Chapter 5
The Development of Mathematics Practices in the Mesopotamian Scribal Schools

Tablets and tokens, lists and tables, wedges and digits, a complex system of artefacts for doing and learning mathematics, 2000 years BCE

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5.1 Introduction

This chapter proposes a view on a particular moment in the learning of mathematics, 2000 BCE in Mesopotamia: a moment particular regarding the medium, with the development of writing and of systems of signs; particular regarding the development of mathematics, with the development of a sexagesimal positional numerical system, and of associated algorithms; particular regarding the places dedicated to learning, with the development of scribal schools; and, last but not least, particular regarding the supports, with the use of clay tablets 'still alive' today.

I will look at this particular moment through the eyes of a contemporary researcher on mathematics education, aware of the difficulty of looking at the past through the eyes of the present, and of the interest of enriching the present didactical questions by an historical lighting.

5.2 A Critical Moment

The period of Mesopotamian mathematics is certainly a critical one: 'The development of scribal schools in the late third millennium and the early second millennium in Mesopotamia corresponds to a switch in the medium used for the accumulation and transmission of knowledge, from memorisation, the medium became essentially written during this period' (Proust, 2012a, p. 161). This switch could be compared to another major one that of the translation from paper to digital era (see Chaps. 2, 11, 13 and 17). This critical period is also a privileged one: 'Concerning Mesopotamian scribal schools, the situation is exceptionally favourable, due to the huge quantity of school tablets handed down to us. No other educational system of the distant past is as well documented as that of Mesopotamia' (Proust, 2012a, p. 162). This situation is due to the material used

J. Monaghan et al., *Tools and Mathematics*, Mathematics Education Library,
DOI 10.1007/978-3-319-02396-0_5

for building the tablets: 'The conservation of the unskilled writings of students is partially accidental. It is due primarily to the nature of the writing medium, the clay, a nearly indestructible material. It also ensues from the reuse of dry and waste tablets as construction material. Trapped in walls, floors or foundations of houses, tablets produced by students and subsequently discarded have escaped other forms of destruction' (Proust, 2012a, p. 163): 4000 years after, clay tablets are still alive, speaking to whom is able to understand them...

I will evoke[1] here four aspects of this rich mathematics teaching context: the computation practices and their support; the set of artefacts necessary for doing computations; the persistence of old artefacts (from the pre-writing period) in the new context of scribal schools; the algorithms for calculating the reciprocal[2] of a regular number, evidencing, in this context, the mastering of a complex and efficient system of artefacts.

5.3 The Computation Practices and Their Support in Scribal Schools

In this section, I will situate the importance of scribal schools as an essential structure for learning/teaching writing,[3] the importance of writing as an essential means for communicating and thinking, and I evidence the importance of artefacts used for writing and computing. These three elements are interrelated: the scribes were the persons mastering the art of writing, essential for writing and reading administrative texts, or for calculating area and taxes; the Sumerian name for 'tablet' is DUB, for 'scribe' is DUB.SAR, meaning 'the one who writes on tablets'; for 'scribal school' is É.DUB.BA, meaning 'the house of the tablets'. The schools are well described by Veldhuis (1997) in his study of Elementary education at Nippur (one of the main cities in this area for this period). From a number of literary texts, scribal schools appear as an institution supported by aristocracy, focusing on

[1] We would like here to greatly thank Christine Proust, historian of mathematics specialist of this period, for her precious advices, particularly about relevant references, and her careful re-reading of this chapter; Ghislaine Gueudet, for her re-reading on an advanced version of the chapter.

[2] Reciprocal of x stands here for $1/x$.

[3] The question of 'who was allowed to attend a scribal school?' is essential to evaluate the scope of writing in society. Veldhuis (1997, p. 27) gives some information about it: 'Admittance was not restricted to members of clerical families. This is shown [...] by two kinds of evidence. First, the teacher was not paid by state or temple, but by the parents of the pupil. Payment by the parents is attested in the literary text called Schooldays. Payment by state or temple [...] would have left traces in official documents, which is not the case. Second, a few girls attended school. Both points are [...] indications of a certain freedom of choice, and a non-mechanistic procedure for admission. One must admit, however, that this freedom of choice must have been restricted to the happy few'. This suggests that the transition from memorisation to writing concerns a quite restricted sphere of the Mesopotamian society, that is not the case for translation era from paper to digital support.

the art of writing, and where the learning of computation is essential, see for example the following text where the king Šulgi describes his childhood:

> When I was young I learned at school the scribal art on the tablets of Sumer and Akkad.
> Among the highborn no one could write like me.
> Where people go for instruction in the scribal art there I mastered completely subtraction, addition, calculating, and accounting.
> The fair Nanibgal Nisaba[4] provided me lavishly with knowledge and understanding.
> I am a meticulous scribe who does not miss a thing! (Veldhuis, 1997, p. 24)

Scribal schools appeared with the development of writing as an essential support for communicating. We know from Goody (1977) the importance of writing for cognitive and intellectual development. Speech has no spatial aspect, but writing has. The writing conditions knowledge into formats in one dimension (list) or two dimensions (tables), leading to what Goody names a 'graphic reason'.[5]

The spatial aspect in this period took the form of *clay tablets* (see an example Fig. 5.1), containing texts, lists and tables. Veldhuis (1997, p. 28) distinguished, for the tablets coming from Nippur and concerning elementary learning, four types: Type I tablets are large tablets containing a long text, continuously and densely inscribed on the obverse and on the reverse[6]; Type II tablets contain different texts on the obverse and on the reverse. On the obverse, a model was noted in an archaic style by a master,[7] and copied once or twice by a student; the copies were sometimes traced and erased repeatedly.[8] On the reverse, a dense text was written by heart by a student; Type III tablets are small rectangular tablets containing a short extract, often a multiplication table; Type IV tablets are small square or round tablets, containing a short exercise.

The set of lists and tables to be learnt constitutes the basis of the Mesopotamian curriculum, as it has been reconstructed by the historians (Table 5.1).

Students began by learning metrological lists[9] and finished by learning the table of roots. The analysis of the structure of clay tablets (see Fig. 5.1) evidences a part

[4] Nisaba is the patroness of the scribal schools and the goddess of writing and mathematics.

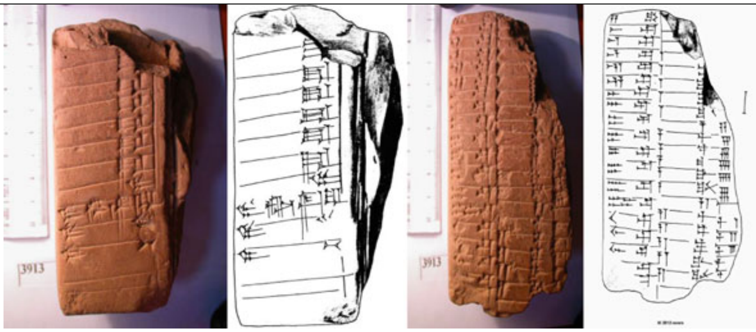
[5] Bachimont (2010) oppose this 'graphic reason' (linked to the writing era) to the 'digital reason' of the digital era. The digital reason allows the gathering in the same space of heterogeneous contents, and a multidimensional writing and reading (thanks to hyperlinks). Bachimont underlines the essential function of the supports of knowledge: they are not only the *consequence*, but also the *cause* of knowledge.

[6] Obverse and reverse stand, for the Assyriologists, for front and back of the tablet.

[7] We use the term of master following Proust's choice: 'Since we ignore the exact nature of the scholars' charge, I prefer to refer to them as 'masters' rather than as 'teachers', a term which could implicitly suggest that teaching at the elementary level was their unique activity' (Proust 2012a, 2012b, p. 163). The persons learning in scribal schools are called in this paper 'students', for reasons of facility, instead of apprentice scribes.

[8] In order to erase signs impressed in wet clay, scribes simply rub them lightly with their finger. Tablets bear often fingerprints and erased signs covered by others.

[9] The metrological lists are enumerations of measures of weight, area or length. The metrological tables consist of the same items in the same order, but each measurement is associated with a number written in sexagesimal place value notation: they constitute tables of conversion between quantities and 'pure' numbers.



The obverse contains a Sumerian lexical list, including mathematical terms regarding volume calculations. The reverse contains a list of measures of capacity. The right side of the tablet, which contained student copies, is lost. Note the characteristic appearance of the fracture, which results from the fact that the right columns have been written and erased several times, becoming thinner and forming a ledge.

Fig. 5.1 School tablet (Type II) from Nippur, courtesy Istanbul Archaeological Museum (Proust, 2012a, p. 168)

Table 5.1 Mathematical curriculum in Nippur (Proust, 2012a, p. 170)

Metrological lists	Capacity list
	Weight list
	Surface list
	Length list
Metrological tables	Capacity table
	Weight table
	Surface table
	Length table
	Height table
Division/multiplication tables	Reciprocal table
	Multiplication tables
	Square table
Tables of roots	Square root table
	Cubic root table

allocated to the master (resp. to the student) and gives access to the mode of learning lists and tables: ‘In a first step, the students learnt to write short excerpts, reproducing a model on the obverse of tablets, then they memorised the pronunciation, they recited the excerpt, and, in the last step, they reproduced by heart a large part of the list by writing it on the reverse of a tablet. Learning therefore inextricably combined writing and memorisation’ (Proust, 2012a, p. 171).

Let us analyse this crucial importance of the clay tablets (in addition of the conservation for historians that I have mentioned in the introduction).

It appears clearly that the nature of this writing support conditions the student’s work: the still fresh clay allows the student to write and erase what s/he wants to change (see Fig. 5.1). The dimensions of the tablets of Type IV (from 6 to 8 cm^2), dedicated to the work at home are called ‘im-šu’ (meaning tablets for hand) allow the student to bring them at home.[10] Kramer gave access to a text evidencing the importance of the tablet for student's work:

> Schoolboy, where did you go from earliest days
> I went to school.
> What did you do in school?
> I read my tablet, ate my lunch,
> prepared my tablet, wrote it, finished it; then
> my prepared lines were prepared for me
> (and in) the afternoon, my hand copies were prepared for me.
> Upon the school's dismissal, I went home,
> entered the house, (there) was my father sitting.
> I spoke to my father of my hand copies, then
> read the tablet to him, (and) my father was pleased;
> truly I found favour with my father. (Kramer, 1949, p. 205)

This text shows that one of the first things that a student had to do at school was the preparation of the tablet.[11] This tablet was also an essential support for the interaction between master and student, and between the student and his/her father.

For deepening this analysis, we have to consider, instead of *one* artefact, a *duo of artefacts*: a clay tablet, support of the writing and a *calame*[12] (in sumerian GI.DUB. BA, in akkadian *qan tuppi*(*m*), meaning 'reed of/for tablet'). Unlike tablets, no calame has been found till now. The existence of this artefact is attested by literary texts:

> You who speak as sweet as honey, whose name suits the mouth, longed-for husband of Inana, to whom Enki gave broad wisdom as a gift! Nisaba, the woman radiant with joy, the true woman, the scribe, the lady who knows everything, guides your fingers on the clay: she makes them put beautiful wedges on the tablets and adorns them with a golden stylus. Nisaba generously bestowed upon you the measuring rod, the surveyor's gleaming line, the yardstick, and the tablets which confer wisdom. (ETCSL, 2-5-5-2)

The existence of calames and their properties are also attested by the shape of their traces on the tablets themselves. It was probably a piece of reed (Proust, 2007, p. 81), sometimes of bone or ivory, of wood or of metal, especially pointed or rounded at first, then with a flat triangular form, or beveled thereafter. The incision of this artefact in fresh clay makes it difficult to draw lines and curves and encourages the user to draw short segments. This gave the Mesopotamian cuneiform writings a distinctive appearance (see Fig. 5.1). One must first plant a tip,

[10] The importance of such handheld device for appropriation by students is certainly crucial, as evidenced, in a recent period, for the purpose of mathematics teaching, by the use of handheld calculators (Trouche & Drijvers, 2010).

[11] The making of clay tablets, particularly those used in schools, was an important aspect of the technology of writing in this period and this geographical era. Bread of fine clay for tablets have been found, stored in jars (Suse), or cavities (palace of Mari). Clay is an abundant material in the Mesopotamian alluvial plain. But the clay used for writing had to be very pure. They had to be degreased and refined to prevent them from cracking as they dry (Charpin, 2002, p. 408).

[12] *Calame* (*pen* in Arabic) has been chosen by some historians to translate the Akkadian name. Other translation used: *stylus*.

giving the shape of a wedge, and then draw a line generally following (certain signs being simple wedges). The incision of signs on a malleable media finally gives not a flat writing like that obtained with ink and paper, but an embossed writing, and signs should be read with lighting that allows the reader to identify all incisions in order to avoid misinterpretation.[13]

The most used Mesopotamian numeration system, following the system used, in this region, before the writing era, was sexagesimal. In mathematical texts, the numbers are made of sequences of digits following a positional principle in base 60: each sign noted in a given place represents 60 times the same sign noted in the previous place (on its right).[14] Using this duo of artefacts for writing numbers, easily and without ambiguity, leads to the introduction of a minimum number of well-contrasted signs, actually two signs were enough: ones (vertical wedges 𒁹) and tens (oblique wedges 𒌋),[15] concatenated to represent the 59 digits used in the sexagesimal system (as 0 did not exist in this period). Proust (2007) presents the usual layout of these numbers, aggregated by a maximum three figures, to allow for rapid reading (see Table 5.2).

Several wedges are thus combined for writing numbers, with precise rules:

- If vertical wedges are written at the right of oblique wedges, they are at the same position; for example 𒌋𒐖 stands for 12 in our numeration system.[16]
- If vertical wedges are written at the left of oblique wedges, they are at an upper position, for example 𒐖𒌋 stands for 130 ($2 \times 60 + 10$). It is transcribed by the historians as 2.10.
- The concatenation has to be considered carefully: 𒐖 stands for 2, and 𒁹 𒁹 stands for 1.1 (i.e. 61 in our decimal positional system).

We are now able to analyse an exercise written on a tablet (Fig. 5.2).

[13] Lavoie (1994) analyses also the importance of the artefact for writing in another context: the passage of the quill of goose to the quill of iron in the primary schools, at the beginning of the twentieth century, in Québec.

[14] Among the Mesopotamian versions of sexagesimal numeration systems, there is only one which is positional, and this is this one which has been developed/used in the scribal schools.

[15] One can hypothesis that these two figures are the written transpositions of token used for computing before the writing era, see Fig. 5.5.

[16] In the Old Babylonian period, the cuneiform writing did not allow to distinguish 12 and 10.2. This ambiguity of the notation created errors, and was corrected in later period by the use of a new sign, to denote the absence of digit. In this improved system, 𒌋 𒐖 stands for 12, and 𒌋 𒐖 stands for 10.2.)

Table 5.2 The ergonomic display of the numbers (Proust, 2007, p. 74)

Units									
	1	2	3	4	5	6	7	8	9
Tens									
	10	20	30	40	50				

Tablet UM 29-15-192 (Neugebauer & Sachs 1984)

Hand copy made by Proust, personal communication

UM 29-15-192 -Transcription	Translation	Interpretation
[2]0 20 6.40	20 x 20 = 6.40	
2 šu-si ib$_2$-si$_8$ --------------------- a-ša$_3$-bi en-nam --------------------- a-ša$_3$-bi igi-3-gal$_2$ še-kam ==============	2 šu-si the side of the square What is its area? Its surface is 1/3 še [a *šu-si* (= a finger) is a length measuring unit a *še* (= a grain) is an area measuring unit]	2 šu-si → 20 20 x 20 = 6.40 6.40 → 1/3 še

Fig. 5.2 A tablet (type IV), its picture, hand-made copy, translation and interpretation (Proust, 2007, p. 193)

The layout of the tablet (Fig. 5.2) shows two distinct places: a place for *computation* (in the upper left area), following the positional system, and a place for *quantification* (in the lower right area), giving the text of the problem in terms of unit of measure and area. Then the student's work can be reconstituted:

– In the lower right place, *using* the metrological tables of lengths for converting length measurements in numbers.

– In the upper left place, *using* the multiplication tables for making the multiplication.[17]
– Back to the lower right place, *using* the metrological tables of area for converting the number in area measurement.

When I say '*using* metrological, or multiplication tables', it has to be understood in a large way: I have explained above the importance of reading and memorisation of such tables, fundamental elements of scribal school practices. For performing such computations (Fig. 5.2), students had certainly to mobilise memorised results from their learning of tables.

I have described, in this section, a set of artefacts used for learning mathematics: symbolic artefacts (as the sexagesimal positional numeration system), written artefacts (as the wedges), material artefacts, some of them have been preserved for us (as the clay tablets of different types), but evidence for some of them are suggested by their traces (calames). Are we sure that this enumeration is exhaustive? We will see in the following section that the answer is probably no.

5.4 Evidencing Computing Artefacts Complementing the Usage of Tablets and Memory

For Proust (2012a, p. 173), 'the resources of the masters [. . .] might have included a complex system of written texts, memorised texts, calculation devices and various communicational processes, but only the written artefacts reached us. We have then to reconstruct a rich environment from truncated evidence'. This reconstruction can rely on three arguments: the necessity of artefacts outside of the tablets for doing intermediate computations, the interpretation of frequent similar errors in the tablets, and the persistence of artefacts coming from the pre-writing era.

Firstly, the necessity of artefacts dedicated to these intermediate computations, for too big multiplications. Some tablets (see Fig. 5.3) show indeed important multiplications without any intermediate results.

We could imagine that such intermediate computation, supported by a given algorithm, could have been made on a 'draft tablet', but such tablet had never been found. We could also imagine that this draft could have been made on the tablet bearing the problem itself, then erased: a careful analysis of the tablet suggests that this was not the case here. One possibility is the presence of an artefact dedicated to such computation, that is, not a clay tablet, but made of a material, which vanished over time due to its nature (lexical evidence suggests that this device was made of wood, cf. Lieberman, 1980).

[17] An application 'mesocalc' has been developed par Mélès for doing such computations, which is useful for a better understanding of this system and a reading of the tablets: http://baptiste.meles.free.fr/site/mesocalc.html#multiplication.

Fig. 5.3 An example of computation difficult to do mentally (Proust, 2007, p. 168)—picture: courtesy Archaeological Museums of Istanbul

The second argument for the use of such a disappeared artefact comes from the careful analysis of the display of the computation on the tablets. Proust (2000) presents a table of successive doublings of the initial term 2.5 (see Table 5.3).

Something strange appears line 21, i.e. as soon as the writing of the number exceeds 5 positions: this writing is split into two parts, separated by a sign (a vertical wedge and an oblique one), and these two parts are separately doubled. This writing of the big numbers in two parts needs afterwards a *reattachment*, taking into account the relative sexagesimal positions of the digits, and this reattachment could explain a number of errors found through tablets during the whole cuneiform history. For example, the following error has been discovered in a tablet (300 BCE), about the computation of the reciprocal of 1.16.53.12.11.15 (Proust, 2000, p. 4). The result displayed is 46.49.19.**54.58**.53.20, instead of 46.49.19.**40.14.48**.53.20 (the curious reader could use the application Mesocalc, see Footnote 13, to check it. . .). The error could derived from a wrong reattachment of separate reciprocal computation of two parts of the number, leading to 46.49.19.**40 and 14.48**.53.20: instead of concatenate these two numbers, the scribe had added the two proximate digits, 40 and 14, giving 54 (otherwise an error of copy could explain the writing of 58 instead of 48). The repetition of such error could be explained by two computations using an artefact other than a tablet (and therefore without writing), and then performing a mental operation of reattachment of the two numbers leading to the written result (which is sometimes an error).

The third argument suggesting the existence of a non written artefact is the persistence of ancient artefacts, *the tokens*, during the period of the cuneiform writing. This persistence can be supported by the large presence of these artefacts in the Ancient Near East just before—and during—the use of writing on clay tablets. Tokens, that were small objects (Fig. 5.4), made of clay, modelled into many shapes such as cones, spheres, cylinders, disks and tetrahedrons are used for counting. Studying them brings us to a period 5000 years before the present day:

Table 5.3 The successive doubling of 2.5: 2.5 times 2 makes 4.10, etc. (Proust, 2000, p. 300)

Line	Obverse of the tablet	Line	Reverse of the tablet
1	2.5	21	10 + 6.48.53.20
2	4.10	22	12 + 13.37.46.40
3	8.20	23	40 + 27.15.33.20
4	16.40	24	1.20 + 54.31.6.40
5	33.20	25	2.**40** + **1**.49.2.13.20
6	1.6.40	26	5.**20** + **3**.38.4.26.40
7	2.13.20	27	10.**40** + **7**.16.8.53.20
8	4.26.40	28	21.**20** + **14**.32.17.46.40
9	8.53.20	29	42.**40** + **29**.4.35.33.20
10	17.46.40	30	1.25.**20** + **58**.9.11.6.40
11	35.33.20	31	2.50.**40**(+)1.**56**.18.22.13.20
12	1.11.6.40	32	5.41.**20**(+)3.**52**.36.44.26.40
13	2.22.13.20	33	11.22.**40**(+)7.**45**.13.28.53.20
14	4.44.26.40	34	22.45.**20**(+)15.**30**.26.57.46.40
15	9.28.53.20	35	45.30.**40**(+)31.**0**.53.55.33.20
16	18.57.46.40	36	1.13.1.**20**(+)1.2.**1**.47.51.6.40
17	37.55.33.20	37	
18	1.15.51.6.40	38	
19	2.31.42.13.20	39	
20	5.3.24.26.40	40	

Fig. 5.4 Complex tokens representing (*above*, from *right* to *left*) one sheep, one jar of oil, one ingot of metal, one garment, (*Below*, from *right* to *left*) one garment, one honeycomb, from Susa, Iran, ca. 3300 BC Courtesy Musée du Louvre, Département des Antiquités Orientales, Paris (Schmandt-Besserat, 2009, p. 148)

> Tokens started to appear in the Fertile Crescent of the Near East, from Syria to Iran, around 7500 BC. This means that counting coincided with farming, and in particular, the redistribution economy that derived from agriculture. Tokens were probably used to pool together community surpluses for the preparation of the religious festivals that constituted the lynchpin of the redistribution economy. The tokens helped leaders to keep track of the goods in kind collected and their redistribution as offerings to the gods and the various community needs. (Schmandt-Besserat, 2009, p. 146)

One could distinguish two major trends in the evolution of tokens:

– A first period of diversification, the tokens having to represent, on a symbolic and imaginative way, the variety of 'things' to be counted (Fig. 5.4):

> the number of token shapes, which was limited to about 12 around 7500 BC, increased to some 350 around 3500 BC, when urban workshops started contributing to the redistribution economy. Some of the new tokens stood for raw materials such as wool and metal while others represented finished products, among them textiles, garments, jewelry, bread, beer and honey. (Schmandt-Besserat, 2009, p. 148)

– A second period of abstraction, around 3000 BCE, linked to the emerging of writing (Fig. 5.5):

> plurality was no longer indicated by one-to-one correspondence. The number of jars of oil was not shown by repeating the sign for "jar of oil" as many times as the number of units to record. The sign for "jar of oil" was preceded by numerals—signs indicating numbers. Surprisingly, no new signs were created to symbolize the numerals but rather the impressed signs for grain took on a numerical value. The wedge that formerly represented a small measure of grain came to mean "1" and the circular sign, formerly representing a large measure of grain meant "10". (Schmandt-Besserat, 2009, p. 148)

The shape of the signs, sketched with a pointed calame, is obviously very close to the shape of the vertical and oblique wedge characteristics of the cuneiform writing, the vertical wedges standing for one, and the oblique wedges standing for ten. Nevertheless, it should be a mistake to imagine that the token had progressively vanished for leaving room to writing on clay tablets. Till now the researchers hypothesise that various forms of cohabitation had existed between token and clay tablets. Some traces of this cohabitation had been evidenced: material cohabitation as for these kinds of spherical envelop (Fig. 5.6) containing inside circular and wedge tokens, and keeping their traces on its surface[18]; symbolic cohabitation

[18] The interpretation of this envelop cannot be done out of its cultural environment. A modern eye could interpret this sphere full of token as a typical artefact for learning numbers in schools. At the opposite, these clay purses have been interpreted by historians as accounting artefacts: 'By 3300 BC, tokens were still the only accounting device to manage the redistribution economy that was now administered at the temple by priestly rulers. The communal offerings in kind for the preparation of festivals continued, but the types of goods, their amounts, and the frequency of delivery to the temple became regulated, and non-compliance was penalized. The response to the new challenge was the invention of envelopes where tokens representing a delinquent account could be kept safely until the debt was paid. The tokens standing for the amounts due were placed in hollow clay balls and, in order to show the content of the envelopes, the accountants created

Fig. 5.5 Pictographic tablet featuring an account of 33 measures of oil, (circular = 10, wedges = 1) from Godin Tepe, Iran, ca. 3100 BC Courtesy Dr. T. Cuyler Young, Royal Ontario Museum, Toronto, Canada (Schmandt-Besserat, 2009, p. 150)

as evidenced by a double system of computation on clay tablets (see Fig. 5.2, abstract numbers vs. quantities).

This hypothesis has been recently validated by a very important discovery. Excavations in South eastern Turkey have uncovered a corpus of tokens dating to the first millennium BCE:

> These tokens are found in association with a range of other artefacts of administrative culture—tablets, dockets, sealings and weights—in a manner which indicates that they had cognitive value concurrent with the cuneiform writing system and suggests that tokens were an important tool in Neo-Assyrian imperial administration. (MacGinnis, Willis Monroe, Wicke, & Matney, 2014, p. 289)

MacGinnis et al. (2014) show how these tokens, under different forms (Fig. 5.7) could intervene in working with tablets, for administrative purposes, in a complementary way.

> They represent a system of accounting that worked in conjunction with tablets to allow for a more flexible type of record keeping that could be achieved by the use of tablets alone. Specifically they provided a system of movable numbers that allowed for stock to be moved and accounts to be modified and updated without committing anything to writing. At the same time, because these tokens exist alongside a contemporary cache of administrative documents, they illustrate the concurrent use of clay tokens and tablets. (MacGinnis et al., 2014, p. 303)

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markings by impressing the tokens on the wet clay surface before enclosing them' (Schmandt-Besserat, 2009, p. 149).

Fig. 5.6 Envelope, tokens and corresponding markings, from Susa, Iran (Courtesy Musée du Louvre, Département des Antiquités Orientales)

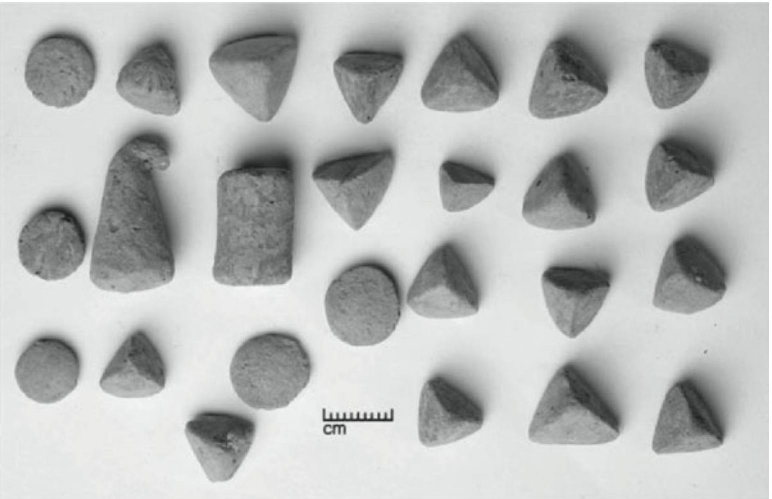

Fig. 5.7 A diversity of tokens intervening in computation (MacGinnis et al., 2014, p. 294)

It appears here two types of *conjunction* between tablets and token use: firstly the tokens provide a system of *movable numbers* allowing *updating without writing*. Secondly they could intervene for supporting, articulated with the use of tablets, a given computation. This articulation could be mastered by a single agent, or via the collaboration of different agents. MacGinnis et al. (p. 302) 'assume that under the trained scribes who wrote the cuneiform tablets were assistants helping to load and unload the grain and counting out transaction', using tokens. Finally, the computation results of a *flexible* combination of artefacts and agents.[19]

This discovery constitutes clearly a proof of the co-existence of written (clay tablets) and not written (token) artefacts for working with numbers during this period. The hypothesis of a device combining wood and clay token is still to be proved, beyond the different clues we have evoked in this section, but the reality of the system of artefacts, mainly tablets and tokens, supporting the practices and the learning of computation in scribal schools seems to be established. A last evidence comes from the analysis of a crucial algorithm, this of reciprocal computation, what we will examine in the next section.

5.5 Analysing the Algorithm for Calculating a Reciprocal, a Way for Entering the Spirit of Mesopotomian Computation

Calculating the reciprocal of A is essential for performing the division B/A as the multiplication $B \times A^{-1}$. Analysing the algorithm supporting this calculation opens an enlightening window on the Mesopotamian mathematics practices and knowledge. The following example is extracted from the tablet CBS 1215 (Fig. 5.9), which could come, according to an estimation of Proust (2012b), from the scribal schools of southern Mesopotamia, during the Old Babylonian period (beginning of the second millennium BCE). This is a multi-column tablet containing advanced mathematics (which is therefore out of the classification in four types, see Sect. 5.3). The existence of such tablets, in scribal schools, evidences the fact that the 'masters' (see Footnote 5) were not only teaching elementary mathematics, but worked also as scholars, for developing mathematics, exchanging texts between masters across the different schools.

I reconstitute below, in modern terms, the computation displayed on this tablet, analysed by Proust (2012b).[20]

The computation of a reciprocal only concerns regular numbers, i.e. in the sexagesimal numeration, numbers that are products of powers of 2, 3, and 5: only such numbers are present in the tablets displaying such a computation. The goal of the algorithm is to decompose the regular number at stake as the

(continued)

[19] We have to keep in mind that the context described by MacGinnis et al. is an administrative one. The computation, in such a context, can be based on highly specialised tasks assigned to different agents, as the learning is not an objective of this activity.

[20] For following the development of the computation, the reader could again use the application Mesocalc (see Footnote 14).

Table 5.4 Table of reciprocal of usual regular numbers, underlining the couples used in the computation of the reciprocal of 25.18.45

n	inv(n)
2	30
3	20
4	15
5	12
6	10
8	7.30
9	6.40
10	6
12	5
15	4
16	3.45
18	3.20
20	3
24	2.30
25	2.24
27	2.13.20
30	2
32	1.52.30
36	1.40
40	1.30
45	1.20
48	1.15
50	1.12
54	1.6.40
1.4	56.15
1.21	44.26.40

product (non unique) of regular numbers whose reciprocal is well known (this algorithm lies therefore on the property: 'the reciprocal of a product of numbers is the product of the reciprocals of these numbers').

The 'well known reciprocals' come from a table (Table 5.3) part of the curriculum (see Table 5.1, Sect. 5.2). Note that the digit 0 is not used in this sexagesimal numeration, therefore $2 \times 30 = 60$, i.e. 1.0, is noted as 1. The reciprocal of 2 is therefore 30.

Let us calculate, following the tablet (Fig. 5.9), the reciprocal of $A = 25.18.45$.

The second property supporting the algorithm is: 'if a regular number terminates the writing of A, then it is a regular factor in one decomposition of A'.

(continued)

Knowing that, we can now begin the computation (For following the development of the computation, the reader could again use the application Mesocalc (see Footnote 14)):

– First step, we isolate, in the final digits of A (thinking A as $25.15 + 3.45$), a number present in the table (3.45), which reciprocal is 16 (see Table 5.4).
– Second step, we try to write A as a product of n and 3.45; the number n is therefore equal to $A \times 16$ (which is the reciprocal of 3.35), i.e. $n = 6.45\ldots$

And we apply again the same technic for 6.45.

– First step, we isolate, in the final digits of this number, a number present in the table: 45, which reciprocal is 1.20 (see Table 5.4).
– Second step, we try to write 6.45 as a product of m and 45; the number m is therefore equal to 6.45×1.20 (which is the reciprocal of 45), i.e. $m = 9$.

The number 9 is present in the table of reciprocals, here is therefore the end of the algorithm.

Finally, the number A has been written as a product of three numbers belonging to the table: $A = 3.45 \times 45 \times 9$, and the reciprocal of A is the product of the reciprocal of these three numbers:

$1/A = 16 \times 1.20 \times 6.40 = 2.22.13.20$.

Analysing the way of applying this algorithm (personal communication of C. Proust), we could again question the presence of hidden artefacts, mobilising tokens. Understanding the cutting of the number 25.18.45 is indeed easier if we have in mind[21] a ‘token-based representation’ instead of a written representation (Fig. 5.8), i.e. if we consider a number, not as a succession of digits (here three positions of wedges) but as a grouping of tokens.

This hypothesis of a hidden artefact is supported also by the examination of the corresponding tablet displaying the computation (Fig. 5.9) in a very ergonomic layout. This extract details the calculation of the reciprocal of 5.55.57.25.18.45 (the curious reader could try to apply the algorithm from this number). The computation I have presented above is just a part of this calculation (see inside the highlighted rectangular). Following our observations in the previous section, we can imagine that this kind of sophisticated computation is supported par ‘an artefact outside the tablet’.

Finally, for performing efficient computations of this kind, the scribal school masters and advanced students had to combine a set of material and symbolic

[21] *Having in mind* could mean ‘using tokens, eventually integrated in a wooden device, to assist the computation’, as shown in Fig. 5.8; or ‘keeping the memory of old practices of computation based on token’. Remember, in another context, the Bachelard’s sentence, concerning the man of the twentieth century: ‘Même chez l’homme nouveau, il reste des vestiges du vieil homme. En nous, le XVIIIe siècle continue sa vie sourde…’ (Bachelard, 1934).

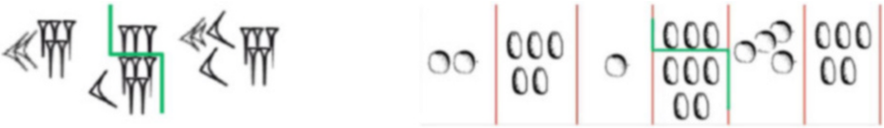

Fig. 5.8 Cutting of 25.18.45 in 25.15 + 3.35, in the sexagesimal numeration (*left*) and in a token representation (*right*)

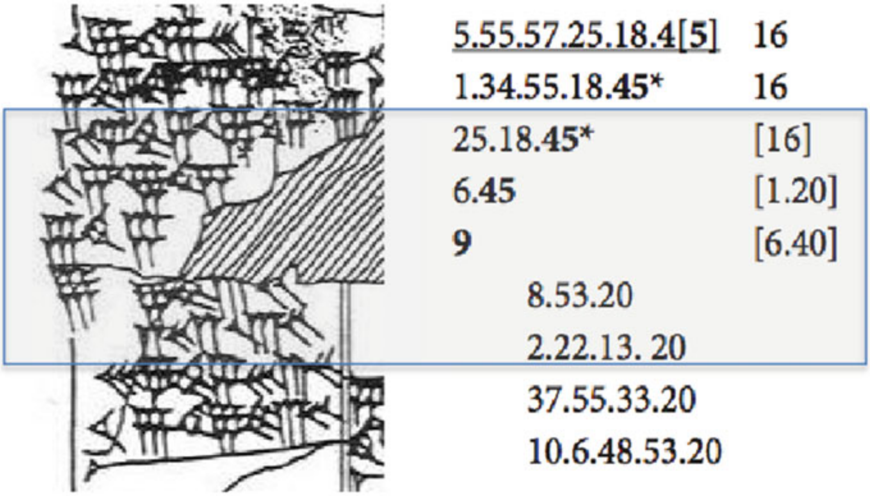

Fig. 5.9 Extract of tablet CBS 1215 (Sachs, 1947, copy Robson, 2000, p. 23), *left*, and its transcription, *right*, by Proust (2012b)

artefacts in their minds or/and in their hands. Learning to use these artefacts was, for them, a part of learning mathematics, the two modes of learning supporting one another (see the discussion of techniques and schemes in Chap. 10): conceptualisation and instrumentation are completely nested (Trouche, 2000).

5.6 Conclusion and Discussion

We have proposed in this chapter to have a look on a very rich period for the development of: mathematics; for learning mathematics; and for the learning on how mathematics was learnt and taught. We can draw, from this examination, supported by the historic research literature on this period, several observations.

Firstly, what appears clearly is *the importance of artefacts for supporting mathematics practices and learning*. We could say that the process of creating artefacts and the process of creating mathematics feed one another. The close analysis, from their traces on clay tablets, of the mathematics practices leads to

Fig. 5.10 The cohabitation between computation with Indian digits, and computation with abacus, during several centuries in France (Hébert, 2004)

the conjecture of the existence of disappeared wooden artefacts allowing to directly manipulate numbers through tokens. The combination of these artefacts allowed for the development of efficient methods of computation.

The second observation is that *new artefacts do not necessarily make old ones redundant*. We know from history that *phases of transition* between an old and a new artefact are *phases of cohabitation*, see the transition between abacus and Indian digits in France (Fig. 5.10), or the shorter transition between slide rule and calculator (Fig. 5.11). Once said that, it remains an important issue: is the use of old artefacts a brake, delaying the integration of new artefacts (i.e. does the death of the former condition the integration of the new artefact)? Or does the integration of a new artefact lead to the establishment of a new equilibrium in the conduct of computations? What we learn from the material of scribal schools, more than 1000 years after the invention of writing, is that writing did not replace, in schools, memorisation and that tablets did not replace tokens: on the contrary the combination of different means supporting calculations seems to have led to a constitution of a few articulated levels of mathematics practices: manipulating numbers through tokens, memorising tables and intermediate results, developing and using highly structured algorithms dedicated to specific mathematical tasks, expressed in a very few lines for saving place on clay tablets. In a very different context, that of the modern period of integration of powerful calculators in mathematics teaching, we

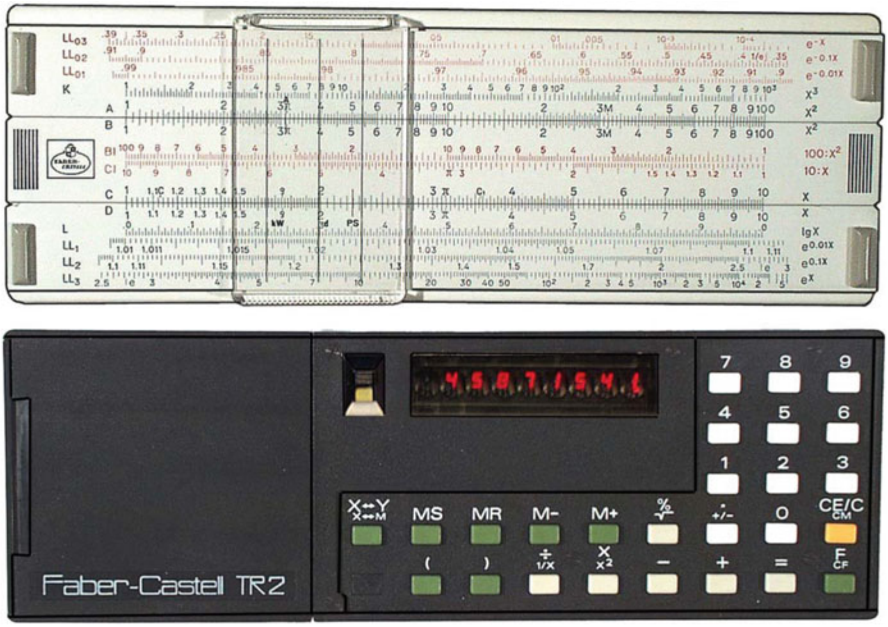

Fig. 5.11 The cohabitation between the slide rule and the calculators, two faces of a same artefact, during several years in France (Trouche, 2005)

find again what Artigue (2005) named 'the intelligence of computing', connected, for her, to three structuring abilities:

– The *relevant use of given repertoires*, in the case of scribal schools, the tables.
– The *flexibility of computing*, that is the ability of switching between several *frames*, *semiotic registers* (Mariotti and Maracci, 2012) or *points of view* (see Chap. 8), in the case of scribal schools the switching between the writing on clay tablets and the manipulation of tokens via wooden device; or the switching between computations on numbers, and computations on quantities; or the switching between a wedges-based representation and a token-based representation.
– The *ability to combine genericity and specificity*, that is the ability, for each kind of computation, to use both global properties of the computation, and specific properties linked to the domain, what we have observed in the case computing a reciprocal (see also, for the modes of reasoning beyond a given computation, Høyrup, 2002).

This intelligence of computation may also be developed by the combination of artefacts artificially reconstructed for pedagogical purpose: Maschietto and Soury-Lavergne (2013) evidenced the interest of introducing, for studying the *decimal* positional numeration system in primary schools, both a physical artefact (a reconstruction of the first calculator designed by Pascal in 1652) and its digital

Fig. 5.12 A combination of two twin artefacts for learning mathematics, manipulation through a digital tablet on the one side, direct manipulation on the other side (Maschietto & Soury-Lavergne, 2013)

counterpart (see Fig. 5.12). There is on one side a tablet—here a digital one—allowing to turn the gear representing the units by clicking on an arrow, and on the other side a tangible device allowing to directly manipulate the gear. The digital tablet allows to combine two semiotic registers: digits (driven by the gears), and tokens (see Fig. 5.12, 12 tokens displayed on the screen). The tablet and the tangible device, even if they come from the same gear principle, lead to different gestures, and then different representations of the process of constructing numbers. These combinations of registers and gestures aim to support an essentially difficult transition in the learning of the decimal positional numeration (Bednarz & Janvier, 1984), the transition between the conception of 'a number as a sequence of digits read from left to right', and a conception of 'a number as a sequence of digits giving from right to left, the units, tens, hundreds, thousands, constituting the given number'. When history and didactics meet. . .

The third observation concerns the analysis of the masters 'resource system' (see Chap. 13), i.e. the set of 'things' which support their work in scribal schools. We have some information on this system, considering two faces of the masters' work: the face 'the master as a teacher', via the students' resources, a great variety of tables enlightening the curriculum and the way of learning basic computation; the face 'the master as a scholar', via tablets as CBS1215 (see Fig. 5.9), enlightening the type of sophisticated mathematical work the master could perform and share. There are a relatively few such tablets that had been found: that is easily understandable, as there are essentially tablets of unskilled students that had been trapped in wall, and that had been then preserved till now (see Sect. 5.2). The tablets integrating a rich mathematical content that had probably travelled in the whole region, from school to school, what could explain the very standardised character of the scribal school curriculum and teaching material. As for the missing artefacts that the historical research supports the existence, we could pledge the reality of missing 'lived resources' (Gueudet, Pepin, & Trouche, 2012) for scribal school masters, including tablets aiming to generate tablets of exercises, tablets describing the mode of combining artefacts, tablets describing the art of teaching, tablets with masters epistolary correspondence. . .

There are probably, on this subject, fruitful possible interactions between historians and researchers in mathematics education, the study of the masters resources in scribal schools enlightening the study of the master's resources in today schools, and vice versa, even if the contexts deeply differ.

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