Chapter 14
Tools and Mathematics in the Real World

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14.1 Introduction

This chapter has two main foci: (1) the use of mathematics in out-of-school[1] mathematical practices; (2) making school mathematics relevant to activities beyond mathematics classrooms (which I shall call 'out-of-school' mathematics/ practices). Both foci are important issues in mathematics education and both are problematic issues. Both foci, of course, will be investigated with special regard to tool use. This chapter has four sections. The two central sections address the two main foci. The opening section sets the scene with an historical account of ways that mathematics has been subdivided with regard to its application(s). The last section considers problem issues.

14.2 Can Mathematics Be Subdivided with Regard to Its Application(s)?

Mathematics has been subdivided in various ways over two millennia. A subdivision, with regard to the application of mathematics, that has been used in Western mathematics for over 100 years, is that between pure and applied mathematics. This sounds like a promising way into the two main foci of this chapter so I start by considering this division: to what extent is it a real division; does tool use vary over this division?

There is a sense in which the division pure and applied mathematics is a real division between mathematical activity for intrinsic or extrinsic purposes. To take an elementary example, if I am teaching (or writing about the teaching of) 456 + 78 using the standard written algorithm, then I will pay careful attention to the fact that

[1] I use the term 'school' loosely to denote any educational institution.

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the ‘1’ I carry into the tens column (when I add 6 and 8) is not the number 1 but 1 unit of 10. But if I am doing this addition for a purpose other than doing mathematics, say, in checking my accounts, then this explicit attention to place value within a calculation is not important (in activity-theoretic terms, see Sect. 9.2, the object of the activity is different). This has immediate implications for tool use in mathematical activity. In the first activity the focus is on the correct use of a specific mathematical tool (a specific algorithm). In the second activity the focus is on obtaining the correct mathematical result and the tool I use to get the answer, as long as it gives the correct answer, does not matter a great deal (an abacus or a calculator or the standard written algorithm will do).[2] But there is also a sense in which the division between pure and applied mathematics is not a ‘natural’ division but a cultural–historical division; and this leads to a brief tour into the history of mathematics and mathematics education: the Ancient Greeks to the nineteenth century (Sect. 14.2.1); the twentieth century (Sect. 14.2.2).

14.2.1 Subdivision of Mathematics over Time

Fauvel and Gray (1987, p. 56) write of the ancient Greek *quadrivium*, ‘the four-part classification of mathematical sciences ... into arithmetic, music, geometry and astronomy’ and claim that this ‘came to constitute a major part of the liberal arts curriculum of medieval universities’. This four-part classification, however, did concern ‘pure’ mathematics for Plato writes (see Fauvel & Gray, 1987, p. 69) of arithmetic, ‘what a subtle and useful instrument it is for our purpose, if one studies it for the sake of knowledge and not for commercial ends’. In seventeenth century Europe the division of mathematics was between pure and mixed mathematics. Francis Bacon wrote in 1603 (see Fauvel & Gray, 1987, pp. 290–291):

> To the pure mathematics are those sciences belonging which handle quantity determinate, merely severed from any axioms of natural philosophy; and these are two, geometry and arithmetic ... Mixed hath for subject some axioms or parts of natural philosophy ... For many parts of nature can neither be invented with sufficient subtility ... nor accommodated unto use with sufficient dexterity, without the aid and intervening of mathematics; of which sort are perspective, music, astronomy, cosmography, architecture, enginery, and divers others ... there cannot fail to be more kinds of them, as nature grows more disclosed.

This division was a part of Bacon’s tree of knowledge where the natural sciences were divided into physics and metaphysics and metaphysics divided into pure and mixed mathematics. Academic debate in the succeeding two centuries, according to Brown (1991), was subject to local variation as new areas of mathematics developed but largely retained Bacon’s distinction. For example, in mid-eighteenth century France, D’Alembert, writing in Diderot’s *Encyclopédie*, placed the new

[2] I think this ‘use of a specific tool in pure mathematics’ and ‘use of a range of tools in applied mathematics’ is a fairly common phenomena but I do not claim that it is a universal phenomenon.

field of probability (analysis of games of chance) into mixed mathematics but he placed the new field of calculus within pure mathematics.

The decline of the term 'mixed mathematics' occurred in the nineteenth century; Brown (1991) notes that the eighth edition of the *Encyclopedia Britannica* (1853–1860) used the pure-mixed classification but in the ninth edition (1875–1889) this was changed to 'pure' and 'applied' mathematics. Behind these classifications is ideology. There is a strong elitist ideology of 'learned men' behind writings from Plato to D'Alembert. Brown (1991, p. 84) writes:

> The mathematician was concerned with doing mathematics; the philosophe with analysing its importance to society. Who best to write about "mixed mathematics" than the scholar who was both a "geometer" and a "philosopher"? Neither Daniel Bernoulli, Euler, Lagrange, nor Laplace could be considered men of letters. That left only Condorcet and D'Alembert.

But ideology and mathematics shifted their foundations in Europe during the nineteenth century. Non-Euclidean geometries emerged which eroded geometry's claim as an a priori constructive field and science was viewed through positivist empirical eyes. 'By 1875 theories were no longer "mixed" with experience, they were "applied" to experience' (Brown, 1991, p. 102).

14.2.2 Subdivisions of Mathematics in the Twentieth Century

So we enter the twentieth century with a division, in the West, between pure and applied mathematics. In the Soviet Union, however, Vygotsky (cf Sect. 7.3) introduced a division between *everyday* and *scientific* concepts. Vygotsky did not introduce this distinction with mathematics in mind but it is, from a Vygotskian perspective, applicable to mathematics; in practical everyday mathematical activity an addition such as 456 + 78 will likely involve 'things' (such as units of currency) but in *mathematics* addition comes with a history (the culture of mathematics) and mathematicians add numbers, not things. As noted in Sect. 7.3, Scott, Mortimer, and Ametller (2011, p. 6), in writing of Vygotsky's distinction, note 'scientific concepts are taken to be the products of specific scientific communities and constitute part of the disciplinary knowledge of that community'; 'the world is flat' was once a scientific concept. In a cultural vein similar (but not identical) to Vygotsky's, Bishop's (1988) study of mathematical enculturation differentiated between 'mathematics' and 'Mathematics':

> the mathematics which is exemplified by Kline's *Mathematics in Western Culture* is a particular variant of mathematics, developed through the ages by various societies. I shall characterise it as 'Mathematics' with a capital 'M'. (Bishop, 1988, p. 19)

These cultural approaches do not directly address the distinction between pure and applied mathematics but are concerned with the division between types of mathematical activity. By the end of the twentieth century, with mathematics education established as an academic discipline (see Sect. 7.2), scholars in this field made further divisions within mathematical activity. Blum and Niss (1991) is an interesting example of this because it represents the collective thoughts of a conference working group and presents 'a pragmatic attempt to give some working definitions' (Blum & Niss, 1991, p. 37). It considers two types of mathematical problems:

> It is characteristic of an applied mathematical problem that the situation and the questions defining it belong to some segment of the real world and allow some mathematical concepts, methods and results to become involved. By real world we mean the "rest of the world" outside mathematics, i.e. school or university subjects or disciplines different from mathematics, or everyday life and the world around us. In contrast, with a purely mathematical problem the defining situation is entirely embedded in some mathematical universe. This does not prevent pure problems from arising from applied ones, but as soon as they are lifted out of the extra-mathematical context which generated them they are no longer applied. (Blum & Niss, 1991, pp. 37–38)

The starting point for Blum and Niss (1991) is a 'real problem situation':

> This situation has to be simplified, idealized, structured, . . . This leads to a *real model* of the original situation . . . [which has to be] has to be mathematized, i.e. its data, concepts, relations, conditions and assumptions are to be translated into mathematics . . . [and] results have to be re-translated into the real world . . . real problem situations can also be called applications. . . mathematical models . . . can be seen as belonging to applied mathematics. Of course, this definition does not imply a strict separation between "pure" and "applied" mathematics. (Blum & Niss, 1991, pp. 38–40)

So, we are back to the division between pure and applied mathematics but the division is not a strict one and there is also a slight difference between 'applications' and 'modelling'. The translation and re-translation that Blum and Niss speak of is often presented in a diagram, like the two leftmost columns in Fig. 14.1, in mathematics education literature on applications and modelling of mathematics. With reference to these two columns Fig. 14.1, the left column represents 'reality' (the real world) and the right column represents 'mathematics' (the mathematical world). The diagram represents a cycle: situation → mathematical model → mathematical results → real result → compare with the situation and

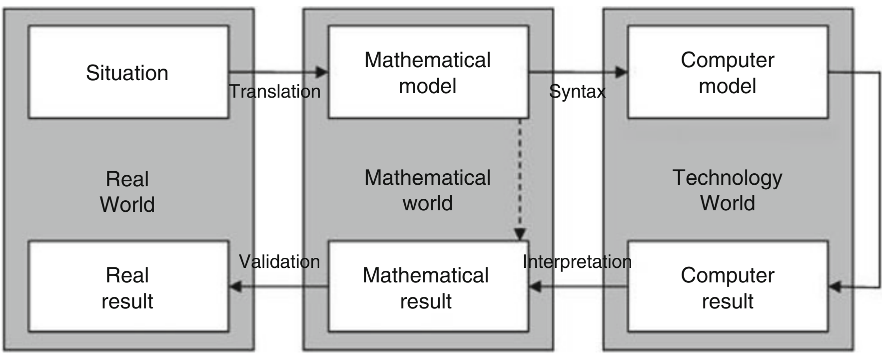

Fig. 14.1 Siller and Greefrath's (2010) extended modelling cycle

note discrepancies → adjust mathematical model → mathematical results → etc. Of the many comments that can be made on this modelling cycle, I make three. First, it does appear (albeit in an oversimplified way) to approximate what goes on in applied mathematical problem solving. But, second, what does it represent? Is it supposed to be descriptive (of the work of mathematicians) or prescriptive (for educational purposes)? Blum and Niss (1991) appear to regard it as descriptive, 'This leads to a *real model* of the original situation' (Blum & Niss, 1991, p. 38), but my own experience of the first step is more akin to 'situation ↔ mathematical model' than it is to 'situation → mathematical model' (i.e. there is a lot of 'fiddling' with the mathematical model). My third point regards tools and is simply that the two column modelling cycle, which is what is usually offered, does not include tools and I consider this further below.

It is curious, from the point of view of tool use in mathematics, that the word 'tool' does not feature in Blum and Niss' account except within the context of talking about computers as a tool. I use the word 'curious' in the sense that Arthur Conan Doyle ascribes to Sherlock Holmes in the *Memoirs of Sherlock Homes* where the detective is talking to a police inspector:

> 'Is there any point to which you would wish to draw my attention?'
> 'To the curious incident of the dog in the night-time.'
> 'The dog did nothing in the night-time.'
> 'That was the curious incident,' remarked Sherlock Holmes.

Computers are very interesting tools but it is curious that tools, other than computers, are not mentioned in Blum and Niss (1991) when tools (measuring artefacts, machinery for experiments, formulas as tools, etc.) are clearly important in modelling and applications of mathematics. I shall mention similar omissions at other points in this chapter, so I give it a name, 'tool blindness'—not seeing something until it *hits you in the face*. Computers are mentioned in the second part of Blum and Niss (1991) in relation to 'trends' and 'obstacles':

- With regard to professional modelling 'For several years it has been evident that computers form a highly powerful tool for the numerical and graphical treatment of mathematical applications and models' (Blum & Niss, 1991, p. 52).
- With regard to mathematical education, computers allow 'More complex applied problems . . . relief from tedious routine . . . Problems can be analysed and understood better by varying parameters . . . [and] Problems which are inaccessible from a given theoretical basis . . .may be simulated numerically or graphically' (Blum & Niss, 1991, p. 58).

Siller and Greefrath's (2010) also focus on the place of computers in modelling (for educational purposes). They offer the first representation of the modelling cycle (to my knowledge) that includes tools (computers), see Fig. 14.1.

Whilst it is nice, from my tool perspective, to see a recognition of the place of technology in modelling, I am sceptical that there are three distinct worlds (as the presentation may suggest). The practice of modelling (be it in-school of out-of-school) is a reality. These three worlds seem to impose an 'unreal' partition of this practice.[3] The resolution of this problem issue may be simply to jettison the modelling cycle and look to real practice (in which tool use will be an integral part of the practice). I return to this point in the final section of this chapter and now consider out-of-school mathematical practices.

14.3 Out-of-School Mathematical Practices

Out-of-school mathematical practices cover an extensive field of activities and I must pare this field to keep this section manageable. Section 14.3.1 sets the scene by mapping the field. This mapping includes constructors (people who design technology) and operators (people who use technology) and Sects. 14.3.2 and 14.3.3 consider constructors and operators in turn. The final subsection looks at the place of computers in out-of-school mathematical practices because computers hold a prominent position in many of these practices in the twenty-first century.

14.3.1 Varieties of Out-of-School Mathematical Practices

There are many out-of-school mathematical practices—certainly too many to list. I will first attempt a map of the field and then consider a subset of Western workplace practices which have been a focus of research and address tool use and mathematics. My map of the field includes three divisions: leisure and work practices; levels of involvement with tools; and Western vs. ‘other’ practices.

The distinction between leisure and work practices is not a precise one since there are instances where such practices overlap (e.g. voluntary work). Leisure, considered as non-paid activity, includes domestic and recreational activity. Domestic activity includes practices which can have mathematical aspects such as: cooking, following a new recipe (which is an artefact which is used and is thus a tool by my Sect. 1.3 definition)—cooking also involves using a great many utensils (tools), some of which (e.g. weighing scales) are ‘pre-mathematicised’; monitoring household accounts, which is often facilitated by the tools available in e-banking; and domestic repairs such as drilling a hole (finding the right drill size and ‘feeling the right angle’ with your body). Recreational activity includes practices which can have mathematical aspects includes: travelling, buying e-tickets online and co-ordinating rail and flight schedules (artefacts); programming the recorder on your TV-media unit; performing music; and playing games (see considered in Chap. 19). Gameplay always has a mathematical aspect as games have rules (which are artefacts, ‘mediational means’ in the language of Wertsch—see

[3] Siller and Greefrath’s (2010, p. 2138) note, ‘The three different worlds shown in Fig. 2 are idealized; they influence each other.’

Sect. 13.3) and these rules include sequencing actions. Although some games (e.g. soccer) can be played with only an epsilon of mathematical activity in a similar way to which they were played 100 years ago, gameplay has increasingly been influenced by digital technology. This is certainly so in the case of digital games but it is also sometimes the case in games such as soccer where even some amateur teams use performance analysis software, which provide statistics on video-recorded motion analyses, to improve their gameplay.

My second division concerns levels of involvement with tools and I employ the language of Skovsmose (2005). Skovsmose is interested in ‘critical mathematics’ and technology and distinguishes between ‘constructors’, ‘operators’ and ‘consumers’. With regard to technology, constructors are professionals who design/develop technology, operators are those who use/manipulate the technology and consumers are people not involved in the construction or operation of the technology but are affected by it. For example, a manager and a computer scientist (constructors) may design/implement a new system of calculating wages, computer operators run the wage system and the consumer is affected with wages and a pay statement. My consideration of leisure practices above concern the consumer level of involvement with tools but this level is also common in workplace practices, especially amongst low paid workers. People’s encounters with tools/technology at the consumer level is typically as ‘black-boxes’, a term originally from cybernetics that refers to artefacts where the input–output relationship is hidden from the user. I write at a time when an international banking crisis is having a profound negative effect on the quality of life of many consumers of banking technology. Enabling people to critically engage with black-boxes is important to critical mathematics. This is one reason why mathematical modelling is considered important.

My third division concerns Western vs. ‘other’ practices. Western research in mathematics education is dominated by Western researchers researching Western contexts. A partial exception is what is commonly referred to as ‘ethnomathematics’. This was a new but rising area of research at the turn of the Millennium but it met problems. Ethnomathematics investigates ‘indigenous, socio-, informal, spontaneous, oral, hidden, implicit, and people’s mathematics’ (Gerdes, 1996, p. 909). Activities investigated, such as basket weaving in Mozambique, are characterised as being both highly context bound and highly creative (Gerdes, 1997). Ethnomathematics is not a non-Western phenomenon but concerns traditions in any locality, though reports on ethnomathematical activities are often written by people with a Western education reporting on the practices of those who have not received a Western education. Dowling (1998, p. 14) considers that these studies succeed in ‘celebrating non-European cultural practices only by describing them in European mathematical terms, that is, by depriving them of their social and cultural specificity’. Pais (2011) considers this and other criticisms of ethnomatics as it has been researched. This and my Western background lead me to leave an account of tool use in non-Western out-of-school practices to a more capable author.

14.3.2 Tool Use in Constructors' Mathematical Practices

Frejd and Bergsten (2016) focus on constructors with specific regard to modelling. They interviewed nine professionals (in both the commercial and academic sectors) with Ph.D.s (all in the sciences, five in mathematics) and focused on three phases: pre-construction, the reason for the modelling activity; construction, how the model is developed; post-construction, the consequences of using the model. The analysis reveals three types of modelling which they call 'empirical', theoretical' and 'applicational'; the use of computers was an essential feature in each type of modelling. A defining characteristic of empirical modelling is data from empirical observations though the data, of course, varied over contexts (e.g. financial risk and workforce scheduling). The mathematical model in every case was implemented in a computer system. Issues with the data (to feed into the computerised mathematical model) included: getting sufficient data; cleaning data; dealing with gaps (e.g. for time series analysis); locating errors in the data. In my experience of such things the time actually using the tool (computer) is a tiny proportion of the time preparing the data for tool use but the tool is central to the activity.

Theoretical modelling involves:

> . . . setting up new equations based on already theorised and established physical equations. This is followed by the activation of computer resources for computational purposes to solve the new equations with aim to get information about the 'theorised' equations. (Frejd & Bergsten, 2016, p. 24)

Example problems in theoretical modelling include predicting climate change and the design of a new material. At the heart of theoretical modelling is the mathematical model and its implementation on a computer. In the problems cited in Frejd and Bergsten (2016) this computer had to be 'powerful', as the designer of models for new materials said, 'The computer is our big tool, not least when it comes to solving these quantum mechanics equations' (Frejd & Bergsten, 2016, p. 26).

Applicational modelling refers to 'identifying situations where some mathematics or some established mathematical models can be directly applied' (Frejd & Bergsten, 2016, p. 26); this was an aspect of the work of all nine modellers. For example, one of the nine modellers was a biologist who was looking into the spread of diseases between oak trees. His starting point was differential equations:

> Fourier transformations are really good and you can then rewrite anything as a sum of sine functions. [. . .] This has been used by people at the department of systems control [. . .] Basically it is knowledge about mathematical methods that do the work, and sometimes you start with the problem and then you add a method [. . .] It is basically the same thing if bugs fly between oak trees or if animals are transported in trucks. (Frejd & Bergsten, 2016, p. 27).

In the above summary of Frejd and Bergsten (2016) I focused on the construction phase where a mathematical model (an artefact) and various other mathematical tools, especially computers, were central features of the activity in all cases. The discussion of the pre- and post-construction phases in Frejd and Bergsten (2016), however, highlight that mathematical tool use is but a part of the activity of modelling. These modellers serve clients who are not necessarily able to understand the model or computer use. In the pre-construction phase the client needs to be convinced that the model will be useful for his/her purpose and in the post-construction phase the client needs to be made aware, by the modeller, of the potential and the limitations of the model. Communication/dialogue is also an essential feature of professional modelling activity. I now turn to Skovsmose's (2005) 'operators'.

14.3.3 Tool Use in Operators' Mathematical Practices

Operators, people who use/manipulate rather than design/develop technology, are a very large class in themselves which includes technicians (manual, e.g. plumbers, and blue collar, e.g. insurance clerks), social service workers (e.g. nurses and police), sales people and teachers (e.g. a mathematics teacher using mathematical software). Skovsmose's (2005) three categories (constructors, operators and consumers) are wide categories and there are 'grey areas'. For example technicians may adapt given tools to their needs in a specific activity and clerks who operate payroll systems are themselves customers of a payroll system. The categories are nevertheless useful for focusing on tool use in out-of-school mathematical practices.

Noss and Hoyles (1996) distinguish between 'visible' and 'invisible' mathematics in out-of-school activity. Visible mathematics is that which is immediately recognised as being mathematics. This distinction is clearly context/person specific. The mathematics in the work of the constructors considered in Frejd and Bergsten (2016) was visible but it is common, when you ask an operator 'What mathematics is involved in your job?', that they reply 'None' or 'Very little'. Very often there is mathematics in this job but they do not see it as mathematics, it is invisible to them, often 'hidden' in tools they use. I shortly explore these general statements in some detail in the context of research I was involved in but I first outline research by a group that provides themes for a discussion of issues related to operators, mathematics and tools.

Hoyles, Noss and Pozzi focused on operators in a series of publications (see Noss, Hoyles, & Pozzi, 1998, for a summary) which examined mathematics in nursing, banking and flying workplaces. In a report of nursing practice (Pozzi, Noss, & Hoyles, 1998, considered in Sect. 9.2) they focus on drug administration and fluid balance monitoring aspects of patient care. The research team made multiple hospital visits to 12 experienced nurses over 4 months which resulted in 80 h of observation.

They set out to observe activities which involved visible mathematics and the mathematisation of the nurses' professional practice. In all cases they

> attempted to delve beyond simple arithmetic procedures to try to understand more complex, but perhaps less visible parts of decision–making on the ward ... [by separating] out

episodes involving routine behaviour from those involving a *breakdown* in the normal habits of nursing practice (Pozzi et al., 1998, pp. 107–108, italics added).

For example, drug administration appears to involve ratio and proportion but proportional reasoning was replaced in routine practice by arithmetic rubrics. They provide an example of a nurse preparing 85 mg of an antibiotic from a vial containing 100 mg of the drug in 2 mL and using the formula (an artefact) $\frac{Amount\ you\ want}{Amount\ you\ have\ got} \times Volume\ \ it\ \ is\ \ in$. The formula, however, is not conceived 'mathematically' but as a strategy for calculations for specific drugs, 'It was often heard that "with amikacin you can double it and divide by a hundred" or "with ondansetron, you only need to half it"' (Pozzi et al., 1998, p. 110). I now turn to an example where Pozzi et al. (1998) interpret a 'breakdown' situation. For reasons of space I omit many of the details which can be found in Pozzi et al. (1998).

Two nurses are discussing a fluid balance chart (an artefact) of a patient who has recently had a kidney transplant. The chart is not questioned by the nurses and it comes with a mathematical structure: the rows record times; the columns record fluids in and fluids out. Sam, an experienced nurse who is new to the ward asks 'why are you recording the difference between these two?' and Al, the nurse who is not new to the ward replies 'Because then when I come to add it up, I add my hourly totals. To get this one, that's why I need to know that figure' (Pozzi et al., 1998, p. 113). Further dialogue around the numbers in the chart ensues, basically along the lines of Al providing rationales for the calculations in the given chart and Sam questioning how the numbers relate to the patient's situation. Eventually Al sees Sam's point and concedes 'I suppose you should write down the rate'.

In their summary, Pozzi et al. (1998) note:

Professional cultures contain a huge number of artefacts which are, like the nurses' chart or the nursing rule, already mathematised . . . workers rarely think mathematically without an artefact to help them to organise or compute the data. In routine use, this mathematics is invisible, and remains so—indeed, the functionality of artefacts often crucially depends on this invisibility. But at times, people will need to understand the models which underlie their artefacts, to sort out what is happening or what has gone amiss . . . As we saw in the fluid balance episode, this typically occurs when there is a breakdown, and in such a situation, people need to represent to themselves how the underlying structures work (Pozzi et al., 1998, p. 118)

I think Pozzi et al. (1998) provides both a well-grounded evidence-base for its claims and insightful comments on tool use in the practice of a group of operators (nurses). But, taken alone, there is a danger that these claims for one practice may be viewed as generic for operators in general. I now consider Magajna and Monaghan (2003) which has similarities and differences to Pozzi et al. (1998).

Magajna and Monaghan (2003) is a case study of the mathematics and tool use of six computer aided design and manufacture (CAD-CAM) technicians. It reports on these technicians' calculations of the internal volume of moulds they produce for glass factories. The six technicians work as a team but three of them (constructors) liaise with clients and three (technologists) liaise with machinists in their factory who produce the metal moulds for glass bottles. The technicians were observed for

60 h over 3 weeks. Constructors evaluate whether a mould for a bottle can be manufactured, define the dimensions of the mould, design the bottle and the mould and make technical drawings. Technologists define the surfaces to be cut, write the programs for computer-numerically-controlled machines and, independently from the constructors, determine the inner volume of the mould.

A mould consists of three parts which close around a piece of molten glass of a given weight/volume. Compressed air is pumped into the molten glass which adheres to the inner shape of the mould. When the glass is cooled, the three pieces of the mould are separated and the bottle is released. The important job is to define the inner shape of the mould and to cut them out using appropriate machines. There are several volumes: of the bottle filled; of the glass; and the inner volume of the mould. The relations between these volumes is obvious to the technicians and the only volume they are concerned with is the inner volume of the mould. When they spoke about a volume related to a bottle they meant the inner volume of the mould. Getting the inner volume correct to a high degree of precision is essential for client satisfaction. The technicians do not distinguish between exact and approximate volumes as all calculations are approximate to them. Six methods of calculating the volume of a shape were observed:

1. The constructors drew the 2D-profile on a computer system and then used a program which automatically calculated the volume of the rotated shape.
2. The constructors represented the shape of the bottle in terms of horizontal cross-sections at various heights and drew a sequence of cross-sections. The volume of the part of the bottle between two horizontal sections with respective areas A and B and the height h between the sections was calculated using the formula $V = h\left(A + \sqrt{AB} + B\right)/3$. The constructors did not know where this formula—it was ‘a shop-floor tradition’.
3. The constructors calculated the volume of a bottle using a 3D-solid CAD.
4. The volumes of standard geometric shapes were calculated using school-learnt formulae, e.g. to calculate the volume of a prism of height h, the constructor drew its base on a computer to obtain its area, A, and then used the formula $V = hA$.
5. The technologists obtained the volume of a shape using a 3D-surface modeller integrated into the CAM software they used. The program they used calculated the volume of a polyhedron with the vertices on the mesh points, but the technologists ignored this.
6. Once the mould was made, its volume was measured by weighing the water it held.

I now consider similarities and differences between this research and that of Pozzi et al. (1998). Both studies provide evidence that the operators under scrutiny rarely engage in mathematics without the use of an artefact/tool. Pozzi et al. (1998, p. 115) add, ‘the use of artefacts never fully structures activity. People are not necessarily slaves to the tools they use’ but Magajna and Monaghan (2003, p. 119) state, of the technicians in their study:

> The mathematics they were really doing, their work mathematics, was inextricably joined with the technology they used. The geometry elements in their designs always represented technological entities and the calculations they performed were grounded in technology. Our practitioners used mathematical tools, including software, as 'black-boxes'. They were not observed to reason about the mathematics hidden in these tools and if a tool-based method did not work, they simply chose another method or overcame the problem by technological means.

A second difference is that Pozzi et al. (1998) concerns 'breakdowns'; as cited above, 'at times, people will need to understand the models which underlie their artefacts . . . this typically occurs when there is a breakdown' (Pozzi et al., 1998, p. 118). Magajna and Monaghan do not dispute that this did occur in observations of the nurses but did not find this to be the case in their study. In breakdown situations their practitioners either chose another method to overcame the problem by technological means:

> participants' reactions to 16 cases of non-trivial mathematics-related errors were observed. In 14 out of the 16 cases the error was due to a mistake in a computer generated geometric construction. Analysis revealed the following causes of errors: poor understanding of some detail in a construction command (11 cases), undocumented details about construction in the software (3 cases), a misunderstanding between participants (1 case) and difficulty in visualising the shape (1 case). In such breakdown situations the participants never reasoned about possible mathematics-related causes of the error, e.g. whether they understood the mathematical aspect of the applied construction. In most cases the geometric error was left unresolved and a solution was found by technological means (8 cases). (Pozzi et al., 1998, pp. 113–114)

This comparison of these two studies suggests that the differences observed/ interpreted in these two studies are likely to result from the particularities of the different workplaces observed and that further studies on tool use in other workplaces are needed.

14.3.4 Computers in Out-of-School Mathematical Practices

I end this section with a consideration of the place of computers in workplace mathematics. Although a computer is just another tool (or 'set of tools'—see Sect. 1.2), the prominence of computers in the workplace in the twenty-first century merits special consideration. This prominence is evident in the above discussion of constructors. It is also evident in Magajna and Monaghan's (2003) study of CAD-CAM technicians/operators. Studies in trends in workplace skills provide evidence that ICT is an increasingly important part of employment:

> There has been a striking and continued increase since 1986 in the number of jobs in which advanced technology is used. There has also been a marked increase over the last four years in the proportion of jobs in which computing is considered to be an essential or very important component of the work. Over 70 percent of people in employment now make use of some type of automated or computerised equipment, and computerised equipment is seen by 40 percent as essential to their work. (Felstead, Gallie, & Green, 2002, p. 12)

Researchers who have addressed this issue are, again, Celia Hoyles and Richard Noss (with colleagues). Noss and Hoyles (2009) reconsider modelling to address the advance of ICT in twenty-first century work practices, 'With the ubiquity of IT, employees now require new kinds of mathematical knowledge that are shaped by the systems that govern their work' (Noss & Hoyles, 2009, p. 76). Behind this paper are two reports which I now consider.

Hoyles, Wolf, Molyneux-Hodgson, and Kent (2002) reports on research into mathematical skills used/needed in seven areas of employment spanning engineering, financial services and health care. It coined the term 'mathematical literacy' which arose from the required skills and made four recommendations. ICT ('IT' in the language of the Report) is not the sole focus but it is a major focus: mathematical literacy is defined by a list of 12 skills of which the first 2 are 'Integrated mathematics and IT skills; an ability to create a formula (using a spreadsheet if necessary)' (Hoyles et al., 2002, p. 5); all the recommendations bar the last one on communication refer to IT:

> That IT and mathematical skills are interdependent . . .Developing models of new forms of training for all employees which reflect mathematical literacy that is integrated with IT competence . . . To investigate the development of training programmes which will be effective in the workplace by achieving a balance between physical experiences and software packages (Hoyles et al., 2002, pp. 3–4)

The IT dimension of Hoyles et al. (2002) was further developed in Hoyles (2007) under the term, 'Technomathematical literacies (TmL), that is, being able to reason with quantitative or symbolic data processed by information technology as part of decision-making or the communication process' (Hoyles, 2007, p. 16). A construct introduced in Hoyles (2007) is 'technology-enhanced boundary object' (TEBO). The construct 'boundary object' was introduced in Star and Griesemer (1989) and has been widely used in social science research since its introduction. A boundary object is an artefact created in one community of practice and travels to a distinct community of practice. Boundary objects abound in all practices including mathematics (e.g. Sloane's online *Encyclopedia of Integer Sequences*, see Chap. 3) and mathematics education (a new version of a mathematics curriculum devised by Ministry workers and sent to teachers). An interesting feature of boundary objects is that the meanings ostensibly embedded in them by their creators are re-interpreted by members of the receiving community. Hoyles (2007) TEBOs were linked to TmLs and workplace learning opportunities, 'Learning opportunities incorporated interactive software tools that modelled elements of the work process, or were reconstructions of the symbolic artefacts from workplace practice . . . TEBOs . . . involving many cycles of collaborative design' (Hoyles, 2007, p. 18).

Hoyles (2007) provides an example of a packaging factory making plastic film by an extrusion process. The computer control and monitoring system, it is claimed, served as a boundary object between managers, engineers and shop-floor machine operators. The computer system captures data on the stages in the process and presents this data in graphical form but shop-floor operators rarely looked at them. The research team identified a TmL:

> Understanding systematic measurement, data collection and display; appreciation of the complex effects of changing variables on the production system as a whole; being able to identify key variables and relationships in the work flow; and reading and interpreting time series data, graphs and charts (Hoyles, 2007, p. 21)

The research team in collaboration with employees developed a TEBO, a computer simulation of the production process with a goal to achieve stable running of the extrusion process. The hands-on TEBO training was viewed by a process engineer as a superior learning opportunity for operators than prior observational style training. From the perspective of tool use in mathematics it is an interesting case of using a tool (a computer model) in workplace training to simulate another workplace tool (the computer system that monitors the actual process).

14.4 Links Between In-School and Out-of-School Mathematical Practices

This section has two subsections. The first presents a case that linking in-school and out-of-school mathematical practices is an incredibly difficult undertaking. The second looks at research that has sought ways into making links between in-school and out-of-school mathematical practices.

14.4.1 Difficulties in Linking In-School and Out-of-School Mathematical Practices

The application of school mathematics to everyday and work settings is one of the main rationales for the place of mathematics in national curricula: 'This fact in itself could be thought to provide a sufficient reason for teaching mathematics' (Cockcroft, 1982, paragraph 1). Last century there was a perception, that I believe was widespread, that people, as students, learnt mathematics in school and applied this same mathematics, when appropriate, in out-of-school settings. In the UK, for example, the Mathematical Association wrote, concerning the teaching of arithmetic in schools, 'The arithmetic rules and processes needed in the practice of double entry book-keeping are, in the main, those with which the pupils of secondary schools are familiar ... The corresponding arithmetic work may be [there follows a list of topics]' (Mathematical Association, 1952, p. 73).

This perception commonly goes by the name 'transfer' (of knowledge or of learning). In the late twentieth century a number of studies presented data and theories of learning/doing mathematics that ranged from regarding transfer as highly problematic to rejecting it outright as a myth. In this section I first consider two studies/theories that question transfer. I then consider the school mathematics and attempts to make links between in-school and out-of-school mathematical practices.

A book that generated a great deal of interest (ranging from revelation to outrage) in the mathematics education community is Lave (1988). Lave presented data that people could do mathematics 'better' in supermarkets than in a test; her examples were arithmetic, the cost of items in a supermarket and equivalent 'sums' in a paper and paper test. Lave had a theory, which came to be called 'situated cognition', that supported her data, that claimed that how one thinks is tied to the practice one is engaged in. 'Situated cognition' is probably an unfortunate name from Lave's viewpoint as she is scathing of traditional cognitive research on 'knowledge':

> the effect on cognitive research of "locating" problems in "knowledge domains" has been to .separate the study of problem solving from analysis of the situations in which it occurs . . . "knowledge domain" is a socially constructed *exoticum*, that is, it lies at the intersection of the myth of decontextualized understanding and professional/academic specializations (Lave, 1988, p. 42)

To Lave (1988) learning in and out of school are different social practices and there is no reason to expect learning in one social practice to influence another social practice. But Lave's, 1988 exposition is not illuminating from the point of tool use in mathematics because tool use in learning does not feature in this account. Indeed, of the wider literature on *communities of practice* (which includes Lave's, 1988 account), Kanes and Lerman (2008) write, 'a theory of mediation is needed . . ..The nature and . . ..role of artefacts and tools is hazy' (Kanes & Lerman, 2008, p. 320).

Lave (1988) regards the perception of transfer of learning across social practice as a myth. Around the same time as Lave developed her theory, Saxe (1991) developed an approach that viewed transfer as problematic but not necessarily impossible.

Saxe (1991) uses a model, developed in ethnographic research into the transformation of mathematical practices of Papua New Guinean tribespeople, to examine the candy-selling practices of Brazilian street children, and then to explore links between in-school and out-of-school mathematical practices. Saxe's model has three components: analysis of practice-linked goals; form-function shifts in cognitive development; the interplay of learning across contexts (i.e. 'transfer'). It suffices for this section of this chapter to focus on the first component where of 'practice-linked goals' means 'emergent goals'—'must do' things that arise in practice and can interrupt that practice is they are not resolved. For example, buying something in a shop in a foreign country may induce the emergent goal 'determine the values of these coins in my wallet'. Emergent goals may or may not be related to mathematics; the agent is not necessarily aware of emergent goals. Saxe claims that four 'parameters' impinge on the resolution of emergent goals:

- *Activity structures*, 'general tasks that must be accomplished in the practice- and task-linked motives' (Saxe, 1991, p. 17)
- *Social interactions*, relationships between participants

- *Conventions and artefacts*, ‘the cultural forms that have emerged over the course of social history’ (Saxe, 1991, p. 18)
- *Prior understandings, which* ‘constrain and enable the goals they construct in practices’ (Saxe, 1991, p. 18)

I have found Saxe’s model useful in examining in-school and out-of-school mathematical practices (see Magajna & Monaghan, 2003; Monaghan, 2004; Monaghan, 2007b) since it affords an analysis of practice to consider the dialectic between mathematics, tool use, networks of artefacts and social activity (which was discussed in Part I of this book). When Saxe (1991) gets to examining learning across contexts (transfer), this model allows him to explore aspects of ‘transfer’ rather than make general claims as to its existence or not.

I provided these brief accounts of two late twentieth century frameworks related to the perception of transfer to establish a background assumption in twenty-first century academic mathematics education, that transfer, and making links between in-school and out-of-school mathematical practices, is problematic. The issue is ongoing. Although Lave’s (1988) ‘situated view’ of transfer was that it is a myth, Engle (2006) presents a situated view of transfer as ‘framing’—‘making references to both past contexts and imagined future ones . . .[to] make it clear to students that they are not just getting current tasks done, but are preparing for future learning’ (Engle, 2006, p. 456), and forms of learner participation. A research question awaiting a researcher is whether framing tool use in mathematics learning can be used to promote intercontextuality.

Neither Lave (1988) nor Saxe (1991) explore school mathematics classroom practices to any depth and it is appropriate to consider this practice at this juncture. There are differences between countries (Mullis, Martin, Foy, & Arora, 2012), within countries (Noyes, 2012) and within schools (Noyes, 2012) in school mathematics classroom practices but a common feature of mathematics classrooms is that they consist of a set of learners and a teacher (or teachers) who have come together, ostensibly for the teacher to help the learners engage in mathematics. It is important to that the age/experience of the children is taken into account though it should not be assumed that young children cannot engage in ‘applied mathematics’. Mathematical practices in classrooms are distinct from those of mathematicians (pure or applied). Sect. 10.4 details Chevallard’s notion of ‘didactical transposition’; this is neatly encapsulated by Lagrange (2005, p. 69) ‘mathematics in research and in school can be seen as a set of knowledge and practices in transposition between two institutions, the first one aiming at the production of knowledge and the other at its *study*.’ Strange things such as *the suspension of sense-making* (see, for example, Verschaffel, Greer, & De Corte, 2000) can happen in mathematics classrooms. For example studies in various countries have presented primary school children with ‘There are 26 sheep and 10 goats on a ship. How old is the Captain?’ and a common answer is ‘36’. There is, of course, a sense to this answer, ‘this is a mathematics class and there are two numbers in this question, I’ll add them’, but this sense works against the sense needed to link in-school and out-of-school mathematics. School mathematics here is seen as a sort of ‘game’.

Verschaffel, Greer, and de Corte (2002, p. 262) cite a 13-year-old student who, when asked by an interviewer why she did not make use of realistic consideration in her solution to a problem, responded:

> I know all these things but I would never think to include them in a math problem. Math isn't about things like that. Its about getting sums right and you don't need to know outside things to get sums right.

And school mathematics is often such a game. In disturbing research by Cooper and Dunne (2000) the researchers presented upper primary and lower secondary school children with 'esoteric' (e.g. $2x+1=17$, find x), realistic mathematics questions and analysed responses with regard to the children's social class. The working class children held their own very well, against children with parents in the professions, in the esoteric questions but performed comparatively poorly in the realistic questions. An interpretation of this data is that working class children took the realistic questions seriously but the other children knew it was *just a game* and this disadvantaged the working class children. For example, a question about the price of a soft drink and a bag of popcorn in a cinema was a disguised simultaneous equations question and children drawing on knowledge of actual cinema prices would get the answer wrong. But even when it is not seen as a game, school mathematics is almost always done in a mathematics lesson and this 'situation' appears to matter. Monaghan (2007b) reports on a study where a company director came into a mathematics class (students aged 14–15) and gave them a problem he was working on (about how to use a GPS position to register when one of his haulage vehicles had arrived at its destination). The research picked up the following exchange between two students:

Student 1	Shall we draw this as a graph?
Student 2	Why?
Student 1	'Cos that's normally what you do with co-ordinates.

The company director wanted a solution to a real problem. He expected that mathematics could be used in the solution but Student 1 expected to use a particular approach due to the mathematical content.

14.4.2 Attempts at Linking In-School and Out-of-School Mathematical Practices

I now focus discussion towards artefact/tool use in attempts to link in-school and out-of-school mathematics. The widespread use of artefact/tools in out-of-school mathematics documented in the first half of this chapter suggests that this focus may have potential to make links between in-school and out-of-school mathematics. I first note, in my experience, a restrictive vision with regard to tools in the applications of mathematics in schools. In 2005, at the outset of a project in my locality concerned with linking in- and out-of-school mathematics, I sent out a questionnaire to local schools with a series of questions on this topic (this is reported in Monaghan & Sheryn, 2006). One question was:

> Does your department use any special resources for linking school mathematics to out-of-school mathematical activity? Anything from surveying equipment, to catalogues to computer software, please specify.

Fifty-two percent of the schools stated that they did not use any special resources. The remainder mentioned occasional use of resources. Twelve percent stated that they had holidays and shopping catalogues. Other resources mentioned were trundle wheels and clinometers. There appears to be a bit of a *tool blindness* (not consciously recognising the use of tools in activity) here as none of the schools mentioned software and I knew that many of them did use spreadsheets in mathematics work. Nevertheless, it does not appear that artefact/tools (resources) are viewed as important in the applications of mathematics. Tool blindness (or, at least, partial vision) appears in research too. Masingila, Davidenko, and Prus-Wisniowska (1996) employs Saxe's framework; it reports on three workplace mathematics studies (dietetics, carpet laying and restaurant management). Selected problems from these contexts were given to pairs of secondary students who were observed and questioned as they solved the problems. They found differences in 'the goals of the activity, the conceptual understanding of persons in each context, and flexibility in dealing with constraints'. Although the paper discusses the role of artefacts/tools in workplace mathematics in its presentation of the theoretical framework, it says surprisingly little about artefacts/tools in its comparison of workplace and in-school problem solving and when it does, it does so in quite general terms, for example:

> For both the restaurant manager and the interior designer, solving the problems were necessary parts of their jobs. They used mathematics as a tool to help them solve problems and not as the goal of the problem. The students, however, seemed to view the problems as mathematical exercises and immediately started using algorithms that they thought would be appropriate. (Masingila et al., 1996, p. 182)

Even when they explicitly consider Saxe's parameter concerned with artefacts they merely mention, with regard to carpet laying, 'students may invent notation to indicate when objects are the same size and shape, in the course of working in a measurement context, before they have formalised the concept of congruence' (Masingila et al., 1996, p. 196).

Two school-based studies that do focus on genuine artefacts are Lowrie (2011) and Bonotto (2013), though the artefacts in question in both papers are not mathematical tools.[4] Lowrie's focus is twofold, the use of genuine artefacts and collaborative learning in solving realistic mathematics problems. The children were a Grade 6 class (11–12 years old) from an Australian primary school. The artefacts

[4] This is not meant to belittle the mathematical potential of artefacts that are not mathematical tools. Many artefacts of this kind enable what the Freudenthal school (see Freudenthal, 1991) call 'horizontal mathematization'; mathematics can be extracted from the artefact and the artefact can be mathematically structured by the agent.

used were brochures, menus, bus timetables, photographs and a real map from a local theme park. The children worked in small groups to plan a group trip to the theme park, 'plan the day's events with appropriate details and budgetary considerations . . . use the map as your main reference point' (Bonotto, 2013, pp. 4–5). The artefacts were judged to have learning potential, they:

> Encouraged the children to make connections to real-life experiences . . . [children] sourced a great deal of visual, spatial and graphical information from the artefacts . . . established a strong motivational intention for the open-ended task. (Lowrie, 2011, pp. 7–8)

With regard to collaborative learning, however, there was considerable variation in the: quality of the solution; the authenticity of the solution; the manner in which the group work (collaboratively or with one student dominating). Lowrie's interpretation of this is interesting:

> These artefacts establish a sense of problem solving 'integrity' . . . helps to establish meaningful engagement between peers . . .However, as the students accessed and used personal knowledge to solve problems, they were less likely to monitor and manage collaborative group goals. (Lowrie, 2011, p. 14)

So the use of genuine artefacts has great potential for applied problem solving in schools but the solutions by individuals in groups are often rich, complex and varied and many of these students found it difficult to simultaneously focus on the complexity of their own solution and that of their peers. It is useful when research alerts us to matters such as these which may not be obvious.

Bonotto (2013) has similarities to Lowrie (2011), the age of the children and the types of artefacts, but focuses on artefacts as a source of real-life problem-posing (I will only refer to problem-posing when necessary as it is not my focus in this chapter). The study was in two parts. The first part was exploratory, 'to evaluate . . . the products of the problem-posing process when it is implemented in situation involving the use of suitable artifacts, with its related mathematics, and particular teaching methods' (Bonotto, 2013, p. 42). The evaluation was largely positive:

> children had no difficulty translating typical everyday data, present in the artefacts, into problems suitable for mathematical treatment . . . [but] it was decided to modify some of the data of the problem in order to render the resolution of the problem more straightforward. (Bonotto, 2013, p. 43)

The second study had three phases: presentation of the artefact (a brochure for an amusement park); a problem-posing activity; a problem-solving activity. Two classes from different primary schools participated. There were similarities and differences between these classes. Whilst all but one of the 189 mathematical problems posed were mathematical one school generated 58 problems whilst the other generated 131 problems. About three quarters of problems from each school were solvable. The school which generated more problems also had a greater variety of types of problems and more 'original problems', 'Original problems include inverse problems, and problems containing almost all the information on the artefact' (Bonotto, 2013, p. 50). This may suggest that problem-posing from an artefact is related to the nature of the teaching children have experience (and this seems a reasonable hypothesis).

Lowrie (2011) and Bonotto's (2013) study provide evidence that artefacts can be useful in generating links between in-school and out-of-school mathematics, though both raise issues for further research. A further aspect of artefacts in generating such links is the production of artefacts. This, as we saw in Chap. 8, ties in with constructionist thinking, 'we extend the idea of manipulative materials to the idea that learning is most effective when part of an activity the learner experiences as constructing is a meaningful product' (Papert, 1987, abstract). Monaghan (2007a) reports on the production of artefacts within secondary school mathematical activities designed to link in-school and out-of-school mathematics activities. In this 2-year study eight teachers worked with the researchers (and often an out-of-school expert) to co-design school-based projects on out-of-school themes. Of 20 project designs 13 were implemented and in 4 of these the production of artefacts was the student outcome: 'designing a mathematical garden' involved transforming a garden including making a sundial; 'designing shelf-ready packaging' involved making a cardboard template of the packaging which was suitable for assembly on a production line; 'writing a rap song' with specialist music software resulted in electronic music; 'setting up your own business' involved producing a business plan on a spreadsheet which was suitable to send to a bank. In all of these projects, artefacts were used to produce new artefacts, which is the case in many out-of-school practices.

14.5 A Consideration of the Issues

The distinction between pure and applied mathematics, the use of mathematics in out-of-school practices and linking in-school and out-of-school mathematical practices are substantial and ongoing issues and it would be foolish of me to expect my tool-focused consideration of these issues in this chapter to bring a resolution to any of them. But a lot of detail has been presented in the three sections above and it is appropriate to consider the *punch line* of this scholarship and research. I structure this section by considering the 'problem issues' (problems of interpretation and problems of apparent gaps in understanding/research).

I started this chapter suggesting that there is a sense in which the division between pure and applied mathematics is a real division between mathematical activity for intrinsic or extrinsic purposes but also a sense in which the division between pure and applied mathematics is a cultural–historical division (and then I took a quick historic tour of ways of conceptualising divisions in mathematical activity). This is a problem issue—what, if anything, is the distinction between pure and applied mathematics?

I think the key to understanding this problem is recognition that there exists the practice of doing mathematics and interpretations of this practice. If we return to Jon's Chap. 3, an account of the practice of doing mathematics, we can see what might be called ‘pure’ (Case Study Ia: Iterative Reflections) and what might be called ‘applied’ (Case Study Ib: Protein Confirmation) mathematics but Jon does not employ these terms in his discussion of these case studies, he just reports on mathematics research (and the significant use of tools in this practice). This appears to be true of non-experimental mathematicians as it is for experimental mathematicians. This ‘problem issue’ is a problem issue for (some) interpreters (philosophers, historians and mathematics educators), not for (most) practitioners. But the interpreters, it seems, do play a role in determining the education (and thus the practice) of future practitioners in as much as they play a significant role in determining the structure of mathematics curricula (from Greek to medieval to modern times). With regard to tool use in mathematics, there appears, to use a term I coined above, *tool blindness* in many practitioners and interpreters. This does not appear important in the case of practitioners as they will use tools whether they realise it or not but it is important in the case of interpreters who have a say in structuring curricula (if their interpretations of the appropriate tools for doing mathematics are out of synch with the tools future practitioners need).

The second problem issue I raise is ‘the modelling cycle’ discussed in Sect. 14.2. I outlined my problems with this cycle above: it is oversimplified; it is not clear what it represents; and the usual presentation of this cycle does not attend to tool use. The Siller and Greefrath (2010) version of this cycle partially attends to tool use but raises complications by positing three worlds. I suggest that it may be useful to ignore the modelling cycle and simply look to practice and this appears to be an approach of current research, for example Noss and Hoyles (2009). I hope that this book will contribute to a focus on tool use (including computers but not just computers) in these practices. But if this leads to a new interpretation of out-of-school mathematical practices, then we should not view this as the final interpretation. Any interpretation will, from a cultural–historical perspective, be an interim interpretation in the developmental path of our understanding of the divide between ‘pure’ and ‘applied’ mathematics, a step on the way from the ancient Greek *quadrivium*, to pure and mixed mathematics, to pure and applied mathematics to . . . another understanding.

The third problem issue I raise is the difficulty of characterising tool use in out-of-school mathematical practices. Contributing to this problem issue are: the sheer number of out-of-school mathematical practices and the variation in both the tools used and the way tools are used in these practices; research into these practices requires contextual data, often obtained by time consuming ethnographic methods, so surveys of tool use may have limited value; mathematical tools are often invisible to practitioners and researchers. Section 14.3 only considered three studies in any depth. These studies all pointed to the importance of tool use in out-of-school mathematical practices but they do offer differing interpretations of tool use in practice. I am not worried about these differences but they suggest that we have barely scratched the surface of understanding tool use in practice.

Finally I raise a set of problem issues related to school mathematics. School mathematics is/can be viewed as a game and when this game is applied to linking mathematics to the real world it often results in *the suspension of sense making* and can disadvantage certain classes of children (Cooper & Dunne, 2000). Schools and classrooms are institutions and we should not expect 'real-life' reasoning to arise 'naturally' (Monaghan, 2007b) in them or that learning will 'naturally' transfer out of them. Tools and genuine artefacts appear to hold some hope that school mathematics can be related to real-life activities but many teachers and some researchers appear to have a form of *tool blindness*. But an awareness of problem issues can be a prelude to attempts to address problem issues.

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