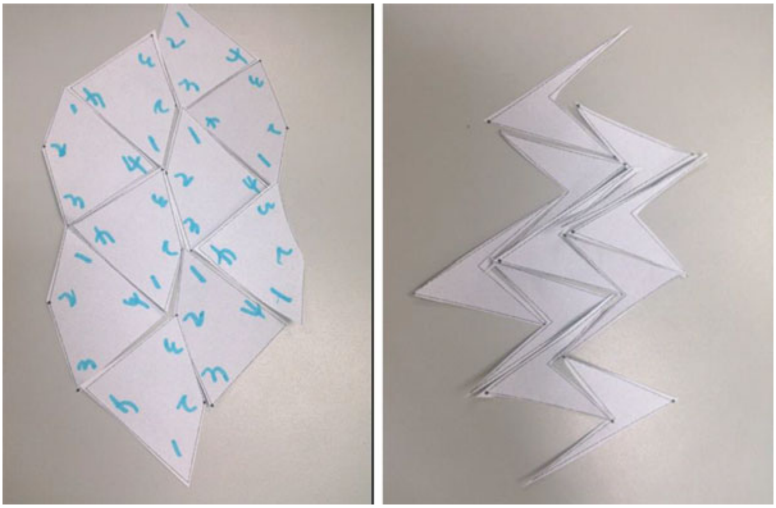

Fig. 6.2 Experiencing the tiling of a plan with given quadrilaterals

In these two situations (proving a possibility vs. proving an impossibility), the necessity of the result emerges from the acting, encompassing the seeing, but not reduced to it.

What comments, John and Jon, do these examples inspire you?

Experiencing, Seeing, Proving

Jon: In the Collin's Cobuild Dictionary (http://www.collinsdictionary.com/dictionary/english-cobuild-learners), the verb to see has many meanings and the optical is far from the most common usage. So image for me is really subordinate to visualisation in a much broader sense. In that spirit we have not exploited either haptics or movement nearly enough, nor indeed auditory data. Moreover, there are times when a movie shows much more than a still picture and others when it distracts. Sometimes the best visualisation is a potato.

John: Manipulatives can be good for fun and for getting mathematical ideas across to learners but DGS need not be that hard. Figure 6.3 shows a *GeoGebra* screenshot of the mid-stage of the construction of a tessellation (a stage that shows all the essential construction points, lines and vectors).

I've modelled my construction on Fig. 6.3 (think of C and E_1' being vertices labelled '1' in Fig. 6.2). E is the mid-point of AB and I've translated the point E by the vector CE to get E_1'. What you give the learner (the bare problem or a partially constructed tessellation as in Fig. 6.4) depends on their DGS and mathematics proficiencies.

(continued)

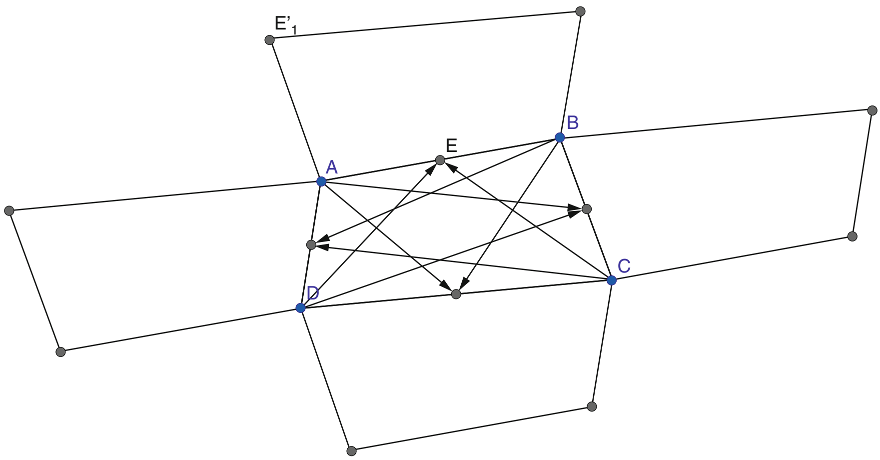

Fig. 6.3 GeoGebra screenshot of the mid-stage of the construction of a tessellation

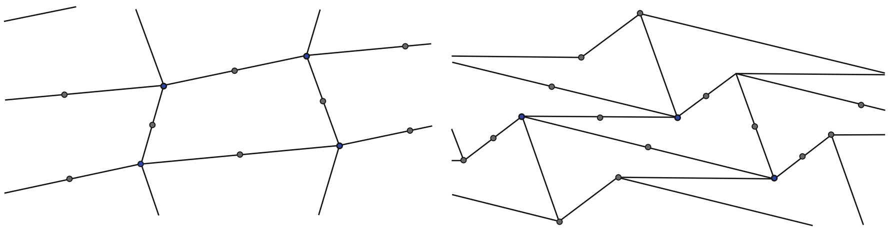

Fig. 6.4 The final stage of the construction and one result of dragging

But whatever one gives the learner as a starting point, the final results, a tessellation that 'sways' as you drag any of the points *A*, *B*, *C* or *D*, is virtually guaranteed to inspire 'awe' in the learner. There can be a kind of 'experimental proof' in this dragging/swaying: try to drag a point so that the quadrilateral does not tessellate. Trying to do this (which, of course, cannot be done) can also link the two sets of manipulatives in Fig. 6.2. Figure 6.4 shows the final stage of my construction and (on the left) one result of dragging.

I now move on to 'seeing'/visualisation. The philosophy and the psychology of perception are specialist and complex fields. They are important for this book on tools and mathematics (and not only for Jon's 'visual theorems'). A problem for us is that we three are not experts in these fields. My own views on perception have been influenced by the work of the philosopher Marx Wartofsky and the psychologists Eleanor and James Gibson. I came to reading works of these scholars from my interest in tools and mathematics and I report on the importance of their work for understanding tool use in Sect. 7.2.1, so I do not repeat it here.

(continued)

Work on visualisation has featured in mathematics education research and scholarship for many decades. An early and, in my opinion, an important paper is Eisenberg and Dreyfus (1991) which shows how diagrams can structure students' epistemic processes. The authors argue, however, that this is only possible for students who are able to perceive the abstract structure permeating the diagram. Visualisation thus benefits high achieving students more than low achieving students.

To return to 'seeing' the theorem that any quadrilateral can produce a tessellation, when one drags a construction like mine above in a DGS, visual images can be 'seen' but mathematical relationships may not be seen; 'seeing in a mathematical way' is, I hold, usually (always?) mediated by an artefact or a teacher. An advantage of a DGS for 'mathematical seeing' is that physical actions (what the hand drags) and what is seen are co-ordinated. The DGS thus affords (in Gibson's sense of the word, see Sect. 7.2.1) seeing mathematical relationships.

6.4 Finally, How Mathematics Teaching Could Develop Interest in Proving?

Following the previous discussions with John and Jon, I would like to evoke, among other possible perspectives, two answers that support my own teaching.

6.4.1 Understanding Being and Reasons of Being

The first answer is: Provoking students' curiosity *for a result* (is it true?) and *its reasons of being* (why is it true?)

Jon evokes such an example, in the rubric 'Proofs without words' (Chap. 3, Fig. 3.5). The objective is to find a synthetic formula for the sum of the first odd numbers. The result emerge from a number pattern: $1+3=4$; $1+3+5=9$; $1+3+5+7=16$... The sum of the n first odd numbers seems to be n^2. The questions are both: is it always true, and what is the reason of such an amazing result? There is of course a lot of proofs, but the geometrical one is very interesting because it evidences the law of building this result. To build a new border of a square (Fig. 6.5), I need at each step two more tiles, i.e. I need to add the following odd number (after 9, it will be 11). It may be a proof without any *written* words, but certainly a proof combining a lot of mental words and written images (i.e. artefacts) leading to this certainty: the result is true, because I have discovered its reason of being.

I would like to confront this first example with a second one that I had found in a book of a man fascinated by numbers (Le Lyonnais, 1983). The geometrical context is: given n distinct points on a circle, draw all the possible chords. These chords determine a certain numbers A_n of regions on the circle. For example (see Fig. 6.6):

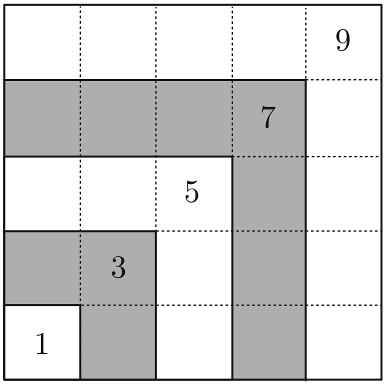

Fig. 6.5 The sum of the first odd numbers (see also Fig. 3.5)

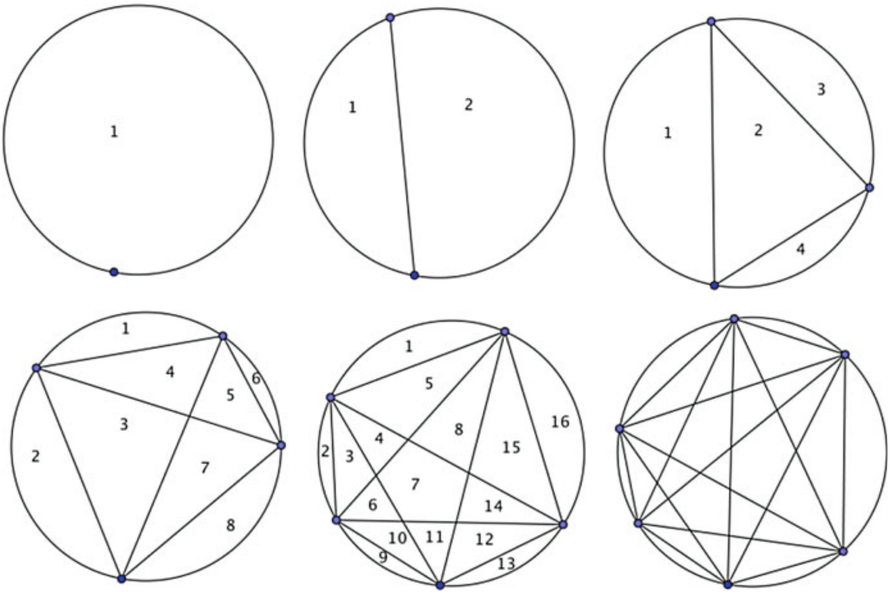

Fig. 6.6 Counting the number of sectors determined by chords in a disc

$A_1 = 1, A_2 = 2, A_3 = 4, A_4 = 8\ldots$ A conjecture quite 'natural' is that, at each step, A_n is multiplied by two. In other terms, (A_1) should be a geometrical sequence whose reason should be 2.

	Number of points on the circle	Number A_n of regions in the circle
	1	1
	2	2
	3	4
	4	8
	5	16
	6	?
	7	?

Then the two interrelated questions are: is it always true? And why, if I add a point, is the number of regions multiplied by two? These two questions feed one another. If the curious reader counts the number of sectors for 6 points, s/he would find that, instead of 32, the number is 31, leading to a reformulation of the two questions at the heart of the proving process: what is the general formula, and what is the rule of building such a sentence, its reason of being?

We will not answer here these questions, letting the reader thinking alone, or looking for the book of Le Lyonnais… But what is important, at this step, is to underline the complementary aspect of the two previous examples (Figs. 6.5 and 6.6): in the first case, what appears at once (the general rule) is true; in the second case, it is wrong. In the two cases, looking for the reasons of being of the potential rule is the motor of the mathematical activity. Meeting these two kinds of situation, for a student, is certainly necessary to avoid the feeling that 'what is true a sufficient number of times is probably always true'. As long as the reasons of being of a result have not been elucidated… the reasonable doubt remains.

6.4.2 *Extending the Domain of the Validity of a Given Result*

The second answer is: *engaging students in a path* allowing them *to enlarge*, *at each step*, *their view on a given mathematical landscape*.

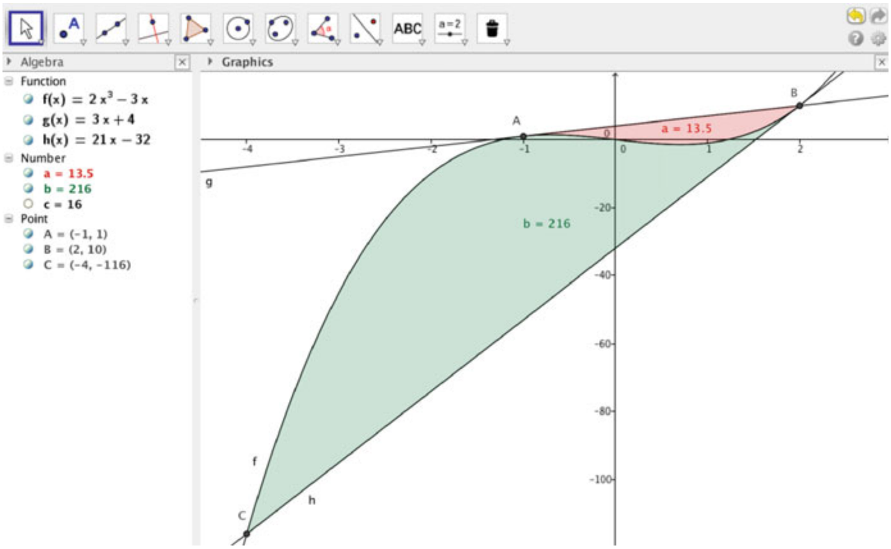

Fig. 6.7 Comparing two areas, and looking for invariants (Trouche, 1998b)

I draw the following example from the work I did, during a whole year, with a 12th grade class experiencing an environment of symbolic calculators (Trouche, 1998a, 1998b). The question at stake was (Fig. 6.7):

One considers the function $f : x \mapsto f(x) = 2x^3 - 3x$, its curve F, and the tangent T to its curve at the point A $(-1, 1)$. T crosses again the curve F at a point B. The tangent T′ to the curve F crosses again the curve F at a point C (we do not examine here the existence and uniqueness of the points B and C, which are far from being obvious). The students were asked to calculate two areas:

- The area a of the surface comprised between the straight lines $x = -1$, $x = 2$, the tangent T and the curve C.
- The area b of the surface comprised between the straight lines $x = -4$, $x = 2$, the tangent T′ and the curve C.

There is a certain relationship between a and b. Roughly speaking, b seems much bigger than a. We could model this relationship with the quotient b/a. Is this quotient a feature of the departure point A? The amazing thing is that the quotient does not depend on A... Then, is it a feature of the third degree polynomial *f*? The amazing thing is that the quotient does not depend on the coefficients of the third degree polynomial *f*... Then, is it a feature of a polynomial function?... I let the reader to engage herself in the exploring.

Advancing in the way of studying the property, extending its domain of validity, the students are led to move from calculating to reasoning again on the shape and property of mathematical objects.

6.4.3 Interacting with Objects and People

The third answer is: *creating conditions for fruitful interactions between students themselves* as well as *between students and mathematical objects*.

I draw the following example from the work I did, in 2015, during a school of high studies[8] in Recife (Brazil, reference to come), with a group of master students. During two months, a productive atmosphere develops, evidencing the potential of social interactions for proving. The mathematical question was formulated in a metaphorical way: 'Find the mother inside the daughter'... In other terms: the triangle 'mother' ABC (see Fig. 6.8) generates the triangle 'daughter' A′B′C′ by three reflections about the points A, B and C (A′ being the symmetrical of A through the reflection about B, etc.). Once removed ABC, is it possible to design the reverse process, constructing ABC from A′B′C′, i.e. to find the mother inside the daughter?

This geometrical situation is quite well known, and there are many ways for developing a given solution. The environment where the resolution took place, the

[8] Escola de Altos Estudos: Dos artefatos aos instrumentos do trabalho matemático: a dualidade essencial instrumentação-instrumentalização (http://lematec.net/EAE/).

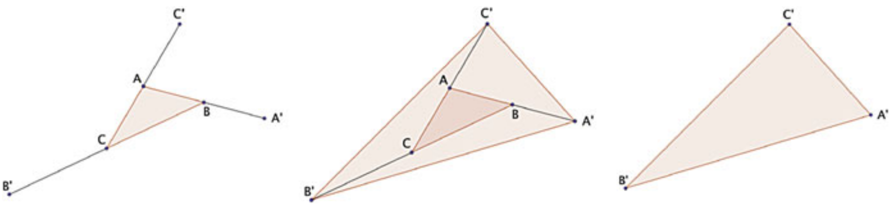

Fig. 6.8 From the mother triangle to the daughter triangle

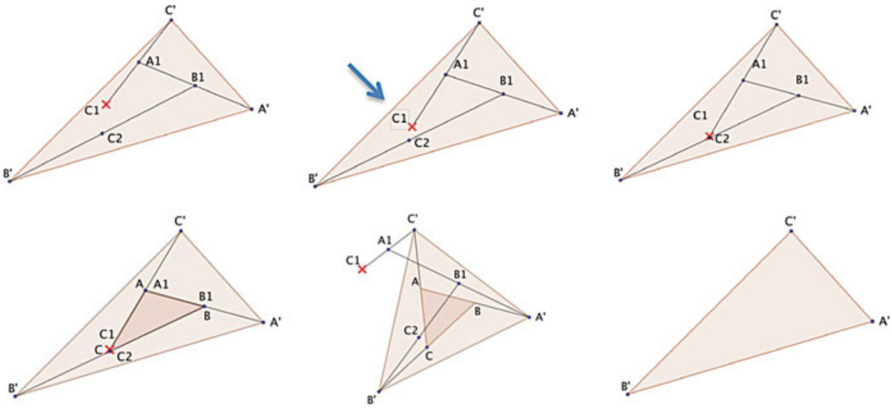

Fig. 6.9 From the mother triangle to the daughter triangle

dynamic geometrical software Geogebra, offers a lot of opportunities for interacting with the mathematical objects, for example (see Fig. 6.9): after hiding ABC, choosing randomly a point C1, one can construct A1, the mid-point of C1C′, then B1, the mid-point of A1A′, then C2, the mid-point of B1B′. Of course, C2 is not on C1. But it is possible to drag C1 to get it on C2. If we make the original mother triangle ABC appear again, we can check that ABC and A1B1C1 are alike. Obviously, the result is not stable: if we move the initiating point A, then A1 will break away from A. Therefore, if the objective was to find a solution that resists movement, the problem is not yet solved. Besides, this first construction opens perspectives for going further (using successive reflections for example. . .)

The discussions among students lead to the emergence of number of solutions. The Rodrigo's solution came from an extension of the initial metaphor: after the daughter triangle (see Fig. 6.10, what happens if one draws the grand-daughter triangle? It appears (and the property resists if we move the point A) that the sides of the mother triangle are respectively parallel to the sides of the grand-daughter triangle. If it is true, it opens a way for constructing from A1B1C1 the triangle A2B2C2, then the triangle ABC.

This new point of view gave students the idea of drawing parallel straight lines in the original figure, splitting each side of the mother triangle in three segments of the same length (Fig. 6.11). Extending this lattice outside the triangle gives the whole

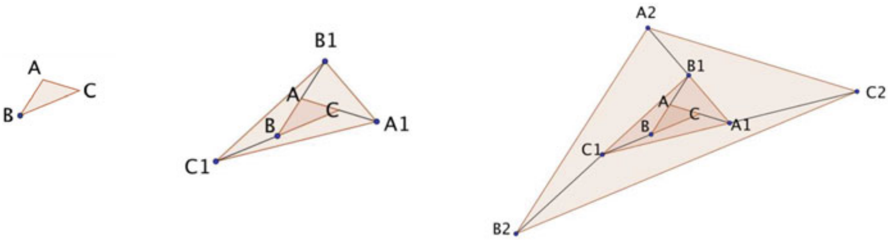

Fig. 6.10 From the mother triangle to the grand-daughter triangle

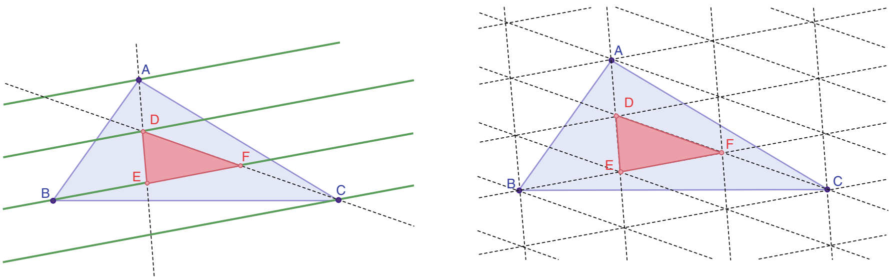

Fig. 6.11 *Parallel straight lines*, inside and outside the initial figure, structuring a set of geometrical properties

plan a structure allowing to understand the links mother–daughter triangle (and recalling a tessellation point of view (see Figs. 6.2, 6.3 and 6.4).

We have proposed in this section some examples and guidelines for designing a possible 'interesting route' in re-thinking mathematics teaching: provoking students' curiosity *for a result* and *its reasons of being*, *engaging students in a path* allowing them *to enlarge*, *at each step*, *their view on a given mathematical landscape*; *creating conditions for fruitful interactions between students themselves* as well as *between students and mathematical objects*. Situating 'Constructing, computing, proving, and understanding' at the heart of the mathematics curriculum seems to be nowadays an object of attention and research, as noticed by Hanna and de Villiers (2012): 'there has been an upsurge in research on the teaching and learning of proof at all grade levels, leading to a re-examination of the role of proof in the curriculum and of its relation to other forms of explanation, illustration and justification'.

We will focus, in the following chapters, on some crucial aspects of this research, regarding the integration of tools in mathematics education, tools seen as critical elements of *mathematics laboratories* (Maschietto & Trouche, 2010). New technological environments lead indeed to *new constraints*, *new opportunities* for the teacher (Trouche, 2000). It is particularly clear, in the previous examples, for the dynamic geometry software (DGS). But… we have to keep in mind, as stated by John (Sect. 6.2.4), that 'when one drags a construction […] in a DGS, *visual images* can be seen, but *mathematical relationships* may not be seen; "seeing in a mathematical way" is […] usually (always?) mediated by an artefact or a teacher.

An advantage of a DGS for 'mathematical seeing' is that physical actions (what the hand drags) and what is seen are co-ordinated'.

The following chapters will provide developments relevant to the use of tools in mathematics (Chap. 7), offering some theoretical approaches allowing to analyse students' activity in advanced technological environments, and teachers' role for *orchestrating* (Sect. 10.4) situations of research.

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