A Symbolic Approach to Boundary Problems for Linear Partial Differential Equations

Applications to the Completely Reducible Case of the Cauchy Problem with Constant Coefficients

Markus Rosenkranz and Nalina Phisanbut*

University of Kent, Canterbury, Kent CT2 7NF, United Kingdom
{M.Rosenkranz,N.Phisanbut}@kent.ac.uk

Abstract. We introduce a general algebraic setting for describing linear boundary problems in a symbolic computation context, with emphasis on the case of partial differential equations. The general setting is then applied to the Cauchy problem for completely reducible partial differential equations with constant coefficients. While we concentrate on the theoretical features in this paper, the underlying operator ring is implemented and provides a sufficient basis for all methods presented here.

1 Introduction

A symbolic framework for boundary problems was built up in [11,13] for linear ordinary differential equations (LODEs); see also [15,6,7] for more recent developments. One of our long-term goals is to extend this to boundary problems for linear partial differential equations (LPDEs). Since this is a daunting task in full generality, we want to tackle it in stages of increasing generality. In the first instance, we restrict ourselves to *constant coefficients*, where the theory is quite well-developed [2]. Within this class we distinguish the following three stages:

1. The simplest is the Cauchy problem for *completely reducible* operators.
2. The next stage will be the *Cauchy problem* for general hyperbolic LPDEs.
3. After that we plan to study *boundary problems* for elliptic/parabolic LPDEs.

In this paper we treat the first case (Section 4). But before that we build up a *general algebraic framework* (Sections 2 and 3) that allows a symbolic description for all boundary problems (LPDEs/LODEs, scalar/system, homogeneous/inhomogeneous, elliptic/hyperbolic/parabolic).[1] Using these concepts and tools we develop a general solution strategy for the Cauchy problem in the case (1). See the Conclusion for some thoughts about the next two steps.
The passage from LODEs to LPDEs was addressed at two earlier occasions:

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[1] Due to space limitations we omit proofs; they can be found in `arXiv:1304.7380`.

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- An *abstract theory of boundary problems* was developed in [9], including LODEs and LPDEs as well as linear systems of these. The concepts and results of Sections 2 and 3 are built on this foundation, adding crucial concepts whose full scope is only appreciated in the LPDE setting: boundary data, semi-homogeneous problem, state operator.
- An algebraic language for *multivariate differential and integral operators* was introduced in Section 4 of [14], with a prototype implementation described in Section 5 of the same paper. This language is generalized in the PIDOS algebra[2] of Section 4, and it is also implemented in a Mathematica package.

In this paper we will not describe the current state of the *implementation* (mainly because of space limitations). Let us thus say a few words about this here. A complete reimplementation of the PIDOS package described in [14] is under way. The new package is called OPIDO (Ordinary and Partial Integro-Differential Operators), and it is implemented as a standalone Mathematica package unlike its predecessor, which was incorporated into the THEOREMA system. In fact, our reimplementation reflects several important design principles of THEOREMA, emphasizing the use of functors and a strong support for modern two-dimensional (user-controllable) parsing rules. We have called this programming paradigm FUNPRO, first presented at the Mathematica symposium [12]. The last current stable version of the (prototype) package can be found at `http://www.kent.ac.uk/smsas/personal/mgr/index.html`.

While the interested reader will find all the relevent details (in particular the rather large rewrite system) in this package, here are a few *remarks on the current state of the implementation*:

- The FUNPRO paradigm emphasizes the a functional style of programming while at the same time encapsulating all mathematical domains similar to object-oriented languages. For example, a ring $\mathcal{R}$ would come along with its various operations like $+_{\mathcal{R}}$ and $*_{\mathcal{R}}$.
- We make heavy use of two-dimensional syntax, using parsing and formatting rules that allow us to write integro-differential operators in a notation that is close to the one we use on paper.
- At this stage we do not aim at efficiency. Whenever there was a conflict between speed and conceptual clarity, we gave preference to the latter. The reason is that we view the current package as a prototype, with the goal of writing a more efficient (and stable) MAPLE package once the content is sufficiently matured.
- Since we rely on Mathematica for computing integrals in closed form, we inherit all the usual limitations in that respect. However, this is not a real issue in the current scope (constant coefficients) since here we can stay within the class of exponential polynomials.

We plan to describe more details of the implementation at another occasion.

[2] PIDOS = Partial Integro-Differential Operator System.

At the time of writing, the ring of *ordinary integro-differential operators* is completed and the ring of *partial integro-differential operators* is close to completion (for two independent variables). Compared to [14], the new PIDOS ring contains several crucial new rewrite rules (instances of the substitution rule for resolving multiple integrals). Our conjecture is that the new rewrite system is noetherian and confluent but this shall also be analyzed at another occasion.

Notation. The algebra of $m \times n$ matrices over a field K is written as K_n^m, where $m = 1$ or $n = 1$ is omitted. Thus we identify $K^m = K \oplus \cdots \oplus K$ with the space of column vectors and $K_n = (K^n)^*$ with the space of row vectors. More generally, we have $K_n^m \cong K^n \to K^m$. As usual we write $C^\omega(\mathbb{R}^n)$ for the algebra of (complex-valued) analytic functions with real arguments is denoted by.

2 An Algebraic Language for Boundary Data

As mentioned in the Introduction, we follow the *abstract setting* developed in [9]. We will motivate and recapitulate some key concepts here, but for a fuller treatment of these issues we must refer the reader to [9] and its references.

Let us recall the notion of boundary problem. Fix vector spaces $\mathcal{F}$ and $\mathcal{G}$ over a common ground field K of characteristic zero (for avoiding trivialities one may assume $\mathcal{F}$ and $\mathcal{G}$ to be infinite-dimensional). Then a *boundary problem* $(T, \mathcal{B})$ consists of an epimorphism $T\colon \mathcal{F} \to \mathcal{G}$ and a subspace $\mathcal{B} \subseteq \mathcal{F}^*$ that is orthogonally closed in the sense defined below. We call T the differential operator and $\mathcal{B}$ the *boundary space*.

Similar to the correspondence of ideals/varieties in algebraic geometry, we make use of the following Galois connection [9, A.11]. If $\mathcal{A}$ is any subspace of the space $\mathcal{F}$, its *orthogonal* $\mathcal{A}^\perp \leq \mathcal{F}^*$ is defined as $\{\varphi \in \mathcal{F}^* \mid \varphi(a) = 0 \text{ for all } a \in \mathcal{A}\}$. Dually, for a subspace $\mathcal{B}$ of the dual space $\mathcal{F}^*$, the orthogonal $\mathcal{B}^\perp \leq \mathcal{F}$ is defined by $\{f \in \mathcal{F} \mid \beta(f) = 0 \text{ for all } \beta \in \mathcal{B}\}$. If we think of $\mathcal{F}$ as "functions" and of $\mathcal{F}^*$ as "boundary conditions", then $\mathcal{A}^\perp$ is the space of valid conditions (the boundary conditions satisfied by the given functions) while $\mathcal{B}^\perp$ is the space of admissible functions (the functions satisfying the given conditions).

Naturally, a subspace of either $\mathcal{S}$ of $\mathcal{F}$ or $\mathcal{F}^*$ is called *orthogonally closed* if $\mathcal{S}^{\perp\perp} = \mathcal{S}$. But while any subspace of $\mathcal{F}$ itself is always orthogonally closed, this is far from being the case of the subspaces of the dual $\mathcal{F}^*$. Hence the condition on boundary spaces $\mathcal{B}$ to be orthogonally closed is in general not trivial. However, if $\mathcal{B}$ is finite-dimensional as in boundary problems for LODEs (as in Example 1 below), then it is automatically orthogonally closed. For LPDEs, the condition of orthogonal closure is important; see Example 2 for an intuitive explanation.

In [11,13] and also in the abstract setting of [9] we have only considered what is sometimes called the *semi-inhomgeneous boundary problem* [16], more precisely the semi-inhomogeneous incarnation of $(T, \mathcal{B})$; see Definition 7 for the full picture. This means we are given a *forcing function* $f \in \mathcal{G}$ and we search for a solution $u \in \mathcal{F}$ with

$$\boxed{\begin{array}{l} Tu = f, \\ \beta(u) = 0 \ (\beta \in \mathcal{B}). \end{array}} \tag{1}$$

In other words, u satisfies the inhomogeneous "differential equation" $Tu = f$ and the homogeneous "boundary conditions" $\beta(u) = 0$ given in $\mathcal{B}$.

A boundary problem which admits a unique solution $u \in \mathcal{F}$ for every forcing function $f \in \mathcal{G}$ is called *regular*. In terms of the spaces, this condition can be expressed equivalently by requiring that $\operatorname{Ker} T \dotplus \mathcal{B}^\perp = \mathcal{F}$; see [9] for further details. In this paper we shall deal exclusively with regular boundary problems. For singular boundary problems we refer the reader to [6] and [5].

For a regular boundary problem, one has a linear operator $G\colon \mathcal{G} \to \mathcal{F}$ sending f to u is known as the *Green's operator* of the boundary problem $(T, \mathcal{B})$. From the above we see that G is characterized by $TG = 1$ and $\operatorname{Im} G = \mathcal{B}^\perp$.

Example 1. A classical example of this notion is the two-point boundary problem. As a typical case, consider the simplified model of *stationary heat conduction* described by

$$\boxed{\begin{array}{l} u'' = f, \\ u(0) = u(1) = 0. \end{array}}$$

Here we can choose $\mathcal{F} = \mathcal{G} = C^\infty(\mathbb{R})$ for the function space such that the differential operator is given by $T = D^2\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ and the boundary space by the two-dimensional subspace of $C^\infty(\mathbb{R})^*$ spanned by the linear functionals $L\colon u \mapsto u(0)$ and $R\colon u \mapsto u(1)$ for evaluation on the left and right endpoint. In the sequel we shall write $\mathcal{B} = [L, R]$, employing an important generalization for LPDEs, described in the next eamples. We can express its Green's operator in the language of integro-differential operators as explained in [13].

Example 2. As a typical counterpart in the world of LPDEs, consider the *equation for waves in an inhomogeneous medium*, described by

$$\boxed{\begin{array}{l} u_{xx} - u_{tt} = f(x,t) \\ u(x,0) = u_t(x,0) = u(0,t) = u(1,t) = 0 \end{array}}$$

in one space dimension. In this case we choose $\mathcal{F} = \mathcal{G} = C^\omega(\mathbb{R} \times \mathbb{R}^+)$; again one could choose much larger spaces of functions (or distributions) in analysis and in the applications. Here the differential operator is $T = D_{xx} - D_{tt}\colon C^\omega(\mathbb{R} \times \mathbb{R}^+) \to C^\omega(\mathbb{R} \times \mathbb{R}^+)$ while the boundary space $\mathcal{B}$ is the orthogonal closure of the linear span of the families of functionals β_x, γ_x $(x \in \mathbb{R})$ and $\kappa_t, \lambda_t; (t \in \mathbb{R}^+)$ defined by $\beta_x(u) = u(x,0), \gamma_x(u) = u_t(x,0)$ and $\kappa_t(u) = u(0,t), \lambda_t(u) = u(1,t)$. Using the notation $[\ldots]$ for denoting the orthogonal closure of the linear span, we can thus write $\mathcal{B} = [\beta_x, \gamma_x, \kappa_t, \lambda_t \mid x \in \mathbb{R}, t \in \mathbb{R}^+]$ for the boundary space under consideration.

The point of the *orthogonal closure* is that the given conditions imply other conditions not in their span, for example $u_x(1/2, 0) = 0$ or $\int_{-3}^{5} u(0,\tau)\, d\tau = 0$. Rather than being linear consequences, these two examples are differential and integral consequences. (Of course the full boundary space also contains many functionals without a natural analytic interpretation.)

In the problems above, the differential equation is inhomogeneous while the boundary conditions are homogeneous. A *semi-homogeneous boundary problem* is the opposite, combining a homogeneous differential equation with inhomogeneous boundary conditions. While this is a simple task for LODEs (as always we assume that the fundamental system is available to us in some form!), it is usually a nontrivial problem for LPDEs (even when they have constant coefficients). We will give the formal definition of a semi-inhomogeneous boundary problem in the next section (Definition 7). Here it suffices to consider an example for developing the necessary auxiliary notions.

Example 3. The *Cauchy problem for the wave equation* in one dimension is

$$\boxed{\begin{array}{l} u_{xx} - u_{tt} = 0, \\ u(x,0) = f(x), u_t(x,0) = g(x). \end{array}}$$

Being a hyperbolic problem, we could use rather general function spaces for the "boundary data" f, g. For reasons of uniformity we will nevertheless restrict ourselves here to the analytic setting, so assume $f, g \in C^\omega(\mathbb{R})$. Note that the association of u to (f, g) is again a linear operator mapping (two univariate) functions to a (bivariate) function; we will come back to this point in Definition 7.

Going back to the abstract setting, one is tempted to define the notion of boundary data as some kind of functions depending on "fewer" variables. But the problem with this approach is that—abstractly speaking—we are not dealing with any functions depending on any number of variables (but see below). Moreover, the inhomogeneous boundary conditions in the form $u(0, x) = f(x), u_t(0, x) = g(x)$ are basis-dependent while the whole point of the abstract theory is to provide a *basis-free description* (which leads to an elegant setting for describing composition and factorization of abstract boundary problems); see Proposition 9. We shall therefore develop a basis-independent notion of boundary data (we can go back to the traditional description by choosing a basis).

We define first the *trace map* $\operatorname{trc}\colon \mathcal{F} \to \mathcal{B}^*$ as sending $f \in \mathcal{F}$ to the functional $\beta \mapsto \beta(f)$. In Example 3 this would map the function $u(x, t)$ to its position and velocity values on $\mathbb{R} \times \{0\}$. We call $\operatorname{trc}(f)$ the trace of f and write it as f^*. Moreover, we denote the image of the map trc by $\mathcal{B}'$ and refer to its elements as *boundary data*. Note that $\mathcal{B}'$ is usually much smaller than the full dual $\mathcal{B}^*$ since a continuous function (let alone an analytic one) cannot assume arbitrary values. (This situation is vaguely reminiscent of the algebraic and continuous dual of a topological vector space.)

Since by definition the trace map is surjective from $\mathcal{F}$ to $\mathcal{B}'$, it has some right inverse $\mathcal{B}^\diamond\colon \mathcal{B}' \to \mathcal{F}$. We refer to $\mathcal{B}^\diamond$ as an *interpolator* for $\mathcal{B}$ since it constructs a "function" $f = \mathcal{B}^\diamond(B) \in \mathcal{F}$ from given boundary values $B \in \mathcal{B}'$ such that $\beta(f) = B(\beta)$. Of course, the choice of f is usually far from being unique. Apart from its use for describing boundary data (see at the end of this section), the notion of interpolator will turn out to be useful for solving the semi-homogeneous boundary problem (see Proposition 10).

Let us now describe how to relate these abstract notions to the usual setting of initial and boundary values problems as they actually in analysis: essentially by *choosing a basis*. However, we have to be a bit careful since we must deal with the orthogonal closure.

Definition 4. If $\mathcal{B} \leq \mathcal{F}^*$ is any orthogonally closed subspace, we call a family $(\beta_i \mid i \in I)$ a *boundary basis* if $\mathcal{B} = [\beta_i \mid i \in I]$, meaning $\mathcal{B}$ is the orthogonal closure of the span of the β_i.

Note that a boundary basis is typically smaller than a K-linear basis of $\mathcal{B}$. All traditional boundary problems are given in terms of such a boundary basis. In Example 3, the boundary basis could be spelled out by using $I = \mathbb{R} \uplus \mathbb{R}$ with $\beta_{(x,0)}(u) = u(x,0)$ and $\beta_{(x,1)}(u) = u_t(x,0)$. Relative to a boundary basis $(\beta_i \mid i \in I)$, we call $f = \beta_i(f)_{i \in I} \in K^I$ the *boundary values* of $f \in \mathcal{F}$. As we can see from the next proposition, we may think of the trace as a basis-free description of boundary values. Conversely, one can always extract from any given boundary data $B \in \mathcal{B}'$ the boundary values $B(\beta_i)_{i \in I}$ as its coordinates relative to the boundary basis (β_i).

Lemma 5. *Let $\mathcal{B} \leq \mathcal{F}^*$ be a boundary space with boundary basis $(\beta_i \mid\in I)$. If for any $B, \tilde{B} \in \mathcal{B}'$ one has $B(\beta_i)_{i \in I} = \tilde{B}(\beta_i)_{i \in I}$ then also $B = \tilde{B}$. In particular, for any $f \in \mathcal{F}$, the trace f^* depends only on the boundary values $f(\beta_i)_{i \in I}$.*

The *analytic interpretation* of this proposition is clear in concrete cases like Example 2: Once the values $u(x,0), u_t(x,0)$ and $u(0,t), u(1,t)$ are fixed, all differential and integral consequences, as in the above examples $u_x(1/2,0) = 0$ or $\int_{1/4}^{3/4} u(0,\tau)\, d\tau$, are likewise fixed. It is therefore natural that an interpolator need only consider the boundary values rather than the full trace information. This is the contents of the next lemma.

Lemma 6. *Let $\mathcal{B} \leq \mathcal{F}^*$ be a boundary space with boundary basis $(\beta_i \mid\in I)$ and write $f_I = f(\beta_i)_{i \in I} \in K^I$ for the boundary values of any $f \in \mathcal{F}$ and $\mathcal{B}'_I$ for the K-subspace of K^I generated by all boundary values f_I. Then any linear map $J \colon \mathcal{B}'_I \to \mathcal{F}$ with $J(f_I)_I = f_I$ induces a unique interpolator $\mathcal{B}^\diamond \colon \mathcal{B}' \to \mathcal{F}$ defined by $B \mapsto J(B(\beta_i)_{i \in I})$.*

As noted above, we can always extract the boundary values $B(\beta_i)_{i \in I} \in K^I$ of some boundary data $B \in \mathcal{B}'$ relative to fixed basis (β_i) of $\mathcal{B}$. However, since one normally has got *only* the boundary values (coming from some function), where does the corresponding $B \in \mathcal{B}'$ come from? By definition, it has to assign values to all $\beta \in \mathcal{B}$, not only to the β_i making up the boundary basis. As suggested by the above lemmata, for actual computations those additional values will be irrelevant. Nevertheless, it gives a feeling of confidence to provide these values: If $\mathcal{B}^\diamond$ is any *interpolator*, we have $B(\beta) = \beta(\mathcal{B}^\diamond(B_i)_{i \in I})$. This follows immediately from the fact that $\mathcal{B}^\diamond$ is a right inverse of the trace map and that it depends only on the boundary values $(B_i)_{i \in I}$ by Lemma 6. In the analysis setting this means we interpolate the given boundary value and then do with the resulting function whatever is desired (like derivatives and integrals in Example 2).

3 Green's Operators for Signals and States

Using the notion of boundary data developed in the previous section, we can now give the formal definition of the *semi-homogeneous boundary problem.* In fact, we can distinguish three different incarnations of a "boundary problem" (as we assume regularity, the *fully homogeneous problem* is of course trivial).

Definition 7. Let $(T, \mathcal{B})$ be a regular boundary problem with $T \colon \mathcal{F} \to \mathcal{G}$ and boundary space $\mathcal{B} \subseteq \mathcal{F}^*$. Then we distinguish the following problems:

Given $(f, B) \in \mathcal{G} \oplus \mathcal{B}'$, find $u \in \mathcal{F}$ with	Given $f \in \mathcal{G}$, find $u \in \mathcal{F}$ with	Given $B \in \mathcal{B}'$, find $u \in \mathcal{F}$ with
$Tu = f$, $\beta(u) = B(\beta)$ $(\beta \in \mathcal{B})$.	$Tu = f$, $\beta(u) = 0$ $(\beta \in \mathcal{B})$.	$Tu = 0$, $\beta(u) = B(\beta)$ $(\beta \in \mathcal{B})$.

They are, respectively, called the *fully inhomogeneous*, the *semi-inhomogeneous* and the *semi-homogeneous* boundary problem for $(T, \mathcal{B})$. The corresponding linear operators will be written as $F \colon \mathcal{G} \oplus \mathcal{B}' \to \mathcal{F}$, $(f, B) \mapsto u$ and $G \colon \mathcal{G} \to \mathcal{F}$, $f \mapsto u$ and $H \colon \mathcal{B}' \to \mathcal{F}$, $B \mapsto u$.

Lemma 8. *Each of the three problems in Definition 7 has a unique solution for the respective input data,so the operators F, G, H are well-defined.*

The terminology for the operators F, G, H is not uniform in the literature. In the past, we have only considered G and called it the "Green's operator" acting on a "forcing function" f. While this is in good keeping with the engineering tradition and large parts of the standard mathematical culture [16], it is difficult to combine with suitable terminology for F and H. In this paper, we shall follow the systems theory jargon [8] and refer to F as the (full) *transfer operator*, to G as the (zero-state) signal transfer operator or briefly *signal operator*, and to H as the (zero-signal) state transfer operator or briefly *state operator*. This terminology reflects the common view of forcing functions $f \in \mathcal{F}$ as "signals" and boundary data $B \in \mathcal{B}'$ as (initial) "states".

One of the advantages of the abstract formulation is that it allows us to describe the *product of boundary problems* in a succinct, basis-free manner (and it includes LODEs and LPDEs as well as systems of these). The composite boundary problem can then be solved, both in its semi-inhomogneous and its semi-homogeneous incarnation (the latter is presented here for the first time).

Proposition 9. *Define the product of two boundary problems $(T, \mathcal{B})$ and $(\tilde{T}, \tilde{\mathcal{B}})$ with $\mathcal{F} \xrightarrow{\tilde{T}} \mathcal{G} \xrightarrow{T} \mathcal{H}$ and $\mathcal{B} \subseteq \mathcal{G}^*$, $\tilde{\mathcal{B}} \subseteq \mathcal{F}$ by*

$$(T, \mathcal{B})(\tilde{T}, \tilde{\mathcal{B}}) = (T\tilde{T}, \mathcal{B}\tilde{T} + \tilde{\mathcal{B}}).$$

Then $(T, \mathcal{B})(\tilde{T}, \tilde{\mathcal{B}})$ is regular if both factors are. In that case, if $(T, \mathcal{B})$, $(\tilde{T}, \tilde{\mathcal{B}})$ have, respectively, the signal operators G, $\tilde{G}$ and the state operators H, $\tilde{H}$, then $(T, \mathcal{B})(\tilde{T}, \tilde{\mathcal{B}})$ has the signal operator $\tilde{G}G$ and the state operator $(\mathcal{B}\tilde{T} + \tilde{\mathcal{B}})' \to \mathcal{F}$ acting by $B + \tilde{B} \mapsto \tilde{G}H(B\tilde{T}^*) + \tilde{H}(\tilde{B})$.

As detailed in [11,9], the computation of the *signal operator* G can be decomposed in two parts: (1) Finding a *right inverse* $T^\diamond$ of the differential operator T, which involves only the differential equation without boundary conditions (so we may replace the boundary by intial conditions, thus having again a unique solution: this is the so-called fundamental right inverse). (2) Determining the *projector* onto the homogeneous solution space along the space of functions admissible for the given boundary conditions—the projector "twists" the solutions coming from the right inverse into satisfying the boundary conditions. An analogous result holds for the computation of the *state operator* H if we replace the right inverse $T^\diamond$ of T by the interpolator $\mathcal{B}^\diamond$ for the boundary space $\mathcal{B}$.

Proposition 10. *Let $(T, \mathcal{B})$ be regular with operators F, G, H as in Definition 7. Then we have $G = (1-P)\,T^\diamond$ and $H = P\mathcal{B}^\diamond$, hence $F = (1-P)\,T^\diamond \oplus P\mathcal{B}^\diamond$ for the transfer operator. Here $T^\diamond\colon \mathcal{G} \to \mathcal{F}$ is any right inverse of the differential operator $T\colon \mathcal{F} \to \mathcal{G}$ and $\mathcal{B}^\diamond\colon \mathcal{B}' \to \mathcal{F}$ any interpolator for $\mathcal{B}$ while $P\colon \mathcal{F} \to \mathcal{F}$ is the projector determined by* $\operatorname{Im} P = \operatorname{Ker} T$ *and* $\operatorname{Ker} P = \mathcal{B}^\perp$.

If $T \in \mathbb{C}[D]$ is a completely reducible differential operator[3] with constant coefficients in $\mathbb{C}$, the determination of $T^\diamond$ reduces to solving an inhomogeneous first-order equation with constant coefficients—which is of course straightforward (Lemma 14). Also the determination of the interpolator $\mathcal{B}^\diamond$ turns out to be easy for a Cauchy problem since it is essentially given by the corresponding Taylor polynomial (3). Hence it remains to find some means for computing the *kernel projector* P for a boundary problem $(T, \mathcal{B})$.

In the case of a LODE of order n, the method for computing P given in the proof of Theorem 26 of [13] and in Section 6 of [9] is essentially a Gaussian elimination on the so-called *evaluation matrix* $\beta(u) = [\beta_i(u_j)]_{ij} \in K^{n \times n}$ formed by evaluating the i-th boundary condition β_i on the j-th fundamental solution u_j. So here we assume $u_1, \dots, u_n$ is a basis of $\operatorname{Ker} T$ and $\beta_1, \dots, \beta_n$ a basis of $\mathcal{B}$. Unfortunately, this is not a very intuitive description of P, and it is not evident how to generalize it to the LPDE case. We have to gain a more conceptual perspective at $\beta(u)$ for making the generalization transparent.

Let us write $\operatorname{Ev}\colon \mathcal{B} \oplus \operatorname{Ker} T \to K$ for the bilinear operation of evaluation $(\beta, u) \mapsto \beta(u)$. Choosing bases $\beta_1, \dots, \beta_n$ for $\mathcal{B}$ and $u_1, \dots, u_n$ for $\operatorname{Ker} T$, the coordinate matrix of Ev is clearly $\beta(u)$. By the usual technique of dualization, we can also think of Ev as the map $\mathcal{B}\colon \operatorname{Ker} T \to \mathcal{B}^*$ that sends $u \in \operatorname{Ker} T$ to the functional $\beta \mapsto \beta(u)$. But this map is nothing else than the *restriction of the trace map* $\operatorname{trc}\colon \mathcal{F} \to \mathcal{B}'$ to $\operatorname{Ker} T \subset \mathcal{F}$. It is easy to check that the restricted trace is bijective and that its inverse gives rise to the projector.

Proposition 11. *Let $(T, \mathcal{B})$ be a regular boundary problem with $E\colon \operatorname{Ker} T \to \mathcal{B}'$ being the restricted trace map. Then E is bijective with the state operator H as its inverse, and $P = H \circ \operatorname{trc}$ is the projector with* $\operatorname{Im} P = \operatorname{Ker} T$ *and* $\operatorname{Ker} P = \mathcal{B}^\perp$.

[3] By definition, this means its characteristic polynomial $T(\lambda) \in \mathbb{C}[\lambda] = \mathbb{C}[\lambda_1, \dots, \lambda_n]$ splits into linear factors.

We observe that the formula $P(u) = H(u^*)$ has a very *natural interpretation*: The kernel projector picks up the boundary data u^* of an arbitrary function $u \in \mathcal{F}$ and then constructs the required kernel element $H(u^*) \in \operatorname{Ker} T$ by solving the semi-homogeneous boundary problem with boundary data u^*. In the LODE case, the relation $P = H \circ \operatorname{trc}$ reduces to the aforementioned formulae (see [9] after Proposition 6.1) after choosing bases $u_1, \ldots, u_n$ for $\operatorname{Ker} T$ and $\beta_1, \ldots, \beta_n$ for $\mathcal{B}$. Apart from its conceptual clarity, the advantage of Proposition 11 is that it can also be used in the LDPE case (see after Lemma 14).

4 The Cauchy Problem for Analytic Functions

At this point we switch from the abstract setting of Sections 2 and 3 to the concrete setting of *analytic functions*. Note that we are dealing with complex-valued functions of real arguments. This means the ground field is $K = \mathbb{C}$, and $\mathcal{F}$ is the integro-differential algebra of entire functions restricted to real arguments.

More precisely, we shall employ the following conventions for easing the burden of book-keeping: As elements of $\mathcal{F}$ we take all holomorphic functions $\mathbb{R}^n \to \mathbb{C}$ for any $n \in \mathbb{N}$, including the constant functions $u \in \mathbb{C}$ for $n = 0$. In other words, $\mathcal{F}$ is a direct limit of algebras. Moreover, we have derivations D_n and integrals A_n for all $n > 0$, namely $D_n(u) = \partial u / \partial x_n$ and

$$A_n(u) = \int_0^{x_n} u(\ldots, \xi, \ldots)\, d\xi,$$

where ξ occurs at the n-th position. Clearly, we have then integro-differential algebras $(\mathcal{F}, D_n, A_n)$ for every $n > 0$. In fact, $\mathcal{F}$ has the structure of a *hierarchical integro-differential algebra*. This notion will be made precise at another occasion; for the moment it suffices to make the following observations. If $\mathbb{N}^{\circledast}$ is the sublattice of the powerset $\mathcal{P}(\mathbb{N}^+)$ that consists of finite sets $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ and the full set $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$, we define for $\alpha \in \mathbb{N}^{\circledast}$ the subalgebras

$$\mathcal{F}_\alpha = \{f \in \mathcal{F} \mid D_i f = 0 \text{ for all } i \notin \alpha\},$$

consisting of the functions depending (at most) on $x_{\alpha_1}, \ldots, x_{\alpha_k}$. Then $(\mathcal{F}_\alpha, \subseteq)$ is a sublattice of $(\mathcal{F}, \subseteq)$ that is isomorphic to the lattice $(\mathbb{N}^{\circledast}, \subseteq)$. The bottom element is of course $\mathcal{F}_\emptyset = \mathbb{C}$, the top element $\mathcal{F}_{\mathbb{N}^+} = \mathcal{F}$. We write $\mathcal{F}_n$ as an abbreviation for $\mathcal{F}_{\{1,\ldots,n\}}$.

As in the earlier paper [14], we add to this algebraic structure all linear *substitution operators*. In accordance with the above hierarchical structure, we use the ring $\mathbb{C}^*_*$ of row and column finite matrices with complex entries.[4] This means

[4] The *usage of complex substitutions* in functions of a real argument may sound strange at first. But an analytic function on $\mathbb{R}^n$ is of course also analytic on $\mathbb{C}^n$ with values in $\mathbb{C}$, so there is no problem with this view. For example, the substitution $(1, i)^*$ sends $f(x) = e^x \in \mathcal{F}_1$ to $f(x + iy) = e^x \cos y + ie^y \sin y \in \mathcal{F}_2$. Moreover, complex substitutions are indispensible for specifying the general solution of elliptic equations like the Laplace equation.

any $M \in \mathbb{C}_*^*$ can actually be seen as a finite matrix $M \in \mathbb{C}_n^m$ with m rows and n columns, extended by zero rows and columns. As usual, we identify $M \in \mathbb{C}_n^m$ with the linear map $M\colon \mathbb{C}^n \to \mathbb{C}^m$, yielding the substitution operator $M^*\colon \mathcal{F}_m \to \mathcal{F}_n$ defined by $u(x) \mapsto u(Mx)$.

We write $\mathcal{F}[D, A]$ for the *PIDOS algebra* generated over $\mathbb{C}$ by the operators D_n, A_n $(n > 0)$, the substitutions M^* induced by $M \in \mathbb{C}_*^*$ and the exponential basis polynomials $x^\alpha e^{\lambda x} \in \mathcal{F}$. Here x denotes the arguments $x = (x_1, \dots, x_n)$ for any $n \geq 0$, with exponents $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}$ and frequencies $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}$. Obviously, $\mathcal{F}[D, A]$ acts on $\mathcal{F}$, with $D = (D_1, D_2, \dots)$ denoting the differential operators and $A = (A_1, A_2, \dots)$ the integral operators, similar to the univariate case in the older notation of [11]. Here we avoid the notation $\int$ for the integrals since the powers $\int^n$ might be mistaken as integrals with upper bound n.

The algebra $\mathcal{F}[D, A]$ can be described by a *rewrite system* (PIDOS = partial integro-differential operator system), analogous to the one given in [14]. We will present this system in more detail—in particular proofs of termination and confluence—at another occasion.

Since in this paper we restrict ourselves to the analytic setting, we can appeal to the well-known *Cauchy-Kovalevskaya theorem* [10, Thm. 2.22] for ensuring the existence and uniqueness of the solution of the Cauchy problem. While the theorem in its usual form yields only local results, there is also a global version [4, Thm. 7.4] that provides a good foundation for our current purposes.[5] Since this form of the theorem is not widely known, we repeat the statement here.

As usual, we designate one *lead variable* t, writing the other ones $x_1, x_2, \dots$ as before. Note that in applications t is not necessarily time. The apparently special form of the differential equation $Tu = 0$ implies no loss of generality: Whenever $T \in \mathbb{C}[D]$ is a differential operator of order $\deg T = m$, the change of variables $\bar{t} = t, \bar{x}_i = x_i + t$ leads to an equation of the required form.

Theorem 12 (Global Cauchy-Kovalevskaya). *Let $T \in \mathbb{C}[D_t, D_1, \dots, D_n]$ be a differential operator in Caucy-Kovalevskaya form with respect to t, meaning $T = D_t^m + \tilde{T}$ with $\deg(\tilde{T}, t) < m$ and $\deg(\tilde{T}) \leq m$. Then the Cauchy problem*

$$\left.\begin{aligned} &Tu = 0 \\ &D_t^{i-1} u(0, x_1, \dots, x_n) = f_i(x_1, \dots, x_n) \text{ for } i = 1, \dots, m \end{aligned}\right\} \tag{2}$$

has a unique solution $u \in \mathcal{F}_{n+1}$ for given $(f_1, \dots, f_m) \in \mathcal{F}_n^m$.

In the *abstract language* of Sections 2 and 3 this is the semi-homogeneous boundary problem $(T, \mathcal{B})$ with boundary space

$$\mathcal{B} = [L_{0,\xi} D_t^i \mid i = 0, \dots, m-1 \text{ and } \xi \in \mathbb{R}^m],$$

where the evaluation $u(t, x_1, \dots, x_n) \mapsto u(0, \xi_1, \dots, \xi_n)$ is written as the substitution $L_{0,\xi} = \operatorname{diag}(0, \xi_1, \dots, \xi_m)^*$ denotes . Hence the solution of (2) is given

[5] Of course the problem may still be *ill-posed*; we will not treat this issue here.

by the state operator $(f_1, \dots, f_m) \in \mathcal{F}_n^m \mapsto u$ if we identify the boundary data $B \in \mathcal{B}'$ with its coordinate representation $(f_1, \dots, f_m) \in \mathcal{F}_n^m$ relative to the above boundary basis $(L_{0,\xi} D_t^i)$. In detail, $B \colon \mathcal{B} \to \mathbb{C}$ is the unique linear map sending $L_{0,\xi} D_t^i \in \mathcal{B}$ to $f(\xi) \in \mathbb{C}$; confer Lemma 5 for the uniqueness statement. In the sequel these identifications will be implicit.

For future reference, we mention also that the usual Taylor polynomial allows one to provide a natural *interpolator* for the initial data, namely

$$\mathcal{B}^\diamond(f_1, \dots, f_m) = f_1(x) + t\, f_2(x) + \cdots + \tfrac{t^{m-1}}{(m-1)!}\, f_m(x), \tag{3}$$

which we will not need here because compute the kernel projector directly from its first-order factors.

In this paper, we will study the Cauchy problem (2) for a *completely reducible operator* $T(D)$. Hence assume $T = T_1^{m_1} \cdots T_k^{m_k}$ with first-order operators $T_1, \dots, T_k \in \mathbb{C}[D]$. By a well-known consequence of the Ehrenpreis-Palamodov theorem, the general solution of $Tu = 0$ is the sum of the general solutions of the factor equations $T_1^{m_1} u = 0, \dots, T_k^{m_k} u = 0$; see the Corollary on [1, p. 187]. Hence it remains to consider differential operators that are powers of first-order ones (we may assume all nonconstant coefficients are nonzero since otherwise we reduce n after renaming variables).

Lemma 13. *Let $T = a + a_0 D_t + a_1 D_1 + \cdots + a_n D_n \in \mathbb{C}[D]$ be a first-order operator with all $a_i \neq 0$. Order the variables such that all cumulative sums $a_0 + a_1 + \cdots + a_{i-1}$ are nonzero. Then the general solution of $T^m u = 0$ is given by*

$$u(t, x_1, \dots, x_n) = \sum_{i=1}^{m} c_i(\bar{x}_1, \dots, \bar{x}_n)\, \tfrac{t^{i-1} e^{-at/a_0}}{(i-1)!}, \tag{4}$$

$$\bar{x}_i = t + x_1 + \cdots + x_{i-1} - (a_0 + a_1 + \cdots + a_{i-1})\, x_i / a_i, \tag{5}$$

where $(f_1, \dots, f_m) \in \mathcal{F}_{n-1}^m$ are arbitrary functions of the indicated arguments.

In principle, one could now combine the general solutions (4) for each factor, substitute them into the initial conditions of (2) and then solve for the c_i in terms of the prescribed boundary data $(f_1, \dots, f_m)$. With this choice of c_i, the general solution will become the state operator for the Cauchy problem. However, this is a very laborious procedure, and therefore we prefer to use another route. Since we assume a completely reducible operator, we can employ the product representation of Proposition 9. In that case, it remains to consider the case of a *single first-order factor*.

Lemma 14. *Let $T = a + a_0 D_t + a_1 D_1 + \cdots + a_n D_n \in \mathbb{C}[D]$ be a first-order operator with all $a_i \neq 0$. Then the Cauchy problem $Tu = 0$, $u(0, x_1, \dots, x_n) = f(x_1, \dots, x_n)$ has the state operator $H(f) = e^{-at/a_0}\, Z^* \tilde{Z}_x^* f$ and the signal operator $G = a_0^{-1}\, e^{at/a_0}\, Z^*\, A_t\, e^{-at/a_0}\, \tilde{Z}^*$, where $Z \in \mathbb{C}_{n+1}^{n+1}$ is the transformation (5) with $\bar{t} = t$, and $\tilde{Z}$ is its inverse (written as $\tilde{Z}_x$ when restricted to the arguments $x_1, \dots, x_n$).*

By Proposition 11, we can determine the *kernel projector* for the Cauchy problem of Lemma 14 as $P = H \circ \operatorname{trc}$, where $\operatorname{trc}(u) = u(0, x_1, \dots, x_n)$ in this simple case. Having the kernel projector and the right inverse $T^\diamond$ in Lemma 14, the *signal operator* is computed by $G = (1-P)T^\diamond$ as usual. Now we can tackle the general Cauchy problem (2) by a simple special case of Proposition 9.

Proposition 15. *Let $T_1, T_2 \in \mathbb{C}[D]$ be two first-order operators with nonzero coefficients for D_t. If $L_{0,\xi}$ is the evaluation defined after Theorem. 12, then we have*

$$(T_1, [L_{0,\xi} \mid \xi \in \mathbb{R}])\,(T_2, [L_{0,\xi} \mid \xi \in \mathbb{R}]) = (T_1 T_2, [L_{0,\xi}, L_{0,\xi} D_t \mid \xi \in \mathbb{R}]$$

for the product of the Cauchy problems.

This settles the completely reducible case: Using Proposition 15 we can break down the *general Cauchy problem* (2) into first-order factors with single initial conditions. For each of these we compute the state and signal operator via Lemma 14, hence the state and signal operator of (2) by Proposition 9.

Example 16. As a typical example, let us consider a *modified wave equation* as treated in [3, §5.2] within the general class of higher-order hyperbolic equations with constant coefficients. We want to solve the initial value problem given by

$$\boxed{\begin{array}{l} u_{tt} - 4\,u_{tx} + 4\,u_{xx} - 9\,u_{yy} = f, \\ u(0,x,y) = f_1(x,y), \quad u_t(0,x,y) = f_2(x,y) \end{array}} \tag{6}$$

for a given forcing function f and initial data f_1, f_2. Note that the differential operator $T = D_t^2 - 4\,D_t D_x + 4\,D_x^2 - 9\,D_y^2$ factors completely as $T = T^- T^+$ with the two first-order factors $T^\pm = D_t - 2\,D_x \pm 3\,D_y$. In view of Proposition 9, it suffices to consider the two first-order factor problems

$$\boxed{\begin{array}{l} u_t - 2\,u_x \pm 3\,u_y = f, \\ u(0,x,y) = f^\pm(x,y). \end{array}} \tag{7}$$

Using Lemma 14, we obtain $H^\pm f^\pm\,(t,x,y) = f^\pm(x+2t, y \mp 3t)$ for the state operators and

$$G^\pm f\,(t,x,y) = \textstyle\int_0^t f(\tau, x + 2t - 2\tau, y \mp 3t \pm 3\tau)\,d\tau$$

for the signal operators. By Proposition15 the composition of the two semi-inhomogeneous boundary problems for (7) yields the semi-inhomogeneous problem for (6), hence we may compute the signal operator as $G = G^+ G^-$ with

$$Gf\,(t,x,y) = \textstyle\int_0^t \int_0^\sigma f(\tau, x + 2t - 2\tau, y - 3t - 3\tau + 6\sigma)\,d\tau\,d\sigma.$$

Using Proposition 9 now for computing the composite state operator one obtains at first $u(t,x,y) = H(f^-, f^+) = G^+ H^- f^- + H^+ f^+$ with

$$u(t,x,y) = f^+(x+2t, y-3t) + \textstyle\int_0^t f^-(x+2t, y-3t+6\tau)\,d\tau.$$

But this result should be interpreted cautiously: It solves the differential equation of (6) for the initial conditions $u(0,x,y) = f^+(x,y)$ and $T^+u\,(0,x,y) = f^-(x,y)$. But since $T^+u(0,x,y) = u_t(0,x,y) - (2D_x - 3D_y)f_1(x,y)$, we just have to pick $f^+(x,y) = f_1(x,y)$ and $f^-(x,y) = f_2(x,y) - 2D_xf_1(x,y) + 3D_yf_1(x,y)$ so that $H(f_1, f_2) = u$ with

$$u(t,x,y) = f_1(x+2t, y-3t) + \int_0^t (f_2 - 2\,D_xf_1 + 3\,D_yf_1)(x+2t, y-3t+6\tau)\,d\tau$$

is the state operator for the given initial value problem (6).

5 Conclusion

As explained in the Introduction, we see the framework developed in this paper as the first stage of a more ambitious endeavor aimed at boundary problems for general constant-coefficient (and other) LPDEs. Following the enumeration of the Introduction, the next steps are as follows:

1. Stage (1) was presented in this paper, but the *detailed implementation* for some of the methods explained here is still ongoing. The crucial feature of this stage is that it allows us to stay within the (rather narrow) confines of the PIDOS algebra. In particular, no Fourier transformations are needed in this case, so the analytic setting is entirely sufficient.
2. As we enter Stage (2), it appears to be necessary to employ stronger tools. The most popular choice is certainly the *framework of Fourier transforms* (and the related Laplace transforms). While this can be algebraized in a manner completely analogous to the PIDOS algebra, the issue of choosing the right function space becomes more pressing: Clearly one has to leave the holomorphic setting for more analysis-flavoured spaces like the Schwartz class or functions with compact support. (As of now we stop short of using distributions since that would necessitate a more radical departure, forcing us to give up rings in favor of modules.)
3. For the treatment of genuine boundary problems in Stage (3) our plan is to use a powerful generalization of the Fourier transformation—the *Ehrenpreis-Palamodov integral representation* [1], also applicable to systems of LPDEs.

Much of this is still far away. But the *general algebraic framework* for boundary problems from Sections 2 and 3 is applicable, so the main work ahead of us is to identify reasonable classes of LPDEs and boundary problems that admit a symbolic treatment of one sort or another.

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