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Geometric Control Theory and Sub-Riemannian Geometry

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To Andrei Agrachev,
on the occasion of his 60^{th} birthday

Preface

Geometric Control Theory and sub-Riemannian geometry are two areas whose fruitful interaction has been witnessed over the last decades.

On the one hand Geometric Control Theory used the differential geometric and Lie algebraic language for studying controllability, motion planning, stabilizability and optimality for nonlinear and linear control systems. Reflected in one of the contributions to the volume is the fact that the foundational result of optimal control theory – Pontryagin Maximum Principle – has differential geometric/Lie algebraic interpretation. The geometric approach turned out to fruitful in applications to robotics, vision modeling, mathematical physics etc. Current research in geometric control theory is concerned with polydynamic models, described by systems of nonlinear ODEs or PDEs with (control) parameters, or, geometrically speaking, by linear or affine subdistributions of tangent/cotangent bundles of manifolds of finite or infinite dimension.

On the other hand Rimemannian geometry and its generalizations, like sub-Riemannian, semi-Riemannian, Finslerian geometry etc., have been actively adopting methods developed in the scope of geometric control. Application of these methods has led to important results regarding geometry and topology of sub-Riemannian spaces, regularity of sub-Riemannian distances, properties of the group of diffeomorphisms of sub-Riemannian manifolds, local geometry and equivalence of distributions and sub-Riemannian structures, regularity of the Hausdorff volume etc.

Directions of active studies, partially reflected in the present collection are sketched below.

Geometric optimal control. This area is naturally drawn to invariant Hamiltonian formulations and use of the concepts and methods of symplectic geometry. Of particular use are the notions of Jacobi curve and Maslov cycle in Lagrangian Grassmanian, which are central for studying sub-Riemannian length minimization problems. Hamiltonian lifts to cotangent bundle allow for establishing second-order optimality conditions for extremals in optimal control problem with parameters. The same Hamiltonian approach, together with numerical schemes, is used for computation of conjugate and cut loci of metrics on Riemannian surfaces. Some integrability problems are addressed for Pontryagin-Hamilton optimality conditions for high-dimensional generalizations of Euler elastica and of Dubins's minimal path problem. Study of topology of configuration space of complex robotic systems allows to discover topological obstructions to continuous feedback stabilizability.

Geometry of sub-Riemannian manifolds. Natural range of issues is an extension of concepts and results of Riemannian geometry for sub-Riemannian manifolds. Those include, for example, curvature-dimension inequalities and Li-Yau-type estimates, as well as smoothness of length-minimizing curves in sub-Riemannian geometry, which is a long standing open problem. Curvature-type (feedback) invariants (introduced by A.A. Agrachev) can be computed for extremals of "least action principle" for natural mechanical systems on sub-Riemannian manifolds.

Classification problem for distributions is a long time challenge. It has been discovered some decades ago, that the geometry of distributions on manifolds can be characterized via Hamiltonian flows which define abnormal sub-Riemannian geodescis. In several symmetric cases the analysis of the abnormal geodesic flow can lead to a construction of canonical moving frames and to description of the moduli spaces for the rank-2 distributions in $\mathbb{R}^n$. Some conjectures regarding classification of (affine) line fields in $\mathbb{C}^n$ with transitive symmetry algebrae, are confirmed for $n = 2, 3$.

Analysis and topology of Carnot-Caratheodory spaces. Analysis in Carnot-Caratheodory spaces is a well established area, whose interaction with sub-Riemannian geometry and the geometric control theory is natural. Topics, illustrating such interaction include: intrinsic notions of Lipschitz maps and Lipschitz domains in Carnot groups, computation of Hausdorff dimension of sub-Riemannian manifolds and of Hausdorff volume of small balls in sub-Riemannian metrics, local approximation theorem for Carnot-Caratheodory spaces, comparison of various topologies in sub-Lorentzian manifolds.

Controllability and optimal control problems for PDEs. Geometric control for infinite-dimensional systems and PDE's is a rather new research area, whose differential geometric/Lie algebraic apparatus is yet to be created. Recent progress concerns the approximate controllability of the viscous Burger's equation on a line by means of trigonometric polynomial control, the null controllability property for parabolic Grusin equation with singular potential, the optimality of steady state modes for a model of exploitation of size-structured population.

The volume contains contributions of the participants of the Meeting on Geometric Control Theory and sub-Riemannian Geometry, which took place in Cortona on May 21–25, 2012. The Meeting has been kindly sponsored by the Istituto Nazionale di Alta Matematica "F. Severi" (INdAM).

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The Workshop has been dedicated to 60th anniversary of professor Andrei A. Agrachev, whose ideas are deeply influential in geometric control and adjacent areas. Many contributors of the present volume are his coauthors, former and current students, scholars inspired by his work in the above mentioned fields. On request of the editors professor A.Agrachev contributed a survey of some open problems in geometric control theory and sub-Riemannian geometry.

July 2013 The Editors

Contents

Some open problems

Andrei A. Agrachev

Abstract We discuss some challenging open problems in the geometric control theory and sub-Riemannian geometry.

1 Singularities of time-optimal trajectories

It is getting harder to prove theorems and easier to force other people to prove them when you are sixty. Some colleagues asked me to describe interesting open problems in geometric control and sub-Riemannian geometry. Here I list few really challenging problems; some of them are open for a long time and were publicly or privately stated by well-known experts: J.-M. Coron, I. Kupka, R. Montgomery, B. Shapiro, H. Sussmann, and others.

Let f, g be a pair of smooth (i. e. C^∞) vector fields on a n-dimensional manifold M. We study time-optimal trajectories for the system

$$\dot{q} = f(q) + ug(q), \quad |u| \leq 1, \ q \in M,$$

with fixed endpoint. Admissible controls are just measurable functions and admissible trajectories are Lipschitz curves in M. We can expect more regularity from time-optimal trajectories imposing reasonable conditions on the pair of vector fields.

A. *(f, g) is a generic pair of vector fields.* Optimal trajectories cannot be all smooth; are they piecewise smooth? This is true for $n = 2$. More precisely, if $\dim M = 2$, then any point of M has a neighborhood such that any contained in the

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neighborhood time-optimal trajectory is piecewise smooth with atmost 1 switching point (see [11, 32]). According to the control theory terminology, a switching point of an admissible trajectory is a point where the trajectory is not smooth.

The question is open for $n = 3$. What is known? Let $\mathrm{sw}(q)$ be minimal among numbers k such that any contained in a sufficiently small neighborhood of $q \in M$ time-optimal trajectory has no more than k switching points. We set $\mathrm{sw}(q) = \infty$ if any neighborhood of q contains a time-optimal trajectory with an infinite number of switching points. It is known that $\mathrm{sw}(q) = 2$ for any q out of a 2-dimensional Whitney stratified subset of the 3-dimensional manifold M (see [28, 33]) and $\mathrm{sw}(q) \leq 4$ for any q out of a 1-dimensional Whitney stratified subset of M (see [7]). Some further results in this direction can be found in [31]. We do not know if $\mathrm{sw}(q) < \infty$ for any $q \in M$. We also do not know if a weaker property, the finiteness of the number of switching points for any individual time-optimal trajectory is valid.

Higher dimensions. There is a common opinion that starting from some (not very big) dimension, time-optimal trajectories with accumulating switching points cannot be eliminated by a C^∞-small perturbation of the system and thus survive any genericity conditions. However, to my knowledge, this opinion was never supported by a proof. There are very interesting examples of extremals with accumulating switching points whose structure survives small perturbations (see [21, 37]) but nobody knows if these extremals are optimal.

B. f, g are real analytic vector fields. Let M be a real analytic manifold and f, g analytic vector fields, not necessary generic. Here we cannot expect any regularity of an arbitrary time-optimal trajectory. Indeed, it is possible, even for linear systems, that all admissible trajectories are time-optimal. We can however expect that among all time-optimal trajectories connecting the same endpoints there is at least one not so bad.

If $n = 2$, then any two points connected by a time-optimal trajectory can be connected by a time-optimal trajectory with a finite number of switching points (see [35,36]). This is not true for $n \geq 3$. Indeed, classical Fuller example with accumulating switching points [17] can be easily reformulated as a 3-dimensional time-optimal problem. Main open question here is as follows: Given two points connected by a time-optimal trajectory, can we connect them by a time-optimal trajectory with no more than a countable number of switching points?

What is known? The points can be connected by a time-optimal trajectory whose set of switching points is nowhere dense [34], is not a Cantor set (can be derived from [1]), and satisfies some additional restrictions [31]. We do not know if we can avoid a positive measure set of switching points.

All mentioned open questions are not easy to answer. In my opinion, the most interesting is one on generic 3-dimensional systems.

2 Cutting the corners in sub-Riemannian spaces

Unlikely the just discussed problems, optimal paths in sub-Riemannian geometry are usually smooth. However we do not know if they are always smooth. A natural open question here is as follows. Let $\gamma_i : [0,1] \to M$ be two smooth admissible paths of a sub-Riemannian structure on the manifold M, $\gamma_0(0) = \gamma_1(0) = q_0$, $\dot{\gamma}_0(0) \wedge \dot{\gamma}_1(0) \neq 0$. Does there exist an admissible path connecting $\gamma_0(1)$ with $\gamma_1(1)$ that is strictly shorter than the concatenation of the curves γ_0 and γ_1?

Admissible paths are integral curves of a bracket generating vector distribution $\Delta \subset TM$. It is easy to show that positive answer to the question for rank 2 distributions implies positive answer in the general case. Let $\Delta = span\{f_0, f_1\}$, where f_0, f_1 are smooth vector fields on M, $\dim M = n$. We set

$$n_k(q) = \dim span \left\{[f_{i_1}, [\cdots, f_{i_j}]\cdots](q) : i_j \in \{0,1\},\ j \leq k\right\},$$

$m = \min\{k : n_k(q_0) = n\}$. If $m \leq 4$, then the answer to our question is positive: it is proved in [22]. Moreover, an example studied in [24] supports the conjecture that the answer is perhaps positive for $m = 5$, $n \leq 4$ as well. Any improvement of the estimates for m and n would be very interesting. We still know very little about sub-Riemannian structures with big m and it may happen that the answer is negative for some m and n.

3 "Morse–Sard theorem" for the endpoint maps

We continue to consider admissible paths of a sub-Riemannian structure on M. Given $q_0 \in M$, the space of starting at q_0 admissible paths equipped with the H^1-topology forms a smooth Hilbert manifold. The endpoint map is a smooth map from this Hilbert manifold to M; it sends a path $\gamma : [0,1] \to M$ to the point $\gamma(1)$. Critical points of the endpoint map are called *singular curves* of the distribution.

The Morse–Sard theorem for a smooth map defined on a finite dimensional manifold states that the set of critical values of the map has zero measure. It is not true in the infinite dimensional case: there are smooth surjective maps without regular points from any infinite dimensional Banach space to $\mathbb{R}^2$ (see [9]).

The endpoint maps have plenty of regular points but we do not know if they always have regular values. This is an interesting open question. We can reformulate the question as follows: is it possible that starting from q_0 singular curves fill the whole manifold M?

Optimal (i. e. length minimizing) singular curves are better controlled; we know that starting from q_0 optimal singular curves fill a nowhere dense subset of M (see [2]). An important open question: can they fill a positive measure subset of M?

4 Unfolding the sub-Riemannian distance

The problem concerns singularities of the distance function for generic sub-Riemannian structures. Let $q_0 \in M$ and $S_{q_0} : M \to [0, +\infty)$ be the sub-Riemannian distance from the point q_0. Sufficiently small balls $S^{-1}([0, \varepsilon])$ are compact. Let $q \in M$ be a point from such a ball. Then q is connected with q_0 by an optimal path. If this optimal path is not a singular curve and any point from a neighborhood of q is connected with q_0 by a unique optimal path, then S_{q_0} is smooth at q.

The points connected with q_0 by more than one optimal path form the *cut locus*. The function S_{q_0} is not smooth in the points of the cut locus and it is not smooth at the points connected with q_0 by optimal singular curves but these two types of singularities are very different.

If all connecting q_0 and q optimal paths are not singular curves, then the singularity of S_{q_0} at q is similar to singularities of Riemannian distances and, more generally, to singularities of the optimal costs of regular variational problems. The function S_{q_0} is semiconcave [12] and typical singularities in low dimensions are well-described by the theory of Lagrangian and Legendrian singularities [8, Ch. 3] developed by V. Arnold and his school.

On the other hand, if q_0 is connected with q by an optimal singular curve, then S_{q_0} is not even locally Lipschitz at q_0 (see [3, Ch. 10]); moreover, classical singularities theory does not work and the structure of typical singularities is totally unknown. There are few studied models [4, 27] but they are too symmetric to be typical and the structure of their singularities is easily destroyed by small perturbations.

Let us consider, in particular, the *Martinet distribution* that is a rank 2 distribution in $\mathbb{R}^3$ in a neighborhood of a point q_0 such that $n_2(q_0) = 2,\ n_3(q_0) = 3$ (see **II.** for the definition of $n_i(q_0)$). The points q in a neighborhood of q_0 where $n_2(q) = 2$ form a smooth 2-dimensional submanifold $N \subset M$, the *Martinet surface*. Moreover, the distribution Δ is transversal to N and $\Delta_q \cap T_q N,\ q \in N$ is a line distribution on N. Integral curves of this line distribution are singular curves whose small segments are optimal. There are no other singular curves for such a distribution.

Example. Let $f_1 = \frac{\partial}{\partial x_1},\ f_2 = \frac{\partial}{\partial x_2} + x_1^2 \frac{\partial}{\partial x_3}$, then $\Delta = span\{f_1, f_2\}$ is a Martinet distribution and the Martinet surface is a coordinate plane defined by the equation $x_1 = 0$. The fields f_1, f_2 form an orthonormal frame of the so called 'flat' sub-Riemannian metric on the Martinet distribution. Let $q_0 = 0$, singularities of S_0 are well-known (see [4]). The cut locus has the form: $\{x \in \mathbb{R}^3 : x_1 = 0,\ x_2 \neq 0\}$, the Martinet surface with the removed singular curve through q_0. The singular locus of a sphere $S_{q_0}^{-1}(\varepsilon)$ is a simple closed curve and its complement (the smooth part of the sphere) is diffeomorphic to the disjoint union of two discs.

The 'flat' metric is rather symmetric, in particular, it respects the orthogonal reflection of $\mathbb{R}^3$ with respect to the Martinet plane. Simple topological arguments show that for generic metric with a broken symmetry, the smooth part of a sphere is connected and is not contractible. The singular locus of the sphere should be cut at the points where the sphere intersects the optimal singular curve but the shape of the sphere near these points is unknown.

An important open question is to find a C^1-classification of the germs of spheres at the points of optimal singular curves for generic metrics. Here we say that two germs are C^1-equivalent if one can be transformed into another by a germ of C^1-diffeomorphism of $\mathbb{R}^3$.

The next step is the Engel distribution, i. e. a rank 2 distribution in $\mathbb{R}^4$ such that $n_3(q) = 3,\ n_4(q) = 4$. There is exactly one singular curve through any point and small segments of singular curves are optimal. We repeat our question for this case; the spheres are now 3-dimensional hypersurfaces in $\mathbb{R}^4$.

Generic germ of a rank 2 distribution in $\mathbb{R}^n$ possesses a $(n-4)$-dimensional family of singular curves through $q_0 \in \mathbb{R}^n$ (see, for instance, [25]). Take a generic curve from this family; its small segments are optimal. Take a point q where the selected singular curve intersects the sphere $S_{q_0}^{-1}(\varepsilon)$. The points of singular curves from the family in a small neighborhood of q in our sphere form a smooth $(n-4)$-dimensional submanifold $\Sigma \subset \mathbb{R}^n$. The intersection of the sphere with a transversal to Σ smooth 4-dimensional submanifold should have a shape similar to the germ of the sphere in the Engel case. A neighborhood of q in the sphere is fibered by such intersections. Hence the solution of the problem in the 4-dimensional Engel case is a very important step in the unfolding of the sphere for any $n \geq 4$.

The desired classification seems to be complicated. There is a 2-dimensional modification of the problem that, in my opinion, already contains the main difficulty. Ones resolved, it will reduce the study of higher dimensional problems to the conventional singularities theory techniques. Consider the germ at $q_0 \in \mathbb{R}^2$ of a pair of smooth vector fields f_0, f_1 such that $f_0(q_0) \wedge f_1(q_0) = 0,\ n_2(q_0) = 1,\ n_3(q_0) = 2$. The almost Riemannian distance $S_{q_0}(q)$ is the optimal time to get q from q_0 by an admissible trajectory of the system

$$\dot{q} = u_0 f_0(q) + u_1 f_1(q), \quad u_1^2 + u_2^2 = 1.$$

The question is to find a C^1-classification of the germs of distance functions S_{q_0} for generic pairs of vector fields f_0, f_1 among the pairs that satisfy conditions $n_2(q_0) = 1,\ n_3(q_0) = 2$. See [10] for some partial results.

5 Symmetries of vector distributions

A symmetry of a distribution $\Delta \subset TM$ is a diffeomorphism $\Phi : M \to M$ such that $\Phi_* \Delta = \Delta$. The differential geometry appeals to search most symmetric objects in the class, those with a maximal symmetry group. The singularities theory, on the contrary, encourages the study of less symmetric generic objects. Both paradigms have their reasons and complement each other. Anyway, a fundamental problem is to characterize objects whose symmetry groups are finite-dimensional Lie groups.

Our objects are vector distributions. Any symmetry transfers singular curves of the distribution in singular curves and these curves often play a key role in the calculation of symmetry groups (see [15, 20]). We say that a distribution is singular transitive if any two points of M can be connected by a concatination of singular curves. A natural open question is as follows: Is it true that singular transitivity of the distribution implies that its symmetry group is a finite dimensional Lie group?

All known examples support the positive answer to this question. Moreover, the group of symmetries is infinite dimensional for many popular classes of not singular transitive distribution: codimension 1 distributions, involutive distributions, Goursat–Darboux distributions. We can even expect that any not singular transitive rank 2 distribution has an infinite dimensional symmetry group. Some results of [15] seem to be rather close to this statement.

6 Closed curves with a nondegenerate Frenet frame

Let $\gamma : S^1 \to \mathbb{R}^n$ be a smooth closed curve in $\mathbb{R}^n$. We say that γ is degenerate at $t \in S^1$ if $\dot\gamma(t) \wedge \cdots \wedge \gamma^{(n)}(t) = 0$. Degeneracy points are the points where velocity or curvature of the curve vanishes if $n = 2$, where velocity or curvature or torsion vanishes if $n = 3$ e. t. c. The curve is nondegenerate if it has no degeneracy points. Any nondegenerate curve admits the orthonormal Frenet frame $E(t) = (e_1(t), \dots, e_n(t))$, $t \in S^1$, that is a smooth closed curve in the orthogonal group $O(n)$.

Now let $n = 3$ and γ be a plane convex curve, $\gamma(t) \in \mathbb{R}^2 \subset \mathbb{R}^3,\ \forall t \in S^1$. Then any small perturbation of γ as a spatial curve is degenerate in some points. On the other hand, an appropriate small perturbation of a plane convex curve run twice (say, of the curve $t \mapsto \gamma(2t),\ t \in S^1$) makes it a nondegenerate curve in $\mathbb{R}^3$. Everyone can get evidence of that playing with a cord on the desk. This is also a mathematical fact proved in [16, 23].

Frenet frame of the plane convex curve treated as a spatial curve is a one-parametric subgroup $SO(2) \subset O(3)$, a shortest closed geodesic in $O(3)$ equipped with a standard bi-invariant metric. It is proved in [23] that the length of the Frenet frame of any regular curve in $\mathbb{R}^3$ is greater than the double length of $SO(2)$.

Come back to an arbitrary n. Let $\mu(n)$ be minimal m such that run m times convex plane curves have regular small perturbations in $\mathbb{R}^n$. We know that $\mu(2) = 1,\ \mu(3) = 2$. An important open problem is to find $\mu(n)$ for $n > 3$ and to check if the length of the Frenet frame of any regular curve in $\mathbb{R}^n$ is greater than the length of $SO(2) \subset O(n)$ multiplied by $\mu(n)$.

Let me explain why this problem is a challenge for the optimal control theory and why its study may bring important new tools to the theory. The Frenet structural equations for a regular curve γ in $\mathbb{R}^n$ have a form:

$$\dot\gamma = e_1, \quad \dot e_i = u_k(t)e_{i+1} - u_{i-1}(t)e_{i-1}, \quad i = 1, \dots, n-1, \tag{1}$$

where $u_0 = u_n = 0,\ u_i(t) > 0,\ i = 1, \dots, n-1,\ t \in S^1$.

In other words, regular curves together with there Frenet frames are periodic admissible trajectories of the control system (1) with positive control parameters $u_1, \dots, u_n$. The length of the Frenet frame on the segment $[0, t_1]$ is equal to $\int_0^{t_1} \left(u_1^2(t) + \cdots + u_{n-1}^2(t)\right)^{\frac{1}{2}} dt$. We are looking for a periodic trajectory with shortest Frenet frame.

A shortest frame is unlikely to exists since control parameters belong to an open cone. It is reasonable to expect that minimizing sequences converge to a solution of (1) with $u_2(t) \equiv \cdots \equiv u_{n-1}(t) \equiv 0$, while $u_1(t)$ stays positive to guarantee the periodicity of γ. In other words, the infimum is most likely realized by a plane convex curve run several times. Obviously, the length of the Frenet frame does not depend on the shape of the convex curve.

So we have to take the m times run circle: $u_1(t) = 1,\ u_2(t) = \cdots = u_{n-1}(t) = 0,\ 0 \leq t \leq 2\pi m$, and try to find small positive perturbations of control parameters in such a way that the perturbed curve stays periodic. Then $\mu(n)$ is minimal among m for which such a perturbation does exist. Unfortunately, we cannot use typical in geometric control sophisticated two-side variations that produce iterated Lie brackets: only one-side variations are available. I think, it is a very good model to understand high order effects of time-distributed one-side variations.

The study of the 3-dimensional case by Milnor in [23] was not variational; it was a nice application of the integral geometry. However, the integral geometry method is less efficient in higher dimensions (see [26] for some partial results).

7 Controllability of the Navier–Stokes equations controlled by a localized degenerate forcing

We consider the Navier–Stokes equation of the incompressible fluid:

$$\frac{\partial u}{\partial t} + (u, \nabla)u - \nu\Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0, \tag{2}$$

with periodic boundary conditions: $x \in \mathbb{T}^d/2\pi\mathbb{Z}^d,\ d = 2, 3$. Here $u(t, x) \in \mathbb{R}^d$ is the velocity of the fluid at the point x and moment t; ν is a positive constant (viscosity), p is the pressure and η external force.

We treat (2) as an evolution equation in the space of divergence free vector fields on the torus $\mathbb{T}^d$ controlled by the force. In other words, $u(t, \cdot)$ is the state of our infinite dimensional control system and η is a control. These notations are against the control theory tradition where u is always control but we do not want to violate absolutely standard notations of the mathematical fluid dynamics. By the way, symbol u for the control was introduced by Pontryagin as the first letter of the Russian word "upravlenie" that means control.

The state space is $V = \left\{u \in H^1(\mathbb{T}^d, \mathbb{R}^d) : \operatorname{div} u = 0\right\}$, control parameters $\eta(t, \cdot)$ belong to a subspace $E \subset V$. We say that the system is approximately controllable (controllable in finite dimensional projections) in any time if for any $u_0, u_1 \in V,\ t_1 > 0$ and any $\varepsilon > 0$ (any finite dimensional subspace $F \subset V$) there exists a bounded control $\eta,\ \eta(t, \cdot) \in E,\ 0 \leq t \leq t_1$, and a solution u of (2) such that

$u(0,\cdot) = u_0,\ \|u(t_1,\cdot) - u_1\|_{L_2} < \varepsilon\ \ \big(P_F(u(t_1,\cdot) - u_1) = 0$, where P_F is the L_2-orthogonal projector on $F\big)$.

Of course, controllability properties depend on the choice of the space of control parameters E. It is known that the systems is controllable in both senses by a *localized forcing* when $E = \{u \in V : \operatorname{supp} u \subset \bar{\mathcal{D}}\}$ and $\mathcal{D}$ is an arbitrary open subset of $\mathbb{T}^d$. Moreover, such E provides a much stronger exact controllability (see [13, 14, 18, 19]).

On the other hand, the system is approximately controllable and controllable in finite-dimensional projections by a *degenerate forcing* (or forcing with a localized spectrum) when E is a finite dimensional space of low frequency trigonometric polynomials (see [5,6,29,30]). This kind of controllability illustrates a mechanism of the energy propagation from low to higher frequencies that is a necessary step in the long way towards a reliable mathematical model for the well-developed turbulence.

It is important that the control parameters space E does not depend on the viscosity ν. Moreover, if $d = 2$, then the described controllability properties are valid also for the Euler equation (i. e. for $\nu = 0$); the Cauchy problem for the Euler equation is well-posed in this case.

Now an important open question: is the system approximately controllable and (or) controllable in the finite dimensional projections by a *localized degenerate forcing* when E is a finite dimensional subspace of the space $\{u \in V : \operatorname{supp} u \subset \bar{\mathcal{D}}\}$? The question is about existence and effective construction of such a space E that does not depend on the viscosity ν.

The independence on ν is important for eventual applications to the well-developed turbulence that concerns the case of very small ν (or very big Reynold number). Of course, similar problems for other boundary conditions and other functional spaces are also very interesting.

We have arrived to a sacral number of seven problems and can relax a little bit. To conclude, I would like to discuss one more problem; it is less precise than already stated questions but, to my taste, is nice and fascinating. The problem concerns contact 3-dimensional manifolds and is inspired by the Ricci flow story.

8 Diffusion along the Reeb field

I recall that a contact structure on a 3-dimensional manifold M is a rank 2 distribution $\Delta \subset M$ such that $n_2(q) = 3,\ \forall q \in M$. According to a classical Martinet theorem, any orientable 3-dimensional manifold admits a contact structure. I am going to introduce some dynamics on the space of sub-Riemannian metrics on a fixed compact contact manifold (M, Δ).

First, to any sub-Riemannian metric on (M, Δ) we associate a transversal to Δ Reeb vector field e on M. In what follows, we assume that Δ is oriented; otherwise $e(q)$ is defined up-to a sign but further considerations are easily extended to this case. Let ω be a nonvanishing differential 1-form on M that annihilates Δ. The condition $n_2(q) = 3$ is equivalent to the inequality $\omega_q \wedge d_q\omega \neq 0$. The form ω is defined up-to the multiplication by a nonvanishing function; the sign of the 3-form $\omega_q \wedge d_q\omega$ does not depend on the choice of ω and defines an orientation on M. We have: $d_q\omega\big|_{\Delta_q} \neq 0$; moreover, $d_q(a\omega)\big|_{\Delta_q} = a(q)d_q\omega\big|_{\Delta_q}$ for any smooth function a of M.

Given a sub-Riemannian metric on Δ, there exists a unique annihilating Δ form ω such that the 2-form $d_q\omega\big|_{\Delta_q}$ coincides with the area form on Δ_q defined by the inner product and the orientation. The kernel of $d_q\omega$ is a 1-dimensional subspace of T_qM transversal to Δ_q, and $e(q)$ is an element of this kernel normalized by the condition $\langle \omega_q, e(q)\rangle = 1$.

In other words, the Reeb vector field is defined by the conditions: $i_e\omega = 1,\ i_e d\omega = 0$. Hence $L_e\omega = 0$, where L_e is the Lie derivative along e, and the generated by e flow on M preserves ω. In general, this flow does not preserves the sub-Riemannian metric. We may try to classify contact structures by selecting best possible sub-Riemannian metrics on them.

Assume that there exists a metric preserved by the flow generated by the Reeb vector field. Take a standard extension of the sub-Riemannian metric to a Riemannian metric on M: simply say that e is orthogonal to Δ and has length 1. The generated by e flow preserves this Riemannian metric as well. So our compact Riemannian space admits a one-parametric group of isometries without equilibria. Hence M is a Seifert bundle. Do not care if you do not remember what is Seifert bundle: it is sufficient to know that they are classified as well as invariant contact structures on them.

The invariant with respect to the Reeb field sub-Riemannian metric gives a lot of information about the manifold. Let $q \in M$; our sub-Riemannian metric induces a structure of Riemannian surface on a neighborhood of q factorized by the trajectories of the local flow generated by the restriction of e to the neighborhood. Let $\kappa(q)$ be the Gaussian curvature of this Riemannian surface at the point q; then κ is a well-defined smooth function on M, a differential invariant of the sub-Riemannian metric. Moreover, κ is a first integral of the flow generated by the Reeb field e. If $\kappa = 0$, then universal covering of the sub-Riemannian manifold is isometric to the Heisenberg group endowed with the standard left-invariant metric. If κ is a negative (positive) constant, then universal covering of the sub-Riemannian manifold is isometric to the universal covering of the group SL(2) (group SU(2)) equipped with a left-invariant sub-Riemannian metric induced by the Killing form.

Assume that function κ is not a constant and $c \in \mathbb{R}$ is it regular value. Then $\kappa^{-1}(c)$ is a compact 2-dimensional submanifold of M; we treat it as a 2-dimensional Riemannian submanifold of the Riemannian manifold M equipped with the standard extension of the sub-Riemannian structure. It is easy to see that $\kappa^{-1}(c)$ is isometric to a flat torus. Indeed, $T(\kappa^{-1}(c))$ contains the field $e|_{\kappa^{-1}(c)}$ and is transversal to Δ; the field $e|_{\kappa^{-1}(c)}$ and the unit length field from the line distribution $T(\kappa^{-1}(c)) \cap \Delta$ commute and form an orthonormal frame.

So preserved by the Reeb field sub-Riemannian metrics have plenty of nice properties. Unfortunately, not any compact contact manifold admits such a metric because not any compact 3-dimensional manifold admits a structure of Seifert bundle.

I am going to discuss a natural procedure that may lead to a generalized version of such a metric with reasonable singularities.

It is more convenient to work in the cotangent bundle than in the tangent one. A sub-Riemannian metric is an inner product on $\Delta \subset TM$; let us consider the dual inner product on $\Delta^* = T^*M/\Delta^\perp$, where $\Delta^\perp$ is the annihilator of Δ. This is a family of positive definite quadratic forms on $\Delta^* = T_q^*M/\Delta_q^\perp$, $q \in M$, or, in other words, a family of nonnegative quadratic forms h_q on T_q^*M such that $\ker h_q = \Delta_q^\perp$. The function

$$h : T^*M \to \mathbb{R}, \text{ where } h(\xi) = h_q(\xi), \quad \forall \xi \in T_q^*M, \ q \in M,$$

is the *Hamiltonian of the sub-Riemannian metric*. Hamiltonian vector field on T^*M associated to h generates the sub-Riemannian geodesic flow.

The Hamiltonian h determines both the vector distribution and the inner product. We denote by $u_h : T^*M \to \mathbb{R}$ the Hamiltonian lift of the Reeb field e,

$$u_h(\xi) = \langle \xi, e(q) \rangle, \quad \forall \xi \in T_q^*M, \ q \in M,$$

and by $\mathcal{U}_h^t : T^*M \to T^*M$, $t \in \mathbb{R}$, the Hamiltonian flow generated by the Hamiltonian field associated to u_h. The flow $\mathcal{U}_h^t$ is a lift to the cotangent bundle of the flow on M generated by e. Let $P_t : M \to M$ be such a flow, $\frac{\partial P_t(q)}{\partial t} = e \circ P_t(q)$, $P_0(q) = q$, $q \in M$; then $\mathcal{U}_h^t = P_{-t}^*$. The flow P_t preserves the sub-Riemannian metric if and only if the flow $\mathcal{U}_h^t$ preserves h; in other words, if and only if $\{u_h, h\} = 0$, where $\{\cdot,\cdot\}$ is the Poisson bracket. Note that $h\big|_{T_q^*M}$ is a quadratic form and $u_h\big|_{T_q^*M}$ is a linear form, $\forall q \in M$; hence $\{u_h, h\}\big|_{T_q^*M}$ is a quadratic form.

Recall that the flow $\mathcal{U}_h^t = P_{-t}^*$ preserves the 1-form ω, and ω is a non-vanishing section of the line distribution $\Delta^\perp$. Hence $\Delta^\perp$ is contained in the kernel of the quadratic forms $h \circ \mathcal{U}_h^t\big|_{T_q^*M}$ and $\underbrace{\{u_h, \{\cdots\{u_h}_{i}, h\}\cdots\}\big|_{T_q^*M} = \frac{d^i}{dt^i}\big|_{t=0} \left(h \circ \mathcal{U}_h^t\right)\big|_{T_q^*M}$.

We are now ready to introduce the promised dynamics on the space of sub-Riemannian metrics on Δ, where metrics are represented by their Hamiltonians. Let ε be a positive smooth function on M. A discrete time dynamical system transforms a Hamiltonian h_n into the Hamiltonian

$$h_{n+1} = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} h_n \circ \mathcal{U}_{h_n}^t \, dt, \quad n = 0, 1, 2, \ldots,$$

a partial average of h_n with respect to the flow $\mathcal{U}_h^t$.

The Hamiltonian h_{n+1} is equal to h_n if and only if $\{u_{h_n}, h_n\} = 0$. Indeed, let $\langle \cdot, \cdot \rangle_q$ be an inner product in Δ_q^* and $H_q^t : \Delta_q^* \to \Delta_q^*$ the symmetric operator associated to the quadratic form $h_n \circ \mathcal{U}_{h_n}^t\big|_{\Delta_q^*}$ by this inner product: $h_n \circ \mathcal{U}_{h_n}^t(\cdot) = \langle H_n^t \cdot, \cdot \rangle_q$. Recall that the flow $\mathcal{U}_{h_n}^t$ is generated by the Reeb field of h_n, hence the area form on Δ_q^* defined by $h_n \circ \mathcal{U}_{h_n}^t\big|_{\Delta_q^*}$ does not depend on t; in other words, $\det H_n^t = const$. The equation $\det H = const$ defines a strongly convex hyperboloid in the

3-dimensional cone of positive definite symmetric operators on the plane, and H_n^t is a curve in such a hyperboloid; hence $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} H_n^t\,dt = H_n^0$ if and only if $H_\varepsilon^t \equiv H_n^0$, i. e. $h_n \circ \mathcal{U}_{h_n}^t \equiv h_n$.

If the sequence h_n converges, then its limit is the Hamiltonian of a sub-Riemannian metric on Δ preserved by the Reeb field. Otherwise we may modify the sequence and take scaled averages:

$$h_{n+1} = c_n \int_{-\varepsilon_n}^{\varepsilon_n} h_n \circ \mathcal{U}_{h_n}^t\,dt.$$

There is a good chance to arrive to a nonzero limiting Hamiltonian h_∞ by a clever choice of the sequences of positive functions ε_n, c_n. Then $h_\infty\big|_{T_q^*M}$ is a nonnegative quadratic form and $\Delta_q^\perp \subset \ker h_\infty\big|_{T_q^*M}$ for any $q \in M$. It may happen however that $rank\big(h_\infty\big|_{T_q^*M}\big) < 2$ for some $q \in M$ and h is not the Hamiltonian of a contact sub-Riemannian metric; we can treat it as a generalized version of such a metric.

A continuous time analogue of the introduced dynamics is a "heat along the Reeb field" equation

$$\frac{\partial h}{\partial t} = c\{u_h, \{u_h, h\}\}$$

in the space of sub-Riemannian metrics on the given contact distribution. It is easy to show that the equality $\{u_h, \{u_h, h\}\} = 0$ implies $\{u_h, h\} = 0$ and stationary solutions of this equation are exactly the metrics preserved by the Reeb fields. I conclude with an explicit expression for this nice and mysterious evolution equation in the appropriate frame.

All contact distributions are locally equivalent according to the Darboux theorem. Let f_1, f_2 be a basis of the contact distribution Δ such that f_1, f_2 generate a Heisenberg Lie algebra: $[f_1, [f_1, f_2]] = [f_2, [f_2, f_1]] = 0$. We set $v_i(\xi) = \langle \xi, f_i(q)\rangle$, $\xi \in T_q^*M$, $q \in M$, the Hamiltonian lift of the field f_i, $i = 1, 2$; then

$$\{v_1, \{v_1, v_2\}\} = \{v_2, \{v_2, v_1\}\} = 0. \tag{3}$$

Hamiltonian of any sub-Riemannian metric on Δ has a form:

$$h = a_{11}v_1^2 + 2a_{12}v_1v_2 + a_{22}v_2^2, \tag{4}$$

where a_{ij} are smooth functions on the domain in M where f_1, f_2 form a basis of Δ, and the quadratic form defined by the matrix $A(q) = \begin{pmatrix} a_{11}(q) & a_{12}(q) \\ a_{12}(q) & a_{22}(q) \end{pmatrix}$ is positive definite for any q from this domain. Let $\delta = \det A$; this is a function on M and we treat it as a constant on the fibers function on T^*M. A key for us function u_h depends only on δ and has a form:

$$-u_h = \delta\{v_1, v_2\} + v_1\{v_2, \delta\} + v_2\{\delta, v_1\}. \tag{5}$$

The relations (3)–(5) give an explicit expression for the equation $\frac{\partial h}{\partial t} = c\{u_h, \{u_h, h\}\}$ as a system of third order partial differential equations for the functions a_{ij}.

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Geometry of Maslov cycles

Davide Barilari and Antonio Lerario

Abstract We introduce the notion of *induced Maslov cycle*, which describes and unifies geometrical and topological invariants of many apparently unrelated situations, from real algebraic geometry to sub-Riemannian geometry.

1 Introduction

In this paper, dedicated to Andrei A. Agrachev in the occasion of his 60th birthday, we survey and develop some of his ideas on the theory of quadratic forms and its applications, from real algebraic geometry to the study of second order conditions in optimal control theory. The investigation of these problems and their geometric interpretation in the language of symplectic geometry is in fact one of the main contributions of Agrachev's research of the 80s-90s (see [1, 5, 6]) and these techniques are still at the core of his more recent research (see the preprints [7, 8]).

Also, this survey can be interpreted as an attempt of the authors to give a unified presentation of the two a priori unrelated subjects of their dissertations under Agrachev's supervision, namely sub-Riemannian geometry and the topology of sets defined by quadratic inequalities. The unifying language comes from symplectic geometry and uses the notion of *Maslov cycle*, as we will discuss in a while.

To start with we introduce some notation. The set $L(n)$ of all n-dimensional Lagrangian subspaces of $\mathbb{R}^{2n}$ (endowed with the standard symplectic structure) is called the *Lagrangian Grassmannian*; it is a compact submanifold of the ordinary Grass-

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mannian and once we fix one of its points Δ, we can consider the algebraic set

$$\Sigma = \{\Pi \in L(n) \mid \Delta \cap \Pi \neq 0\},$$

(this is what is usually referred to as the *train of* Δ, or the *universal Maslov cycle*). The main idea of this paper is to study *generic* maps $f : X \to L(n)$, for X a smooth manifold, and the geometry of the preimage under f of the cycle Σ. Such a preimage $f^{-1}(\Sigma)$ is what we will call the *induced* Maslov cycle.

It turns out that many interesting problems can be formulated in this setting and our goal is to describe a kind of *duality* that allows to get geometric information on the map f by replacing its study with the geometry of $f^{-1}(\Sigma)$.

To give an example, the Maslov cycle already provides information on the topology of $L(n)$ itself. In fact Σ is a *cooriented algebraic hypersurface* smooth outside a set of codimension *three* and its intersection number with a generic map $\gamma : S^1 \to L(n)$ computes $[\gamma] \in \pi_1(L(n)) \simeq \mathbb{Z}$.

The theory of quadratic forms naturally appears when we look at the local geometry of the Lagrangian Grassmannian $L(n)$: it can be seen as a compactification of the space $Q(n)$ of real quadratic forms in n variables and, using this point of view, the Maslov cycle Σ is a compactification of the space of *degenerate* forms.

Given k quadratic forms $q_1, \dots, q_k \in Q(n)$ we can construct the map:

$$f : S^{k-1} \to L(n), \quad (x_1, \dots, x_k) \mapsto x_1 q_1 + \dots + x_k q_k.$$

In fact the image of this map is contained in the *affine* part of $L(n)$ and its homotopy invariants are trivial. Neverthless the induced Maslov cycle $f^{-1}(\Sigma)$ has a nontrivial geometry and can be used to study the topology of:

$$X = \{[x] \in \mathbb{R}\mathrm{P}^{n-1} \mid q_1(x) = \dots = q_k(x) = 0\}.$$

More specifically, it turns out that as a first approximation for the topology of X we can take the "number of holes" of $f^{-1}(\Sigma)$. Refining this approximation procedure amounts to exploit how the coorientation of Σ is pulled-back by f.

In some sense this is the idea of the study of (locally defined) families of quadratic forms and their degenerate locus, and the set of Lagrange multipliers for a variational problem admits the same description. One can consider two smooth maps between manifolds $F : U \to M$ and $J : U \to \mathbb{R}$ and ask for the study of critical points of J on level sets of F. With this notation the manifold of *Lagrange multipliers* is defined to be:

$$C_{F,J} = \{(u, \lambda) \in F^*(T^*M) \mid \lambda D_u F - d_u J = 0\}.$$

Attached to every point $(u, \lambda) \in C_{F,J}$ there is a quadratic form, namely the Hessian of $J|_{F^{-1}(F(u))}$ evaluated at u, and using this family of quadratic forms we can still define an induced Maslov cycle $\Sigma_{F,J}$ (the definition we will give in the sequel is indeed more intrinsic).

This abstract setting includes for instance the geodesic problem in Riemannian and sub-Riemannian geometry (and even more general variational problems). In this case the set U parametrizes the space of admissible curves, F is the *end-point map* (i. e. the map that assigns to each admissible curve its final point), and J is the *energy* of the curve. The problem of finding critical points of the energy on a fixed level set of F corresponds precisely to the geodesic problem between two fixed points on the manifold M. In this context $\Sigma_{F,J}$ corresponds to points where the Hessian of the energy is degenerate and its geometry is related to the structure of *conjugate locus* in sub-Riemannain geometry. Moreover the way this family of quadratic forms (the above mentioned Hessians) degenerates translates into optimality properties of the corresponding geodesics.

Rather than a systematic and fully detailed treatment we present here the main ideas, giving only some sketches of the proofs (providing references where possible) and offering a different perspective in these well-estabilshed research fields.

In Sect. 2 we introduce the basic terminology: the geometry and topology of Lagrangian Grassmannians and the *universal Maslov cycle* are discussed. In Sec. 3 we present the "local" part of the theory: we define the *induced Maslov cycle* and we use it to study the topology of *intersections of real quadrics*. In Sect. 4 we introduce Lagrangian maps and extend the definition of induced Maslov cycle. In Sect. 5 we introduce a special class of Lagrangian maps: the projection of the manifold of Lagrange mutlipliers of an extremal problem to the variable space; we discuss how *families of Morse functions* can be handled using this setting; finally we translate the geodesic problem in subriemannian geometry in the language of Lagrange multipliers and show that the structure of the Maslov cycle gives in this case partial answers to optimality.

Our presentation is strongly influenced by the deep insight and the ideas of A. A. Agrachev. We are extremely grateful to him for having shown us, both in mathematics and in life, the elegance of simpleness.

2 Lagrangian Grassmannian and universal Maslov cycles

2.1 The Lagrangian Grassmannian

Let us consider $\mathbb{R}^{2n}$ with its standard symplectic form σ. A vector subspace Λ of $\mathbb{R}^{2n}$ is called *Lagrangian* if it has dimension n and $\sigma|_\Lambda \equiv 0$. The *Lagrangian Grassmannian* $L(n)$ in $\mathbb{R}^{2n}$ is the set of its n-dimensional Lagrangian subspaces.

Proposition 1. *$L(n)$ is a compact submanifold of the Grassmannian of n-planes in $\mathbb{R}^{2n}$; its dimension is $n(n+1)/2$.*

Consider the set $\Delta^{\pitchfork} = \{\Lambda \in L(n) \mid \Lambda \cap \Delta = 0\}$ of all Lagrangian subspaces that are transversal to a given one $\Delta \in L(n)$. Clearly $\Delta^{\pitchfork} \subset L(n)$ is an open subset and

$$L(n) = \bigcup_{\Delta \in L(n)} \Delta^{\pitchfork}. \tag{1}$$

It is then sufficient to find some coordinates on these open subsets. Let us fix a Lagrangian complement Π of Δ (which always exists but is not unique). Every n-dimensional subspace $\Lambda \subset \mathbb{R}^{2n}$ which is transversal to Δ is the graph of a linear map from Π to Δ. Choosing a Darboux basis on $\Sigma = \Delta \oplus \Pi$ adapted to the splitting, this linear map is represented in coordinates by a symmetric matrix S such that:

$$\Lambda \cap \Delta = 0 \Leftrightarrow \Lambda = \{(x, Sx),\ x \in \Pi \simeq \mathbb{R}^n\}.$$

Hence the open set $\Delta^{\pitchfork}$ of all Lagrangian subspaces that are transversal to Δ is parametrized by the set of symmetric matrices, that gives coordinates on $\Delta^{\pitchfork}$. This also proves that the dimension of $L(n)$ is $n(n+1)/2$. Notice finally that, being $L(n)$ a closed set in a compact manifold, it is itself compact.

Fix now an element $\Lambda \in L(n)$. The tangent space $T_\Lambda L(n)$ to the Lagrangian Grassmannian at the point Λ can be canonically identified with set of quadratic forms on the space Λ itself:

$$T_\Lambda L(n) \simeq Q(\Lambda).$$

Indeed consider a smooth curve $\Lambda(t)$ in $L(n)$ such that $\Lambda(0) = \Lambda$ and denote by $\dot{\Lambda} \in T_\Lambda L(n)$ its tangent vector. For any point $x \in \Lambda$ and any smooth extension $x(t) \in \Lambda(t)$ we define the quadratic form:

$$\dot{\Lambda} : x \mapsto \sigma(x, \dot{x}), \quad \dot{x} = \dot{x}(0).$$

An easy computation shows that this map is indeed well defined; moreover writing $\Lambda(t) = \{(x,\ S(t)x),\ x \in \mathbb{R}^n\}$ then the quadratic form $\dot{\Lambda}$ associated to the tangent vector of $\Lambda(t)$ at zero is represented by the matrix $\dot{S}(0)$, i. e. $\dot{\Lambda}(x) = x^T \dot{S}(0)x$.

We stress that this representation using *symmetric* matrices works only for coordinates induced by a Darboux basis associated with a *Lagrangian* splitting $\mathbb{R}^{2n} = \Pi \oplus \Delta$, i. e. Π and Δ are both Lagrangian.

Example 1 (The Lagrangian Grassmannians $L(1)$ and $L(2)$). Since every line in $\mathbb{R}^2$ is Lagrangian (the restriction of a skew-symmetric form to a one-dimensional subspace must be zero), then $L(1) \simeq \mathbb{R}\mathrm{P}^1$.

The case $n = 2$ is more interesting. Each 2-plane W in $\mathbb{R}^4$ defines a unique (up to a multiple) degenerate 2-form ω in $\Lambda^2\mathbb{R}^4$, by $W = \ker \omega$. Thus there is a map:

$$p : G(2,4) \to \mathbb{P}(\Lambda^2\mathbb{R}^4) \simeq \mathbb{R}\mathrm{P}^5.$$

This map is called the *Plücker embedding*; its image is a projective quadric of signature $(3,3)$. The restriction of p to $L(2)$ is

$$p(L(2)) = \{[\omega] \mid \ker \omega \neq 0 \quad \text{and} \quad \omega \wedge \sigma = 0\},$$

which is the intersection of the image of p with an hyperplane in $\mathbb{R}\mathrm{P}^5$ (i. e. the zero locus of the restriction of the above projective quadric to such hyperplane). In particular $L(2)$ is diffeomorphic to a smooth quadric of signature $(2,3)$ in $\mathbb{R}\mathrm{P}^4$.

2.2 Topology of Lagrangian Grassmannians

It is possible to realize the Lagrangian Grassmannian as a homogeneous space, through an action of the unitary group $U(n)$. In fact we have a homomorphism of groups $\phi : GL(n,\mathbb{C}) \to GL(2n,\mathbb{R})$ defined by:

$$\phi : A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and the image of the unitary group is contained in the symplectic one. In particular for every Lagrangian subspace $\Lambda \subset \mathbb{R}^{2n}$ and every M in $U(n)$ the vector space $\phi(M)\Lambda$ is still Lagrangian. This defines the action of $U(n)$ on $L(n)$; the stabilizer of a point is readily verified to be the group $O(n)$ and we get:

$$L(n) \simeq U(n)/O(n).$$

The cohomology of $L(n)$ can be studied applying standard techniques to the fibration $U(n) \to L(n)$ and working with $\mathbb{Z}_2$ coefficients (unless differently stated, we always make this assumption) we get a ring isomorphism $H^*(L(n)) \simeq H^*(S^1 \times \cdots \times S^n)$; we refer the reader to [13] for more details.
For our purposes we need an explict description of the fundamental group of $L(n)$ and this can be obtained as follows. We first consider the map $\det^2 : U(n) \to S^1$ defined by $M \mapsto \det^2(M)$. Multiplication by a matrix of $O(n)$ does not change the value of the square of the determinant, thus we get a surjective map $\det^2 : L(n) \to S^1$. This map also is a fibration, its fibers are contractible (in fact they are diffeomorphic to $SU(n)/SO(n)$) and it realizes an isomorphism of fundamental groups:

$$\pi_1(L(n)) \simeq \mathbb{Z}.$$

2.3 The universal Maslov cycle

Since the fundamental group of $L(n)$ is $\mathbb{Z}$, then the 1-form $d\theta/2\pi$ on S^1 (the class of this form generates the first cohomology group with integer coefficients) pulls-back via $\det^2$ to a 1-form on $L(n)$ whose cohomology class μ generates $H^1(L(n),\mathbb{Z})$:

$$\mu = \left[\frac{1}{2\pi}(\det^2)^*\, d\theta\right] \in H^1(L(n),\mathbb{Z}).$$

Such a class is usually referred to as the *universal Maslov class* (see [9, 11]). Once we fix a Lagrangian space $\Delta \in L(n)$ it is possible to define a cooriented algebraic cycle in $L(n)$ which is Poincaré dual to μ; such cycle is called the *train of* Δ and is defined as follows:

$$\Sigma_\Delta = \{\Lambda \in L(n) \mid \Lambda \cap \Delta \neq 0\} = L(n)\backslash\Delta^{\pitchfork}.$$

Here the subscript denotes the dependence on Δ and when no confusion arises we will omit it: a different choice of Δ produces an homologous train (in fact just differing by a symplectic transformation). We discuss the geometry of Σ in greater detail in the next section; what we need for now is that Σ is an *algebraic* hypersurface whose singularities have codimension *three* and is *cooriented*. The fact that Σ is an algebraic set makes it a cycle, the fact that it is an hypersurface whose singularities have codimension three allows to define intersection number with it and the fact that is cooriented makes this intersection number an integer. Here coorientation means that Σ is two-sided in $L(n)$, i. e. there is a canonical orientation of its normal bundle along its smooth points. Using the above diffeomorphism $L(n) \simeq U(n)/O(n)$ it is easy to choose a positive normal at a smooth point $\Lambda \in \Sigma$: we represent Λ as $[M]$ for a unitary matrix M and we take the velocity vector at zero of the curve $t \mapsto [e^{it}M]$.

Example 2 (The train in $L(2)$). We have seen that $L(2)$ is diffeomorphic to a quadric of signature $(2,3)$ in $\mathbb{R}\mathrm{P}^4$; thus it is double covered by $S^1 \times S^2$ (i. e. the set of points in $S^4 \subset \mathbb{R}^5$ satisfying the equation $x_0^2 + x_1^2 = x_2^2 + x_3^2 + x_4^2$).

We fix now a plane Δ and study the geometry of the train Σ_Δ. We let Π be a Lagrangian complement to Δ and using symmetric matrices charts on $\Pi^{\pitchfork}$ we have:

$$\Sigma_\Delta \cap \Pi^{\pitchfork} \simeq \{S \mid \det(S) = 0\}.$$

The set of symmetric matrices with determinant zero is a quadratic cone in $\mathbb{R}^3$ with a singular point at the origin; to get Σ_Δ we have to add its limit points in $L(2)$ and this results into an identification of the two boundaries components of such a cone. What we get is a Klein bottle with one cycle collapsed to a point. More generally one can show that $H^*(L(n)) = H^*(S^1 \times \cdots \times S^n)$.

The main idea of this paper is to study *generic* maps $f : X \to L(n)$, for X a smooth manifold, and the geometry of the preimage under f of the cycle Σ (together with its coorientation). Such a preimage $f^{-1}(\Sigma)$ is what we call the *induced* Maslov cycle. Sometimes in the sequel the map f will be defined only locally but it will still produce a *Maslov type* cycle on X. Our goal is to describe a kind of *duality* that allows to get geometric information on the map f by replacing its study with the one of the geometry of $f^{-1}(\Sigma)$. We discuss these ideas in greater detail in the next section.

Example 3 (Generic loops). Consider a smooth map:

$$\gamma : S^1 \to L(n)$$

transversal to the smooth points of Σ. Such a property is generic and we might ask for the meaning of the number of points in $\gamma^{-1}(\Sigma)$. Since the intersection number with Σ computes the integer $[\gamma] \in \pi_1(L(n))$, in a very rough way we can write:

$$|[\gamma]| \leq b(\gamma^{-1}(\Sigma)), \tag{2}$$

where the r.h.s. denotes the sum of the Betti numbers, which in this case coincides with the number of connected components (i. e. number of points). This inequality is simply what we obtain by forgetting the coorientation in the sum defining the intersection number. The comparison through the inequality between what appears on the l.h.s. and what on the r.h.s. is the first mirror of the mentioned duality between the geometric properties of γ and the topological ones of $\gamma^{-1}(\Sigma)$.

Remark 1 (Schubert varieties). It is indeed possible to give $L(n)$ a cellular structure using *Schubert varieties* in a fashion similar to the ordinary Grassmannian: the cells are in one to one correspondence with *symmetric* Young diagrams (we refer the reader to [15] for more details). Given one of such diagram the corresponding Schubert cell is the one obtained by considering a flag that is *isotropic* with respect to the symplectic form. More precisely let $\{0\} \subset V_1 \subset V_2 \cdots \subset V_{2n} = \mathbb{R}^{2n}$ be a complete flag such that $\sigma(V_j, V_{2n-j}) = 0$ for every $j = 1, \ldots, n$ (this means the flag is isotropic; in particular V_n is Lagrangian). If now we let a be the partition $a : n \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then the corresponding Schubert variety is:

$$Y_a = \{\Lambda \in L(n) \mid \dim(\Lambda \cap V_{n+i-a_i}) \geq i \quad \text{for} \quad i = 1, \ldots, n\}.$$

The codimension of Y_a is $(|a| + l(a))/2$, where $l(a)$ is the number of boxes on the main diagonal of the associated Young diagram (such a diagram has a_i boxes in its i-th row). Since this diagram must be symmetric along its diagonal we see that there are only 2^n possible good partitions. Geometrically this shows that the combinatorics of the cell structure of the Grassmannian $G(n, 2n)$ descends (by intersection) to the one of $L(n)$. Moreover, since the incidence maps have even degree, cellular homology with $\mathbb{Z}_2$ coefficients gives again the above formula for $H^*(L(n))$.

Notice in particular that Σ is a Schubert variety: letting the n-th element of the isotropic flag to be Δ itself, then:

$$\Sigma_\Delta = \{\Lambda \in L(n) \mid \dim(\Lambda \cap \Delta) \geq 1\} = Y_{(1,0,\ldots,0)}.$$

Example 4 (Schubert varieties of $L(2)$). We consider again the case of $L(2)$ and fix an isotropic flag $\{0\} \subset V_1 \subset \Delta \subset V_3 \subset \mathbb{R}^4$. The cell structure is given by the four following possible partitions $(0,0)$, $(1,0)$, $(2,1)$, $(2,2)$. Let us see how the corresponding Schubert varieties look like. To this end let us write $\mathbb{R}^{2n} = \Delta \oplus \Pi$, where Π is a Lagrangian complement to Δ. In this way every Λ in $\Pi^{\pitchfork}$ is of the form $\Lambda = \{(x, Sx) \mid S = S^T\}$.

We immediately get $Y_{(0,0)} = L(n)$; moreover $\Sigma_\Delta = Y_{(1,0)}$. The Schubert variety $Y_{(2,2)}$ equals Δ itself (in the symmetric matrices coordinates it is the zero matrix). Finally we have $Y_{(2,1)} = \{\Lambda \mid \Lambda \supset V_1,\ \Lambda \subset V_3\}$. The intersection of this variety with $\Pi^{\pitchfork}$ equals all the symmetric matrices S whose kernel contains $V_1 \subset \Delta$: such matrices are all multiple one of the other and they form a line, thus $Y_{(2,1)} \simeq \mathbb{R}\mathrm{P}^1$.

3 Linear systems of quadrics

3.1 Local geometry and induced Maslov cylces

In this section we study in more detail the local geometry of the Lagrangian Grassmannian. If no data are specified, being a differentiable manifold, each one of its points looks exactly like the others. Once we fix one of them, say Δ, the situation drastically enriches: we have seen, for example, that we can choose a cycle Σ_Δ representing the generator of the first cohomology group.

The following proposition gives a more precise structure of the local geometry we obtain on $L(n)$ after we have fixed one of its points Δ.

Proposition 2. *Let Δ in $L(n)$ be fixed. Every $\Lambda \in L(n)$ has a neighborood U such that if $\Delta \cap \Lambda \simeq \mathbb{R}^k$ there is a smooth algebraic submersion:*

$$\phi : U \to W,$$

where W is an open set of the space of quadratic forms on $\mathbb{R}^k$; moreover the following properties are satisfied:

(i) $(d_\Lambda \phi)\dot{\Lambda} = \dot{\Lambda}|_{\Delta \cap \Lambda}$;
(ii) $\dim(\ker \phi(\Pi)) = k - \dim(\Delta \cap \Pi)$ *for every* Π *in* U;
(iii) *for every* Π *in* W *the fiber* $\phi^{-1}(\Pi)$ *is contractible.*

Let Δ' be a Lagrangian complement to Δ transversal to Λ. Then, giving coordinates to the open set $\{\Pi \in L(n) \,|\, \Pi \pitchfork \Delta'\}$ using symmetric matrices, this proposition is just a reformulation of Lemma 2 from [1].

The fact that ϕ is a *submersion* allows to reduce the study of properties of $L(n)$ to smaller Grassmannians, via the Implicit Function Theorem. For the first property, recall that we have a natural identification of the vector space $T_\Lambda L(n)$ with the space of quadratic forms on Λ; each one of these quadratic forms can be restricted to the subspace $\Delta \cap \Lambda$ and this restriction operation is what $d_\Lambda \phi$ does. The second property says that ϕ transforms the combinatorics of intersections with Δ with the one of the kernels of the corresponding quadratic forms.

Thus locally Σ_Δ looks like the space of degenerate quadratic forms and it is interesting to see how all these local charts are glued together. Let us consider Λ in Σ_Δ and some Π_1 Lagrangian complement to Δ such that $\Pi_1 \pitchfork \Lambda$. Given a symplectyc transformation $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserving Δ, the matrix T representing it in the coordinates given by the Lagrangian splitting $\Delta \oplus \Pi_1$ has the form:

$$T = \begin{pmatrix} A^{-1} & BA^T \\ 0 & A^T \end{pmatrix} \quad \text{with} \quad B = B^T.$$

If Λ is represented by the symmetric matrix S, the change of coordinates ψ changes its representative to $(A^T SA)(I + BA^T SA)^{-1}$ (indeed this formula works for every Λ transversal to Π).

Remark 2 (Local topology of the train). The local topology of Σ_Δ can be described using Proposition 2. Let B_Λ be a small ball centered at $\Lambda \in \Sigma_\Delta$ with boundary ∂B_Λ. Then the intersection $B_\Lambda \cap \Sigma_\Delta$ is contractible: it is a cone over the intersection $\partial B_\Lambda \cap \Sigma_\Delta$; moreover $\partial B_\Lambda \cap \Sigma_\Delta$ is Spanier-Whitehead dual to a union of ordinary Grassmannians[1] and:

$$H^*(\partial B_\Lambda \cap \Sigma_\Delta) \simeq \bigoplus_{j=0}^{k} H_*(G(j,k)), \quad k = \dim(\Lambda \cap \Delta) \tag{3}$$

Theorem 3 from [16] gives the statement for $\Lambda = \Delta$ and the general result follows by applying Proposition 2. This means that if we pick a $\Lambda \in \Sigma_\Delta$ with $\dim(\Lambda \cap \Delta) = k$, then a sufficiently small neighborhood of Λ in Σ_Δ is homeomorphic to a cone with vertex Λ and base a space whose cohomology is given by (3), i. e. this base space has 2^k "holes":

$$2^k = \sum_{j=0}^{k} \binom{k}{j} = \sum_{j=0}^{k} \mathrm{rk}\,(H_*(G(j,k)))\,.$$

For every $r \geq 1$ we can define the sets:

$$\Sigma_\Delta^{(r)} = \{\Lambda \in L(n) \mid \dim(\Lambda \cap \Delta) \geq r\} \quad \text{and} \quad Z_r = \Sigma_\Delta^{(r)} \backslash \Sigma_\Delta^{(r+1)}.$$

Using this notation, Proposition 2 implies that Σ_Δ is stratified by $\bigcup_r Z_r$ and the codimension of Z_r in $L(n)$ is $\binom{r+1}{2}$ (the reader is referred to [12] for properties of such stratifications).

Remark 3 (Cooorientation revised). Let Λ be a smooth point of Σ_Δ and $\gamma : (-\epsilon, \epsilon) \to L(n)$ be a curve transversal to all strata of Σ_Δ and with $\gamma(0) = \Lambda$ (the transversality condition ensures that γ meets only $\Sigma_\Delta \backslash \Sigma_\Delta^{(2)}$, i. e. the set of smooth points of Σ). Since $T_\Lambda L(n) \simeq Q(\Lambda)$, the velocity $\dot{\gamma}(0)$ can be interpreted (by restriction) as a quadratic form on $\Lambda \cap \Delta$. Proposition 2 together with the transversality condition ensures that this restriction is nonzero. We say that the curve γ is positively oriented at zero if $\dot{\gamma}(0)|_{\Lambda \cap \Delta} > 0$. Since this definition is intrinsic, it gives a coorientation on Σ and it is not difficult to show that it coincides with the one described before.

Definition 1 (Induced Maslov cycle). Let X be a smooth manifold and $f : X \to L(n)$ be a map transversal to all strata of $\Sigma = \Sigma_\Delta$. The cooriented preimage $f^{-1}(\Sigma)$ will be called the Maslov cycle induced by f.

A generic map $f : X \to L(n)$ is indeed transversal to all strata of Σ and $f^{-1}(\Sigma)$ is itself stratified[2](its strata being the preimage of the strata of Σ); the transversality condition ensures that the the normal bundle of the smooth points of $f^{-1}(\Sigma)$ (which is the pull-back of the normal bundle of Σ) has a nonvanishing section, i. e. the induced Maslov cycle also has a coorientation.

[1] This simply means that its complement on the sphere is homotopy equivalent to a union of Grassmannians; in particular these spaces have the same cohomology, see [14].

[2] In fact these stratifications are also "good" in the sense of Whitney and Nash, see [12].

3.2 Linear systems of quadrics

We turn now to the above mentioned duality between the geometry of a map $f : X \to L(n)$ transversal to all strata of Σ_Δ and the cooriented cycle induced by f. We consider a specific example, namely the case of a map from the sphere, whose image is contained in one coordinate chart.
More precisely let $\Delta \oplus \Pi \simeq \mathbb{R}^{2n}$ be a Lagrangian splitting and $W \simeq \mathbb{R}^k$ be a linear subspace of $\Pi^{\pitchfork} \simeq Q(\Delta)$ (the space of quadratic forms on Δ):

$$W = \text{span}\{q_1, \ldots, q_k\} \quad \text{with} \quad q_1, \ldots, q_k \in Q(\Lambda) \simeq Q(n)$$

(here $Q(n)$ denotes the space of quadratic forms in n variables).
Notice that the above isomorphism is defined once a scalar product on Δ is given: this allows to identify symmetric matrices with quadratic forms.

In this context W is called a *linear system* of real quadrics; the inclusion $S^{k-1} \hookrightarrow W$ defines a map:

$$f : S^{k-1} \to Q(n)$$

and for a generic choice of W such a map is transversal to all strata of $\Sigma = \Sigma_\Delta$. Notice that Σ equals the discriminant of the set of quadratic forms in n variables and Eq. (3) gives a descritpion of its cohomology.

To every linear space W as above we can associate an algebraic subset X_W of the real projective space $\mathbb{R}\mathrm{P}^{n-1} = \mathbb{P}(\Delta)$ (usually referred to by algebraic geometers as the *base locus* of W):

$$X_W = \{[x] \in \mathbb{R}\mathrm{P}^{n-1} \,|\, q_1(x) = \cdots = q_k(x) = 0\}.$$

The study of the topology of X_W was started by Agrachev in [1, 5] and continued by Agrachev and the second author in [6].

Remark 4 (The spectral sequence approach). The main idea of Agrachev's approach is to study the Lebesgue sets of the *positive inertia index* function on W, i.e. the number of positive eigenvalues $\mathrm{i}^+(q)$ of a symmetric matrix representing q. More specifically we can consider:

$$W^j = \{q \in W \,|\, \mathrm{i}^+(q) \geq j\}, \quad j \geq 1,$$

and Theorem A from [6] says that roughly we can take the homology of these sets as the homology of X_W:

$$\bigoplus_{j=1}^{n} H^*(W, W^j) \quad \text{“approximates”} \quad H^*(X_W).$$

The cohomology classes from $H^*(W, W^j)$ are just the *canditates* for the homology of X_W. The requirements they have to fulfill in order to represent effective classes in $H^*(X_W)$ are algebro-topological conditions. The way to make these statements precise is to use the language of spectral sequences (the above conditions on the canditates translate into them being in the kernels of the differentials of the spectral sequence). The reader is referred to [6] for a detailed treatment.

Going back to the map $f : S^k \to Q(n)$ defined by W, for simplicity of notation we set:

$$\Sigma_W^{(r)} = S^{k-1} \cap \Sigma^{(r)}.$$

Thus to all these data there correspond two objects: $X_W \subset \mathbb{R}\mathrm{P}^{n-1}$ and $\Sigma_W^{(1)} \subset S^{k-1}$. The induced Maslov cycle is Σ_W: notice that the cohomology class it represents in $H^1(S^{k-1})$ is clearly zero, though its geometry has a nontrivial meaning. In fact we can relate the sum of the Betti numbers of X_W to the ones of Σ_W and its singular points:

$$b(X_W) \leq n + \frac{1}{2}\sum_{r\geq 1} b(\Sigma_W^{(r)}) \quad \text{for a generic } W. \tag{4}$$

This formula is one of the expressions of the above mentioned duality: the l.h.s. is the *homological complexity* of the intersection of k quadrics in $\mathbb{R}\mathrm{P}^{n-1}$, the r.h.s. is the complexity of the Maslov cycle induced on the span of these quadrics. The reader should compare (4) with (2): in both cases the complexity of the induced Maslov cycle gives a restriction (in the form of an upper bound) on some geometrical invariants associated to f.

Example 5 (The intersection of three quadrics). Let us consider the intersection X of *three* quadrics in $\mathbb{R}\mathrm{P}^{n-1}$. Such intersection arises by considering a three dimensional space $W = \mathrm{span}\{Q_1, Q_2, Q_3\}$ in a coordinate chart $\Pi^{\pitchfork} \simeq Q(\Delta)$. Hence the Q_i are symmetric matrices and X is given by the equations $q_1 = q_2 = q_3 = 0$ on $\mathbb{P}(\Delta)$; notice that the definition of each q_i depends on the choice of a scalar product on Δ but two different choices give the same X up to a projective equivalence. The induced Maslov cycle is the curve Σ_W on S^2 given by the equation:

$$\det(x_1Q_1 + x_2Q_2 + x_3Q_3) = 0, \quad (x_1, x_2, x_3) \in S^2 \subset W.$$

This is a degree n curve on S^2 and for a generic choice of W it is smooth: in fact $\Sigma_W = S^2 \cap \Sigma_\Delta$ and since the codimension of $\mathrm{Sing}(\Sigma_\Delta)$ is *three*, by slightly perturbing W this singular locus can be avoided on the sphere.
The curve Σ_W has at most $O(n^2)$ components and the manifold X at most $O(n^2)$ "holes" (the sum of its Betti numbers is less than $n^2 + O(n)$) ; in this case Eq. (4) tells that:

$$|b(X) - b_0(\Sigma)| \leq O(n),$$

i. e. if we replace the homology of X with the one of the associated Maslov cycle the error of such replacement has order $O(n)$. The coorientation of the induced Maslov cycle in this case assigns a number ± 1 to each oval of the curve Σ_W: this number is obtained by looking at the change of the number of positive eigenvalues when crossing the oval. The knowledge of the coorientation on each oval allows to compute the error term in (4); the reader is referred to [1,6,16].

4 Geometry of Gauss maps

4.1 Lagrange submanifolds of $\mathbb{R}^{2n}$

Consider a Lagrangian submanifold M of the symplectic space $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$. The *Gauss map* of M is:

$$\nu : M \to L(n)$$

and associates to each point $x \in M$ the tangent space $T_x M$ (which is by definition a Lagrangian subspace of $\mathbb{R}^{2n}$).

We consider the Lagrangian splitting $\mathbb{R}^{2n} = \Pi \oplus \Delta$ and we are interested in the description of the induced Maslov cycle $\nu^{-1}(\Sigma_\Delta)$ on M. To this end we consider the projection on the first factor $\pi : \mathbb{R}^{2n} \to \Pi$ and its restriction to M:

$$\pi|_M : M \to \Pi.$$

The critical points of $\pi|_M$ are those points x in M such that the tangent space $T_x M$ does not intersect Δ transversally; in other words:

$$\text{Crit}(\pi|_M) = \nu^{-1}(\Sigma_\Delta). \tag{5}$$

Thus the induced Maslov cycle in this case coincides with the set of critical points of a map from M to $\mathbb{R}^n$: this critical set represents the Poincaré dual of $w_1(TM)$, the first Stiefel-Whitney class of TM (see Remark 5 below). In fact ν pulls-back the tautological bundle $\tau(n)$ of $L(n)$ to the tangent bundle of M and, by functoriality of characteristic classes, it also pulls-back the first Stiefel-Whitney class of $\tau(n)$ to $w_1(TM)$. Notice that $w_1(\tau(n))$ equals the modulo two reduction of the universal Maslov class μ defined above.

Remark 5 (Characteristic classes revised). Consider an n-dimensional manifold M and a smooth function $f : M \to \mathbb{R}^{n-k+1}$. For a generic f we can relate the k-th Stiefel-Whitney class of M to the critical points of f by:

$$w_k(TM) = \text{Poincaré dual of Crit}(f). \tag{6}$$

For $k = n$ the generic f is a Morse function and $w_n(TM) \in H^n(M) \simeq \mathbb{Z}_2$ is the Euler characteristic of M modulo two, thus the previous equations reads $\chi(M) \equiv \text{Card}(\text{Crit}(f)) \mod 2$.

In the case $k = 1$ we can apply (6) to:

$$f = \pi|_M : M \to \mathbb{R}^n,$$

and Eq. (5) implies that the Maslov cycle induced by ν represents the Poincaré dual of $w_1(TM)$.

We know from Remark 1 that the cohomology of $L(n)$ is generated by the Poincaré duals of its Schubert varieties. Each of these varieties is labelled using *symmetric Young diagrams* and their intersections are computed using Schubert calculus. The variety corresponding to the diagram having only one box is $Y_{(1,0,\dots,0)}$: this is the train of Δ (the middle space in the isotropic flag) and it represents the Poincaré dual of $\mu = w_1(\tau(n))$ (again reduction modulo two is considered).

4.2 Lagrangian maps

Generalizing the construction of the previous section, we consider a Lagrangian submanifold M of the symplectic manifold T^*N (with the standard symplectic structure); we denote by $\pi : T^*N \to N$ the bundle projection. In this case we do not have a global Gauss map, but in analogy with (5) we can still define the *induced Maslov cycle* as:

$$\Sigma_M = \text{Crit}(\pi|_M).$$

The case of a submanifold M of T^*N projecting to N is itself a special case of a *Lagrangian map*; this is defined as follows. First we say that a fibration $\pi : E \to N$ is Lagrangian if E is a symplectic manifold and each fiber is Lagrangian. A Lagrangian map is thus a smooth map $f : M \to N$ between manifolds of the same dimension obtained by composition of a Lagrangian inclusion $i : M \to E$ followed by π :

$$f : M \stackrel{i}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} N.$$

We refer the reader to [10] for more details and examples.

Example 6 (Normal Gauss maps of hypersurfaces). Consider a smooth oriented hypersurface M in $\mathbb{R}^{n+1}$; the *normal Gauss map* of M is the map:

$$f : M \to S^n, \quad x \mapsto \text{oriented normal of } M \text{ at } x.$$

This map is Lagrangian; in fact we can set $E = T^*S^n \simeq TS^n$ with projection $\pi : E \to S^n$ and define the Lagrangian inclusion $i : M \to E$ as $x \mapsto (f(x), \text{proj}_{T_xM}x)$. The image in S^n of the induced Maslov cycle under f is called the *focal surface* of M.

Thus a Lagrangian map $f : M \to N$ is a special case of map between two manifolds of the same dimension; the set of its critical values is called a *caustic*. Proposition 2 allows to give a local description of the set of critical points of a Lagrangian map.

Proposition 3. *The set of critical points of a Lagrangian map is a cooriented hypersurface, smooth outside a set of codimension three.*

5 Lagrange multipliers

Let U be an open set in a Hilbert space (or a finite dimensional manifold) and let M be a smooth n-dimensional manifold. Assume we have a pair of smooth maps

$F : U \to M$, and $J : U \to \mathbb{R}$. We want to characterize critical points of the functional J when restricted to level sets of F:

$$\min_{F^{-1}(x)} J, \quad x \in M. \tag{7}$$

Recall that for a smooth function $f : M \to \mathbb{R}$ and a smooth submanifold $N \subset M$ a point $x \in N$ is said a *critical point* of $f\big|_N$ if $d_x f\big|_{T_x N} = 0$. We state the geometric version of the Lagrange multipliers rule, which characterizes regular constrained critical points.

Proposition 4 (Lagrange multipliers rule). *Assume $u \in U$ is a regular point of $F : U \to M$ such that $F(u) = x$. Then u is a critical point of $J\big|_{F^{-1}(x)}$ if and only if:*

$$\exists\, \lambda \in T_x^* M \quad s.t. \quad d_u J = \lambda\, D_u F. \tag{8}$$

The above discussion suggests to consider pairs (u, λ) such that the identity $d_u J = \lambda\, D_u F$ holds true. More precisely we should consider the pair (u, λ) as an element of the pullback bundle $F^*(T^*M)$, and set

$$C_{F,J} = \{(u, \lambda) \in F^*(T^*M) |\, d_u J = \lambda\, D_u F\}.$$

Notice that by definition of pullback bundle, if $(u, \lambda) \in F^*(T^*M)$, then $F(u) = \pi(\lambda)$ ($\pi : T^*M \to M$ is the bundle projection). The study of the geometry of the set $C_{F,J}$ leads us to investigate the constrained critical points for the whole family of problems (7), as x varies on M. The following regularity condition ensures that $C_{F,J}$ has nice properties: the pair (F, J) is said to be a *Morse problem* if the function

$$\theta : F^*(T^*M) \to T^*U, \qquad (u, \lambda) \mapsto d_u J - \lambda\, D_u F. \tag{9}$$

is transversal to the zero section in T^*U. Notice that, if $M = \{0\}$, then F is the trivial map and with this definition we have that (F, J) is a Morse problem if and only if J is a Morse function.

If (F, J) defines a Morse problem, then $C_{F,J}$ is a smooth n-dimensional manifold in $F^*(T^*M)$. In the case U is a finite dimensional manifold this is easy to show it, since by a standard transversality argument:

$$\begin{aligned} \dim C_{F,J} &= \dim F^*(T^*M) - \dim U \\ &= (\dim U + \operatorname{rank} T^*M) - \dim U \\ &= \operatorname{rank} T^*M = n. \end{aligned}$$

The above argument is no more valid in the infinite dimensional case but one can show that the same result holds (under some additional technical assumptions, see [4]).

Let us now consider the map $\overline{F} : F^*(T^*M) \to T^*M$ given by $(u, \lambda) \mapsto \lambda$. We can consider the set $\overline{C}_{F,J} = \overline{F}(C_{F,J})$ in T^*M :

$$\begin{array}{ccc} C_{F,J} & \xrightarrow{\overline{F}} & T^*M \\ {\scriptstyle \overline{\pi}} \downarrow & & \downarrow {\scriptstyle \pi} \\ U & \xrightarrow[F]{} & M \end{array} \tag{10}$$

It turns out that $\overline{F}$ is an exact Lagrangian immersion, i. e. it pulls-back the Liouville form $p\,dq$ to an exact form.

We assume now that $\overline{C}_{F,J}$ is an embedded submanifold (and not only immersed).

Theorem 1. *Let (F, J) be a Morse problem and assume (u, λ) is a Lagrange multiplier such that u is a regular point for F, where $F(u) = x$. The following properties are equivalent:*
(i) *$\text{Hess}_u\, J\big|_{F^{-1}(x)}$ is degenerate;*
(ii) *(u, λ) is a critical point for the map $\pi \circ \overline{F} : C_{F,J} \to M$.*

Moreover $\pi \circ \overline{F} : C_{F,J} \to M$ is a Lagrangian map; the induced Maslov cycle $\Sigma_{F,J}$, i. e. the set of critical points of this map, coincides with the set of those (u, λ) such that the Hessian of $J|_{F^{-1}(F(u))}$ is degenerate at u.

We discuss the proof in the special case of Morse functions in the next section; the general proof follows the same line.

5.1 Morse functions

Let us consider two Morse functions $f_0, f_1 : M \to \mathbb{R}$ and an homotopy of maps $f_t : M \to \mathbb{R}$. Then we define $U = [0, 1] \times M$ and:

$$F : [0,1] \times M \to \mathbb{R}, \quad F(t, x) = t$$
$$J : [0,1] \times M \to \mathbb{R}, \quad J(t, x) = f_t(x).$$

We have that $J\big|_{F^{-1}(t)} = f_t$ and we can study the critical points of the family of maps $\{f_t\}_{t\in[0,1]}$ with the Lagrange multipliers technique. If $u = (t, x)$, writing $d_u J = (\partial_t J, \partial_x J)$ and $D_u F = (1, 0)$ the Lagrange multipliers rule reads

$$\begin{cases} \lambda = \partial_t J(t, x) \\ \partial_x J(t, x) = 0. \end{cases} \tag{11}$$

Namely $C_{F,J}$ is the set of (λ, t, x) such that (11) holds true (the second identity is equivalent to the fact that x is a critical point of f_t). This is a system of $n + 1$ equations in a $n + 2$-dimensional space and $C_{F,J}$ defines a 1-dimensional manifold if the problem is Morse, i. e. the linearized system in the variables (λ', t', x')

$$\begin{cases} \lambda' = \partial^2_{tt} J(t,x)t' + \partial^2_{xt} J(t,x)x' \\ \partial^2_{tx} J(t,x)t' + \partial^2_{xx} J(t,x)x' = 0 \end{cases} \tag{12}$$

is regular, that means $\operatorname{rank}\{\partial^2_{tx} f, \partial^2_{xx} f\} = n$. In particular this condition is satisfied if the function f_t is Morse for every $t \in [0,1]$, i. e. $\partial^2_{xx} f_t$ is non degenerate. The tangent space to $C_{F,J}$ at the point (λ, t, x) is the set of (λ', t', x') such that (12) are satisfied.

$$\begin{array}{ccc} (\lambda, t, x) & \xrightarrow{\overline{F}} & (\lambda, t) \\ \big\downarrow{\scriptstyle \overline{\pi}} & & \big\downarrow{\scriptstyle \pi} \\ (t, x) & \xrightarrow[F]{} & t \end{array} \tag{13}$$

Hence the point $(\lambda, t, x) \in C_{F,J}$ is critical for the map if and only if there exists a nonzero element (λ', t', x') such that $\pi_* \circ \overline{F}_*(\lambda', t', x') = t' = 0$. From (12) it is easy to see that this is equivalent to $x' \neq 0$ and $\partial^2_{xx} J(t,x)x' = 0$.

Let now f_t be a generic homotopy between two Morse functions f_0 and f_1. Then the corresponding pair (F, J) defines a Morse problem and the above discussion holds. Moreover the genericity assumption on the homotopy ensures that if f_{t_0} has a critical point at x_0, the Hessian of f_{t_0} at x_0 has a one-dimensional kernel. It is indeed possible to show that near the point (t_0, x_0) the family f_t can be written in coordinates as:

$$f_t(x) = c_0 + x_1^3 \pm t x_1 \pm x_2^2 + \cdots \pm x_n^2, \quad t \in [t_0 - \epsilon, t_0 + \epsilon].$$

As t passes from $t_0 - \epsilon$ to $t_0 + \epsilon$ two critical points merge or vanish, according to the sign of $\pm tx$ (see [18]).

The induced Maslov cycle $\Sigma_{F,J}$ in this case consists of those points (λ, t, x) on $C_{F,J}$ such that f_t is not a Morse function. If (λ, t, x) is in $C_{F,J} \backslash \Sigma_{F,J}$, then in a neighborood $[a, b]$ of t the function t is a coordinate for $C_{F,J}$ and we can "follow" the critical point $x(t)$. Property 1 of Proposition 1 implies that as long as t varies on $[a, b]$, the index of such critical point never changes. The genericity assumption on the homotopy implies that if two critical points merge, their indices *must* differ by one. If $(\lambda(s), t(s), x(s))$ is a parametrization of $C_{F,J}$ near a point $(\lambda(0), t(0), x(0)) \in \Sigma_{F,J}$, the change in the *sign* of the determinant of the Hessian of $f_{t(s)}$ at $x(s)$ when passing through $s = 0$ is determined by the coorientation of $\Sigma_{F,J}$ at $(\lambda(0), t(0), x(0))$.

In this case the number of points of $\Sigma_{F,J}$ tells how many functions in our family are not Morse; the coorientation tells how the Morse index changes when two critical points merge or vanish.

Example 7 (Depth of Morse functions). Assume M is a smooth hypersurface in $\mathbb{R}^n$ defined by a polynomial of degree d and p_0, p_1 are two Morse functions obtained by restricting to M two polynomials of degree $k \geq d$. Using the above technique it is possible to prove that p_0 and p_1 can be joined by a homotopy $p_t : M \to \mathbb{R}$ such that:

$$\mathrm{Card}\{t \in [0,1] \mid p_t \text{ is not Morse}\} \le dk^n(d+nk).$$

In the case $k \le d$ the bound is $d^{n+2}(n+1)$.

5.2 Riemannian and sub-Riemannian geometry

In this section we discuss how the problem of finding geodesics in Riemannian or sub-Riemannian geometry fits in the above setting. For a comprehensive presentation of Riemannian and sub-Riemannian geometry see for instance [4, 17].

A sub-Riemannian manifold is a triple $(M, \mathcal{D}, g)$ where M is a smooth manifold and $\mathcal{D}$ is a constant rank $k \le n$ distribution endowed with a scalar product g on it. The case $k = n$, i. e. when $\mathcal{D} = TM$, corresponds to Riemannian geometry.

A curve on M defined on the interval $[0, 1]$ is said horizontal if it is almost everywhere tangent to the distribution. Once fixed a local orthonormal basis of vector fields $f_1, \ldots, f_k$ on $\mathcal{D}$, every horizontal curve is described by the dynamical system:

$$\dot{x}(t) = \sum_{i=1}^{k} u_i(t) f_i(x(t)), \quad x(0) = x_0, \tag{14}$$

for some choice of the control u; the length of such horizontal curve is defined by:

$$\ell(u) = \int_0^1 \sqrt{g(\dot{x}(t), \dot{x}(t))}\, dt = \int_0^1 \sqrt{\sum_{i=1}^{k} u_i^2(t)}\, dt.$$

It is well known that the problem of minimizing the length with fixed final time is equivalent, by Cauchy-Schwartz inequality, to the minimization of the energy

$$J(u) = \frac{1}{2} \int_0^1 \sum_{i=1}^{k} u_i^2(t)\, dt.$$

For this reason it is convenient to parametrize horizontal curves by admissible controls $u \in L^2([0,1], \mathbb{R}^k)$. By the classical theory of ODE, for every such control u and every initial condition $x_0 \in M$, there exists a unique solution x_u to the Cauchy problem (14), defined for small time (see for instance [2] for a proof).

The resulting non autonomous local flow defined on M by the ODE associated with u, i. e. the family of diffeomorphisms $P_{0,t} : M \to M$, defined by $P_{0,t}(x) := x_u(t)$ is smooth in the space variable and Lipschitz in the time variable. Analogously one can define the flow $P_{s,t} : M \to M$ fixing the initial condition at time s, i. e. $x(s) = x_0$ ($P_{s,t}$ is defined for s, t close enough).

Fix a point $x_0 \in M$. The *end-point map* of the system (14) is the map

$$F : U \to M, \quad u \mapsto x_u(1),$$

where $U \subset L^2([0,1],\mathbb{R}^k)$ is the open subset of controls u such that the solution $t \mapsto x_u(t)$ to the Cauchy problem (14) exists and is defined on the whole interval $[0,1]$. The end-point map is a smooth map. Moreover its differential $D_uF : L^2([0,1],\mathbb{R}^k) \to T_xM$ at a point $u \in U$ is computed by the following well-known formula (see [2])

$$D_uF(v) = \sum_{i=1}^{k} \int_0^1 v_i(s)(P_{s,1})_* f_i(x_u(s))ds, \quad v \in L^2([0,1],\mathbb{R}^k). \tag{15}$$

where $x_u(t)$ is the trajectory associated with u and $x = x_u(1)$.

Notice that when $u = 0$ we have rank $D_0F = \operatorname{rank} \mathcal{D} = k$. Indeed $x_u(t) \equiv x_0$ and the above formula reduces to:

$$D_0F(v) = \sum_{i=1}^{k} \alpha_i f_i(x_0), \quad \alpha_i = \int_0^1 v_i(s)\, ds.$$

In this framework, the problem of finding constrained critical points of the functional $J : U \to \mathbb{R}$ on the level set $F^{-1}(x)$ is equivalent to find critical points of the energy among those curves that join x_0 to x in fixed final time equal to 1.

Hence the solutions of the problem (7) represent exactly sub-Riemannian *geodesics* starting at x_0 and ending at x.

Notice that in the Riemannian case the map F is always a submersion, while in the sub-Riemannian case it can happen that $\operatorname{rank}(D_uF) < n$ for some u (this is the case for the control $u = 0$ as we explained above). In this case u is said *abnormal* and x_u is an *abnormal geodesic*. If u satisfies the Lagrange multipliers rule $\lambda D_uF = D_uJ$ for some λ, then u is said *normal* and x_u is a *normal geodesic* (this happens in particular at regular point of F). A control u can be at the same time normal and abnormal.

In what follows we focus our attention to *strongly normal* controls, i. e. those controls such that all the family $u_s(t) := su(st)$ is not abnormal for all $s \in]0,1]$. Notice that, by the linearity of (14) with respect to u, we have $x_{u_s}(t) = x_u(st)$. Notice also that in Riemannian geometry all geodesics are strongly normal.

Given a sub-Riemannian structure on a manifold M it is natural to build the *sub-Riemannian Hamiltonian* $H : T^*M \to \mathbb{R}$ defined by

$$H(\lambda) = \frac{1}{2}\|\lambda\|^2, \quad \|\lambda\| = \sup_{v \in \mathcal{D}_q, |v| \leq 1} |\langle \lambda, v \rangle|.$$

This is a smooth function on T^*M which is quadratic on fibers. The canonical symplectic structure allows to define a vector field $\vec{H}$ by the identity $\sigma(\cdot, \vec{H}) = dH$. The flow of $\vec{H}$ defines the normal geodesic flow and characterizes the manifold of Lagrange multipliers as follows (see [3,4]).

Proposition 5. *The sub-Riemannian pair (F, J) defines a Morse problem. Moreover the manifold of Lagrange multipliers satisfies $\overline{C}_{F,J} = e^{\vec{H}}(T^*_{x_0}M)$.*

We discuss some related ideas, giving an outline of the proof. Let $x \in M$ and $(u, \lambda) \in C_{F,J}$ associated with a critical point of $J\big|_{F^{-1}(x)}$. Then for every $v \in \ker D_u F$:

$$\mathrm{Hess}_u J\big|_{F^{-1}(x)}(v) = \|v\|^2_{L^2} - \Big\langle \lambda, \iint\limits_{0\leq\tau\leq t\leq 1} [(P_{\tau,1})_* f_{v(\tau)}, (P_{t,1})_* f_{v(t)}] d\tau dt \Big\rangle. \tag{16}$$

Indeed one can compute that in coordinates $\mathrm{Hess}_u J\big|_{F^{-1}(x)} = d^2_u J - \lambda D^2_u F$ and that the second differential of the end-point map is expressed as the commutator

$$D^2_u F(v, v) = \iint\limits_{0\leq\tau\leq t\leq 1} [(P_{\tau,1})_* f_{v(\tau)}, (P_{t,1})_* f_{v(t)}] d\tau dt,$$

where $P_{s,t}$ is the flow associated with u and $f_v = \sum_{i=1}^k v_i f_i$. Let $(u, \lambda) \in C_{F,J}$. The relation $d_u J = \lambda D_u F$ can be rewritten as follows, using the fact that $J(u) = \frac{1}{2}\|u\|^2_{L^2}$:

$$u_i(t) = \langle \lambda(t), f_i(x(t)) \rangle, \quad \lambda(t) := (P_{t,1})^* \lambda \in T^*_{x(t)}M. \tag{17}$$

Moreover the curve $\lambda(t) \in T^*_{x(t)}M$ is a solution of the Hamiltonian system $\dot{\lambda}(t) = \vec{H}(\lambda(t))$ and $\lambda(1) = \lambda$. This allows to parametrize geodesics via their initial covector rather than the final one. We define the *exponential map* starting from x_0 as:

$$\mathcal{E} : T^*_{x_0}M \to M, \quad \mathcal{E}(\lambda_0) = \pi \circ e^{\vec{H}}(\lambda_0).$$

Since $e^{\vec{H}}(T^*_{x_0}(M)) = \overline{C}_{F,J}$, then this map is Lagrangian. Moreover, by homogeneity of the Hamiltonian, for all $t > 0$ we have $\mathcal{E}(t\lambda_0) = \pi \circ e^{t\vec{H}}(\lambda_0) = x_u(t)$, which permits to recover the whole normal geodesic associated with λ_0 (here u is the control defined by (17) and $\lambda(t) = (P_{t,0})^*\lambda_0$). Thus the exponential map parametrizes normal geodesics starting from a fixed point with covectors attached to the fiber $T^*_{x_0}M$. If λ_0 is a critical point of $\mathcal{E}$ then the point $x = x_u(1) = \mathcal{E}(\lambda_0)$ is said to be *conjugate* to x_0 along the geodesic $x_u(t)$.

Theorem 2. *Let $x_u(t)$ be a strongly normal geodesic joining x_0 to x. The following are equivalent:*
(i) *$\mathrm{Hess}_u\, J\big|_{F^{-1}(x)}$ is degenerate;*
(ii) *x is conjugate to x_0 along $x_u(t)$.*

Moreover the geodesic $x_u(t)$ loses its local optimality at its first conjugate point.

The induced Maslov cycle Σ_{x_0}, i. e. the set of critical points of $\mathcal{E}$, coincides with the set of those $\lambda \in T^*_{x_0}M$ such that the Hessian of $J|_{F^{-1}(F(u))}$ at the corresponding geodesic is degenerate.

By the homogeneity of the Hamiltonian, to study the local optimality of a piece $x_u|_{[0,s]}$ of the fixed trajectory x_u it is enough to apply the functional J to the control $u_s(t) = su(st)$, whose final point is $x_{u_s}(1) = x_u(s)$.

Thus we have the following picture: the map $\mathcal{E} : T^*_{x_0}M \to M$ is a Lagrangian map with the property that $\mathcal{E}(\lambda_0)$ is the final point of a geodesic x starting at x_0; this geodesic is the one associated to the control u defined by Eq. (17).

We can indeed consider the all ray $\{s\lambda\}_{s>0}$: the image of such ray is the geodesic associated with λ. For small $s > 0$ the Hessian $\mathrm{Hess}_{u_s} J\big|_{F^{-1}(x_u(s))}$ is positive definite (as a consequence of formula (16)), and it becomes degenerate exactly when $s\lambda$ belongs to the induced Maslov cycle Σ_{x_0} (in particular the first degeneracy point coincide with the first conjugate point).

With a normal geodesic $x(t)$ (with lift $\lambda(t)$) one can associate also the so-called *Jacobi curve*:

$$\Lambda(t) = e_*^{-t\vec{H}} T_{\lambda(t)}(T^*_{x(t)}M),$$

which is a curve of Lagrangian subspaces in the symplectic space $T_{\lambda_0}(T^*_{x_0}M)$. Using this curve we can compute the index of the Hessian: in fact if c is a cocycle representing the *Maslov class* μ, we have:

$$c(\Lambda(s)) = -\mathrm{Ind}\,\mathrm{Hess}_{u_s} J\big|_{F^{-1}(x_u(s))}.$$

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How to Run a Centipede: a Topological Perspective

Yuliy Baryshnikov and Boris Shapiro

Abstract In this paper we study the topology of the configuration space of a device with d legs ("centipede") under some constraints, such as the impossibility to have more than k legs off the ground. We construct feedback controls stabilizing the system on a periodic gait and defined on a 'maximal' subset of the configuration space.

A centipede was happy quite!
Until a toad in fun
Said, "Pray, which leg moves after which?"
This raised her doubts to such a pitch,
She fell exhausted in the ditch
Not knowing how to run.

Katherine Craster

1 Introduction

How the centipedes move? This question becomes nontrivial once one starts to think about it, or when one is designing a multi-legged robotic device [2]. Indeed, the motivation for this work comes from a class of agile robotic devices, *RHex* [3]. Our take on the centipede's quandary is that it is caused by essentially *topological reasons*, preventing continuous feedback controls.

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Fig. 1 A specimen of the RHex family of legged robots, designed in University of Pennsylvania. Reproduced with permission from [4]

In this note we consider a caricature of an automotive robot moving around using rotating "legs", making the configuration space a torus $\mathbb{T}^d$, i. e. a d-fold product of the circle, $\mathbb{T}^1$. The similarities with the wheeled vehicles end here: for obvious reasons, there exist regions in the configuration space, a "forbidden" subset, where the system should avoid at any cost. The picture below, taken from `http://kodlab.seas.upenn.edu/RHex/Home` illustrate the kind of systems we are dealing here.

As an example, the configuration where all the legs point up should be forbidden. Of course the forbidden configurations are design specific: thus in RHex, the forbidden configurations also include those with all legs up on one side of the robot, or those with just two legs (out of six) pointing down.

Excluding the forbidden regions makes the topology of the configuration space interesting, and the control problems (even in the fully actuated setting) nontrivial. Typically, the control design problem aims at a closed-loop feedback control that stabilizes the system on a (say, periodic) trajectory, a *gait*. As the homotopy type of the configuration space differs from that of the limiting attractor, a continuous feedback control is impossible, and a locus of discontinuity emerges. This locus of discontinuity is *not* canonical, and depends on the realization of the feedback control, but its topology is, as it turns out, more or less fixed by the mismatch of the homotopy types of the configuration space and the attractor.

This motivates our attention to the topology of the configuration space and constructions of the minimal, in a suitable sense, discontinuity loci.

In this paper our objective is to analyze from this viewpoint the topology of the configuration spaces of *RHex*-like robots which we will be referring to as the *centipedes*. To do this we:

- describe the topology of the discontinuity loci;
- present an explicit construction of the discontinuity loci for a large class of robots (and their corresponding forbidden regions), and
- find a feedback control for rotation of centipede's legs stabilizing the system on a prespecified (diagonal) gait.

1.1 Setup

Let us fix the notation. We denote the total number of legs as d, which are fully actuated and can (apriori) take all possible positions. The space of legs positions, the d-dimensional torus $\mathbb{T}^d$ is coordinatized by the *angles* $\phi_i, i = 1, \ldots, d, \phi_i \in \mathbb{T}^1 = [0, 2\pi]/\langle 0 = 2\pi \rangle$. We will assume that $\phi_i = 0$ corresponds to the position of the i-th leg pointing vertically up.

To describe the class of forbidden configurations, we will need the notion of *coordinate toric arrangements*. Let I be an *ideal* in the Boolean lattice B_d of subsets of $\{1, \ldots, d\}$ (i.e. if $A \in I$, and $B \subset A$ then $B \in I$).

The coordinate toric arrangement $\mathcal{A}_I$ is the union of all coordinate tori T_A, $A \in I$:

$$\mathcal{A}_I = \bigcup_{A \in I} \mathbb{T}_A,$$

where $\mathbb{T}_A = \{\phi_i = 0 \text{ for } i \notin A\}$ (the size of A is the dimension of $\mathbb{T}_A$). We remark that the tori $\mathbb{T}_A$ provide a natural stratification of the arrangement $\mathcal{A}_I$. The inclusion $I_1 \subset I_2$ implies $\mathcal{A}_{I_1} \subset \mathcal{A}_{I_2}$.

One typical example is $I = \{A : |A| \geq k\}$, the configurations with at least $k \leq d$ legs are pointing up. In this case, the corresponding toric arrangement is just the k-skeleton of the torus.

A toric arrangement is good approximation for a forbidden region: the fact that a whole coordinate torus is forbidden is equivalent to the natural assumption, that if having some collection of legs up causes failure of the device when the rest of the legs point down, then bringing these remaining legs into *any* configuration still will result in a failure. Thus, for the original *RHex*, having three right legs up, and three left legs down is a failure, and any other position of the left legs will still be a failure.

Of course, having the forbidden set a toric arrangement is merely a caricature of the *physical* set of forbidden configurations: clearly, the stability of a robotic device cannot fail exactly when some collection of legs is pointing upwards, and not in nearby points. However, from the topological perspective, this assumption is rather reasonable, if one adopts its softer version.

1.2 Conventions

We will be assuming (relying on the intuition outlined above, and developed in the literature on RHex, see, e. g. [3,4]) that *set of failure positions* $\mathtt{Fb}_I \subset \mathbb{T}^d$ is an open domain containing $\mathcal{A}_I$ with *smooth boundary*, such that $\mathcal{A}_I \subset \mathtt{Fb}_I$ is a *deformation retract*. Its complement $\mathtt{Fr}_I = \mathbb{T}^d \setminus \mathtt{Fb}_I$ is the *set of safe configurations*.

Further, we assume that $\mathtt{Fb}_I$ is an open and $\mathtt{Fr}_I$ is a closed manifold with a smooth boundary $\partial\mathtt{Fr}_k$.

We are interested in *closed loop feedback stabilization*, that is in vector fields $\mathbf{v}$ defined at least in $\mathtt{Fr}_I$ (including its boundary $\partial\mathtt{Fr}_I$) and such that the field $\mathbf{v}$ points into $\mathtt{Fr}_I$ on $\partial\mathtt{Fr}_I$. The vector field $\mathbf{v}$ should have as an attractor a periodic trajectory (gait) γ.

If (as is typical) $\mathtt{Fb}_I$ does not have the homotopy type of a circle, it is impossible to have $\mathtt{Fb}_I$ as the basin of attraction for γ. Hence, we need to find a subset $\mathtt{Bs}_I \subset \mathtt{Fr}_I$ which contains the attractor γ, is as large as possible and is a basin of attraction for γ. (We are deliberately vague here about the meaning of the expression "as large as possible" which will be clarified below.)

The complement to such a basin will be called a *cut*. The fact that a continuous feedback stabilization is impossible if the topologies of the configuration space and the attractor do not match has been noticed long ago (see, e. g. [6]). What we emphasize here is the nontrivial topology of the cuts (implying that it has to be non-empty), and some useful criteria for its minimality.

1.3 Outline

The general theory of the topologically forced cuts in the closed loop feedback stabilization will be addressed elsewhere; this note serves as an extended example of the stabilization in nontrivial configuration spaces, rich and relevant to applications yet simple to be analyzed completely.

The structure of the paper is as follows. In Sect. 2 we describe some relevant topological preliminaries. In Sect. 3 we introduce a construction of a cut that is optimal for all ideals I. In Sect. 4 we describe a vector field stabilizing the system to a periodic trajectory on the optimal $\mathtt{Bs}_I$. Finally, in Appendix we describe an intriguing discrete dynamical system associated with our choice of the basin and cut.

2 Topology of $\mathcal{A}_I$ and $\mathtt{Fb}_I$

2.1 Topology of forbidden set

By assumption, the set $\mathtt{Fb}_I$ of forbidden configurations is retractable to the toric arrangement $\mathcal{A}_I$ so that the embedding of its complement $\mathtt{Fr}_I$ to $T^d \setminus \mathcal{A}_I$ is a homotopy equivalence.

The space $T^d \setminus \mathcal{A}_I$ is in its turn is retractable to a certain toric arrangement. We refer for the detailed exposition to, e. g. [1], and present here just the result.

An ideal I (of the partition lattice) can be considered as a non-increasing Boolean function f_I: of the vector of 0, 1's is the indicator function of A, then $f_I(A) = 1$ iff $A \in I$. The function

$$f_{I^\circ} : (x_1, \dots, x_d) \mapsto 1 - f_I(1 - x_1, \dots, 1 - x_d)$$

is also non-increasing and therefore defines a Boolean ideal I°; we call it the *dual ideal* to I.

The toric arrangement corresponding to I° on which $T^d \setminus \mathcal{A}_I$ retracts can be described as

$$\mathcal{A}_I^\circ = \bigcup_{B \in I^\circ} \mathbb{T}_B^\circ,$$

where $\mathbb{T}_B^\circ = \{\phi_j = \pi \text{ for } j \notin B\}$.

In particular, if I° contains all singletons (or, equivalently, if *each* leg can make a full turn avoiding forbidden configurations, with the remaining legs in some fixed positions), then the first homology of $\mathtt{Fr}_I$ coincides with that of the torus. More generally, if I° contains the all subsets of size k (or I does not contain subset of size $(d-k)$ or more), then the integer (co)homology groups of $\mathtt{Fr}_I$ coincide with these of T^d up to the dimension $d-k-1$ and the isomorphism of (co)homology groups is induces by the inclusion $\mathtt{Fr}_I \subset T^d$.

Also the fundamental group $\pi_1(\mathtt{Fr}_I)$ of $\mathtt{Fr}_I$ is isomorphic to that of T^d and thus coincides with $\mathbb{Z}^d$ if I does not contain subsets of size $(d-2)$.

2.2 Feedback stabilization

2.2.1 Attractors

We are concerned primarily with the stabilization on a specific gait, a periodic trajectory representing the diagonal homology class in $H_1(\mathbb{T}^d, \mathbb{Z})$. (Note that in principle other classes are possible, for example a multiple of the diagonal class, corresponding to a periodic gait.)

Remark 1. Knotted attractors present a potential complicating twist. If the number of legs is three, there are infinitely many nonequivalent (under an ambient isotopy) trajectories representing the same (free) homotopy class in the space $\mathtt{Fr}_I$. We will be ignoring this problem – there are few plausible engineering designs with mere three legs, and in $d \geq 4$ piece-wise smoothly embedded closed curves are isotopic when they represent the same homotopy class.

However, it would be interesting to try to construct a knotted gait for three-legged robots, and a feedback control stabilizing on such a gait.

We fix this closed simple oriented curve γ in $\mathtt{Fr}_I$, the attractor convergence to which we are seeking, representing the diagonal homology class in $H_1(\mathbb{T}^d, \mathbb{Z})$ (which means, in words, that over the trajectory, each leg makes exactly one turn around).

If $\mathtt{Fb}_I$ is a sufficiently small neighborhood of $\mathcal{A}_I$ we can choose γ among the geodesics of the flat metrics on $\mathbb{T}^d$, i. e. among $\gamma_\phi = \phi + t(1, ..., 1)$ on $\mathbb{T}^d$ where $t \in \mathbb{R}$ and γ is a sufficiently generic point in $\mathbb{T}^d$. (This does not reduce generality as by assumption, one can always find a diffeomorphism – fixing $\mathcal{A}_I$ – that would shrink $\mathtt{Fb}_I$ to a small vicinity of $\mathcal{A}_I$.)

2.3 Vector fields and their basins

As we mentioned above, the closed loop feedback stabilization of $\mathtt{Fr}_I$ on γ is impossible in nontrivial situations: there is no vector field $\mathbf{v}$ on $\mathtt{Fr}_I$, pointing inward $\mathtt{Fr}_I$ on the boundary, such that all solutions tend to the attractor γ. This means one need to reduce the domain where the vector field is defined.

Definition 1. We will be calling an open subset $\mathtt{Bs} \subset \mathtt{Fr}_I$ an *admissible basin*, if there exists a smooth vector field on $\mathtt{Fr}$ such that:

- the gait γ is an attractor of the positive time flow $g_{\mathbf{v}}^t$ defined by $\mathbf{v}$;
- the negative trajectories $g_{\mathbf{v}}^t x, t < 0$ starting outside γ leave $\mathtt{Bs}$ in finite time (depending on the starting point $x \notin \gamma$).

The complement $\mathtt{Ct}_I$ to the admissible basin $\mathtt{Bs}_I$ will be called an *admissible cut*, or simply a cut.

We will call an admissible basin $\mathtt{Bs}_I$ *set maximal* in $\mathtt{Fr}_I$ if no proper superset of $\mathtt{Bs}_I$ in $\mathtt{Fr}_I$ has the same homotopy type as $\mathtt{Bs}_I$.

The set-maximality property of $\mathtt{Bs}_I$ is rather basic and departs from the natural geometric characteristics like volume of $\mathtt{Ct}$ or its dimension, Hausdorff measure and suchlike. The reason is obvious: the definition is universal, and independent of any extraneous data save the topological ones.

3 Universal cut

One of the main contributions of this paper is the construction of a universal cut, that is one that serves *all* arrangements $\mathcal{A}_I$.

3.1 Main construction

Represent $\mathbb{T}^d$ as the d-dimensional cube $K_d = [-\pi, \pi]^d$ with its parallel sides identified in the standard way. We use the system of coordinates ψ, with $\psi_i = \pi - \phi_i$, so that the origin $O = (0, 0, ..., 0)$ corresponds now to the position 'all legs down'.

The tori $\mathbb{T}_A, A \subset \{1, \ldots, d\}$ introduced above define a stratification of $\mathbb{T}^d$. Its open strata are cells of different dimensions, again indexed by the subsets A. We will be referring to these open cells as the *cubes* $\mathtt{Cb}_A$. The union of the cells of the stratification of dimensions $\leq k$ - the k-skeleton - is denoted as $\mathtt{Sk}_k$.

Consider the cone $\mathtt{Co}_d$ in $\mathbb{T}^d$ over the $(d-2)$-skeleton $\mathtt{Sk}_{d-2}$ with the vertex at O. This cone is a singular hypersurface in $\mathbb{T}^d$ stratified by the cones over different coordinate subtori contained in $\mathtt{Sk}_{d-2}$. Notice that $\mathtt{Co}_d - \mathtt{Sk}_{d-2}$ contains $2^k\binom{d}{k}$ strata of codimension $(k-1)$ (the factor 2^k comes from various ways to connect the torus $\mathbb{T}_A, |A| = d-k$ with O), so that the total number of cones over $(d-2)$-dimensional cubes in $\mathtt{Co}_d$, that is flats of codimension 1 is $2d(d-1)$.

The complement $\mathbb{T}^d \setminus \mathtt{Co}_d$ consists of d open polyhedra, each being the union of two pyramids over the $(d-1)$-dimensional open cube $\mathtt{Cb}_{-i}$, the open cell in $\mathbb{T}_{-i} = \{\phi_i = 0\}$.

Let us denote these polytopes as $\mathtt{Pyr}_i, i = 1, ..., d$: here i is the coordinate missing in the $(d-1)$-dimensional cube $\mathtt{Cb}_{-i}$ which is coned. The gait γ intercepts the boundary of each $\mathtt{Pyr}_i$ at two points belonging to some faces $\mathtt{Fc}_i^+, \mathtt{Fc}_i^-$ on its boundary. Each such face is the interior of a cone (one of 4 possible), still with apex at O, over some $(d-2)$-dimensional cube $\mathtt{Cb}_{-i-j}$.

The face where the trajectory γ enters (resp. leaves) $\mathtt{Pyr}_i$ is called the *i-th entrance face* $\mathtt{Fc}_i^+$ (resp. the ith exit face $\mathtt{Fc}_i^-$) and the corresponding points are called

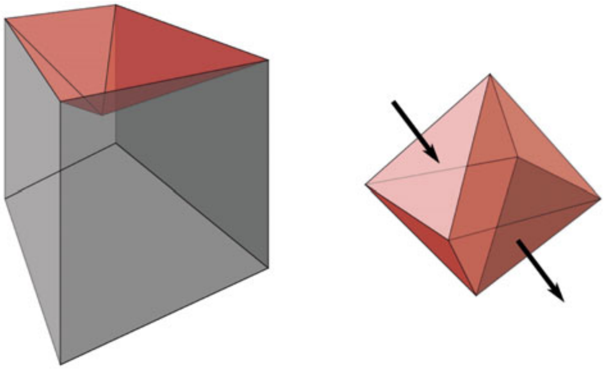

Fig. 2 Left: some of the strata of the cone $\mathtt{Co}_3$. Right: a pyramid

the entrance/exit points. Another important point within $\mathtt{Pyr}_i$ besides the entrance and exit points is the point where γ intersects the base of $\mathtt{Pyr}_i$, i. e. the corresponding $(d-1)$-dimensional cube, see Fig. 2.

We remark that γ defines a *cyclic order* on the set of all $\mathtt{Pyr}_i$ according to the order in which the trajectory hits them, see Fig.1. Note that the exit face for any pyramid is at the same time the entrance face of the next one in this cyclic order. Without loss of generality, we can assume that this cyclic order is $1 < 2 < 3 < \ldots < d < 1$.

In the configuration space, the exit face $\mathtt{Fc}_i^-$ of the pyramid $\mathtt{Pyr}_i$ is identified with the entrance face of $\mathtt{Pyr}_{i+1}$. We will be calling this face, which is, again, a cone over $\mathtt{Cb}_{-i-(i+1)}$, the *i-th door*.

Finally, we define

$$\mathtt{Ct}_d = \mathtt{Co}_d \setminus \bigcup_{i=1}^{d} \mathtt{Fc}_i^+ = \mathtt{Co}_d \setminus \bigcup_{j=1}^{d} \mathtt{Fc}_j^-$$

to be the union of the cones (with the apex O) over all codimension 2 cubes in the $(d-2)$ skeleton of $\mathbb{T}^d$ *with exception of the doors*. Equivalently, it is the cone over the full $(d-2)$-skeleton with the doors removed.

Theorem 1. *The stratified hypersurface* $\mathtt{Ct}_d$ *is a set-minimal cut for any* $\mathtt{Fb}_I$*, as long as the boundary of* $\mathtt{Fb}_I$ *is transversal to* $\mathtt{Ct}_d$*.*

The transversality required in the theorem is automatic if, for example, $\mathtt{Fb}_I$ is a small enough tubular neighborhood of $\mathcal{A}_I$.

Before moving to the proof of the Theorem 1, we will describe the cut in more “engineering” terms.

3.2 Forbidden leg positions

For the sake of clarity let us present a simple description of $\mathtt{Ct}_d$ in terms of configurations of legs. Let $(\psi_1, ..., \psi_d)$, $-\pi \leq \psi_i \leq \pi$ be the usual angular coordinates on the torus $\mathbb{T}^d$, $\psi = 0$ corresponding to the "leg down" position.

The i-th open pyramid $\mathtt{Pyr}_i$ consists then of exactly those leg positions, for which the i-th leg has the maximal height, i. e. $1 - \cos\psi_i > 1 - \cos\psi_j, \forall j \neq i$.

Its entrance face is the set of all leg positions when exactly the $(i-1)$-st leg and the i-th leg are at the maximal height among all legs. Additionally, their positions are not allowed to coincide ($\psi_i \neq \psi_j$) and the corresponding angles are in the correct cyclic position (i. e. $\psi_i > \psi_{i+1}$).

The Fig. 3 illustrates the positions in the cut and outside it.

Proof (Proof of the Theorem 1). The torus $\mathbb{T}^d$ with the cut $\mathtt{Co}_d$ deleted can be constructed by identifying the d pyramids $\mathtt{Pyr}_i, i = 1, \dots, d$ along the pairs of exit-entrance faces: the exit face of the pyramid $\mathtt{Pyr}_i$ is identified with the entrance face of $\mathtt{Pyr}_{i+1}$. This immediately implies that the admissible basin $\mathtt{Bs}_d = \mathbb{T}^d - \mathtt{Co}_d$ is homeomorphic to the d-dimensional solid torus, the product of $(d-1)$-dimensional (open) ball and $\mathbb{T}^1$. The trajectory γ is embedded into the *basin* and, again by construction, generates $H_1(\mathtt{Bs}_d, \mathbb{Z})$. Now, the assumption of unknottedness implies immediately that γ is a deformation retract of $\mathtt{Bs}_d$. (In fact, we will construct an explicit flow on $\mathtt{Bs}_d$ realizing such a deformation.)

Now, it remains to show that the cut is set-minimal. Assume that a superset S of $\mathtt{Bs}_d$ contains a point $x \in \mathtt{Co}_d \cap \mathtt{Fr}_I$. As the intersection of a small ball around x in $\mathbb{T}_d$ intersected with $\mathtt{Bs}_d$ contains more than one connected components (corresponding to different pyramids), one can choose (piecewise-linear) curves that connects x to some points x_1, x_2 on the segments of the gait γ in the corresponding pyramids. Combining these curves with a segment of γ connecting x_1 and x_2 one obtains a closed curve that represents a class β in $H_1(\mathbb{T}^d, \mathbb{Z})$ different from the diagonal class $\delta = [\gamma]$. Hence, $H_1(S, \mathbb{Z})$ has rank at least two, and the homotopy type of S cannot be that of $\mathbb{T}^1$. □

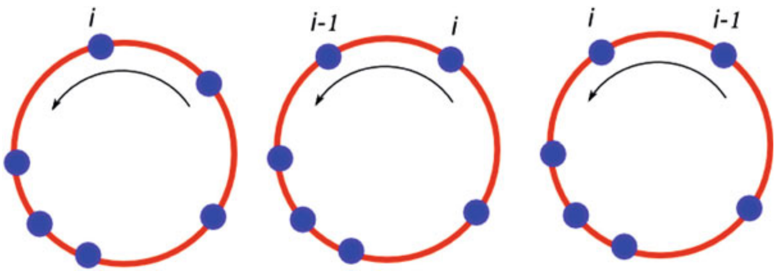

Fig. 3 Left: a typical configuration inside $\mathtt{Pyr}_i$; middle and right: configurations on the entrance and exit faces of $\mathtt{Pyr}_i$

4 Feedback stabilization on γ

In this section we will construct two explicit vector fields on $\mathtt{Bs}_I$ for $\mathtt{Fb}_I = \mathcal{A}_I$, such that applying one for a short period of time (one full rotation of a leg) and then switching on the other, all trajectories will converge to a prespecified gait (for example, the equispaced gait γ_s with the phases ϕ_i of the d legs uniformly spaced over the circle and moving with constant speed. This particular trajectory is not necessarily a realistic one and is chosen just to simplify the presentation.

We remark that any control mechanism that stabilizes on the equispaced ordered gait γ_s can be considered as a continuous-time *sorting algorithm*: starting with any leg configuration, we align them, after some time, in a prearranged cyclic order. In fact, this is precisely the task that the first vector field will perform: we will show that after one period, all the legs are cyclically ordered (say, in the standard order assumed above). The second stage is then a straightforward synchronization, locking the gait on the exponentially stable period trajectory γ_s.

Not to overload the exposition, we consider just the case where $\mathtt{Fb} = \mathcal{A}_I$, although quite general sets of forbidden configurations (tubular neighborhoods of $\mathcal{A}_I$ can be handled in a similar fashion.

4.1 Rearranging the legs

It is piece-wise smooth and analytic in each of the open pyramids where the single leg is the highest one. (In principle, the idea behind this dynamics is very similar to that of the time-dependent dynamics described in the next section.) Take a pyramid $\mathtt{Pyr}_i$ where the i-th leg has the strictly largest height among all legs, i. e. $h_i = 1 - \cos\psi_i$ is greater than all the other h_j's. (Recall that $\psi_j,\ j = 1, ..., d$ are the angle coordinates on our torus $\mathbb{T}^d$ normalized so that $\psi = 0$ corresponds to the "leg down" position.)

Define $\mathbf{v}$ on $\mathtt{Pyr}_i$ as

$$\begin{cases} \dot\psi_j = 1 \text{ for } j \neq i+1, \\ \dot\psi_{i+1} = (h_i - \max_{j\neq i,i+1} h_j)^{-1/2}. \end{cases}$$

This vector field is well defined outside of the "diagonals" $\Delta_{kl} = \{h_k = h_l\}, 1 \leq k < l \leq d$ (in fact, it is real-analytic on the complement to the union of these diagonals).

Conceptually, on $\mathtt{Pyr}_i$, where the leg i is at the highest position, the $(i+1)$-th coordinate accelerates so that it overtakes all other coordinates while i is still the highest height leg - that is while still in $\mathtt{Pyr}_i$.

The structure of the trajectories on $\mathtt{Bs}_I$ is given by the following

Proposition 1. *The vector field* $\mathbf{v}$ *defines a continuous flow on each pyramid* $\mathtt{Pyr}_i$. *Furthermore:*

- *for any point inside* $\mathtt{Pyr}_i$, *the forward trajectory reaches the exit door (a point on the exit face* $\{h_i = h_{i+1}, \phi_i < 0 < \phi_{i+1}\}$*) of the pyramid* $\mathtt{Pyr}_i$ *in finite time;*

- *moreover, for any point on the entrance door of* $\mathtt{Pyr}_i$ *(that is a point with* $h_i = h_{i-1}, h_i > h_j, j \neq i, i+1, \phi_{i-1} < 0 < \phi_i$*) there exists a unique trajectory of* $\mathbf{v}$ *on* $\mathtt{Pyr}_i$ *having that point as its initial value.*

Proof. The proof of these claims is pretty straightforward. The first statement follows from the evident fact that $\mathbf{v}$ is smooth as long as $(i+1)$-th leg is not the second in height, and near the diagonal $h_i > h_{i+1} = h_k > h_l, l \neq i, i+1, k$ (where $\mathbf{v}$ loses smoothness - but not continuity), the flow can be constructed explicitly.

The second statement follows from the fact as long as the $(i+1)$-st leg is not the second in height after i-st leg, the velocity of ψ_{i+1} behaves like $(t_* - t)^{-1}$ (where t_* is the instant when the the height of i-th leg equals to the height of some of the other legs with index $\neq i+1$ - recall that on $\mathtt{Pyr}_i$, all legs but $(i+1)$-st have constant velocity). It follows that $(i+1)$-st leg becomes the closest competitor to the leader i overtaking all other legs.

Once the $(i+1)$-st leg become the competitor to i-th one, it remains second in height, eventually taking over the leadership, as can be computed explicitly, again.

The sorting to which we alluded above is achieved after just one full rotation (of the initial leader leg).

4.2 Asymptotic stability

Once we know that the legs are in a required cyclic order, it is a routine matter to stabilize them on a desired trajectory γ_s: as an example, one can consider the following vector field,

$$\dot{\psi}_i = 1 - (\phi_{i-1} - \psi_i)^{-2} + (\psi_i - \psi_{i+1})^{-2}.$$

Note that the phase differences are well defined as the phases are cyclically ordered.

This system can be interpreted as d particles constrained to the circle, under the Coulomb's repulsive force between nearby particles and constant drift. It is immediate to see that the flow preserves the cyclic order, and has the gait γ_s as the global asymptotically stable attractor.

Remark 2. The above dynamics consists of two phases: the 1st turn of the legs and the remaining motion. During the first turn all the legs are placed in the clockwise order coinciding with their cyclic order. This is done within a rather small time interval and might be difficult to technically realize in practice since it requires quick motions of legs and quick stops. One observes that small measurement mistakes can result in the instability of the motion since the order of leading legs can experience big changes. The second phase, on the other hand, presents no difficulties, and the motion quickly converges to the rotation of the equally spaced legs with the unit speed.

5 Further remarks and speculations

In the present note we introduced and discussed the notion of a set-theoretical maximality of the set $\mathtt{Bs}_I$. Obviously, this is a rather weak notion: there are many set maximal basins (just act by a diffeomorphism of the torus identical near $\mathtt{Fb}_I$), and our definition does not single out any of them. To do so one needs some alternative notions of minimality for the cuts (on top of set-minimality). As an example of another notion of maximality that makes sense one can suggest the $(d-1)$-*volume* of the cut $\mathtt{Ct}_I \subset \mathbb{T}^d$.

While in our situation, the cut is always a (singular) hypersurface, there are similar models, where the cut has higher co-dimension. In such cases one should consider the volume form of the appropriate dimension.

We remark that the set $\mathtt{Bs}_I$ which was constructed above is *not* volume minimal in the above sense: the easiest way to see it is to remember that in the minimal soap films, the codimension 1 sheets come together at a codimension 2 strata in triples, at the angle of 120°. The problem of finding of the set $\mathtt{Bs}_k$ of the minimal volume is interesting even in the standard case $\mathtt{Fb}_k$ of the configuration "no more than k legs up"...

Appendix

A Discrete autonomous control

A.1 Entrance-Base-Exit Flows

Below we describe an interesting discrete dynamical system associated with our construction above. It addresses a somewhat different problem - not the stabilization on a single attractor, but rather generating a simple flow with piece-wise linear trajectories, but its nice mathematical features compelled us to present it here.

We construct a flow through the union of the pyramids $\mathtt{Pyr}_i$ such that on each of them this flow enters only through its entrance face, $F := \mathtt{Fc}_i^+$ and leaves through the exit face, $G := \mathtt{Fc}_i^-$.

Both faces are cones over certain $(d-2)$-dimensional cubes (corresponding to the legs i, $(i-1)$ and i, $(i+1)$ being simultaneously leaders, in the proper order). The flow we are looking for should move from the entrance face F through the $(d-1)$-cube $B := \mathtt{Cb}_{-i}$ of the whole pyramid and then further to the exit face G.

A.2 Birational mappings

Let us define two natural maps from the (open) entrance face F to the (open) base cube B and then from B to the (open) exit face G. Each such map can be transformed into a (continuous) flow by connecting the preimage and its image by a straight line within the pyramid. (Thus each trajectory of such a flow within $\mathtt{Pyr}_i$ will be the union of two straight segments.)

The most natural way to do it is by using the so-called blow-up/blow down rational transformations [5]. We present these transformations explicitly below for the cases $d = 3$ and $d \geq 4$. (The essential distinction of these two cases is explained by the fact that for $d = 3$ the entrance/exit faces are the usual triangles and, therefore, they allow additional symmetry transformations unavailable for $d \geq 4$.)

Case $d = 3$

The entrance/exit faces F and G are usual triangles and the base cube B is a usual square. Let us identify the entrance triangle F with the triangle with the vertices $(0, 0), (1, 0), (1, 1)$ in $\mathbb{R}^2$; the base square B with the square whose vertices are $(0, 0), (1, 0), (0, 1), (1, 1)$ and, finally, the exit triangle G with the triangle with the vertices $(0, 0), (0, 1), (1, 1)$.

The *blow-up* map $\Phi : (x, y) \to (x, \frac{y}{x})$ sends F to B. (It sends the pencil of lines through the origin to the pencil of horizontal lines.) Its inverse *blow-down* map $\Psi : (s, t) \to (st, t)$ maps B to G. It sends the pencil of vertical lines to the pencil of lines through the origin. Their composition $\chi = \Psi \circ \Phi : (x, y) \to (y, \frac{y}{x})$ sends F to G, see Fig.1

To get the whole discrete dynamical system assume that the three (since $d = 3$) pyramids $\mathtt{Pyr}_1, \mathtt{Pyr}_2, \mathtt{Pyr}_3$ are cyclically ordered as $1 < 2 < 3 < 1$ by the choice of Γ_γ. Denote their entrance faces as F_1, F_2, F_3 and their exit faces as G_1, G_2, G_3. Notice that $F_1 = G_2, F_2 = G_3, F_3 = G_1$. Assume now that we apply our transformation χ three times consecutively, i.e first from F_1 to $G_1 = F_2$, then from F_2 to $G_2 = F_3$, and, finally back to $G_3 = F_1$. The resulting self-map $\Theta : F_1 \to F_1$ is classically referred to as the *Poincare return map* of the dynamical system. To calculate it explicitly we need to find a suitable affine transformation A sending G back to F in the above example. Then we get the self-map Θ by composing χ with A and taking the 3-rd power of the resulting composition. As such a map A one can choose $A : (u, v) \to (1 - u, 1 - v)$ which implies that the required Poincare return map is the third power of $\Theta = A \circ \chi$ where:

$$\Theta : (x, y) \to \left(1 - y, 1 - \frac{y}{x}\right).$$

Lemma 1. *The above map Θ has a unique fixed point within the triangle F_1 and its fifth power is identity.*

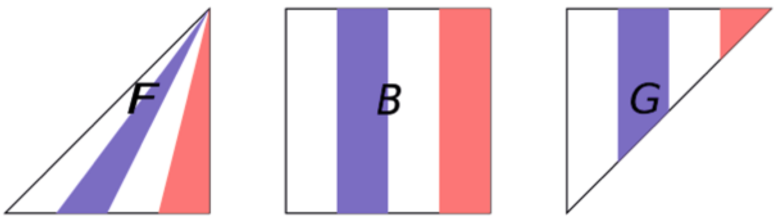

Fig. 1 Birational transformation from the entrance face to base to exit face

Proof. The system of equations defining fixed points reads as

$$\begin{cases} x = 1 - y, \\ y = 1 - \frac{y}{x} \end{cases}$$

and its two solutions are $\psi_1 = \frac{2\sqrt{2}-1}{2}$, $y_1 = \frac{3-2\sqrt{2}}{2}$ and $\psi_2 = -\frac{1+2\sqrt{2}}{2}$, $y_2 = \frac{3+2\sqrt{2}}{2}$. One can easily check that only the first solution belongs to F_1. Direct calculations show that

$$\Theta^2 : (x, y) \to \left(\frac{y}{x}, \frac{y(1-x)}{x(1-y)}\right), \quad \Theta^3 : (x, y) \to \left(\frac{x-y}{x(1-y)}, \frac{x-y}{(1-x)}\right)$$

$$\Theta^4 : (x, y) \to \left(\frac{1-x}{1-y}, 1-x\right), \quad \Theta^5 : (x, y) \to (x, y).$$

The Poincare return map is thus equals to $\Theta^3 : (x, y) \to \left(\frac{x-y}{x(1-y)}, \frac{x-y}{(1-x)}\right)$.

Case $d \geq 4$

Analogously, we have d pyramids each being a cone over a $(d-1)$-cube. Their entrance and exit faces are cones over a square respectively. The map Φ sends the open entrance face F to the open base $(d-1)$-cube B and the map Ψ sends the open base cube B to the open exit face G. They can be given explicitly as follows. Let us identify F with the domain $\{0 < \psi_2 < \psi_1 < 1; 0 < \psi_3 < \psi_1 < 1; ... 0 < \psi_{d-1} < \psi_1 < 1\}$, i.e. with the cone over the square $\{0 < \psi_2 < 1,\ 0 < \psi_{d-1} < 1\}$ with the vertex at the origin. The base B will be identified with the cube $\{0 < \psi_1 < 1,\ 0 < \psi_2 < 1, 0 < \psi_{d-1} < 1\}$, and, finally, the exit face G with $\{0 < \psi_1 < \psi_2 < 1;\ 0 < \psi_3 < \psi_2 < 1, ..., 0 < \psi_{d-1} < \psi_2 < 1\}$. Then the *blow-up map* Φ and the *blow-down map* Ψ can be chosen as follows:

$$\Phi : (\psi_1, \psi_2, ..., \psi_{d-1}) \to \left(\psi_1, \frac{\psi_2}{\psi_1}, \frac{\psi_3}{\psi_1}, ..., \frac{\psi_{d-1}}{\psi_1}\right)$$

$$\Psi : (y_1, y_2, ..., y_{d-1}) \to (y_1 y_2, y_2, ..., y_{d-1} y_2).$$

Their composition $\chi : F \to G$ coincides with

$$\chi : (\psi_1, \psi_2, ..., \psi_{d-1}) \to \left(\psi_2, \frac{\psi_2}{\psi_1}, \frac{\psi_2\psi_3}{\psi_1^2}, ..., \frac{\psi_2\psi_{d-1}}{\psi_1^2}\right).$$

An appropriate linear map A sending G back to F is just a cyclic permutation of coordinates:

$$A : (z_1, z_2, ..., z_{d-1}) \to (z_2, z_3, ..., z_1).$$

Thus we get the composition $\Theta = A \circ \chi : F \to F$ (whose d-th power is the Poincare return map) given by:

$$\Theta : (\psi_1, \psi_2, ..., \psi_{d-1}) \to \left(\frac{\psi_2}{\psi_1}, \frac{\psi_2 \psi_3}{\psi_1^2}, ..., \frac{\psi_2 \psi_{d-1}}{\psi_1^2}, \psi_2 \right).$$

Proposition 2. *The above map Θ has a curve of fixed points parameterized by $(t, t^2, t^2, ..., t^2)$, $t \in \mathbb{R}$. Moreover, for any $d \geq 3$ one has that $\Theta^{d-1} = id$.*

Proof. Indeed, the system of equations defining fixed points reads as

$$\psi_1 = \frac{\psi_2}{\psi_1}, \quad \psi_2 = \frac{\psi_2 \psi_3}{\psi_1^2}, \quad \psi_3 = \frac{\psi_2 \psi_4}{\psi_1^2}, \quad \cdots \psi_{d-2} = \frac{\psi_2 \psi_{d-1}}{\psi_1^2}, \quad \psi_{d-1} = \psi_2.$$

which immediately implies $\psi_1^2 = \psi_2 = \psi_3 = ... = \psi_{d-1}$. To show that $\Theta^{d-1} = id$ notice that since Θ is a monomial map it suffices to show that $M_d^{d-1} = id_{d-1}$ where M_d is the matrix of exponents of the map Θ and id_{d-1} is the identity matrix of size $d-1$. (Indeed, the matrix of exponents for Θ^i coincides with M_d^i.) This is done in the following lemma.

Lemma 2. *The characteristic polynomial of the $(d-1) \times (d-1)$-matrix M_d equals $(-1)^d (1 - t^{d-1})$. Therefore, by the Hamilton-Cayley theorem $M_d^{d-1} = id_{d-1}$.*

Proof. Looking at the exponents of Θ we see that the matrix M_d has the form

$$M_d = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ -2 & 1 & 1 & 0 & \dots & 0 \\ -2 & 1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2 & 1 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

To make our calculations easy we introduce two families of $(k \times k)$-matrices D_k and E_k given by:

$$D_k = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -t & 1 & 0 & \dots & 0 \\ 1 & 0 & -t & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \cdots & -t & 1 \\ 1 & 0 & 0 & \cdots & \cdots & -t \end{pmatrix}, \quad E_k = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 2 & -t & 1 & 0 & \dots & 0 \\ 2 & 0 & -t & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 0 & 0 & \cdots & -t & 1 \\ 0 & 0 & 0 & \cdots & 0 & -t \end{pmatrix}.$$

Expanding by the first row one obtains the following recurrences

$$Det(D_k) = (-t)^{k-1} - Det(D_{k-1}) \quad Det(E_k) = 2(-t)^{k-1} - Det(E_{k-1})$$

resulting in the formulas

$$Det(D_k) = (-1)^{k-1}(t^{k-1} + t^{k-2} + ... + 1),$$
$$Det(E_k) = (-1)^{k-1}2(t^{k-1} + t^{k-2} + ... + t).$$

Expanding now the characteristic polynomial $Ch_d(t)$ of M_d by the first row (after the sign change in the first row) we get the relation

$$-Ch_d(t) = (t + 1)[(1 - t)(-t)^{d-3} - Det(D_{d-3})] - Det(E_{d-2}).$$

Substituting of the expressions for $Det(D_{d-3})$ and $Det(E_{d-2})$ in the latter formula one gets $Ch_d(t) = (-1)^d(1 - t^{d-1})$.

This completes the proof.

Corollary 1. *The Poincare return map equals* $\Theta^d = \Theta$.

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Geometric and numerical techniques to compute conjugate and cut loci on Riemannian surfaces

Bernard Bonnard, Olivier Cots, and Lionel Jassionnesse

Abstract We combine geometric and numerical techniques – the Hampath code – to compute conjugate and cut loci on Riemannian surfaces using three test bed examples: ellipsoids of revolution, general ellipsoids, and metrics with singularities on $\mathbf{S}^2$ associated to spin dynamics.

1 Introduction

On a Riemannian manifold (M, g), the cut point along the geodesic γ emanating from q_0 is the first point where γ ceases to be minimizing, while the first conjugate point is where it ceases to be minimizing among the geodesics C^1-close to γ. Considering all the geodesics starting from q_0 they will form respectively the cut locus $C_{\text{cut}}(q_0)$ and the conjugate locus $C(q_0)$. The computations of the conjugate and cut loci on a Riemannian surface is an important problem in global geometry [1] and it can be extended to optimal control with many important applications [4]. Also convexity property of the injectivity domain of the exponential map is related to the continuity property of the Monge transport map T on the surfaces [6]. The structure of the conjugate and cut loci on surfaces diffeomorphic to $\mathbf{S}^2$ was investigated in details by Poincaré and Myers [9, 10]. In the analytic case, the cut locus is a finite tree and the extremity of each branch is a cusp point. But the explicit computation

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of the number of branches and cusps points is a very complicated problem and only very recently was proved the four cusp Jacobi conjecture on ellipsoids [7,12].

The aim of this article is to present techniques which lead to the explicit computation of the cut and conjugate loci based on three examples, combining geometric techniques and numerical simulations using the Hampath code [5]. Geometry is used in a first step to choose appropriate coordinates to analyze the metric (for instance the computation of curvature and principal lines of curvature) and the geodesic flow. Also the explicit computations will be related to the micro-local complexity of this flow. This is clear in the example of an ellipsoid of revolution: geodesics can be meridians, the equator and a family of geodesics such that representing the metric in the normal form $g = \mathrm{d}\varphi^2 + m(\varphi)\mathrm{d}\theta^2$, θ increases or decreases monotonously while φ oscillates between φ^- and φ^+. The important task is to evaluate the first conjugate point t_{1c} which corresponds to the existence of a solution of the Jacobi equation $\ddot{J}(t) + G(\gamma(t))J(t) = 0$ such that $J(0) = J(t_{1c}) = 0$, G being the Gauss curvature. Since in our case the usual Sturm theorem [8] is not very helpful to estimate conjugate points, our approach is to compute them in relation with the period mapping T of the φ-variable.

In the case of an ellipsoid of revolution it can be shown that conjugate and cut loci can be computed with only the first and second order derivative of the period mapping [3].

The Hampath code is useful to analyze the geodesics and to evaluate conjugate points and the conjugate locus, using Jacobi fields and continuation method. In particular the analysis of the case of revolution can be easily extended to a general ellipsoid.

The time optimal transfer of three linearly coupled spins with Ising coupling described in [13] leads to study a one parameter Riemannian metric on $\mathbf{S}^2$ with equatorial singularity which is a deformation of the Grushin case $g = \mathrm{d}\varphi^2 + \tan^2\varphi \mathrm{d}\theta^2$. Again the analysis of the flow and conjugate points computation lead to describe the conjugate and cut loci for various values of the parameter.

2 Riemannian metrics on surfaces of revolution

We briefly recall the general tools to handle the analysis of surfaces of revolution with applications to the ellipsoids [3,11].

2.1 Generalities

Taking a chart (U, q) the metric can be written in polar coordinates as

$$g = \mathrm{d}\varphi^2 + m(\varphi)\mathrm{d}\theta^2.$$

We use Hamiltonian formalism on T^*U, $\frac{\partial}{\partial p}$ is the vertical space, $\frac{\partial}{\partial q}$ is the horizontal space and $\alpha = p\mathrm{d}q$ is the (horizontal) Liouville form. The associated Hamiltonian is

$$H = \frac{1}{2}\left(p_\varphi^2 + \frac{p_\theta^2}{m(\varphi)}\right)$$

and we denote $\exp t\vec{H}$ the one-parameter group. Parameterizing by arc length amounts to fix the level set to $H = 1/2$. Extremals solution of $\vec{H}$ are denoted $\gamma : t \to (q(t, q_0, p_0), p(t, q_0, p_0))$ and fixing q_0 it defines the exponential mapping $\exp_{q_0} : (t, p_0) \to q(t, q_0, p_0) = \Pi(\exp t\vec{H}(q_0, p_0))$ where $\Pi : (q, p) \to q$ is the standard projection. Extremals are solutions of the equations

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = p_\varphi, \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{p_\theta}{m(\varphi)}, \quad \frac{\mathrm{d}p_\varphi}{\mathrm{d}t} = \frac{1}{2}p_\theta^2\frac{m'(\varphi)}{m^2(\varphi)}, \quad \frac{\mathrm{d}p_\theta}{\mathrm{d}t} = 0.$$

Definition 1. The relation $p_\theta = Constant$ is called **Clairaut** relation on surfaces of revolution. We have two types of specific solutions: **meridians** for which $p_\theta = 0$ and $\theta(t) = \theta_0$ and **parallels** for which $\frac{d\varphi}{dt}(0) = p_\varphi(0) = 0$ and $\varphi(t) = \varphi(0)$.

To analyze the extremal behaviours, we fix $H = 1/2$ and we consider the mechanical system

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + V(\varphi, p_\theta) = 1$$

where $V(\varphi, p_\theta) = p_\theta^2/m(\varphi)$ is the **potential** mapping depending upon the parameter p_θ and parallels correspond to local extrema.

Assumptions 1. *In the sequel we shall assume the following:*

(A1) $\varphi = 0$ *is a parallel solution with a local minimum of the potential and the corresponding parallel is called the* **equator***;*
(A2) *the metric is* **reflectionally symmetric** *with respect to the equator:* $m(-\varphi) = m(\varphi)$.

Micro-local behaviors of the extremals

We describe a set of solutions confined to the segment $[-\varphi^{\max}, +\varphi^{\max}]$ where $\varphi^{\max}$ is the local maximum of V closest to 0. Let I be the open interval $p_\theta \in (\sqrt{m(\varphi^{\max})}, \sqrt{m(\varphi(0))})$. Taking such an extremal, φ oscillates periodically between φ^- and φ^+. The dynamics is described by:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \pm\frac{1}{g}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{p_\theta}{m(\varphi)},$$

where

$$g(\varphi, p_\theta) = \sqrt{\frac{m(\varphi)}{m(\varphi) - p_\theta^2}}$$

and for an increasing branch one can parameterize θ by φ and we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{g(\varphi, p_\theta)p_\theta}{m(\varphi)} = f(\varphi, p_\theta),$$

where

$$f(\varphi, p_\theta) = \frac{p_\theta}{\sqrt{m(\varphi)}\sqrt{m(\varphi) - p_\theta^2}}.$$

The trajectory $t \mapsto \varphi(t, p_\theta)$ is periodic and one can assume $\varphi(0) = 0$. The period of oscillation T is given by

$$T = 4\int_0^{\varphi^+} g(\varphi, p_\theta)\mathrm{d}\varphi$$

and the first return to the equator is at time $T/2$ and the variation of θ at this time is given by

$$\Delta\theta = 2\int_0^{\varphi^+} f(\varphi, p_\theta)\mathrm{d}\varphi.$$

Definition 2. The mapping $p_\theta \in I \to T(p_\theta)$ is called the **period mapping** and $R : p_\theta \to \Delta\theta$ is called the **first return mapping**.

Definition 3. The extremal flow is called **tame** on I if the first return mapping R is such that $R' < 0$.

Proposition 1. *For extremal curves with $p_\theta \in I$, in the tame case there exists no conjugate times on* $(0, T/2)$.

Proof. If $R' < 0$, the extremal curves initiating from the equator with $p_\theta \in I$ are not intersecting before returning to the equator. As conjugate points are limits of intersecting extremals curves, conjugate points are not allowed before returning to the equator.

Assumptions 2. *In the tame case we assume the following*

(A3) *at the equator the Gauss curvature $G = -\frac{1}{\sqrt{m(\varphi)}}\frac{\partial^2\sqrt{m(\varphi)}}{\partial\varphi^2}$ is positive and maximum.*

Using Jacobi equation we deduce:

Lemma 1. *Under assumption (A3), the first conjugate point along the equator is at time $\pi/\sqrt{G(0)}$ and realizes the minimum distance to the cut locus $C_{\mathrm{cut}}(\theta(0) = 0, \varphi(0) = 0)$. It is a cusp point of the conjugate locus.*

Parameterization of the conjugate locus under assumptions (A1-2-3) for $p_\theta \in I$

Fixing a reference extremal γ, Jacobi equation is the variational equation:

$$\delta\dot{z}(t) = \frac{\partial\vec{H}(\gamma(t))}{\partial z}\delta z(t), \quad \delta z = (\delta q, \delta p)$$

and a Jacobi field $J(t)$ is a non trivial solution of Jacobi equation. According to standard theory on surfaces, if γ is parametrized by arc length, let $J_1(t) = (\delta q(t), \delta p(t))$ denotes the Jacobi field vertical at time $t = 0$, that is $\delta q(0) = 0$ and such that $\langle p(0), \delta p(0)\rangle = 0$. Since $J_1(0)$ is vertical, $\alpha(J_1(0)) = 0$ and then $\alpha(J_1(t)) = 0$.

We have [8, 11]:

Proposition 2. *Conjugate points are given by the relation* $\mathrm{d}\Pi(J_1(t)) = 0$ *and*

$$\mathrm{d}\Pi(J_1(t)) = \left(\frac{\partial\varphi(t, p_\theta)}{\partial p_\theta}, \frac{\partial\theta(t, p_\theta)}{\partial p_\theta}\right).$$

In particular we have at any time the collinearity condition:

$$p_\varphi\frac{\partial\varphi}{\partial p_\theta} + p_\theta\frac{\partial\theta}{\partial p_\theta} = 0.$$

The conjugate locus will be computed **by continuation**, starting from the cusp point at the equator. Let $p_\theta \in I$ and $t \in (T/2, T/2 + T/4)$. One has the formula

$$\theta(t, p_\theta) = \Delta\theta(p_\theta) + \int_{T/2}^{t} \frac{p_\theta}{m(\varphi)}\mathrm{d}t$$

and on $[T/2, t]$, $\frac{\mathrm{d}\varphi}{\mathrm{d}t} < 0$, $\varphi < 0$. Hence

$$\int_{T/2}^{t} \frac{p_\theta}{m(\varphi)}\mathrm{d}t = \int_{\varphi(t,p_\theta)}^{0} f(\varphi, p_\theta)\mathrm{d}\varphi.$$

We have:

Lemma 2. *For* $p_\theta \in I$ *and conjugate times between* $(T/2, T/2 + T/4)$ *the conjugate locus is solution of*

$$\frac{\partial\theta(\varphi, p_\theta)}{\partial p_\theta} = 0, \tag{1}$$

where $\theta(\varphi, p_\theta) = \Delta\theta(p_\theta) + \int_\varphi^0 f(\varphi, p_\theta)\mathrm{d}\varphi$.

This gives a simple relation to compute the conjugate locus by continuation. One notes $p_\theta \to \varphi_{1c}(p_\theta)$ the solution of Eq. (1) initiating from the equator. Differentiating one has

$$\Delta\theta' + \int_\varphi^0 \frac{\partial f}{\partial p_\theta}\mathrm{d}\varphi = 0$$

at $\varphi_{1c}(p_\theta)$. Differentiating again one obtains

$$\Delta\theta'' + \int_{\varphi_{1c}}^0 \frac{\partial^2 f}{\partial p_\theta^2}\mathrm{d}\varphi - \frac{\partial\varphi_{1c}}{\partial p_\theta}\cdot\frac{\partial f}{\partial p_\theta} = 0.$$

One can easily check that $\frac{\partial f}{\partial p_\theta} > 0$ and $\frac{\partial^2 f}{\partial p_\theta^2} > 0$. In particular

$$\frac{\partial\varphi_{1c}}{\partial p_\theta} = \left(\Delta\theta'' + \int_{\varphi_{1c}}^0 \frac{\partial^2 f}{\partial p_\theta^2}\mathrm{d}\varphi\right)\left(\frac{\partial f}{\partial p_\theta}\right)^{-1}$$

and one deduces the following.

Proposition 3. *If $\Delta\theta'' > 0$ on I, then $\frac{\partial\varphi_{1c}}{\partial p_\theta} \neq 0$ and the curve $p_\theta \to (\varphi_{1c}(p_\theta), \theta_{1c}(p_\theta))$ is a curve defined for $p_\theta \in I$ and with no self-intersection in the plane (φ, θ). In particular it is without cusp point.*

Remark 1. Self-intersections are depending upon the parameterization of the conjugate locus but not cusp points of the conjugate locus.

To simplify the computations we use the following lemma:

Lemma 3. *We have the relation*

$$R'(p_\theta) = \frac{T'(p_\theta)}{2p_\theta}.$$

2.2 Ellipsoids of revolution

The ellipsoid of revolution is generated by the curve

$$y = \sin\varphi, \quad z = \varepsilon\cos\varphi$$

where $0 < \varepsilon < 1$ corresponds to the oblate (flattened) case while $\varepsilon > 1$ is the prolate (elongated) case. The restriction of the Euclidian metric is

$$g = F_1(\varphi)\mathrm{d}\varphi^2 + F_2(\varphi)\mathrm{d}\theta^2$$

where $F_1 = \cos^2\varphi + \varepsilon^2\sin^2\varphi$, $F_2 = \sin^2\varphi$. The metric can be written in the normal form setting:

$$\mathrm{d}\Phi = F_1^{1/2}(\varphi)\mathrm{d}\varphi.$$

Observe that φ oscillates periodically and θ is monotonous. Hence the period mapping can be computed in the (ψ, θ)-coordinate, $\psi = \pi/2 - \varphi$ and $\psi = 0$ is the equator. The Hamiltonian is

$$H = \frac{1}{2}\left(\frac{p_\varphi^2}{F_1(\varphi)} + \frac{p_\theta^2}{F_2(\varphi)}\right)$$

and with $H = 1/2$, one gets

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{(\cos^2\psi - p_\theta^2)^{1/2}}{\cos\psi(\sin^2\psi + \varepsilon^2\cos^2\psi)^{1/2}}.$$

Denoting $1 - p_\theta^2 = \sin^2\psi_1$ and making the rescaling $Y = \sin\psi_1 Z$, where $Y = \sin\psi$, one gets

$$\frac{(\varepsilon^2 + Z^2\sin^2\psi_1^2(1-\varepsilon^2))^{1/2}}{(1-Z^2)^{1/2}}\mathrm{d}Z = \mathrm{d}t.$$

Hence the formula for the period mapping is

$$\frac{T}{4} = \int_0^1 \frac{(\varepsilon^2 + Z^2\sin^2\psi_1(1-\varepsilon^2))^{1/2}}{(1-Z^2)^{1/2}}\mathrm{d}Z$$

which corresponds to an elliptic integral. The discussion is the following.

Oblate Case

In this case the Gauss curvature is increasing from the north pole to the equator and the problem is tame and the period mapping is such that

$$T'(p_\theta) < 0 < T''(p_\theta)$$

for each admissible $p_\theta > 0$. The cut point of $q(0) = (\varphi(0), 0)$ is given by $t_0(p_\theta) = T(p_\theta)/2$ and corresponds to the intersection of the two extremal curves associated to $\dot\varphi(0)$, and $-\dot\varphi(0)$. The cut locus $C_{\text{cut}}(q(0))$ of a point different of a pole is a segment of the antipodal parallel. If $q(0)$ is not a pole nor on the equator, the distance to the cut locus is the half-period of the extremal starting from $\varphi(0)$ with $\dot\varphi(0) = 0$ and the injectivity radius is realized for $\varphi(0) = \pi/2$ on the equator, and is given by $\pi/\sqrt{G(\pi/2)}$ where G is the Gauss curvature. The conjugate locus $C(q(0))$ of a point different of a pole has exactly four cusps, two on the antipodal parallel which are the extremities of the cut locus segment and two on the antipodal meridian.

Prolate Case

In this case, the Gauss curvature is decreasing from the north pole to the equator and the first return mapping to the equator is an increasing function of $p_\theta \geq 0$. Let a geodesic being not a meridian circle, the cut point $t_0(p_\theta)$ is given by solving $\theta(t_0, p_\theta) = \pi$ and corresponds to the intersection of the two extremal curves associated respectively to p_θ and $-p_\theta$. The cut locus of a point which is not a pole is a segment of the antipodal meridian. The conjugate locus $C(q(0))$ of such a point has exactly four cusps, two on the antipodal meridian which are the extremities of the cut locus and two on the antipodal parallel.

Conclusion

To resume both cases are distinguished by the monotonicity property of the Gauss curvature or equivalently of the first return mapping. The cut loci are computed using the symmetric property of the extremal curves: in the oblate case, the symmetry of the metric with respect to the equator and in the prolate case the symmetry of the metric with respect to the meridian. Additionaly to this discrete symmetry, the symmetry of revolution ensures the existence of an additionnal one-dimensional group of symmetry which gives according to Noether theorem the first integral p_θ linear with respect to the adjoint vector and corresponds to a Clairaut metric [2].

3 General Ellipsoids

We shall extend the result on ellipsoids of revolution to general ellipsoids. Roughly speaking, the general case intertwines the oblate and the prolate case, which will be easily seen in the classification of the extremal flow.

3.1 Geometric Properties [7]

A general ellipsoid E is defined by the equation

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \frac{x_3^2}{a_3} = 1, \quad a_1 > a_2 > a_3 > 0$$

and we use the double covering parameterization of $E(\theta_1, \theta_2) \in T^2 = S^1 \times S^1 \to E$:

$$\begin{aligned} x_1 &= \sqrt{a_1} \cos\theta_1 \sqrt{(1-\beta)\cos^2\theta_2 + \sin^2\theta_2} \\ x_2 &= \sqrt{a_2} \sin\theta_1 \sin\theta_2 \\ x_3 &= \sqrt{a_3} \cos\theta_2 \sqrt{\beta \cos^2\theta_1 + \sin^2\theta_1} \end{aligned}$$

where $\beta = (a_2 - a_3)/(a_1 - a_3) \in (0, 1)$ and the (θ_1, θ_2)-coordinates are related to the elliptic coordinates (λ_1, λ_2) by

$$\lambda_1 = a_1 \sin^2\theta_1 + a_2 \cos^2\theta_1, \quad \lambda_2 = a_2 \cos^2\theta_2 + a_3 \sin^2\theta_2.$$

In the (θ_1, θ_2)-coordinates the restriction of the euclidian metric on $\mathbf{R}^3$ takes the form

$$g = (\lambda_1 - \lambda_2)\left(\frac{\lambda_1}{\lambda_1 - a_3}\mathrm{d}\theta_1{}^2 + \frac{\lambda_2}{a_1 - \lambda_2}\mathrm{d}\theta_2{}^2\right).$$

The metric has two main discrete symmetries defined for $i = 1, 2$ by the change of variables: $\theta_i \to \pi - \theta_i$ and $\theta_i \to -\theta_i$. The associated Hamiltonian is

$$2H = \frac{1}{\lambda_1 - \lambda_2}\left(\frac{\lambda_1 - a_3}{\lambda_1} p_{\theta_1}^2 + \frac{a_1 - \lambda_2}{\lambda_2} p_{\theta_2}^2\right)$$

and an additional first integral quadratic in $(p_{\theta_1}, p_{\theta_2})$ is given by

$$F = \frac{1}{\lambda_1 - \lambda_2}\left(\frac{\lambda_1 - a_3}{\lambda_1}(a_2 - \lambda_2) p_{\theta_1}^2 - \frac{a_1 - \lambda_2}{\lambda_2}(\lambda_1 - a_2) p_{\theta_2}^2\right).$$

According to Liouville theory [2], the metric can be written in the normal form

$$g = (F_1(u_1) + F_2(u_2))\left(\mathrm{d}u_1^2 + \mathrm{d}u_2^2\right),$$

where u_1, u_2 are defined by the quadratures

$$\mathrm{d}u_1 = \sqrt{\frac{\lambda_1}{\lambda_1 - a_3}}\mathrm{d}\theta_1, \quad \mathrm{d}u_2 = \sqrt{\frac{\lambda_2}{a_1 - \lambda_2}}\mathrm{d}\theta_2$$

and see [8] for the relation with elliptic coordinates. The third fondamental form is given for $x_3 \neq 0$ by

$$III(\mathrm{d}x_1, \mathrm{d}x_2) = \frac{x_1x_2x_3}{a_1a_2a_3}\left(\frac{1}{a_3}-\frac{1}{a_1}\right)\mathrm{d}x_1^2 - \frac{x_1x_2x_3}{a_1a_2a_3}\left(\frac{1}{a_2}-\frac{1}{a_3}\right)\mathrm{d}x_2^2 + \frac{x_3}{a_3}\left(\left(\frac{1}{a_1}-\frac{1}{a_2}\right)\left(\frac{x_3}{a_3}\right)^2 - \left(\frac{1}{a_2}-\frac{1}{a_3}\right)\left(\frac{x_1}{a_1}\right)^2 - \left(\frac{1}{a_3}-\frac{1}{a_2}\right)\left(\frac{x_2}{a_2}\right)^2\right)\mathrm{d}x_1\mathrm{d}x_2$$

and we get the four umbilical points $\left(\pm\sqrt{a_1}\sqrt{1-\beta}, 0, \pm\sqrt{a_3\beta}\right)$. Besides in elliptic coordinates the lines $\lambda_i = Constant$ are the curvature lines. Finally, taking the Liouville normal form, the Gauss curvature is given by

$$G(u_1,u_2) = \frac{F_1'(u_1)^2 + F_2'(u_2)^2}{2\left(F_1(u_1)+F_2(u_2)\right)^3} - \frac{F''(u_1)+F''(u_2)}{2\left(F_1(u_1)+F_2(u_2)\right)^2}$$

or similarly, in the elliptic coordinates, one has

$$G(\lambda_1,\lambda_2) = \frac{a_1a_2a_3}{\lambda_1^2\lambda_2^2}, \quad (\lambda_1,\lambda_2) \in [a_2,a_1]\times[a_3,a_2].$$

We represent in Fig. 1 the Gauss curvature in the (θ_1,θ_2)-coordinates restricted to $(\theta_1,\theta_2) \in [0,\pi/2]\times[0,\pi/2]$ by symmetry.

Remark 2. We have the following correspondences with the ellipsoid of revolution: in the limit case $a_1 = a_2 > a_3$ (oblate case), the metric g reduces to the form presented in 2.2 where $(\varphi,\theta) \equiv (\theta_2,\theta_1)$ and we have $F \geq 0$. In the limit case $a_1 > a_2 = a_3$ (prolate case), the metric g reduces to the form presented in 2.2 where $(\varphi,\theta) \equiv (\theta_1,\theta_2)$ and we have $F \leq 0$.

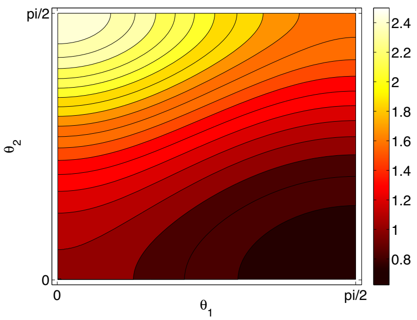

Fig. 1 Gauss curvature in (θ_1,θ_2)-coordinates for $(\theta_1,\theta_2) \in [0,\pi/2]\times[0,\pi/2]$. The maximum $a_1/(a_2a_3)$ of G is at $(0,\pi/2)$ which is the intersection of the longest and the middle ellipses while the minimum $a_3/(a_2a_1)$ is at $(\pi/2, 0)$, the intersection of the shortest and the middle ellipses

3.2 Geodesic Flow [7]

Parameterizing by arc length $H = 1/2$ and setting $F = c$, the extremal equations are described by

$$\frac{\varepsilon_1\sqrt{\lambda_1}\mathrm{d}\theta_1}{\sqrt{\lambda_1 - a_3}\sqrt{\lambda_1 - a_2 + c}} = \frac{\varepsilon_2\sqrt{\lambda_2}\mathrm{d}\theta_2}{\sqrt{a_1 - \lambda_2}\sqrt{a_2 - c - \lambda_2}}$$

and

$$\mathrm{d}t = \frac{\varepsilon_1\sqrt{\lambda_1}\sqrt{\lambda_1 - a_2 + c}}{\sqrt{\lambda_1 - a_3}}\mathrm{d}\theta_1 + \frac{\varepsilon_2\sqrt{\lambda_2}\sqrt{a_2 - c - \lambda_2}}{\sqrt{a_1 - \lambda_2}}\mathrm{d}\theta_2$$

where $\varepsilon_i = \pm 1$ is the sign of $\mathrm{d}\theta_i/\mathrm{d}t$, $i = 1, 2$. The value c of F varies between $-(a_1 - a_2)$ and $(a_2 - a_3)$ and the behavior of the extremals depends on the sign of c.

- If $0 < c < a_2 - a_3$, then $\theta_1(t)$ increases or decreases monotonously and $\theta_2(t)$ oscillates between $\nu_2(c)$ and $\pi - \nu_2(c)$, where $\nu_2(c)$ is defined by

  $$\sin \nu_2(c) = \sqrt{\frac{c}{a_2 - a_3}}, \quad 0 < \nu_2(c) < \frac{\pi}{2}.$$

  These trajectories do not cross transversely the segments $\theta_2 = 0$ and $\theta_2 = \pi$ which degenerate into two poles in the oblate case. Here the longest ellipse $\theta_2 = \pi/2$ plays the role of the equator from the oblate case.

- If $-(a_1 - a_2) < c < 0$, then $\theta_2(t)$ increases or decreases monotonously and $\theta_1(t)$ oscillates periodically between $\nu_1(c)$ and $\pi - \nu_1(c)$ where $\nu_1(c)$ is defined by

  $$\sin \nu_1(c) = \sqrt{\frac{-c}{a_1 - a_2}}, \quad 0 < \nu_1(c) < \frac{\pi}{2}.$$

  These trajectories do not cross transversely the segments $\theta_1 = 0$ and $\theta_1 = \pi$ which degenerate into two poles in the prolate case. Here the longest ellipse plays the role of the meridian circle from the prolate case.

- The separating case $c = 0$ is the level set containing the umbilical points.

Arc length geodesic curves $\gamma(t) = (\theta_1(t), \theta_2(t))$ can be parameterized by c but we introduce the parameter η defined by:

$$\nu(\eta) = \cos \eta e_1 + \sin \eta e_2$$

where (e_1, e_2) is a orthonormal basis

$$e_1 = \left(\frac{(\lambda_1 - \lambda_2)\lambda_1}{\lambda_1 - a_3}\right)^{-1/2} \frac{\partial}{\partial \theta_1}, \quad e_2 = \left(\frac{(\lambda_1 - \lambda_2)\lambda_2}{a_1 - \lambda_2}\right)^{-1/2} \frac{\partial}{\partial \theta_2}.$$

3.3 Results on the Conjugate and Cut Loci

According to [7] we have the following proposition which generalizes the case of an ellipsoid of revolution.

Proposition 4. *The cut locus of a non-umbilical point is a subarc of the curvature lines through its antipodal point and the conjugate locus has exactly four cusps.*

The Analysis

Fixing the initial point to $(\theta_1(0), \theta_2(0))$, the relation between η and c is given by:

$$c(\eta) = \big(a_2 - \lambda_2(\theta_2(0))\big)\cos^2\eta - \big(\lambda_1(\theta_1(0)) - a_2\big)\sin^2\eta$$

and let η_0 be the unique η such that $c(\eta_0) = 0$, $0 \le \eta_0 \le \frac{\pi}{2}$.

- The case $c > 0$ (cf. Fig. 2). We use the parameterization (θ_1, θ_2) with $\theta_1 \in T^1$ and $0 \le \theta_2 \le \pi$.
  - For $\eta \in (0, \eta_0) \cup (\pi - \eta_0, \pi)$ the value of θ_2 along the geodesic increases until it reaches a maximum θ_2^+ and then it decreases. The cut time $t_0(\eta)$ is the second positive time such that θ_2 takes the value $\pi - \theta_2(0)$.
  - For $\overline{\eta} = 2\pi - \eta$, the value of θ_2 along the geodesic decreases until it reaches a minimum θ_2^- and then it increases. The cut time $t_0(\overline{\eta})$ is the first positive time such that θ_2 takes the value $\pi - \theta_2(0)$.
  - Besides, we have $t_0(\eta) = t_0(\overline{\eta})$ and $\gamma_\eta(t_0(\eta)) = \gamma_{\overline{\eta}}(t_0(\overline{\eta}))$.

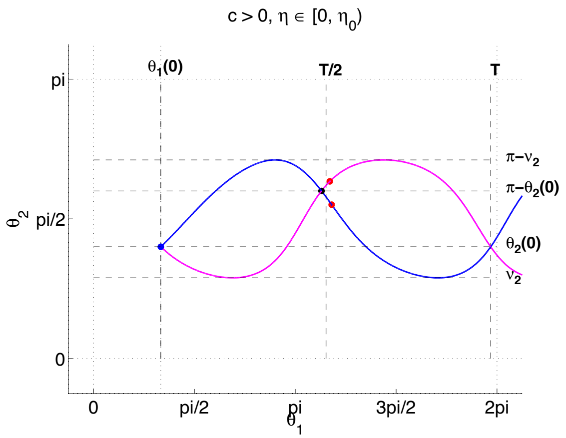

Fig. 2 Trajectories, cut and conjugate points in the case $c > 0$. The trajectory with $\dot{\theta}_2(0) > 0$ corresponds to $\eta \in (0, \eta_0)$ while the other corresponds to $\overline{\eta} = 2\pi - \eta$. The two conjugate points are plotted in red and come after the cut point in black. The period T of the θ_2-variable is equal for each trajectory and is represented with the half-period

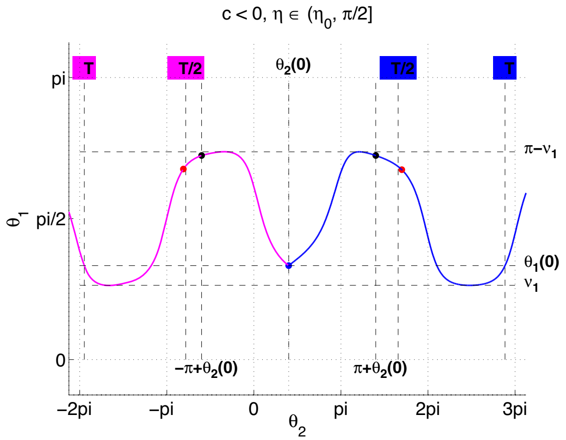

Fig. 3 Trajectories, cut and conjugate points in the case $c < 0$. The trajectory with $\dot{\theta}_2(0) > 0$ corresponds to $\eta \in (0, \eta_0)$ while the other corresponds to $\overline{\eta} = 2\pi - \eta$. The two conjugate points are plotted in red and come after the cut point in black. The periods of the θ_2-variable are not equal for each trajectory and are represented with the half-period

- The case $c \leq 0$ (cf. Fig. 3). We use the parameterization (θ_1, θ_2) with $0 \leq \theta_1 \leq \pi$ and $\theta_2 \in T^1$.
  - For $\eta \in (\eta_0, \pi - \eta_0)$, θ_2 increases monotonously and let $t_0(\eta)$ be the first positive time t such that θ_2 takes the value $\theta_2(0) + \pi$. The cut time is given by $t_0(\eta)$.
  - For $\overline{\eta} = 2\pi - \eta$, θ_2 decreases monotonously and let $t_0(\eta)$ be the first positive time t such that θ_2 takes the value $\theta_2(0) - \pi$. The cut time is given by $t_0(\eta)$.
  - Besides, we have $t_0(\eta) = t_0(\overline{\eta})$ and $\theta_{1,\eta}(t_0(\eta)) = \theta_{1,\overline{\eta}}(t_0(\overline{\eta}))$.

Numerical Computation of Conjugate and Cut Loci

We fix the parameters of the ellipsoid $a_1 > a_2 > a_3 > 0$ such that $a_1 - a_2 \neq a_2 - a_3$ to avoid any additionnal symmetry. We take $(a_1, a_2, a_3) = (1.0, 0.8, 0.5)$ and $(\theta_1(0), \theta_2(0)) = (\pi/3, 2\pi/5)$ for the computations, which correspond to a generic situation. Indeed, if the initial point is on $\theta_2 = 0$ or π, $c(\eta) \leq 0$ then there are only oblate-like extremals. Similarly, if $\theta_1(0) = 0$ or π, there are only prolate-like extremals.

First of all we represent the conjugate and cut loci in Fig. 4 using the double covering parameterization, with several trajectories for $\eta \in [0, 2\pi)$ on the top subplot. The conjugate and cut loci are given on Fig. 5 in (x_1, x_2, x_3)-coordinates. Finally, the cut time t_0, the first conjugate time t_1 and the half-period $T/2$ of the oscillating variable is plotted in Fig. 6.

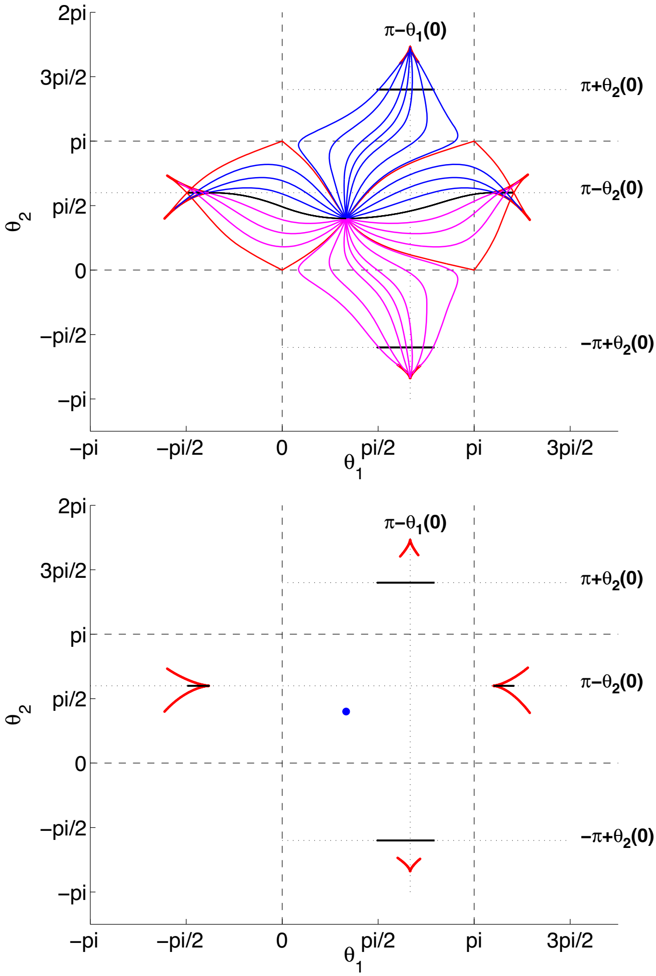

Fig. 4 On the top subplot is represented several trajectories γ_η, $\eta \in [0, 2\pi)$, with the conjugate (in red) and cut (in black) loci, using the double covering parameterization. In blue are plotted the trajectories for $\eta \in (0, \pi)$ and in magenta for $\overline{\eta} = 2\pi - \eta$. The four trajectories in red such that $c(\eta) = 0$ pass through an umbilical point. They separate oblate-like ($c > 0$) behaviour from prolate-like ($c < 0$) one. The two intersections of these trajectories are junction of parts of the cut locus coming from oblate-like extremals and the cut locus coming from prolate-like extremals. One should notice that in red are plotted the four trajectories passing through the umbilical points with the parameterization (θ_1, θ_2), $\theta_1 \in T^1$ and $0 \leq \theta_2 \leq \pi$. (Bottom) Conjugate and cut loci

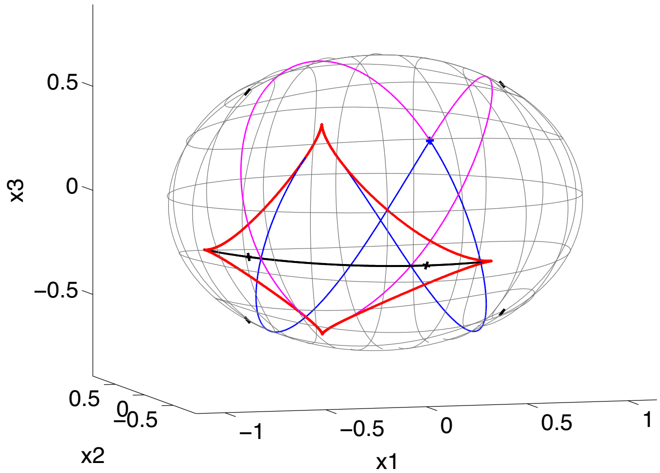

Fig. 5 Conjugate (in red) and cut (in black) loci in (x_1, x_2, x_3)-coordinates obtained from Fig. 4. The center segment of the cut locus corresponds to prolate-like case while the two extreme parts come from oblate-like case

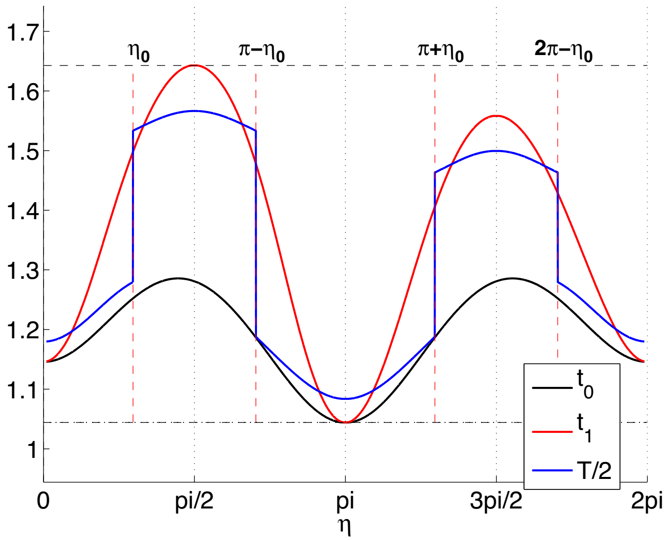

Fig. 6 The cut time t_0, the first conjugate time t_1 and the half-period $T/2$ of the oscillating variable, with respect to the parameter $\eta \in [0, 2\pi]$. In the generic case, the half-period is not equal to the cut time, even for oblate-like trajectories. This is still true for an initial point on $\theta_1 = 0$ or π. The period is discontinuous when $c(\eta) = 0$ since the oscillating variable changes. The only relevant symmetry is on the period mapping. Indeed, for η such that $c(\eta) > 0$, $T(\eta) = T(2\pi - \eta)$

4 Dynamics of spin particles

The problem fully described in [13, 14] arises in the case of a spin chain of three linearly coupled spins with Ising coupling. Using appropriate coordinates the dynamics takes the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 & -\cos\theta(t) & 0 \\ \cos\theta(t) & 0 & -k\sin\theta(t) \\ 0 & k\sin\theta(t) & 0 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

where $k = 1$ corresponds to equal coupling. Setting $u_3 = -\cos\theta$, $u_1 = -k\sin\theta$, the dynamics is:

$$\dot{r}_1 = u_3 r_2, \quad \dot{r}_2 = -u_3 r_1 + u_1 r_3, \quad \dot{r}_3 = -u_1 r_2.$$

The optimal problem is transferring the system from $r_0 = (1, 0, 0)$ to $r(T) = (0, 0, 1)$ and minimizing the functional:

$$\int_0^T \left(I_1 u_1^2 + I_3 u_3^2\right) \mathrm{d}t \longrightarrow \min, \quad k^2 = \frac{I_1}{I_3}.$$

We introduce the metric:

$$g = I_1 u_1^2 + I_3 u_3^2 = I_3\left(\frac{\mathrm{d}r_1^2 + I_1 I_3^{-1}\mathrm{d}r_3^2}{r_2^2}\right)$$

and this defines an almost Riemannian metric on the sphere $\mathbf{S}^2$:

$$g = \frac{\mathrm{d}r_1^2 + k^2\mathrm{d}r_3^2}{r_2^2}, \quad k^2 = \frac{I_1}{I_3}.$$

Lemma 4. *In the spherical coordinates $r_2 = \cos\varphi$, $r_1 = \sin\varphi\cos\theta$, $r_3 = \sin\varphi\sin\theta$ the metric g takes the form:*

$$\begin{aligned} g = {} & (\cos^2\theta + k^2\sin^2\theta)\mathrm{d}\varphi^2 + 2(k^2-1)\tan\varphi\sin\theta\cos\theta\mathrm{d}\varphi\mathrm{d}\theta \\ & + \tan^2\varphi(\sin^2\theta + k^2\cos^2\theta)\mathrm{d}\theta^2, \end{aligned}$$

while the associated Hamiltonian function is given by

$$\begin{aligned} H = {} & \frac{1}{4k^2}\left(p_\varphi^2(\sin^2\theta + k^2\cos^2\theta) + p_\theta^2\cot^2\varphi(\cos^2\theta + k^2\sin^2\theta)\right. \\ & \left. - 2\,(k^2-1)p_\varphi p_\theta\cot\varphi\sin\theta\cos\theta\right). \end{aligned}$$

We deduce the following

Lemma 5. *For $k = 1$*

$$H = \frac{1}{4}\left(p_\varphi^2 + p_\theta^2\cot^2\varphi\right),$$

and it corresponds to the so-called Grushin case $g = \mathrm{d}\varphi^2 + \tan^2\varphi\mathrm{d}\theta^2$.

The Grushin case is analyzed in details in [3]. Moreover, we have

Lemma 6. *The family of metrics g depending upon the parameter k have a fixed singularity on the equator* $\varphi = \pi/2$ *and a discrete symmetry group defined by the two reflexions:* $H(\varphi, p_\varphi) = H(\pi - \varphi, -p_\varphi)$ *and* $H(\theta, p_\theta) = H(-\theta, -p_\theta)$.

Numerical Computation of Conjugate and Cut Loci

Next the conjugate and cut loci are computed for the fixed initial conditions: $\varphi(0) = \pi/2$, $\theta(0) = 0$, and are represented via the deformation of the parameter k starting from $k = 1$. There are two different cases to be analyzed: $k > 1$ and $k < 1$. Starting from the axis of symmetry, the Hamiltonian reduces to $H(\theta(0), \varphi(0), p_\theta(0), p_\varphi(0)) = p_\varphi^2(0)/4$, and restricting the extremals to $H = 1$, we can parameterize the geodesics by $p_\varphi(0) = \pm 2$, $p_\theta(0) \in \mathbf{R}$. By symmetry we can fix $p_\varphi(0) = -2$ and consider $p_\theta(0) \geq 0$. For any k, the conjugate locus has a contact of order two at the initial point, as $p_\theta(0) \to \infty$.

- $\mathbf{k \geq 1}$. We study the deformation of the conjugate locus for $k \geq 1$ in Figs. 7–9.

The key point is: when $k > 1$, θ is not monotonous for all the trajectories. This is true even for small k, like $k = 1.01$, taking $p_\theta(0) = 0.1$ and $t_f > 14$.

We denote $t_1(p_\theta, k)$ the first conjugate time and $q_1(p_\theta, k) = (\theta, \varphi)_{|t=t_1(p_\theta,k)}$ the associated conjugate point. In Fig. 7, we represent the map $k \in [1, 1.5] \mapsto q_1(k)$ for p_θ fixed to 10^{-4}. The value 1.5 is heuristically chosen to simplify the analysis. We can notice that $\theta(t_1(k))$ only takes approximately the values 0 and π and so it is on the same meridian as the initial point. It switches three times at $1 < k_1 < k_2 < k_3 < 1.5$, with $k_2 - k_1 \neq k_3 - k_2$. We then restrict the study of the conjugate locus to $k \leq k_3$ to simplify.

We can see in Fig. 8, three subplots which represent the deformation of one branch ($p_\varphi(0) = -2$ and $p_\theta(0) \geq 0$) of the conjugate locus resp. for k in $[1, k_1]$, $[k_1, k_2]$ and $[k_2, k_3]$. For any $k \in [1, k_3]$, the branch is located in the half-plane $\theta \geq 0$. If

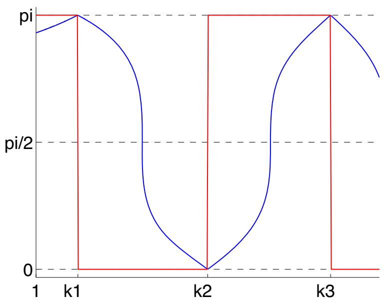

Fig. 7 The first conjugate point with respect to k, for $p_\theta(0)$ fixed to 10^{-4}. In red is plotted $\theta(t_1(p_\theta, k))$ while we have in blue $\varphi(t_1(p_\theta, k))$. The θ-variable takes the values 0 and π. The values k_1, k_2, k_3 are approximately and respectively 1.061, 1.250, 1.429

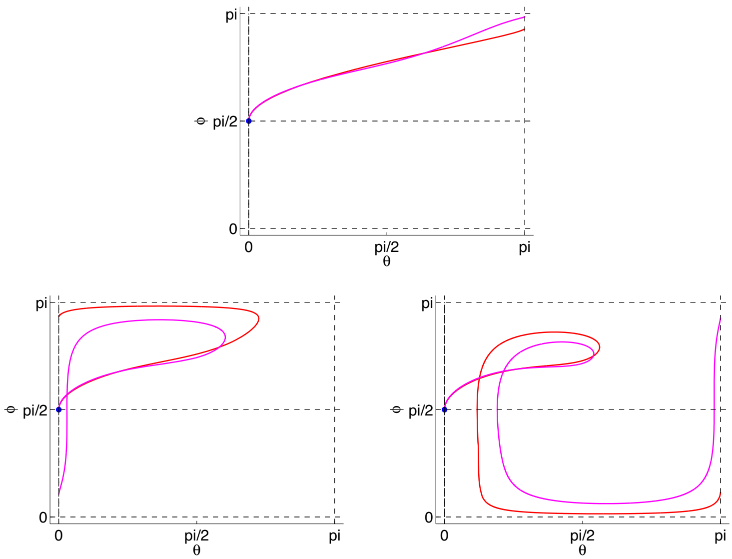

Fig. 8 The deformation of one branch ($p_\varphi(0) = -2$ and $p_\theta(0) \geq 0$) of the conjugate locus with respect to the parameter $k \in [1, k_3]$. (top) $k = 1.0, 1.05$. (left) $k = 1.1, 1.2$. (right) $k = 1.3, 1.4$

we denote $k_1 < \overline{k} < k_2$, the parameter value such that $\varphi(t_1(\overline{k})) = \pi/2$, then the branch form a loop for $\overline{k} \leq k \leq k_3$.

The deformation of the conjugate locus can be explained analysing the behaviors of the trajectories. We describe four types of trajectories in (θ, φ)-coordinates (see Fig. 9), limiting the study to $k \leq k_3$ to simplify and $p_\theta(0) \geq 0$ by symmetry. These trajectories clarify the evolution of the conjugate locus.

– The first type occuring for any k such that $1 \leq k \leq k_3$, is represented in the top left subplot of Fig. 9. Its characteristic is that the θ-variable is monotonous non-decreasing on $[0, t_1]$.

  The three others trajectories do not have a monotonous θ-variable on $[0, t_1]$. We denote $\overline{t}$ the first time when the trajectory leaves the domain $0 \leq \theta \leq \pi$.

– The second type (top right) existing for $k_1 \leq k \leq k_3$ has no self-intersection on $[0, \overline{t}]$ and is such that $\theta(\overline{t}) = 0$.

  The last types of extremals have a self-intersection in the state-space in $[0, \overline{t}]$.

– The third kind of trajectories (bottom left) is such that $\theta(\overline{t}) = 0$ and occurs for $\overline{k} \leq k \leq k_3$.

– The last one (bottom right) exists only for $k_2 \leq k \leq k_3$ and has $\theta(\overline{t}) = \pi$.

• **k** $\leq$ **1**. The deformation of the conjugate locus in the case $k < 1$ is easier to analyze. We give on Fig. 10 the conjugate locus for $k \in \{0.8, 0.5, 0.2, 0.1\}$ with 15 chosen trajectories. The key point is the non-monotony of the θ-variable for $k < 1$.

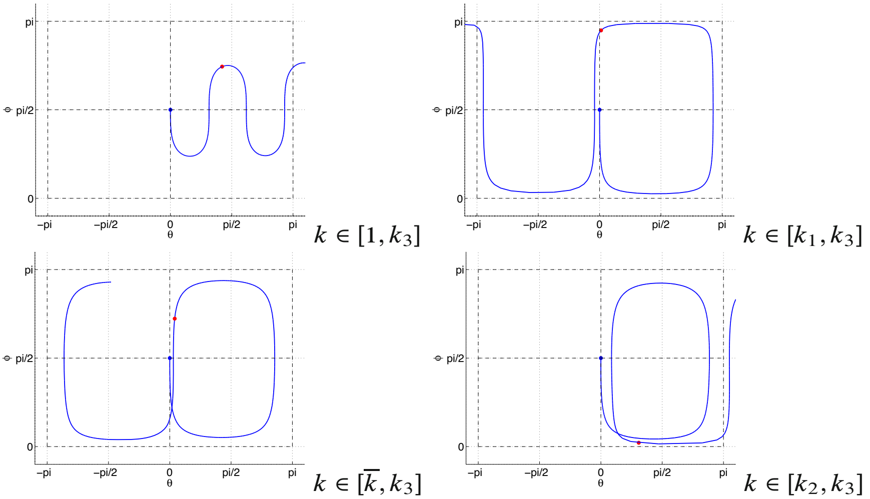

Fig. 9 The four types of trajectories which clarify the evolution of the conjugate locus

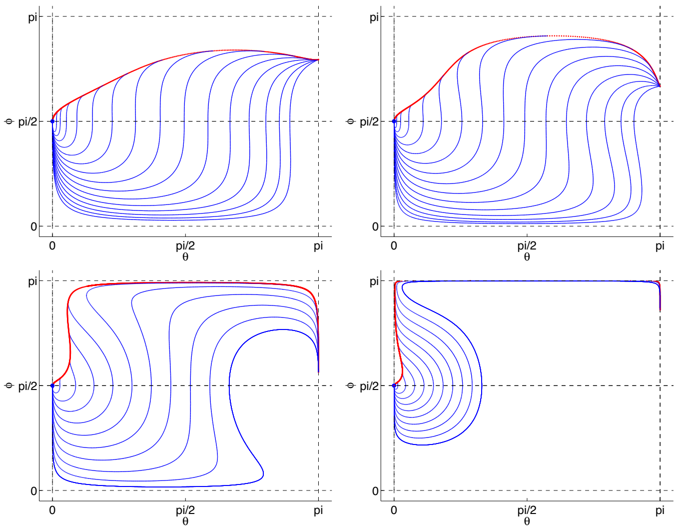

Fig. 10 Conjugate locus with 15 trajectories for $k = 0.8, 0.5, 0.2, 0.1$ from top left-hand to bottom right-hand

The deformation of the conjugate locus on the sphere is given Fig. 11. Only the half: $p_\varphi(0) = -2$, $p_\theta(0) \in \mathbf{R}$ is plotted to clarify the figures. The deformation is clear: the cusp moves along the meridian with respect to the parameter k. It does not cross the equator for $k < 1$ while for $k > 1$ it first crosses the North pole ($k = k_1$), then the equator ($k = \overline{k}$). For $k \geq \overline{k}$, the conjugate locus has self-intersections. Then,

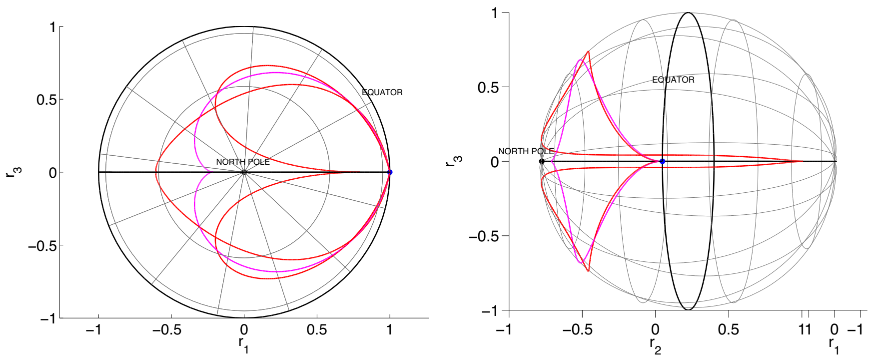

Fig. 11 Half of the conjugate locus on the sphere. (left) For $k = 1.0$ in magenta and $k = 0.8$, 1.15 in red. (right) For $k = 1.0$ in magenta and $k = 1.18$ in red

it crosses poles again for $k = k_2$ and k_3. This is repeated for greater values of k making the loops smaller and smaller.

We give a preliminary experimental result about the cut loci to conclude these numerical computations. We denote $p_\theta(0) > 0 \mapsto \Delta\theta_k(p_\theta) \in (0, \pi)$ the variation of θ at the first return to the equator (or first return mapping) as in §2.1. The previous numerical simulations show that $\Delta\theta_k$ is well defined for $k \in [0, k_3]$. The Fig. 12 indicates that for any k, the first return mapping is monotonous non-increasing and surjective. As a consequence, for a fixed k and starting from $\varphi(0) = \pi/2, \theta(0) = 0$, if there is no intersection between trajectories before the first return to the equator, then the cut locus is the equator minus the initial point. The Fig. 10 shows that there is no intersection before the first return to the equator for $k < 1$. Similar computations for $k \in [1, k_3]$ lead to the same conclusion.

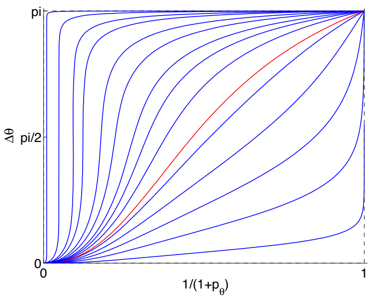

Fig. 12 First return mapping for different values of the parameter $k \in [0.1, 50]$. In red is plotted the curve for $k = 1$

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On the injectivity and nonfocal domains of the ellipsoid of revolution

Jean-Baptiste Caillau and Clément W. Royer

Abstract In relation with regularity properties of the transport map in optimal transportation on Riemannian manifolds, convexity of injectivity and nonfocal domains is investigated on the ellipsoid of revolution. Building upon previous results [4, 5], both the oblate and prolate cases are addressed. Preliminary numerical estimates are given in the prolate situation.

Introduction

It is known after the work of Brenier [7] and McCann [12] that, under suitable assumptions, the optimal transport map between two probability measures on a compact Riemannian manifold X exists and is unique when the cost is the square of the geodesic distance, d. The issue of the continuity of this map is addressed in a series of papers of Figalli *et al.* (*cf.* [9, 10] and references therein). A crucial object in this respect is the Ma-Trudinger-Wang tensor,

$$\sigma_{(x,v)}(\xi,\eta) := -\frac{3}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\,\frac{1}{2}d^2(\exp_x(t\xi),\exp_x(v+s\eta))$$

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defined at $x \in X$, $v \in I(x)$ and $(\xi, \eta) \in T_x X \times T_x X$, where $I(x) \subset T_x X$ denotes the injectivity domain of x (see §1). On surfaces, positivity of this tensor, namely

$$\xi \perp \eta = 0 \implies \sigma_{(x,v)}(\xi, \eta) \geq 0, \quad (x, v) \in TX, \quad v \in I(x), \quad (\xi, \eta) \in T_x X \times T_x X,$$

together with convexity of the injectivity domain $I(x)$ for all points x are proved to be necessary and sufficient for the continuity of the optimal transport map. (A gap exists in dimension greater than two [10].) Using the fact that the exponential mapping is a local diffeomorphism prior to the first conjugate time, the tensor can be extended to the nonfocal domain of x, $\mathrm{NF}(x) \supset I(x)$ (see §1), and similar results involving the convexity of the nonfocal domain can also be formulated: On surfaces, positivity of the extended tensor on nonfocal domains together with convexity of all these domains are sufficient for the continuity of the optimal transport map. The ellipsoid of revolution provides a one-parameter example whose geometry is rich enough to illustrate the change in convexity of the two types of domains, injectivity and nonfocal. It has been considered in the oblate case (ellipsoid squeezed along its axis of revolution) in [4,5]. As a deformation of the round sphere, it paves the way for a systematic study of surfaces of revolution whose integrable geodesic flow has a prescribed transcendency. On the ellipsoid of revolution, the quadratures are parameterized by a complex curve of genus one, and only elliptic functions (and primitives) are required (see also [6] for the general ellipsoid).

The paper is organized as follows: In Sect. 1, the main definitions are recalled; a unified framework using a parameterization by an elliptic curve is provided, which lays the emphasis on the role of singularities of this curve to understand convexity properties of the domains. It is moreover important to use a Hamiltonian point of view that allows to interpretate the limit case of the oblate ellipsoid flattened onto a two-sided disk in connection with almost-Riemannian metrics [1,3]. Sects. 2 and 3 are devoted to the oblate and prolate cases, respectively. It is proven that the nonfocal domain of a point on the equator is not convex for an oblate enough ellipsoid. In the prolate case, numerical estimates of the curvature are given using a suitable compactification suggesting that, for a sufficiently large semi-major axis, convexity holds for injectivity domain, not for nonfocal ones.

1 Preliminaries

For $\mu > 0$, consider the ellipsoid of revolution with z-axis embedded in $\mathbf{R}^3$, $x^2 + y^2 + z^2/\mu^2 = 1$. For $\mu < 1$ (*resp.* $\mu > 1$), one has an *oblate* (*resp.* prolate) ellipsoid, while for $\mu = 1$ the round sphere is retrieved. For $(\theta, \varphi) \in \mathbf{R} \times (0, \pi)$,

$$x = \sin\varphi \cos\theta, \quad y = \sin\varphi \sin\theta, \quad z = \mu \cos\varphi,$$

is the universal covering of the ellipsoid minus its poles. In the associated coordinates (θ, φ), the metric reads $X \mathrm{d}\theta^2 + (1 - X/\lambda)\mathrm{d}\varphi^2$ with $X := \sin^2\varphi$ and $\lambda := 1/(1-\mu^2)$. We set $\lambda = \infty$ when $\mu = 1$ (round sphere), and use indifferently μ or λ to specify the geometry of the surface in the sequel. From the Hamiltonian point of view, one sets

$$H(\theta, \varphi, p_\theta, p_\varphi) := \frac{1}{2}\left(\frac{p_\theta^2}{X} + \frac{p_\varphi^2}{1 - X/\lambda}\right).$$

Because of the symmetry of revolution, θ is a cyclic variable so p_θ is a linear first integral (Clairaut constant); the geodesic flow is integrable and arc length geodesics are Hamiltonian integral curves on $\{H = 1/2\}$.

Proposition 1. *The quadrature on φ is parameterized by the complex curve*

$$Y^2 = 4(X - p_\theta^2)(X-1)(X-\lambda), \quad X = \sin^2\varphi, \quad Y = \frac{\dot{X}(\lambda - X)}{\sqrt{\lambda}},$$

which is elliptic outside singularities.

Proof. On $\{H = 1/2\}$, $p_\varphi^2/(1 - X/\lambda) = 1 - p_\theta^2/X$ and one has

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{1 - X/\lambda}.$$

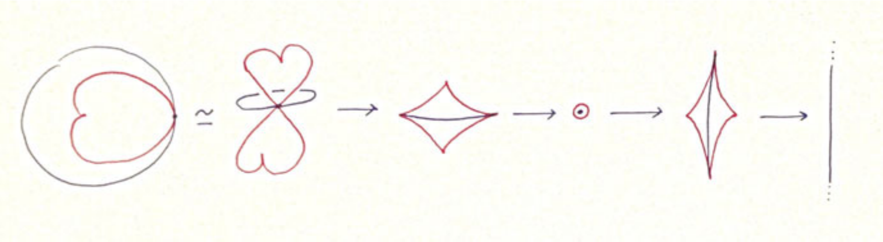
Since $\dot{X}^2 = 4X(1-X)\dot{\varphi}^2$, the result follows. □

When $\mu < 1$ (oblate ellipsoid), λ is positive and the real cubic ($Y \in \mathbf{R}$) has to be used; on the converse, when $\mu > 1$ (prolate ellipsoid), λ is negative and the parameterization is obtained considering the imaginary cubic ($Y \in i\mathbf{R}$). In both cases, as $p_\theta^2 \le X = \sin^2\varphi \le 1$, the bounded component of the cubic is used. The complex curve is homeomorphic to some torus $\mathbf{C}/\Lambda$ where $\Lambda = \omega\mathbf{Z} + \omega'\mathbf{Z}$ is the real-rectangular lattice of periods. In the oblate (*resp.* prolate) case, X is ω-periodic (*resp.* ω'-periodic) as a function on the torus. (The period of φ is twice the period of $X = \sin^2\varphi$, and the period as a function of time is given by some time law).

The singularities are the following. When $\mu = 0$, $\lambda = 1$ and the elliptic curve degenerates to a rational one; geometrically, the ellipsoid is flat and the resulting singular metric is simply the flat metric on a two-sided disk (see Proposition 3). When $\mu = 1$, $\lambda = \infty$ and the curve also degenerates for all p_θ; one has the round sphere whose geodesics are indeed rational curves. For any μ, $p_\theta = \pm 1$ (allowed only when $X = 1$) corresponds to the equator and is also a degeneracy of the elliptic curve. Finally, when $\mu = \infty$, $\lambda = 0$ and the curve degenerates for $p_\theta = 0$ (meridians); one may expect to use this, together with some compactification, to establish convexity properties in the prolate case, μ big enough (see the preliminary discussion §3). The bifurcations occuring in the cut and conjugate loci when going from $\mu = 0$ to $\mu = 1$, then to $\mu = \infty$ are portrayed Fig. 1. (See Sects. 2 and 3 on the structure of these sets in the oblate and prolate settings).

Given an initial point x_0 on the ellipsoid, consider the geodesic γ defined by $p_0 \in H^{-1}(x_0, \cdot)(\{1/2\})$; as the manifold is compact,

$$t_{\text{cut}}(x_0, p_0, \mu) := \sup\{t > 0 \mid \gamma \text{ is minimizing on } [0, t]\}$$



$\mu = 0$ $\mu \in (0,1)$ $\mu = 1$ $\mu \in (1,\infty)$ $\mu = \infty$

Fig. 1 Bifurcation of the cut and conjugate loci of $\varphi_0 = \pi/2$ when μ goes from $\mathbf{0}$ to $\mathbf{1}$, then to ∞. See §2 for the interpretation when $\mu = \mathbf{0}$. When $\mu = \infty$, the cut locus is the vertical line antipodal on the cylinder to the initial point (not a pole), and the conjugate locus is empty (see §3)

is finite, and is called the cut time along γ. As a subspace of the cotangent space at x_0, the injectivity domain of x_0 is defined according to

$$I(x_0) := \{t_{\text{cut}}(x_0, p_0, \mu) p_0 \mid H(x_0, p_0) = 1/2\}.$$

As convexity is invariant by linear transformations, whether the injectivity domain is defined as a subspace of the tangent or cotangent fibre does not matter. The exponential mapping is

$$\exp_{x_0}(t, p_0) := x(t, x_0, p_0), \quad (t, p_0) \in \mathbf{R} \times H^{-1}(x_0, \cdot)(\{1/2\}),$$

where $(x(., x_0, p_0), p(., x_0, p_0))$ is the integral curve of H for initial condition (x_0, p_0) (globally defined on the compact manifold). Along γ, the time t is said to be conjugate if (t, p_0) is a critical point of $\exp_{x_0}$; the first of such times, if any, is called the (first) conjugate time along γ and is denoted $t_c(x_0, p_0, \mu)$. The corresponding critical value is the (first) conjugate point. One defines the nonfocal domain of x_0 as

$$\text{NF}(x_0) := \{t_c(x_0, p_0, \mu) p_0 \mid H(x_0, p_0) = 1/2\}.$$

Up to the dilation $(x, y) \mapsto (x/\sqrt{X_0}, y/\sqrt{1 - X_0/\lambda})$ which does not change convexity, the boundary of $I(x_0)$ is parameterized by

$$\mathbf{S}^1 \ni \alpha \to t_{\text{cut}}(x_0, p_0, \mu) \exp(i\alpha), \quad \alpha = \arg\left(\frac{p_\theta}{\sqrt{X_0}} + i\frac{p_{\varphi 0}}{\sqrt{1 - X_0/\lambda}}\right). \tag{1}$$

One can also parameterize by $p_\theta = \cos\alpha\sqrt{X_0}$, allowing a ramification above $p_{\varphi 0} = 0$ since

$$p_{\varphi 0} = \pm\sqrt{1 - X_0/\lambda}\,\sqrt{1 - p_\theta^2/X_0}.$$

(Two distinct geodesics are generated depending on the sign.) For the sake of simplicity, we denote $\tau(p_\theta) := t_{\text{cut}}(x_0, p_0, \mu)$ and $' := \mathrm{d}/\mathrm{d}p_\theta$. Convexity of the injectivity domain holds if and only if the curvature of its boundary (provided the boundary is regular enough) is nonnegative.

Proposition 2. *The curvature of the injectivity domain of x_0 is*

$$K = X_0^{3/2} \frac{\tau(\tau + p_\theta \tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'')}{[(X_0 - p_\theta^2)(\tau + p_\theta\tau')^2 + (p_\theta\tau - (X_0 - p_\theta^2)\tau')^2]^{3/2}}, \quad p_\theta^2 \le X_0,$$

whose sign is given by

$$\kappa := \tau(\tau + p_\theta\tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'').$$

Proof. In cartesian coordinates, $K = (x''y' - x'y'')/(x'^2 + y'^2)^{3/2}$ whenever defined, hence the result. □

2 Oblate case

Let $x_0 = (\theta_0, \varphi_0)$; thanks to the symmetry of revolution, one can set $\theta_0 = 0$. The initial condition is thus reduced to φ_0, that is to $X_0 = \sin^2\varphi_0$.

Lemma 1. *The cut time along a geodesic (not a meridian) is equal to the half-period of the φ-coordinate. As such, $\tau = \tau(p_\theta, \mu)$ is independent of X_0, and of the sign of $p_{\varphi 0}$ (no ramification[1]). The injectivity domain has two axial symmetries, and convexity can be checked on a quarter of the domain.*

Proof. When $\mu < 1$, cut points are generated by the discrete symmetry $p_{\varphi 0} \mapsto -p_{\varphi 0}$: the associated geodesics intersect at $t = T/2$ where T is the period of φ. The period does not depend on the initial condition since, up to a translation on θ, any geodesic can be seen as a geodesic with initial condition $\varphi_0 = \pi/2$. The limit case $p_{\varphi 0} = 0$ (where the cut point is a conjugate point) is obtained letting p_θ tend to $\pm\sqrt{X_0}$. Because of the symmetry involved, $\tau(p_\theta, -p_{\varphi 0}, \mu) = \tau(p_\theta, p_{\varphi 0}, \mu)$ and one has an x-axis symmetry on the injectivity domain. Obviously, $p_\theta \mapsto -p_\theta$ induces another symmetry (wrt. y-axis) on the domain. □

When $\mu = 0$ ($\lambda = 1$), the metric is singular at $X = 1$ (that is $\varphi = \pi/2$). Setting $\rho = \sin\varphi$, one gets

$$X d\theta^2 + (1 - X/\lambda) d\varphi^2 = \sin^2\varphi d\theta^2 + d\rho^2 = dx^2 + dy^2$$

which is the flat metric on the Poincaré disk $\mathbf{D}$. Geometrically, the ellipsoid is collapsed on the unit disk and the equatorial singularity corresponds to the boundary. Crossing $\partial\mathbf{D}$ is interpretated as going from one side of the disk to the other, that is crossing the equator to go from one hemisphere to the other on the flat ellipsoid. As

[1] This is not true anymore for conjugate times outside polar or equatorial points; only one axial symmetry is preserved, see Fig. 4.

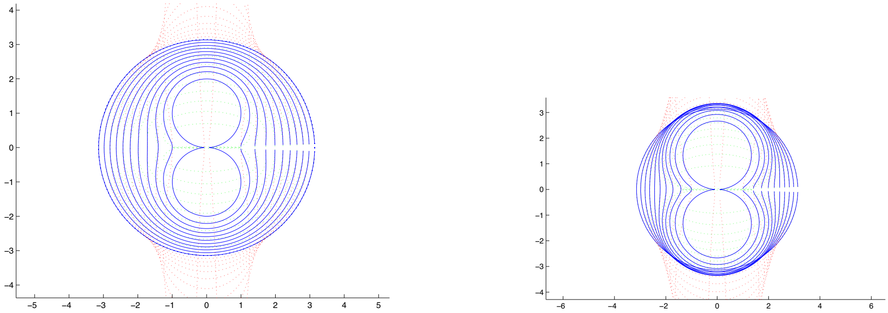


Fig. 2 Injectivity and nonfocal domains (left and right, respectively) of $\varphi_0 = \pi/2$ in the oblate case when $\mu \to \mathbf{0}$. For $\mu = \mathbf{1}$, both domains are disks, while for $\mu = \mathbf{0}$ both are union of tangent disks (of radii $\mathbf{1}$ and $\mathbf{4/3}$, respectively)

the metric is flat, geodesics are straight lines, in accordance with the degeneracy of the parameterization by the elliptic curve in Proposition 1 (double root $X = 1$ when $\lambda = 1$), which trivializes the computation of the cut locus.

Proposition 3. *For $\mu = 0$ and $\varphi_0 = \pi/2$, the cut locus is the equator minus the initial point. The injectivity domain is the union of two unit disks both tangent to the x-axis at the origin, and is not convex (Fig. 2).*

Proof. The geodesic from any point on the boundary is a straight line segment which meets again the boundary; the resulting point is a cut point as is intersects the geodesic from the other side of the disk, and the cut time is just given by the common length of these segments. The whole boundary but the initial the point is so made of cut points. In parameterization (1), $\tau(\alpha) = 2\sin\alpha$ and $\alpha \mapsto \pm\tau(\alpha)\exp(i\alpha)$, $\alpha \in (0, \pi)$, is the union of two circles tangent at the origin and of radii one. □

Remark 1. When $\mu = 0$ and $X_0 = 1$,

$$p_{\varphi_0} = \pm\sqrt{1 - X_0}\,\sqrt{1 - p_\theta^2/X_0} = 0,$$

so the dilation used in (1) desingularizes the parameterization of the boundary of the injectivity domain (which would otherwise collapse on a segment), revealing its non-convexity.

By continuity, $I(\varphi_0 = \pi/2)$ cannot be convex for μ small enough; conversely, when $\mu = 1$, the injectivity domain of any point (including equatorial ones) are circles of radius π (the cut locus of any point on the round sphere is the antipodal point, at distance π), therefore convex. There is so some threshold between $\mu = 1$ and $\mu = 0$ regarding convexity. Besides, for any fixed $\mu \in [0, 1]$, the injectivity domains of poles are circles (as on the round sphere), and the same must hold for initial conditions $\varphi_0 \in (0, \pi/2)$ close enough to 0 (by symmetry, one can restrict to $\varphi < \pi/2$). See Fig. 4. The following is proved in [4,5].

Theorem 1. *There is a nondecreasing function* $\varphi_0 \mapsto \mu(\varphi_0)$ *from* $[0, \pi/2]$ *to* $\mathbf{R}$ *such that the injectivity domain* $I(\varphi_0)$ *on an oblate ellipsoid of semi-minor axis* $\mu \leq 1$ *is convex if and only if* $\mu \geq \mu(\varphi_0)$*. One has* $\mu(0) = 0$ *(pole) and* $\mu(\pi/2) = 1/\sqrt{3}$ *(equator).*

The proof makes an essential use of the two following facts: (i) the degeneracy at $p_\theta = \pm 1$ of the elliptic curve to obtain $\mu(\pi/2) = 1/\sqrt{3}$; (ii) the fact that the cut time is given by the period of φ (this is not true anymore in the prolate case, see §3), which allows to derive analytic expressions of τ' and τ'' using derivatives of the periods of Weierstraß functions with respect to their invariants. As the threshold $\mu(\varphi_0)$ is monotonic, injectivity domains of any point on an oblate ellipsoid with $1/\sqrt{3} \leq \mu \leq 1$ are convex. The determination of $\mu(\varphi_0)$ for $\varphi_0 \in (0, \pi/2)$ is an open problem. (Numerical estimates are available, though.)

When $\mu = 0$, since crossing the boundary is changing hemisphere, one can also interpretate the geodesic continuing on the other side as a reflection on $\partial\mathbf{D}$ (with the usual rule on the angles). As a result, the conjugate locus is obtained as a catacaustic of the circle (see Fig. 3).

Proposition 4. *For* $\mu = 0$ *and* $\varphi_0 = \pi/2$*, the conjugate locus is a cardioid deprived of the initial point. The nonfocal domain is the union of two disks of radii* $4/3$ *both tangent to the* x*-axis at the origin, and is not convex (Fig. 2).*

Proof. The catacaustic of the unit circle with a source on the boundary is known to be the cardioid $z(\beta) := (2/3)(1 + \cos\beta)\exp(i\beta) - 1/3$ (see [2]). To prove that the associated nonfocal domain is the union of two circles, consider the ray generated by some $\alpha \in (0, \pi)$ in the parameterization (1) (considering $\alpha \in (-\pi, 0)$, that is p_{φ_0} negative, gives the other disk); it is enough to check that $w(\alpha) := 1 + 2\cos\alpha\exp(i(\pi - \alpha)) + (2/3)\cos\alpha\exp(-3i\alpha)$ (see construction on Fig. 3) belongs to the cardioid, which is clear. □

Remark 2. When $\mu = 0$, the metric is conformal to an almost-Riemannian metric with a singularity at the equator since

$$X\mathrm{d}\theta^2 + (1 - X)\mathrm{d}\varphi^2 = (1 - X)(XR(X)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2)$$

with $R(X) = 1/(1 - X)$ having a pole of order one at $X = 1$ ($\varphi = \pi/2$). Such metrics are particular cases of sub-Riemannian metrics and are considered in [1,3]. Here, the conformal coefficient itself is singular, but the analysis is obvious because of the flatness of the metric. Note that the cut locus of an equatorial point is the equator minus a point, and that the contact of the conjugate locus with the initial point is of order one (compare Theorem 1 and 2 in [3]).

As for the injectivity domain, there exists some threshold phenomenon for the loss of convexity of the nonfocal domain of $\varphi_0 = \pi/2$ when μ goes to zero (see Fig. 2). Conversely, for a fixed $\mu \leq 1$, convexity of nonfocal domains is retrieved when φ_0 tends to zero (see Fig. 4). Although numerical investigation suggests that some result similar to Theorem 1 may hold for nonfocal domains, the problem is open.

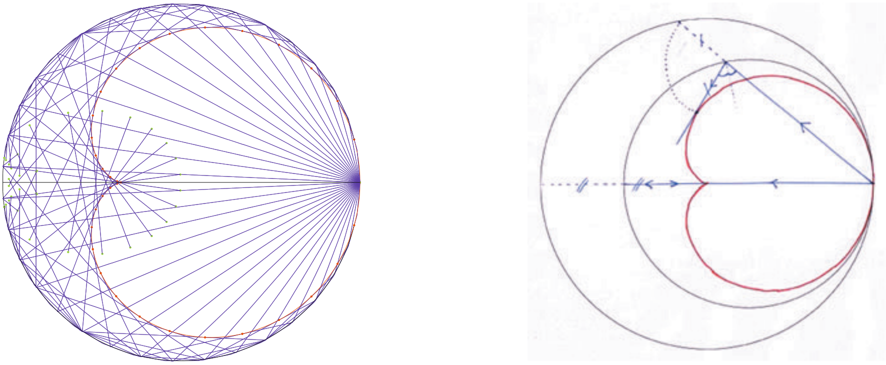


Fig. 3 Conjugate locus and nonfocal domain for $\mu = 0$ and $\varphi_0 = \pi/2$. On the disk, geodesics are straight lines starting from a point on the boundary and crossing $\partial\mathbf{D}$ when changing hemisphere can be seen as reflecting on $\partial\mathbf{D}$. The envelope of reflected rays forms the conjugate locus obtained as a catacaustic of the circle with source point on its boundary (leftmost graph). The geometric construction of the nonfocal domain from the cardioid is illustrated on the righmost picture

3 Prolate case

When $\mu > 1$, some loss of symmetry occurs, except when $X_0 = 1$.

Lemma 2. *The cut time along a geodesic (not a meridian) is obtained solving $\theta = \pi$. As such, $\tau = \tau(p_\theta, \mu)$ is independent of X_0 but depends on the sign of p_{φ_0}. The injectivity domain has just one axial symmetry wrt. y-axis, and convexity can be checked on a half of the domain.*

Proof. The situation in the prolate case is reversed compared to the oblate one: The symmetry $p_\theta \mapsto -p_\theta$ now generates intersections between geodesics emanating from the same point at length shorter than those generating by $p_{\varphi_0} \mapsto -p_{\varphi_0}$. Along every geodesic not a meridian, the cut is thus obtained at $\theta = \pi$ (while the meridian case is obtained as an envelope, letting p_θ tend to 0, providing a point both in the cut and conjugate loci). Clearly, $\pm p_\theta$ provide the same cut time, so the symmetry wrt. the y-axis of the injectivity domain is preserved. On the contrary, for $X_0 \neq 1$, geodesics with same p_θ but opposite p_{φ_0} do not cross $\theta = \pi$ at the same time, so that τ actually depends on the sign of p_{φ_0}; it has to be thought of as a function ramified above p_{φ_0} when parameterizing by p_θ alone. To prove that $\tau = \tau(p_\theta, X_0, \mu)$ actually does not depend on the initial condition, define $\Delta\theta$ the quasi-period of θ, that is the increment from $\theta_0 = 0$ on a period of φ. (According to Proposition 1, the period of X, and so of φ, is given in the complex parameterization by the imaginary period of the lattice). As in the oblate case, the period of φ only depends on p_θ, and so does $\Delta\theta$. Given $p_\theta > 0$, as $\mathrm{d}t/\mathrm{d}\theta = 1/\dot\theta = X/p_\theta > 0$, one can reparametrize using θ instead of t; since $\dot\theta$ and φ have the same period, X remains periodic as a

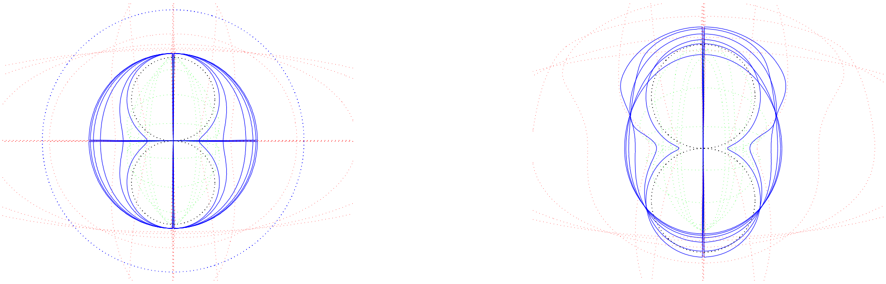


Fig. 4 Injectivity and nonfocal domains (left and right, respectively) for $\mu < 1/\sqrt{3}$ when $\varphi_0 \to \mathbf{0}$. For $\varphi_0 = \mathbf{0}$, both domains are disks, though for $\varphi_0 = \pi/2$ none are convex. Observe the loss of the axial symmetry wrt. the x-axis for the nonfocal domain

function of θ (with period $\Delta\theta(p_\theta)$), and

$$\tau = \int_0^\pi \frac{X(\theta)}{p_\theta}\,\mathrm{d}\theta.$$

Let $t_1 > 0$ be the first intersection of the geodesic with $\varphi = \pi/2$, assuming for the sake of simplicity $\varphi_0 < \pi/2$ and $p_{\varphi_0} > 0$ (the same kind of argument works for $p_{\varphi_0} < 0$). The geodesic of initial condition $(\theta(t_1), \pi/2)$ with same p_θ (and positive p_{φ_0}) has cut time

$$\tilde{\tau} = \int_{\theta(t_1)}^{\theta(t_1)+\pi} \frac{X(\theta)}{p_\theta}\,\mathrm{d}\theta = \int_0^\pi \frac{X(\theta)}{p_\theta}\,\mathrm{d}\theta$$

by periodicity of $X(\theta)$, so $\tau = \tilde{\tau}$, cut time associated with initial condition $\pi/2$, whatever φ_0. □

Up to translation, X is given by some Weierstraß function, $\wp$, whose invariants depend only on p_θ and λ (that is on p_θ and μ – see Proposition 1). In the parameterization by $z \in \mathbf{C}/\Lambda$, one checks that the resulting quadrature on θ involves integrating rational fractions in $\wp$ such as

$$\int \frac{\wp'(a)\,\mathrm{d}z}{\wp(z)-\wp(a)} = 2\zeta(a)z + \ln\frac{\sigma(z-a)}{\sigma(z+a)}$$

where $\zeta' = -\wp$ and $\sigma'/\sigma = \zeta$. Studying the roots of an equation with such transcendence is a complicated task. We provide a preliminary analysis trying to take advantage of the degeneracy for $\mu = \infty$ when $p_\theta = 0$, and using numerical estimates.

Proposition 5. *The metric of the prolate ellipsoid converges pointwisely outside poles to the metric of the flat cylinder of revolution when $\mu \to \infty$. All injectivity and nonfocal domains of the cylinder are convex.*

Proof. Recalling that $z = \mu \cos\varphi$, the metric on the ellipsoid writes

$$\left(1 - \frac{z^2}{\mu^2}\right) \mathrm{d}\theta^2 + \left(1 + \frac{z^2}{\mu^2(\mu^2 - z^2)}\right) \mathrm{d}z^2$$

and convergence is clear. The geodesics on the cylinder of revolution are either vertical lines ($p_\theta = 0$) without cut points, or helices ($\mathrm{d}z/\mathrm{d}\theta = p_z/p_\theta = \mathrm{cst}$); in the second case, the cut time is $\pi/|p_\theta|$ (cut point on the antipodal vertical line). Injectivity domains are therefore all equal to a vertical strip $[-\pi, \pi] \times \mathbf{R}$, and convex. The metric is flat and there are no conjugate points, so nonfocal domains are the whole fiber ($\simeq \mathbf{R}^2$) at any point, also convex. □

In contrast with the oblate case, another complication is so that there is no obvious obstruction to convexity arising from the asymptotic behavior when $\mu \to \infty$. With implicit function use on $\theta = \pi$ in mind, we recall the computation of the sensitivities wrt. initial condition of first (Jacobi fields) and second order for a Hamiltonian system.

Remark 3. The fact that the metric converges towards a flat metric (previous Proposition) does not even entail that the limit, after some compactification, of the nonfocal domains must be convex (see Fig. 5).

Let $\dot{z} = \overrightarrow{H}(z)$ be a smooth Hamiltonian system, with $z = (x, p) \in \mathbf{R}^{2n}$ and $\overrightarrow{H} = (\partial_p H, -\partial_x H)$. The solution $z(\cdot, z_0)$ with initial condition $z(0) = z_0$ depends smoothly on z_0, and for any $\delta z_0 \in \mathbf{R}^{2n}$ one has

$$\frac{\partial z}{\partial z_0}(t, z_0)\delta z_0 = \delta z(t), \quad \frac{\partial^2 z}{\partial z_0^2}(t, z_0)(\delta z_0, \delta z_0) = \delta_2 z(t),$$

where δz and $\delta_2 z$ are solutions of, respectively (by $H[t]$ we mean $H(z(t, z_0))$, *etc.*),

$$\dot{\delta z} = \overrightarrow{H}'[t]\delta z, \quad \delta z(0) = \mathrm{id},$$

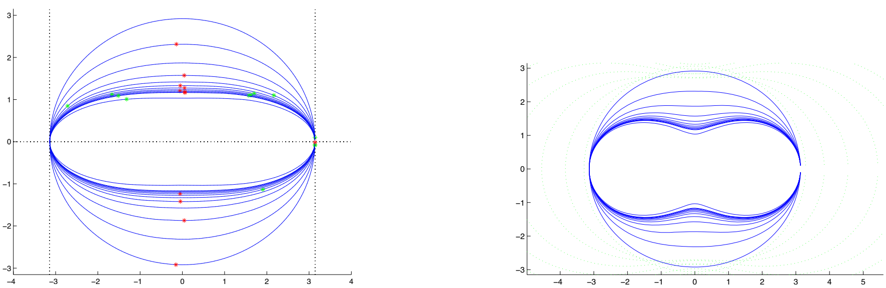
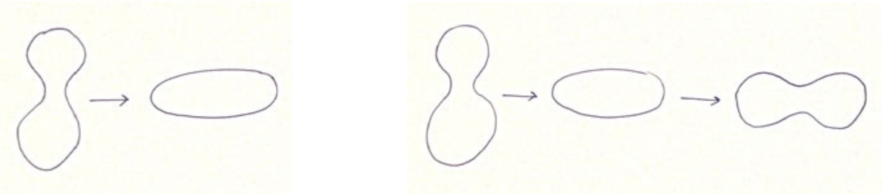


Fig. 5 Injectivity and nonfocal domains (left and right, respectively) of $\varphi_0 = \pi/2$ in the prolate case when $\mu \to \infty$. While convexity seems to hold at $\mu = \infty$ (and before) for the injectivity domain, nonfocal domains are clearly not convex for μ large enough, suggesting a threshold phenomenon as in the oblate case when $\mu \to \mathbf{0}$



$\mu < \mu(\varphi_0)$ $\mu(\varphi_0) \le \mu$ $\mu < \tilde{\mu}(\varphi_0)$ $\tilde{\mu}(\varphi_0) \le \mu \le \hat{\mu}(\varphi_0)$ $\hat{\mu}(\varphi_0) < \mu$

Fig. 6 Conjecture on the bifurcation of convexity of injectivity and nonfocal domains on the ellipsoid of revolution for a given point (φ_0) not a pole when μ goes from $\mathbf{0}$ to ∞. Leftmost graph: For the injectivity domain, there might be only be one threshold $\mu(\varphi_0) < \mathbf{1}$. Rightmost graph: For the nonfocal domain, there might be two thresholds $\tilde{\mu}(\varphi_0) < \mathbf{1}$ and $\hat{\mu}(\varphi_0) > \mathbf{1}$, as convexity might be retrieved for μ close to $\mathbf{1}$, and lost again for μ large enough

$$\dot{\delta_2 z} = \vec{H}'[t]\delta_2 z + \vec{H}''[t](\delta z(t), \delta z(t)), \quad \delta_2 z(0) = 0.$$

The numerical computation of these sensitivities, up to order two, is performed by the `cotcot` software[2] combining automatic differentiation and numerical integration of ordinary differential equations. In our case, $z = (\theta, \varphi, p_\theta, p_\varphi)$ and, for $p_\theta \neq 0$, τ is implicitely defined by $\theta(\tau, p_\theta) = \pi$. As previously mentioned, there is a dependence of the geodesic not only on p_θ but also on the sign of p_{φ_0}. The initial condition on $\{H = 1/2\}$ writes

$$z_0(p_\theta) := \left(p_\theta, \pm\sqrt{1 - X_0/\lambda}\,\sqrt{1 - p_\theta^2/X_0}\right).$$

Proposition 6. *For* $0 < p_\theta^2 < X_0$, *the derivatives of first and second order are*

$$\tau' = -\frac{1}{\dot{\theta}}\delta\theta, \quad \tau'' = -\frac{1}{\dot{\theta}}(\ddot{\theta}\tau'^2 + 2\delta\dot{\theta}\tau' + \delta_2\theta + \widetilde{\delta\theta}),$$

where $\delta\theta$ *(resp.* $\delta_2\theta$*) is the first (resp. second) variation associated with* $\delta z_0 = z_0'(p_\theta)$, $\widetilde{\delta\theta}$ *the first variation associated with* $\widetilde{\delta z_0} = z_0''(p_\theta)$, *and where all functions are evaluated at* $\tau(p_\theta)$.

Proof. Apply implicit function theorem to $\theta(\tau, p_\theta) = \pi$ noting that $\dot{\theta} = p_\theta/X \neq 0$ whenever $p_\theta \neq 0$. □

Whereas the worst case for curvature on an oblate ellipsoid, whatever the point, is given by the equator ($p_\theta^2 = X_0$), numerical simulations below indicate that the worst case in the prolate situation is given by meridians, $p_\theta = 0$, at the apparent singularity of the expressions before. Worst cases for curvature of injectivity domains (and, seemingly, of nonfocal domains – see Fig. 5) actually occur along geodesics where cut points are conjugate ones (equator in the oblate case, meridian in the prolate one).

To achieve numerical convergence of the domains, and of the curvature, we use a second dilation: $(x, y) \mapsto (x, y/\mu)$. The curvature is thus renormalized according

[2] `apo.enseeiht.fr/cotcot`

to

$$\widetilde{K} = (1/\mu)X_0^{3/2}\frac{\tau(\tau + p_\theta\tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'')}{[(X_0 - p_\theta^2)(\tau + p_\theta\tau')^2 + (1/\mu)^2(p_\theta\tau - (X_0 - p_\theta^2)\tau')^2]^{3/2}}.$$

This provides a heuristical compactification of domains and curvature, but has the effect that the parameterization by $p_\theta = \cos\alpha\sqrt{X_0}$ becomes singular when $\mu \to \infty$ as

$$\alpha = \arg\left(\frac{p_\theta}{\sqrt{X_0}} + i\frac{p_{\varphi 0}}{\sqrt{1 + (\mu^2 - 1)X_0}}\right) \to 0$$

whenever $p_\theta \neq 0$. One parameterizes instead $\widetilde{K}$ using $\beta := \arg(\cos\alpha + (i/\mu)\sin\alpha)$. On the basis of numerical estimates computed as in Proposition 6, the following observations can be made: (i) For $\varphi_0 = \pi/2$, numerical convergence of the (renormalized) injectivity domain is obtained (see Fig. 5); the limit domain seems to be convex, which suggests that convexity holds for equatorial points and μ large enough. A stronger conjecture would be convexity for all $\mu > 1$, or even for all $\mu > 1$ whatever φ_0 (see also Fig. 8 in this respect). (ii) For $\varphi_0 = \pi/2$, an estimation of the curvature $\widetilde{K}$ of the (renormalized) injectivity domain is obtained (see Fig. 7), not contradicting (i). (iii) For $\varphi_0 = \pi/2$, numerical convergence of the (renormalized) nonfocal domain is also obtained (see Fig. 5), which suggests that convexity does not hold for large enough μ; one can conjecturate a threshold phenomenon as in the oblate situation. (iv) The dependence of the convexity on the initial condition for $\mu > 1$ seems to be more complicate than in the oblate case, both for injectivity and nonfocal domains, as no monotonic behaviour seems to hold (see Fig. 8). For a fixed φ_0, Fig. 6 summarizes the previous conjectures on the bifurcation of the domains in terms of convexity.

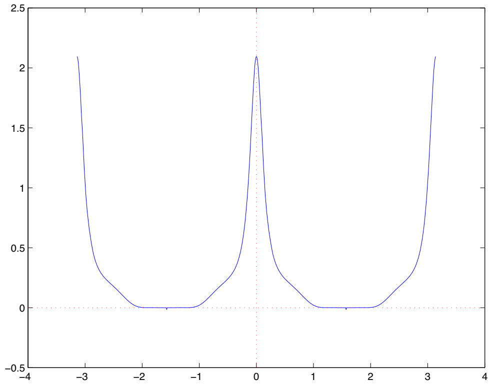


Fig. 7 Renormalized curvature $\widetilde{K}$ of the injectivity domain for $\mu = \infty$ and $\varphi_0 = \pi/2$. The parameter in abscissa is $\beta = \arg(\cos\alpha + (i/\mu)\sin\alpha)$ with $p_\theta = \cos\alpha\sqrt{X_0}$ so meridians are retrieved for $\beta = \pm\pi/2$. They actually correspond to the minimum estimated value of the curvature, in accordance with Fig. 5

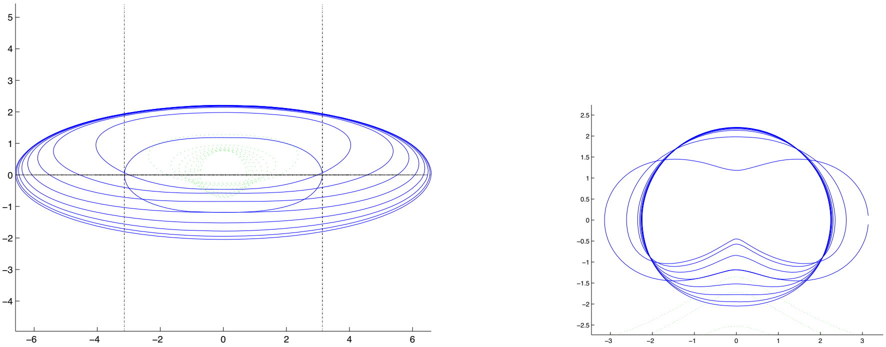


Fig. 8 Injectivity and nonfocal domains (left and right, respectively) for $\mu > 1$ when $\varphi_0 \to 0$. For $\varphi_0 = 0$, both domains are disks; for $\varphi_0 = \pi/2$, the injectivity domain remains convex but not the nonfocal domain. For $\varphi_0 \in (0, \pi/2)$, domains only have one axial symmetry. Monotonic dependence of the curvature on φ_0 does not seem to hold, either for the injectivity domain, or for the nonfocal one

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Null controllability in large time for the parabolic Grushin operator with singular potential

Piermarco Cannarsa and Roberto Guglielmi

Abstract We investigate the null controllability property for the parabolic Grushin equation with an inverse square singular potential. Thanks to a Fourier decomposition for the solution of the equation, we can reduce the problem to the validity of a uniform observability inequality with respect to the Fourier frequency. Such an inequality is obtained by means of a suitable Carleman estimate, with an adapted spatial weight function. We thus show that null controllability holds in large time, as in the case of the Grushin operator without potential.

1 Introduction

The work [2] provides a complete analysis of the null controllability properties (with respect to the values of $\gamma > 0$ and $T > 0$) of the generalized Grushin operator

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = f(x,y,t)1_\omega(x,y) & (x,y,t) \in D \times (0,T)\,, \\ u(x,y,t) = 0 & (x,y,t) \in \partial D \times (0,T)\,, \\ u(x,y,0) = u_0(x,y) \in L^2(D)\,, \end{cases} \tag{1}$$

where $D := (-1,1) \times (0,1)$ and $\omega \subset (0,1) \times (0,1)$. We can summarize the controllability result in [2] as follows:

1) if $\gamma \in (0,1)$, then system (1) is null controllable in any time $T > 0$;
2) if $\gamma = 1$ and $\omega = (a,b) \times (0,1)$ where $0 < a < b \leqslant 1$, then there exists $T^* \geq a^2/2$ such that

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– for every $T > T^*$ system (1) is null controllable in time T;
– for every $T < T^*$ system (1) is not null controllable in time T;

3) if $\gamma > 1$, then (1) is not null controllable.

On the other hand, the controllability property for the operator in system (1) is in general sensitive to lower order perturbations. Indeed, a result in [3] shows that, for all $\gamma \geq 1$, the dynamics ruled by the operator

$$Lu = \partial_x^2 u + |x|^{2\gamma}\partial_y^2 u - \frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{1}{x^2}u \tag{2}$$

separates the two connected component of $D \setminus \{0\} \times [0,1]$, where $\{0\} \times [0,1]$ is the singular set for the γ-Grushin metric generated by the vector fields

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ |x|^\gamma \end{pmatrix}, \quad \gamma \geq 1\,.$$

Thus, there is no trasmission of information across the singular set. In turn, this implies that in the case of $\gamma \geq 1$ no controllability result can be expected for the equation

$$\begin{cases} \partial_t u - Lu = f(x,y,t)1_\omega(x,y) & \text{in } D \times (0,\infty)\,, \\ u(x,y,t) = 0 & \text{on } \partial D \times (0,\infty)\,, \end{cases} \tag{3}$$

when ω lies in only one connected component of $D \setminus \{0\} \times [0,1]$, that is the case accounted for in [2].

Thus, we are naturally led to face the following question: which controllability properties do hold for the operator L?

In this paper we establish a partial (positive) answer to the above question. Indeed, we show a null controllability result for all sufficiently large times, in the case $\gamma = 1$, restricting the domain to one side only of the singular set. More precisely, setting $\Omega := (0,1) \times (0,1)$, we will address the null controllability problem for the equation

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^2\partial_y^2 u - \dfrac{\lambda}{x^2}u = f(x,y,t)1_\omega(x,y) & \text{in } \Omega \times (0,T)\,, \\ u(x,y,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u(x,y,0) = u_0(x,y) \in L^2(\Omega)\,, \end{cases} \tag{4}$$

where $T > 0$, $\lambda \in \mathbb{R}$ and ω is an open subset of Ω. The following result holds.

Theorem 1. *Let* $\omega = (a,b) \times (0,1)$ *for some* $0 < a < b \leqslant 1$ *and* $\lambda < 1/4$. *Then there exists* $T^* > 0$ *such that for every* $T > T^*$ *system* (4) *is null controllable in time* T.

Thus, also for the Grushin operator with singular potential, the case $\gamma = 1$ seems to be a transition regime, needing a minimum time for the null controllability, as in the problem addressed in [2]. Indeed, the same methods developed in [2] should apply, in order to prove also the necessity of such a minimum time. By a standard duality argument, Theorem 1 is equivalent to the observability in large time in $\omega \times (0, T)$ for the adjoint system

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^2 \partial_y^2 g - \dfrac{\lambda}{x^2} g = 0 & \text{in } \Omega \times (0,T)\,, \\ g(x,y,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ g(x,y,0) = g_0(x,y) \in L^2(\Omega)\,. \end{cases} \tag{5}$$

Thanks to a suitable Carleman estimate (see Proposition 5), we will prove the following result.

Theorem 2. *Let* $\omega = (a,b) \times (0,1)$ *for some* $0 < a < b \leqslant 1$ *and* $\lambda < 1/4$. *Then there exists* $T^* > 0$ *such that for every* $T > T^*$ *system* (5) *is observable in* $\omega \times (0,T)$.

As a consequence, we deduce null controllability in large times also for Eq. (3), with a control region located on both sides of the degeneracy, of the type $\omega = (a_1, b_1) \cup (a_2, b_2) \times (0,1)$, with $-1 \le a_1 < b_1 < 0 < a_2 < b_2 \le 1$.

For the time being, the Carleman estimate that we prove in this paper allows to obtain the observability only in the case $\gamma = 1$, though we expect a similar result (without minimum time) also in the case $0 < \gamma < 1$, just like in [2].

For future reference, in Sects. 2 and 3 we will treat the general case of an operator $Lu = \partial_x^2 u + |x|^{2\gamma} \partial_y^2 u + \frac{\lambda}{x^2} u$, with $\gamma > 0$. From Sect. 4 on we will focus on the case $\gamma = 1$.

2 Well-posedness and Fourier decomposition

2.1 Well-posedness of the Cauchy-problem

Let $H := L^2(\Omega)$, and denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|_H$, respectively, the scalar product and norm in H. We recall the well-known Hardy's inequality [5]

$$\int_0^1 \frac{z^2}{x^2} dx \le 4 \int_0^1 z_x^2 dx \quad \forall z \in H_0^1(0,1)\,. \tag{6}$$

Thanks to (6), the scalar product

$$(u,v) := \int_\Omega \left(u_x v_x + |x|^{2\gamma} u_y v_y - \frac{\lambda}{x^2} uv \right) dx dy \quad \forall u,\ v \in C_0^\infty(\Omega) \tag{7}$$

is positive for every $\lambda < 1/4$ (as we will assume from now on). Set

$W := \overline{C_0^\infty(\Omega)}^{|\cdot|_W}$, where $|u|_W := (u,u)^{1/2}$, and observe that $H_0^1(\Omega) \subset W \subset H$, thus W is dense in H. Introduce the space $V := \overline{C_0^\infty(\Omega)}^{|\cdot|_V}$ (see [2]), where $|u|_V := ((u,u))^{1/2}$ and

$$((u,v)) := \int_\Omega \left(u_x v_x + |x|^{2\gamma} u_y v_y\right) dxdy \quad \forall u,\, v \in C_0^\infty(\Omega)\,.$$

Hardy's inequality (6) ensures that $(z,z) \geq C_\lambda((z,z))$ for all $z \in C_0^\infty(\Omega)$, with $C_\lambda := 1 - 4\lambda > 0$, thus $W \subset V$. Let introduce the space $L^2_\gamma(\Omega)$ of all square-integrable functions with respect to the measure $d\mu = |x|^{2\gamma} dxdy$. Then (see [2, Lemma 1]), for all elements g of W there exist $\partial_x g \in L^2(\Omega)$, $\partial_y g \in L^2_\gamma(\Omega)$ such that for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega \left(g(x,y)\partial_x\phi(x,y) + |x|^{2\gamma} g(x,y)\partial_y\phi(x,y)\right) dxdy$$
$$= -\int_\Omega \left(\partial_x g(x,y) + |x|^{2\gamma}\partial_y g(x,y)\right)\phi(x,y)dxdy\,. \quad (8)$$

Define now

$$D(A) = \{u \in W : \exists\, c > 0 \text{ such that } |(u,h)| \leq c|h|_H \ \forall h \in W\}\,, \quad (9)$$

$$\langle Au, h\rangle := -(u,h) \quad \forall h \in W\,. \quad (10)$$

Then (see [12, Theorem 1.18]), the operator $(A, D(A))$ generates an analytic semigroup $S(t)$ of contractions on H. Note that A is selfadjoint on H, and (10) implies that

$$Au = \partial_x^2 u + |x|^{2\gamma}\partial_y^2 u + \frac{\lambda}{x^2}u \quad \text{a.e. in } \Omega\,.$$

So, system (4) can be recast in the form

$$\begin{cases} u'(t) = Au(t) + f(t) & t \in [0,T]\,, \\ u(0) = u_0\,, \end{cases} \quad (11)$$

where $T > 0$, $f \in L^2(0,T;H)$ and $u_0 \in H$.

Definition 1 (Weak solution). A function $u \in C([0,T];H) \cap L^2(0,T;W)$ is a weak solution of system (11) if for every $h \in D(A)$ the function $\langle u(t), h\rangle$ is absolutely continuous on $[0,T]$ and for a.e. $t \in [0,T]$

$$\frac{d}{dt}\langle u(t), h\rangle = \langle u(t), Ah\rangle + \langle f(t), h\rangle\,. \quad (12)$$

In [8], it is shown that the equivalence between condition (12) and the definition of solution by transposition, that is,

$$\int_\Omega [u(x,y,t^*)\varphi(x,y,t^*) - u_0(x,y)\varphi(x,y,0)]dxdy \\ = \int_0^{t^*}\int_\Omega \left\{u\left(\partial_x^2\varphi + |x|^{2\gamma}\partial_y^2\varphi + \frac{\lambda}{x^2}\varphi\right) + f\varphi\right\} dxdydt$$

for every $\varphi \in C^2([0,T]\times\Omega)$ and $t^* \in (0,T)$.

Moreover, the unique weak solution of (11) in the sense of Definition 1 is given by the variations-of-constants formula (see [1])

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds\,, \quad t \in [0,T]\,. \tag{13}$$

The following existence and uniqueness result follows.

Proposition 1. *For every $u_0 \in H$, $T > 0$ and $f \in L^2(0,T;H)$, there exists a unique weak solution of the Cauchy problem* (11). *This solution satisfies*

$$|u(t)|_H \leqslant |u_0|_H + \sqrt{T}\|f\|_{L^2(0,T;H)} \quad \forall t \in [0,T]\,. \tag{14}$$

Moreover, $u(t) \in D(A)$ and $u'(t) \in H$ for a.e. $t \in (0,T)$.

2.2 Fourier decomposition of the solution

Let $g \in C([0,T];H) \cap L^2(0,T;W)$ be the solution of Eq. (5) in the sense of Definition 1. Thus, the function $y \mapsto g(x,y,t)$ belongs to $L^2(0,1)$ for a.e. $(x,t) \in (0,1)\times(0,T)$, and we can develop g in Fourier series with respect to y

$$g(x,y,t) = \sum_{n\in\mathbb{N}^*} g_n(x,t)\varphi_n(y)\,, \tag{15}$$

where for all $n \in \mathbb{N}^*$ we set $\varphi_n(y) := \sqrt{2}\sin(n\pi y)$ and

$$g_n(x,t) := \int_0^1 g(x,y,t)\varphi_n(y)dy\,. \tag{16}$$

Proposition 2. *For every $n \geq 1$, g_n is the unique weak solution of*

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right]g_n = 0 & (x,t) \in (0,1)\times(0,T)\,, \\ g_n(0,t) = g_n(1,t) = 0 & t \in (0,T)\,, \\ g_n(x,0) = g_{0,n}(x) & x \in (0,1)\,, \end{cases} \tag{17}$$

where $g_{0,n} \in L^2(0,1)$ is given by $g_{0,n}(x) := \int_0^1 g_0(x,y)\varphi_n(y)dy$.

Proof. First, observe that, for any $n \geq 1$, system (17) is a first order Cauchy problem, that admits the unique weak solution

$$\tilde{g}_n \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H^1_0(0,1))$$

which satisfies

$$\frac{d}{dt}\left(\int_0^1 \tilde{g}_n(x,t)\psi(x)dx\right) + \int_0^1 \left[\tilde{g}_{n,x}(x,t)\psi_x(x) + \left((n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right)\tilde{g}_n(x,t)\psi(x)\right]dx = 0 \quad (18)$$

for every $\psi \in H^1_0(0,1)$.

In order to verify that the nth Fourier coefficient of g, defined by (16), satisfies system (17), observe that

$$g_n(\cdot,0) = \int_0^1 g_0(y,\cdot)dy = g_{n,0}(\cdot), \quad g_n(0,t) = g_n(1,t) = 0 \quad \forall t \in (0,T)$$

and

$$g_n \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H^1_0(0,1)).$$

Thus, it is sufficient to prove that g_n fulfills condition (18). Indeed, using the identity (16), for all $\psi \in H^1_0(0,1)$ we obtain, for a.e. $t \in [0,T]$,

$$\frac{d}{dt}\left(\int_0^1 g_n\psi dx\right) + \int_0^1 \left(g_{n,x}\psi_x + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right]g_n\psi\right)dx$$
$$= \int_0^1\int_0^1 \left\{g_t\varphi_n\psi + g_x\varphi_n\psi_x + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right]g\varphi_n\psi\right\}dydx. \quad (19)$$

Observe that Proposition 1 ensures $g_t(\cdot,t) \in L^2(\Omega)$ and $g(\cdot,t) \in D(A)$ for a.e. $t \in (0,T)$. So, we multiply $g_t = Ag$ by $h(x,y) = \psi(x)\varphi_n(y) \in W$ and integrate over Ω, in order to obtain, for a.e. $t \in (0,T)$,

$$\int_0^1\int_0^1 g_t\psi\varphi_n dxdy = \int_0^1\int_0^1 Ag\psi\varphi_n dxdy$$
$$= -\int_0^1\int_0^1 \left(g_x\psi_x\varphi_n + |x|^{2\gamma}g_y\psi\varphi_{n,y} - \frac{\lambda}{x^2}g\psi\varphi_n\right)dxdy$$
$$= -\int_0^1\int_0^1 \left(g_x\psi_x\varphi_n + (n\pi)^2|x|^{2\gamma}g\psi\varphi_n - \frac{\lambda}{x^2}g\psi\varphi_n\right)dxdy, \quad (20)$$

where (in the last identity) we have used relation (8). Combining identities (19) and (20) completes the proof. □

The unique continuation result for the adjoint system (5) can be readily derived.

Proposition 3. *Let $T > 0$, $\gamma > 0$, $\lambda < 1/4$, ω an open subset of $(0, 1) \times (0, 1)$, and let $g \in C([0, T]; H) \cap L^2(0, T; W)$ be a weak solution of system (5). If $g \equiv 0$ on $\omega \times (0, T)$, then $g \equiv 0$ on $\Omega \times (0, T)$.*

Proof. Let $\epsilon > 0$ be such that $\omega \subset (\epsilon, 1) \times (0, 1)$. In the rectangle $(\epsilon, 1) \times (0, 1)$, Eq. (5) has neither degenerate coefficients nor singular potential, so we are in the position to apply the unique continuation for uniformly parabolic 2−D equation. Thus, the hypothesis $g \equiv 0$ on $\omega \times (0, T)$ implies that $g \equiv 0$ on $(\epsilon, 1) \times (0, 1) \times (0, T)$. Then, relation (16) ensures that $g_n \equiv 0$ on $(\epsilon, 1) \times (0, T)$ for every $n \in \mathbb{N}^*$. Moreover, since $g_n \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H_0^1(0, 1))$, in particular, for a.e. $t \in (0, T)$, we have $g_n(\cdot, t) \in H_0^1(0, 1) \subset C([0, 1])$. Thus, by continuity, we conclude that $g_n \equiv 0$ on $(0, 1) \times (0, T)$ for every $n \in \mathbb{N}^*$ (compare also with the observability inequality in [10, Lemma 3.2(ii)]). Therefore, back to Eq. (15), we conclude that $g \equiv 0$ on $\Omega \times (0, T)$. □

Remark 1. Thanks to Proposition 3, we derive that the Grushin operator with singular potential (4) is approximately controllable by a locally distributed control in an arbitrary open subset ω of Ω, for every $T > 0$, $\gamma > 0$ and $\lambda < 1/4$. In particular, the condition $\lambda < 1/4$ embraces the case of the operator (2) accounted in [3], whose potential coefficient $-\gamma/2(\gamma/2 + 1)$ is smaller than $1/4$ for every $\gamma \neq -1$.

3 Spectral analysis for the 1−D problem

Aiming at proving the null controllability for Eq. (4), we now focus on the asymptotic behaviour (with respect to n) of the one dimensional eigenvalue problem associated with system (17). For this reason, let us introduce, for every $n \in \mathbb{N}^*$, $\gamma > 0$ and $\lambda < 1/4$, the operator $A_{n,\gamma,\lambda}$ on $L^2(0, 1)$ by

$$\begin{aligned} D(A_{n,\gamma,\lambda}) &:= \Big\{\varphi \in H_0^1(0, 1) \ : \ \varphi' \in AC_{loc}((0, 1]) \text{ and.} \\ &\qquad -\varphi'' + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right]\varphi \in L^2(0, 1)\Big\}, \\ A_{n,\gamma,\lambda}\varphi &:= -\varphi'' + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right]\varphi \in L^2(0, 1). \end{aligned} \tag{21}$$

The least eigenvalue of $A_{n,\gamma,\lambda}$ is given by

$$\mu_{n,\gamma,\lambda} = \min_{\substack{v \in H_0^1(0,1) \\ v \neq 0}} \left\{ \frac{\int_0^1 \left\{v'(x)^2 + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right] v(x)^2\right\} dx}{\int_0^1 v(x)^2 dx} \right\}. \tag{22}$$

For simplicity, from now on we will refer to $A_{n,\gamma,\lambda}$ and $\mu_{n,\gamma,\lambda}$ just as A_n and μ_n. We mention that the case $n = 0$ has been investigated in [11], where well-posedness and observability are proven for the operator A_0. Here we would achieve a similar observability result for the general operator A_n, uniformly in n (and in γ and λ as well). We start by characterizing the behaviour of μ_n as $n \to +\infty$, that quantifies the dissipation speed of the solution of (17).

Lemma 1. *Problem*

$$\begin{cases} -v''_{n,\gamma,\lambda}(x) + \left[(n\pi)^2|x|^{2\gamma} - \dfrac{\lambda}{x^2}\right] v_{n,\gamma,\lambda}(x) = \mu_n v_{n,\gamma,\lambda}(x) \ x \in (0,1) \\ v_{n,\gamma,\lambda}(0) = v_{n,\gamma,\lambda}(1) = 0\,, \end{cases} \tag{23}$$

admits a unique positive solution with $L^2(0,1)$-norm one.

Proof. Observe that the domain $D(A_n)$ of A_n is compactly embedded in $L^2(0,1)$, thus the resolvent operator of A_n is a compact operator. Then, there exists an orthonormal basis of $L^2(0,1)$ consisting of eigenvectors of A_n, and the first eigenvalue μ_n is simple. Moreover, the associated eigenfunction v is positive. Indeed, if not so, let us consider the function $w(x) = |v(x)|$. Then, w still belongs to $H^1_0(0,1)$, it is a weak solution of (23) and it does not increase the functional in (22). □

We next provide a precise growth condition for the eigenvalue μ_n, with respect to $n \in \mathbb{N}^*$.

Proposition 4. *For every $\gamma > 0$ and $\lambda < 1/4$, there exist two constants $C_* = C_*(\gamma,\lambda)$, $C^* = C^*(\gamma) > 0$ such that*

$$C_* n^{\frac{2}{1+\gamma}} \leqslant \mu_n \leqslant C^* n^{\frac{2}{1+\gamma}} \quad \forall n \in \mathbb{N}^*\,.$$

Proof. We prove first the lower bound. Let $\tau_n := n^{\frac{1}{1+\gamma}}$. With the change of variable $\phi(x) = \sqrt{\tau_n}\varphi(\tau_n x)$, we get

$$\begin{aligned} \mu_n &= \inf_{\phi \in C_c^\infty(0,1)} \left\{ \int_0^1 \left(\phi'(x)^2 + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right]\phi(x)^2 \right) dx : \|\phi\|_{L^2(0,1)} = 1 \right\} \\ &= \tau_n^2 \inf_{\varphi \in C_c^\infty(0,\tau_n)} \left\{ \int_0^{\tau_n} \left(\varphi'(y)^2 + \left[\pi^2|y|^{2\gamma} - \frac{\lambda}{y^2}\right]\varphi(y)^2 \right) dy : \|\varphi\|_{L^2(0,\tau_n)} = 1 \right\} \\ &\geqslant C_* \tau_n^2 \end{aligned}$$

where

$$C_* := \inf_{\varphi \in C_c^\infty(0,+\infty)} \left\{ \int_0^{+\infty} \left(\varphi'(y)^2 + \left[\pi^2|y|^{2\gamma} - \frac{\lambda}{y^2}\right]\varphi(y)^2 \right) dy : \|\varphi\|_{L^2(0,+\infty)} = 1 \right\}$$

is positive since, owing to Hardy's inequality, it is greater than $(1 - 4\lambda)c_*$, where c_* is the positive constant (see [9] for the case $\gamma = 1$)

$$c_* := \inf_{\varphi \in C_c^\infty(0,+\infty)} \left\{ \int_0^{+\infty} \left(\varphi'(y)^2 + \pi^2|y|^{2\gamma}\varphi(y)^2\right) dy : \|\varphi\|_{L^2(0,+\infty)} = 1 \right\}.$$

Moreover, observe that C_* goes to 0 as $\lambda \to 1/4$.

Now we prove the upper bound for μ_n. For every $k > 1$ we define the function $\varphi_k \in H_0^1(0,1)$ by

$$\varphi_k(x) = \begin{cases} kx & \text{for } x \in [0, 1/k)\,, \\ 2 - kx & \text{for } x \in [1/k, 2/k)\,, \\ 0 & \text{for } x \in [2/k, 1]\,. \end{cases} \tag{24}$$

Straightforward computations show that

$$\int_0^1 \varphi_k(x)^2 dx = \frac{2}{3k}\,, \int_0^1 |x|^{2\gamma} \varphi_k(x)^2 dx = c(\gamma) k^{-1-2\gamma}\,,$$
$$\int_0^1 \varphi_k'(x)^2 dx = 2k\,, \int_0^1 \frac{1}{x^2}\varphi_k(x)^2 dx = 4(1-\ln 2)k\,,$$

where

$$c(\gamma) := \frac{2^{2\gamma+3}}{2\gamma+3} + 4\frac{2^{2\gamma+1}-1}{2\gamma+1} - 2\frac{2^{2\gamma+2}-1}{\gamma+1}\,.$$

Thus, $\mu_n \leqslant f_{n,\gamma,\lambda}(k) := 3k^2 + 3/2(\pi n)^2 c(\gamma) k^{-2\gamma} - 6\lambda(1-\ln 2)k^2$ for all $k > 1$. Since $f_{n,\gamma,\lambda}$ attains its minimum at $\bar{k} = \tilde{c}(\gamma,\lambda) n^{\frac{1}{\gamma+1}}$, we have that

$$\mu_n \leqslant f_{n,\gamma,\lambda}(\bar{k}) = C(\gamma,\lambda) n^{\frac{2}{\gamma+1}}\,.$$

Moreover, since

$$C(\gamma,\lambda) = 3\left(\frac{\pi^2\gamma c(\gamma)}{2}\right)^{1/(\gamma+1)} \frac{\gamma+1}{\gamma}\,[1-2\lambda(1-\ln 2)]^{\gamma/(\gamma+1)}\,,$$

the constant C^* can be chosen independent from λ; indeed, $1-2\lambda(1-\ln 2) > 0$ for every $\lambda < 1/4$, and the exponent $\gamma/(\gamma+1)$ of the rightmost term is smaller than one. □

4 A global Carleman inequality

We want to prove that, if $\gamma = 1$ and $\omega = (a,b) \times (0,1)$ with $0 < a < b \leq 1$, then there exists a positive time $T^* > 0$ such that system (4) is null controllable in any time $T > T^*$, or, equivalently, system (5) is observable in any time $T > T^*$. For this purpose, we will implement a global Carleman inequality for solutions of (17).

For every $n \in \mathbb{N}^*$, we introduce the operator

$$P_n g = g_t - g_{xx} + \left[(n\pi)^2 x^2 - \frac{\lambda}{x^2}\right] g$$

and the functions $\theta(t) = [t(T-t)]^{-k}$, $t \in (0,T)$, for some $k > 2$, and

$$\beta(x) := \frac{2-x^2}{4}\,, \quad x \in [0,1]\,. \tag{25}$$

We then consider the weight function

$$p(x,t) = M\theta(t)\beta(x)\,, \quad (x,t) \in Q := (0,1)\times(0,T) \tag{26}$$

for a sufficiently large constant M.

Proposition 5. *There exist positive constant C_1, C_2 and $\eta \in (0,2)$ such that for every $n \in \mathbb{N}^*$, $T > 0$ and $g \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))$ we have*

$$C_1 \int_Q \left[M\theta(g_x^2 - \frac{\lambda}{x^2}g^2) + M^3\theta^3 x^2 g^2 + M\theta\frac{g^2}{x^\eta}\right] e^{-2p}dQ \\ \leq \int_Q |P_n g|^2 e^{-2p}dQ + \int_0^T M\theta(g_x^2 e^{-2p})_{|x=1}dt\,, \tag{27}$$

where $M := C_2 \max(T^{k/2} + T^{2k}, T^{2k}n)$.

Remark 2. In the following proof, in order to ensure the regularity of the function g needed for all integrations by parts, namely, that $g \in H^2(0,1) \cap H_0^1(0,1)$, we will regularize the operator P_n with the relaxed operator $P_{n,\delta}$ with potential $\frac{\lambda}{(x+\delta)^2}g$, and then pass to the limit as $\delta \to 0$. For simplicity, we will perform computations directly on P_n.

Proof. Let $g \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))$, and define

$$z(x,t) := g(x,t)e^{-p(x,t)}\,, \tag{28}$$

with weight function $p(x,t)$ defined as in (26). First, note that

$$\begin{cases} z(0,t) = z(1,t) = z_t(0,t) = z_t(1,t) = 0 \quad \text{for all } t \in (0,T)\,, \\ \theta^2 z,\ \theta_t z \text{ and } z_x \to 0 \quad \text{as } t \to 0^+ \text{ or } t \to T^-\,. \end{cases} \tag{29}$$

Moreover, one verifies that

$$e^{-p}P_n g = P_n^+ z + P_n^- z\,,$$

where $P_n^+ z = (p_t - p_x^2)z - z_{xx} + \left[(n\pi)^2 x^2 - \frac{\lambda}{x^2}\right] z$ and $P_n^- z = z_t - 2p_x z_x - p_{xx}z$. Thus, we have

$$\langle P_n^- z, P_n^+ z\rangle \leq \frac{1}{2}\int_Q e^{-2p}|P_n g|^2 dQ \tag{30}$$

and $\langle P_n^- z, P_n^+ z\rangle = D + B$, where (after several integration by parts we have that) the distributed part D is given by

$$D = -2\int_Q p_{xx}z_x^2 dQ - \int_Q p_{xxx}zz_x dQ - \int_Q \frac{1}{2}(p_{tt} - 2p_x p_{xt})z^2 dQ \\ + \int_Q (p_t - p_x^2)_x p_x z^2 dQ + \int_Q \left[(n\pi)^2 x^2 - \frac{\lambda}{x^2}\right]_x p_x z^2 dQ \tag{31}$$

and the boundary terms are

$$B = \left[\int_0^1 \frac{1}{2}\left(p_t - p_x^2 + (n\pi)^2 x^2 - \frac{\lambda}{x^2}\right) z^2 dx\right]_0^T$$
$$+ \left[\int_0^T \left(p_x z_x^2 + p_{xx} z z_x - [p_t - p_x^2 + (n\pi)^2 x^2 - \frac{\lambda}{x^2}] p_x z^2\right) dt\right]_0^1 . \quad (32)$$

Observe that, thanks to hypotheses (29), the boundary contribution reduces to

$$B = \left[\int_0^T p_x z_x^2 dt\right]_0^1 .$$

In order to cope with the singular potential, we will adapt the choice of the spatial weight β. As proposed in [4] and later in [10], we choose $\beta(x) := (2 - x^2)/4$, as in (25). Recalling that $p(x,t) = M\theta(t)\beta(x)$, the distributed part becomes

$$D = \int_Q M\theta z_x^2 dQ + \int_Q \frac{M^3}{4} x^2 \theta^3 z^2 dQ$$
$$+ \int_Q \left[\frac{M^2}{2} x^2 \theta\theta_t - \frac{M}{8}(2 - x^2)\theta_{tt} - (n\pi)^2 M\theta x^2 - M\theta\frac{\lambda}{x^2}\right] z^2 dQ , \quad (33)$$

and

$$B = \int_0^T -\frac{1}{2} M\theta z_x^2(1) dt . \quad (34)$$

We now estimate from below the distributed component D, taking advantage of the two coercive terms in the first line of Eq. (33). To this aim, we need an improved version of Hardy's inequality, namely, the so-called Hardy-Poincaré inequality: for all $m > 0$ and $\eta < 2$ there exists a positive constant $C_0 = C_0(\eta, m)$ such that

$$\int_0^1 \left(z_x^2 - \frac{z^2}{4x^2}\right) dx \geq m \int_0^1 \frac{z^2}{x^\eta} dx - C_0 \int_0^1 z^2 dx . \quad (35)$$

Since $\lambda < 1/4$, applying the Hardy-Poincaré inequality with $m = 2$, we deduce that

$$D \geq \frac{M}{2} \int_Q \theta\left(z_x^2 - \frac{\lambda}{x^2} z^2\right) dQ + \frac{M^3}{4} \int_Q x^2 \theta^3 z^2 dQ + \int_Q M\theta \frac{z^2}{x^\eta} dQ$$
$$-\frac{C_0}{2} \int_Q M\theta z^2 dQ + \int_Q \left[-\frac{M}{8}(2 - x^2)\theta_{tt} + \frac{M^2}{2} x^2 \theta\theta_t - (n\pi)^2 x^2 M\theta\right] z^2 dQ ,$$

where the three terms on the first line are positive, whereas the integrals in the second line need to be evaluated. Observe that

$$|\theta_t(t)| \leq c_1(T)\theta^{1+1/k} \quad \text{and} \quad |\theta_{tt}(t)| \leq c_2(T)\theta^{1+2/k} \quad \forall t \in (0,T) , \quad (36)$$

where $c_1(T) = kT$ and $c_2(T) = k(k+1)T + k/2T^2$. Moreover,

$$|\theta^{2+1/k}| \le c_3(T)\theta^3$$

with $c_3(T) = cT^{2(k-1)}$, where here and in the following c stands for a generic constant independent of n and T. Thus,

$$\left|\int_Q \frac{M^2}{2} x^2 \theta\theta_t z^2 dQ\right| \le \frac{c_1 c_3}{2}\int_Q M^2 x^2 \theta^3 z^2 dQ\,.$$

So, for $M \ge C_1(T) = cT^{2k-1}$, we deduce that

$$D \ge \frac{M}{2}\int_Q \theta\left(z_x^2 - \frac{\lambda}{x^2}z^2\right)dQ + \frac{M^3}{8}\int_Q x^2\theta^3 z^2 dQ + \int_Q M\theta\frac{z^2}{x^\eta}dQ$$
$$-\frac{C_0}{2}\int_Q M\theta z^2 dQ + \int_Q\left[-\frac{M}{8}(2-x^2)\theta_{tt} - (n\pi)^2 x^2 M\theta\right]z^2 dQ\,.$$

On the other hand, fix $k = 1 + 2/\eta$ and consider the conjugate exponents $q = k$ and $q' = k/(k-1)$. Then, taking $c_4(T) = c(T + T^4)$, for every $\varepsilon > 0$,

$$\left|\int_Q\left[-\frac{C_0}{2}M\theta - \frac{M}{8}(2-x^2)\theta_{tt}\right]z^2 dQ\right| \le c_4 M\int_Q \theta^{1+2/k}z^2 dQ$$
$$= c_4 M\int_Q\left(\frac{1}{\varepsilon}\theta^{1+2/k-1/q'}x^{\eta/q'}z^{2/q}\right)(\varepsilon\theta^{1/q'}x^{-\eta/q'}z^{2/q'})dQ$$
$$\le \frac{cc_4 M}{\varepsilon^q}\int_Q \theta^{q(1+2/k-1/q')}x^{\eta q/q'}z^2 dQ + \varepsilon^{q'}c_4 M\int_Q \theta\frac{z^2}{x^\eta}dQ$$
$$= \frac{cc_4 M}{\varepsilon^q}\int_Q \theta^{q(1+2/k-1/q')}x^{\eta q/q'}z^2 dQ + \varepsilon^{q'}c_4 M\int_Q \theta\frac{z^2}{x^\eta}dQ\,.$$

Note that

$$q(1 + 2/k - 1/q') = 3 \quad\text{and}\quad \eta q/q' = 2\,.$$

Thus,

$$\left|\int_Q\left[-\frac{C_0}{2}M\theta - \frac{M}{8}(2-x^2)\theta_{tt}\right]z^2 dQ\right| \le \frac{c_4 M}{\varepsilon^q}\int_Q \theta^3 x^2 z^2 dQ$$
$$+ \varepsilon^{q'}c_4 M\int_Q \theta\frac{z^2}{x^\eta}dQ\,.$$

Now, choose $\varepsilon > 0$ such that $1 - \varepsilon^{q'}c_4 = 1/2$. So, for all $M \ge c(T^{k/2} + T^{2k})$, we have that

$$D \ge \frac{M}{2}\int_Q \theta\left(z_x^2 - \frac{\lambda}{x^2}z^2\right)dQ + \frac{M^3}{16}\int_Q x^2\theta^3 z^2 dQ + \frac{1}{2}\int_Q M\theta\frac{z^2}{x^\eta}dQ$$
$$-\int_Q (n\pi)^2 x^2 M\theta z^2 dQ\,.$$

Finally, we estimate the last integral, which depends on n. Since $\theta \leq c_5\theta^3$, with $c_5(T) = cT^{4k}$,

$$\left|\int_Q (n\pi)^2 x^2 M\theta z^2 dQ\right| \leq c_5 n^2 M \int_Q x^2\theta^3 z^2 dQ\,,$$

so, for every $M \geq c\max(T^{k/2} + T^{2k}, T^{2k}n)$, we conclude that

$$D \geq \frac{M}{2}\int_Q \theta\left(z_x^2 - \frac{\lambda}{x^2}z^2\right)dQ + \frac{M^3}{32}\int_Q x^2\theta^3 z^2 dQ + \frac{1}{2}\int_Q M\theta\frac{z^2}{x^\eta}dQ\,. \quad (37)$$

Thanks to relation (28) and estimates (30)–(34)–(37), we can easily complete the proof of (27). □

5 Uniform observability

Thanks to the Carleman estimate of Proposition 5, we can prove a uniform observability result for the adjoint system (17).

Proposition 6. *Let $a, b \in \mathbb{R}$ such that $0 < a < b \leq 1$. Then there exist $C > 0$, $k > 2$ and $T^* > 0$ such that for every $T > T^*$, $n \in \mathbb{N}^*$ and $g_{0,n} \in L^2(0,1)$ the solution of (17) for $\gamma = 1$ satisfies*

$$\int_0^1 g_n(x,T)^2 dx \leqslant T^{2k-1} e^{C(1+T^{-3k/2})}\int_0^T\int_a^b g_n(x,t)^2 dxdt\,. \quad (38)$$

Let us recall that explicit bounds on the observability constant of the heat equation with a potential are already known (see [6] and [7]). However, such results are of no use in the present context where the main point is uniform (in n) observability.

Proof (of Proposition 6). Let $(a', b') \subset\subset (a, b)$, $0 \leq \chi \leq 1$ such that $\chi(x) \equiv 1$ on $(0, a')$ and $\chi(x) \equiv 0$ on $(b', 1)$, and define

$$w(x,t) := \chi(x)g(x,t) \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1))\,.$$

Observe that $supp(\chi_{xx}) \subset supp(\chi_x) \subset (a', b')$, and $P_n w = \chi_{xx}g + 2\chi_x g_x$. By definition of w we deduce that

$$\int_0^T\int_0^a \theta g^2 e^{-2p} dxdt \leq \int_Q \theta w^2 e^{-2p} dQ\,. \quad (39)$$

Moreover, since $w_x(1) = 0$, the Carleman estimate in Proposition 5 ensures that for every $n \in \mathbb{N}^*$, $T > 0$ and for some $\eta \in (0,2)$ we have

$$\begin{aligned} M\int_Q \theta w^2 e^{-2p} dQ &\leq M\int_Q \theta\frac{w^2}{x^\eta}e^{-2p}dQ \\ &\leq c\int_Q |P_n w|^2 e^{-2p}dQ \leq c\int_0^T\int_{a'}^{b'}(g^2 + g_x^2)e^{-2p}dxdt\,, \end{aligned}$$

where $M{:=}\, C_2 \max(T^{k/2}{+}T^{2k}, T^{2k}n)$. Thanks to Caccioppoli's inequality (see [4])

$$\int_0^T \int_{a'}^{b'} g_x^2 e^{-2p}\, dx dt \le c \int_0^T \int_a^b g^2 dx dt\,,$$

so

$$M \int_Q \theta w^2 e^{-2p} dQ \le c \int_0^T \int_a^b g^2 dx dt\,. \tag{40}$$

Combining Eqs. (39)–(40), we have that

$$M \int_0^T \int_0^a \theta g^2 e^{-2p} dx dt \le c \int_0^T \int_a^b g^2 dx dt\,. \tag{41}$$

By the same argument, choosing a cut-off function that vanishes in a neighbourhood of 0 and equals 1 near the point $x = 1$, we deduce a similar inequality and conclude that

$$M \int_0^T \int_0^1 \theta g^2 e^{-2p} dx dt \le c \int_0^T \int_a^b g^2 dx dt\,. \tag{42}$$

Note that, for every $t \in (T/3, 2T/3)$,

$$\left(\frac{4}{T^2}\right)^k \leqslant \theta(t) \leqslant \left(\frac{9}{2T^2}\right)^k$$

and

$$\int_0^1 g^2(x,T) dx \leqslant e^{-\frac{2}{3}\mu_n T} \int_0^1 g^2(x,t) dx\,.$$

Integrating over $(T/3, 2T/3)$, we deduce that

$$\begin{aligned}\frac{T}{3}\int_0^1 g^2(x,T)dx &\le e^{-\frac{2}{3}\mu_n T}\int_{T/3}^{2T/3}\int_0^1 g^2(x,t) dx dt\\ &\le e^{-\frac{2}{3}\mu_n T}\left(\frac{T^2}{4}\right)^k e^{\left(\frac{9}{2}\right)^k \frac{M}{T^{2k}}} \int_{T/3}^{2T/3}\int_0^1 \theta g^2(x,t) e^{-2p} dx dt\,.\end{aligned}$$

Thanks to relation (42) and Proposition 4, we conclude that

$$\int_0^1 g^2(x,T)dx \le \frac{c_1}{T}\left(\frac{T^2}{4}\right)^k e^{-c_2 n T + \left(\frac{9}{2}\right)^k \frac{M}{T^{2k}}} \int_0^T \int_a^b g^2 dx dt\,, \tag{43}$$

for some constants $c_1, c_2 > 0$ (independent of n, T and g). Recalling that $M := C_2 \max(T^{k/2} + T^{2k}, T^{2k} n)$, we consider two different cases.

First case: $n < 1 + \frac{1}{T^{3k/2}}$**.** Then $M = C_2(T^{k/2} + T^{2k})$, thus

$$\int_0^1 g^2(x,T)dx \leqslant c T^{2k-1} e^{c_1\left(1+\frac{1}{T^{3k/2}}\right)} \int_0^T \int_a^b g^2(x,t) dx dt\,.$$

Second case: $n \geqslant 1 + \frac{1}{T^{3k/2}}$. Then $M = C_2 n T^{2k}$, and

$$\int_0^1 g^2(x,T)dx \leqslant cT^{2k-1} e^{\left(\frac{9}{2}\right)^k n - \frac{2}{3}cnT} \int_0^T \int_a^b g^2(x,t)dxdt\,.$$

Finally, observe that $\left(\frac{9}{2}\right)^k n - \frac{2}{3}cnT < 0$ as soon as $T \geq T^* := \left(\frac{9}{2}\right)^k \frac{3}{2c}$, completing the proof of (38).

Finally, Proposition 6 ensures that Theorem 2 holds, so system (5) is observable in $\omega \times (0,T)$ with $\omega = (a,b) \times (0,1)$ and $T \geq T^* > 0$, thus, equivalently, system (4) is null controllable in large time with a control located in ω.

6 Open problems and perspectives

In this paper we have shown a first positive controllability result for the Grushin operator with a singular potential in the square $\Omega = (0,1) \times (0,1)$: approximate controllability holds for every $\gamma > 0$ and every $\lambda < 1/4$; moreover, exploiting the spectral analysis provided in Sect. 3, we have proven null controllability in large time in the case $\gamma = 1$ and $\lambda < 1/4$. By analogy with the theory in [2], it should be possible to obtain a negative controllability result for (4) if T is too small, as well as positive and negative results depending on the value of the parameter γ. Indeed, for subcritical values of the coefficient of the inverse square potential ($\lambda < 1/4$), we expect a behaviour similar to the case of the generalized Grushin operator without singular potential studied in [2]: null controllability should hold in every time for $\gamma \in (0,1)$, whereas it should fail for $\gamma > 1$. Widely open is the case of a potential term with the critical coefficient $\lambda = 1/4$. In this case, one has to adapt the functional setting in order to compensate for the lack of coercivity of the associated bilinear form (see [11]). Furthermore, completely open is the controllability problem for the Grushin operator with singular potential in the domain $D = (-1,1) \times (0,1)$, that is, with degeneracy of the diffusion coefficient and singularity of the potential occurring at the interior of the domain.

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The rolling problem: overview and challenges

Yacine Chitour, Mauricio Godoy Molina, and Petri Kokkonen

Abstract In the present paper we give a historical account –ranging from classical to modern results– of the problem of rolling two Riemannian manifolds one on the other, with the restrictions that they cannot instantaneously slip or spin one with respect to the other. On the way we show how this problem has profited from the development of intrinsic Riemannian geometry, from geometric control theory and sub-Riemannian geometry. We also mention how other areas –such as robotics and interpolation theory– have employed the rolling problem.

1 Introduction

Differential geometry has been inextricably related to classical mechanics, since its very conception in the 18th century. As a matter of fact, back in the days, this area of research was referred to as rational mechanics. The basic idea of this point of view is reasonably simple: to a given mechanical system $\mathbb{M}$, one can associate a differentiable manifold M in such a way that each possible state of the system corresponds to a unique point in M. In this way, each possible velocity vector of $\mathbb{M}$ at a given configuration is represented as a tangent vector to M at the corresponding point. The classical dictionary goes as follows:

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1) physical data (such as masses, lengths, etc.) of elements in $\mathbb{M}$ induce a Riemannian metric in M representing the kinetic energy;
2) linear restrictions imposed on the positions of $\mathbb{M}$ (or that can be integrated to such) translate to submanifolds of M.

In the late 19th century, physicists noted there were plenty of mechanical systems not considered by the above dictionary. These systems were named non-holonomic, opposed to holonomic systems which are defined in the second point of the dictionary above. A mechanical system $\mathbb{M}$ is non-holonomic if its dynamics has linear restrictions that cannot be integrated to constraints of the position. For various examples and a brief historical bibliography, we refer the interested reader to the survey [8]. A well-known early example of these systems is the sphere rolling on the plane without sliding or spinning, studied (with some variants) by Chaplygin in the seminal works [16, 17]. Our aim in this paper is to give a general look at some of the most important breakthroughs in mathematics that gave us some understanding of the generalized version of this system consisting on two Riemannian manifolds M and $\hat{M}$ of the same dimension rolling one on the other, not allowing spins or slips. Nowadays these systems are often studied in connection to sub-Riemannian and Riemannian geometry [43, 48] and geometric control theory [3].

The structure of the paper is the following. In Sect. 2 we recall two major players in the study of the mechanical system described above and early differential geometry: Chaplygin and Cartan. Chaplygin studies for the first time the problem from a mechanical point of view and finds first integrals of motion in different situations. Cartan's development and his celebrated "five variables" paper were not evidently connected to the rolling problem at the time of their publication, see [10], nevertheless we present them from our point of view. In Sect. 3, we briefly present Nomizu's breakthrough introduction of the dynamics of rolling in higher dimensions, through embedded submanifolds of Euclidean space and its relation to Cartan's development. In Sect. 4 we present how the problem was brought back to life when control theory sees in differential geometry a useful tool to treat the controllability issue of the rolling problem in two dimensions and some geometric consequences of optimality conditions. Sect. 5 surveys how the higher dimensional rolling problem was re-discovered and how it appears naturally in geometric interpolation. Finally in Sect. 6 we present the latest results that have been obtained concerning the controllability of the system and its symmetries. We conclude with a brief discussion on some generalizations and open problems.

2 The early years: Mechanics and the new differential geometry

The first time the problem of a ball rolling on the plane was considered as worthy of study was in the seminal papers of Chaplygin [16, 17], one of the fathers of non-holonomic mechanics. The results were considered surprisingly difficult at the time, and for [16] Chaplygin won the Gold Medal of Russian Academy of Sciences. The main results he obtained were first integrals of motion for the system in several geometric situations. Even these seemingly elementary problems contains unexpected difficulties and bottlenecks when trying to obtain closed formulae for the dynamics. Essentially at the same time, Cartan was developing his coordinate-free differential geometry. With this new language he was able to propose and study many problems, most often related to the search of invariants of geometric systems. In this survey, we will only focus in two of his many ideas: the search for invariants and symmetries for control systems with two controls and five degrees of freedom, and the definition of affine Riemannian holonomy through the development of a curve. Both of this ideas will appear several other times in this survey.

2.1 Chaplygin's ball

In the year 1897 the work [16] written by Chaplygin was published. This papers is one of a series of research articles in which Chaplygin analyzed non-holonomic systems. Also of particular relevance to this survey is another paper [17]. In particular he was interested in studying first integrals and equations of motion for different systems of rolling balls.

To illustrate his results, Chaplygin was able to find an integral of motion for the system of a homogeneous small ball of mass m_1 and a homogeneous sphere of mass m_2, in which the ball rolls without slipping inside the sphere. We will think of the dynamics occurring in Euclidean 3-space. Let O be the center of the sphere, let G be the center of the moving ball and A the point of contact between the two. Introducing the quantities $a = \mathrm{dist}(O, G)$ and $b = \mathrm{dist}(O, A)$, then one has the integrals of motion:

$$\sum_{i=1}^{2} m_i \left(y_i \frac{dx_i}{dt} - x_i \frac{dy_i}{dt} \right) + M \left(\frac{b}{a} - 1 \right) \left(\beta \frac{d\alpha}{dt} - \alpha \frac{d\beta}{dt} \right) = \text{const.}$$

Where $A = (\alpha, \beta, \gamma)$ with respect to a fixed frame $OX'Y'Z'$, and the points $G = (x_1, y_1, z_1)$ and $O = (x_2, y_2, z_2)$ with respect to a moving frame $AXYZ$, with axes at all times parallel to those in $OX'Y'Z'$. The total mass is $M = m_1 + m_2$.

The equations of motion are complicated and it serves little purpose to write them down here. Nevertheless, there is an interesting historical remark at this point. After arriving at a very complicated differential equation to describe the dynamics of the system, Chaplygin observes it can be written in the form

$$\frac{dv}{d\zeta} + v\Phi(\zeta) + \Psi(\zeta) = 0,$$

for some functions Φ and Ψ after changing variables. The integration of differential equations connected to the problem of rolling balls is still an area of active research, see for example [13].

2.2 Cartan's "five variables" paper

A rank l vector distribution D on an n-dimensional manifold M or (l, n)-distribution (where $l < n$) is, by definition, an l-dimensional subbundle of the tangent bundle TM, i. e., a smooth assignment $q \mapsto D|_q$ defined on M where $D|_q$ is an l-dimensional subspace of the tangent space $T|_q M$. Two vector distributions D_1 and D_2 are said to be equivalent, if there exists a diffeomorphism $F : M \to M$ such that $F_* D_1|_q = D_2|_{F(q)}$ for every $q \in M$. Local equivalence of two distributions is defined analogously.

Cartan's equivalence problem consists in constructing invariants of distributions with respect to the equivalence relation defined above. A seminal contribution by Cartan in [14] was the introduction of the "reduction-prolongation" procedure for building invariants and the characterization for $(2, 5)$-distributions via a functional invariant (Cartan's tensor) which vanishes precisely when the distribution is flat, that is, when it is locally equivalent to the (unique) graded nilpotent Lie algebra $\mathfrak{h}$ of step 3 with growth vector $(2, 3, 5)$.

In the same paper, Cartan also proved that in this system there is hidden a realization of the 14-dimensional exceptional Lie algebra $\mathfrak{g}_2$. To explain where does it appear, let us recall that an infinitesimal symmetry of an $(l, n)-$distribution D is a vector field $X \in \mathrm{VF}(M)$ such that $[X, D] \subseteq D$. Now consider the (unique) connected and simply connected nilpotent Lie group H with Lie algebra $\mathfrak{h}$. The two dimensional subspace of $\mathfrak{h}$ that Lie generates it, can be seen as a $(2, 5)-$distribution on H. In general, a $(2, 5)-$distribution that is bracket generating is nowadays known as a Cartan distribution. In this setting, the following theorem takes place.

Theorem 1 (Cartan 1910). *The Lie algebra of symmetries of the flat Cartan distribution is precisely* $\mathfrak{g}_2$*, and this situation is maximal, that is, for general Cartan distributions the dimension of the Lie algebra of symmetries is* ≤ 14.

Moreover, Cartan gave a geometric description of the flat G_2-structure as the differential system that describes space curves of constant torsion 2 or $1/2$ in the standard unit 3-sphere (see Sect. 53 in Paragraph XI in [14].)

The connection between this studies by Cartan and the rolling problem comes from the fact that the flat situation described above occurs in the problem of two 2-dimensional spheres rolling one on the other without slipping or spinning, assuming that the ratio of their radii is $1 : 3$, see [12] for some historical notes and a thorough attempt of an explanation for this ratio. In fact, whenever the ratio of their radii is different from $1 : 3$, the Lie algebra of symmetries becomes $\mathfrak{so}(3) \times \mathfrak{so}(3)$, thus dropping its dimension to 6. A complete answer to this strange phenomenon as well as a geometric reason for Cartan's tensor was finally given in two remarkable papers [52, 53] (cf. also [4]), where a geometric method for construction of functional invariants of generic germs of $(2, n)$-distribution for arbitrary $n \geq 5$ is developed. It has been recently observed in [5] that the Lie algebra of symmetries of a system of rolling surfaces can be $\mathfrak{g}_2$ in the case of non-constant Gaussian curvature.

2.3 Cartan's development

Cartan in [15] defined a geometric operation, that he called development of a manifold on a tangent space, in order to define holonomy in terms of "Euclidean displacements", i. e., elements of $\mathrm{E}(n)$. In his own words:

> Quand on développe l'espace de Riemann sur l'espace euclidien tangent en A le long d'un cycle partant de A et y revenant, cet espace euclidien subit un déplacement et tous les déplacements correspondant aux différents cycles possibles forment un groupe, appelé groupe d'holonomie.

An interpretation of this quote in terms of manifolds rolling follows naturally. For a given loop $\gamma\colon [0,\tau] \to M$ on an n dimensional Riemannian manifold M, one can roll M on the Euclidean space $\mathbb{R}^n$ obtaining a new curve $\hat{\gamma}\colon [0,\tau] \to \mathbb{R}^n$, called the development of γ. By parallel transporting along γ any orthonormal frame of $T|_{\gamma(0)}M$, we obtain a rotation $R_\gamma \in \mathrm{O}(n)$. The fact that $\hat{\gamma}$ is not necessarily a loop induces a translation T_γ corresponding to the vector $\hat{\gamma}(\tau) - \hat{\gamma}(0)$. We conclude that we can associate to γ an element (R_γ, T_γ) of the Euclidean group of motions $\mathrm{E}(n)$. The subgroup $\mathrm{Hol}^{aff}(M)$ of $\mathrm{E}(n)$ consisting of all such (R_γ, T_γ) obtained by rolling along all absolutely continuous loops γ is known as the affine holonomy group of M and the orthogonal part $\mathrm{Hol}(M) \subseteq \mathrm{O}(n)$ of it is the holonomy group of M.

It is known that if M is complete and with irreducible Riemannian holonomy group, the affine holonomy group contains all translations of $T|_x M$, see [37, Corollary 7.4, Chap. IV]. In other words, under the irreducibility hypothesis, the rotational part of the affine holonomy permits to recover the translational part, and this consists of all the possible translations in $T|_x M$.

Perhaps something that might have been not expected by Cartan is that this concept of development would play a fundamental role in the definition of Brownian motion on a manifold, and the subsequent explosion of interest that stochastic analysis in Riemannian manifolds has had in later decades, see [29]. For a long time, mathematicians have had the intuition that by rolling an n-dimensional manifold M along a given curve $y(t)$ in $\mathbb{R}^n$ with the Euclidean structure, one would obtain a curve in M which resembles the original curve $y(t)$, see [27]. The main outstanding idea (as far as we know due to Malliavin) was to use Cartan's development through the orthonormal frame bundle and Wiener's measure, see [50].

The idea of how to define Brownian trajectories on manifolds is similar to the interpretation given above. Intuitively, one can consider a Brownian path $B(t)$ in $\mathbb{R}^n$, and then roll M on $\mathbb{R}^n$ following the path $B(t)$. The precise definition uses a less regular version of Cartan's development and parallel transport.

This naive notion allows one to recover the Laplace-Beltrami operator Δ_M of the manifold. It is often interpreted as if Brownian paths are the "integral curves" for Δ_M. Of course this assertion lacks of mathematical precision, but it introduces the idea that second order differential operators induce "diffusions" on the manifold. This point of view has been exploited significantly in the study of stochastic differential equations on manifolds, see [7].

3 A "forgotten" breakthrough

An important contribution to the understanding of the problem of rolling without slips or spins came to light in the paper [45] by Nomizu. His aim was to give a mechanical interpretation of certain differential geometric invariants using this system. He mainly focuses in submanifolds of $\mathbb{R}^N$ with the usual Euclidean structure, and so will we along this section.

He begins with a simple general consideration: as a motion occurring in a Euclidean space $\mathbb{R}^N$ without deforming objects, a rolling can be seen as a curve in the Euclidean group $\mathrm{E}(N)$, that is a function $[0,\tau] \ni t \mapsto f_t \in \mathrm{E}(N)$ given by

$$f_t = \begin{pmatrix} C_t & c_t \\ 0 & 1 \end{pmatrix}, \tag{1}$$

where $f_0 = \mathrm{Id}$ is the identity matrix of $(N+1)\times(N+1)$, $C_t \in \mathrm{O}(N)$ and $c_t \in \mathbb{R}^N$. He calls such types of curves 1-parametric motions.

For a given 1-parametric motion $\{f_t\}$, he observed that there is a natural time-dependent vector field X_t associated to it. For an arbitrary point $y \in \mathbb{R}^N$ we define $(X_t)_y := \frac{df_u(x)}{du}\Big|_{u=t}$, where $x = f_t^{-1}(y)$, with the inversion taking place in $\mathrm{E}(N)$. Using Eq. (1), one can see that $(X_t)_y = S_t y + v_t$, where $S_t = \frac{dC_t}{dt} C_t^{-1} \in \mathfrak{o}(N)$ and $v_t = -S_t c_t + \frac{dc_t}{dt} \in \mathbb{R}^N$ are both completely determined by $\{f_t\}$. The corresponding element of the Lie algebra $\mathfrak{e}(N)$

$$\frac{df_t}{dt} f_t^{-1} = \begin{pmatrix} S_t & v_t \\ 0 & 0 \end{pmatrix} \tag{2}$$

is called the instantaneous motion. Slips and spins can now be encoded in terms of the vector field X_t and the instantaneous motion.

Definition 1. The instantaneous motion (2) is called an instantaneous:

- standstill if $S_t = 0$ and $v_t = 0$;
- translation if $S_t = 0$ and $v_t \neq 0$;
- rotation if there exists a point $y_0 \in \mathbb{R}^N$ such that $(X_t)_{y_0} = 0$ and $S_t \neq 0$.

With this at hand, it is possible to define rolling without slipping (*skidding* in Nomizu's terminology) nor spinning between M^n and $\hat{M}^n$, Riemannian submanifolds of $\mathbb{R}^N$. We denote by $\langle\cdot,\cdot\rangle$ the standard Riemannian structure in $\mathbb{R}^N$.

Definition 2. Let $\{f_t\}$ be a 1-parametric motion such that $f_t(M)$ is tangent to $\hat{M}$ at a point $y_t \in \hat{M}$. Assume that $(X_t)_{y_t} = 0$ and $S_t \neq 0$. The motion f_t is a rolling if for any pair of tangent vectors $X, Y \in T|_{y_t}\hat{M}$ and for any pair of normal vectors $U, V \in T|_{y_t}^{\perp}\hat{M}$

$$\langle S_t(X), Y\rangle = 0, \qquad \langle S_t(U), V\rangle = 0. \tag{3}$$

An equivalent way of stating conditions (3) is that S_t maps $T|_{y_t}\hat{M}$ to $T|_{y_t}^{\perp}\hat{M}$ and also maps $T|_{y_t}^{\perp}\hat{M}$ to $T|_{y_t}\hat{M}$.

This definition allowed Nomizu to find a very concrete realization of Cartan's development. For the case of surfaces rolling on the plane, his result reads

Theorem 2 (Nomizu 1978). *Let x_t be a smooth curve on a surface M which does not go through a flat point of M. There exists a unique rolling $\{f_t\}$ of M on the tangent plane Σ at x_0 such that $y_t = f_t(x_t)$ is the locus of points of contact. The curve y_t is the development of the curve x_t into Σ.*

As a consequence of this result, Nomizu noticed that there is a natural kinematic interpretation of the Levi-Civita connection for a surface M, coming from the rolling formulation: a vector field $U(t)$ along the curve x_t is parallel with respect to the Levi-Civita connection of M if and only if $C_t(U(t))$ is a constant vector for all t.

As a matter of fact, he was able to extend this result to higher dimensions and gave conditions under which rollings exist in terms of the shapes of the submanifolds, that is, in terms of both intrinsic and extrinsic data.

For reasons unknown to us, this paper seems to have been forgotten over the years. Nomizu's definition of higher dimensional rolling is equivalent to Sharpe's one in Sect. 5.1 and many of his observations have been rediscovered in [48, Appendix B]. Nevertheless, there is no reference to the paper [45] in Sharpe's book.

4 Revival: The two dimensional case and robotics

The aim of this section is to put in context the study of the rolling problem for the case of two dimensional manifolds, and how they appeared naturally in problems of sub-Riemannian geometry, robotics and geometric control theory.

4.1 Rigidity of integral curves in Cartan's distribution

In the celebrated paper [11], Bryant and Hsu studied curves on a manifold Q of dimension $n \geq 3$ tangent to a $(2, n)$–distribution D. The idea was to analyze the space $\Omega_D(p, q)$ of differentiable curves in Q connecting two points $p, q \in Q$ and being tangent to D (called D-curves by them). The space $\Omega_D(p, q)$ is endowed with its natural C^1 topology. The idea that D-curves can be "rigid" plays a fundamental role in their paper.

Definition 3. A D-curve $\gamma\colon [0, \tau] \to Q$ is *rigid* if there is a C^1-neighborhood $\mathscr{U}$ of γ in $\Omega_D(\gamma(0), \gamma(\tau))$ so that every $\gamma_1 \in \mathscr{U}$ is a reparametrization of γ. We say that γ is *locally rigid* if every point of $I = [0, \tau]$ lies in a subinterval $J \subset I$ so that γ restricted to J is rigid.

Their main result goes as follows.

Theorem 3 (Bryant & Hsu 1993). *Let D be a non-integrable rank 2 distribution on a manifold Q of dimension $(2 + s) \geq 3$. Suppose further that the distribution $D_1 = [D, D]$ (which has rank 3) is nowhere integrable. Then there always exist D-curves that are locally rigid.*

They give a more precise description of such curves in terms of projections of characteristic curves in a dense subset of the annihilator of D_1, but stating it precisely would not serve the purposes of this exposition.

For us, the most relevant part of their work is their section on examples, in particular their study of systems of Cartan type and of rolling surfaces.

Recall that a bracket generating $(2,5)-$distribution is said to be of Cartan type. In other words D is a Cartan distribution if D_1 has rank 3 and $D_2 = [D_1, D]$ has rank 5. As a consequence of Theorem 3, they observe that there is exactly a 5-parameter family of locally rigid D-curves. In fact they briefly discuss a remarkable geometric behavior occurring in this situation: if M is connected, then any two points of M can be joined by a piecewise smooth D-curve, whose smooth segments are rigid.

After all these observations, they devote themselves to the analysis of two oriented surfaces M and $\hat{M}$ endowed with Riemannian metrics rolling one over another without slipping or twisting. Let F and $\hat{F}$ be the oriented orthonormal frame bundles of M and $\hat{M}$. Bryant and Hsu considered the "state space" manifold $Q = Q(M, \hat{M}) = (F \times \hat{F})/\mathrm{SO}(2)$, where SO(2) acts diagonally on the Cartesian product. An element in Q is a triple $(x, \hat{x}; A)$, where $x \in M$, $\hat{x} \in \hat{M}$ and $A\colon T|_x M \to T|_{\hat{x}}\hat{M}$ is an oriented isometry. Their formulation is as follows. Consider a curve $\gamma\colon [0,\tau] \to Q$ given by $\gamma(t) = (x(t), \hat{x}(t); A(t))$, then the no-slip condition reads $A(t)(\dot{x}(t)) = \dot{\hat{x}}(t)$. The no-twist condition requires some more care. Let $e_1, f_1 : [0,\tau] \to TM$ be a parallel orthonormal frame along the curve $x(t)$ and let

$$e_2(t) = A(t)(e_1(t)), \quad f_2(t) = A(t)(f_1(t)),$$

be the orthonormal frame along $\hat{x}(t)$ obtained via A. The rolling has no-twist whenever the moving frame e_2, f_2 is also parallel (along $\hat{x}$).

An important insight for the problem was expressing the no-twist and no-slip conditions in terms of a $(2,5)-$distribution D on Q. Let $\alpha_1, \alpha_2, \alpha_{21}$ be the canonical 1-forms of M on F and similarly $\beta_1, \beta_2, \beta_{21}$ for $\hat{M}$, see [49]. Recall that these forms satisfy the so-called structure equations

$$\begin{aligned} d\alpha_1 &= \alpha_{21} \wedge \alpha_2, & d\beta_1 &= \beta_{21} \wedge \beta_2, \\ d\alpha_2 &= -\alpha_{21} \wedge \alpha_1, & d\beta_2 &= -\beta_{21} \wedge \beta_1, \\ d\alpha_{21} &= \kappa\, \alpha_1 \wedge \alpha_2, & d\beta_{21} &= \hat{\kappa}\, \beta_1 \wedge \beta_2, \end{aligned}$$

where κ and $\hat{\kappa}$ are the Gaussian curvatures of M and $\hat{M}$ respectively. With all of this, one can consider the distribution $\tilde{D}$ on $F \times \hat{F}$ defined by the Pfaffian equations

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \alpha_{21} - \beta_{21} = 0.$$

The distribution they were looking for corresponds to the "push-down" image of $\tilde{D}$ under the submersion $F \times \hat{F} \to Q$. A smooth curve $\gamma\colon [0,\tau] \to Q$ describes a rolling without slipping or twisting if and only if γ is a D-curve.

A remarkable fact is that the distribution D is of Cartan type whenever $\kappa - \hat{\kappa} \neq 0$, which is an open set in Q. On this set, the corresponding 5-parameter family of rigid curves describes the rolling of M on $\hat{M}$ following geodesics.

4.2 Non-holonomy in robotics

The traditional modeling of a mechanical system considers configurations (or states) of this mechanical system as points q of a smooth n-dimensional manifold M, and the corresponding velocities $\dot{q} \in T|_q M$ are subject to locally independent constraints in the Pfaffian form

$$A(q)\dot{q} = 0, \tag{4}$$

where $A(\cdot)$ is an $m \times n$ matrix of real-valued analytic functions, where $m < n$. Constraints are said to be *holonomic* if their differential form given by (4) is integrable. In this case, there exist integral submanifolds of dimension $n - m$ that are invariant. If the constraints are not holonomic at some $q_0 \in M$, then there will exist an integral submanifold containing q_0 of dimension $n - m + k$ with $0 < k \leq m$. The integer k is referred to as degree of non-holonomy. If $k = m$, the constraints, and by extension the system, are said to be maximally non-holonomic (see [44]).

There is a more convenient way for control theory to describe the constrained system. If $G(q)$ denotes a matrix whose columns form a basis for the annihilating distribution of $A(q)$, then all admissible velocities $\dot{q} \in A(q)^{\perp} \subset T|_q M$ can be written as linear combinations of the columns of $G(q)$,

$$\dot{q} = G(q)w = \sum_{i=1}^{n-m} g_i(q)w_i, \tag{5}$$

where w is a vector of *quasivelocities* taking values in $\mathbb{R}^{n-m}$. When quasivelocities can be assigned values at will in time, functions can be regarded as *control* inputs of the driftless, linear-in-the-control, nonlinear system defined by (5). A physical *actuator* is associated to each control input , e. g. a motor for electromechanical systems. The issue of non-holonomy of the original system, i. e. non-integrability of (4), can be addressed by studying the distribution Δ spanned by the the vector fields $g_1, \ldots, g_{n-m}$ and, more precisely, the corresponding Lie algebra generated by them. If the system is maximally non-holonomic (or completely controllable), any two configurations q and q' of its n-dimensional manifold can be connected along the flows of $n - m$ vector fields. From an utilitarian engineer's viewpoint, the latter definition may be rephrased as an n-dimensional non-holonomic system that can be steered at will using less than n actuators. This formulation underscores the appealing fact that devices with reduced hardware complexity can be used to perform nontrivial tasks, if non-holonomy is introduced on purpose, and cleverly exploited, in the device design (see [44]).

Non-holonomy of rolling is particularly relevant to robotic manipulation, one of the main goals of which is to manipulate an object grasped by a robot end-effector so as to relocate and re-orient it arbitrarily, the so-called dexterity property. Dexterous robotic hands developed so far according to an anthropomorphic paradigm employ far too many joints and actuators (a minimum of nine) to be a viable industrial solution. Non-holonomy of rolling can be used to alleviate this limitation. In fact, while rolling between the surfaces of the manipulated object and that of fingers has been previously regarded as a complication to be neglected, or compensated for, some works (see, in particular, [1, 6, 18, 24, 39, 40] and the references therein) tried to exploit rolling for achieving dexterity with simpler mechanical hardware.

Introducing non-holonomy on purpose in the design of robotic mechanisms can be regarded as a means of lifting complexity from hardware to the software and control level of design. In fact, planning and controlling non-holonomic systems is in general a considerably more difficult task than for holonomic systems. The very fact that there are fewer degrees-of-freedom available than there are configurations implies that standard motion planning techniques can not be directly adapted to non-holonomic systems. From the control viewpoint, non-holonomic systems are intrinsically nonlinear systems, in the sense that they are not exactly feedback linearizable, nor does their linear approximation retain the fundamental characteristics of the system, such as controllability (see [44]).

The system of rolling bodies considered here differs substantially from the class of chained form systems or differentially flat systems (see Rouchon [46]). Consider, for example, the case of the *plate-ball system* (i. e. a ball rolling on a plane without slipping or spinning, studied by Chaplygin in [17]), which is a classical problem in rational mechanics, brought to the attention of the control community by Brockett and Dai [9]. Montana [42] derived a differential-geometric model of the rolling constraint between general bodies, and discussed applications to robotic manipulation. Li and Canny [38] showed that the plate-ball system is controllable, and that the same holds for two rolling spheres, provided that their radii are different.

We close this subsection by mentioning the beautiful works of Jurdjevic [34, 35] who studied the problem of finding the path that minimizes the length of the curve traced out by the sphere on the fixed plane. It turns out that optimal paths also minimize the integral of their geodesic curvature, so that solutions are those of Euler's *elastica* problem. For the higher dimensional cases of this problem, see [36, 54].

4.3 Orbits and complete answer for controllability

The point of view adopted by Bryant and Hsu was improved significantly by Agrachev and Sachkov in [2] employing tools in geometric control theory.

Two innocent, yet powerful, changes in perspective made the problem more accessible for the application of the orbit theorem of Sussmann [51]. These modifications consist of rewriting the state space of the rolling and, most importantly, to prefer the use of vector fields (written in local coordinates) instead of differential forms (written without using coordinates).

Let M and $\hat{M}$ be smooth two-dimensional connected oriented Riemannian surfaces. The new version of the state space is given by

$$\begin{aligned} Q &= Q(M, \hat{M}) \\ &= \{A\colon T|_x M \to T|_{\hat{x}} \hat{M} \mid x \in M, \hat{x} \in \hat{M}, A \text{ is an oriented isometry}\}. \end{aligned}$$

It is an easy exercise to see that Q is indeed diffeomorphic to the manifold Q introduced in Sect. 4.1. The natural projection $Q \to M \times \hat{M}$ is a principal SO(2)-bundle. As before, a curve $\gamma\colon [0, \tau] \to Q$ describes a rolling motion if there is no slipping, that is, if $A(t)(\dot{x}(t)) = \dot{\hat{x}}(t)$ and there is no twisting (see [2])

$$A(t)\big(\text{vector field parallel along } x(t)\big) = \big(\text{vector field parallel along } \hat{x}(t)\big).$$

Let us now give expressions of the rolling distribution in local coordinates about a point $(x, \hat{x}; A) \in Q$. Let us consider local orthonormal frames e_1, e_2 for M and $\hat{e}_1, \hat{e}_2$ for $\hat{M}$. They define their structure constants $c_1, c_2 \in C^\infty(M)$ and $\hat{c}_1, \hat{c}_2 \in C^\infty(\hat{M})$ by the equations $[e_1, e_2] = c_1 e_1 + c_2 e_2$ on M and $[\hat{e}_1, \hat{e}_2] = \hat{c}_1 \hat{e}_1 + \hat{c}_2 \hat{e}_2$ on $\hat{M}$.

Since Q is a principal SO(2) bundle over $M \times \hat{M}$, in the natural trivialization, there is a well defined angular direction $\frac{\partial}{\partial \theta}$ and we can identify the isometry A with an angle θ. With these notations, the rolling distribution D_{R} is spanned by the vector fields

$$X_1 = e_1 + \cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2 + \big(-c_1 + \hat{c}_1 \cos\theta + \hat{c}_2 \sin\theta\big)\frac{\partial}{\partial\theta},$$

$$X_2 = e_2 - \sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_2 + \big(-c_2 - \hat{c}_1 \sin\theta + \hat{c}_2 \cos\theta\big)\frac{\partial}{\partial\theta}.$$

The main controllability theorem for the system of two Riemannian surfaces rolling, as presented in [3, Chap. 24], is the following.

Theorem 4 (Agrachev & Sachkov 1999). *Let $\mathcal{O} = \mathcal{O}_{D_{\mathrm{R}}}(q)$ be the orbit of the rolling distribution starting at $q \in Q$ and let κ and $\hat{\kappa}$ be the Gaussian curvatures of M and $\hat{M}$ respectively. Then:*

1) *the orbit $\mathcal{O}$ is a immersed connected submanifold of Q of dimension 2 or 5. More precisely, one has that if $(\kappa - \hat{\kappa})|_{\mathcal{O}}$ is identically zero, then $\dim \mathcal{O} = 2$; and if $(\kappa - \hat{\kappa})|_{\mathcal{O}}$ is not identically zero, then $\dim \mathcal{O} = 5$;*
2) *there is an injective correspondence between isometries $\iota\colon M \to \hat{M}$ and two dimensional orbits of the rolling problem. In particular, if the manifolds M and $\hat{M}$ are isometric, then the rolling problem is not completely controllable;*
3) *if M and $\hat{M}$ are complete and simply connected, then the correspondence between isometries $\iota\colon M \to \hat{M}$ and two dimensional orbits of the rolling problem is bijective. In particular, the rolling problem is completely controllable if and only if the manifolds M and $\hat{M}$ are not isometric.*

5 Re-discovery of the higher dimensional case and interpolation

Here we briefly review the way the higher dimensional problem of rolling manifolds presented to the control theory community and we explain how this was employed in geometric interpolation theory.

5.1 Sharpe's definition

Here we present the definition of rolling maps found in the Appendix B of Sharpe's book [48] with some minor modifications.

Definition 4. Let M, $\hat{M}$ be n-dimensional Riemannian submanifolds of $\mathbb{R}^{n+\nu}$. Then, a differentiable map $g : [0,\tau] \to \mathrm{Isom}(\mathbb{R}^{n+\nu}) \cong \mathrm{O}(n+\nu) \ltimes \mathbb{R}^{n+\nu}$ satisfying the following conditions:

- there is a piecewise smooth curve $x : [0,\tau] \to M$, such that
  - $g(t)x(t) \in \hat{M}$;
  - $T|_{g(t)x(t)}(g(t)M) = T|_{g(t)x(t)}\hat{M}$;
- furthermore, the curve $\hat{x}(t) := g(t)x(t)$ satisfies the following conditions
  - No-slip: $\dot{g}(t)g(t)^{-1}\hat{x}(t) = 0$;
  - No-twist, tangential part: $d(\dot{g}(t)g(t)^{-1})T|_{\hat{x}(t)}\hat{M} \subseteq T|_0(\dot{g}(t)g(t)^{-1}\hat{M})^{\perp}$;
  - No-twist, normal part: $d(\dot{g}(t)g(t)^{-1})T|_{\hat{x}(t)}\hat{M}^{\perp} \subseteq T|_0(\dot{g}(t)g(t)^{-1}\hat{M})$.

for any $t \in [0,\tau]$ is called a rolling map of M on $\hat{M}$ without slipping or twisting. We say that M rolls on $\hat{M}$ along the curve $x(t)$.

We do not know whether Sharpe was aware of the existence of the paper [45] at the time of the publication of his book, but his deduction of the "correct" definition rolling maps follows the same structure as Nomizu's. Nevertheless, Sharpe does obtain plenty of extra information. For example he shows that in the imbedded rolling problem there is a deep relation with the Levi-Civita connections of the manifolds and the normal connections to the imbeddings. Besides this, he is able to prove precisely that rolling is transitive, that is

Theorem 5 (Sharpe 1997). *Let $M_0, M_1, M_2 \subset \mathbb{R}^{n+\nu}$ be three n-dimensional Riemannian submanifolds, such that they are tangent to each other at a common point $p \in M_0 \cap M_1 \cap M_2$. Let $\gamma\colon [0,\tau] \to M$ be given such that $\gamma(0) = p$. Assume that M_1 rolls on M_0 along the curve γ, with rolling map g_1, and similarly let M_2 roll on M_1 along the curve $\hat{\gamma} = g_1\gamma$, with rolling map g_2. Then M_2 rolls on M_0 along the curve γ, with rolling map g_2g_1 and with image curve $\tilde{\gamma} = g_2g_1\gamma = g_2\hat{\gamma}$.*

5.2 *Applications to geometric interpolation*

An interesting application of the rolling problem has been in interpolation. The article where this idea appeared for the first time is [33] for the case of the two dimensional sphere. Afterward it was extended successfully to arbitrary dimensional spheres, Grassmanians and to the special orthogonal groups in [31]. This last application was employed in [32] to study the motion planning of a rotating satellite. Later on in [30] the idea was also shown to work on Stiefel manifolds.

The setting of the interpolation problem seems quite innocent. Let $x_0, x_1, \ldots, x_N \in M$ be measurements at times $0 = t_0 < t_1 < \cdots < t_N = \tau$, and consider given initial and final velocities $v \in T|_{x_0}M$ and $w \in T|_{x_N}M$. The interpolation problem consists in finding a C^2 curve $x\colon [0,\tau] \to M$ satisfying

$$x(t_i) = x_i, \quad \dot{x}(0) = v,\ \dot{x}(\tau) = w, \qquad \text{(INTERP)}$$

and γ minimizes the functional

$$J(x) = \frac{1}{2}\int_0^\tau \left\langle \frac{D}{dt}\dot{x}, \frac{D}{dt}\dot{x} \right\rangle dt, \tag{ENERGY}$$

where $\frac{D}{dt}$ is the covariant derivative. Let $\Omega = \{x \in C^2 \mid x \text{ satisfies (INTERP)}\}$. Then

Theorem 6 (Crouch & Silva Leite 1991). *If $x \in \Omega$ minimizes* (ENERGY)*, then*

$$\frac{D^3}{dt^3}\dot{x} + R\left(\frac{D}{dt}\dot{x}, \dot{x}\right)\dot{x} = 0,$$

on each $[t_i, t_{i+1}]$, where R is the curvature tensor of M.

The curves satisfying the differential equation in Theorem 6 are called geometric cubic splines, and they are in general quite hard to find. Nevertheless, in the cases described above, the authors were able to find a surprising relation between the rolling dynamics and geometric interpolation. The idea is to transform the interpolation problem in M to a classical cubic splines problem in $\mathbb{R}^n \cong T|_x M$ for some $x \in M$, and then go back to M appropriately. For simplicity of exposition, we only present the relevant results for the case of the n dimensional sphere S^n. A first observation that takes place is the following.

Theorem 7 (Jupp & Kent 1987, Hüper & Silva Leite 2007). *Consider S^n rolling on its tangent space $\mathbb{R}^n$, both imbedded in $\mathbb{R}^{n+1}$, along the curve $x\colon [0,\tau] \to S^n$ with rolling map $g(t) = (R^\top(t), s(t)) \in \mathrm{O}(n+1) \ltimes \mathbb{R}^{n+1}$. For all $t \in [0,\tau]$ and all $j \in \mathbb{N}$,*

$$R^\top(t)\,\frac{D^j}{dt^j}\dot{x}(t) = x_{\mathrm{dev}}^{(j+1)}(t),$$

where x_{dev} is the development of x, and $x_{\mathrm{dev}}^{(j+1)}$ is its $(j+1)$-st derivative in $\mathbb{R}^n$.

A consequence of the above is the following application to interpolation in S^n.

Corollary 1 (Jupp & Kent 1987, Hüper & Silva Leite 2007). *If the development $t \mapsto x_{\mathrm{dev}}(t)$ is an Euclidean cubic spline, then $t \mapsto x(t)$ is a geometric cubic spline on S^n if and only if it is a re-parameterized geodesic.*

6 Nowadays: The coordinate-free approach

The intrinsic definition of the rolling problem in higher dimensions was presented for the first time in [22,26]. It is clearly motivated by the definition given by Agrachev and Sachkov in [2].

Let (M, g) and $(\hat{M}, \hat{g})$ be two oriented n-dimensional Riemannian manifolds. The state space of the rolling problem is the manifold

$$\begin{aligned} Q &= Q(M, \hat{M}) \\ &= \{A : T|_x M \to T|_{\hat{x}}\hat{M} \mid x \in M,\ \hat{x} \in \hat{M},\ A \text{ is an oriented isometry}\}. \end{aligned}$$

An absolutely continuous curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in Q is a rolling curve if $A(t)X(t)$ is parallel along $\hat{\gamma}(t)$ for every vector field $X(t)$ that is parallel along $\gamma(t)$ (no twist condition) and if $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$ (no slip condition). These rolling curves are the *intrinsic* versions of the rolling maps introduced by Sharpe.

A counting argument shows that Q has dimension $\frac{1}{2}n(n+3)$. Over this manifold there is an n-dimensional distribution D_{R}, called the rolling distribution, such that the rolling curves in Q are exactly the integral curves of D_{R}. Let us describe this distribution briefly as given in [22]. For a configuration $q = (x, \hat{x}; A) \in Q$, and an initial velocity $X \in T|_x M$, we define the *rolling lift* $\mathscr{L}_{\mathrm{R}}(X)|_q \in T|_q Q$ as

$$\mathscr{L}_{\mathrm{R}}(X)|_q = \frac{d}{dt}\Big|_0 (P_0^t(\hat{\gamma}) \circ A \circ P_t^0(\gamma)), \tag{6}$$

where $\gamma, \hat{\gamma}$ are any smooth curves in $M, \hat{M}$, respectively, such that $\dot{\gamma}(0) = X$ and $\dot{\hat{\gamma}}(0) = AX$, and $P_a^b(\gamma)$ (resp. $P_a^b(\hat{\gamma})$) denotes the parallel transport along γ from $\gamma(a)$ to $\gamma(b)$ (resp. along $\hat{\gamma}$ from $\hat{\gamma}(a)$ to $\hat{\gamma}(b)$).

Definition 5. (cf. [22]). The *rolling distribution* D_{R} on Q is the n-dimensional smooth distribution defined, for $q = (x, \hat{x}; A) \in Q$, by $D_{\mathrm{R}}|_q = \mathscr{L}_{\mathrm{R}}(T|_x M)|_q$.

An interpretation of the rolling lift $\mathscr{L}_{\mathrm{R}}(X)|_q$ of $X \in T|_x M$ at $q = (x, \hat{x}; A)$ is as follows. Let γ be a curve in M such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$ then, by the general theory of ordinary differential equations, for short times there is a rolling curve $q(t)$ of M on $\hat{M}$ satisfying $q(0) = q$. The rolling lift is precisely $\dot{q}(0)$.

6.1 The controllability problem

The orbit $\mathcal{O}_{D_{\mathrm{R}}}(q)$ of the rolling problem described above passing through $q \in Q$ consists of all the states $\tilde{q}$ that can be connected to q via a rolling curve. The (complete) controllability problem asks for conditions on the geometries of M and $\hat{M}$ such that $\mathcal{O}_{D_{\mathrm{R}}}(q) = Q$. One way of addressing this problem is via Sussmann's orbit theorem, that is, by showing that all the Lie brackets of the vector fields steering the dynamics have to span the tangent bundle of the state space. For the rolling problem, this Lie brackets are expressed in terms of the curvature tensors R and $\hat{R}$ associated to the Riemannian metrics g on M and $\hat{g}$ on $\hat{M}$ respectively, together with the covariant derivatives of R and $\hat{R}$. It seems therefore impossible to solve for general dimension n the controllability issue on the sole knowledge of the Lie algebraic structure of D_{R}, except for low dimensions. Indeed, in the case for instance where $(\hat{M}, \hat{g})$ is the n-dimensional Euclidean space, it would amount to determining $\mathrm{Hol}(M)$, see Sect. 2.3, with the only knowledge of its curvature tensor and its covariant derivatives. Instead, the latter issue can be successfully addressed by resorting to group theoretic and algebraic arguments, see [22]. For specific examples, using extra knowledge of the problem at hand, see [26, 36, 54].

In general, one can define a notion of curvature especially adapted to the rolling problem, see [23]. For $q = (x, \hat{x}; A) \in Q$, the *rolling curvature* is the linear map

$$\mathrm{Rol}_q : \bigwedge^2 T|_x M \to T^*|_x M \otimes T|_{\hat{x}} \hat{M}; \ \mathrm{Rol}_q(X \wedge Y) := AR(X, Y) - \hat{R}(AX, AY)A.$$

The normalized map $\widetilde{\mathrm{Rol}}_q(X \wedge Y) = R(X, Y) - A^{-1}\hat{R}(AX, AY)A$ is an endomorphism of $\bigwedge^2 T|_x M$. This map permits to give a first sufficient condition for the rolling problem to be controllable, see [22, 28].

Theorem 8 (Chitour & Kokkonen 2011, Grong 2012). *If* $\widetilde{\mathrm{Rol}}_q$ *is an isomorphism for every* $q \in Q$*, then the rolling problem is completely controllable.*

The above condition is very hard to deduce directly from conditions on the geometry of M and $\hat{M}$. It is therefore necessary to reduce the problem to a simpler one. One possible way to do this is to give some extra structure to the manifold $\hat{M}$. In this vein, it was possible to give controllability conditions "without Lie brackets" for the case in which $(\hat{M}, \hat{g}) = (\mathbb{F}_c^n, \mathrm{g}_c^n)$ is the space form of constant sectional curvature c, see [37, 47]. To state these, let us first introduce some terminology.

Definition 6. Consider the vector bundle $\pi_{TM \oplus \mathbb{R}} : TM \oplus \mathbb{R} \to M$. The rolling connection ∇^c is the vector bundle connection on $\pi_{TM \oplus \mathbb{R}}$ defined by

$$\nabla_X^c(Y, s) = \Big(\nabla_X Y + s(x)X, X(s) - cg\big(Y|_x, X\big)\Big), \tag{7}$$

for every $x \in M$, $X \in T|_x M$, $(Y, s) \in \mathrm{VF}(M) \times C^\infty(M)$; where we have canonically identified the space of smooth sections $\Gamma(\pi_{TM \oplus \mathbb{R}})$ of $\pi_{TM \oplus \mathbb{R}}$ with $\mathrm{VF}(M) \times C^\infty(M)$.

When $c \neq 0$, the connection ∇^c is a vector bundle connection which is metric with respect to the fiber inner product h_c on $TM \oplus \mathbb{R}$ defined by

$$h_c((X, r), (Y, s)) = g(X, Y) + c^{-1} rs,$$

where $X, Y \in T|_x M$, $r, s \in \mathbb{R}$. The holonomy group of the connection ∇^c with respect to h_c is denoted by $\mathcal{H}^c(M)$. In this language, we have the following result, see [23].

Theorem 9 (Chitour & Kokkonen 2012). *Let* (M, g) *be a complete, oriented and simply connected Riemannian manifold. The rolling problem of* M *rolling on* $\mathbb{F}_c^n$ *is completely controllable if and only if*

$$\mathcal{H}^c(M) = \begin{cases} \mathrm{SO}(n+1), & c > 0; \\ \mathrm{SE}(n), & c = 0; \\ \mathrm{SO}_0(n, 1), & c < 0. \end{cases}$$

Here the Lie group $\mathrm{SO}_0(n, 1)$ *represents the identity component of the group* $\mathrm{O}(n, 1)$ *of linear transformations that preserve the quadratic form* $F_{n,1}(x_1, \dots, x_{n+1}) = x_1^2 + \dots + x_n^2 - x_{n+1}^2$.

Wanting to fully understand these cases, it is important to remark some structure theorems encoded in Theorem 9. Observe that up to rescaling, it is sufficient to study when $c = 0$, 1 and -1. In the Euclidean situation, i. e. $c = 0$, the condition $\mathscr{H}^0(M) = \mathrm{SE}(n)$ is equivalent to that M has full Riemannian holonomy. In the case $c = 1$, if the action of $\mathscr{H}^1(M)$ on the unit sphere is not transitive, then (M, g) is the unit sphere. As a consequence, it holds that, for $n \geq 16$ and even, the rolling problem $Q = Q(M, S^{n-1})$ is completely controllable if and only if (M, g) is not isometric to the unit sphere. Both theses cases were analyzed in [23], and the remaining cases are currently under investigation. The hyperbolic case presented a more difficult challenge, see [19].

Theorem 10 (Chitour, Godoy & Kokkonen 2012). *Let (M, g) be a complete, oriented and simply connected Riemannian n-manifold rolling on the space form $(\mathbb{H}^n, \mathsf{g}^n_{-1})$ of curvature -1. Then the associated rolling problem is completely controllable if and only if (M, g) is not isometric to a warped product of the form*

(WP1) $(\mathbb{R} \times M_1, ds^2 \oplus_{e^{cs}} g_1)$, *or*
(WP2) $(\mathbb{H}^k \times M_1, \mathsf{g}^k_{-1} \oplus_{\cosh(\sqrt{-c}\, d)} g_1)$, *where $1 \leq k \leq n$ and for each $x \in \mathbb{H}^k$, $d(x)$ is the distance between x and an arbitrary fixed point $x_0 \in \mathbb{H}^k$.*

In both situations, (M_1, g_1) is some complete simply connected Riemannian manifold. As usual, the term ds^2 represents the usual Riemannian metric on $\mathbb{R}$.

6.2 Symmetries of the rolling problem

The idea developed in Sect. 6.1 of setting $\hat{M}$ to be a space form has a beautiful geometric consequence on the bundle structure of the natural projection $\pi_{Q,M} \colon Q(M, \mathbb{F}^n_c) \to M$. Let us explain what this is.

In general, it is not clear if there is a G-principal bundle structure on $Q = Q(M, \hat{M})$ making D_{R} a G-principal bundle connection for some Lie group G. This is indeed the case if the manifolds are of dimension 2, in which case the projection $Q \to M \times \hat{M}$ is a principal SO(2) bundle with D_{R} as its connection, but an analogous statement in higher dimensions does not hold, see [22, Proposition 3.4]. In order to find a Lie group G acting on Q so that D_{R} is a G-principal bundle connection, we need to consider space forms $\mathbb{F}^n_c$ in the place of $\hat{M}$.

For $c \neq 0$, let $G_c(n)$ be the identity component of the Lie group of linear maps that leave invariant the bilinear form $\langle x, y\rangle^n_c := \sum_{i=1}^n x_i y_i + c^{-1} x_{n+1} y_{n+1}$, for $x = (x_1, \ldots, x_{n+1})$, $y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$. Observe that $G_1(n) = \mathrm{SO}(n + 1)$ and $G_{-1}(n) = \mathrm{SO}_0(n, 1)$. For $c = 0$, we set $G_0(n) = \mathrm{SE}(n)$. Recall that, with this notation, the identity component of the isometry group of $(\mathbb{F}^n_c, \mathsf{g}^n_c)$ is equal to $G_c(n)$ for all $c \in \mathbb{R}$ (cf. [37]).

The fundamental result concerning rolling on a space form lies in the fact that there is a $G_c(n)$-principal bundle structure for the state space compatible with the distribution D_{R}, i. e., D_{R} is a $G_c(n)$-principal bundle connection, see [23]. The precise result follows.

Theorem 11 (Chitour & Kokkonen 2012). *Let $Q = Q(M, \mathbb{F}_c^n)$ be the state space of rolling M on the space form $\mathbb{F}_c^n$. Then we have:*

(i) *The projection $\pi_{Q,M} : Q \to M$ is a principal $G_c(n)$-bundle with a left action $\mu : G_c(n) \times Q \to Q$ defined for every $q = (x, \hat{x}; A)$ by*

$$\mu((\hat{y}, C), q) = (x, C\hat{x} + \hat{y}; C \circ A), \text{ if } c = 0,$$
$$\mu(B, q) = (x, B\hat{x}; B \circ A), \text{ if } c \neq 0.$$

Moreover, the action μ preserves the distribution D_{R}, i. e., for any $q \in Q$ and $B \in G_c(n)$, $(\mu_B)_* D_{\mathrm{R}}|_q = D_{\mathrm{R}}|_{\mu(B,q)}$, where $\mu_B : Q \to Q; q \mapsto \mu(B, q)$.

(ii) *For any given $q = (x, \hat{x}; A) \in Q$, there is a unique subgroup $\mathcal{H}_q^c$ of $G_c(n)$, called the holonomy group of D_{R} at $q \in Q$, such that*

$$\mu(\mathcal{H}_q^c \times \{q\}) = \mathcal{O}_{D_{\mathrm{R}}}(q) \cap \pi_{Q,M}^{-1}(x).$$

Also, if $q' = (x, \hat{x}'; A') \in Q$ is in the same $\pi_{Q,M}$-fiber as q, then $\mathcal{H}_q^c$ and $\mathcal{H}_{q'}^c$ are conjugate in $G_c(n)$ and all conjugacy classes of $\mathcal{H}_q^c$ in $G_c(n)$ are of the form $\mathcal{H}_{q'}^c$.

The holonomy group $\mathcal{H}_q^c$ of D_{R} at $q \in Q$ is in fact isomorphic to the holonomy group $\mathcal{H}^c(M)$ of the rolling connection ∇^c, see [23].

A natural question to ask is whether a converse of the theorem above holds, in other words, does the existence of a G-principal bundle structure on Q such that D_{R} is a connection imply that $\hat{M}$ must have constant sectional curvature? The answer is generically yes, but we need to introduce some more terminology.

Recall that in Sect. 2.2 we defined the Lie algebra of symmetries $\mathrm{Sym}(D)$ of a distribution D on a manifold $\tilde{M}$ as the set of vector fields $X \in \mathrm{VF}(\tilde{M})$ that satisfy $[X, D] \subseteq D$. For the case of the rolling distribution, we will focus our attention to the symmetries of the rolling distribution that are annihilated by the projection $\pi_{Q,M} \colon Q \to M$, that is, in the Lie algebra

$$\mathrm{Sym}_0(D_{\mathrm{R}}) := \{S \in \mathrm{Sym}(D_{\mathrm{R}}) \mid (\pi_{Q,M})_* S = 0\}.$$

With this at hand, the mentioned converse takes the following form, see [20].

Theorem 12 (Chitour, Godoy & Kokkonen 2012). *If there is an open dense set $O \subset Q$ such that $R|_x : \bigwedge^2 T|_x M \to \bigwedge^2 T|_x M$ is invertible on $\pi_{Q,M}(O)$ and $\widetilde{\mathrm{Rol}}_q$ is invertible for all $q \in O$, then, up to an isomorphism of Lie-algebras,*

$$\mathrm{Sym}_0(D_{\mathrm{R}}) = \mathrm{Isom}(\hat{M}, \hat{g})$$

and therefore all the elements of $\mathrm{Sym}_0(D_{\mathrm{R}})$ are induced by Killing fields of $(\hat{M}, \hat{g})$.

In particular, under the above assumptions, if there is a principal bundle structure on $\pi_{Q,M} : Q \to M$ that renders D_{R} to a principal bundle connection, then $(\hat{M}, \hat{g})$ is a space of constant curvature.

6.3 Generalizations and perspectives

Two natural questions to ask concern the extension of the rolling problem to the situation in which the manifolds involved have different dimension and to to extend the classification result in Sect. 4.3 to other cases. For the first question, one needs to consider curves of isometric injections instead of isometries. This change introduces many difficulties in understanding the controllability problem, and in fact many tools that work well in the classical situation can not be generalized. The second question has a satisfactory answer for the three dimensional case, see [22]. There it is shown that the orbits can have dimensions 3, 6, 7, 8 and 9.

A question that has been in our minds for a while is to actually compare the manifolds via the rolling problem. This idea of comparison is naively evident in the rolling curvature tensor: one is actually subtracting the Riemannian curvatures of the manifolds. In fact, rolling should provide a framework for the isometric characterization of manifolds by using curvature tensor spectrum information (as in Osserman-type conditions for instance, cf. [25]).

Finally, we have noticed that the problem of rolling manifolds can be generalized far beyond than allowing arbitrary connections, as in [28, Sect. 7], or to pseudo-Riemannian manifolds, as in [41]. This extension consists of rolling so-called Cartan geometries, see [48], and it includes as particular cases both of the situations mentioned above, together with the problem of rolling manifolds of different dimensions, see [21]. The main idea behind this is that Cartan geometries are the most general framework for a notion of development to exists, which underlies the very definition of the rolling dynamics. So far this generalized model has resisted a thorough study of controllability.

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Optimal stationary exploitation of size-structured population with intra-specific competition

Alexey A. Davydov and Anton S. Platov

Abstract We analyze an exploitation of size-structured population in stationary mode and prove the existence of stationary state of population for a given stationary control. The existence of an optimal control is proved and the necessary optimal condition is found.

1 Introduction

Modeling of exploitation of size-structured populations and search for an optimal exploitation mode lead to interesting mathematical problems, reasonable results in which are in demand in economic and environmental applications. Even for a one-specy population these problems are essentially nonlinear since the effect of intra-specific competition in the growth and mortality of individuals usually differs significantly for individuals of different sizes. This makes the task to develop methods for optimizing heterogeneous distributed systems (see, for example, [1]).

In this paper we study steady-state modes of operation of common types, and analyze the corresponding stationary states of the population. In the model under consideration the dynamics of population is described by the equation

$$\frac{\partial x(t,l)}{\partial t} + \frac{\partial\big[g(l,E(t))x(t,l)\big]}{\partial l} = -[\mu(l,E(t)) + u(l)]x(t,l), \qquad (1)$$

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where $x(t,l)$ is the resource density, namely, the average density of size l individuals at the moment t, g and μ are respectively the growth and mortality rates, and a control function u accounts for the exploitation intensity, namely, it measures the proportion of the population withdrawn per unit time. The function E characterizes the intra-specific competition and has the form

$$E(t) = \int_0^L \chi(l)x(t,l)dl, \tag{2}$$

where χ is a continuous increasing positive function on the interval $[0, L]$, $L > 0$. This is the interval of the sizes we manage and exploit in the population. Similar functions of growth and mortality rates were used in [2].

We assume that the functions g and μ are continuous and positive on the interval $[0, L]$ for all non-negative values of the second arguments.

The inflow of new individuals is defined by the boundary condition which is the sum of the natural reproduction and the density p of the industrial renewal population

$$x(t,0) = \int_0^L r(l, E(t))x(t,l)dl + p(t). \tag{3}$$

Here r is the birth rate. It is natural to assume that the rate r is a non-negative continuous function which is positive for sufficiently big sizes l. Note that this inflow of individuals is different from the one in [3], where in the integrand instead $x(t,l)$ we had $(x(t,l))^\beta$ with $0 < \beta < 1$.

We assume that for $E_1 < E_2$ the growth, birth and mortality rates satisfy the following conditions:

$$g(., E_1) \geq g(., E_2), \quad r(., E_1) \geq r(., E_2), \quad \mu(., E_1) \leq \mu(., E_2), \tag{4}$$

$$\frac{g(0, E_1)}{g(l, E_1)} \geq \frac{g(0, E_2)}{g(l, E_2)}. \tag{5}$$

These conditions are coming from natural constraints. Namely, inequalities (4) mean that, with increase of the intra-specific competition, the growth and birth rates cannot go up while the mortality rate cannot go down. The sense of condition (5) is that the influence of smaller size individuals does not go down when the exponent E increases.

We prove that, under these conditions on a selected measurable intensity of exploitation and for a given control u and a constant positive planting $p(t) \equiv p_0 > 0$, there exists a nontrivial stationary solution $x = x(l, E)$ to the model (1)–(3). We also show that a control providing the maximum profit exists, and find a corresponding necessary optimality condition. The formulation of the results and their proofs are respectively in Sects. 2 and 3.

2 Main results

2.1 Existence of a stationary solution

A measurable control u is called *admissible* if

$$u_1(l) \le u(l) \le u_2(l), \quad l \in [0, L], \tag{6}$$

where u_1 and u_2 are some positive continuous functions on $l \in [0, L]$. These functions characterize technological or ecological constraints.

Theorem 1. *Let* $p(t) = p_0 > 0$ *be constant, and* g, μ *and* r *be continuous functions satisfying conditions* (4) *and* (5)*. Then, for any given admissible control* u*, there exists a unique positive stationary solution to the problem* (1)–(3) *provided the inequality*

$$\int_0^L r(l,0)\frac{g(0,0)}{g(l,0)}e^{-\int_0^l \frac{\mu(s,0)+u(s)}{g(s,0)}ds}dl < 1 \tag{7}$$

takes place.

Theorem 1 is proved in the Sect. 3.1, where the stationary solution is also found. It has the form

$$x(l, E) = \frac{x_0 g(0, E)}{g(l, E)}e^{-\int_0^l m(s,E)ds} \quad \text{with} \quad m(s, E) = \frac{\mu(s, E) + u(s)}{g(s, E)}, \tag{8}$$

and with the value x_0 defined by the formula

$$x_0 = p_0 \Big/ \left(1 - \int_0^L r(l, E)\frac{g(0, E)}{g(l, E)}e^{-\int_0^l m(s,E)ds}dl\right). \tag{9}$$

Remark 1. For the growth and mortality rates satisfying the Lipschitz condition and $u \equiv 0$, the existence of a stationary solution was proved in [4].

2.2 Optimal stationary solution

Objective functionals to define optimal stationary controls may be different in different settings. In our case the objective functional has the form

$$\int_0^L c(l)u(l)x(l, E)dl + c_L x(L, E) - p_0 c_0, \tag{10}$$

and it accounts for the economic and environmental costs and benefits in terms of the aggregated prices c, c_L, c_0. The last term in (10) is control-independent, and we remove it because it has no influence on the selection of an optimal control. Now substitution of solution (8) into (10) brings the objective functional to the form

$$x_0 g(0,E)\left[\int_0^L c(l)e^{H(l,E)-\phi(l,E)}\,d\phi(l,E)+\frac{c_L e^{H(L,E)-\phi(L,E)}}{g(L,E)}\right], \tag{11}$$

where

$$H(l,E)=-\int_0^l \frac{\mu(s,E)}{g(s,E)}dl, \quad \phi(l,E)=\int_0^l \frac{u(s)}{g(s,E)}ds. \tag{12}$$

Thus our task is to find an admissible control which maximizes the functional (11).

Theorem 2. *Assume the price c is a continuous function, the continuous growth, birth and mortality rates g, r, μ satisfy conditions* (4), (5) *and* (7)*, and the functions g and μ are positive. Then there exists an admissible control providing a maximum of the functional* (11).

Theorem 2 is proved in Sect. 3.2.

2.3 Necessary optimality condition

An appropriate necessary optimality condition is one of the best tools for constructive search for an optimal control. To describe our optimality condition we introduce the following functions:

$$M(l_1,l_2,E)=\int_{l_1}^{l_2} \chi(l)\frac{g(0,E)}{g(l,E)}e^{-\int_0^l m(s,E)ds}dl, \tag{13}$$

$$H(l_1,l_2,E)=\int_{l_1}^{l_2} r(l,E)\frac{g(0,E)}{g(l,E)}e^{-\int_0^l m(s,E)ds}dl, \tag{14}$$

$$F(x_0,E)=E-x_0 M(0,L,E), \tag{15}$$

$$G(x_0,E)=x_0-p_0-x_0 H(0,L,E), \tag{16}$$

$$I(l_1,l_2,E)=\int_{l_1}^{l_2} c(l)u(l)\frac{g(0,E)}{g(l,E)}e^{-\int_0^l m(s,E)ds}dl + c_L\frac{g(0,E)}{g(L,E)}e^{-\int_0^L m(s,E)ds}, \tag{17}$$

$$\mathfrak{M}(x_0,E):=\begin{pmatrix} F'_E(x_0,E) & F'_{x_0}(x_0,E)\\ G'_E(x_0,E) & G'_{x_0}(x_0,E)\end{pmatrix}. \tag{18}$$

Here $m(s,E)$ is defined by (8), $[l_1,l_2]\subset[0,L]$, and the values x_0 and E are independent.

Theorem 3. *Suppose that, under the assumptions of Theorem 2 and of differentiability of functions r,μ,g, an admissible control u maximizes the functional* (10)*. Then for any point $l_0\in[0,L]$, at which $u_1(l_0)\neq u_2(l_0)$ and the control u is the* *derivative of its integral, the value*

$$2\frac{I(0,L,E)}{\det\mathfrak{M}}\left(F'_E H(l_0,L,E) - G'_E M(l_0,L,E)\right) + 2\frac{x_0 I'_E(0,L,E)}{\det\mathfrak{M}}\left(G'_{x_0} M(l_0,L,E) - F'_{x_0} H(l_0,L,E)\right) + c(l_0)g(0,E)e^{-\int_0^{l_0} m(s,E)ds} - I(l_0,L,E) \quad (19)$$

is non-positive, non-negative or equal to zero, if $u(l_0)$ *is respectively equal to* $u_1(l_0)$ *or* $u_2(l_0)$*, or belongs to the interval* $(u_1(l_0), u_2(l_0))$.

Theorem 3 is proved in Sect. 3.3.

Remark 2. The function (19) plays the role of a switching function. It is not convenient to calculate an optimal control because its value at l_0 depends on the integrals over the segment $[l_0, L]$. However, after simple transformation it may be rewritten in the form

$$A + B\cdot H(0,l_0,E) + C\cdot M(0,l_0,E) + I(0,l_0,E) + c(l_0)g(0,E)e^{-\int_0^{l_0} m(s,E)ds}, \quad (20)$$

where

$$\begin{aligned} A &= \frac{2}{\det\mathfrak{M}}\Big(I(0,L,E)\big(F'_E H(0,L,E) - G'_E M(0,L,E)\big) \\ &\quad + x_0 I'_E(0,L,E)\big(G'_{x_0} M(0,L,E) - F'_{x_0} H(0,L,E)\big)\Big) - I(0,L,E), \\ B &= \frac{-2}{\det\mathfrak{M}}\left(F'_E I(0,L,E) - x_0 F'_{x_0} I'_E(0,L,E)\right), \\ C &= \frac{2}{\det\mathfrak{M}}\left(G'_E I(0,L,E) - x_0 G'_{x_0} I'_E(0,L,E)\right). \end{aligned} \quad (21)$$

This form of a switching function is suitable for creating numerical algorithm to search for an optimal control.

3 Proof of the theorems

Here we prove Theorems 1–3 in turns.

3.1 Proof of Theorem 1

The stationary solution x, $x = x(l,E)$, satisfies Eq. (1) in the form

$$\frac{d\left[g(l,E)x(l,E)\right]}{dl} = -[\mu(l,E) + u(l)]x(l,E), \quad (22)$$

where E is the corresponding constant value of the exponent calculated according to (2). Though E depends on the solution x, let us first consider E as an independent parameter. In such case the solution of the last equation is easy to find. It has the form

$$x(l, E) = \frac{x(0, E)g(0, E)}{g(l, E)} e^{-\int_0^l m(s,E)ds}, \tag{23}$$

where m is defined by (8). Substitution of the expression (23) into (3) leads to an equation for the value $x_0 := x(0, E)$:

$$x_0 = x_0 \int_0^L r(l, E) \frac{g(0, E)}{g(l, E)} e^{-\int_0^l m(s,E)ds} dl + p_0. \tag{24}$$

From the expression (24) we immediately get (9). The value x_0 is positive, and so it makes sense as an initial population density, once the denominator in the expression (9) is positive. This is always true under conditions (4) and (5) provided the inequality (7) holds. In such a case the solution has the form (8) where x_0 is defined by (9).

The following two statements are useful.

Lemma 1. *Assume conditions* (4)–(5) *and* (7) *hold. Then for any* $l \in [0, L]$ *solution* (8) *is a non-increasing function of* $E \in [0, \infty)$.

Corollary 1. *Assume conditions (4)-(5) and (7) hold. Then the function*

$$f(E) := \int_0^L \chi(l)x(l, E)dl$$

is a continuous non-increasing positive function of $E \in [0, \infty)$.

We shall now finish the proof of the theorem and then prove the lemma and its corollary. The function f is non-increasing on the interval $[0, f(0)]$, and its value varies between $f(0)$ and $f(f(0))$. Hence the difference $E - f(E)$ increases on this interval, and is negative at $E = 0$ and non-negative at $E = f(0)$. Hence there is only one value $E_0 > 0$ at which the difference vanishes. It is clear that E_0 belongs to the interval $[0, f(0)]$.

As it is easy to see, for the solution $x(., E_0)$ we have $E = E_0$. Therefore $x(., E_0)$ is a stationary solution required. It is clear that this solution is uniquely defined.

Theorem 1 is proved modulo Lemma 1 and Corollary 1. The statement of this corollary follows immediately from the lemma. Let us prove the lemma.

Due to the conditions, the integrand in Eq. (9) is a non-increasing function of E. Hence the value x_0 is also a non-increasing function of E on the interval $[0, \infty)$. This immediately implies that the solution is also a non-increasing function on E on this interval. Thus, Lemma 1 is valid.

3.2 Proof of Theorem 2

Firstly, we prove the following useful statement

Lemma 2. *Under the conditions of Theorem 2, profit* (11) *is a bounded functional on the space of admissible controls.*

Proof. Indeed, we have

$$\left| x_0 g(0,E)\left[\int_0^L c(l)e^{H(l,E)-\phi(l,E)}d\phi(l,E)+\frac{c_L e^{H(l,E)-\phi(L,E)}}{g(L,E)}\right]\right|$$
$$\le x_0 g(0,0)\left(\left|\int_0^L c(l)e^{-\phi(l,E)}d\phi(l,E)\right|+\frac{c_L}{g(L,f(0))}\right)$$
$$\le x_0 g(0,0)\left(C+\frac{c_L}{g(L,f(0))}\right)<\infty\,.$$

It is clear that the value x_0 here is bounded, and the values $g(0,0)$, $g(L, f(0))^{-1}$ and $C = \max\{c(l) : l \in [0, L]\}$ are finite due to the constraints imposed. Hence the statement of the lemma follows.

Consider now the exact upper bound of possible values of the objective functional and take a sequence $u = v_k$ of admissible controls, for which the values of the functional converge to this bound when $k \to \infty$. Denote the corresponding values of the competition parameter by E_k.

All possible values of the competition parameter are also bounded and hence there exists a subsequence $E_{k_j} \to E_\infty$ with $k_j \to \infty$. Without loss of generality we assume that $E_k \to E_\infty$ as $k \to \infty$.

Thus we have pairs $\{v_k, E_k\}$ whose second component has a limit as $k \to \infty$.

For the controls v_k and any $l_1, l_2 \in [0, L]$, $l_1 \le l_2$, the corresponding sequence ϕ_k satisfies the inequalities

$$\int_{l_1}^{l_2}\frac{u_1(l)}{g(l,E_k)}dl \le \phi_k(l_2,E_k)-\phi_k(l_1,E_k)\le \int_{l_1}^{l_2}\frac{u_2(l)}{g(l,E_k)}dl\,. \qquad (25)$$

This is easy to see. In particular, all the ϕ_k satisfy the Lipschitz condition with the constant equal to the maximum of the function $u_2(\cdot)/g(\cdot, f(0))$ on the interval $[0, L]$. Consequently, the set of the functions ϕ_k is bounded and equicontinuous on this interval. Hence, due to the Arzela-Ascoli theorem [5], there exists a subsequence $\{\phi_{k_n}\}$ that converges uniformly to some function ϕ_∞ when $k_n \to \infty$.

The profit functional (11) depends continuously on ϕ and E. Hence this functional attains its maximum value at $\phi = \phi_\infty$ and $E = E_\infty$.

To complete the proof we have to find an admissible control u_∞ which provides the limit function ϕ_∞ by the formula (12). This function satisfies the inequalities (25) and is absolutely continuous. Hence its derivative exists almost everywhere on the interval $[0, L]$. The derivative satisfies the inequality

$$\frac{u_1(l)}{g(l, E_\infty)} \le \phi'_\infty(l, E_\infty) \le \frac{u_2(l)}{g(l, E_\infty)}$$

wherever it exists. Consequently, we can define the control u_∞ by the formula

$$u_\infty(l) = g(l, E_\infty)\phi'_\infty(l, E_\infty)$$

at any such point, and assign u_∞ any value between the values of u_1 and u_2 at any other point of this interval.

Theorem 2 is proved.

3.3 *Proof of Theorem 3*

The proof is based on the direct calculation of the first variation for the functional (10).

Consider an exploitation intensity providing the maximum of the functional (10), and a point $l_0 \in (0, L)$ at which the intensity is the derivative of its integral and $u_1(l_0) \ne u_2(l_0)$. Choose a sufficiently small $\delta > 0$ so that the interval $[l_0, l_0 + \delta] \subset [0, L]$, and take a perturbed exploitation intensity $\tilde{u}$ so that the difference $\tilde{u} - u$ is zero outside the interval $[l_0, l_0 + \delta]$ and has a small value h inside it.

The corresponding variations ΔE and Δx_0 may be found from the equations

$$\begin{aligned} F'_E \Delta E + F'_{x_0} \Delta x_0 + F'_\delta \delta + F'_h h + o(h\delta)&=0, \\ G'_E \Delta E + G'_{x_0} \Delta x_0 + G'_\delta \delta + G'_h h + o(h\delta)&=0. \end{aligned} \tag{26}$$

Here and below the $o(h\delta)$ stay for functions infinitesimal compared with $h\delta$ as $h\delta \to 0$. The third and fourth terms in these equations are

$$\begin{aligned} F'_\delta \delta = F'_h h &= -\frac{x_0 h\delta}{g(l_0, E)} M(l_0, L, E) + o(h\delta), \\ G'_\delta \delta = G'_h h &= -\frac{x_0 h\delta}{g(l_0, E)} H(l_0, L, E) + o(h\delta). \end{aligned} \tag{27}$$

The following statement is useful.

Lemma 3. *For differentiable functions r, μ, g, the matrix* (18) *is non-degenerate for all positive values E and x_0, if conditions* (4), (5) *and* (7) *hold.*

Proof. As it is easy to see,

$$\mathfrak{M}(x_0, E) = \begin{pmatrix} 1 - x_0 M'_E(0, L, E) & -M(0, L, E) \\ -x_0 H'_E(0, L, E) & 1 - H(0, L, E) \end{pmatrix}. \tag{28}$$

Due to the conditions (4), (5) and (7), the entries on the main diagonal are positive and the product of the other two is non-negative. Hence $\det \mathfrak{M} > 0$. Therefore, the matrix is non-degenerate.

Lemma 3 is proved. □

Hence the system (26) may be solved for ΔE and Δx_0. The solution is

$$\Delta E = \frac{2x_0 h\delta [G'_{x_0} M(l_0, L, E) - F'_{x_0} H(l_0, L, E)]}{g(l_0, E) \det \mathfrak{M}} + o(h\delta), \tag{29}$$

$$\Delta x_0 = \frac{2x_0 h\delta [F'_E H(l_0, L, E) - G'_E M(l_0, L, E)]}{g(l_0, E) \det \mathfrak{M}} + o(h\delta). \tag{30}$$

We are now ready to calculate the first variation of our objective functional. The variation has the form

$$\Delta x_0 \cdot I(0, L, E) + x_0 \Delta E \cdot I_E(0, L, E) + \\ + \frac{h\delta x_0}{g(l_0, E)} \left(c(l_0) g(0, E) e^{-\int_0^{l_0} m(s,E)ds} - I(l_0, L, E) \right) + o(h\delta). \tag{31}$$

Substitution into (31) of the expressions (29) and (30) gives

$$\frac{x_0 h\delta}{g(l_0, E)} \left[\frac{2I'_E(0, L, E) x_0 [G'_{x_0} M(l_0, L, E) - F'_{x_0} H(l_0, L, E)]}{g(l_0, E) \det \mathfrak{M}} + \right. \\ + \frac{2I(0, L, E)[F'_E H(l_0, L, E) - G'_E M(l_0, L, E)]}{g(l_0, E) \det \mathfrak{M}} + \\ \left. + c(l_0) g(0, E) e^{-\int_0^{l_0} m(s,E)ds} - I(l_0, L, E) \right] + o(h\delta). \tag{32}$$

Since $\frac{x_0 \delta}{g(l_0, E)} > 0$, the sign of the first variation is determined by the sign of h once the quantity in the square brackets in (32) is not zero. For a control u providing the maximum of the objective functional and its admissible perturbation done, this difference must be non-positive. Hence the quantity in the square brackets must be non-positive when $u(l_0) = u_1(l_0)$, non-negative when $u(l_0) = u_2(l_0)$, and zero if $u_1(l_0) < u(l_0) < u_2(l_0)$. Indeed, in these three cases an admissible value of h can respectively be any sufficiently small positive, negative or both positive and negative.

Theorem 3 is proved.

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On geometry of affine control systems with one input

Boris Doubrov and Igor Zelenko

Abstract We demonstrate how the novel approach to the local geometry of structures of nonholonomic nature, originated by Andrei Agrachev, works for rank 2 distributions of maximal class in $\mathbb{R}^n$ with additional structures such as affine control systems with one input spanning these distributions, sub-(pseudo)Riemannian structures etc. In contrast to the case of an arbitrary rank 2 distribution without additional structures, in the considered cases each abnormal extremal (of the underlying rank 2 distribution) possesses a distinguished parametrization. This fact allows one to construct the canonical frame on a $(2n-3)$-dimensional for arbitrary $n \geq 5$. The moduli spaces of the most symmetric models are described as well.

1 Introduction

About seventeen years ago Andrei Agrachev proposed the idea to study the local geometry of control systems and geometric structures on manifolds by studying the flow of extremals of optimal control problems naturally associated with these objects [1–3]. Originally he considered situations when one can assign a curve of Lagrangian subspaces of a linear symplectic space or, in other words, a curve in a Lagrangian Grassmannian to an extremal of these optimal control problems. This curve was called the Jacobi curve of this extremal, because it contains all information about the solutions of the Jacobi equations along it. Agrachev's constructions of Jacobi curves worked in particular for normal extremals of sub-Riemannian structures and abnormal extremals of rank 2 distributions. Similar idea can be used for

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abnormal extremals of distribution of any rank, resulting in more general curves of coisotropic subspaces in a linear symplectic space [11,15].

The key point is that the differential geometry of the original structure can be studied via differential geometry of such curves with respect to the action of the linear symplectic group. The latter problem is simpler in many respects than the original one. In particular, any symplectic invariants of the Jacobi curves produces the invariant of the original structure.

This idea proved to be very prolific. For the geometry of distributions, first it led to a new geometric-control interpretation of the classical Cartan invariant of rank 2 distributions on a five dimensional manifold, relating it to the classical Wilczynski invariants of curves in projective spaces [4,23,24]. It also gave a new effective method of the calculation of the Cartan tensor and the generalization of the latter invariant to rank 2 distributions on manifolds of arbitrary dimensions. These new invariants are obtained from the Wilczynski invariants of curves in projective spaces, induced from the Jacobi curves by a series of osculations together with the operation of taking skew symmetric complements. They are called the *generalized Wilczynski invariants of rank 2 distributions* (see Sect. 5 for details).

Later on, we used this approach for the construction of the canonical frames for rank 2 distributions on manifolds of arbitrary dimension [9,10], and, in combination with algebraic prolongation techniques in a spirit of N. Tanaka, for the construction of the canonical frames for distributions of rank 3 [11] and recently of arbitrary rank [15,16] under very mild genericity assumptions called maximality of class. Remarkably, these constructions are independent of the nilpotent approximation (the Tanaka symbol) of a distribution at a point and even independent of its small growth vector. This extends significantly the scope of distributions for which the canonical frames can be constructed explicitly and in an unified way compared to the Tanaka approach ([6,18,20,26]).

Perhaps the case of rank 2 distributions of so-called maximal class in $\mathbb{R}^n$ with $n > 5$ provides the most illustrative example of the effectiveness of this approach, because the construction of the canonical frame in this case needs nothing more than some simple facts from the classical theory of curves in projective spaces such as the existence of the canonical projective structure on such curves, i. e. a special set of parametrizations defined up to a Möbius transformation (see Sect. 5 below). The canonical frame for such distributions is constructed in a unified way on a bundle of dimension $2n - 1$ and this dimension cannot be reduced, because there exists the unique, up to a local equivalence, rank 2 distribution of maximal class in $\mathbb{R}^n$ with the pseudo-group of local symmetries of dimension equal to $2n - 1$. For this most symmetric rank 2 distribution of maximal class all generalized Wilczynski invariants are identically zero.

However, under some additional assumptions, the canonical parametrization, up to a shift, on abnormal extremals can be distinguished instead of the canonical projective structure and one would expect that the canonical frame can be constructed on a bundle of smaller dimension.

What are these additional assumptions? One possibility is to consider rank 2 distributions of maximal class such that at least one of its generalized Wilczynski invariant does not vanish. Due to the size limits for the paper we postpone the treatment of this case to another paper (see also preprint [13]).

Another possibility is to consider a rank 2 distribution D with the additional structures defining a control system with one input satisfying certain regularity assumptions. A *control system with one input on a distribution D in the manifold M* is given by choosing a one-dimensional submanifold $\mathcal{V}_q$ on each fiber $D(q)$ of the distribution D for any point $q \in M$ (smoothly depending on q). The set $\mathcal{V}_q \subset D(q)$ is called the *set of admissible velocities of the control system at q*.

Let us introduce several natural notions of equivalence of control systems. We say that two control systems given by one-dimensional submanifolds $\mathcal{V}_q$ and $\widetilde{\mathcal{V}_q}$ on each fiber $D(q)$ are *(state-feedback) equivalent* if there exists a diffeomorphism F of M such that

$$F_*(\mathcal{V}_q) = \widetilde{\mathcal{V}}_{F(q)} \tag{1}$$

for any $q \in M$. These control systems are called *locally equivalent* at the points q_0 and $\tilde{q}_0$ of M, respectively, if there exists neighborhoods U and $\widetilde{U}$ of q_0 and $\tilde{q}_0$ in M, respectively, and a diffeomorphism $F\colon\ : U \to \widetilde{U}$ such that (1) holds for any $q \in U$. Finally, these control systems are called *micro-locally equivalent* at (q_0, v_0) and $(\tilde{q}_0, \tilde{v}_0)$, where the points q_0 and $\tilde{q}_0$ belong to M, $v \in V_q$, and $\tilde{v} \in V_q$, if there exist neighborhoods $\mathfrak{U}$ and $\widetilde{\mathfrak{U}}$ of (q_0, v_0) and $(\tilde{q}_0, \tilde{v}_0)$ in the set $\mathfrak{V} = \{(q, v) : q \in M, v \in V_q\}$ and a diffeomorphism $F\colon\ : \mathrm{pr}(\mathfrak{U}) \to \mathrm{pr}(\widetilde{\mathfrak{U}})$, where $\mathrm{pr}\colon \mathfrak{V} \to M$ is the canonical projection, such that $F_* v \in \mathcal{V}_{F(q)} \cap \widetilde{\mathfrak{U}}$ for any $(q, v) \in \mathfrak{U}$. From these notions of equivalence one can define the group of symmetries and pseudo-groups of local and micro-local symmetries of a control system accordingly. In the paper we mainly work with the micro-local equivalence but if one restricts himself to affine control systems only, then in all formulations the micro-local equivalence can be replaces by the local one.

Definition 1. Consider a control system with one input on a distribution D with the set of admissible velocities $\mathcal{V}_q$ at a point q. A line in $D(q)$ (through the origin) intersecting the set $\mathcal{V}_q \backslash \{\text{the origin of } D(q)\}$ in a finite number of points is called a *regular line* of the control system at the point q.

Definition 2. We say that a control system with one input on a rank 2 distribution D is *regular* if for any point q the sets of regular lines is a nonempty open subset of the projectivization $\mathbb{P} D(q)$.

An important particular class of examples of such control systems is when $\mathcal{V}_q$ is an affine line. In this case we get an *affine control system with one input and with a non-zero drift*. Another examples are sub-(pseudo)Riemannian structures, when the curves are ± 1-level sets of non-degenerate quadrics. For affine control systems with a non-zero drift and sub-Riemannian structures all lines in $D(q)$ are regular, while for sub-pseudo-Riemannian case all lines except the asymptotic lines of the quadrics are regular.

The goal of this paper is to demonstrate the approach, originated by Andrei Agrachev, in this simplified but still important situation of regular control system with one input on rank 2 distributions of maximal class. We show that *in this situations the canonical frame can be constructed in a unified way on a bundle of dimension* $2n-3$ *for all* $n \geq 5$ (Theorem 3, Sect. 7). We also describe all models with the pseudo-group of micro-local symmetries of dimension $2n-3$. i. e. the most symmetric ones, among the considered class of objects (Theorem 1 below and its reformulation in Theorem, Sect. 9).

The most symmetric models depend on continuous parameters. Let us describe these models. Given a tuple of $n-3$ constants $(r_1,\ldots,r_{n-3})$ let $A_{(r_1,\ldots,r_{n-3})}$ be the following affine control system in R^n taken with coordinates $(x, y_0, \ldots, y_{n-3}, z)$:

$$\dot{q} = X_1\big(q\big) + uX_2(q), \tag{2}$$

where

$$X_1 = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y_0} + \cdots + y_{n-3}\frac{\partial}{\partial y_{n-4}} + \big(y_{n-3}^2 + r_1 y_{n-4}^2 + r_2 y_{n-5}^2 + \ldots r_{n-3} y_0^2\big)\frac{\partial}{\partial z}, \tag{3}$$

$$X_2 = \frac{\partial}{\partial y_{n-3}}. \tag{4}$$

and denote by $D_{(r_1,\ldots,r_{n-3})}$ the corresponding rank 2 distribution generated by the vector fields X_1 and X_2 as in (3)-(4). Note that, as shown in [9, 10], the most symmetric rank 2 distribution in R^n of maximal class with $n \geq 5$ is locally equivalent to $D_{(0,\ldots,0)}$. In the case of regular control systems we prove the following

Theorem 1. *A regular control systems with one input on a rank 2 distribution of maximal class in* $\mathbb{R}^n$ *with* $n \geq 5$ *has the pseudo-group of micro-local symmetries of dimension not greater than* $2n-3$. *If this dimension is equal to* $2n-3$, *then the control system is micro-locally equivalent to the system* $A_{(r_1,\ldots,r_{n-3})}$ *for some constants* $r_i \in \mathbb{R}$, $1 \leq i \leq n-3$. *The affine control systems* $A_{(r_1,\ldots,r_{n-3})}$ *corresponding to the different tuples* $(r_1,\ldots,r_{n-3})$ *are not equivalent.*

Rephrasing the last sentence of the Theorem 1, the map $(r_1,\ldots,r_{n-3}) \mapsto A_{(r_1,\ldots,r_{n-3})}$ identifies the space $\mathcal{A}_n$ of the most symmetric, up to a micro-local equivalence, regular control systems on rank 2 distributions of maximal class in $\mathbb{R}^n$ with $\mathbb{R}^{n-3}$.

Remark 1 (see [13] for more detail). Note that the underlying distributions $D_{(r_1,\ldots,r_{n-3})}$ might be equivalent for different tuples $(r_1,\ldots,r_{n-3})$. Among all distributions of the type $D_{(r_1,\ldots,r_{n-3})}$ there is a one-parametric family of distributions which are locally equivalent to $D_{(0,\ldots,0)}$. To describe this family we say that a tuple of m numbers $(r_1,\ldots,r_m)$ is called exceptional if the roots of the polynomial

$$\lambda^{2m} + \sum_{i=1}^{m}(-1)^i r_i \lambda^{2(m-i)} \tag{5}$$

constitute an arithmetic progression (with the zero sum in this case). Equivalently, $(r_1, \dots, r_m)$ is exceptional if $r_i = \alpha_{m,i}\left(\frac{r_1}{\alpha_{m,1}}\right)^i$, $1 \le i \le m$, where the constants $\alpha_{m,i}$, $1 \le i \le m$, satisfy the following identity

$$x^{2m} + \sum_{i=1}^{m}(-1)^i \alpha_{m,i} x^{2(m-i)} = \prod_{i=1}^{m}\left(x^2 - (2i-1)^2\right). \tag{6}$$

The distribution $D_{(r_1,\dots,r_{n-3})}$ is locally equivalent to the distribution $D_{(0,\dots,0)}$ (or, equivalently, has the algebra of infinitesimal symmetries of the maximal possible dimension among all rank 2 distributions of maximal class in $\mathbb{R}^n$) if and only if the tuple $(r_1, \dots, r_{n-3})$ is exceptional. The distribution $D_{(\tilde r_1,\dots,\tilde r_{n-3})}$ is locally equivalent to the distribution $D_{(r_1,\dots,r_{n-3})}$, where the tuple $(r_1, \dots, r_{n-3})$ is not exceptional, if and only if

$$\text{there exists } c \neq 0 \text{ such that } \tilde r_i = c^{2i} r_i, \quad 1 \le i \le n-3. \qquad \square$$

Finally note that affine control systems with one input were considered also in [5], but the genericity assumptions imposed there are much stronger than our genericity assumptions here.

The paper is organized as follows. The main results are given in Sects. 7 and 9 (Theorem 3 and Theorem 4, which are reformulations of Theorem 1 above). Sects. 2–6 are preparatory for Sect. 7, Sect. 8 is preparatory for Sect. 9. In Sects. 2–5 we list all necessary facts about abnormal extremals of rank 2 distributions, their Jacobi curves and describe the canpnical projective structure on a unparametrized curve in projective spaces. The details can be found in [9,22,23]. In Sect. 6 we summarize the main results of [9, 10] about canonical frames for rank 2 distributions of maximal class in order to compare them with the analogous results of Sects. 7 and 9. In Sect. 8 we list all necessary facts about the invariants of parametrized self-dual curves in projective spaces.

2 Abnormal extremals of rank 2 distributions

Let D be a rank 2 distribution on a manifold M. A smooth section of a vector bundle D is called a *horizontal vector field of* D. Taking iterative brackets of horizontal vector fields of D, we obtain the natural filtration $\{\dim D^j(q)\}_{j\in\mathbb{N}}$ on each tangent space T_qM. Here D^j is the j-th power of the distribution D, i. e., $D^j = D^{j-1} + [D, D^{j-1}]$, $D^1 = D$, or , equivalently, $D^j(q)$ is a linear span of all Lie brackets of the length not greater than j of horizontal vector fields of D evaluated at q.

Assume that $\dim D^2(q) = 3$ and $\dim D^3(q) > 3$ for any $q \in M$. Denote by $(D^j)^\perp \subset T^*M$ the annihilator of the jth power D^j, namely

$$(D^j)^\perp = \{(p, q) \in T^*M : \ p \cdot v = 0 \ \forall v \in D^j(q)\}.$$

Recall that abnormal extremals of D are by definition the Pontryagin extremals with the vanishing Lagrange multiplier near the functional for any extremal problem with constrains, given by the distribution D. They depend only on the distribution D and not on a functional.

It is easy to show (see, for example, [10, 22]) that for rank 2 distributions all abnormal extremals lie in $(D^2)^\perp$ and that through any point of the codimension 3 submanifold $(D^2)^\perp \backslash (D^3)^\perp$ of T^*M passes exactly one abnormal extremal or, in other words, $(D^2)^\perp \backslash (D^3)^\perp$ is foliated by the characteristic 1-foliation of abnormal extremals. To describe this foliation let $\pi : T^*M \mapsto M$ be the canonical projection. For any $\lambda \in T^*M$, $\lambda = (p,q)$, $q \in M$, $p \in T_q^*M$, let $\mathfrak{s}(\lambda)(\cdot) = p(\pi_* \cdot)$ be the canonical Liouville form and $\sigma = d\mathfrak{s}$ be the standard symplectic structure on T^*M. Since the submanifold $(D^2)^\perp$ has odd codimension in T^*M, the kernels of the restriction $\sigma|_{(D^2)^\perp}$ of σ on $(D^2)^\perp$ are not trivial. At the the points of $(D^2)^\perp \backslash (D^3)^\perp$ these kernels are one-dimensional. They form the *characteristic line distribution* in $(D^2)^\perp \backslash (D^3)^\perp$, which will be denoted by $\mathcal{C}$. The line distribution $\mathcal{C}$ defines the desired *characteristic 1-foliation* on $(D^2)^\perp \backslash (D^3)^\perp$ and the leaf of this foliation through a point is exactly the abnormal extremal passing through this point. From now on we shall work with abnormal extremals which are integral curves of the characteristic distribution $\mathcal{C}$.

The characteristic line distribution $\mathcal{C}$ can be easily described in terms of a local basis of the distribution D, i.e. two horizontal vector fields X_1 and X_2 such that $D(q) = \text{span}\{X_1(q), X_2(q)\}$ for all q from some open set of M. Denote by

$$X_3 = [X_1, X_2],\ X_4 = \big[X_1, [X_1, X_2]\big], \quad X_5 = \big[X_2, [X_1, X_2]\big]. \tag{7}$$

Let us introduce the "quasi-impulses" $u_i : T^*M \mapsto \mathbb{R}$, $1 \le i \le 5$,

$$u_i(\lambda) = p \cdot X_i(q), \quad \lambda = (p,q),\ q \in M, \quad p \in T_q^*M. \tag{8}$$

Then by the definition

$$(D^2)^\perp = \{\lambda \in T^*M : u_1(\lambda) = u_2(\lambda) = u_3(\lambda) = 0\}. \tag{9}$$

As usual, for a given function $h : T^*M \mapsto \mathbb{R}$ denote by $\overrightarrow{h}$ the corresponding Hamiltonian vector field defined by the relation $i_{\overrightarrow{h}}\sigma = -d\,h$. Then by the direct computations (see, for example, [10]) the characteristic line distribution $\mathcal{C}$ satisfies

$$\mathcal{C} = \text{span}\{u_4 \overrightarrow{u}_2 - u_5 \overrightarrow{u}_1\}. \tag{10}$$

3 Jacobi curves of abnormal extremals

Now we are ready to define the Jacobi curve of an abnormal extremal of D. For this first lift the distribution D to $(D^2)^\perp$, namely considered the distribution $\mathcal{J}$ on $(D^2)^\perp$

such that

$$\mathcal{J}(\lambda) = \{v \in T_\lambda (D^2)^\perp : d\pi(v) \in D(\pi(\lambda))\}. \tag{11}$$

Note that $\dim \mathcal{J} = n - 1$ and $\mathcal{C} \subset \mathcal{J}$ by (10) . The distribution $\mathcal{J}$ is called the *lift of the distribution* D *to* $(D^2)^\perp \backslash (D^3)^\perp$.

Given a segment γ of an abnormal extremal (i. e. of a leaf of the 1-characteristic foliation) of D, take a sufficiently small neighborhood O_γ of γ in $(D^2)^\perp$ such that the quotient $N = O_\gamma/($*the characteristic one-foliation*$)$ is a well defined smooth manifold. The quotient manifold N is a symplectic manifold endowed with the symplectic structure $\bar{\sigma}$ induced by $\sigma|_{(D^2)^\perp}$. Let

$$\phi : O_\gamma \to N \tag{12}$$

be the canonical projection on the factor. Define the following curves of subspaces in $T_\gamma N$:

$$\lambda \mapsto \phi_*\big(\mathcal{J}(\lambda)\big), \quad \forall \lambda \in \gamma. \tag{13}$$

Informally speaking, these curves describe the dynamics of the distribution $\mathcal{J}$ w.r.t. the characteristic 1-foliation along the abnormal extremal γ.

Note that there exists a straight line, which is common to all subspaces appearing in (13) for any $\lambda \in \gamma$. So, it is more convenient to get rid of it by a factorization. Indeed, let e be the Euler field on T^*M, i. e., the infinitesimal generator of homotheties on the fibers of T^*M. Since a transformation of T^*M, which is a homothety on each fiber with the same homothety coefficient, sends abnormal extremals to abnormal extremals, we see that the vector $\bar{e} = \phi_* e(\lambda)$ is the same for any $\lambda \in \gamma$ and lies in any subspace appearing in (13). Let

$$J_\gamma(\lambda) = \phi_*\big(\mathcal{J}(\lambda)\big)/\{\mathbb{R}\bar{e}\}, \quad \forall \lambda \in \gamma. \tag{14}$$

The (unparametrized) curve $\lambda \mapsto J_\gamma(\lambda)$, $\lambda \in \gamma$ is called the *Jacobi curve of the abnormal extremal* γ. It is clear that all subspaces appearing in (14) belong to the space

$$W_\gamma = \{v \in T_\gamma N : \bar{\sigma}(v, \bar{e}) = 0\}/\{\mathbb{R}\bar{e}\}. \tag{15}$$

and that

$$\dim J_\gamma(\lambda) = n - 3. \tag{16}$$

The space W_γ is endowed with the natural symplectic structure $\tilde{\sigma}_\gamma$ induced by $\bar{\sigma}$. Also $\dim W_\gamma = 2(n-3)$.

Given a subspace L of W_γ denote by $L^\angle$ the skew-orthogonal complement of L with respect to the symplectic form $\tilde{\sigma}_\gamma$, $L^\angle = \{v \in W_\gamma, \sigma_\gamma(v, \ell) = 0 \quad \forall \ell \in L\}$. Recall that the subspace L is called *isotropic* if $L \subseteq L^\angle$, *coisotropic* if $L^\angle \subseteq L$, and *Lagrangian*, if $L = L^\angle$. Directly from the definition, the dimension of an isotropic subspace does not exceed $\frac{1}{2}\dim W_\gamma$, and a Lagrangian subspace is an isotropic subspace of the maximal possible dimension $\frac{1}{2}\dim W_\gamma$. The set of all Lagrangian subspaces of W_γ is called the *Lagrangian Grassmannian of* W_γ.

It is easy to see ([10, 23]) that the Jacobi curve of an abnormal extremal consists of Lagrangian subspaces, i. e. it is a curve in the Lagrangian Grassmannian of W_γ. In the case $n \geq 5$ (equivalently, dim $W_\gamma \geq 4$) curves in the Lagrangian Grassmannian of W_γ have a nontrivial geometry with respect to the action of the linear symplectic group and any symplectic invariant of Jacobi curves of abnormal extremals produces an invariant of the original distribution D.

4 Reduction to geometry of curves in projective spaces

In the earlier works [3, 23] invariants of Jacobi curves were constructed using the notion of the cross-ratio of four points in Lagrangian Grassmannians analogous to the classical cross-ratio of four point in a projective line. Later, we developed a different method, leading to the construction of canonical bundles of moving frames and invariants for quite general curves in Grassmannians and flag varieties [12, 14]. The geometry of Jacobi curves J_γ in the case of rank 2 distributions can be reduced to the geometry of the so-called self-dual curves in the projective space $\mathbb{P}W_\gamma$.

For this first one can produce a curve of flags of isotropic/coisotropic subspaces of W_γ by a series of osculations together with the operation of taking skew symmetric complements. For this, denote by $C(J_\gamma)$ the *tautological bundle* over J_γ: the fiber of $C(J_\gamma)$ over the point $J_\gamma(\lambda)$ is the linear space $J_\gamma(\lambda)$. Let $\Gamma(J_\gamma)$ be the space of all smooth sections of $C(J_\gamma)$. If $\psi : (-\varepsilon, \varepsilon) \mapsto \gamma$ is a parametrization of γ such that $\psi(0) = \lambda$, then for any $i \geq 0$ define

$$J_\gamma^{(i)}(\lambda) := \text{span}\left\{\frac{d^j}{d\tau^j}\ell\big(\psi(t)\big)\Big|_{t=0} : \ell \in \Gamma(J_\gamma), 0 \leq j \leq i\right\} \tag{17}$$

$$J_\gamma^{(-i)}(\lambda) = \big(J_\gamma^{(i)}(\lambda)\big)^{\angle}. \tag{18}$$

For $i > 0$ we say that the space $J_\gamma^{(i)}(\lambda)$ is the *i-th osculating space of the curve J_γ at λ*.

Note that $J_\gamma = J_\gamma^{(0)}$. Directly from the definitions the subspaces $J_\gamma^{(i)}(\lambda)$ are coisotropic for $i > 0$ and isotropic for $i < 0$ and the tuple $\{J_\gamma^{(i)}(\lambda)\}_{i\in\mathbb{Z}}$ defines a filtration of W_γ. In other words, the curve $\lambda \mapsto \{J_\gamma^{(i)}(\lambda)\}_{i\in\mathbb{Z}}$ is a curve of flags of W_γ. Besides, it can be shown [23] that

$$\dim J^{(1)}(\lambda) - \dim J^{(0)}(\lambda) = \dim J^{(0)}(\lambda) - \dim J^{(-1)}(\lambda) = 1,$$

which in turn implies that $\dim J^{(i)}(\lambda) - \dim J^{(i-1)}(\lambda) \leq 1$, i. e. the jump of dimensions between the consecutive subspaces of the filtration $\{J_\gamma^{(i)}(\lambda)\}_{i\in\mathbb{Z}}$ is at most 1. This together with (16) implies that $\dim J_\gamma^{(i)}(\lambda) \leq n - 3 + i$ for $i > 0$.

We say that λ is a *regular point of* $(D^2)^\perp \backslash (D^3)^\perp$ if $\dim J_\gamma^{(i)}(\lambda) = n - 3 + i$ for $0 < i \leq n - 3$ or, equivalently, if $J_\gamma^{(n-3)}(\lambda) = W_\gamma$. A rank 2 distribution D is called of *maximal class at a point* $q \in M$ if at least one point in $\pi^{-1}(q) \cap (D^2)^\perp$

is regular. Since by (10) the characteristic distribution $\mathcal{C}$ generated by a vector field depending algebraically on the fibers $(D^2)^{\perp}$, if D is of maximal class at a point $q \in M$, then the set of all regular points of $\pi^{-1}(q) \cap (D^2)^{\perp}$ is non-empty open set in Zariski topology. The same argument is used to show that the set of germs of rank 2 distributions of maximal class is generic.

If D is of maximal class at q and $n \geq 5$, then by necessity $\dim D^3(q) = 5$. The following question is still open: Does there exist a rank 2 distribution with $\dim D^3 = 5$ such that it is not of maximal class on some open set of M? We proved that the answer is negative for $n \leq 8$ and we have strong evidences that the answer is negative in general.

Remark 2. Note that from (10) it follow that if a rank 2 distribution D is of maximal class at a point $q \in M$ then the set of all lines $\{d\pi\big(\mathcal{C}(\lambda)\big) : \lambda \in \mathcal{R}_D \cap \pi^{-1}(q)\}$ is an open and dense subset of the projectivization $\mathbb{P}D(q)$ of the plane $D(q)$, where, as before, $\pi : T^*M \to M$ is the canonical projection. □

From now on we will work with rank 2 distributions of maximal class. In this case $\dim J_\gamma^{(4-n)}(\lambda) = 1$, i. e. the curve $J_\gamma^{(4-n)}$ is a curve in the projective space $\mathbb{P}W_\gamma$. Moreover, the curve of flags $\lambda \mapsto \{J_\gamma^{(i)}(\lambda)\}_{i=3-n}^{n-3}$, $\lambda \in \gamma$ is the curve of complete flags and the space $J_\gamma^{(i)}(\lambda)$ is the $(i+n-4)$th-osculating space of the curve $J_\gamma^{(4-n)}$. In other words, the whole curve of complete flags $\lambda \mapsto \{J_\gamma^{(i)}(\lambda)\}_{i=3-n}^{n-3}$, $\lambda \in \gamma$ can be recovered from the curve $J_\gamma^{(4-n)}$ and the differential geometry of Jacobi curves of abnormal extremals of rank 2 distributions is reduced to the differential geometry of curves in projective spaces.

5 Canonical projective structure on curves in projective spaces

The differential geometry of curves in projective spaces is the classical subject, essentially completed already in 1905 by E.J. Wilczynski ([21]). In particular, it is well known that these curves are endowed with the canonical projective structure, i. e., there is a distinguished set of parameterizations (called projective) such that the transition function from one such parametrization to another is a Möbius transformation. Let us demonstrate how to construct it for the curve $\lambda \mapsto J_\gamma^{(4-n)}(\lambda)$, $\lambda \in \gamma$.

As before, let $C(J_\gamma^{(4-n)})$ be the tautological bundle $C(J_\gamma^{(4-n)})$ over $J_\gamma^{(4-n)}$. Set $m = n - 3$. Here we use a "naive approach", based on reparametrization rules for certain coefficient in the expansion of the derivative of order $2m$ of certain sections of $C(J_\gamma^{(4-n)})$ w.r.t. to the lower order derivatives of this sections. For the more algebraic point of view, based on a Tanaka-like theory of curves of flags and $\mathfrak{sl}_2$-representations see [8, 12].

Take some parametrization $\psi : I \mapsto \gamma$ of γ, where I is an interval in $\mathbb{R}$. By above, for any section ℓ of $C(J_\gamma^{(4-n)})$ one has that

$$\operatorname{span}\Big\{\frac{d^j}{dt^j}\ell\big(\psi(t)\big) \mid 0 \leq j \leq 2m-1\Big\} = W_\gamma. \qquad (19)$$

A curves in the projective space $\mathbb{P}W_\gamma$ satisfying the last property is called *regular* (or *convex*). It is well known that there exists the unique, up to the multiplication by a nonzero constant, section E of $C(J_\gamma^{(4-n)})$, called a *canonical section of* $C(J_\gamma^{(4-n)})$ *with respect to the parametrization* ψ, such that

$$\frac{d^{2m}}{dt^{2m}}E\big(\psi(t)\big) = \sum_{i=0}^{2m-2} B_i(t)\frac{d^i}{dt^i}E\big(\psi(t)\big), \tag{20}$$

i.e. the coefficient of the term $\frac{d^{2m-1}}{dt^{2m-1}}E\big(\psi(t)\big)$ in the linear decomposition of $\frac{d^{2m}}{dt^{2m}}E\big(\psi(t)\big)$ w.r.t. the basis $\big\{\frac{d^i}{dt^i}E\big(\psi(t)\big) : 0 \le i \le 2m-1\big\}$ vanishes.

Further, let ψ_1 be another parameter, $\widetilde{E}$ be a canonical section of $C(J_\gamma^{(4-n)})$ with respect to the parametrization ψ_1, and $\upsilon = \psi^{-1} \circ \psi_1$. Then directly from the definition it easy to see that

$$\widetilde{E}\big(\psi_1(\tau)\big) = c(\upsilon'(\tau))^{\frac{1}{2}-m}E(\psi(t)) \tag{21}$$

for some non-zero constant c.

Now let $\widetilde{B}_i(\tau)$ be the coefficient in the linear decomposition of $\frac{d^{2m}}{d\tau^{2m}}\widetilde{E}\big(\psi_1(\tau)\big)$ w.r.t. the basis $\big\{\frac{d^i}{d\tau^i}\widetilde{E}\big(\psi_1(t)\big) : 0 \le i \le 2m-1\big\}$ as in (20). Then, using the relation (21) it is not hard to show that the coefficients B_{2m-2} and $\widetilde{B}_{2m-2}$ in the decomposition (20), corresponding to parameterizations ψ and ψ_1, are related as follows:

$$\widetilde{B}_{2m-2}(\tau) = \upsilon'(\tau)^2 B_{2m-2}(\upsilon(\tau)) - \frac{m(4m^2-1)}{3}\mathbb{S}(\upsilon)(\tau), \tag{22}$$

where $\mathbb{S}(\upsilon)$ is the Schwarzian derivative of υ, $\mathbb{S}(\upsilon) = \frac{d}{d\tau}\Big(\frac{\upsilon''}{2\upsilon'}\Big) - \Big(\frac{\upsilon''}{2\upsilon'}\Big)^2$.

From the last formula and the fact that $\mathbb{S}\upsilon \equiv 0$ if and only if the function υ is Möbius it follows that *the set of all parameterizations φ of γ such that*

$$B_{2m-2} \equiv 0 \tag{23}$$

defines the canonical projective structure on γ. Such parameterizations are called the *projective parameterizations of the abnormal extremal* γ. If ψ and ψ_1 are two projective parametrizations, then there exists a Möbius transformation υ such that $\psi_1 = \psi \circ \upsilon$.

Note that the curve $J_\gamma^{(4-n)}$ is not an arbitrary regular curve in the projective space $\mathbb{P}W$. It satisfies the following additional property:

(S1) *The* $(n-4)$*th-osculating space of* $J_\gamma^{(4-n)}$ *at any point* λ *is Lagrangian.*

As shown already by Wilczynski [21] such curves are *self-dual* in the following sense:

(S2) *The curve* $(J_\gamma^{(n-4)})^*$ *in the projectivization* $\mathbb{P}W_\gamma^*$ *of the dual space* W_γ^**, which is dual to the curve of hyperplanes* $J_\gamma^{(n-4)}$ *obtained from the original curve*

$J_\gamma^{(4-n)}$ *by the osculation of order* $2(n-4)$, *is equivalent to the original curve* $J_\gamma^{(4-n)}$, *i. e. there is a linear transformation* $A : W \mapsto W^*$ *sending* $J_\gamma^{(n-4)}$ *onto* $(J_\gamma^{(n-4)})^*$.

Note that in contrast to property (S1) the formulation of property (S2) does not involve a symplectic structure on W_γ. However, it can be shown [17, 21] that if the property (S2) holds then there exists a unique, up to a multiplication by a nonzero constant, symplectic structure on W_γ such that the property (S1) holds (here it is important that dim W_γ is even; similar statement for the case of odd dimensional linear space involves nondegenerate symmetric forms instead of skew-symmetric ones). Since in our case the symplectic structure on W_γ is a priori given, in the sequel we will consider projective spaces of linear symplectic spaces only and by self-dual curves we will mean curves satisfying property (S1).

Using the coefficients of the decomposition (20) w.r.t. a projective parameter t one can construct the (relative) invariants of the unparametrized curve $J_\gamma^{(4-n)}$, called the Wilczynski invariants. Since we shall not use these invariants in the sequel, we will not give here their construction referring the interested reader to [8, 12]. Note only that in the case of a self-dual curve in such decomposition also $B_{2m-3}(t) \equiv 0$ and the first nontrivial Wilczynski invariant is $B_{2m-4}(t)dt^4$, i. e. this is the homogeneous function of degree 4 on each tangent line to our curve. As shown in [24], for rank 2 distributions in $\mathbb{R}^5$ with maximal possible small growth vector (2, 3, 5), this invariant, calculated along each abnormal extremal, gives the classical Cartan invariant of [7].

6 Canonical frames for rank 2 distributions of maximal class

Now let $\mathcal{R}_D$ be the set if all regular points of $(D^2)^\perp \backslash (D^3)^\perp$. Denote by $\mathfrak{P}_\lambda$ the set of all projective parameterizations ψ on the characteristic curve γ , passing through λ, such that $\psi(0) = \lambda$. Let

$$\Sigma_D = \{(\lambda, \psi) : \lambda \in \mathcal{R}_D, \psi \in \mathfrak{P}_\lambda\}.$$

Actually, Σ_D is a principal bundle over $\mathcal{R}_D$ with the structural group of all Möbius transformations, preserving 0 and dim $\Sigma_D = 2n-1$. The main results of [9, 10] can be summarized in the following:

Theorem 2. *For any rank* 2 *distribution in* $\mathbb{R}^n$ *with* $n > 5$ *of maximal class there exists the canonical, up to the action of* $\mathbb{Z}_2$, *frame on the corresponding* $(2n-1)$-*dimensional manifold* Σ_D *so that two distributions from the considered class are equivalent if and only if their canonical frames are equivalent. The group of symmetries of such distributions is at most* $(2n-1)$-*dimensional and this upper bound is sharp. All distributions from the considered class with* $(2n-1)$-*dimensional Lie algebra of infinitesimal symmetries is locally equivalent to the distribution* $D_{((0,\ldots 0)}$ *generated by the vector fields* X_1 *and* X_2 *from* (3)-(4) *with all* r_i *equal to* 0 *or, equivalently, associated with the underdetermined ODE* $z'(x) = \left(y^{(n-3)}(x)\right)^2$. *The* *symmetry algebra of this distribution is isomorphic to a semidirect sum of* $\mathfrak{gl}(2,\mathbb{R})$ *and* $(2n-5)$*-dimensional Heisenberg algebra* $\mathfrak{n}_{2n-5}$ *such that* $\mathfrak{gl}(2,\mathbb{R})$ *acts irreducibly on a complement of the center of* $\mathfrak{n}_{2n-5}$ *to* $\mathfrak{n}_{2n-5}$ *itself*.

7 Canonical frames for rank 2 distributions of maximal class with distinguish parametrization on abnormal extremals

Let us show that for regular control systems on rank 2 distributions in the sense Definition 2 a special parametrization, up to a shift, can be distinguished on each abnormal extremal lying in $\mathcal{R}_D$. Let $\mathcal{V}_q$ be the set of the admissible velocities of the control system under consideration at the point $q \in M$. Let $\widehat{\mathcal{R}}$ be a subset of $\mathcal{R}_D$ consisting of all points λ such that the image under $d\pi$ of the tangent line at λ to the abnormal extremal passing through λ is a regular line in $D\big(\pi(\lambda)\big)$ in the sense of Definition 1 (here , as before $\pi : T^*M \to M$ is the canonical projection). Then by Definition 2 and Remark 2 the set $\widehat{\mathcal{R}}$ is a non-empty open subset of $(D^2)^\perp$. Given a regular line L in $D(q)$ let $w(L)$ be the admissible velocity in L of the smallest norm. Clearly $w(L)$ does not depend on the choice of a norm in $D(q)$, but in general it may be defined up to a sign (for example, in the sub-(pseudo) Riemannian case).

A parametrization $\psi : I \mapsto \gamma$ of an abnormal extremal γ living in $\widehat{\mathcal{R}}$ is called *weakly canonical (with respect to the regular control system given by the set of admissible velocities* $\{\mathcal{V}_q\}_{q\in M}$*)* if

$$d\pi\big(\frac{d}{dt}\gamma(\psi(t))\big) = w\left(\operatorname{span} d\pi\left(\frac{d}{dt}\gamma(\psi(t))\right)\right) \tag{24}$$

This parametrization is defined up to a shift and maybe up to the change of orientation. In the case when the orientation is not fixed by (24) we can fix the orientation as follows: Since the curve $J_\gamma^{(4-n)}$ is self-dual, given a parametrization ψ on γ, among all canonical sections of the tautological bundle $C(J_\gamma^{(4-n)})$ (defined up to the multiplication by a nonzero constant) there exists the unique, up to a sign, section E of such that (20) holds and

$$\left|\tilde\sigma_\gamma\left(\frac{d^{n-3}}{dt^{n-3}}E\big(\psi(t)\big), \frac{d^{n-4}}{dt^{n-4}}E\big(\psi(t)\big)\right)\right| \equiv 1. \tag{25}$$

This section E will be called the *strongly canonical section* of $C(J_\gamma^{(4-n)})$ with respect to the parametrization ψ. The parametrization ψ is called the *canonical parametrization of the abnormal extremal* γ if (24) holds and

$$\tilde\sigma_\gamma\left(\frac{d^{n-3}}{dt^{n-3}}E\big(\psi(t)\big), \frac{d^{n-4}}{dt^{n-4}}E\big(\psi(t)\big)\right) \equiv 1. \tag{26}$$

We finally obtain the parametrization of γ defined up to a shift only.

Finally let $\widetilde{\mathcal{R}}$ be a subset of $\widehat{\mathcal{R}}$ where the vector field consisting of the tangent vectors to the abnormal extremals parameterized by the canonical parameter is smooth.

Note that $\widetilde{\mathcal{R}}$ is an open and dense subset of $\widehat{\mathcal{R}}$. For affine control systems with one input and a non-zero drift and for sub-Riemannian structures $\widetilde{\mathcal{R}}$ coincides with the set $\mathcal{R}_D$ of the regular points in $(D^2)^\perp \backslash (D^3)^\perp$.

Note that the canonical parametrization is preserved by the homotheties of the fibers of $(D^2)^\perp$. Namely, if δ_s is the flow of homotheties on the fibers of T^*M: $\delta_s(p,q) = (e^s p, q), \quad q \in M, \; p \in T_q^*M$ or, equivalently, the flow generated by the Euler field e generates this flow, then $\psi : I \mapsto \gamma$ is the canonical parametrization on an abnormal extremal γ if and only if $\delta_s \circ \psi$ is the canonical parametrization on the abnormal extremal $\delta_s \circ \gamma$.

The main goal of this section is to prove the following

Theorem 3. *Given a regular control system on a rank 2 distribution D of maximal class one can assign to it a canonical, up to the action of $\mathbb{Z}_2$, frame on the set $\widetilde{\mathcal{R}}$ defined above so that two objects from the considered class are micro-locally equivalent if and only if their canonical frames are equivalent.*

Proof. First, let h be the vector field consisting of the tangent vectors to the abnormal extremals parameterized by the canonical parameter.

Second, given $\lambda \in (D^2)^\perp$ denote by $V(\lambda)$ the tangent space to the fiber of the bundle $\pi : (D^2)^\perp \mapsto M$ (the vertical subspace of $T_\lambda (D^2)^\perp$),

$$V(\lambda) = \{v \in T_\lambda (D^2)^\perp, \pi_* v = 0\}. \tag{27}$$

It is easy to show ([10, 23]) that

$$d\phi\big(V(\lambda) \oplus \mathcal{C}(\lambda)\big) = J_\gamma^{(-1)}(\lambda) \mod \mathbb{R}\bar{e}, \tag{28}$$

where ϕ is as in (12), $\bar{e} = \phi_* e$ with e being the Euler field, and γ is the abnormal extremal passing through λ. Define also the following subspaces of $T_\lambda (D^2)^\perp$:

$$\mathcal{J}^{(i)}(\lambda) = \{w \in T_\lambda (D^2)^\perp : d\phi(w) \in J_\gamma^{(i)}(\lambda) \mod \mathbb{R}\bar{e}\}. \tag{29}$$

Directly from the definition, if $\lambda \in \mathcal{R}_D$, then

$$[\mathcal{C}, \mathcal{J}^{(i)}](\lambda) = \mathcal{J}^{(i+1)}(\lambda). \tag{30}$$

Also, if $V^{(i)}(\lambda) = V(\lambda) \cap \mathcal{J}^{(i)}(\lambda)$, then

$$\mathcal{J}^{(i)}(\lambda) = V^{(i)}(\lambda) \oplus \mathcal{C}(\lambda) \quad \forall i \leq 0. \tag{31}$$

Moreover, it can be shown ([10, Lemma 2]) that

$$[V^{(i)}, V^{(i)}] \subseteq V^{(i)}, \quad [V^{(i)}, \mathcal{J}^{(i)}] \subseteq \mathcal{J}^{(i)}, \quad \forall i \leq 0. \tag{32}$$

Let E be the strongly canonical section of $C(J_\gamma^{(4-n)})$ with respect to the canonical parametrization ψ of the abnormal extremal γ (as defined by (25)). Then (28) implies that a vector field ϵ_1 such that

(A1) $d\phi\big(\varepsilon_1(\lambda)\big) \equiv E \mod \bar{e}$;
(A2) ε_1 is the section of the vertical distribution V

is defined modulo the Euler field e. Note that conditions (A1) and (A2) also imply that ε_1 is the section of $V^{(4-n)}$.

Lemma 1. *Among all vector fields ε_1 satisfying conditions (A1) and (A2), there exists the unique, up to a multiplication by -1, vector field such that*

$$\big[\varepsilon_1, [h, \varepsilon_1]\big](\lambda) \in \operatorname{span}\{e(\lambda), h(\lambda), \varepsilon_1(\lambda)\}. \tag{33}$$

Proof. Let $\tilde{\varepsilon}_1$ be a vector field satisfying the conditions (A1) and (A2). Then $\tilde{\varepsilon}_1$ is the section of $V^{(4-n)}$. Using (31) and (32) for $n > 5$ and also the definition of $\mathcal{J}$ given by (11) in the case $n = 5$, we get

$$\big[\tilde{\varepsilon}_1, [h, \tilde{\varepsilon}_1]\big] \equiv k[h, \tilde{\varepsilon}_1] \operatorname{mod} \operatorname{span}\{e, h, \tilde{\varepsilon}_1\} \tag{34}$$

for some function k. Now let ε_1 be another vector field satisfying conditions (A1) and (A2). Then by above there exists a function μ such that

$$\varepsilon_1 = \pm\tilde{\varepsilon}_1 + \mu e. \tag{35}$$

From the fact that the canonical parametrization is preserved by the homotheties of the fibers of $(D^2)^\perp$ it follows that $[e, h] = 0$. Also from the normalization condition (25) it is easy to get that

$$[e, \varepsilon_1] = -\frac{1}{2}\varepsilon_1 \operatorname{mod} \operatorname{span}(e). \tag{36}$$

Then

$$\big[e, [h, \varepsilon_1]\big] = -\frac{1}{2}[h, \varepsilon_1] \operatorname{mod}(e, h). \tag{37}$$

From this and (35) it follows that

$$\big[\varepsilon_1, [h, \varepsilon_1]\big] \equiv \big(k \mp \frac{\mu}{2}\big)[h, \varepsilon_1] \operatorname{span}\{e, h, \varepsilon_1\}, \tag{38}$$

which implies the statement of the lemma: the required vector $\tilde{\varepsilon}_1$ is obtained by taking $\mu = \pm 2k$. □

Now we are ready to construct the canonical frame on the set $\widetilde{\mathcal{R}}$. One option is to take as a canonical frame the following one:

$$\big\{e, h, \varepsilon_1, \{(\operatorname{ad} h)^i \varepsilon_1\}_{i=1}^{2n-7}, [\varepsilon_1, (\operatorname{ad} h)^{2n-7} \varepsilon_1]\big\}, \tag{39}$$

where ε_1 is as in Lemma 1. Let us explain why it is indeed a frame. First the vector fields $\big\{e, h, \varepsilon_1, \{(\operatorname{ad} h)^i \varepsilon_1\}_{i=1}^{2n-7}\big\}$ are linearly independent on $\widetilde{\mathcal{R}}$ due to the relation (30). Besides $[\varepsilon_1, (\operatorname{ad} h)^{2n-7}\varepsilon_1](\lambda) \notin \mathcal{J}^{(n-3)}(\lambda)$. Otherwise, $\varepsilon_1(\lambda)$ belongs to the kernel of the form $\sigma(\lambda)|_{(D^2)^\perp}$ and therefore it must be collinear to h. We get a contradiction. Therefore the tuple of vectors in (39) constitute a frame on $\widetilde{\mathcal{R}}$.

The construction of the frame (39) is intrinsic. However, in order to guaranty that two objects from the considered class are equivalent if and only if their canonical frames are equivalent, we have to modify this frame such that it will contain the basis of the vertical distribution V (defined by 27). For this, replace the vector fields of the form $(adh)^i\varepsilon_1$ for $1 \le i \le n-4$ by their projections to $V^{(i)}$ with respect to the splitting (31), i. e. their vertical components with respect to this splitting. This completes the construction of the required canonical frame (defined up to the action of the required finite groups). The proof of Theorem 3 is completed.

As a direct consequence of Theorem 3 we have

Corollary 1. *For a regular control system on a rank 2 distribution D of maximal class the dimension of pseudo-group of micro-local symmetries does not exceed* $2n-3$.

8 Symplectic curvatures for the structures under consideration

Before proving Theorem 1 about the most symmetric models for geometric structures under consideration, we want to reformulate this theorem in more geometric terms. For this we distinguish special invariants for this structures called the *symplectic curvatures*. They are functions on the open subset $\widetilde{\mathcal{R}}$ of $\mathcal{R}_D$, defined in the beginning of the previous section.

From the construction of the previous section all curves $J_\gamma^{(4-n)}$ are parameterized by the canonical (up to a shift) parametrization ψ given by (24) (and maybe also by (26)). The geometry of parameterized regular self-dual curves in projective spaces is simpler than of unparametrized ones: instead of forms (relative invariants) on the curve we obtain invariant, which are scalar-valued function on the curve ([25]). The main result of [25] (Theorem 2 there) can be reformulated as follows (see also [17]): if E is a (strongly) canonical section of $C(J_\gamma^{(4-n)})$ with respect to the (canonical) parametrization ψ, then there exist m functions $\rho_1(t), \dots, \rho_m(t)$ such that

$$E^{(2m)}\big(\psi(t)\big) = \sum_{i=1}^{m}(-1)^{i+1}\frac{d^{m-i}}{dt^{m-i}}\Big(\rho_i(t)\frac{d^{m-i}}{dt^{m-i}}E\big(\psi(t)\big)\Big). \tag{40}$$

Note that formula (40) resembles the classical normal form for the formally self-adjoint linear differential operators [19][§1].

By constructions, the functions $\rho_1(t), \dots, \rho_m(t)$ are invariants of the parameterized curve $t \mapsto J_\gamma^{(4-n)}\big(\psi(t)\big)$ with respect to the action of the linear symplectic group on W_γ. We call the function $\rho_i(t)$ the *ith symplectic curvature of the parametrized curve* $t \mapsto J_\gamma^{(4-n)}\big(\psi(t)\big)$. Besides, the functions $\rho_1(t), \dots, \rho_m(t)$ constitute the fundamental system of symplectic invariant of the parametrized curve $t \mapsto J_\gamma^{(4-n)}\big(\psi(t)\big)$, i. e. they determine this curve uniquely up to a symplectic transformation. Moreover, these invariants are independent: for any tuple of m functions $\rho_1(t), \dots, \rho_m(t)$ on the interval $I \subseteq R$ there exists a parameterized regular self-dual curve $t \mapsto \Lambda(t)$, $t \in I$, in the projective space of dimension $2m-1$ with the ith symplectic curvature equal to $\rho_i(t)$ for any $1 \le i \le m$.

Also in the sequel we will need the following

Remark 3. Assume that E is the strongly canonical section of $C(J_\gamma^{(4-n)})$ with respect to the parametrization ψ. Using the fact that the spaces $\mathrm{span}\Big\{\frac{d^j}{dt^j}E\big(\psi(t)\big)\Big\}_{j=1}^m$ are Lagrangian and the condition (25), it is easy to show that

$$\tilde{\sigma}_\gamma\Big(\frac{d^j}{dt^j}E\big(\psi(t)\big), \frac{d^i}{dt^i}E\big(\psi(t)\big)\Big)$$

are either identically equal to 0, if $i + j < 2m - 1$ or to ± 1, if $i + j = 2m - 1$, or they are polynomial expressions (with universal constant coefficients) with respect to the symplectic curvatures $\rho_1(t), \dots, \rho_m(t)$ and their derivatives, if $i + j > 2m$. □

Taking the ith symplectic curvature for Jacobi curves (parameterized by the canonical parameter) of all abnormal extremals living in $\widetilde{\mathcal{R}}$, we obtain the invariants of the regular control systems, called the *ith symplectic curvature* and denoted also by ρ_i. The symplectic curvatures are scalar valued functions on the set $\widetilde{\mathcal{R}}$.

9 The maximally symmetric models

Now we will find all structures from the considered classes having the pseudo-group of micro-local symmetries of dimension equal to $2n-3$. As a consequence of Corollary 1 if an object from the considered class has the pseudo-group of micro-local symmetries of dimension equal to $2n - 3$ then all structure functions of the canonical frame (39) must be constant. Note that formula (40) can be rewritten in terms of the canonical frame (39) as follows

$$[h, \varepsilon_{2m}] = \sum_{i=1}^{m}(-1)^{i+1}(\mathrm{ad}\, h)^{m-i}\Big(\rho_i\big(\mathrm{ad}\, h^{m-i}\varepsilon_1\big)\Big) \mod \mathrm{span}\{e, h\}, \qquad (41)$$

where ρ_i are the ith symplectic curvatures of a structures under consideration. This implies that the symplectic curvatures of all order must be constant for any structure from the considered classes having $2n-3$-dimensional pseudo-group of micro-local symmetries . This implies that the following theorem is equivalent to Theorems 1

Theorem 4. *Given any tuples of $n-3$ numbers $(r_1, \dots, r_{n-3})$ there exists the unique, up to micro-local equivalence, regular control system on a rank 2 distribution of maximal class in $\mathbb{R}^n$ with $n \geq 5$ having the group of micro-local symmetries of dimension $2n - 3$ and the ith symplectic curvature identically equal to r_i for any $1 \leq i \leq n-3$. Such regular control system is micro-locally equivalent to the system $A_{(r_1,\dots,r_{n-3})}$ defined by* (2)-(4).

Proof. First, let us prove the uniqueness. Take a structure from the considered class having the pseudo-group of micro-local symmetries of dimension $2n-3$ and the ith symplectic curvature identically equal to r_i for any $1 \leq i \leq m$, where, as before,

$m = n - 3$. Then, as was already mentioned, all structure functions of the canonical frame (39) must be constant. The uniqueness will be proved if we will show that all nontrivial structure function (i. e. those that are not prescribed by the normalization conditions for the canonical frame) are uniquely determined by the tuple $(r_1, \dots, r_{n-3})$.

Let ε_1 be as in the Lemma 1. Denote

$$\varepsilon_{i+1} := (\operatorname{ad} h)^i \varepsilon_1, \quad \nu = [\varepsilon_1, \varepsilon_{2m}] \tag{42}$$

In this notations the canonical frame (39) is $\{e, h, \varepsilon_1, \dots, \varepsilon_{2m}, \eta\}$.

1) Let us prove that

$$[e, \varepsilon_1] = -\frac{1}{2}\varepsilon_1 \tag{43}$$

where, as before e is the Euler field. Indeed, from (36)

$$[e, \varepsilon_1] = -\frac{1}{2}\varepsilon_1 + ae \tag{44}$$

where a is constant by our assumptions. Then, using the Jacobi identity and the fact that

$$[e, h] = 0 \tag{45}$$

we get that

$$[e, \varepsilon_2] = \big[e, [h, \varepsilon_1]\big] = \big[h, [e, \varepsilon_1]\big] = \left[h, -\frac{1}{2}\varepsilon_1 + ae\right] = -\frac{1}{2}\varepsilon_2. \tag{46}$$

Further, from the normalization condition (33) and formula (44) it follows that

$$\big[e, [\varepsilon_1, \varepsilon_2]\big] \in \operatorname{span}\{e(\lambda), h(\lambda), \varepsilon_1(\lambda)\}. \tag{47}$$

On the other hand, using the Jacobi identity and formulas (44),(45),(46), we get that

$$\begin{aligned}\big[e, [\varepsilon_1, \varepsilon_2]\big] &= \big[[e, \varepsilon_1], \varepsilon_2\big] + \big[\varepsilon_1, [e, \varepsilon_2]\big] = \left[-\frac{1}{2}\varepsilon_1 + ae, \varepsilon_2\right] - \frac{1}{2}[\varepsilon_1, \varepsilon_2] \\ &\equiv -\frac{a}{2}\varepsilon_2 \mod \operatorname{span}\{e(\lambda), h(\lambda), \varepsilon_1(\lambda)\},\end{aligned}$$

which together with (47) implies that $a = 0$.

2) By analogy with the chain of the equalities (47) we can prove that

$$[e, \varepsilon_i] = -\frac{1}{2}\varepsilon_i, \quad \forall 1 \le i \le 2m, \tag{48}$$

which in turn implies by the Jacobi identity that

$$\big[e, [\varepsilon_i, \varepsilon_j]\big] = -[\varepsilon_i, \varepsilon_j], \quad \forall 1 \le i, j \le 2m. \tag{49}$$

In particular, $[e, \eta] = -\eta$.

3) Let us show that

$$[h, \varepsilon_{2m}] = \sum_{i=1}^{m-1} (-1)^{i+1} r_i \varepsilon_{2(m-i)}. \tag{50}$$

From (41) and our assumptions it follows that

$$[h, \varepsilon_{2m}] = \sum_{i=1}^{m-1} (-1)^{i+1} r_i \varepsilon_{2(m-i)} + \gamma e + \delta h \tag{51}$$

for some constants γ and δ. Applying $\operatorname{ad} e$ to both sides of (51) and using the Jacobi identity and formulas (45) and (48), we will get that $\gamma = \delta = 0$, which implies (50).

4) Let us prove that

$$[\varepsilon_i, \varepsilon_j] = d_{ij} \eta \tag{52}$$

for some constants d_{ij} Indeed, in general

$$[\varepsilon_i, \varepsilon_j] = b_{ij} e + c_{ij} h + d_{ij} \eta + \sum_{k=1}^{2m} a_{ij}^k \varepsilon_k \tag{53}$$

where a_{ij}^k, b_{ij}, c_{ij} and d_{ij} are constant by our assumptions. Applying $\operatorname{ad} e$ to both sides of (53) and using the Jacobi identity and the formulas (45), (48), and (49), we get

$$-[\varepsilon_i, \varepsilon_j] = -d_{ij} \eta - \frac{1}{2} \sum_{k=1}^{2m} a_{ij}^k \varepsilon_k. \tag{54}$$

Comparing (53) and (54) we get that $a_{ij}^k = b_{ij} = c_{ij} = 0$, which implies (52).

5) Moreover, by Remark 3 and the definition of the vector field η (see (42)) the constants d_{ij} from (52) are either identically equal to 0, if $i + j < 2m$ or equal to $(-1)^{i-1}$, if $i + j = 2m + 1$, or they are polynomial expressions (with universal constant coefficients) with respect to the constant symplectic curvatures $r_1 \dots, r_m$, if $i + j > 2m$.

6) The remaining brackets of the canonical frame are obtained iteratively from the brackets considered in the previous items.

Therefore all nontrivial structure functions of the canonical frame are determined by the tuple $(r_1, \dots, r_{n-3})$, which completes the proof of uniqueness.

To prove the existence one checks by the direct computations that the models $A_{(r_1,\dots,r_m)}$ have the prescribed symplectic curvatures and that all structure functions of their canonical frame are constant similarly to the proof of the existence part of Theorem 3 in [10], devoted to the computation of the canonical frame for $D_{(0,\dots,0)}$. □

Remark 4. As a matter of fact it can be shown that Theorem 3 (with a modified set $\widetilde{\mathcal{R}}$), Corollary 1, and Theorem 4 are true if we replace the regularity condition for control systems given in Definition 2 by the following weaker one: for any point q the curve of admissible velocities $\mathcal{V}_q$ does not belong entirely to a line through the origin. One only needs more technicalities in the description of the set $\widetilde{\mathcal{R}}$ in Theorem 3. □.

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Remarks on Lipschitz domains in Carnot groups

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Abstract In this Note we present the basic features of the theory of Lipschitz maps within Carnot groups as it is developed in [8], and we prove that intrinsic Lipschitz domains in Carnot groups are uniform domains.

1 Introduction

The aim of this note is to provide a gist of few very basic points of the theory of Lipschitz maps within Carnot groups as it is developed in [8], and to present some applications to the study of the geometry of subsets of the groups.

Let us first establish a few notations concerning Carnot groups. For a general account, we refer, e. g. to [3, 7, 15].

A *graded group* of step κ is a connected, simply connected Lie group $\mathbb{G}$ whose finite dimensional Lie algebra $\mathfrak{g}$ is the direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa$$

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of κ subspaces $\mathfrak{g}_i$, $i = 1, \dots, \kappa$ such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

for $1 \le i, j \le \kappa$ and $\mathfrak{g}_i = 0$ for $i > \kappa$. We denote by n the dimension of $\mathfrak{g}$ and by m_j the dimension of $\mathfrak{g}_j$, for $1 \le j \le \kappa$.

A *Carnot group* $\mathbb{G}$ of step κ is a graded group of step κ, where $\mathfrak{g}_1$ generates all of $\mathfrak{g}$. That is $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, for $i = 1, \dots, \kappa$. We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i. e. we set

$$Q := \sum_{i=1}^{\kappa} i \dim(\mathfrak{g}_i).$$

If e is the unit element of $(\mathbb{G}, \cdot)$, we remind that the map $X \to X(e)$, that associates with a left-invariant vector field X its value at e, is an isomorphism from $\mathfrak{g}$ to $T\mathbb{G}_e$, in turn identified with $\mathbb{R}^n$. From now on, we shall use systematically these identifications. Thus, the horizontal layer defines, by left translation, a fiber bundle $H\mathbb{G}$ over $\mathbb{G}$ (the *horizontal bundle*). Its sections are the *horizontal vector fields*.

A Carnot group $\mathbb{G}$ can be always identified, through exponential coordinates, with the Euclidean space $(\mathbb{R}^n, \cdot)$, where n is the dimension of $\mathfrak{g}$, endowed with a suitable group operation. The explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula. From now on, $\mathbb{G}$ will be always a Carnot group written in exponential coordinates.

We choose a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ adapted to the stratification of $\mathfrak{g}$, i. e. such that

$$e_{h_{j-1}+1}, \dots, e_{h_j} \quad \text{is a basis of } \mathfrak{g}_j$$

where $h_0 = 0$ and $h_j = m_1 + \dots + m_j$, for each $j = 1, \dots, \kappa$. Then, we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathfrak{g}$ making the adapted basis $\{e_1, \dots, e_n\}$ orthonormal. Finally $\{X_1, \dots, X_n\}$ is the family of left invariant vector fields such that $X_i(e) = e_i$, for $i = 1, \dots, n$.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{d_1} x_1, ..., \lambda^{d_n} x_n),$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [7] Chap. 1) and is defined as

$$d_j = i \quad \text{whenever} \quad h_{i-1} + 1 \le j \le h_i,$$

hence $1 = d_1 = ... = d_{m_1} < d_{m_1+1} = 2 \le ... \le d_n = \kappa$.

Through this note, homogeneity will be always meant with respect to group dilations δ_λ (see again [7], Chap. 1).

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$.

Definition 1. An absolutely continuous curve $\gamma : [0,T] \to \mathbb{G}$ is a sub-unit curve with respect to $X_1,\dots,X_{m_1}$ if there exist measurable real functions $c_1(s),\dots,c_{m_1}(s)$, $s \in [0,T]$ such that $\sum_j c_j^2 \le 1$ and

$$\dot\gamma(s) = \sum_{j=1}^{m_1} c_j(s) X_j(\gamma(s)), \quad \text{for a.e. } s \in [0,T].$$

Definition 2. If $p, q \in \mathbb{G}$, we define their Carnot-Carathéodory distance as

$$d_c(p,q) := \inf\{T > 0 : \text{ there exists a sub-unit curve } \gamma \text{ with } \gamma(0) = p,\ \gamma(T) = q\}.$$

By Chow's Theorem, the set of sub-unit curves joining p and q is not empty for all $p, q \in \mathbb{G}$, furthermore d_c is a distance on $\mathbb{G}$ that induces the Euclidean topology (see Chap. 19 in [3] or Theorem 1.6.2 in [26]). It is also important to stress that $(\mathbb{G}, d_c)$ is a metric space *with geodesics*.

More generally, given any homogeneous norm $\|\cdot\|$, it is possible to define a distance in $\mathbb{G}$ as

$$d(x,y) = d(y^{-1}\cdot x, 0) = \left\|y^{-1}\cdot x\right\|, \quad \text{for all } x, y \in \mathbb{G}. \tag{1}$$

The distance d in (1) is comparable with the Carnot-Carathéodory distance of $\mathbb{G}$ and

$$d(z\cdot x, z\cdot y) = d(x,y), \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y)$$

for all $x, y, z \in \mathbb{G}$ and all $\lambda > 0$.

A possible convenient homogeneous norm (described in [10, Theorem 5.1]) is the following one, if $p = (p^1,\dots,p^\kappa) \in \mathbb{R}^n = \mathbb{G}$, with $p^j \in \mathbb{R}^{m_j - m_{j-1}}$, for $j = 1,\dots,\kappa$, then

$$d_\infty(p,0) := \|p\| := \max_{j=1,\dots,\kappa}\{\varepsilon_j \left\|p^j\right\|_{\mathbb{R}^{m_j-m_{j-1}}}^{1/d_j}\},$$

where $\varepsilon_1 = 1$, and $\varepsilon_2,\dots\varepsilon_\kappa \in (0,1]$ are suitable positive constants depending on $\mathbb{G}$. For $r > 0$ and $p \in \mathbb{G}$, we denote by $B_c(p,r)$ the open balls associated with the Carnot-Carathéodory distance d_c, and by $B(p,r)$ the ones associated with d or d_∞.

The following results will be used throughout this note.

Proposition 1 ([10], Proposition 2.4). *Let d be a distance in $\mathbb{G}$ such that*

$$d(z\cdot x, z\cdot y) = d(x,y), \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y)$$

for $x, y, z \in \mathbb{G}$ and $\lambda > 0$, and denote by B_d the closed d-balls. Then

$$\operatorname{diam}_d(B_d(x,r)) = 2r, \quad \textit{for } r > 0.$$

Lemma 1 ([8], Lemma 2.2.12). *Let $\mathbb{G}$ be a step κ group. There is $C = C(\mathbb{G}) > 0$ such that*

$$\left\| y^{-1} \cdot x \cdot y \right\| \le \|x\| + C\left(\|x\|^{\frac{1}{\kappa}} \|y\|^{\frac{\kappa-1}{\kappa}} + \|y\|^{\frac{1}{\kappa}} \|x\|^{\frac{\kappa-1}{\kappa}}\right), \quad \textit{for all } x, y \in \mathbb{G}.$$

Definition 3. A *homogeneous subgroup* of a Carnot group $\mathbb{G}$ (see [30, 5.2.4]) is a Lie subgroup $\mathbb{M}$ of $\mathbb{G}$ such that $\delta_\lambda g \in \mathbb{M}$, for all $g \in \mathbb{M}$ and for all $\lambda > 0$.

Remark 1. Homogeneous subgroups are linear subspaces of $\mathbb{G}$, when $\mathbb{G}$ is identified with $\mathbb{R}^n$ with exponential coordinates. Moreover, an homogeneous subgroup $\mathbb{M}$ is stratified, that is $\mathbb{M} = \mathbb{M}^1 \oplus \cdots \oplus \mathbb{M}^\kappa$, where $\mathbb{M}^i \subset \mathbb{G}^i := \exp(\mathfrak{g}_i)$ and $\mathbb{M}^i$ is a linear subspace of $\mathbb{G}^i$.

Definition 4. Let $\mathbb{M}$, $\mathbb{N}$ be homogeneous subgroups of $\mathbb{G}$. We say that $\mathbb{M}, \mathbb{N}$ are *complementary homogeneous subgroups* or, briefly, *complementary subgroups* in $\mathbb{G}$, if $\mathbb{M} \cap \mathbb{N} = \{e\}$ and

$$\mathbb{G} = \mathbb{M} \cdot \mathbb{N},$$

that is for each $g \in \mathbb{G}$, there are $m \in \mathbb{M}$ and $n \in \mathbb{N}$ such that $g = m \cdot n$.

If $\mathbb{M}$ and $\mathbb{N}$ are complementary subgroups of $\mathbb{G}$, the elements $m \in \mathbb{M}$ and $n \in \mathbb{N}$ such that $g = m \cdot n$ are unique because $\mathbb{M} \cap \mathbb{N} = \{e\}$ and are denoted as *components* of g along $\mathbb{M}$ and $\mathbb{N}$ or as *projections* of g on $\mathbb{M}$ and $\mathbb{N}$. We write $m := \mathbf{P}_{\mathbb{M}} g$, $n := \mathbf{P}_{\mathbb{N}} g$.

If $\mathbb{M}$, $\mathbb{N}$ are complementary subgroups of $\mathbb{G}$ and one of them is a normal subgroup then $\mathbb{G}$ is said to be the *semi-direct product* of $\mathbb{M}$ and $\mathbb{N}$.

Example 1. Let $\mathbb{G}$ be the Heisenberg group $\mathbb{H}^n$. Then all the possible couples of complementary subgroups of $\mathbb{H}^n$ contain a horizontal subgroup $\mathbb{V}$ of dimension $k \le n$, isomorphic and isometric to $\mathbb{R}^k$ and a normal subgroup $\mathbb{W}$ of homogeneous dimension $2n + 1 - k$, containing the centre $\mathbb{T}$. Moreover $\mathbb{W}^1 \oplus \mathbb{V} = \mathbb{G}^1$.

Very similar splittings exist in a general Carnot group $\mathbb{G}$. Indeed, choose any homogeneous horizontal subgroup $\mathbb{N}$, i. e. an homogeneous subgroup $\mathbb{N}$ contained in the horizontal layer $\mathbb{G}^1$, and a subgroup $\mathbb{M}$ such that

$$\mathbb{N} \oplus \mathbb{M}^1 = \mathbb{G}^1, \tag{i}$$

$$\mathbb{G}^j \subset \mathbb{M} \quad \text{for all } 2 \le j \le \kappa, \tag{ii}$$

then $\mathbb{M}$ and $\mathbb{N}$ are complementary subgroups in $\mathbb{G}$ and $\mathbb{M}$ is a normal subgroup.

Example 2. Decomposition of a Carnot group $\mathbb{G}$ as in Definition 4 are canonically associated with left invariant covectors of Rumin's complex (E_0^*, d_c). Necessarily, we must be very sketchy here. For further details we refer the reader to [2,8,13,28,29]. Following the notations of [17], p.90, if X is a vector field, we denote by $i(X)$ the interior product with X. Then we have:

Theorem 1. *If $1 \le h < n$, $\xi \in E_0^h$ and $\omega \in E_0^{n-h}$ are simple covectors such that*

$$\xi \wedge \omega \neq 0,$$

we set

$$\mathfrak{m} := \{X \in \mathfrak{g} \,:\, i(X)\xi = 0\}, \quad \mathfrak{h} := \{X \in \mathfrak{g} \,:\, i(X)\omega = 0\}.$$

Then both $\mathfrak{m}$ *and* $\mathfrak{h}$ *are Lie subalgebras of* $\mathfrak{g}$. *Moreover* $\dim \mathfrak{m} = n - h$, $\dim \mathfrak{h} = h$ *and* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$.

If, in addition, $\xi = \xi_1 \wedge \cdots \wedge \xi_h$, $\omega = \omega_1 \wedge \cdots \wedge \omega_{n-h}$, *where all the* ξ_i*'s and the* ω_i *have pure weights* $\mathbf{p}_i$ *and* q_i, *respectively, then both* $\mathfrak{m}$ *and* $\mathfrak{h}$ *are homogeneous Lie subalgebras of* $\mathfrak{g}$. *Thus, if we set*

$$\mathbb{M} := \exp(\mathfrak{m}) \quad \textit{and} \quad \mathbb{N} := \exp(\mathfrak{h}),$$

then $\mathbb{M}$ *and* $\mathbb{N}$ *is a couple of complementary subgroups as in Definition 4.*
In particular, since $*E_0^h = E_0^{n-h}$, *if* $\xi \in E_0^h$, *we can choose* $\omega := *\xi$. *In this case,* $\mathfrak{m}$ *and* $\mathfrak{h}$ *are orthogonal.*
Reciprocally, suppose $\mathfrak{m}$ *and* $\mathfrak{h}$ *are two homogeneous Lie subalgebras of* $\mathfrak{g}$ *such that* $\dim \mathfrak{m} = n - h$, $\dim \mathfrak{h} = h$, *and* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$.
Then there exist a scalar product $\langle \cdot, \cdot \rangle_0$ *in* $\mathfrak{g}$, $\xi \in E_0^h$ *and* $\omega \in E_0^{n-h}$ *such that* $\xi \wedge \omega \neq 0$ *and*

$$\mathfrak{m} := \{X \in \mathfrak{g} \,:\, i(X)\xi = 0\}, \quad \mathfrak{h} := \{X \in \mathfrak{g} \,:\, i(X)\omega = 0\}.$$

Proposition 2. *If* $\mathbb{M}, \mathbb{N}$ *are complementary subgroups in* $\mathbb{G}$ *there is* $c_0 = c_0(\mathbb{M}, \mathbb{N}) > 0$ *such that for all* $g = m \cdot n$

$$c_0 \left(\|m\| + \|n\|\right) \leq \|g\| \leq \|m\| + \|n\| \,.$$

The sizes of the components $\mathbf{P}_{\mathbb{M}} g$ and $\mathbf{P}_{\mathbb{N}} g$ control the distance of $g \in \mathbb{G}$ from the complementary subspaces $\mathbb{M}$ and $\mathbb{N}$. The control is different when considering the distance of g from the first component $\mathbb{M}$ or from the second component $\mathbb{N}$.

Proposition 3. *Let* $\mathbb{G}$ *be a step* κ *group with* $\mathbb{M}$ *and* $\mathbb{N}$ *complementary subgroups. Then,*

$$c_0 \left\|\mathbf{P}_{\mathbb{N}} g\right\| \leq \operatorname{dist}(g, \mathbb{M}) \leq \left\|\mathbf{P}_{\mathbb{N}} g\right\|, \quad \textit{for all } g \in \mathbb{G},$$

where c_0 *is the constant in Proposition 2. Moreover, there is* $c_1 = c_1(\mathbb{M}, \mathbb{N}) > 0$ *such that*

$$\frac{1}{c_1} \left\|\mathbf{P}_{\mathbb{M}} g\right\|^{\kappa} \leq \operatorname{dist}(g, \mathbb{N}) \leq c_1 \left\|\mathbf{P}_{\mathbb{M}} g\right\|^{1/\kappa}, \quad \textit{if } \|g\| = 1.$$

Proposition 4. *Let* $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$ *then the projection maps* $\mathbf{P}_{\mathbb{M}} : \mathbb{G} \to \mathbb{M}$ *and* $\mathbf{P}_{\mathbb{N}} : \mathbb{G} \to \mathbb{N}$ *are polynomial maps.*

2 Graphs and Lipschitz graphs

If a Carnot group $\mathbb{G}$ admits a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$ as a product of complementary homogeneous subgroups, then we give a natural notion of graph within $\mathbb{G}$.

Definition 5. Suppose $\mathbb{G}$ admits a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$ as a product of complementary homogeneous subgroups. We say that $S \subset \mathbb{G}$ is a *(left)* $\mathbb{N}$*-graph* (or a left graph in direction $\mathbb{N}$) if there is $\varphi : \mathcal{E} \subset \mathbb{M} \to \mathbb{N}$ such that

$$S = \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\}.$$

We write $S = \text{graph}\,(\varphi) := \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\}$.

By uniqueness of the components along $\mathbb{M}$ and $\mathbb{N}$, if $S = \text{graph}\,(\varphi)$ then φ is uniquely determined among all functions from $\mathbb{M}$ to $\mathbb{N}$.

This notion of graph is intrinsic, in the sense that is invariant under group translations and dilations (we refer again to [8] for an exhaustive presentation). Indeed

Proposition 5. *If* $\mathbb{M}, \mathbb{N}$ *are complementary subgroups in* $\mathbb{G}$*, if* $S = \text{graph}\,(\varphi)$ *with* $\varphi : \mathcal{E} \subset \mathbb{M} \to \mathbb{N}$*, then*

$$\text{for all } \lambda > 0,\ \delta_\lambda S = \text{graph}\,(\varphi_\lambda), \quad \text{with } \varphi_\lambda : \delta_\lambda \mathcal{E} \subset \mathbb{M} \to \mathbb{N} \quad \text{and}$$
$$\varphi_\lambda(m) := \delta_\lambda \varphi(\delta_{1/\lambda} m), \quad \text{for } m \in \delta_\lambda \mathcal{E};$$

for all $q \in \mathbb{G}$, $q \cdot S = \text{graph}\,(\varphi_q)$*, where* $\varphi_q : \mathcal{E}_q := \{m : \mathbf{P}_\mathbb{M}(q^{-1} \cdot m) \in \mathcal{E}\} \to \mathbb{N}$*, and*

$$\varphi_q(m) = \big(\mathbf{P}_\mathbb{N}(q^{-1} \cdot m)\big)^{-1} \cdot \varphi\big(\mathbf{P}_\mathbb{M}(q^{-1} \cdot m)\big), \quad \text{for all } m \in \mathcal{E}_q.$$

Remark 2. In this paper we consider only graphs of functions acting between complementary subgroups. Nevertheless, it is relevant to mention that it is possible to give a more general notion of $\mathbb{N}$-graph also when $\mathbb{N}$ fails to admit a complementary subgroup. For instance, $\mathbb{T}$-graphs, where $\mathbb{T}$ is the centre of the Heisenberg group $\mathbb{H}^N$, have been studied recently. We refer to [8] for details.

Given a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$ there are natural notions of intrinsic cones in $\mathbb{G}$. We refer also to [8] and to [5] for different but related definitions.

Definition 6. If $\mathbb{M}, \mathbb{N}$ are complementary subgroups in $\mathbb{G}$, $q \in \mathbb{G}$ and $\beta \geq 0$ the cones $C_{\mathbb{M},\mathbb{N}}(q, \beta)$, with base $\mathbb{M}$, axis $\mathbb{N}$, vertex q, opening β are defined as

$$C_{\mathbb{M},\mathbb{N}}(e, \beta) = \{p : \|\mathbf{P}_\mathbb{M} p\| \leq \beta \|\mathbf{P}_\mathbb{N} p\|\},$$
$$C_{\mathbb{M},\mathbb{N}}(q, \beta) = q \cdot C_{\mathbb{M},\mathbb{N}}(e, \beta).$$

We say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{N}$ is *intrinsic L-Lipschitz* in $\mathcal{E}$ if there is $L > 0$ such that

$$C_{\mathbb{M},\mathbb{N}}(p, 1/L) \cap \text{graph}\,(f) = \{p\}, \quad \text{for all } p \in \text{graph}\,(f). \tag{2}$$

The Lipschitz constant of f in $\mathcal{E}$ is the infimum of the $L > 0$ such that (2) holds.

The notion of intrinsic Lipschitz graph is invariant under group translations and group dilations. Indeed we have:

Theorem 2. *Let* $\mathbb{M}, \mathbb{N}$ *be complementary subgroups in a Carnot group* $\mathbb{G}$*:*

1) *if* $S \subset \mathbb{G}$ *is an intrinsic Lipschitz* $\mathbb{N}$*-graph then* $q \cdot S$ *is an intrinsic Lipschitz* $\mathbb{N}$*-graph, for all* $q \in \mathbb{G}$*;*
2) *if* $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{N}$ *is intrinsic L-Lipschitz in* $\mathcal{E}$*, then* $f_q : \mathcal{E}_q \subset \mathbb{M} \to \mathbb{N}$ *is intrinsic L-Lipschitz in* $\mathcal{E}_q$*, for all* $q \in \mathbb{G}$*.*

The geometric definition of intrinsic Lipschitz graphs has equivalent algebraic forms (see [8] and also [1, 11, 12]).

Proposition 6. *Let* $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$*,* $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{N}$ *and* $L > 0$*. Then are equivalent*

$$f \text{ is intrinsic } L\text{-Lipschitz in } \mathcal{E}. \tag{i}$$

$$\left\| \mathbf{P}_{\mathbb{N}}(\bar{q}^{-1} \cdot q) \right\| \le L \left\| \mathbf{P}_{\mathbb{M}}(\bar{q}^{-1} \cdot q) \right\|, \quad \text{for all } q, \bar{q} \in \operatorname{graph}(f). \tag{ii}$$

Moreover, the distance of two points $q, \bar{q} \in \operatorname{graph}(f)$*, is bounded by the norm of their projection on the domain* $\mathbb{M}$*. Precisely*

$$\left\| \bar{q}^{-1} \cdot q \right\| \le c_0(1 + L) \left\| \mathbf{P}_{\mathbb{M}}(\bar{q}^{-1} \cdot q) \right\| \implies \left\| \mathbf{P}_{\mathbb{N}}(\bar{q}^{-1} \cdot q) \right\| \le L \left\| \mathbf{P}_{\mathbb{M}}(\bar{q}^{-1} \cdot q) \right\|,$$

where $c_0 < 1$ *is the constant in Proposition 2, and conversely*

$$\left\| \mathbf{P}_{\mathbb{N}}(\bar{q}^{-1} \cdot q) \right\| \le L \left\| \mathbf{P}_{\mathbb{M}}(\bar{q}^{-1} \cdot q) \right\| \implies \left\| \bar{q}^{-1} \cdot q \right\| \le (1 + L) \left\| \mathbf{P}_{\mathbb{M}}(\bar{q}^{-1} \cdot q) \right\|,$$

for all $q, \bar{q} \in \operatorname{graph}(f)$*.*

Remark 3. Intrinsic Lipschitz functions between complementary homogeneous subgroups are extensively studied in [8] (see also [12]). In particular, in [8] the authors show that:

- the boundary of a positive intrinsic cone is an intrinsic Lipschitz graph;
- an extension theorem for 1-codimensional intrinsic Lipschitz graphs;
- a Rademacher's type Theorem for 1-codimensional intrinsic Lipschitz graphs in a large class of Carnot groups.

3 Intrinsic Lipschitz domains

From now on, we assume that $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$, where as usual $\mathbb{M}$ and $\mathbb{N}$ are complementary homogeneous subgroups and $\mathbb{N}$ is *one dimensional* and *horizontal*. Precisely we assume the existence of $V \in \mathfrak{g}_1$ such that $\mathbb{N} = \{\exp(tV) : t \in \mathbb{R}\}$. We notice that under these assumptions, $\mathbb{M}$ is always a normal subgroup.

When dealing with $\mathbb{N}$-valued functions, using that $\mathbb{N}$ is one dimensional, we can characterize an intrinsic Lipschitz function by the fact that its subgraph and its supergraph contain half cones.

More precisely for $f : \mathcal{U} \subset \mathbb{M} \to \mathbb{N}$, with $f(m) = \exp(\varphi(m)V)$ and $\varphi : \mathcal{U} \to \mathbb{R}$, we define the *supergraph* E_f^+ and the *subgraph* E_f^- of f as

$$E_f^- := \{m \cdot \exp(tV) : \ m \in \mathcal{U},\ t < \varphi(m)\},$$
$$E_f^+ := \{m \cdot \exp(tV) : \ m \in \mathcal{U},\ t > \varphi(m)\}.$$

Notice that, if $f : \mathbb{M} \to \mathbb{N}$ is continuous,

$$\overline{E_f^-} = \{m \exp(tV) : \ m \in \mathbb{M},\ t \leq \varphi(m)\},$$
$$\overline{E_f^+} = \{m \exp(tV) : \ m \in \mathbb{M},\ t \geq \varphi(m)\}.$$

We also define the half cones $C^+_{\mathbb{M},\mathbb{N}}(p,\beta)$ as $C^+_{\mathbb{M},\mathbb{N}}(p,\beta) := p \cdot C^+_{\mathbb{M},\mathbb{N}}(e,\beta)$, where

$$C^+_{\mathbb{M},\mathbb{N}}(e,\beta) := \{p \in \mathbb{G} : \mathbf{P}_{\mathbb{N}}\, p = \exp(tV), \text{with } t \geq 0\}.$$

The definition of $C^-_{\mathbb{M},\mathbb{N}}(p,\beta)$ is analogous.

Lemma 2. $f : \mathbb{M} \to \mathbb{N}$ *is intrinsic L-Lipschitz if and only if,*

$$C^+_{\mathbb{M},\mathbb{N}}(mf(m), 1/L) \subset \overline{E_f^+} \quad \textit{and} \quad C^-_{\mathbb{M},\mathbb{N}}(mf(m), 1/L) \subset \overline{E_f^-},$$

for all $m \in \mathbb{M}$.

Intrinsic Lipschitz domains are domains whose boundaries are locally graphs of intrinsic Lipschitz functions acting between homogeneous subgroups of $\mathbb{G}$ and such that the domain lies on one side of the graph.

Definition 7. Let Ω be a domain of $\mathbb{G}$. We say that Ω is an *intrinsic Lipschitz domain* if, for each $z \in \partial\Omega$, there are $r_0 > 0$, a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$, with $\mathbb{N}$ one dimensional and horizontal, an intrinsic Lipschitz map $f : \mathcal{E} \subset \mathbb{M} \longrightarrow \mathbb{N}$, with $\mathcal{E}$ relatively open in $\mathbb{M}$, such that

$$\overline{\Omega} \cap B(z, r_0) \ = \ \overline{E_f^-} \cap B(z, r_0).$$

In the sequel, we prove that bounded intrinsic Lipschitz domains are *uniform domains* according to the following definition.

Definition 8. Let $\Omega \subset \mathbb{G}$ be a bounded connected open set. We say that Ω is a *uniform domain* if there exists $\varepsilon > 0$ such that for every $x, y \in \Omega$ there is a continuous rectifiable curve $\gamma : [0, 1] \longrightarrow \Omega$ joining x to y with

$$\text{length}(\gamma) \ \leq \ \frac{1}{\varepsilon}\, d_c(x, y), \tag{3}$$

and for every $t \in [0, 1]$

$$\text{dist}\,(\gamma(t), \partial\Omega) \ \geq \ \varepsilon \min\left\{\text{length}(\gamma_{|[0,t]}), \text{length}(\gamma_{|[t,1]})\right\}. \tag{4}$$

Uniform domains (also known as (ε, δ)-domains) are a sub-class of John domains and have been introduced by Martio and Sarvas [23] and Jones [19]. But we refer also the reader to the thoughtful review contained in Monti's PhD thesis ([27]), that, at many points, was a precious help for the authors of this paper.

It is well known that, up to a reparametrization, rectifiable curves are 1-Lipschitz continuous and therefore are sub-unit curves by [16], Proposition 11.4. Thus, Definition 8 can be rephrased in terms of horizontal curves. In other words, our notion of uniform domain is *intrinsic*, in the sense that depends only on the structure of the Lie algebra $\mathfrak{g}$. For a careful discussion of the relationships between the notion of intrinsic uniform domain and that of Euclidean uniform domain we refer the reader to [5].

Our main result reads as follows:

Theorem 3. *Let $\Omega \subset \mathbb{G}$ be a bounded intrinsic Lipschitz domain. Then Ω is a uniform domain.*

First of all, following [31], the problem can be localized thanks to the following result:

Lemma 3. *Let $\Omega \subset \mathbb{G}$ be a bounded open set and let* $0 < r < \mathrm{diam}(\Omega)$. *If there is $\varepsilon > 0$ such that for every $z \in \partial\Omega$ and for every $x, y \in \Omega \cap B(z, r)$, there exists a continuous rectifiable curve $\gamma : [0, 1] \longrightarrow \Omega$, joining x and y, such that* (3) *and* (4) *hold, then Ω is a uniform domain.*

The proof is based upon ideas found in [33], where the author gives a characterization of intrinsic Lipschitz domains.

Theorem 4 (see [33]). *A bounded open set $\Omega \subset \mathbb{G}$ is an intrinsic Lipschitz domain if and only if for each $z \in \partial\Omega$ there exists a neighbourhood of z, say $\mathcal{U}$, a metric Lipschitz function $F : \mathcal{U} \longrightarrow \mathbb{R}$ and an $X \in \mathfrak{g}_1$ such that:*

1) $\Omega \cap \mathcal{U} = \{x \in \mathcal{U} : F(x) < 0\}$;
2) *there exists $l > 0$ such that $XF \geq l$, $\mathcal{L}^n$-a.e. on $\mathcal{U}$, where XF has to be meant in the distributional sense. Notice that $XF \in L^\infty(\mathcal{U})$ by [14], Theorem 1.3.*

Proof (Proof of Theorem 3). Let $z \in \partial\Omega$ be fixed. From Theorem 4, there exist an open neighbourhood $\mathcal{U}$ of z, an $X \in \mathfrak{g}_1$ (identified with a first order differential operators), a metric Lipschitz function $F : \mathcal{U} \subset \mathbb{G} \longrightarrow \mathbb{R}$ such that $XF \geq l$ $\mathcal{L}^n$-a.e. on $\mathcal{U}$ and

$$\partial\Omega \cap \mathcal{U} = \{x \in \mathcal{U} : F(x) = 0\}, \quad \Omega \cap \mathcal{U} = \{x \in \mathcal{U} : F(x) < 0\}.$$

Let $x, y \in \Omega \cap \mathcal{U}$. We shall construct a rectifiable curve $\Gamma : [0, 1] \longrightarrow \Omega \cap \mathcal{U}$ such that properties (3) and (4) are satisfied. We divide the proof in a number of small steps.

Step 1. Let $R > 0$ be such that the ball $B(z, 2R)$ is entirely contained in the open set $\mathcal{U}$. Take $p \in B(z, R/2) \cap \Omega$. If $t \in (0, R/2)$, then

$$p \cdot \exp(tX) \in B(z, R).$$

Indeed, since the exponential map is an isometry along horizontal directions,

$$\begin{aligned} d_c(p\cdot\exp(tX),z) &\le d_c(p\cdot\exp(tX),p) + d_c(p,z) \\ &< t + \frac{R}{2} < R. \end{aligned}$$

Step 2. Take again $p \in B(z,R/2)\cap\Omega$ and consider a point $q\in\partial\Omega$ which realizes the distance of p from $\partial\Omega$. We take $\sigma:[0,1]\longrightarrow\mathbb{G}$ to be a geodesic joining p to q. If $\xi\in\sigma([0,1])$, then

$$\begin{aligned} d_c(\xi,z) &\le d_c(\xi,p) + d_c(p,z) \\ &\le d_c(p,q) + d_c(p,z) \\ &\le 2\,d_c(p,z) < R. \end{aligned}$$

This chain of inequalities implies that the support of σ is entirely contained in $B(z,R)$.

Step 3. The idea now is to use the function F to measure how much points inside Ω are far from the boundary. First, since F is metric L-Lipschitz continuous, we can write

$$|F(p)| = |F(p)-F(q)| \le L\,d_c(p,q) = L\,\mathrm{dist}(p,\partial\Omega). \tag{5}$$

On the other hand, let us assume that $p\in B(z,\varepsilon R)\cap\Omega$ for some $\varepsilon\in(0,1/2)$. We aim to prove that there is $l>0$ such that

$$|F(p)| \ge l\,\mathrm{dist}(p,\partial\Omega). \tag{6}$$

Using the classical technique of convolution in homogeneous groups we set $\tilde\varepsilon := \varepsilon R$ and, since $B(p,\tilde\varepsilon)$ is entirely contained in $B(z,2R)\subset\mathcal{U}$, we can estimate, for $x\in B(z,R)$:

$$\begin{aligned} X(\eta_{\tilde\varepsilon}*F)(x) &= \int_{B(p,\tilde\varepsilon)} \eta_{\tilde\varepsilon}(x\cdot q^{-1})(XF)(q)dq \\ &\ge l\int_{B(p,\tilde\varepsilon)} \eta_{\tilde\varepsilon}(x\cdot q^{-1})dq \\ &\ge l. \end{aligned}$$

Then, for $t\in(0,1/2)$,

$$\begin{aligned} F(p\cdot\exp(tX)) - F(p) &= \lim_{\varepsilon\to 0}\left((\eta_{\tilde\varepsilon}*F)\,(p\cdot\exp(tX)) - (\eta_{\tilde\varepsilon}*F)(p)\right) \\ &= \lim_{\varepsilon\to 0}\int_0^t X(\eta_{\tilde\varepsilon}*F)(p\cdot\exp(sX))ds \\ &\ge lt. \end{aligned}$$

Therefore

$$F(p \cdot \exp(tX)) \geq lt + F(p) \geq lt + \min_{\bar{\Omega} \cap B(z,\varepsilon R)} F,$$

and, since $F(z) = 0$, we can choose $\varepsilon > 0$ such that

$$l\frac{R}{3} + \min_{\bar{\Omega} \cap B(z,\varepsilon R)} F > 0.$$

So there exists $t_p \in (0, R/3)$ such that $p \cdot \exp(t_p X) \in \partial\Omega$. Now,

$$\operatorname{dist}(p, \partial\Omega) \leq d_c(p, p \cdot \exp(t_p X)) \leq t_p;$$

hence, if $p \in B(z, \varepsilon R) \cap \Omega$, using the same technique as above, we conclude

$$-F(p) = F(p \cdot \exp(t_p X)) - F(p) \geq l\, t_p \geq l \operatorname{dist}(p, \partial\Omega).$$

Step 4. To prove the Theorem we use Lemma 3 and we construct a rectifiable curve with the required properties. Let us assume that $x, y \in \mathbb{B}(z, \delta)$, where $\delta < \varepsilon R$. We denote $d := d_c(x, y)$, $x' = x \cdot \exp(-dMX)$, $y' = y \cdot \exp(-dMX)$, for some constant $0 < M < R/4$ to be determined. We consider the curve

$$\Gamma(t) := \begin{cases} x \cdot \exp(-tX), & t \in [0, dM], \\ \gamma(t), & t \in [dM, dM + d_c(x', y')], \\ y \cdot \exp\left((t - \tilde{M})X\right), & t \in [dM + d_c(x', y'), \tilde{M}], \end{cases}$$

where γ is a geodesic joining x' to y' and $\tilde{M} := 2dM + d_c(x', y')$. Let us prove that this curve is the one we are looking for.

Step 5. Using Lemma 1, one has:

$$\begin{aligned} \operatorname{length}(\Gamma) &\leq 2dM + d_c(x', y') \\ &\leq 2dM + C \,\|\exp(dMX) \cdot y^{-1} \cdot x \cdot \exp(-dMX)\| \\ &\leq 2dM + C\,\|y^{-1} \cdot x\| + C_1 \Big(\|y^{-1} \cdot x\|^{\frac{1}{\kappa}} \,\|\exp(dMX)\|^{\frac{\kappa-1}{\kappa}} \\ &\qquad\qquad + \|y^{-1} \cdot x\|^{\frac{\kappa-1}{\kappa}} \,\|\exp(dMX)\|^{\frac{1}{\kappa}} \Big) \\ &\leq d_c(x, y) \left(2M + C_2 \left(1 + M^{\frac{\kappa-1}{\kappa}} + M^{\frac{1}{\kappa}}\right)\right), \end{aligned}$$

and this provides inequality (3).

Step 6. Let us prove inequality (4). We need to distinguish points in the three pieces of Γ. Let $t \in [0, dM]$ be fixed. First, we notice that

$$\begin{aligned} -F(\Gamma(t)) &\geq -F(x \cdot \exp(-tX)) + F(x) \\ &\geq l\, t. \end{aligned}$$

We combine this inequality with (5) and (6) of step 3 in order to obtain

$$\mathrm{dist}(\Gamma(t), \partial\Omega) \geq \frac{l}{L} t \geq \frac{l}{L} \mathrm{length}\left(\Gamma_{|[0,t]}\right).$$

In the same way, one can prove the inequality for $t \in [dM + d_c(x', y'), \tilde{M}]$.

Step 7. Let us consider $t \in [dM, dM + d_c(x', y')]$ and denote $\xi := \Gamma(t)$. If $\eta \in \partial\Omega$, we have, mimicking computations that we already did,

$$\begin{aligned} d_c(\xi, \eta) &\geq d_c(x', \eta) - d_c(\xi, x') \\ &\geq \mathrm{dist}(x', \partial\Omega) - d_c(x', y') \\ &\geq d_c(x, y) \left(\frac{l}{L} M - C_2 \left(1 + M^{\frac{\kappa-1}{\kappa}} + M^{\frac{1}{\kappa}} \right) \right). \end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned} \mathrm{length}\left(\Gamma_{|[0,t]}\right) &\leq d_c(x', y') + dM \\ &\leq d_c(x, y) \left(C_2 \left(1 + M^{\frac{\kappa-1}{\kappa}} + M^{\frac{1}{\kappa}} \right) + M \right) \\ &= C_3 d_c(x, y). \end{aligned} \tag{8}$$

Therefore, if we choose $M > 0$ such that $\frac{l}{L} M - C_2 \left(1 + M^{\frac{\kappa-1}{\kappa}} + M^{\frac{1}{\kappa}} \right) > 1$, we can combine (7) and (8) and the assertion follows. □

Remark 4. In fact, the proof of Theorem 3 yields a slightly stronger result, namely that Ω is a NTA-domain in the sense of [4,18]. We refer to [4] and [5] for the notion of NTA-domains in Carnot groups, as well as for several examples and contrexamples. The crucial point is that, by Proposition 1, uniform domains in Carnot groups satisfy the so-called *Harnack chain condition*: see for instance [27], Proposition 3.1.18.

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Differential-geometric and invariance properties of the equations of Maximum Principle (MP)

Revaz V. Gamkrelidze

Abstract An invariant formulation of the Pontryagin Maximum Principle (PMP) is given. It is proved that the Pontryagin derivative $\mathcal{P}_X$ coincides on vector fields $X \in \operatorname{Vect} M$, ($M$ – the configuration space of the problem), with the Lie bracket ad_X, and the flow generated on the cotangent bundle T^*M by the vector field $\mathcal{P}_X$ is bundle-preserving.

1 Introduction

I think, I should start my contribution by asking the audience not to consider it as a report on some latest news in optimal control, but rather as a lecture on its foundations, on differential-geometric and invariance properties of the equations of MP, and general consequences, which these properties imply.

And certainly, as a modest offering to Andrey's 60th anniversary jubilee.

It is a common mathematical evidence that tells us to expect every system of differential equations, which proved its general mathematical validity, to have interesting differential-geometric and invariance properties, and equations of MP should not be, presumably, an exception. Certainly, we should not expect that already in the first order we will be led to new geometric invariants, though in the second order we really come to a new curvature type invariant of a Hamiltonian system.

I shall only discuss the first order invariants generated by the equations of MP, in fact, objects of common and "everyday" mathematical usage, and still manifesting some unexpected relations to our subject, at least for me.

I shall be concerned with the case of time-optimal problem only, which is no restriction in generality, since an arbitrary optimal problem with a minimized integral type functional is canonically reduced to the time optimal problem by considering

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the minimized integral with free upper limit as a new time variable. The time-optimal problem has not only the advantage of being simpler exposed than the general problem. Its conceptual advantages are much deeper and explained by the fact that the formulation of the time-optimal problem is itself invariant-geometric from the beginning, therefore the problem of invariant representation of MP in the time-optimal case has an exact meaning without further reductions.

The general optimal problem obtains its "intrinsic" invariant form after it is canonically transformed to the associated time-optimal problem, which makes possible to investigate the invariance properties of necessary conditions of the initial problem.

2 MP for the time-optimal problem

Let me start now with formulating MP in a form suitable for further exposition.

Consider a controlled equation on the configuration space M, an n-dimensional smooth manifold,

$$\frac{dx}{dt} = X(x,u), \quad x \in M, \quad u \in U,$$

where U is the *space of admissible values of the control parameter* u. *Admissible controls* are arbitrary measurable (essentially) bounded functions $u(t)$ on a time interval J, with values in U, and the *corresponding (admissible) trajectories* $x(t)$, $t \in J$, are absolutely continuous solutions of the nonautonomous differential equation,

$$\frac{dx(t)}{dt} = X(x(t), u(t)), \quad u(t) \in U, \quad t \in J.$$

An admissible trajectory $x(t)$, $t \in J$, is an *(optimal) solution* of the time-optimal problem, if for arbitrary $t_1 < t_2$ in the interval J, the transition time $t_2?t_1$ from $x(t_1)$ to $x(t_2)$ along the trajectory $x(t)$ is minimal with respect to every other choice of the admissible trajectory containing the points $x(t_1)$, $x(t_2)$.

Thus the problem is uniquely defined by the family of vector fields X on M, and the manifold M itself.

To formulate MP, we have to perform the following preliminary procedures:

- in an arbitrary coordinate neighborhood (U, x) of M we represent the vector field X as $X = \sum_\alpha X^\alpha \frac{\partial}{\partial x^\alpha}$;
- introduce n auxiliary variables $\psi = (\psi_1, \ldots, \psi_n)$;
- take the *"Hamiltonian* of the problem" (*linear* in ψ),

$$H(\psi, x, u) = \sum_\alpha \psi_\alpha X^\alpha(x, u); \tag{1}$$

- consider the corresponding Hamiltonian system (containing a parameter u),

$$\frac{dx}{dt} = \frac{\partial H}{\partial \psi}, \; \frac{d\psi}{dt} = -\frac{\partial H}{\partial x}; \tag{2}$$

- supplement the system with the maximum condition

$$H(\psi, x, u) = \max_{v} H(\psi, x, v), \tag{3}$$

  which serves for eliminating the parameter u from the system;
- then, MP asserts:
  For every optimal solution $x(t), t \in J$, of the time-optimal problem corresponding to an admissible control $u(t), t \in J$, there exists a (nonzero) function $\psi(t)$, $t \in J$, such that for (almost) $\forall t \in J$ the functions $\psi(t), x(t), u(t)$, satisfy the Hamiltonian system and the maximum condition.

A certain clumsiness of the formulation is the price we have to pay for its mathematical rigor, whereas the intuitive meaning of the principle is very simple.

It is a method of generating trajectories (extremals) $\psi(t), x(t), t \in J$, with given initial conditions,

$$\psi(t_0) \neq 0,\ x(t_0) = x_0,$$

of the Hamiltonian system (2) *with parameter u, as a result of "dynamical elimination" of the parameter from the system by the maximum condition* (3)*, as we advance along the trajectory. Every solution $x(t)$ of the time-optimal problem is obtained as a projection of an extremal onto the x-space.*

It is important to emphasize that the parameter elimination by the maximum condition is performed "dynamically", simultaneously with our motion along the trajectory under consideration, and not in the "classical sense", as a representation of the control parameter u as a function of the "coordinate variables" (x, ψ), which would be a generalization of the Legendre transformation in classical theory and would represent the family of extremals as a flow of the Hamiltonian system with eliminated parameter. This is impossible unless we impose on the initial equations some strong regularity conditions. Without such conditions, the classical differential equations for extremals may appear with heavy singularities, making impossible to reduce them to the Hamiltonian form, whereas the Hamiltonian form of the equations, though with a parameter, is an inherent part of MP, and all singularities of the problem are in fact relegated to the maximum condtion, which nevertheless successfully eliminates ("dynamically") the parameter in many nontrivial strongly degenerate cases, such as linear systems, completely unamenable for classical methods.

3 The Pontryagin derivative $\mathcal{P}_X$

According to MP, the vector field defined by the Hamiltonian system (2) should be considered as the basic "variational derivative" of the problem that contains, together with the maximum condition (3), complete first order information about the time-optimal problem. We call the vector field the *"Pontryagin derivative"* and denote it $\mathcal{P}_X$. It was introduced by L. S. Pontryagin before MP was formulated, and in fact, was the basic guideline in the discovery of the principle.

The procedure leading us to the Eqs. (1)–(3) provides a simple, almost "tautological", transfer from an arbitrary vector field X on M to the Hamiltonian vector field $\mathcal{P}_X$, defined by the sequence of correspondences

$$\begin{gathered} X \mapsto \sum_\alpha X^\alpha \frac{\partial}{\partial x^\alpha} \mapsto H = \sum_\alpha \psi_\alpha X^\alpha(x,u) \mapsto \mathcal{P}_X, \\ \mathcal{P}_X = \sum_\alpha \frac{\partial H}{\partial \psi_\alpha}\frac{\partial}{\partial x^\alpha} - \sum_\alpha \frac{\partial H}{\partial x^\alpha}\frac{\partial}{\partial \psi_\alpha}. \end{gathered} \tag{4}$$

Though the transfer $X \mapsto \mathcal{P}_X$ provides an explicit expression for $\mathcal{P}_X$, it is formulated not invariantly, depending on the choice of a coordinate neighborhood on M to introduce the Hamiltonian function H.

Our intention is to present the transfer $X \mapsto H$ as a canonically invariant transition from X to a scalar valued fiberwise linear smooth function H_X on the cotangent bundle T^*M. Then the whole sequence (4) will be easily turned into a canonically invariant construction, expressing the Pontryagin derivative $\mathcal{P}_X$ as a canonically invariant $\mathbb{R}$-linear functorial correspondence $X \mapsto \mathcal{P}_X$ – the *"Hamiltonian lift to T^*M over the vector field $X \in$ Vect M"*,

$$\begin{gathered} X \mapsto \mathcal{P}_X \in \text{Vect}\,(T^*M), \\ \mathcal{P}_{\lambda X+\mu Y} \mapsto \lambda\mathcal{P}_X + \mu\mathcal{P}_Y, \ \forall\lambda,\mu \in \mathbb{R}, \ X,Y \in \text{Vect}\,M. \end{gathered}$$

This provids *an invariant representation of MP*, from which the basic properties of $\mathcal{P}_X$ automatically follow.

We carry out this construction in next two sections. In last Sects. 6–7, we "identify" the vector field $\mathcal{P}_X$ and formulate the final result. I shall show that $\mathcal{P}_X$ is not some new exotic invariant on T^*M generated by X, but rather one of the most basic objects of "everyday mathematical usage" in differential geometry.

4 The Hamiltonian lift Vect $M \longrightarrow \mathfrak{H}(T^*M) \subset$ Vect T^*M

We devote this section to some preliminary comments to introduce the correspondence $X \mapsto \mathcal{P}_X$ invariantly.

To take full advantage of the fibered structures of the bundles

$$T^*M \xrightarrow{\pi} M, \quad TM \xrightarrow{pr} M,$$

we shall consider the corresponding $\mathbb{R}$-algebras of smooth scalar-valued functions $C^\infty(T^*M)$, $C^\infty(TM)$ simultaneously as $C^\infty(M)$-modules, the action of the "ring of scalars" $C^\infty(M)$ given, say on $C^\infty(T^*M)$, by the equation

$$\begin{gathered} aH' + bH'' \stackrel{def}{=} \pi^*a \cdot H' + \pi^*b \cdot H'' \\ \forall a,b \in C^\infty(M),\, H',H'' \in C^\infty(T^*M), \end{gathered}$$

where $C^\infty(M) \longrightarrow^{\pi^*} C^\infty(T^*M)$ is the pullback homomorphism of the corresponding algebras generated by the projection π.

Each of the modules $C^\infty(T^*M)$ and $C^\infty(TM)$ contains two submodules, $\mathfrak{Q}, \mathfrak{P}$ and $\mathcal{Q}, \dot{\mathcal{Q}}$, respectively,

$$\mathfrak{Q}, \mathfrak{P} \subset C^\infty(T^*M), \quad \mathcal{Q}, \dot{\mathcal{Q}} \subset C^\infty(TM),$$

which are of special importance for the geometry of the fibrations T^*M and TM.

Submodules $\mathfrak{Q}, \mathcal{Q}$ consist of *fiberwise constant functions* denoted q and represented as

$$\mathfrak{Q} = \left\{ q = \pi^* a \middle| a \in C^\infty(M) \right\},$$
$$\mathcal{Q} = \left\{ q = pr^* a \middle| a \in C^\infty(M) \right\}.$$

Submodules $\mathfrak{P}, \dot{\mathfrak{Q}}$ consist of fiberwise linear functions, denoted respectively $p, \dot{q}$ (the havy influence of Hamiltonian Mechanics on Optimal Control!) and reprsented as fiber sections,

$$\mathfrak{P} = \left\{ p \in C^\infty(T^*M) \middle| \, p|_{T_x^*M} \in L(T_x^*M, \mathbb{R}) \; \forall x \in M \right\},$$
$$\dot{\mathfrak{Q}} = \left\{ \dot{q} \in C^\infty(TM) \middle| \, \dot{q}|_{T_xM} \in L(T_xM, \mathbb{R}) \;\; \forall x \in M \right\}.$$

A straightforward reasoning shows that *the submodule* $\mathfrak{P}$ *is canonically identified with* Vect M, and *the submodule* $\dot{\mathfrak{Q}}$ *is canonically identified with* $\Lambda^1 M$, the $C^\infty(M)$-module of differential 1-forms on M.

For example, we obtain the canonical embedding,

$$\text{Vect}\, M \subset \mathfrak{P} \subset C^\infty(T^*M),$$

if we represent an arbitrary vector field Z on M as a smooth cross-section $Z : x \mapsto Z_x \in T_xM, x \in M$, i. e. as a fiberwise linear function $H_Z = p \in \mathfrak{P}$, defined by the relation

$$H_Z = \left\{ p\Big|_{T_x^*M} = Z_x \,\Big|\, x \in M \right\} \iff H_Z(\sigma) = \langle \sigma, Z_{\pi\sigma} \rangle \;\forall \sigma \in T_x^*M, x \in M, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket between the fibers T_x^*M and T_xM, $x \in M$.

Conversely, every function $p \in \mathfrak{P}$ is uniquely represented as $p = H_Z$, if Z is defined as the family of linear functionals $\left\{ Z_x \middle| x \in M \right\}$ on $C^\infty(M)$ by the equation

$$Z_x a = p\left(da\Big|_x \right) \; \forall a \in C^\infty(M), \quad x \in M.$$

Since the kernel of every Z_x contains functions, which are stationary at x, all Z_x are tangent vectors, hence Z is a (smooth) vector field on M.

We thus obtain a canonically invariant sequence of correspondences – *the Hamiltonian lift to T^*M over Z,*

$$\begin{gathered} Z \mapsto H_Z \mapsto D_Z, \quad i_{D_Z}\omega = -dH_Z, \\ Z \in \operatorname{Vect} M,\ H_Z \in \mathfrak{P},\ D_Z \in \mathfrak{H}, \\ \operatorname{Vect} M \longrightarrow \mathfrak{H} \subset \operatorname{Vect} T^*M, \end{gathered} \tag{6}$$

where ω is the canonical symplectic 2-form on the cotangent bundle T^*M, and the mapping, $\operatorname{Vect} M \longrightarrow \mathfrak{H}$, is a canonical isomorphism from the $\mathbb{R}$-vector space of vector fields on M to the $\mathbb{R}$-vector space $\mathfrak{H}$ of Hamiltonian vector fields on T^*M generated by (fiberwise-linear) Hamiltonians in $\mathfrak{P}$.

General considerations easily imply that *the Hamiltonian lift D_Z preserves the submodule $\mathfrak{P} \subset C^\infty(T^*M)$, and its restriction on $\mathfrak{P}$ is a derivation over Z of the $C^\infty(M)$-module $\mathfrak{P}$, i. e. $D_Z\big|_{\mathfrak{P}}$ is $\mathbb{R}$-linear and satisfies the Leibnitz identity,*

$$D_Z \cdot ap = Za \cdot p + a(D_Z \cdot p) \quad \forall a \in C^\infty(M),\ p \in \mathfrak{P}.$$

Hence the corresponding flow e^{tD_Z} on T^*M is a lift over the flow e^{tZ}, i. e. its restriction on an arbitrary fiber $e^{tD_Z}\big|_{T_x^*M}$ is a linear isomorphism

$$e^{tD_Z}\Big|_{T_x^*M} : T_x^*M \longrightarrow T_{e^{tZ}x}^*M.$$

5 Invariant representation of the sequence (4)

We return now to our basic sequence (4), which led us (for every fixed value of the parameter u) from an arbitrary vector field $X \in \operatorname{Vect} M$ to the Pontryagin derivative $\mathcal{P}_X$,

$$X = \sum_\alpha X^\alpha \frac{\partial}{\partial x^\alpha} \mapsto H = \sum_\alpha \psi_\alpha X^\alpha(x,u) \mapsto \mathcal{P}_X.$$

According to (6), it will yield a *canonically invariant form*,

$$\begin{gathered} X \mapsto H_X \mapsto \mathcal{P}_X,\ i_{\mathcal{P}_X}\omega = -dH_X, \\ X \in \operatorname{Vect} M,\ H_X \in \mathfrak{P},\ \mathcal{P}_X \in \mathfrak{H}, \\ \operatorname{Vect} M \longrightarrow \mathfrak{H} \subset \operatorname{Vect} T^*M, \end{gathered} \tag{7}$$

if we show that, at least in one (hence in all) canonical coordinate system (q, p) on T^*M, the canonically introduced Hamiltonian $H_X \in \mathfrak{P}$ has the expression $H_X = \sum_\alpha p_\alpha X^\alpha$, coinciding with the expression of the Hamiltonian of MP in (4).

The Hamiltonian H_X is easily expressed in canonical coordinates q, p (over an arbitrary coordinate neighborhood (U, x) in M),

$$\left(\pi^{-1}U, (q, p)\right), \quad q = \pi^* x,\ p = \frac{\partial}{\partial x},$$

and indeed coincides with $H = \sum_\alpha p_\alpha X^\alpha$, since, for arbitrary

$$X = \sum_\alpha X^\alpha \frac{\partial}{\partial x^\alpha} \quad \text{and} \quad \sigma = \sum_\alpha p_\alpha(\sigma) dx^\alpha \Big|_{\pi\sigma} \in T^*M$$

we have,

$$\begin{aligned} H_X(\sigma) &= \left\langle \sigma, X\Big|_{\pi\sigma} \right\rangle = \sum_{\alpha,\beta} \left\langle p_\alpha(\sigma) dx^\alpha \big|_{\pi\sigma}, X^\beta \frac{\partial}{\partial x^\beta}\Big|_{\pi\sigma} \right\rangle \\ &= \sum_\alpha p_\alpha(\sigma) X^\alpha(\pi\sigma) = \sum_\alpha p_\alpha X^\alpha \Big|_\sigma \quad \forall \sigma \in T^*M. \end{aligned}$$

Thus we obtain a canonically invariant representation (7) *of MP.*

The vector field $\mathcal{P}_X$ obtains a familiar expression in canonical coordinates q, p, if we write the Hamiltonian system in the canonical form,

$$\mathcal{P}_X = \frac{\partial H_X}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H_X}{\partial q}\frac{\partial}{\partial p}.$$

6 Identification of the Pontryagin derivatrive $\mathcal{P}_X$

To identify the vector field $\mathcal{P}_X$, let me proceed in a straightforward way, as an unsophisticated mind in differential geometry should do in such situations.

Consider the standard object of "everyday usage" – the differential $(e^{tX})_*$ of the flow e^{tX} on M generated by $X \in \operatorname{Vect} M$, which is a *natural lift to* TM *over* e^{tX} – a fiber-preserving flow on TM represented by the family of fiberwise-linear isomorphisms,

$$\left(e^{tX}\right)_*\Big|_{T_xM} : T_xM \longrightarrow T_{e^{tX}x}M,\ x \in M.$$

Denote by $\mathcal{L}_X$ the *Lie derivative over* X – the vector field on TM generated by the flow $(e^{tX})_*$,

$$e^{t\mathcal{L}_X} = \left(e^{tX}\right)_*.$$

The Lie derivative $\mathcal{L}_X$ is a *natural lift to* TM *over* X – it preserves the submodule of differential 1-forms $\dot{\mathfrak{Q}}$ and its restriction on $\dot{\mathfrak{Q}}$ is a module-derivation over X; it is also $\mathbb{R}$-linear in X.

The flow $e^{t\mathcal{L}_X}$ canonically defines two flows on the cotangent bundle: the *adjoint flow* $\left(e^{t\mathcal{L}_X}\right)^{\#} =^{def} e^{t\mathcal{L}_X^{\#}}$ to $e^{t\mathcal{L}_X}$,

$$\left.\begin{aligned} \langle e^{t\mathcal{L}_X^{\#}}\theta_x, Y_{e^{-tX}x}\rangle = \langle \theta_x, e^{t\mathcal{L}_X} Y_{e^{-tX}x}\rangle \\ \forall \theta_x \in T_x^*M,\ Y_{e^{-tX}x} \in T_{e^{-tX}x}M, \end{aligned}\right\} \tag{8}$$

which is a lift to T^*M over e^{-tX}, and the *dual flow* to $e^{t\mathcal{L}_X}$,

$$\left(e^{t\mathcal{L}_X^{\#}}\right)^{-1} = e^{-t\mathcal{L}_X^{\#}} = e^{t\mathcal{L}_{-X}^{\#}}, \tag{9}$$

which is a lift to T^*M over e^{tX}.

Respectively, $\mathcal{L}_{-X}^{\#}$ is a (natural) lift to T^*M over X and we come to our basic conjecture: *is Pontryagin derivative identical with the dual to the Lie derivative, that is,*

$$\mathcal{P}_X = \mathcal{L}_{-X}^{\#} \quad ?$$

The answer is affirmative. To prove it we must compute explicitly the dual field $\mathcal{L}_{-X}^{\#}$, which requires the relation (8) between the adjoint flows to be rewritten as a relation between the corresponding pullback flows of algebra automorphisms $\exp(tX)$, $\exp(t\mathcal{L}_X)$ and $\exp(t\mathcal{L}_X^{\#})$, respectively. This could be easily done and we obtain,

$$\langle \exp(t\mathcal{L}_X)\theta, Y\rangle = \exp(tX)\langle \theta, \exp(t\mathcal{L}_X^{\#})Y\rangle.$$

Differentiating the obtained expression with respect to t and then putting $t = 0$ yields, for every $\theta \in \Lambda^{(1)}(M)$ and $X, Y \in \operatorname{Vect} M$,

$$X\langle \theta, Y\rangle = \langle \mathcal{L}_X\theta, Y\rangle + \langle \theta, \mathcal{L}_{-X}^{\#}Y\rangle.$$

Since

$$X\langle \theta, Y\rangle = \langle \mathcal{L}_X\theta, Y\rangle + \langle \theta, ad_X Y\rangle,$$

we conclude that

$$L_{-X}^{\#} = ad_X.$$

Hence our conjecture is reduced to the equation

$$\mathcal{P}_X\Big|_{\mathfrak{P}} = ad_X.$$

This was an unexpected though easily verifiable conjecture since computations with ad_X are much easier to perform than with $\mathcal{L}_{-X}^{\#}$. A short straightforward calculation gives,

$$\mathcal{P}_X H_Y = H_{ad_X Y} = H_{[X,Y]} \quad \forall Y \in \operatorname{Vect} M,$$

which proves that

$$\mathcal{P}_X\Big|_{\mathfrak{P}} = ad_X.$$

7 Formulation of the final result

The Pontryagin derivative $\mathcal{P}_X$ is a Hamiltonian lift to T^*M over the vector field $X \in \mathrm{Vect}\, M$, invariantly derived from X by the canonical sequence,

$$X \mapsto H_X \in \mathfrak{P} \subset C^\infty(T^*M) \mapsto \mathcal{P}_X,$$
$$i_{\mathcal{P}_X}\omega = -dH_X,$$

and the natural correspondence $X \mapsto \mathcal{P}_X$ is $\mathbb{R}$-linear. The corresponding flow $e^{t\mathcal{P}_X}$ is a Hamiltonian lift to T^*M over e^{tX}.

Furthermore, the Hamiltonian vector field $\mathcal{P}_X$ is dual to the Lie derivative $\mathcal{L}_X \in \mathrm{Vect}\, TM$ and is an extension of the Lie bracket ad_X, where ad_X is considered as a derivation over X on the $C^\infty(M)$-submodule $\mathfrak{P} \subset C^\infty(T^*M)$ of fiberwise linear functions (vector fields on M).

Thus the whole computational power of MP is based, via the "dynamical elimination procedure", on two rudimentary differential-geometric notions – the Lie derivative (the infinitesimal variations of the optimal system), and its dual – the Pontryagin derivative, which, in fact, coincides with the Lie bracket.

I would like to finish my contribution with a final remark concerning the time-optimal problem with restricted phase coordinates.

The problem was already considered in our book in 1962 and the result was formulated in the form of a MP with some modifications, which took into account the motion on the boundary of the region. Since then, the problem was reconsidered several times, the latest publications appearing quite recently, and the search was always directed towards most simple and perfect analogs of MP.

I think, if the "ultimate" form of MP exists for the problem, it could be found by considerations quite similar to those given above. The only essential difference should consist in the assumption that the configuration space of the problem M is not a closed manifold, but rather a manifold with boundary.

Curvature-dimension inequalities and Li-Yau inequalities in sub-Riemannian spaces

Nicola Garofalo

Abstract In this paper we present a survey of the joint program with Fabrice Baudoin originated with the paper [6], and continued with the works [7–9] and [10], joint with Baudoin, Michel Bonnefont and Isidro Munive.

1 Introduction

One of the most exciting aspects of Riemannian geometry consists in the beautiful interplay between global topological and geometric properties of the ambient manifold and properties of solutions of those natural pde's such as the Laplace-Beltrami operator Δ, with its associated heat semigroup $P_t f(x) = e^{t\Delta} f(x)$. In their 1986 Acta Mathematica paper [22] Li and Yau established their celebrated inequalities. Let us just focus on the one concerned with $\mathrm{Ric} \geq 0$.

Theorem 1 (The Li-Yau parabolic gradient estimate). *Suppose that M is a complete, connected n-dimensional Riemannian manifold such that* $\mathrm{Ric} \geq 0$. *Then, for any $f \geq 0$ which solves the heat equation $\Delta f - f_t = 0$ on M one has for $u = \ln f$,*

$$|\nabla u|^2 - u_t \leq \frac{n}{2t}. \tag{1}$$

The motivation for (1) comes from considering the case when M is flat $\mathbb{R}^n$ and $f(x,t) = (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4t})$ is the fundamental solution of the heat equation. In

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such case $u = \ln f$ is easily seen to satisfy

$$|\nabla u|^2 - u_t \equiv \frac{n}{2t}.$$

Understanding the "$\leq$" in (1) requires a deeper analysis of the role played by curvature. Integration of the the Li-Yau inequality (1) along a geodesic path joining (y,t) to (x,s), where $x, y \in M$ and $0 < s < t$, gives the following fundamental result.

Theorem 2 (The Li-Yau Harnack inequality). *Let M be a complete connected n-dimensional Riemannian manifold having Ric ≥ 0. Let $f \geq 0$ be a solution of the heat equation on M. For any $x, y \in M$, $0 < s < t < \infty$, one has*

$$f(x,s) \leq f(y,t)\left(\frac{t}{s}\right)^{\frac{n}{2}} \exp\left(\frac{d(x,y)^2}{4(t-s)}\right).$$

Theorem 2 extends to Riemannian manifolds with Ric ≥ 0 the Harnack inequality for the heat equation independently discovered by B. Pini in [23] and J. Hadamard in [17]. Theorems 1 and 2 provide remarkable evidence of how the geometry of the manifold is intimately connected to the properties of its Laplacian and the associated heat flow. In fact, once Theorem 2 is available one can obtain many fundamental results, such as Liouville type theorems, on and off-diagonal Gaussian upper bounds for the heat kernel, Sobolev and isoperimetric inequalities, etc.

Another beautiful global result which connects the geometry to the topology of M is the Bonnet-Myers theorem which states that if for some $\rho_1 > 0$, Ric$\geq (n-1)\rho_1$, then M with its Riemannian metric is compact, with a finite fundamental group, and diam$(M) \leq \frac{\pi}{\sqrt{\rho_1}}$.

The identity of Bochner and the role of Jacobi fields

The original proof of Li and Yau of Theorem 1 hinges on two basic tools from Riemannian geometry:

(i) the Bochner identity

$$\Delta(|\nabla f|^2) = 2\|\nabla^2 f\|^2 + 2 < \nabla f, \quad \nabla(\Delta f) > +2\,\mathrm{Ric}(\nabla f, \nabla f), \qquad (2)$$

which holds for any $f \in C^3(M)$;

(ii) the Laplacian comparison theorem. When Ric ≥ 0 the latter states that the geodesic distance on M satisfies the following differential inequality outside the cut-locus of a fixed base point (and in the sense of distributions on M)

$$\Delta \rho_M(x) \leq \frac{n-1}{\rho_M(x)}. \qquad (3)$$

As it is well-known, the Laplacian comparison theorem (like other comparison theorems in Riemannian geometry, or like the Bonnet-Myers theorem) uses in an essential way the existence of a rich supply of Jacobi fields.

This paper is devoted to surveying a joint program with Fabrice Baudoin originated with the paper [6], and continued with the works [7–9] and [10]. It is worth emphasizing that our approach allows for the first time to extend the Li-Yau program, and many of its fundamental consequences, to situations which are genuinely non-Riemannian. The original motivation in [6] was generalizing global results such as Theorems 1 and 2 above, or the topological Bonnet-Myers theorem, to smooth manifolds in which the governing operator is no longer the Laplace-Beltrami operator, but rather a smooth locally subelliptic operator L. These operators are typically never elliptic and their natural geometric framework is that of sub-Riemannian manifolds. Such manifolds are a generalization of Riemannian ones and they constitute the appropriate setting for describing phenomena with a constrained dynamic, in which only certain directions in the tangent space are allowed.

We close this introduction by mentioning that, in their interesting preprint [2], Agrachev and Lee have used a notion of Ricci tensor, denoted by $\mathfrak{Ric}$, which was introduced by the first author in [1]. They study three-dimensional contact manifolds and, under the assumption that the manifold be Sasakian, they prove that a lower bound on $\mathfrak{Ric}$ implies the so-called measure-contraction property. In particular, when $\mathfrak{Ric} \geq 0$, then the manifold $\mathbb{M}$ satisfies a global volume growth similar to the Riemannian Bishop-Gromov theorem. An analysis shows that, interestingly, our notion of Ricci tensor coincides, up to a scaling factor, with theirs.

We also mention that the sub-Riemannian geometric invariants for contact manifolds of dimension three were computed by Hughen in his unpublished Ph.D. dissertation, see [19]. In particular, with his notations, the CR Sasakian structure corresponds to the case $a_1^2 + a_2^2 = 0$ and, up to a scaling factor, his K is the Tanaka-Webster Ricci curvature. In such respect, the Bonnet-Myers type theorem obtained by Hughen (Proposition 3.5 in [19]) is the exact analogue (with a better constant) of our Theorem 6, applied to the case of three-dimensional Sasakian manifolds. Finally, it must be mentioned that a Bonnet-Myers type theorem on general three-dimensional CR manifolds was first obtained by Rumin in [24]. The methods of Rumin and Hughen are close as they both rely on the analysis of the second-variation formula for sub-Riemannian geodesics.

2 From Riemannian to sub-Riemannian geometry

A fundamental property of the Laplace-Beltrami operator is *ellipticity*. As we have just said, in sub-Riemannian geometry the relevant partial differential operators, the sub-Laplacians, fail to be elliptic. The moment one gives up coercivity (i.e., control of all directions in the tangent space), new interesting phenomena arise. For instance, the exponential mapping fails to be a local diffeomorphism, and geodesics are no longer locally unique. A rich theory of Jacobi fields is (at least presently) not available and, consequently, results such as the Laplacian comparison theorem, the Bonnet-Myers theorem, or Theorems 1 and 2 seemed to be completely out of reach. Furthermore, it was not clear what one means by “Ricci curvature”.

The paper [6] took a different approach to these questions, based on a new *curvature-dimension inequality* and a systematic use of the heat semigroup. Besides the Riemannian case, the program in [6] presently covers sub-Riemannian spaces of rank two, such as, for instance, Carnot groups of step two, CR manifolds, etc. This is the first genuinely non-Riemannian setting in which a good notion of *Ricci curvature* has been introduced, and we feel it is important to emphasize that the Riemannian approach has been so far mostly unsuccessful to cover the large classes of examples encompassed by [6].

In this connection we stress that, even in the Riemannian framework, the ideas in [6] provide a new and simplified account of the Li-Yau program based on tools which are purely analytical and avoid the use of results which are preeminently based on the theory of Jacobi fields, such as, e. g., the Laplacian or the volume comparison theorem, see [7].

3 The curvature-dimension inequality CD(ρ, n) and the Ricci tensor

Recall that a Riemannian manifold M with Laplacian Δ is said to satisfy the Bakry-Emery *curvature-dimension inequality* CD(ρ, n) if

$$\Gamma_2(f) \geq \frac{1}{n}(\Delta f)^2 + \rho\Gamma(f), \quad \forall f \in C^\infty(M). \tag{4}$$

Here,

$$\begin{aligned} \Gamma(f) &= \frac{1}{2}\left\{\Delta(f^2) - 2f\Delta f\right\} = |\nabla f|^2, \\ \Gamma_2(f) &= \frac{1}{2}\left\{\Delta(\Gamma(f)) - 2\Gamma(f, \Delta f)\right\}. \end{aligned} \tag{5}$$

Using Bochner's identity (2) and Newton's inequality, it is easy to see that if Ric $\geq \rho$, then CD(ρ, n) holds. It is remarkable that the curvature dimension inequality (4) perfectly captures the notion of Ricci lower bound. It was in fact proved by Bakry in Proposition 6.2 in [4] that: *on a n-dimensional Riemannian manifold* $\mathbb{M}$ *the inequality CD*(ρ, n) *implies* Ric $\geq \rho$. In conclusion,

$$\text{Ric} \geq \rho \Longleftrightarrow \text{CD}(\rho, n). \tag{6}$$

This equivalence (6) was the motivation behind the work [6], whose setup we now describe.

We consider a smooth, connected manifold $\mathbb{M}$ endowed with a smooth measure μ and a smooth second-order diffusion operator L with real coefficients, satisfying $L1 = 0$, and which is symmetric with respect to μ and non-positive. By this we mean that

$$\int_{\mathbb{M}} fLg d\mu = \int_{\mathbb{M}} gLf d\mu, \quad \int_{\mathbb{M}} fLf d\mu \leq 0, \tag{7}$$

for every $f, g \in C_0^\infty(\mathbb{M})$. We make the technical assumption that L be locally subelliptic in the sense of [13], and associate with L the following symmetric, first-order, differential bilinear form:

$$\Gamma(f,g) = \frac{1}{2}\{L(fg) - fLg - gLf\}, \quad f,g \in C^\infty(\mathbb{M}). \tag{8}$$

The expression $\Gamma(f) = \Gamma(f,f)$ is known as *le carré du champ*, see (5) above. There is a canonical distance associated with the operator L:

$$d(x,y) = \sup\{|f(x) - f(y)| \mid f \in C^\infty(\mathbb{M}), \|\Gamma(f)\|_\infty \le 1\}, \quad x,y \in \mathbb{M}, \tag{9}$$

where for a function g on $\mathbb{M}$ we have let $||g||_\infty = \text{ess}\sup_{\mathbb{M}}|g|$. A tangent vector $v \in T_x\mathbb{M}$ is called *subunit* for L at x if $v = \sum_{i=1}^m a_i X_i(x)$, with $\sum_{i=1}^m a_i^2 \le 1$, see [13]. A Lipschitz path $\gamma : [0,T] \to \mathbb{M}$ is called subunit for L if $\gamma'(t)$ is subunit for L at $\gamma(t)$ for a.e. $t \in [0,T]$. We then define the subunit length of γ as $\ell_s(\gamma) = T$. Given $x, y \in \mathbb{M}$, we indicate with

$$S(x,y) = \{\gamma : [0,T] \to \mathbb{M} \mid \gamma \text{ is subunit for } L, \gamma(0) = x,\ \gamma(T) = y\}.$$

In this paper we make the assumption that

$$S(x,y) \neq \varnothing, \quad \text{for every } x,y \in \mathbb{M}. \tag{10}$$

Under such hypothesis one verifies that

$$d_s(x,y) = \inf\{\ell_s(\gamma) \mid \gamma \in S(x,y)\}, \tag{11}$$

defines a true distance on $\mathbb{M}$, and that furthermore,

$$d(x,y) = d_s(x,y), \quad x,y \in \mathbb{M}.$$

It follows that one can work indifferently with either one of the distances d in (9), or d_s in (11).

Throughout this paper we assume that the metric space $(\mathbb{M}, d)$ be complete.

We also suppose that $\mathbb{M}$ is equipped with a symmetric, first-order differential bilinear form $\Gamma^Z : C^\infty(\mathbb{M}) \times C^\infty(\mathbb{M}) \to \mathbb{R}$, satisfying $\Gamma^Z(fg,h) = f\Gamma^Z(g,h) + g\Gamma^Z(f,h)$. We assume that $\Gamma^Z(f) = \Gamma^Z(f,f) \ge 0$ (one should notice that $\Gamma^Z(1) = 0$).

Given the sub-Laplacian L and the first-order bilinear form Γ^Z on $\mathbb{M}$, we now introduce the following second-order differential forms:

$$\Gamma_2(f,g) = \frac{1}{2}\big\{L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big\}, \tag{12}$$

$$\Gamma_2^Z(f,g) = \frac{1}{2}\big\{L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf)\big\}. \tag{13}$$

Observe that if $\Gamma^Z \equiv 0$, then $\Gamma_2^Z \equiv 0$ as well. As for Γ and Γ^Z, we will use the notations $\Gamma_2(f) = \Gamma_2(f,f)$, $\Gamma_2^Z(f) = \Gamma_2^Z(f,f)$.

We are ready to introduce the central character of our program, a generalization of the above mentioned curvature-dimension inequality (6).

Definition 1. We say that $\mathbb{M}$ satisfies the *generalized curvature-dimension inequality* $CD(\rho_1, \rho_2, \kappa, d)$ with respect to L and Γ^Z if there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$, and $0 < d \leq \infty$ such that the inequality

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f) \tag{14}$$

hold for every $f \in C^\infty(\mathbb{M})$ and every $\nu > 0$.

It is worth observing explicitly that if in Definition 1 we choose $L = \Delta$, $\Gamma^Z \equiv 0$, $d = n = \dim(\mathbb{M})$, $\rho_1 = \rho$ and $\kappa = 0$, we obtain the Riemannian curvature-dimension inequality $\mathrm{CD}(\rho, n)$ in (6) above. Thus, the case of Riemannian manifolds is trivially encompassed by Definition 1. We also remark that, changing Γ^Z into $a\Gamma^Z$, where $a > 0$, changes the inequality $\mathrm{CD}(\rho_1, \rho_2, \kappa, d)$ into $\mathrm{CD}(\rho_1, a\rho_2, a\kappa, d)$. We express this fact by saying that the quantity $\frac{\kappa}{\rho_2}$ is intrinsic. Hereafter, when we say that $\mathbb{M}$ satisfies the curvature dimension inequality $\mathrm{CD}(\rho_1, \rho_2, \kappa, d)$ (with respect to L and Γ^Z), we will routinely avoid repeating at each occurrence the sentence "for some $\rho_2 > 0$, $\kappa \geq 0$ and $d > 0$". Instead, we will explicitly mention whether $\rho_1 = 0$, or > 0, or simply $\rho_1 \in \mathbb{R}$. The reason for this is that the parameter ρ_1 in the inequality (14) has a special relevance since, in the geometric examples in [6], it represents the lower bound on a sub-Riemannian generalization of the Ricci tensor. Thus, $\rho_1 = 0$ is, in our framework, the counterpart of the Riemannian Ric ≥ 0, whereas when $\rho_1 > 0$ (< 0), we are dealing with the counterpart of the case Ric > 0 (Ric bounded from below by a negative constant).

In addition to (14) we will work with three general assumptions: they will be listed as Hypothesis 1, 2 and 3.

Hypothesis 1. *There exists an increasing sequence* $h_k \in C_0^\infty(\mathbb{M})$ *such that* $h_k \nearrow 1$ *on* $\mathbb{M}$, *and*

$$||\Gamma(h_k)||_\infty + ||\Gamma^Z(h_k)||_\infty \to 0, \qquad \text{as } k \to \infty.$$

We will also assume that the following commutation relation be satisfied.

Hypothesis 2. *For any* $f \in C^\infty(\mathbb{M})$ *one has* $\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f))$.

When $\mathbb{M}$ is a Riemannian manifold, μ is the Riemannian volume on $\mathbb{M}$, and $L = \Delta$, then $d(x, y)$ in (9) above is equal to the Riemannian distance on $\mathbb{M}$. In this situation if we take $\Gamma^Z \equiv 0$, then Hypothesis 1, 2 are fulfilled. In fact, Hypothesis 2 is trivially satisfied, whereas Hypothesis 1 is equivalent to assuming that $(\mathbb{M}, d)$ be a complete metric space, which we are assuming.

Before proceeding with the discussion, we pause to stress that, in the generality in which we work the bilinear differential form Γ^Z, unlike Γ, is not a priori canonical. Whereas Γ is determined once L is assigned, the form Γ^Z in general is not intrinsically associated with L. However, in the geometric examples described in Sect. 2

of the paper [6] the choice of Γ^Z is canonical, as is the case, for instance, for CR Sasakian manifolds. The reader should think of Γ^Z as an orthogonal complement of Γ: the bilinear form Γ represents the square of the length of the gradient in the horizontal directions, whereas Γ^Z represents the square of the length of the gradient along the vertical directions.

We will also need the following assumption which is necessary to rigorously justify the computations in [6] on functionals of the heat semigroup. Hereafter, we will denote by $P_t = e^{tL}$ the semigroup generated by the diffusion operator L.

Hypothesis 3. *The semigroup P_t is stochastically complete that is, for $t \geq 0$, $P_t 1 = 1$ and for every $f \in C_0^\infty(\mathbb{M})$ and $T \geq 0$, one has*

$$\sup_{t\in[0,T]} \|\Gamma(P_t f)\|_\infty + \|\Gamma^Z(P_t f)\|_\infty < +\infty.$$

In the Riemannian setting ($L = \Delta$ and $\Gamma^Z \equiv 0$), Hypothesis 3, is satisfied if one assumes the lower bound Ricci $\geq \rho$, for some $\rho \in \mathbb{R}$. This can be derived from the paper by Yau [30] and Bakry's note [3]. It thus follows that, in the Riemannian case, the Hypothesis 3 is not needed since it can be derived as a consequence of the curvature-dimension inequality CD(ρ, n) in (6) above. More generally, it is proved in [6] that a similar situation occurs in every sub-Riemannian manifold with transverse symmetries of Yang-Mills type, for the relevant definitions see [6]. In that paper it is shown that, in such framework, the Hypothesis 3 is not needed since it follows (in a non-trivial way) from the generalized curvature-dimension inequality CD$(\rho_1, \rho_2, \kappa, d)$ in Definition 1 above.

The above discussion prompts us to underline the distinctive aspect of the theory developed in the papers [6, 8, 9] and [10]: *for the class of complete sub-Riemannian manifolds with transverse symmetries of Yang-Mills type studied in [6], all the results are solely deduced from the curvature-dimension inequality CD$(\rho_1, \rho_2, \kappa, d)$ in* (14).

4 Li-Yau type estimates

In this section, we discuss a generalization of the celebrated Li-Yau inequality in [22] to the heat semigroup associated with the subelliptic operator L. We mention that, in this setting, related inequalities were obtained by Cao-Yau [11]. However, these authors work locally and the geometry of the manifold does not enter in their study. Instead, the analysis in [6] in based on some entropic inequalities which are derived from the curvature-dimension inequality (14) above. We have mentioned in the introduction that, even when specialized to the Riemannian case, the ideas in this section provide a new, more elementary approach to the Li-Yau inequalities. For this aspect we refer the reader to the paper [8].

Theorem 3 (sub-Riemannian Li-Yau gradient estimate). *Assume that the curvature-dimension inequality* (14) *be satisfied for $\rho_1 \in \mathbb{R}$, and that the Hypothesis 1,* *2, 3 hold. Let $f \in C_0^\infty(\mathbb{M})$, $f \geq 0$, $f \not\equiv 0$, then the following inequality holds for $t > 0$:*

$$\Gamma(\ln P_t f) + \frac{2\rho_2}{3} t \Gamma^Z(\ln P_t f) \leq \left(1 + \frac{3\kappa}{2\rho_2} - \frac{2\rho_1}{3} t\right) \frac{LP_t f}{P_t f} + \frac{d\rho_1^2}{6} t - \frac{\rho_1 d}{2}\left(1 + \frac{3\kappa}{2\rho_2}\right) + \frac{d\left(1 + \frac{3\kappa}{2\rho_2}\right)^2}{2t}.$$

Remark 1. We notice that when $\rho_1 \geq \rho_1'$, then one trivially has that:

$$CD(\rho_1, \rho_2, \kappa, d) \implies CD(\rho_1', \rho_2, \kappa, d).$$

As a consequence of this observation, when (14) holds with $\rho_1 > 0$, then also $\mathrm{CD}(0, \rho_2, \kappa, d)$ is true. Therefore, when $\rho_1 \geq 0$, Theorem 3 gives in particular for $f \in C_0^\infty(\mathbb{M})$, $f \geq 0$,

$$\Gamma(\ln P_t f) + \frac{2\rho_2}{3} t \Gamma^Z(\ln P_t f) \leq \left(1 + \frac{3\kappa}{2\rho_2}\right) \frac{LP_t f}{P_t f} + \frac{d\left(1 + \frac{3\kappa}{2\rho_2}\right)^2}{2t}. \tag{15}$$

However, this inequality is not optimal when $\rho_1 > 0$. It leads to a optimal Harnack inequality only when $\rho_1 = 0$.

Remark 2. Throughout the remainder of the paper the symbol D will only be used with the following meaning:

$$D = d\left(1 + \frac{3\kappa}{2\rho_2}\right). \tag{16}$$

With this notation, observing that the left-hand side of (15) is always nonnegative, and that $LP_t f = \partial_t P_t f$, when $\rho_1 \geq 0$ we obtain

$$\partial_t(\ln(t^{D/2} P_t f(x))) \geq 0. \tag{17}$$

By integrating (17) from $t < 1$ to 1 leads to the following on-diagonal bound for the heat kernel,

$$p(x, x, t) \leq \frac{1}{t^{D/2}} p(x, x, 1). \tag{18}$$

The constant $\frac{D}{2}$ in (18) is not optimal, in general, as the example of the heat semigroup on a Carnot group shows. In such case, in fact, one can show that the heat kernel $p(x, y, t)$ is homogeneous of degree $-Q/2$ with respect to the non-isotropic group dilations, where Q indicates the corresponding homogeneous dimension of the group. From such homogeneity of $p(x, y, t)$, one obtains the estimate

$$p(x, x, t) \leq \frac{1}{t^{Q/2}} p(x, x, 1),$$

which, unlike (18), is best possible. In the sub-Riemannian setting it does not seem easy to obtain sharp geometric constants by using only the curvature-dimension inequality (14). This aspect is quite different from the Riemannian case.

5 The parabolic Harnack inequality for Ricci ≥ 0

In this section we discuss a generalization of the celebrated Harnack inequality in [22] to solutions of the heat equation $Lu - u_t = 0$ on $\mathbb{M}$. One should also see the paper [11], where the authors deal with subelliptic operators on a compact manifold. As we have mentioned, these authors do not obtain bounds which depend on the sub-Riemannian geometry of the underlying manifold. Henceforth, we indicate with $C_b^\infty(\mathbb{M})$ the space $C^\infty(\mathbb{M}) \cap L^\infty(\mathbb{M})$.

Theorem 4. *Assume that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 \geq 0$ *and that the Hypothesis 1, 2, 3 hold. Given* $(x,s),(y,t) \in \mathbb{M} \times (0,\infty)$*, with* $s < t$*, one has for any* $f \in C_b^\infty(\mathbb{M})$*,* $f \geq 0$*,*

$$P_s f(x) \leq P_t f(y) \left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{D}{d}\frac{d(x,y)^2}{4(t-s)}\right). \tag{19}$$

Proof. Let $f \in C_0^\infty(\mathbb{M})$ be as in the statement of the theorem, and for every $(x,t) \in \mathbb{M}\times(0,\infty)$ consider $u(x,t) = P_t f(x)$. Since $Lu = \frac{\partial u}{\partial t}$, in terms of u the inequality (15) can be reformulated as

$$\Gamma(\ln u) + \frac{2\rho_2}{3} t \Gamma^Z(\ln u) \leq \left(1 + \frac{3\kappa}{2\rho_2}\right)\frac{\partial \log u}{\partial t} + \frac{d\left(1+\frac{3\kappa}{2\rho_2}\right)^2}{2t}.$$

Recalling (16), this implies in particular,

$$-\frac{\partial \ln u}{\partial t} \leq -\frac{d}{D}\Gamma(\ln u) + \frac{D}{2t}. \tag{20}$$

We now fix two points $(x,s),(y,t) \in \mathbb{M} \times (0,\infty)$, with $s < t$. Let $\gamma(\tau)$, $0 \leq \tau \leq T$ be a subunit path such that $\gamma(0) = y$, $\gamma(T) = x$, and consider the path in $\mathbb{M} \times (0,\infty)$ defined by

$$\alpha(\tau) = \left(\gamma(\tau), t + \frac{s-t}{T}\tau\right), \quad 0 \leq \tau \leq T,$$

so that $\alpha(0) = (y,t)$, $\alpha(T) = (x,s)$. We have

$$\begin{aligned}\ln\frac{u(x,s)}{u(y,t)} &= \int_0^T \frac{d}{d\tau}\ln u(\alpha(\tau))d\tau \\ &\leq \int_0^T \left[\Gamma(\ln u(\alpha(\tau)))^{\frac{1}{2}} - \frac{t-s}{T}\frac{\partial \ln u}{\partial t}(\alpha(\tau))\right]d\tau.\end{aligned}$$

Applying (20) for any $\epsilon > 0$ we find

$$\log \frac{u(x,s)}{u(y,t)} \le T^{\frac{1}{2}} \left(\int_0^T \Gamma(\ln u)(\alpha(\tau))d\tau \right)^{\frac{1}{2}} - \frac{t-s}{T} \int_0^T \frac{\partial \ln u}{\partial t}(\alpha(\tau))d\tau$$
$$\le \frac{1}{2\epsilon} T + \frac{\epsilon}{2} \int_0^T \Gamma(\ln u)(\alpha(\tau))d\tau - \frac{d}{D}\frac{t-s}{T} \int_0^T \Gamma(\ln u)(\alpha(\tau))d\tau$$
$$- \frac{D(s-t)}{2T} \int_0^T \frac{d\tau}{t + \frac{s-t}{T}\tau}.$$

If we now choose $\epsilon > 0$ such that

$$\frac{\epsilon}{2} = \frac{d}{D}\frac{t-s}{T},$$

we obtain from the latter inequality

$$\log \frac{u(x,s)}{u(y,t)} \le \frac{D}{d}\frac{\ell_s(\gamma)^2}{4(t-s)} + \frac{D}{2}\ln\left(\frac{t}{s}\right),$$

where we have denoted by $\ell_s(\gamma)$ the subunitary length of γ. If we now minimize over all subunitary paths joining y to x, and we exponentiate, we obtain

$$u(x,s) \le u(y,t)\left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{D}{d}\frac{d(x,y)^2}{4(t-s)}\right).$$

This proves (19) when $f \in C_0^\infty(\mathbb{M})$. We can then extend the result to $f \in C_b^\infty(\mathbb{M})$ by considering the approximations $h_n P_\tau f \in C_0^\infty(\mathbb{M})$, where $h_n \in C_0^\infty(\mathbb{M})$, $h_n \ge 0$, $h_n \to_{n\to\infty} 1$, and let $n \to \infty$ and $\tau \to 0$.

The following result represents an important consequence of Theorem 4.

Corollary 1. *Suppose that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 \ge 0$, *and that the Hypothesis 1, 2, 3 be valid. Let* $p(x,y,t)$ *be the heat kernel on* $\mathbb{M}$. *For every* $x,y,z \in \mathbb{M}$ *and every* $0 < s < t < \infty$ *one has*

$$p(x,y,s) \le p(x,z,t)\left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{D}{d}\frac{d(y,z)^2}{4(t-s)}\right).$$

6 Off-diagonal Gaussian upper bounds for Ricci ≥ 0

Suppose that the assumption of Theorem 4 are in force. Fix $x \in \mathbb{M}$ and $t > 0$. Applying Corollary 1 to $(y,t) \to p(x,y,t)$ for every $y \in B(x,\sqrt{t})$ we find

$$p(x,x,t) \le 2^{\frac{D}{2}} e^{\frac{D}{4d}}\, p(x,y,2t) = C(\rho_2,\kappa,d)p(x,y,2t).$$

Integration over $B(x, \sqrt{t})$ gives

$$p(x,x,t)\mu(B(x,\sqrt{t})) \le C(\rho_2,\kappa,d)\int_{B(x,\sqrt{t})} p(x,y,2t)d\mu(y) \le C(\rho_2,\kappa,d),$$

where we have used $P_t 1 \le 1$. This gives the on-diagonal upper bound

$$p(x,x,t) \le \frac{C(\rho_2,\kappa,d)}{\mu(B(x,\sqrt{t}))}. \tag{21}$$

Obtaining an off-diagonal upper bound for the heat kernel requires a more delicate analysis. The relevant result is contained in the following theorem, for whose proof we refer the reader to [6].

Theorem 5. *Assume that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 \ge 0$ *and that the Hypothesis 1, 2, 3 be fulfilled. For any* $0 < \epsilon < 1$ *there exists a constant* $C(\rho_2,\kappa,d,\epsilon) > 0$*, which tends to* ∞ *as* $\epsilon \to 0^+$*, such that for every* $x, y \in \mathbb{M}$ *and* $t > 0$ *one has*

$$p(x,y,t) \le \frac{C(d,\kappa,\rho_2,\epsilon)}{\mu(B(x,\sqrt{t}))^{\frac{1}{2}}\mu(B(y,\sqrt{t}))^{\frac{1}{2}}} \exp\left(-\frac{d(x,y)^2}{(4+\epsilon)t}\right).$$

7 A sub-Riemannian Bonnet-Myers theorem

Let $(\mathbb{M}, g)$ be a complete, connected Riemannian manifold of dimension $n \ge 2$. It is well-known that if for some $\rho > 0$ the Ricci tensor of $\mathbb{M}$ satisfies the bound

$$\mathrm{Ric} \ge (n-1)\rho, \tag{22}$$

then $\mathbb{M}$ is compact, with a finite fundamental group, and $\mathrm{diam}(\mathbb{M}) \le \pi/\sqrt{\rho}$. This is the celebrated Myer's theorem, which strengthens Bonnet's theorem.

In what follows we state a sub-Riemannian counterpart of the Bonnet-Myer's compactness theorem, see [6].

Theorem 6. *Assume that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 > 0$*, and that the Hypothesis 1, 2, 3 be valid. Then, the metric space* $(\mathbb{M}, d)$ *is compact and we have*

$$\mathit{diam}\,\mathbb{M} \le 2\sqrt{3}\pi\sqrt{\frac{\rho_2+\kappa}{\rho_1\rho_2}\left(1+\frac{3\kappa}{2\rho_2}\right)d}.$$

8 Global volume doubling when Ricci ≥ 0

Another fundamental tool in Riemannian geometry is the Bishop-Gromov volume comparison theorem. In what follows, given $\kappa \in \mathbb{R}$, we will indicate with $\mathbb{M}_\kappa$ the space of constant sectional curvature κ, and with $V_\kappa(r)$ the volume of the geodesic ball $B_\kappa(r)$ in $\mathbb{M}_\kappa$. Given a Riemannian manifold with measure tensor μ, for $x \in \mathbb{M}$ and $r > 0$ we let $V(x,r) = \mu(B(x,r))$.

Theorem 7 (Bishop-Gromov comparison theorem). *Let $\mathbb{M}$ be a complete n-dimensional Riemannian manifold such that Ric $\geq \rho$, $\rho \in \mathbb{R}$. Then, for every $x \in \mathbb{M}$ and every $r > 0$ the function*

$$r \to \frac{V(x,r)}{V_{\frac{\rho}{n-1}}(r)}$$

is non-increasing.

Corollary 2. *Let $\mathbb{M}$ be a complete n-dimensional Riemannian manifold with Ric $\geq$ 0. Then, for every $x \in \mathbb{M}$ and every $r > 0$ the function*

$$r \to \frac{V(x,r)}{r^n}$$

is non-increasing. As a consequence, one has

$$V(x,2r) \leq 2^n V(x,r), \quad x \in \mathbb{M},\ r > 0, \tag{23}$$

and since $\lim_{r\to 0^+} \frac{Vol(B(x,r))}{\omega_n r^n} = 1$, *we also have the following maximum volume growth estimate*

$$V(x,r) \leq \omega_n r^n, \quad x \in \mathbb{M},\ r > 0. \tag{24}$$

Theorem 7 and Corollary 2 play a pervasive role in the development of analysis on a Riemannian manifold with Ricci ≥ 0. They are important, among other things, in the study of the spectrum of the Laplacian on a manifold, for establishing Gaussian bounds on the heat kernel, isoperimetric theorems, etc.

In this section we intend to discuss a sub-Riemannian generalization of the doubling estimate (23) in Corollary 2 which has been established in [9], but see also [7] for the Riemannian case. Remarkably, our approach shows that an inequality such as (23) above can be exclusively derived from the Bochner identity without a direct use of the theory of Jacobi fields. As a consequence, it provides a very flexible tool for situations in which the tools of Riemannian geometry are not readily available.

We illustrate the main essential point. From the semigroup property and the symmetry of the heat kernel we have for any $y \in M$ and $t > 0$

$$p(y,y,2t) = \int_M p(y,z,t)^2 d\mu(z).$$

Consider now a function $h \in C_0^\infty(M)$ such that $0 \leq h \leq 1, h \equiv 1$ on $B(x,\sqrt{t}/2)$ and $h \equiv 0$ outside $B(x,\sqrt{t})$. We thus have

$$\begin{aligned} P_t h(y) &= \int_M p(y,z,t)h(z)d\mu(z) \\ &\leq \left(\int_{B(x,\sqrt{t})} p(y,z,t)^2 d\mu(z)\right)^{\frac{1}{2}} \left(\int_M h(z)^2 d\mu(z)\right)^{\frac{1}{2}} \\ &\leq p(y,y,2t)^{\frac{1}{2}} \mu(B(x,\sqrt{t}))^{\frac{1}{2}}. \end{aligned}$$

By taking $y = x$, and $t = r^2$ in the latter inequality, we obtain

$$P_{r^2}\left(\mathbf{1}_{B(x,r)}\right)(x)^2 \le P_{r^2}h(x)^2 \le p(x,x,2r^2)\,\mu(B(x,r)). \tag{25}$$

Applying Corollary 1 to $(y,t) \to p(x,y,t)$, for every $y \in B(x,\sqrt{t})$ we find

$$p(x,x,t) \le Cp(x,y,2t).$$

Integration in $y \in B(x,\sqrt{t})$ gives

$$p(x,x,t)\mu(B(x,\sqrt{t})) \le C\int_{B(x,\sqrt{t})} p(x,y,2t)d\mu(y) \le C,$$

where we have used $P_t 1 \le 1$. Letting $t = 4r^2$, we obtain from this the on-diagonal upper bound

$$\mu(B(x,2r)) \le \frac{C}{p(x,x,4r^2)}. \tag{26}$$

At this point we combine (25) with (26) to obtain

$$\begin{aligned}\mu(B(x,2r)) &\le C\frac{p(x,x,2r^2)}{p(x,x,4r^2)}\frac{\mu(B(x,r))}{P_{r^2}\left(\mathbf{1}_{B(x,r)}\right)(x)^2} \\ &\le C^*\frac{\mu(B(x,r))}{P_{r^2}\left(\mathbf{1}_{B(x,r)}\right)(x)^2},\end{aligned} \tag{27}$$

for every $x \in M$ and every $r > 0$.

It is clear that we would obtain a sub-Riemannian counterpart of (23) if we could show that there exists $A \in (0,1)$, $K > 0$, independent of $x \in M$ and $r > 0$, such that

$$P_{Ar^2}\left(\mathbf{1}_{B(x,r)}\right)(x) \ge K.$$

Note: The Harnack inequality in Theorem 4 gives

$$P_{r^2}\left(\mathbf{1}_{B(x,r)}\right)(x) \ge CP_{Ar^2}\left(\mathbf{1}_{B(x,r)}\right)(x). \tag{28}$$

Theorem 8. *Assume that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 \ge 0$ *and that the Hypothesis 1, 2, 3 be fulfilled. There exists a universal constant* $0 < A < 1$ *such that for every* $x \in \mathbb{M}$, *and* $r > 0$,

$$P_{Ar^2}(\mathbf{1}_{B(x,r)})(x) \ge \frac{1}{2}.$$

The proof of Theorem 8 is fairly complicated and it occupies large part of the work [9]. For a much simpler account in the Riemannian setting we refer the reader to [7]. For future reference we record the following consequence of (25), (28), and Theorem 8,

$$p(x,x,2r^2) \ge \frac{C}{\mu(B(x,r))}, \quad x \in \mathbb{M}, r > 0. \tag{29}$$

With Theorem 8 in hands, following the arguments developed above, we obtain the following basic result.

Theorem 9 (Global doubling property). *Assume that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 \geq 0$*, and that the Hypothesis 1, 2, 3 be valid. Then, the metric measure space* $(\mathbb{M}, d, \mu)$ *satisfies the global volume doubling property. More precisely, there exists a constant* $C_1 = C_1(\rho_1, \rho_2, \kappa, d) > 0$ *such that for every* $x \in \mathbb{M}$ *and every* $r > 0$*,* $\mu(B(x, 2r)) \leq C_1 \mu(B(x, r))$.

9 Sharp Gaussian bounds, Poincaré inequality and parabolic Harnack inequality

The purpose of this section is to establish some optimal two-sided bounds for the heat kernel $p(x, y, t)$ associated with the subelliptic operator L. Such estimates are reminiscent of those obtained by Li and Yau for complete Riemannian manifolds having $\mathrm{Ric} \geq 0$. As a consequence of the two-sided Gaussian bound for the heat kernel, we will derive a global Poincaré inequality and a localized parabolic Harnack inequality. Here is our main result.

Theorem 10. *Suppose that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 \geq 0$*, and that the Hypothesis 1, 2, 3 be valid. For any* $0 < \varepsilon < 1$ *there exists a constant* $C(\varepsilon) = C(d, \kappa, \rho_2, \varepsilon) > 0$*, which tends to* ∞ *as* $\varepsilon \to 0^+$*, such that for every* $x, y \in \mathbb{M}$ *and* $t > 0$ *one has*

$$\frac{C(\varepsilon)^{-1}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{D d(x, y)^2}{d(4 - \varepsilon)t}\right) \leq p(x, y, t) \leq \frac{C(\varepsilon)}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{(4 + \varepsilon)t}\right).$$

Proof. We begin by establishing the lower bound. First, from Corollary 1 we obtain for all $y \in \mathbb{M}$, $t > 0$, and every $0 < \varepsilon < 1$,

$$p(x, y, t) \geq p(x, x, \varepsilon t) \varepsilon^{\frac{D}{2}} \exp\left(-\frac{D}{d} \frac{d(x, y)^2}{(4 - \varepsilon)t}\right).$$

We thus need to estimate $p(x, x, \varepsilon t)$ from below. But this has already been done in (29). Choosing $r > 0$ such that $2r^2 = \varepsilon t$, we obtain from that estimate

$$p(x, x, \varepsilon t) \geq \frac{C^*}{\mu(B(x, \sqrt{\varepsilon/2}\sqrt{t}))}, \quad x \in \mathbb{M},\ t > 0.$$

On the other hand, since $\sqrt{\varepsilon/2} < 1$, by the trivial inequality $\mu(B(x, \sqrt{\varepsilon/2}\sqrt{t})) \leq \mu(B(x, \sqrt{t}))$, we conclude

$$p(x, y, t) \geq \frac{C^*}{\mu(B(x, \sqrt{t}))} \varepsilon^{\frac{D}{2}} \exp\left(-\frac{D}{d} \frac{d(x, y)^2}{(4 - \varepsilon)t}\right).$$

This proves the Gaussian lower bound.

For the Gaussian upper bound, we first observe recall that Theorem 5 gives for any $0 < \varepsilon' < 1$

$$p(x,y,t) \le \frac{C(d,\kappa,\rho_2,\varepsilon')}{\mu(B(x,\sqrt{t}))^{\frac{1}{2}}\mu(B(y,\sqrt{t}))^{\frac{1}{2}}} \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon')t}\right). \tag{30}$$

At this point, by the triangle inequality and Theorem 9 we find with $Q = \log_2 C_1$,

$$\begin{aligned}\mu(B(x,\sqrt{t})) &\le \mu(B(y,d(x,y)+\sqrt{t}))\\ &\le C_1\mu(B(y,\sqrt{t}))\left(\frac{d(x,y)+\sqrt{t}}{\sqrt{t}}\right)^{Q}.\end{aligned}$$

This gives

$$\frac{1}{\mu(B(y,\sqrt{t}))} \le \frac{C_1}{\mu(B(x,\sqrt{t}))}\left(\frac{d(x,y)}{\sqrt{t}}+1\right)^{Q}.$$

Combining this with (30) we obtain

$$p(x,y,t) \le \frac{C_1^{1/2}C(d,\kappa,\rho_2,\varepsilon')}{\mu(B(x,\sqrt{t}))}\left(\frac{d(x,y)}{\sqrt{t}}+1\right)^{\frac{Q}{2}} \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon')t}\right).$$

If now $0 < \varepsilon < 1$, it is clear that we can choose $0 < \varepsilon' < \varepsilon$ such that

$$\frac{C_1^{1/2}C(d,\kappa,\rho_2,\varepsilon')}{\mu(B(x,\sqrt{t}))}\left(\frac{d(x,y)}{\sqrt{t}}+1\right)^{\frac{Q}{2}} \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon')t}\right) \le \frac{C^*(d,\kappa,\rho_2,\varepsilon)}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon)t}\right),$$

where $C^*(d,\kappa,\rho_2,\varepsilon)$ is a constant which tends to ∞ as $\varepsilon \to 0^+$. The desired conclusion follows by suitably adjusting the values of both ε' and of the constant in the right-hand side of the estimate.

With Theorems 9 and 10 in hands, we can now appeal to the results in [12, 14, 21, 25, 27–29], see also the books [15, 16]. More precisely, from the developments in these papers it is by now well-known that in the context of strictly regular local Dirichlet spaces we have the equivalence between:

(1) a two sided Gaussian bounds for the heat kernel (like in Theorem 10);
(2) the conjunction of the volume doubling property and the Poincaré inequality (see Theorem 11 below);
(3) the parabolic Harnack inequality (see Theorem 12 below).

Thus, thanks to Theorems 9 and 10, we obtain the following form of Poincaré inequality.

Theorem 11. *Suppose that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 \geq 0$*, and that the Hypothesis 1, 2, 3 be valid. Then, there exists a constant* $C = C(d, \kappa, \rho_2) > 0$ *such that for every* $x \in \mathbb{M}, r > 0$*, and* $f \in C^\infty(\mathbb{M})$ *one has*

$$\int_{B(x,r)} |f(y) - f_r|^2 d\mu(y) \leq C r^2 \int_{B(x,2r)} \Gamma(f)(y) d\mu(y),$$

where we have let $f_r = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu$.

Since thanks to Theorem 9 the space $(\mathbb{M}, \mu, d)$, where $d = d(x, y)$ indicates the sub-Riemannian distance (11), is a space of homogeneous type, and furthermore (10) above guarantees that it is a length-space, then, arguing as in [20], from Theorem 11 we obtain the following result.

Corollary 3. *Under the hypothesis of Theorem 11 there exists a constant* $C^* = C^*(d, \kappa, \rho_2) > 0$ *such that for every* $x \in \mathbb{M}, r > 0$*, and* $f \in C^\infty(\mathbb{M})$ *one has*

$$\int_{B(x,r)} |f(y) - f_r|^2 d\mu(y) \leq C^* r^2 \int_{B(x,r)} \Gamma(f)(y) d\mu(y).$$

Furthermore, the following scale invariant Harnack inequality for local solutions holds.

Theorem 12. *Assume that the curvature-dimension inequality* (14) *be satisfied for* $\rho_1 \geq 0$*, and that the Hypothesis 1, 2, 3 be valid. If* u *is a positive solution of the heat equation in a cylinder of the form* $Q = (s, s + \alpha r^2) \times B(x, r)$ *then*

$$\sup_{Q-} u \leq C \inf_{Q+} u, \tag{31}$$

where for some fixed $0 < \beta < \gamma < \delta < \alpha < \infty$ *and* $\eta \in (0, 1)$*,*

$$Q- = (s + \beta r^2, s + \gamma r^2) \times B(x, \eta r), Q+ = (s + \delta r^2, s + \alpha r^2) \times B(x, \eta r).$$

Here, the constant C *is independent of* x, r *and* u*, but depends on the parameters* d, κ, ρ_2*, as well as on* $\alpha, \beta, \gamma, \delta$ *and* η*.*

10 Negatively curved manifolds

In the previous sections we have exclusively discussed the case of sub-Riemannian manifolds with nonnegative Ricci curvature. In this section we present some of the main results in [10] relative to the case in which Ricci is bounded from below by a number which is allowed to be negative.

Theorem 13. *Suppose that the generalized curvature-dimension inequality* (14) *hold for some* $\rho_1 \in \mathbb{R}$*, and that the Hypothesis 1, 2, 3 be valid. Then, there exist constants*

$C_1, C_2 > 0$, depending only on $\rho_1, \rho_2, \kappa, d$, for which one has for every $x, y \in \mathbb{M}$ and every $r > 0$:

$$\mu(B(x, 2r)) \leq C_1 \exp\left(C_2 r^2\right) \mu(B(x, r)). \tag{32}$$

The constant C_2 tends to zero as $\rho_1 \to 0$, and thus (32) contains in particular the estimate in Theorem 9.

In order to state the next result, we introduce a family of control distances d_τ for $\tau \geq 0$. Given $x, y \in \mathbb{M}$, let us consider

$$S_\tau(x, y) = \{\gamma : [0, T] \to \mathbb{M} \mid \gamma \text{ is subunit for } \Gamma + \tau^2 \Gamma^Z, \gamma(0) = x,\ \gamma(T) = y\}.$$

A curve which is subunit for Γ is obviously subunit for $\Gamma + \tau^2 \Gamma^Z$, therefore thanks to the assumption (10) above we have $S_\tau(x, y) \neq \varnothing$. We can then define

$$d_\tau(x, y) = \inf\{\ell_s(\gamma) \mid \gamma \in S_\tau(x, y)\}. \tag{33}$$

Note that $d(x, y) = d_0(x, y)$ and that, clearly: $d_\tau(x, y) \leq d(x, y)$.

Theorem 14. *Suppose that the generalized curvature-dimension inequality hold for some $\rho_1 \in \mathbb{R}$, and that the Hypothesis 1, 2, 3 be satisfied. Let $\tau \geq 0$. Then, there exists a constant $C(\tau) > 0$, depending only on $\rho_1, \rho_2, \kappa, d$ and τ, for which one has for every $x, y \in \mathbb{M}$:*

$$d\ (x, y) \leq C(\tau) \max\{\sqrt{d_\tau\ (x, y)}, d_\tau\ (x, y)\}. \tag{34}$$

11 Geometric examples

In this section we present several classes of sub-Riemannian spaces satisfying the generalized curvature-dimension inequality in Definition 1 above. These examples constitute the central motivation of the present work.

11.1 Riemannian manifolds

As we have mentioned in the introduction, when $\mathbb{M}$ is a n-dimensional complete Riemannian manifold with Riemannian distance d_R, Levi-Civita connection ∇ and Laplace-Beltrami operator Δ, our main assumptions hold trivially. It suffices to choose $\Gamma^Z = 0$ to satisfy Hypothesis 2 in a trivial fashion. Hypothesis 1 is also satisfied since it is equivalent to assuming that $(\mathbb{M}, d_R)$ be complete (observe in passing that the distance (9) coincides with d_R). Finally, with the choice $\kappa = 0$ and $\rho_1 = \rho$ the curvature-dimension inequality (14) reduces to (6), which, as we have already observed, is implied by (and it is in fact equivalent to) the assumption Ric $\geq \rho$.

11.2 The three-dimensional Sasakian models

The purpose of this section is providing a first basic sub-Riemannian example which fits the framework of the present paper. This example was first studied in [5]. Given a number $\rho_1 \in \mathbb{R}$, suppose that $\mathbb{G}(\rho_1)$ be a three-dimensional Lie group whose Lie algebra $\mathfrak{g}$ has a basis $\{X, Y, Z\}$ satisfying:

(i) $[X, Y] = Z$;
(ii) $[X, Z] = -\rho_1 Y$;
(iii) $[Y, Z] = \rho_1 X$.

A sub-Laplacian on $\mathbb{G}(\rho_1)$ is the left-invariant, second-order differential operator

$$L = X^2 + Y^2. \tag{35}$$

In view of (i)-(iii) Hörmander's theorem, see [18], implies that L be hypoelliptic, although it fails to be elliptic at every point of $\mathbb{G}(\rho_1)$. From (8) we find in the present situation

$$\Gamma(f) = \frac{1}{2}\big(L(f^2) - 2fLf\big) = (Xf)^2 + (Yf)^2.$$

If we define

$$\Gamma^Z(f, g) = ZfZg,$$

then from (i)-(iii) we easily verify that

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

We conclude that the Hypothesis 2 is satisfied. It is not difficult to show that the Hypothesis 1 is also fulfilled.

Using (i)-(iii) we leave it to the reader to verify that

$$[L, Z] = 0. \tag{36}$$

By means of (36) we easily find

$$\begin{aligned}\Gamma_2^Z(f) &= \frac{1}{2}L(\Gamma^Z(f)) - \Gamma^Z(f, Lf) = Zf[L, Z]f + (XZf)^2 + (YZf)^2\\ &= (XZf)^2 + (YZf)^2.\end{aligned}$$

Finally, from definition (12) and from (i)-(iii) we obtain

$$\begin{aligned}\Gamma_2(f) &= \frac{1}{2}L(\Gamma(f)) - \Gamma(f, Lf)\\ &= \rho_1\Gamma(f) + (X^2f)^2 + (YXf)^2 + (XYf)^2 + (Y^2f)^2\\ &\quad + 2Yf(XZf) - 2Xf(YZf).\end{aligned}$$

We now notice that

$$(X^2f)^2 + (YXf)^2 + (XYf)^2 + (Y^2f)^2 = ||\nabla_H^2 f||^2 + \frac{1}{2}\Gamma^Z(f),$$

where we have denoted by

$$\nabla^2_H f = \begin{pmatrix} X^2 f & \frac{1}{2}(XYf + YXf) \\ \frac{1}{2}(XYf + YXf) & Y^2 f \end{pmatrix}$$

the symmetrized Hessian of f with respect to the horizontal distribution generated by X, Y. Substituting this information in the above formula we find

$$\Gamma_2(f) = ||\nabla^2_H f||^2 + \rho_1 \Gamma(f) + \frac{1}{2}\Gamma^Z(f) + 2\big(Yf(XZf) - Xf(YZf)\big).$$

By the above expression for $\Gamma^Z_2(f)$, using Cauchy-Schwarz inequality, we obtain for every $\nu > 0$

$$|2Yf(XZf) - 2Xf(YZf)| \le \nu \Gamma^Z_2(f) + \frac{1}{\nu}\Gamma(f).$$

Similarly, one easily recognizes that

$$||\nabla^2_H f||^2 \ge \frac{1}{2}(Lf)^2.$$

Combining these inequalities, we conclude that we have proved the following result.

Proposition 1. *For every $\rho_1 \in \mathbb{R}$ the Lie group $\mathbb{G}(\rho_1)$, with the sub-Laplacian L in* (35)*, satisfies the generalized curvature dimension inequality $CD(\rho_1, \frac{1}{2}, 1, 2)$. Precisely, for every $f \in C^\infty(\mathbb{G}(\rho_1))$ and any $\nu > 0$ one has:*

$$\Gamma_2(f) + \nu \Gamma^Z_2(f) \ge \frac{1}{2}(Lf)^2 + \left(\rho_1 - \frac{1}{\nu}\right)\Gamma(f) + \frac{1}{2}\Gamma^Z(f).$$

Proposition 1 provides a basic motivation for Definition 1. It is also important to observe at this point that the Lie group $\mathbb{G}(\rho_1)$ can be endowed with a natural CR structure. Denoting in fact with $\mathcal{H}$ the subbundle of $T\mathbb{G}(\rho_1)$ generated by the vector fields X and Y, the endomorphism J of $\mathcal{H}$ defined by

$$J(Y) = X, \quad J(X) = -Y,$$

satisfies $J^2 = -I$, and thus defines a complex structure on $\mathbb{G}(\rho_1)$. By choosing θ as the form such that

$$\operatorname{Ker}\theta = \mathcal{H}, \quad \text{and} \quad d\theta(X, Y) = 1,$$

we obtain a CR structure on $\mathbb{G}(\rho_1)$ whose Reeb vector field is $-Z$. Thus, the above choice of Γ^Z is canonical.

The pseudo-hermitian Tanaka-Webster torsion of $\mathbb{G}(\rho_1)$ vanishes, thus $(\mathbb{G}(\rho_1), \theta)$ is a Sasakian manifold. It is also easy to verify that for the CR manifold $(\mathbb{G}(\rho_1), \theta)$ the Tanaka-Webster horizontal sectional curvature is constant and equals ρ_1. The following three model spaces correspond respectively to the cases $\rho_1 = 1, \rho_1 = 0$ and $\rho_1 = -1$:

1. the Lie group $\mathbb{SU}(2)$ is the group of 2×2, complex, unitary matrices of determinant 1;
2. the Heisenberg group $\mathbb{H}$ is the group of 3×3 matrices:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R};$$

3. the Lie group $\mathbb{SL}(2)$ is the group of 2×2, real matrices of determinant 1.

11.3 Sub-Riemannian manifolds with transverse symmetries

We now turn our attention to a large class of sub-Riemannian manifolds, encompassing the three-dimensional model spaces discussed in the previous subsection. Theorem 15 below states that for these sub-Riemannian manifolds the generalized curvature-dimension inequality (14) does hold under some natural geometric assumptions which, in the Riemannian case, reduce to requiring a lower bound for the Ricci tensor. To achieve this result, some new Bochner type identities were established in [6].

Let $\mathbb{M}$ be a smooth, connected manifold equipped with a bracket generating distribution $\mathcal{H}$ of dimension d and a fiberwise inner product g on that distribution. The distribution $\mathcal{H}$ will be referred to as the set of *horizontal directions*.

We indicate with $\mathfrak{iso}$ the finite-dimensional Lie algebra of all sub-Riemannian Killing vector fields on $\mathbb{M}$ (see [26]). A vector field $Z \in \mathfrak{iso}$ if the one-parameter flow generated by it locally preserves the sub-Riemannian geometry defined by $(\mathcal{H}, g)$. This amounts to saying that:

(1) for every $x \in \mathbb{M}$, and any $u, v \in \mathcal{H}(x)$, $\mathcal{L}_Z g(u, v) = 0$;
(2) if $X \in \mathcal{H}$, then $[Z, X] \in \mathcal{H}$.

In (1) we have denoted by $\mathcal{L}_Z g$ the Lie derivative of g with respect to Z. Our main geometric assumption is the following:

Hypothesis 4. *There exists a Lie sub-algebra $\mathcal{V} \subset \mathfrak{iso}$, such that for every $x \in \mathbb{M}$,*

$$T_x\mathbb{M} = \mathcal{H}(x) \oplus \mathcal{V}(x).$$

The distribution $\mathcal{V}$ will be referred to as the set of *vertical directions*. The dimension of $\mathcal{V}$ will be denoted by $\mathfrak{h}$.

The choice of an inner product on the Lie algebra $\mathcal{V}$ naturally endows $\mathbb{M}$ with a Riemannian extension g_R of g that makes the decomposition $\mathcal{H}(x) \oplus \mathcal{V}(x)$ orthogonal. Although g_R is useful for computational purposes, the geometric objects that introduced in [6], like the sub-Laplacian L, the canonical connection ∇ and the "Ricci" tensor $\mathcal{R}$, do not depend on the choice of an inner product on $\mathcal{V}$. We refer to [6] for a detailed geometric discussion.

Theorem 15. *Suppose that there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$ and $\kappa \geq 0$ such that for every $f \in C^\infty(\mathbb{M})$:*

$$\begin{cases} \mathcal{R}(f) \geq \rho_1 \Gamma(f) + \rho_2 \Gamma^Z(f), \\ \mathcal{T}(f) \leq \kappa \Gamma(f). \end{cases} \tag{37}$$

Then, the sub-Riemannian manifold $\mathbb{M}$ satisfies the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ in (14) *with respect to the sub-Laplacian L and the differential form Γ^Z.*

In [6] it was shown that, remarkably, the generalized curvature-dimension inequality (14) in Definition 1 is equivalent to the geometric bounds (37) above. Here is the relevant result.

Theorem 16. *Suppose that there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$ and $\kappa \geq 0$ such that $\mathbb{M}$ satisfy the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$. Then, $\mathbb{M}$ satisfies the geometric bounds* (37)*. As a consequence of this fact and of Theorem 15 we conclude that*

$$CD(\rho_1, \rho_2, \kappa, d) \Longleftrightarrow \begin{cases} \mathcal{R}(f) \geq \rho_1 \Gamma(f) + \rho_2 \Gamma^Z(f), \\ \mathcal{T}(f) \leq \kappa \Gamma(f). \end{cases}$$

11.4 Carnot groups of step two

Carnot groups of step two provide a natural reservoir of sub-Riemannian manifolds with transverse symmetries. Let $\mathfrak{g}$ be a graded nilpotent Lie algebra of step two. This means that $\mathfrak{g}$ admits a splitting $\mathfrak{g} = V_1 \oplus V_2$, where $[V_1, V_1] = V_2$, and $[V_1, V_2] = \{0\}$. We endow $\mathfrak{g}$ with an inner product $\langle \cdot, \cdot \rangle$ with respect to which the decomposition $V_1 \oplus V_2$ is orthogonal. We denote by $e_1, ..., e_d$ an orthonormal basis of V_1 and by $\varepsilon_1, ..., \varepsilon_{\mathfrak{h}}$ an orthonormal basis of V_2. Let $\mathbb{G}$ be the connected and simply connected graded nilpotent Lie group associated with $\mathfrak{g}$. Left-invariant vector fields in V_2 are seen to be transverse sub-Riemannian Killing vector fields of the horizontal distribution given by V_1. The geometric assumptions of the previous section are thus satisfied.

Proposition 2. *Let $\mathbb{G}$ be a Carnot group of step two, with d being the dimension of the horizontal layer of its Lie algebra. Then, $\mathbb{G}$ satisfies the generalized curvature-dimension inequality $CD(0, \rho_2, \kappa, d)$ (with respect to any sub-Laplacian L on $\mathbb{G}$), with $\rho_2 > 0$ and $\kappa \geq 0$ which solely depend on $\mathbb{G}$.*

In particular, in our framework, every Carnot group of step two is a *sub-Riemannian manifold with nonnegative Ricci tensor.*

11.5 CR Sasakian manifolds

Another interesting class of sub-Riemannian manifolds with transverse symmetries is given by the class of CR Sasakian manifolds. For these manifolds one has the following result, established in [6].

Theorem 17. *Let $\mathbb{M}$ be a Sasakian manifold, having real dimension $2n+1$. Assume that the Tanaka-Webster Ricci tensor is bounded from below by $\rho_1 \in \mathbb{R}$ on smooth functions, that is for every $f \in C^\infty(\mathbb{M})$*

$$Ric(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} f) \geq \rho_1 \|\nabla_{\mathcal{H}} f\|^2.$$

Then, $\mathbb{M}$ satisfies the generalized curvature-dimension inequality $CD(\rho_1, \frac{n}{2}, 1, 2n)$.

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Hausdorff measures and dimensions in non equiregular sub-Riemannian manifolds

Roberta Ghezzi and Frédéric Jean

Abstract This paper is a starting point towards computing the Hausdorff dimension of submanifolds and the Hausdorff volume of small balls in a sub-Riemannian manifold with singular points. We first consider the case of a strongly equiregular submanifold, i. e., a smooth submanifold N for which the growth vector of the distribution D and the growth vector of the intersection of D with TN are constant on N. In this case, we generalize the result in [12], which relates the Hausdorff dimension to the growth vector of the distribution. We then consider analytic sub-Riemannian manifolds and, under the assumption that the singular point p is typical, we state a theorem which characterizes the Hausdorff dimension of the manifold and the finiteness of the Hausdorff volume of small balls $B(p, \rho)$ in terms of the growth vector of both the distribution and the intersection of the distribution with the singular locus, and of the nonholonomic order at p of the volume form on M evaluated along some families of vector fields.

1 Introduction

The main motivation of this paper arises from the study of sub-Riemannian manifolds as particular metric spaces. Recall that a sub-Riemannian manifold is a triplet (M, D, g), where M is a smooth manifold, D a Lie-bracket generating subbundle of TM and g a Riemannian metric on D. The absolutely continuous paths which are almost everywhere tangent to D are called horizontal and their length is ob-

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tained as in Riemannian geometry integrating the norm of their tangent vectors. The sub-Riemannian distance d is defined as the infimum of length of horizontal paths between two given points.

Hausdorff measures and spherical Hausdorff measures can be defined on sub-Riemannian manifolds using the sub-Riemannian distance. It is well-known that for these metric spaces the Hausdorff dimension is strictly greater than the topological one. Although the presence of an extra structure, i. e., the differential one, constitute a considerable help, computing Hausdorff measures and dimensions of sets is a difficult problem. In [5] we study Hausdorff measures of continuous curves, whereas in [1] the authors analyze the regularity of the top-dimensional Hausdorff measure in the equiregular case (see the definition below). In the case of Carnot groups, Hausdorff measures of regular hypersurfaces have been studied in [4] and in a more general context, a representation formula for the perimeter measure in terms of Hausdorff measure has been proved in [2].

In this paper we consider three questions: given a sub-Riemannian manifold (M, D, g), $p \in M$ and a small $\rho > 0$,

1) What is the Hausdorff dimension $\dim_H(M)$?
2) Under which condition is the Hausdorff volume $\mathscr{H}^{\dim_H(M)}(B(p,\rho))$ finite?
3) The two preceding questions when M is replaced by a submanifold N, i. e., what is $\dim_H(N)$ and when is $\mathscr{H}^{\dim_H(N)}(N \cap B(p,\rho))$ finite?

A key feature to be taken into account is whether p is regular or singular for the sub-Riemannian manifold. Given $i \geq 1$, define recursively the submodule D^i of $\mathrm{Vec}(M)$ by $D^1 = D$, $D^{i+1} = D^i + [D, D^i]$. Denote by $D^i_p = \{X(p) \mid X \in D^i\}$. Since D is Lie-bracket generating, there exists $r(p) \in \mathbb{N}$ such that

$$\{0\} = D^0_p \subset D^1_p \subset \cdots \subset D^{r(p)}_p = T_pM.$$

A point p is regular if, for every i, the dimensions $\dim D^i_q$ are constant as q varies in a neighborhood of p. Otherwise, p is said to be singular. A set $S \subset M$ is equiregular if, for every i, $\dim D^i_q$ is constant as q varies in S. For equiregular manifolds, questions 1 and 2 have been answered in [12] (but with an incorrect proof, see [13] for a correct one). In that paper, the author shows that the Hausdorff dimension of an equiregular manifold M is

$$\dim_H(M) = Q, \quad \text{where } Q = \sum_{i=1}^{r(p)} i(\dim D^i_p - \dim D^{i-1}_p), \tag{1}$$

and that the Hausdorff Q-dimensional measure near a regular point is absolutely continuous with respect to any Lebesgue measure on M. As a consequence, when p is regular, the Hausdorff dimension of a small ball $B(p,\rho)$ is Q, and the Hausdorff Q-dimensional measure of $B(p,\rho)$ is finite.

When there are singular points, these problems have been mentioned in [8, Sect. 1.3.A]. In this case, the idea is to compute the Hausdorff dimension using suitable stratifications of M where the discontinuities of the dimensions $q \mapsto \dim D^i_q$ are somehow controlled. Namely, as suggested in [8], we consider stratifications made by submanifolds N which are *strongly equiregular*, i. e., for which both the dimensions $\dim D_q^i$ and $\dim(D_q^i \cap T_q N)$ are constant as q varies in N.

The first part of the paper provides an answer to question 3 when N is strongly equiregular. The first result of the paper (Theorem 1) computes the Hausdorff dimension of a strongly equiregular submanifold N in terms of the dimensions of $\dim(D_q^i \cap T_q N)$, generalizing formula (1) which corresponds to the case $N = M$. More precisely, $\dim_H(N) = Q_N$ where

$$Q_N := \sum_{i=1}^{r(p)} i(\dim(D_p^i \cap T_p N) - \dim(D_p^{i-1} \cap T_p N)).$$

This actually follows from a stronger property: indeed, we show that the Q_N-dimensional spherical Hausdorff measure in N is absolutely continuous with respect to any smooth measure (i. e. any measure induced locally by a volume form) on N. The Radon–Nikodym derivative computed in Theorem 1 generalizes [1, Lemma 32], which corresponds to the case $N = M$. The main ingredient behind the proofs of such results is the fact that for a strongly equiregular submanifold N the metric tangent cone to $(N, d|_N)$ exists at every $p \in N$ and can be identified to $T_p N$ via suitable systems of privileged coordinates (see Lemma 1).

The results for strongly equiregular submanifolds provide a first step towards the answer of questions 1 and 2 in the general case, at least for analytic sub-Riemannian manifolds. This is the topic in the second part of the paper. Indeed, when (M, D, g) is analytic, M can be stratified as $M = \cup_{i \geq 0} M_i$ where each M_i is an analytic equiregular submanifold. Then, the Hausdorff dimension of a small ball B is the maximum of the Hausdorff dimensions of the intersections $B \cap M_i$. To compute the latter ones, we use that each strata M_i can further be decomposed as the disjointed union of strongly equiregular analytic submanifolds. In Lemma 3, using Theorem 1 we compute the Hausdorff dimension of an equiregular (but possibly not strongly equiregular) analytic submanifold and we estimate the density of the corresponding Hausdorff measure. Characterizing the finiteness of the corresponding Hausdorff measure of the intersection of a small ball with an equiregular analytic submanifold is rather involved. Yet this is the main issue in question 2, as whenever the Hausdorff measure of $B(p, \rho) \cap \{\text{regular points}\}$ is infinite at a singular point p then so is $\mathcal{H}^{\dim_H(M)}(B(p, \rho))$. To estimate $\mathcal{H}^{\dim_H(M)}(B(p, \rho) \cap \{\text{regular points}\})$, we assume that the singular point p is "typical", that is, it belongs to a strongly equiregular submanifold N of the singular set. In Theorem 2 we characterize the finiteness of the aforementioned measure at typical singular points through an algebraic relation involving the Hausdorff dimension Q_{reg} near a regular point, the Hausdorff dimension Q_N of N, and the nonholonomic order at p of the volume form on M evaluated along some families of vector fields, given by Lie brackets between generators of the distribution.

The proof of Theorem 2 (and of Proposition 1) will appear in a forthcoming paper.

The structure of the paper is the following. In Sect. 2 we recall shortly the definitions of Hausdorff measures and dimension and some basic notions in sub-Riemannian geometry. Section 3 is devoted to the the definition and the study of strongly equiregular submanifolds and contains the proof of Theorem 1 and the statement of Proposition 1. In Sect. 4 we treat analytic sub-Riemannian manifolds. First, we estimate the Hausdorff dimension $\bar{Q}_N$ of an analytic equiregular submanifold N in Sect 4.1. Then, in Sect. 4.2, we prove that the $\bar{Q}_N$-dimensional Hausdorff measure of the intersection of a small ball $B(p,\rho)$ with N is finite if $p \in N$ and we state Theorem 2. Finally, we end by applying our results to some examples of sub-Riemannian manifolds in Sect. 4.3. In particular, the examples show that when the Hausdorff dimension of a ball centered at a singular point is equal to the Hausdorff dimension of the whole manifold, the corresponding Hausdorff measure can be both finite or infinite.

2 Basic notations

2.1 *Hausdorff measures*

Let (M,d) be a metric space. We denote by $\operatorname{diam} S$ the diameter of a set $S \subset M$, by $B(p,\rho)$ the open ball $\{q \in M \mid d(q,p) < \rho\}$, and by $\overline{B(p,\rho)}$ the closure of $B(p,\rho)$. Let $\alpha \geq 0$ be a real number. For every set $A \subset M$, the *α-dimensional Hausdorff measure* $\mathcal{H}^\alpha$ of A is defined as $\mathcal{H}^\alpha(A) = \lim_{\epsilon\to 0^+} \mathcal{H}^\alpha_\epsilon(A)$, where

$$\mathcal{H}^\alpha_\epsilon(A) = \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} S_i)^\alpha \ : \ A \subset \bigcup_{i=1}^\infty S_i,\ S_i \text{ closed set},\ \operatorname{diam} S_i \leq \epsilon\right\},$$

and the *α-dimensional spherical Hausdorff measure* is defined as $\mathcal{S}^\alpha(A) = \lim_{\epsilon\to 0^+} \mathcal{S}^\alpha_\epsilon(A)$, where

$$\mathcal{S}^\alpha_\epsilon(A) = \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} S_i)^\alpha \ : \ A \subset \bigcup_{i=1}^\infty S_i,\ S_i \text{ is a ball},\ \operatorname{diam} S_i \leq \epsilon\right\}.$$

For every set $A \subset M$, the non-negative number

$$D = \sup\{\alpha \geq 0 \mid \mathcal{H}^\alpha(A) = \infty\} = \inf\{\alpha \geq 0 \mid \mathcal{H}^\alpha(A) = 0\}$$

is called the *Hausdorff dimension of A*. The D-dimensional Hausdorff measure $\mathcal{H}^D(A)$ is called the Hausdorff volume of A. Notice that this volume may be 0, > 0, or ∞.

Given a subset $N \subset M$, we can consider the metric space $(N, d|_N)$. Denoting by $\mathcal{H}^\alpha_N$ and $\mathcal{S}^\alpha_N$ the Hausdorff and spherical Hausdorff measures in this space, by definition we have

$$\begin{aligned} \mathcal{H}^\alpha\llcorner_N(A) &:= \mathcal{H}^\alpha(A\cap N) = \mathcal{H}^\alpha_N(A\cap N),\\ \mathcal{S}^\alpha\llcorner_N(A) &:= \mathcal{S}^\alpha(A\cap N) \leq \mathcal{S}^\alpha_N(A\cap N). \end{aligned} \tag{2}$$

These are a simple consequences of the fact that a set C is closed in N if and only if $C = C' \cap N$, with C' closed in M. Notice that the inequality (2) is strict in general, as coverings in the definition of $\mathcal{S}_N^\alpha$ are made with sets B which satisfy $B = \overline{B(p,\rho)} \cap N$ with $p \in N$, whereas coverings in the definition of $\mathcal{S}^\alpha \llcorner_N$ include sets of the type $\overline{B(p,\rho)} \cap N$ with $p \notin N$. Moreover, by construction of Hausdorff measures, for every subset $S \subset N$, $\mathcal{H}^\alpha(S) \le \mathcal{S}^\alpha(S) \le 2^\alpha \mathcal{H}^\alpha(S)$ and $\mathcal{H}_N^\alpha(S) \le \mathcal{S}_N^\alpha(S) \le 2^\alpha \mathcal{H}_N^\alpha(S)$. Hence

$$\mathcal{H}^\alpha(S) \le \mathcal{S}_N^\alpha(S) \le 2^\alpha \mathcal{H}^\alpha(S),$$

and $\mathcal{S}_N^\alpha$ is absolutely continuous with respect to $\mathcal{H}^\alpha \llcorner_N$.

2.2 Sub-Riemannian manifolds

A *sub-Riemannian manifold of class* $\mathcal{C}^k$ ($k = \infty$ or $k = \omega$ in the analytic case) is a triplet (M, D, g), where M is a $\mathcal{C}^k$-manifold, D is a Lie-bracket generating $\mathcal{C}^k$-subbundle of TM of rank $m < \dim M$ and g is a Riemannian metric of class $\mathcal{C}^k$ on D. Using the Riemannian metric, the length of horizontal curves, i. e., absolutely continuous curves which are almost everywhere tangent to D, is well-defined. The Lie-bracket generating assumption implies that the distance d defined as the infimum of length of horizontal curves between two given points is finite and continuous (Rashewski–Chow Theorem). We refer to d as the *sub-Riemannian distance*. The set M endowed with the sub-Riemannian distance d is a metric space (M, d) (often called *Carnot-Carathéodory space*) which has the same topology than the manifold M.

We denote by $D_q \subset T_qM$ the fiber of D over q. The subbundle D can be identified with the module of sections

$$\{X \in \mathrm{Vec}(M) \mid X(q) \in D_q, \forall\, q \in M\}.$$

Given $i \ge 1$, define recursively the submodule D^i of $\mathrm{Vec}(M)$ by

$$D^1 = D, \quad D^{i+1} = D^i + [D, D^i].$$

Set $D_q^i = \{X(q) \mid X \in D^i\}$. Notice that the identification between the submodule D^i and the distribution $q \mapsto D_q^i$ is no more meaningful when the dimension of D_q^i varies as a function of q (see the discussion in [3, page 48]). The Lie-bracket generating assumption implies that for every $q \in M$ there exists an integer $r(q)$, the *non-holonomy degree at* q, such that

$$\{0\} \subset D_q^1 \subset \cdots \subset D_q^{r(q)} = T_qM. \tag{3}$$

The sequence of subspaces (3) is called the *flag of* D *at* q. Set $n_i(q) = \dim D_q^i$ and

$$Q(q) = \sum_{i=1}^{r(q)} i(n_i(q) - n_{i-1}(q)), \tag{4}$$

where $n_0(q) = 0$.

We say that a point p is *regular* if, for every i, $n_i(q)$ is constant as q varies in a neighborhood of p. Otherwise, the point is said to be *singular*. A subset $A \subset M$ is called *equiregular* if, for every i, $n_i(q)$ is constant as q varies in A. When the whole manifold is equiregular, the integer $Q(q)$ defined in (4) does not depend on q and it is the Hausdorff dimension of (M, d) (see [12]).

Given $p \in M$, let $X_1, \dots, X_m$ be a local orthonormal frame of D. A multiindex I of length $|I| = j \geq 1$ is an element of $\{1, \dots, m\}^j$. With any multiindex $I = (i_1, \dots, i_j)$ is associated an iterated Lie bracket $X_I = [X_{i_1}, [X_{i_2}, \dots, X_{i_j}] \dots]$ (we set $X_I = X_{i_1}$ if $j = 1$). The set of vector fields X_I such that $|I| \leq j$ is a family of generators of the module D^j. As a consequence, if the values of $X_{I_1}, \dots, X_{I_n}$ at $q \in M$ are linearly independent, then $\sum_i |I_i| \geq Q(q)$.

Let Y be a vector field. We define the *length of* Y by

$$\ell(Y) = \min\{i \in \mathbb{N} \mid Y \in D^i\}.$$

In particular, $\ell(X_I) \leq |I|$. Note that, in general, if a vector field Y satisfies $Y(q) \in D_q^i$ for every $q \in M$, Y need not be in the submodule D^i. By an *adapted basis* to the flag (3) at q, we mean n vector fields $Y_1, \dots, Y_n$ such that their values at q satisfy

$$D_q^i = \text{span}\{Y_j(q) \mid \ell(Y_j) \leq i\}, \quad \forall\, i = 1, \dots, r(q).$$

In particular, $\sum_{i=1}^n \ell(Y_i) = Q(q)$. As a consequence, a family of brackets $X_{I_1}, \dots, X_{I_n}$ such that $X_{I_1}(q), \dots, X_{I_n}(q)$ are linearly independent is an adapted basis to the flag (3) at q if and only if $\sum_i |I_i| = Q(q)$.

3 Hausdorff dimensions and volumes of strongly equiregular submanifolds

In this section, we answer question 3 when N is a particular kind of submanifold, namely a strongly equiregular one. These results include the case where M itself is equiregular.

3.1 Strongly equiregular submanifolds

Let $N \subset M$ be a smooth connected submanifold of dimension b. The *flag at* $q \in N$ *of* D *restricted to* N is the sequence of subspaces

$$\{0\} \subset (D_q^1 \cap T_qN) \subset \cdots \subset (D_q^{r(q)} \cap T_qN) = T_qN. \tag{5}$$

Set

$$n_i^N(q) = \dim(D_q^i \cap T_qN) \quad \text{and} \quad Q_N(q) = \sum_{i=1}^{r(q)} i(n_i^N(q) - n_{i-1}^N(q)),$$

with $n_0^N(q) = 0$.

Definition 1. We say that N is *strongly equiregular* if

(i) N is equiregular, that is, for every i, the dimension $n_i(q)$ is constant as q varies in N;
(ii) for every i, the dimension $n_i^N(q)$ is constant as q varies in N.

In this case, we denote by Q_N the constant value of $Q_N(q)$, $q \in N$.

By an *adapted basis* to the flag (5) at $q \in N$, we mean b vector fields $Z_1, \dots, Z_b$ such taht

$$D_q^i \cap T_q N = \operatorname{span}\{Z_j(q) \mid \ell(Z_j) \leq i\}, \quad \forall\, i = 1, \dots, r(q).$$

In particular, when $Z_1, \dots, Z_b$ is adapted to the flag (5), we have

$$T_q N = \operatorname{span}\{Z_1(q), \dots, Z_b(q)\}$$

and $Q_N = \sum_{i=1}^b \ell(Z_i)$.

Recall that the metric tangent cone[1] to (M, d) at any point p exists and it is isometric to $(T_p M, \widehat{d}_p)$, where $\widehat{d}_p$ denotes the sub-Riemannian distance associated with a nilpotent approximation at p (see [3]). The following lemma shows the relevance of strongly equiregular submanifolds as particular subsets of M for which a metric tangent cone exists. Such metric space is isometrically embedded in a metric tangent cone to the whole M at the point.

Lemma 1. *Let $N \subset M$ be a b-dimensional submanifold of M. Assume N is strongly equiregular. Then, for every $p \in N$:*

(i) *there exists a metric tangent cone to $(N, d|_N)$ at p and it is isometric to $(T_p N, \widehat{d}_p|_{T_p N})$;*
(ii) *the graded vector space*

$$\mathfrak{gr}_p^N(D) := \oplus_{i=1}^{r(p)} (D_p^i \cap T_p N)/(D_p^{i-1} \cap T_p N)$$

is a nilpotent Lie algebra whose associated Lie group $\mathrm{Gr}_p^N(D)$ is diffeomorphic to $T_p N$;
(iii) *every b-form $\omega \in \bigwedge^b N$ on N induces canonically a left-invariant b-form $\hat{\omega}^p$ on $\mathrm{Gr}_p^N(D)$. Moreover,*

$$\int_{N \cap B(p,\epsilon)} \omega = \epsilon^{Q_N} \int_{T_p N \cap \widehat{B}_p} \hat{\omega}^p + o(\epsilon^{Q_N}), \tag{6}$$

where $o(\epsilon^{Q_N})$ is uniform as p varies in N and $\widehat{B}_p$ is the ball centered at 0 of radius 1 in the nilpotent approximation at p of the sub-Riemannian manifold.

Remark 1. When N is an open submanifold of M, assuming N strongly equiregular is equivalent to saying that N contains only regular points. In that case, Lemma 1 is well-known (point (i) follows by the fact that the nilpotent approximation is a metric

[1] in Gromov's sense, see [7]

tangent cone, point (ii) says that the tangent cone shares a group structure - which in this case satisfies the additional property $\mathfrak{gr}_p(D) = \text{span}_p\{D^1\}$ - and (iii) has been remarked in [1] using the canonical isomorphism between $\bigwedge^n(\mathfrak{gr}_p(D)^*)$ and $\bigwedge^n(T_p^*M)$.

Proof. Note first that since the result is of local nature, it is sufficient that we prove it on a small neighbourhood $B(p_0,\rho)\cap N$ of a point $p_0 \in N$. For every p in a such a neighbourhood, there exists a coordinate system $\varphi_p : U_p \to \mathbb{R}^n$ on a neighborhood $U_p \subset M$ of p, such that φ_p are privileged coordinates at p, $p \mapsto \varphi_p$ is continuous, and N is rectified in coordinates φ_p, that is $\varphi_p(N \cap U_p) \subset \{x \in \mathbb{R}^n \mid x_{b+1} = \cdots = x_n = 0\}$. The construction is as follows.

Given $\rho > 0$ small enough, we can find b vector fields $Y_1,\dots,Y_b$ defined on $B(p_0,\rho)$ which form a basis adapted to the flag (5) restricted to N at every $p \in B(p_0,\rho) \cap N$. Moreover, up to reducing ρ, we can find $Y_{b+1},\dots,Y_n$ such that $Y_1,\dots,Y_n$ is adapted to the flag (3) of the distribution at every point $p \in B(p_0,\rho)\cap N$. Using these bases, we define for $p \in N \cap B(p_0,\rho)$, a local diffeomorphism $\Phi_p : \mathbb{R}^n \to M$ by

$$\Phi_p(x) = \exp\left(\sum_{i=b+1}^{n} x_i Y_i\right) \circ \exp\left(\sum_{i=1}^{b} x_i Y_i\right)(p). \tag{7}$$

The inverse $\varphi_p = \Phi_p^{-1}$ of Φ_p provides a system of coordinates centered at p which are privileged (see [9]). Moreover, thanks to property (i) in Definition 1, the map from $B(p_0,\rho) \cap N$ to M which associates with p the point $\Phi_p(x)$ is smooth for every $x \in \mathbb{R}^n$. Finally, in coordinates φ_p, the submanifold $N \cap U$ coincides with the set

$$\left\{\exp\left(\sum_{i=1}^{b} x_i Y_i\right)(p) \mid (x_1,\dots,x_b) \in \Omega\right\} \subset \left\{\Phi_p(x) \mid x_{b+1} = \cdots = x_n = 0\right\},$$

where Ω is an open subset of $\mathbb{R}^b$.

Using φ_p we identify M with $T_pM \simeq \mathbb{R}^n$. Since $Y_1(p),\dots,Y_b(p)$ span T_pN, φ_p maps N in T_pN, where T_pN is identified with $\mathbb{R}^b \times \{0\} \subset \mathbb{R}^n \simeq T_pM$. Therefore, whenever $q_1, q_2 \in U \cap N$ we have

$$\widehat{d}_p(q_1,q_2) = \widehat{d}_p|_{T_pN}(q_1,q_2),$$

and obviously $d(q_1,q_2) = d|_N(q_1,q_2)$. Hence estimate (70) in [3, Theorem 7.32] holds when we restrict d to N and $\widehat{d}$ to T_qN. This allows to conclude that a metric tangent cone to $(N, d|_N)$ at p exists and it is isometric to $(T_pN, \widehat{d}_p|_{T_pN})$, where the inclusion of T_pN into T_pM is to be intended via φ_p.

The algebraic structure of $\mathfrak{gr}_p^N(D)$ and the fact that $Gr_p^N(D)$ is diffeomorphic to $\mathbb{R}^b$ are straightforward. As a consequence, there also exists a canonical isomorphism between $\bigwedge^b(\mathfrak{gr}_p^N(D)^*)$ and $\bigwedge^b(T_p^*N)$. Let $\tilde{\omega}_p$ be the image of ω_p under such isomorphism (see the construction in [13, Sect. 10.5]). Then $\hat{\omega}^p$ is defined as the left-invariant b-form on T_pN which coincides with $\tilde{\omega}_p$ at the origin.

Finally, as a consequence of point (i), by definition of metric tangent cone $\varphi_p(B(p,\epsilon)\cap N)$ converges to $\widehat{B}(0,\epsilon)\cap T_pN$ in the Gromov–Hausdorff sense as ϵ goes to 0. By homogeneity of $\widehat{d}_p$ we have $\widehat{B}(0,\epsilon)\cap T_pN = \epsilon^{Q_N}(\widehat{B}_p\cap T_pN)$ and we get (6). Since $p\mapsto\varphi_p$ and $p\mapsto\widehat{B}_p$ are continuous [1, Sect. 4.1], the remainder $o(\epsilon^{Q_N})$ in (6) is uniform with respect to p. □

For the sake of completeness, let us give an explicit formula for $\hat{\omega}^p$. Recall that the construction of the coordinates φ_p involves an adapted basis $Y_1,\dots,Y_b$ to the flag (5) restricted to N at every $p\in B(p_0,\rho)\cap N$. In particular the vector fields $Y_1,\dots,Y_b$ restricted to N form a local frame for the tangent bundle to N and

$$\omega = \omega(Y_1,\dots,Y_b)d(Y_1|_N)\wedge\cdots\wedge d(Y_b|_N).$$

Let $X_1,\dots,X_m$ be a local orthonormal frame for the sub-Riemannian structure in a neighborhood of p, and $X_{I_1},\dots,X_{I_n}$ be an adapted basis to the flag (3) at p, where X_{I_j} is the Lie bracket corresponding to the multi-index I_j. Since $X_{I_1},\dots,X_{I_n}$ is a local frame for the tangent bundle to M, for every $i=1,\dots,b$ we can write Y_i in this basis as

$$Y_i = \sum_{|I|\leq \ell(Y_i)} Y_i^I X_I,$$

where Y_i^I are smooth function (the fact that only multiindices with length smaller than $\ell(Y_i)$ appear in this sum is due to the definition of length of a vector field). Denote by $\widehat{X}_1^p,\dots,\widehat{X}_m^p$ the nilpotent approximation of $X_1,\dots,X_m$ at p obtained in coordinates φ_p, and by $\widehat{X}_{I_j}^p$ the Lie bracket between the $\widehat{X}_1^p,\dots,\widehat{X}_m^p$ corresponding to the multiindex I_j. For every $i=1,\dots,b$ we define the vector field

$$\widehat{Y}_i^p = \sum_{|I|=\ell(Y_i)} Y_i^I(p)\widehat{X}_I.$$

This enables us to compute $\hat{\omega}^p$ as

$$\hat{\omega}^p = \omega_p(Y_1(p),\dots,Y_b(p))d(\widehat{Y}_1^p|_{T_pN})\wedge\cdots\wedge d(\widehat{Y}_b^p|_{T_pN}). \tag{8}$$

The fact that the right-hand side of (8) does not depend on the X_I nor on the Y_i is a consequence of the intrinsic definition of $\hat{\omega}^p$.

3.2 Hausdorff volume

Assume now that N is an orientable submanifold. By a *smooth volume* on N we mean a measure μ associated with a never vanishing smooth form $\omega\in\bigwedge^b N$, i. e., for every Borel set $A\subset N$, $\mu(A)=\int_A\omega$. We will denote by $\hat{\mu}^p$ the smooth volume on T_pN associated with $\hat{\omega}^p$.

We are now in a position to prove the main result.

Theorem 1. *Let $N \subset M$ be a smooth orientable submanifold. Assume N is strongly equiregular. Then, for every smooth volume μ on N,*

$$\lim_{\epsilon \to 0} \frac{\mathcal{S}_N^{Q_N}(B(q,\epsilon))}{\mu(N \cap B(q,\epsilon))} = \frac{\operatorname{diam}_{\widehat{d}_q}(T_q N \cap \widehat{B}_q)^{Q_N}}{\hat{\mu}^q(T_q N \cap \widehat{B}_q)}, \quad \forall\, q \in N, \tag{9}$$

where $\operatorname{diam}_{\widehat{d}_q}$ *denotes the diameter with respect to the distance $\widehat{d}_q$. In particular, $\mathcal{S}_N^{Q_N}$ is absolutely continuous with respect to μ with Radon–Nikodym derivative equal to the right hand side of (9). As a consequence,*

$$\dim_H N = Q_N, \tag{10}$$

and, for a small ball $B(p,\rho)$ centered at a point $p \in N$, the Hausdorff volume $\mathcal{H}^{Q_N}(N \cap B(p,\rho))$ is finite.

Remark 2. When N is an open submanifold of M, e. g., $N = \{p \in M \mid p \text{ is regular}\}$, the computation of Hausdorff dimension is well-known, see [12]. In particular, when p is a regular point the top-dimensional Hausdorff measure $\mathcal{H}^Q(B(p,r))$ is positive and finite. When $N = M$, Eq. (9) gives a new proof to [1, Theorem 1]. This is interesting since the latter was obtained as a consequence of [1, Lemma 32], whose proof is incorrect.

To prove Theorem 1 a fundamental step is the following lemma.

Lemma 2. *Let N and μ be as in Theorem 1. Let $p \in N$. Assume there exists positive constants ϵ_0 and $\mu_+ > \mu_-$ such that, for every $\epsilon < \epsilon_0$ and every point $q \in B(p,\epsilon_0) \cap N$, there holds*

$$\mu_- \operatorname{diam}(B(q,\epsilon) \cap N)^{Q_N} \le \mu(B(q,\epsilon) \cap N) \le \mu_+ \operatorname{diam}(B(q,\epsilon) \cap N)^{Q_N}. \tag{11}$$

Then, for every $\epsilon < \epsilon_0$,

$$\frac{\mu(B(p,\epsilon) \cap N)}{\mu_+} \le \mathcal{S}_N^{Q_N}(B(p,\epsilon)) \le \frac{\mu(B(p,\epsilon) \cap N)}{\mu_-}.$$

Proof. Let $\bigcup_i B(q_i, r_i)$ be a covering of $B(p,\epsilon) \cap N$ with balls of radius smaller than $\delta < \epsilon_0$. If δ is small enough, every q_i belongs to $B(p,\epsilon_0) \cap N$ and, using (11), there holds

$$\mu(B(p,\epsilon) \cap N) \le \sum_i \mu(B(q_i,r_i) \cap N) \le \mu_+ \sum_i \operatorname{diam}(B(q_i,r_i) \cap N)^{Q_N}.$$

Hence, we have $\mathcal{S}_N^{Q_N}(B(p,\epsilon)) \ge \frac{\mu(B(p,\epsilon) \cap N)}{\mu_+}$.

For the other inequality, let $\eta > 0$, $0 < \delta < \epsilon_0$ and let $\bigcup_i B(q_i, r_i)$ be a covering of $B(p,\epsilon) \cap N$ such that $q_i \in B(p,\epsilon) \cap N$ $r_i < \delta$ and $\sum_i \mu(B(q_i,r_i) \cap N) \le \mu(B(p,\epsilon)) + \eta$. Such a covering exists due to the Vitali covering lemma. Using as above (11), we obtain

$$\mu(B(p,\epsilon) \cap N) + \eta \ge \sum_i \mu(B(q_i,r_i) \cap N) \ge \mu_- \sum_i \operatorname{diam}(B(q_i,r_i) \cap N)^{Q_N}.$$

We then have $\mathcal{S}^{Q_N}_{N,\delta}(B(p,\epsilon)) \leq \frac{\mu(B(p,\epsilon)\cap N)}{\mu_-} + \frac{\eta}{\mu_-}$. Letting η and δ tend to 0, we get the conclusion. □

Proof of Theorem 1. Fix $q \in N$. By point (ii) in Lemma 1 $(T_qN, \widehat{d}_q|_{T_qN})$ is a metric tangent cone to $(N, d|_N)$ at q, whence, from the definition of Gromov–Hausdorff convergence we get

$$\lim_{\epsilon\to 0} \frac{\operatorname{diam}(N \cap B(q,\epsilon))}{\epsilon} = \operatorname{diam}_{\widehat{d}_q}(T_qN \cap \widehat{B}_q). \tag{12}$$

By (6) in Lemma 1, for every $q \in N$ there holds

$$\mu(N \cap B(q,\epsilon)) = \epsilon^{Q_N} \hat{\mu}^q(T_qN \cap \widehat{B}_q) + o(\epsilon^{Q_N}). \tag{13}$$

Since N is strongly equiregular, the limits in (12) and (13) hold uniformly as q varies in N.

Moreover, adapting the argument in [1, Sect. 4.1], we deduce that the map $q \mapsto \hat{\mu}^q(\widehat{B}_q \cap T_qN)$ is continuous on N. As a consequence, for any $\eta > 0$ there exists $\epsilon_1 > 0$ such that for every $q \in B(p,\epsilon_1)$ and every $\epsilon < \epsilon_1$ we have

$$\mu_- \leq \frac{\mu(N \cap B(q,\epsilon))}{\operatorname{diam}(N \cap B(q,\epsilon))^{Q_N}} \leq \mu_+$$

with

$$\mu_\pm = \frac{\widehat{\mu}_q(T_qN \cap \widehat{B}_q)}{\operatorname{diam}_{\widehat{d}_q}(T_qN \cap \widehat{B}_q)^{Q_N}} \pm \eta.$$

Therefore, applying Lemma 2 and letting η tend to 0 we deduce (9).

To show (10), notice that the right-hand side of (9) is continuous and positive as a function of q. Hence, for $\mathcal{S}^{Q_N}_N$-almost every $q \in N$ there exists $\rho > 0$ small enough such that

$$0 < \mathcal{S}^{Q_N}(N \cap B(p,\rho)) < \infty. \tag{14}$$

This is equivalent to (10). □

We end this section by stating a result which gives a weak equivalent of the function $\hat{\mu}^q(T_qN \cap \widehat{B}_q)$ appearing in Theorem 1. This will be useful in the following to determine whether the Hausdorff volume of a small ball is finite or not. This result stems from the uniform Ball-Box Theorem, [10] and [11, Th. 4.7].

Proposition 1. *Let M be orientable and ϖ be a volume form on M. Let N be an orientable submanifold of M of dimension b, and let ω be a volume form on N, with associated smooth volume μ. Assume N is strongly equiregular and set $Q[N]$ equal to the constant value of $Q(q)$, for $q \in N$. Then there exists a constant $C > 0$ such that, for every $q \in N$,*

$$\frac{1}{C}\nu_q \leq \hat{\mu}^q(T_qN \cap \widehat{B}_q) \leq C\nu_q \quad (\textit{i. e. } \hat{\mu}^q(T_qN \cap \widehat{B}_q) \asymp \nu_q \textit{ uniformly w.r.t. } q),$$

where $\nu_q = \max\{\left(\omega \wedge dX_{I_{b+1}} \wedge \cdots \wedge dX_{I_n}\right)_q (X_{I_1}(q), \dots, X_{I_n}(q))\}$*, the maximum being taken among all* n*-tuples* $(X_{I_1}, \dots, X_{I_n})$ *in* $\arg\max\{\varpi_q(X_{I'_1}(q), \dots, X_{I'_n}(q)) \mid \sum_i |I'_i| = Q[N]\}$.

In particular, if N *is an open equiregular subset of* M*, i. e.,* $b = n$*, and if* μ *is the smooth measure on* M *associated with* ϖ*, we have*

$$\hat{\mu}^q(\widehat{B}_q) \asymp \max\{\varpi_q(X_{I'_1}, \dots, X_{I'_n}) \mid \sum_i |I'_i| = Q[M]\}, \quad \textit{uniformly w.r.t. } q \in M.$$

This proposition, together with Theorem 1, allows to give an estimate of the Hausdorff volume of a subset of N. If $S \subset N$, then

$$\frac{1}{C'} \int_S \frac{1}{\nu_q} d\mu \le \mathcal{H}^{Q_N}(S) \le C' \int_S \frac{1}{\nu_q} d\mu, \tag{15}$$

where the constant $C' > 0$ does not depend on S.

4 Hausdorff dimensions and volumes of analytic sub-Riemannian manifolds

Let (M, D, g) be an analytic (C^ω) sub-Riemannian manifold. The set Σ of singular points is an analytic subset of M which admits a locally finite stratification $\Sigma = \bigcup_{i \ge 1} M_i$ by analytic and equiregular submanifolds M_i (see for instance [6]). Denoting $M_0 = M \setminus \Sigma$ the set of regular points, we obtain a stratification $M = \bigcup_{i \ge 0} M_i$ of M by analytic and equiregular submanifolds. Note that M_0 is an open and dense subset of M, but it may be disconnected. As a consequence, the Hausdorff dimension of M is

$$\dim_H(M) = \max_{i \ge 0} \dim_H(M_i),$$

and the α-dimensional Hausdorff measure of a ball $B(p, \rho)$, $p \in M$ and $\rho > 0$, is

$$\mathcal{H}^\alpha(B(p, \rho)) = \sum_i \mathcal{H}^\alpha(B(p, \rho) \cap M_i).$$

4.1 Hausdorff dimension

The first problem is then to determine the Hausdorff dimension of an equiregular - possibly not strongly equiregular - submanifold.

Lemma 3. *Let* N *be an analytic and equiregular submanifold of* M*. Set* $\overline{Q}_N := \max_{q \in N} Q_N(q)$*. Then*

$$\dim_H(N) = \overline{Q}_N,$$

and $Q_N(q) = \overline{Q}_N$ *on an open and dense subset of* N*.*

If moreover N is orientable, then for every smooth measure μ on N, $\mathcal{S}_N^{\overline{Q}_N}$ is absolutely continuous with respect to μ with Radon–Nikodym derivative

$$\frac{d\mathcal{S}_N^{\overline{Q}_N}}{d\mu}(q) = \frac{(\mathrm{diam}_{\widehat{d}_q}(T_qN \cap \widehat{B}_q))^{\overline{Q}_N}}{\widehat{\mu}_q(T_qN \cap \widehat{B}_q)}, \quad \textit{for } \mu\textit{-a.e. } q \in N. \tag{16}$$

Proof. Since N is analytic and equiregular, it admits a stratification $N = \bigcup_i N_i$ by strongly equiregular submanifolds N_i of N. By Theorem 1, $\dim_H(N_i) = Q_{N_i}$ and thus $\dim_H(N) = \max_i Q_{N_i}$. In particular, $\dim_H(N) \leq \max_{q\in N} Q_N(q)$.

Now, recall that $Q_N(q) = \sum_{i=1}^{r_N} i(n_i^N(q) - n_{i-1}^N(q))$, where $r_N := r(q)$ is constant since N is equiregular, and $n_{r_N}^N(q) = \dim N$. This may be rewritten as

$$Q_N(q) = \sum_{i=0}^{r_N-1} \mathrm{codim}(D_q^i \cap T_qN), \tag{17}$$

where $\mathrm{codim}(D_q^i \cap T_qN) = n_{r_N}^N(q) - n_i^N(q)$ is the codimension of $D_q^i \cap T_qN$ in T_qN. The submanifold N being equiregular, $Q_N(q)$ is a lower semi-continuous function on N with integer values. Hence $Q_N(q)$ takes its maximal value $\overline{Q}_N$ on the strata N_i which are open in N, and smaller values on non open strata. Since $Q_{N_i}(q) = Q_N(q)$ when N_i is an open subset of N and $Q_{N_i}(q) < Q_N(q)$ when N_i is a non open subset of N, the first part of the lemma follows.

As for the second part, notice that every non open stratum N_i is of μ-measure zero, since N_i is a subset of N of positive codimension, and of $\mathcal{S}_N^{\overline{Q}_N}$-measure zero, since $\dim_H(N_i) = Q_{N_i} < \overline{Q}_N$. A first consequence is that N is strongly equiregular near μ-a.e. point q. Therefore the measure $\widehat{\mu}_q$ on T_qN is defined μ-a.e. – and so is the right-hand side of (16). Applying then Theorem 1 to every open stratum N_i, we get the conclusion. □

Corollary 1. $\dim_H(M) = \max\{Q_{M_i}(q) \,:\, i \geq 0,\, q \in M_i\} = \max\{\overline{Q}_{M_i} \,:\, i \geq 0\}$.

4.2 Finiteness of the Hausdorff volume of balls

Let $p \in M$ and $\rho > 0$ (ρ is assumed to be arbitrarily small). The aim of this section is to determine under which conditions the small ball $B(p,\rho)$ has a finite Hausdorff volume $\mathcal{H}^{\dim_H(B(p,\rho))}(B(p,\rho))$. We make first two preliminary remarks:

- if p is a regular point, then there exists a neighbourhood of p in M which is strongly equiregular, and Theorem 1 implies that $\mathcal{H}^{\dim_H(B(p,\rho))}(B(p,\rho))$ is finite. We then assume in the following that p is a singular point;
- the results of this section are local. Up to reducing to a neighbourhood of p, we can assume that M is an oriented manifold with volume form ϖ.

Recall that, by definition, the stratification $M = \bigcup_{i\geq 0} M_i$ is locally finite. That is, there exists a finite set $\mathcal{I}$ of indices such that $p \in \overline{M_i}$ if and only if $i \in \mathcal{I}$, where

$\overline{M_i}$ denotes the closure of the stratum M_i. Therefore, for ρ small enough, the ball $B(p,\rho)$ admits a finite stratification $B(p,\rho) = \bigcup_{i\in\mathcal{I}}(B(p,\rho)\cap M_i)$. Applying Corollary 1, the Hausdorff dimension D_p of $B(p,\rho)$ is

$$D_p = \max\{Q_{M_i}(q) \ : \ i \in \mathcal{I},\ q \in M_i\}.$$

Let $\mathcal{J} \subset \mathcal{I}$ be the subset of indices i such that $\dim_H(M_i) = D_p$. We have

$$\mathcal{H}^{D_p}(B(p,\rho)) = \sum_{i\in\mathcal{J}} \mathcal{H}^{D_p}(B(p,\rho)\cap M_i).$$

Proposition 2. *Let N be an analytic and equiregular submanifold of M, $\dim_H(N) = \overline{Q}_N$. If $p \in N$ and if $\rho > 0$ is small enough, then the Hausdorff volume $\mathcal{H}^{\overline{Q}_N}(B(p,\rho)\cap N)$ is finite.*

Proof. Up to replacing N with a small neighbourhood of p in N, we assume that N is orientable. We then choose a smooth measure μ on N and we have, for ρ small enough, $\mu(B(p,\rho)\cap N) < +\infty$. From Lemma 3,

$$\mathcal{S}_N^{\overline{Q}_N}(B(p,\rho)\cap N) = \int_{B(p,\rho)\cap N} \frac{(\mathrm{diam}_{\widehat{d}_q}(T_qN\cap \widehat{B}_q))^{\overline{Q}_N}}{\widehat{\mu}_q(T_qN\cap \widehat{B}_q)} d\mu.$$

The submanifold N is strongly equiregular near μ-a.e. $q \in N$. We can then apply Proposition 1 near μ-a.e. $q \in N$ and we get

$$\mathcal{S}_N^{\overline{Q}_N}(B(p,\rho)\cap N) \le C \int_{B(p,\rho)\cap N} \frac{(\mathrm{diam}_{\widehat{d}_q}(T_qN\cap \widehat{B}_q))^{\overline{Q}_N}}{\nu_q} d\mu.$$

The function $q \mapsto \nu_q$ is positive and continuous on N, so the integrand function in the previous formula is finite and continuous on N, and we have $\mathcal{S}_N^{\overline{Q}_N}(B(p,\rho)\cap N) \le \mathrm{Cst}\ \mu(B(p,\rho)\cap N) < +\infty$. Since $\mathcal{H}^{\overline{Q}_N}$ is absolutely continuous with respect to $\mathcal{S}_N^{\overline{Q}_N}$, the conclusion follows. □

As a consequence, the Hausdorff volume $\mathcal{H}^{D_p}(B(p,\rho))$ is finite if and only if $\mathcal{H}^{D_p}(B(p,\rho)\cap M_i)$ is finite for every stratum M_i such that $\dim_H(M_i) = D_p$ and $p \in \partial M_i$. To go further, we will assume that p is a *typical* singular point, that is, that p satisfies the following assumptions for ρ small enough:

(A1) p belongs to a strongly equiregular submanifold N of M, $N \subset \Sigma$, and $B(p,\rho)\cap\Sigma \subset N$;

(A2) for every $q \in N\cap B(p,\rho)$, there exists a family $X_{I_1},\dots,X_{I_n}$ such that $\sum_i |I_i| = Q_{\mathrm{reg}}$ and $\mathrm{ord}_q \varpi(X_{I_1},\dots,X_{I_n}) = \sigma$, where Q_{reg} is the constant value of $Q(q)$ for $q \in M\setminus\Sigma$, and

$$\sigma = \max\{s \in \mathbb{N} \ : \ q \in N\cap B(p,\rho)$$

and

$$\sum_i |I_i| = Q_{\mathrm{reg}} \text{ imply } \mathrm{ord}_q \varpi(X_{I_1},\dots,X_{I_n}) \ge s\}.$$

Let us recall the definition of ord_q (see [3] for details). Given $f \in \mathcal{C}^k(M)$, we say that f has *non-holonomic order at p greater than or equal to s*, and we write $\mathrm{ord}_p f \geq s$ if for every $j \leq s-1$

$$(X_{i_1} \dots X_{i_j} f)(p) = 0 \quad \forall\, (i_1, \dots, i_j) \in \{1, \dots, m\}^j,$$

where $X_i f$ denotes the Lie derivative of f along X_i. Equivalently, $f(q) = O(d(p,q)^s)$. If moreover we do not have $\mathrm{ord}_p f \geq s+1$, then we say that f has *non-holonomic order at p equal to s*, and we write $\mathrm{ord}_p f = s$.

Theorem 2. *Assume p satisfies (A1) and (A2). Let Q_N be the constant value of $Q_N(q)$ for $q \in N$, and $r_{\not N}$ be the maximal integer i such that $n_i(p) - n_{i-1}(p) > n_i^N(p) - n_{i-1}^N(p)$. Then*

$$\mathcal{H}^{Q_{\mathrm{reg}}}(B(p,\rho) \setminus \Sigma) < \infty \quad \Leftrightarrow \quad \sigma \leq Q(p) - Q_N - r_{\not N}.$$

As a consequence,

- *if $Q_{\mathrm{reg}} < Q_N$, then $D_p = Q_N$ and $\mathcal{H}^{D_p}(B(p,\rho))$ is finite;*
- *if $Q_{\mathrm{reg}} \geq Q_N$, then $D_p = Q_{\mathrm{reg}}$ and $\mathcal{H}^{D_p}(B(p,\rho))$ is finite if and only if $\sigma \leq Q(p) - Q_N - r_{\not N}$.*

The proof of this theorem is postponed to a forthcoming paper. It relies on the use of Proposition 1.

Remark 3. Assumption (A2) is actually not necessary for the computations. If p satisfies only (A1), we introduce two integers $\sigma_- \leq \sigma_+$:

$$\begin{aligned}\sigma_+ &= \min\{s \in \mathbb{N} \;:\; \forall q \in N \cap B(p,\rho),\ \exists X_{I_1}, \dots, X_{I_n} \text{ s.t. } \sum_i |I_i| \\ &= Q_{\mathrm{reg}} \text{ and } \mathrm{ord}_q \varpi(X_{I_1}, \dots, X_{I_n}) \leq s\},\\ \sigma_- &= \max\{s \in \mathbb{N} \;:\; \exists \text{ an open subset } \Omega \text{ of } N \cap B(p,\rho) \text{ s.t. } q \in \Omega \text{ and} \\ &\textstyle\sum_i |I_i| = Q_{\mathrm{reg}} \text{ imply } \mathrm{ord}_q \varpi(X_{I_1}, \dots, X_{I_n}) \geq s\}.\end{aligned}$$

Assumption (A2) is equivalent to $\sigma_- = \sigma_+ = \sigma$. The generalization of the criterion of Theorem 2 to the case where p satisfies only (A1) is then:

- if $\sigma_+ \leq Q(p) - Q_N - r_{\not N}$, then $\mathcal{H}^{Q_{\mathrm{reg}}}(B(p,\rho) \setminus \Sigma) < \infty$;
- if $\sigma_- > Q(p) - Q_N - r_{\not N}$, then $\mathcal{H}^{Q_{\mathrm{reg}}}(B(p,\rho) \setminus \Sigma) = \infty$.

Notice that the order σ (and σ_- if p does not satisfies (A2)) always satisfies $\sigma \geq Q(p) - Q_{\mathrm{reg}}$. We thus obtain a simpler criterion for the non finiteness of the Hausdorff volume of a ball.

Corollary 2. *Assume p satisfies (A1). If $0 \leq Q_{\mathrm{reg}} - Q_N < r_{\not N}$, then $\mathcal{H}^{D_p}(B(p,\rho)) = \infty$.*

4.3 Examples

Example 1 (the Martinet space). Consider the sub-Riemannian manifold given by $M = \mathbb{R}^3$, $D = \text{span}\{X_1, X_2\}$,

$$X_1 = \partial_1, \quad X_2 = \partial_2 + \frac{x_1^2}{2}\partial_3,$$

and the metric $dx_1^2 + dx_2^2$. We choose $\varpi = dx_1 \wedge dx_2 \wedge dx_3$, that is, the canonical volume form on $\mathbb{R}^3$.

The growth vector is equal to $(2, 2, 3)$ on the plane $N = \{x_1 = 0\}$, and it is $(2, 3)$ elsewhere. As a consequence, N is the set of singular points. At a regular point, $Q_{\text{reg}} = 4$. Every singular point $p = (0, x_2, x_3)$ satisfies (A1) and we have $Q(p) = 5$, $Q_N = 4$, and $r_{\not{N}} = 1$. Applying Corollaries 1 and 2, we obtain:

$$\dim_H(M) = 4,$$

and

$$\mathcal{H}^4(B(p,\rho)) < \infty \text{ if } p \text{ regular}, \quad \mathcal{H}^4(B(p,\rho)) = \infty \text{ otherwise}.$$

Thus small balls centered at singular points have infinite Hausdorff volume. This result can also be obtained by a direct computation based on the uniform Ball-Box Theorem, see [11].

Note that the only family $(X_{I_1}, X_{I_2}, X_{I_3})$ such that $\sum_i |I_i| = Q_{\text{reg}}$ is $(X_1, X_2, [X_1, X_2])$. The volume form of this family equals x_1 and it is of order 1 at every point of N. Thus every singular point satisfies assumptions (A1) and (A2) with $\sigma = 1$ ($\sigma = Q(p) - Q_{\text{reg}}$ here).

Example 2. Consider the sub-Riemannian manifold given by $M = \mathbb{R}^4$, $D = \text{span}\{X_1, X_2, X_3\}$, where

$$X_1 = \partial_1, \quad X_2 = \partial_2 + \frac{x_1^2}{2}\partial_4, \quad X_3 = \partial_3 + \frac{x_2^2}{2}\partial_4,$$

and $g = dx_1^2 + dx_2^2 + dx_4^2$. We choose ϖ as the canonical volume form on $\mathbb{R}^4$.

At a regular point, $Q_{\text{reg}} = 5$. The set of singular points is $N = \{x_1 = x_2 = 0\}$. Every singular point satisfies (A1) and we have $Q(p) = 6$, $Q_N = 4$, and $r_{\not{N}} = 1$. Thus, by Corollary 1, $\dim_H(M) = 5$. However Corollary 2 does not allow to conclude on the finiteness of the Hausdorff volume.

The only families such that $\sum_i |I_i| = Q_{\text{reg}}$ are $(X_1, X_2, X_3, [X_1, X_2])$ and $(X_1, X_2, X_3, [X_2, X_3])$. The volume form applied to these families is equal to x_1 and x_2 respectively, and both of them are of order 1 at every point of N. Thus every singular point satisfies assumptions (A1) and (A2) with $\sigma = 1$ ($\sigma = Q(p) - Q_{\text{reg}}$ here). Applying Theorem 2, we obtain:

$$\dim_H(M) = 5, \quad \text{and} \quad \mathcal{H}^5(B(p,\rho)) < \infty \quad \text{for any } p \in M.$$

Example 3. Let $M = \mathbb{R}^5$, $D = \text{span}\{X_1, X_2, X_3\}$,

$$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1\partial_3 + x_1^2\partial_5, \quad X_3 = \partial_4 + x_1^k\partial_5,$$

with $k > 2$, and $g = dx_1^2 + dx_2^2 + dx_3^2$. We choose ϖ as the canonical volume form on $\mathbb{R}^5$.

The singular set is $N = \{x_1 = 0\}$. A simple computation shows that every singular point p satisfies (A1) and (A2), and $Q_{\text{reg}} = 7$, $Q(p) = 8$, $Q_N = 7$, $r_{\mathcal{N}} = 1$, and $\sigma = k - 1$. Thus in this example $\sigma > Q(p) - Q_{\text{reg}}$. Now Corollaries 1 and 2 apply and we obtain

$$\dim_H(M) = 7,$$

and

$$\mathcal{H}^7(B(p,\rho)) < \infty \;\text{ if } p \text{ regular,} \quad \mathcal{H}^7(B(p,\rho)) = \infty \;\text{ otherwise.}$$

Example 4. Let $M = \mathbb{R}^5$, $D = \text{span}\{X_1, X_2, X_3\}$,

$$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1\partial_3 + x_1^2\partial_5, \quad X_3 = \partial_4 + (x_1^k + x_2^k)\partial_5,$$

with $k > 2$, and $g = dx_1^2 + dx_2^2 + dx_3^2$. We choose ϖ as the canonical volume form on $\mathbb{R}^5$.

The singular set is $N = \{x_1 = x_2 = 0\}$. Every singular point p satisfies (A1) and (A2) and we have $Q_{\text{reg}} = 7$, $Q(0) = 8$, $Q_N = 6$, $r_{\mathcal{N}} = 1$, and $\sigma = k - 1$. By Corollary 1 and Theorem 2, we obtain

$$\dim_H(M) = 7,$$

and

$$\mathcal{H}^7(B(p,\rho)) < \infty \;\text{ if } p \text{ regular,} \quad \mathcal{H}^7(B(p,\rho)) = \infty \;\text{ otherwise.}$$

Note that in this case we do not have $Q_{\text{reg}} - Q_N < r_{\mathcal{N}}$. This shows that the criterion in Corollary 2 does not provide a necessary condition for the Hausdorff volume to be infinite.

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The Delauney-Dubins Problem

Velimir Jurdjevic

Abstract The problem of Delauney, posed in the middle of the nineteenth century asked for curves of shortest and longest length among all space curves with a given constant curvature that connect two given tangential directions. About a hundred years later, L. Dubins, apparently unaware of the former problem, asked for a curve of minimal length that joins two fixed directions in the space of curves whose curvature is less or equal than a given constant. Dubins showed that the minimizers exist in the class of continuously differentiable curves having Lebesgue integrable second derivative and he characterized optimal solutions in the plane as the concatenations of circles of curvature $\pm c$ and straight lines with at most two switchings from one arc to another ([7]). Remarkably, the key equation in the problem of Delauney, obtained by Josepha Von Schwartz in mid 1930s also appears in the spacial version of the problem of Dubins.

In this paper we will show that the n-dimensional problem of Dubins (called Delauney-Dubins, for historical reasons) is essentially three dimensional on any space form (simply connected space of constant curvature). We also show that the extremal equations are completely integrable and consist of two kinds, switching and non-switching.The non-switching extremals are expressed in terms of elliptic functions obtained by solving the fundamental equation of Josepha Von Schwarz, while the projections of the switching extremals are shown to be the concatenations of arcs of circles (hyperbolas, in the hyperbolic case) and geodesics, exactly as in the two dimensional Dubins' problem ([16]).

1 Introduction

This work was originally motivated by the problem of Delauney, firstly because of its relation to the problem of Dubins and secondly, because of certain ambiguities

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in its presentation in the classical literature. Delauney's problem, dating back to the middle of the nineteenth century, consists of determining the shortest and the longest curves among all space curves with a given constant curvature which join two line elements of the space ([4]). Dubins' problem, posed about a hundred years later, consists of determining the curves of shortest length which join two line elements among all curves whose curvature is less or equal to a given constant ([7]). Rather than following the chronological order, we will begin with the problem of Dubins.

In his remarkable paper of 1958 Dubins proved that minimizers exist in the class of continuously differentiable functions having Lebesgue integrable second derivatives and he characterized optimal solutions in the plane as the concatenations of circles of curvature ± 1 and straight lines with at most two switchings from one arc to another ([7]). Apart from proving the existence of optimizers, Dubins did not go further into the the nature of optimal solutions in dimensions greater than two. It is relevant to point out that, at the time of Dubins' paper, the calculus of variations had no adequate means to deal with variational problems with inequality constraints and Dubins, unaware of control theory and its quest for the Maximum Principle, tackled the problem directly with "bare hands".

Much later, in his Ph.D thesis on three dimensional Dubins' problem, F. Monroy-Pérez showed that this problem was integrable, and he noticed that the key equation for the problem fo Dubins bore a striking resemblance to the key equation for the problem of Delauney ([17]), but did not go further into this phenomenon since his methodology (optimal control on Lie groups) did not readily translate into the classical literature on the problem of Delauney ([4, 5, 21]). To make a segue to this literature, we will go back to Carathéodory's treatise on the calculus of variations ([4]).

Carathéodory ends this book with the problem of Delauney in which he states that the general solution to this problem was not fully known until Weierstrass, who apparently was the first to successfully integrate the associated Euler equation. However, Caratéodory, himself, felt that the Hamiltonian approach was more insightful and he proposed a solution based on Hamiltonian methods. Ultimately, he claimed that the associated Hamiltonian equations are integrable by quadratures in terms of elliptic functions obtained by solving the key equation of the form

$$\dot{u}^2 = \lambda^2[(w - u^2)(u - 1)^2 - k^2], \tag{1}$$

where w and k are constants, and $\lambda = \pm 1$ depending on the sign of u.

However, Carathéodory's key equation is different from the one obtained earlier by Josepha Von Schwarz in 1934 ([21]), and to make the matter even more confusing, Carathéodory does not comment on this discrepancy, even though Schwarz's work is cited in his bibliography (Ref 157). Schwarz's treatment of Delauney's problem, more detailed and more extensive, provides a solution by quadratures based on the equation

$$(u - \mu_0)^2 \left(\frac{du}{ds}\right)^2 = (u - \mu_0)^2 (k^2 - u^2) - h_0^2, \tag{2}$$

with $\mu_0 = -1$ for the minimum length and $\mu_0 = 1$ for the maximum length and h_0 a constant.

Much later, P. Griffiths using completely different methods based on Cartan's exterior calculus ([9]) obtained the Euler equation for the problem of Delauney in the form

$$\frac{d^2\lambda}{ds^2} + \left(\lambda - \frac{c_1^2}{\lambda^3} - 1\right) = 0, \tag{3}$$

where c_1 is a constant. This equation can be written in integrated form

$$\left(\frac{d\lambda}{ds}\right)^2 + \lambda^2 + \frac{c_1^2}{\lambda^2} - 2\lambda = c_2, \tag{4}$$

by multiplying (3) by $2\frac{d\lambda}{ds}$ and then integrating the resulting equation (there is an unfortunate misprint in Eq. (11.b.33) on page 156 of (([9]) that obscures easy comparisons with other sources). Equations (2) and (4) are of the same form as can be easily verified under the identification $\lambda = u - \mu_0$. Oddly enough, Griffiths, like Carathéodory, does not comment on the discrepancy between his equation and that obtained by Carathéodory, even though he quotes Carathéodory for the statement of Delauney's problem.

It is partly for these reasons, but mostly for their own intrinsic interest, that I wanted to take up these problems in more detail. Rather than treating these problems on principal bundles and Lie groups, as I have done in my previous publications, I will take a more direct approach and consider them as variational problems on the tangent bundle of the underlying manifold, analogous to the problems of mechanics.

The problem of Delauney can be rephrased as the problem of finding the minimum (maximum) length of an interval $[0, T]$ on which there exists a curve $f(t)$ in $\mathbb{R}^3$ that satisfies the boundary conditions $f(0) = x_0, f'(0) = y_0$ and $f(T) = x_1, f'(T) = y_1$ and is a subject to to the constraints that $||f'(t)|| = 1$ and $||f''(t)|| = c$ on $[0, T]$ where c is a constant. A curve $f(t)$ parametrized by arc length on an interval $[0, T]$, has length T on that interval, and its geodesic curvature is equal to $||f''(t)||$. In analogy with linear time optimal control theory, the problem of Delauney can be formulated on the cylinder $M = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : ||y|| = 1\}$ as a time optimal problem of transferring an initial state (x_0, y_0) to a terminal state (x_1, y_1) in minimal time along a trajectory of the control system

$$\frac{dx}{dt} = y \, \frac{dy}{dt} = u(t), \quad ||u(t)|| = c$$

where the control $u(t)$ is further constrained by $u(t) \cdot y(t) = 0$.

This formulation makes it transparent that the problem of Delauney is not well posed, a fact that was not noticed before, because the sphere $\{u : ||u|| = c\}$ is not convex. For instance, points which are tangential to the same straight line can not be be connected to each other in a minimum time by a trajectory of the above system.

The reason is simple and rests on the following asymptotic formula:

$$\lim_{n\to\infty} \underbrace{e^{\frac{t}{2n}V_1} \circ e^{\frac{t}{2n}V_2} \circ e^{\frac{t}{2n}V_1} \circ e^{\frac{t}{2n}V_2} \cdots \circ e^{\frac{t}{2n}V_1} \circ e^{\frac{t}{2n}V_2}}_{2n} = e^{t(V_1+V_2)}$$

where V_1 and V_2 are vector fields and where $\{e^{tV_1} : t \in R\}$ and $\{e^{tV_2} : t \in R\}$ are their one-parameter groups of diffeomorphisms. In particular, if V_1 and V_2 are vector fields whose integral curves have curvature equal to $\pm c$ then $V_1 + V_2$ is a vector field whose integral curves have zero curvature, hence, its integral curves are straight lines. So points on straight lines are the limits of the concatenations of arcs with curvature $\pm c$. The optimal time is reached only along the geodesic.

So it is natural to enlarge the set of controls to the convex closure $\{u \in \mathbb{R}^n : ||u|| \leq c\}$. As is well known, the reachable sets by the controls in a set U and the reachable sets by the controls in the convex hull of U have the same topological closure. The above shows that the problem of Dubins can be regarded as the convexified n-dimensional Delauney problem. For this reason, and also because of the overlaps between these two problems, we will rename Dubins' problem as *the Delauney-Dubins problem* which we will treat as the time optimal control problem on the submanifold $N = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : ||y|| = 1\}$ defined by the control system

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = u(t), \quad ||u(t)|| \leq c. \tag{5}$$

where the controls conform to an additional constraint $U_1 = u \cdot y = 0$.

Apart from the constraint $U_1 = 0$, which normally does not appear in the literature on control theory, the above problem, at least in appearance, resembles a linear-time optimal problem, and as such, it naturally draws attention to the corresponding linear-quadratic problem. This "linear-quadratic problem" , known as the Euler-Griffiths problem ([12]), or the elastic ptoblem, is defined as the problem of minimizing the integral $\frac{1}{2}\int_0^T ||u(t)||^2\, dt$ over the solutions of

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = u(t), \quad U_1 = u(t) \cdot y(t) = 0, \tag{6}$$

that connect the given end-points in M in T units of time.

We will consider both of these problems side by side, and for the sake of completeness, we will also include their non-Euclidean versions on the sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : ||x||^2 = 1\}$ and the hyperboloid $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1}^2 = 1 + \sum_{i=1}^n x_i^2, x_{n+1} > 0\}$. Both cases can be handled simultaneously in terms of the parameter $\epsilon = \pm 1$ and the quadratic form

$$(v \cdot w)_\epsilon = \sum_{i=1}^n v_i w_i + \epsilon v_{n+1} w_{n+1}, v \in \mathbb{R}^{n+1}, \quad w \in \mathbb{R}^{n+1}.$$

It follows that $||x||_\epsilon^2 = \epsilon = (x \cdot x)_\epsilon$ coincides with $\mathbb{S}^n$ when $\epsilon = 1$ and the hyperboloid $x_{n+1}^2 = 1 + \sum_{i=1}^{n+1} x_i^2$ when $\epsilon = -1$. We will let $\mathbb{S}_\epsilon^n$ denote the unit sphere $\mathbb{S}^n$

for $\epsilon = 1$ and the one-sheeted hyperboloid $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}$ when $\epsilon = -1$. Then the unit tangent bundle N_ϵ of $\mathbb{S}_\epsilon^n$ is given by

$$N_\epsilon = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : ||x||_\epsilon^2 = \epsilon, x_{n+1} > 0, ||y||_\epsilon^2 = 1,\ (x \cdot y)_\epsilon = 0\}.$$

If $u(t)$ denote the covariant derivative of $\frac{dx}{dt}$ associated with any curve $x(t)$ on $\mathbb{S}_\epsilon^n$ such that $||\frac{dx}{dt}||_\epsilon = 1$, then $||u(t)||^2 = \kappa^2(t)$, where $\kappa(t)$ denotes the geodesic curvature of $x(t)$, and $\frac{dy}{dt} = u(t) - \epsilon x(t)$. Therefore,

$$\frac{dx}{dt} = y(t),\ \frac{dy}{dt} = u(t) - \epsilon x(t), U_1 = (u(t) \cdot x(t))_\epsilon = 0,\ U_2 = (u(t) \cdot y(t))_\epsilon = 0 \tag{7}$$

is the corresponding control system. The elastic problem is defined by the energy $\frac{1}{2}\int_0^T ||u(t)||_\epsilon^2\, dt$ and the Delauney-Dubins problem by the bound $||u(t)||_\epsilon \leq c$.

There is a version of the Maximum Principle, called the Hybrid Maximum Principle, that leads to the correct Hamiltonians for the above problems. We will be able to write the appropriate Hamiltonian equations in terms of the canonical coordinates of the ambient space and express their integrability properties in terms of the relevant symmetries of the system. This “down to earth” approach bridges the gap between the results obtained by Carathéodory and von Schwarz mentioned earlier and the contemporary publications on these topics ([3, 9, 13, 15, 17]).

As expected, the extremal equations for Delauney-Dubins problem may be of two kinds: switching and non-switching. We will show that the non-switching extremals are solvable in terms of elliptic functions and we will also show that the projections of the switching extremals are the concatenations of arcs of circles (hyperbolas, in the hyperbolic case) and geodesics, excactly as in the two dimensional Dubins’ problem ([16]).

2 The Hybrid Maximum Principle and the Extremal curves

The preceding problems are a prototype of the following situation:

a. a control system
$$\frac{dz}{dt} = F(z(t), u(t)), \quad u(t) \in U \tag{8}$$
on a manifold M with U a subset of $\mathbb{R}^m$ possibly equal to it;

b. a submanifold N of M such that its cotangent bundle is embedded in the cotangent bundle of M and is given by $G_1 = G_2 = \ldots G_{2(n-k)} = 0$ for some functionally independent functions $G_1, \ldots G_{2(n-k)}$ on T^*M;

c. additional constraints $U_1, \ldots, U_l$, with each U_i a smooth functions on $M \times U$, such that the restriction of (8) to N with controls $u(t)$ in U and subject to $U_1(z, u) = \cdots = U_l(z, u) = 0$ results in a control system on N
$$\frac{dz}{dt} = F(z(t), u(t)); \tag{9}$$

d. an optimal problem of minimizing a cost functional $\int_0^T f(z(t), u(t))\, dt$ over the trajectories of system (9) in N that satisfy the given boundary conditions $z(0) = z_0$ and $z(T) = z_1$. The time interval $[0, T]$ could be either fixed or variable.

Rather than choosing a system of coordinates for N and proceeding independently of the ambient space, we will find it more convenient to work in the constrained ambient space. Then the task of finding a Hamiltonian on the cotangent bundle of the ambient space whose restriction to the cotangent bundle of the submanifold coincides with the Hamiltonian for the above optimal problem requires modifications in the use of the Maximum Principle because of the state dependent constraints $U_1, \dots, U_l$.

The version of the Maximum Principle that is applicable to this situation will be referred to as the Hybrid Maximum Principle. Its main features are sketched below.

Control system (8) together with the cost f lifts to the cost-extended Hamiltonians $h_{u,\mu}$ on T^*M of the form

$$h_{u,\mu}(\xi) = -\mu f(z, u) + \xi(F(z, u)) + \lambda_1 G_1(\xi) + \cdots + \lambda_{2(n-k)} G_{2(n-k)}(\xi), \quad (10)$$

$\xi \in T_z^*M$, where $\mu = 0, 1$, and $\lambda_1, \dots, \lambda_{2(n-k)}$ so chosen that for any $u \in U$, $\{h_{u,\mu}, G_i\} = 0$, $i = 1, \dots, 2(n-k)$. It follows that the integral curves of $\vec{h}_{u,\mu}$ which originate in T^*N, and conform to the constraints imposed by $U_1, \dots, U_l$, remain there for all t. For that reason, the above Hamiltonian is called the Hamiltonian lift of (9).

An integral curve $\xi(t)$ of $\vec{h}_{u(t),\mu}$ that originates on T^*N at $t = 0$ is called an *extremal curve* if it satisfies the maximality condition

$$h_{u(t),\mu}(\xi(t)) \geq h_{v,\mu}(\xi(t)) \quad (11)$$

on $G_1 = \cdots = G_{2(n-k)} = 0$ for all $v \in U$ subject to the constraints

$$U_1(\pi(\xi(t)), v) = \cdots = U_l(\pi(\xi(t)), v) = 0.$$

Extremal curves which are integral curves of $\vec{h}_{u,\mu}$ with $\mu = 0$ and satisfy the non-degeneracy condition $\xi(t) \neq 0$ are called *abnormal*. Extremal curves which are integral curves of $\vec{h}_{u,\mu}$ with $\mu = 1$ are called *normal*. With this terminology at our disposal we are now ready to state the Hybrid Maximum Principle.

The Hybrid Maximum Principle. Every optimal trajectory $(z(t), u(t))$ in N is the projection of an extremal curve $\xi(t)$.

Because of the space limitations, we will not go into a more detailed discussion of this principle (it will be presented elsewhere). Instead, we will apply it to the optimal problems defined earlier. As an intermediate step, however, let us first illustrate its effectiveness for the geodesic problem on the spaces of constant curvature.

Example 1 (The geodesic problem on $\mathbb{S}^n_\epsilon$, $\epsilon = \pm 1$). The cotangent bundle of the sphere $\mathbb{S}^n_\epsilon$ can be identified with its tangent bundle $\{(x, p) : G_1(x, p) = ||x||_\epsilon^2 - \epsilon = 0,\ G_2(x, p) = (x \cdot p)_\epsilon = 0\}$ via the quadratic form $(\ ,\ )_\epsilon$ of the ambient space $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. The geodesic problem on $\mathbb{S}^n_\epsilon$ can then be phrased as the time optimal problem of connecting a given pair of points on $\mathbb{S}^n_\epsilon$ in the shortest time via the trajectories of $\frac{dx}{dt} = u(t)$, subject to $u \in U = \{u \in \mathbb{R}^{n+1} : ||u||_\epsilon = 1\}$ and $U_1(x, u) = (x \cdot u)_\epsilon = 0$. Since Riemannian problems do not admit abnormal extremals, the Hamiltonian lift is given by

$$h_u = -1 + (p \cdot u)_\epsilon + \lambda_1 G_1 + \lambda_2 G_2.$$

An easy calculation yields $\{G_1, G_2\} = 2||x||^2_\epsilon$, $\{(p \cdot u)_\epsilon, G_2\} = -(u \cdot p)_\epsilon$ and $\{(p \cdot u)_\epsilon, G_1\} = -2(x \cdot u)_\epsilon$. Therefore,

$$\lambda_1 = -\frac{1}{2||x||^2_\epsilon}(u \cdot p)_\epsilon, \quad \lambda_2 = -\frac{1}{||x||^2_\epsilon}(u \cdot x)_\epsilon.$$

The maximality condition implies that the extremal control must be of the form $u = \frac{p}{||p||_\epsilon}$. The corresponding Hamiltonian is given by $H = -1 + ||p||_\epsilon + \lambda_1 G_1 + \lambda_2 G_2$. It follows that

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{||p||_\epsilon}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x} = -2\lambda_1 x = \epsilon ||p||_\epsilon x.$$

are the Hamiltonian equations on T^*N_ϵ. The extremals to our time optimal problem reside on $H = 0$ which implies that $||p||_\epsilon = 1$ along the extremals. Therefore, geodesic curves are the solutions of

$$\frac{d^2x}{dt^2} = \epsilon x,$$

which recovers the well known facts that the geodesics are arcs of the great circles for $\epsilon = 1$ and arcs of the great hyperbolas for $\epsilon = -1$.

3 The Euclidean case

In the Euclidean case, $M = \mathbb{R}^n \times \mathbb{R}^n$ is the ambient space, $N = \{(x, y) : ||y||^2 = 1\}$ and $\frac{dx}{dt} = y$, $\frac{dy}{dt} = \sum_{i=1}^n u_i e_i = u$ is the control system subject to the constraint $U_1 = y \cdot u = 0$. Then T^*N is identified with the tangent bundle of N as the set of all points $(x, y, p, q) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ subject to $G_1 = ||y||^2 - 1 = 0$ and $G_2 = y \cdot q = 0$.

As in the case of linear control theory, where the time optimal problem is more challenging than the linear-quadratic problem, so here too, the Delauney-Dubins problem is more challenging than the Euler-Griffiths problem. For that reason we will begin with the easier problem first.

3.1 Extremals for the Euler-Griffiths problem

The Hamiltonian lift is given by

$$h_{u,\mu} = -\mu \frac{1}{2}||u||^2 + p \cdot y + q \cdot u + \lambda_1 G_1 + \lambda_2 G_2, \quad \mu = 0, 1. \tag{12}$$

Then, $\{h_{u,\mu}, G_1\} = 0$ and $\{h_{u,\mu}, G_2\} = 0$ if and only if

$$\{g_u, G_2\} + \lambda_1\{G_2, G_1\} = 0, \quad \{g_u, G_1\} + \lambda_2\{G_1, G_2\} = 0,$$

where $g_u = -\mu\frac{1}{2}||u||^2 + p \cdot y + q \cdot u$. It follows that $\{G_1, G_2\} = 2$, $\{h_u, G_1\} = -2u \cdot y$, and $\{g_u, G_2\} = p \cdot y - u \cdot q$. Hence,

$$\lambda_1 = -\frac{1}{2}(p \cdot y - u \cdot q),\ \lambda_2 = -2u \cdot y.$$

Suppose now that $(x(t), y(t), p(t), q(t))$ is an extremal curve generated by a control $u(t)$. Then $u(t)$ maximizes $h_{v,\mu}$ in (12) subject to $U_1(y, v) = 0$. According to the Lagrange multiplier rule we can find the maximum by introducing a multiplier λ_0 and then considering the maximum of $h_{v,\mu} + \lambda_0(v \cdot y)$.

In the case $\mu = 0$, the maximality condition yields $q(t) + \lambda_0 y(t) = 0$. Then $q(t) \cdot y(t) + \lambda_0||y(t)||^2 = 0$ implies that $\lambda_0 = 0$, since $y(t) \cdot q(t) = 0$. Thus, $q(t) = 0$. But then,

$$0 = \frac{dq}{dt} = -\frac{\partial h_u}{\partial y} = -p + (p \cdot y)y \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial h_u}{\partial x} = 0.$$

It follows that p is constant, and hence, $(p \cdot y)y$ is also constant. The latter is not equal to zero since then $p = 0$ violates the non-degeneracy condition. Therefore, $y(t)$ must be constant, which implies that $u(t) = 0$. The preceding argument shows that the abnormal extremals project onto the straight lines in N.

Let us now investigate the normal case ($\mu = 1$). In this case (11) yields

$$-u(t) + q(t) + \lambda_0 y(t) = 0, \tag{13}$$

But then $-u(t) \cdot y(t) + q(t) \cdot y(t) + \lambda_0||y(t)||^2 = 0$ implies that $\lambda_0 = 0$ and $u(t) = q(t)$. It follows that the normal extremal curves are the solutions of the restricted Hamiltonian system associated to

$$H = \frac{1}{2}||q||^2 + p \cdot y + \lambda_1 G_1 + \lambda_2 G_2, \quad \lambda_1 = \frac{1}{2}(p \cdot y - ||q||^2), \quad \lambda_2 = \frac{1}{2}(q \cdot y) = 0. \tag{14}$$

It follows that

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = q, \quad \frac{dp}{dt} = 0, \quad \frac{dq}{dt} = -p + (p \cdot y - ||q||^2)y. \tag{15}$$

The projections $x(t)$ of normal extremal curves are called *elastic* ([12]).

It turns out that solutions of (15) share many features with the analogous system associated with the Delauney-Dubins problem and for that reason we will defer our discussion of solutions until we have both sets of equations at our disposal.

3.2 Extremals for the Delauney-Dubins problem

For this problem, the Hamiltonian lift is given by

$$h_{u,\mu} = -\mu + p \cdot y + q \cdot u + \lambda_1 G_1 + \lambda_2 G_2, \quad ||u|| \leq c.$$

Let $g_u = -\mu + p \cdot y + u \cdot q$. Then $\{h_{u,\mu}, G_1\} = \{h_{u,\mu}, G_2\} = 0$ implies that

$$\lambda_1 = -\frac{\{g_u, G_2\}}{\{G_1, G_2\}} = -\frac{1}{2}(p \cdot y - u \cdot q), \quad \lambda_2 = -\frac{\{g_u, G_1\}}{\{G_2, G_1\}} = \frac{1}{2}(y \cdot q)$$

Suppose now that $x(t), y(t), p(t), q(t)$ is an extremal curve. Then $u(t) \cdot y(t) = 0$ and

$$-\mu + p(t) \cdot y(t) + q(t)u(t) \geq -\mu + p(t) \cdot y(t) + v \cdot q(t)$$

for all $v \in \mathbb{R}^n$ such that $||v|| \leq c$ and $v \cdot y(t) = 0$.

On any open interval where $q(t)$ is not equal to zero, $u(t) = c\frac{q(t)}{||q(t)||}$ and the extremal curve is an integral curve of $\vec{h}|_{G_1=G_2=0}$ associated with $h = -\mu + p \cdot y + c||q(t)|| + \lambda_1 G_1 + \lambda_2 G_2$. These extremal curves are the solutions of

$$\frac{dx}{dt} = y(t), \quad \frac{dy}{dt} = c\frac{q(t)}{||q(t)|}, \quad \frac{dp}{dt} = 0, \quad \frac{dq}{dt} = -p + (p \cdot y(t) - c||q(t)||)y(t). \tag{16}$$

Our next proposition deals with the case that $q(t) =$ on an interval.

Proposition 1. *If* $(x(t), y(t), p(t), q(t))$ *is an extremal curve generated by a control* $u(t)$ *such that* $q(t) = 0$ *on an interval* (t_0, t_1) *then* $u(t) = 0$ *on* (t_0, t_1) *and* $x(t)$ *is a straight line on this interval.*

Proof. The extremal $(x(t), y(t), p(t), q(t))$ satisfies

$$\frac{dx}{dt} = y(t), \quad \frac{dy}{dt} = u(t), \quad \frac{dp}{dt} = 0, \quad \frac{dq}{dt} = -p + (p \cdot y(t) - u(t) \cdot q(t))y(t).$$

On an open interval where $q(t) = 0$, $p = (p \cdot y(t))y(t)$. Since the Hamiltonian is equal to zero along an extremal curve, $p \cdot y(t) = \mu$. Then $\mu = 1$ ($\mu = 0$, $p = q = 0$ violates the non-degeneracy condition). Therefore, $p = y$ and hence, $\frac{dy}{dt} = u(t) = 0$. □

3.3 Integrals of motion and integrability

Both the Euler-Griffiths and the Delauney-Dubins problem are invariant under the group of motions of $\mathbb{R}^n$ and that fact accouns for their integrability. To be more explicit, let G denote the semi-direct product $\mathbb{R}^n \ltimes SO_n(R)$. Systems (6) and (7) and the appropriate cost functionals are invariant under the diagonal action $(v, R)(x, y) \rightarrow$

$(v + Rx, Ry)$ of G. That means that each infinitesimal generator

$$V(x,y) = (Ax + a, Ay) = \frac{d}{d\epsilon}(e^{A\epsilon}x + \epsilon a, e^{A\epsilon}y)|_{\epsilon=0}, \quad A \in so_n(R),\ a \in \mathbb{R}^n$$

is a symmetry for the elastic problem. Then according to Noether's theorem, the Hamiltonian $h = p \cdot (Ax + a) + y \cdot Ay$ is constant along the extremal curves. The above implies that p is constant (which we already knew) and that $p \cdot Ax + q \cdot Ay$ is constant for each skew-symmetric matrix A. It follows from ([12], pp. 43) that $p \cdot Ax + q \cdot Ay = \langle A, p \wedge x + q \wedge y \rangle$, where $\langle\ ,\ \rangle$ is the scalar product on the space of skew-symmetric matrices given by $\langle A, B \rangle = -\frac{1}{2}Trace(AB)$ and where $a \wedge b$ is the skew-symmetric matrix defined by $(a \wedge b)x = (b \cdot x)a - (a \cdot x)b$ for each $x \in \mathbb{R}^n$. Therefore,

$$\Lambda = p \wedge x + q \wedge y \tag{17}$$

is constant for each system (15) and (16).

It follows that the spectral invariants of Λ are constants of motion for any Hamiltonian system whose projection is invariant under G. Since Λ is skew-symmetric with a four dimensional range, its non-zero spectrum is given by a polynomial of degree 4 of the form

$$\lambda^4 + a\lambda^2 + b = 0.$$

It turns out (after a somewhat lengthy calculation) that a and b are functionally dependent on two much simpler integrals

$$I_1 = ||p|| \quad \text{and} \quad I_2 = ||p||^2||q||^2 - (q \cdot p)^2 - (y \cdot p)^2||q||^2. \tag{18}$$

The second integral is the square of the volume spanned by y, p, q. In fact,

$$I_2 = ||p - (p \cdot y)y - (p \cdot q)\frac{q}{||q^2||^2}||^2||q||^2$$

so it can be written as $I_2 = h^2$. With these integrals at our disposal we can easily recover the essential properties of the elastic curves reported in ([9, 13, 15]).

Proposition 2. *Let $\kappa(t)$ and $\tau(t)$ denote the curvature and the torsion of an elastic curve. Then:*

1) $\frac{d\xi}{dt}^2 + \xi^3 - 4H\xi^2 - 4(I_1^2 - H^2)\xi + 4I_2 = 0$, *where $\kappa^2(t) = \xi(t)$;*
2) $(\kappa^2\tau)^2 = I_2 = h^2$;
3) *if $T(t), N(t), B(t)$ denote the Serret-Frenet triad defined by*

$$\frac{dx}{dt} = T(t), \quad \frac{dT}{dt} = \kappa N, \quad \frac{dN}{dt} = -\kappa T + \tau B,$$

then $\frac{dB}{dt}(t)$ is contained in the linear span of $T(t), N(t), B(t)$. Hence, the Serret-Frenet frame generated by an elastic curve is at most three dimensional.

Proof. Since $\kappa^2 = ||q||^2$, $\frac{d\kappa^2}{dt} = 2(\frac{dq}{dt} \cdot q) = -2p \cdot q$. Therefore,

$$\begin{aligned}\frac{1}{4}\left(\frac{d\xi}{dt}\right)^2 &= (p\cdot q)^2 = ||q||^2||p||^2 - (y\cdot p)^2||q||^2 - I_2\\ &= \kappa^2||p||^2 - \left(H - \frac{1}{2}||q||^2\right)^2||q||^2 - I_2^2 = \xi I_1^2 - \left(H - \frac{1}{2}\xi\right)^2\xi - I_2\\ &= -\frac{1}{4}\xi^3 + H\xi^2 + (I_1^2 - H^2)\xi - I_2.\end{aligned}$$

Therefore (1) holds.

Since $\frac{dx}{dt} = y$ and $||y|| = 1$, $T(t) = y(t)$. Hence, $\frac{dT}{dt} = \frac{dy}{dt} = q(t)$ and $N = \frac{1}{||q||}q$. Then,

$$\begin{aligned}\frac{dN}{dt} &= -\left(q\cdot\frac{dq}{dt}\right)\frac{1}{||q||^3}q + \frac{1}{||q||}(-p + (p\cdot y - ||q||^2)y)\\ &= \frac{p\cdot q}{||q||^3}q + \frac{1}{||q||}(-p + (p\cdot y - ||q||^2)y)\\ &= -\frac{p\cdot q}{||q||^2}N + \frac{1}{||q||}(-p + (p\cdot y)T) - \kappa T = -\kappa T + \tau B.\end{aligned}$$

The above yields

$$\kappa\tau B = -p + (p\cdot y)T + \frac{p\cdot q}{||q||}N.$$

This implies that $\frac{dB}{dt}$ is contained in the linear span of T, N, B and moreover, it implies that

$$\begin{aligned}(\kappa\tau)^2 &= ||-p + (p\cdot y)y - \frac{p\cdot q}{||q||^2}q||^2\\ &= \frac{1}{||q||^2}(||p||^2||q||^2 - (p\cdot q)^2 - (p\cdot y)^2||q||^2) = \frac{I_2}{\kappa^2}.\end{aligned}$$

The proof is now complete. □

Let us now return to the extremal equations for the Delauney-Dubins problem.

Proposition 3. *Let $\kappa(t)$ and $\tau(t)$ denote the curvature and the torsion associated with an extremal curve $(x(t), y(t), p(t), q(t))$. On any interval (t_0, t_1) such that $q(t) \neq 0$, $\kappa(t) = c$ and $||q||$ is a solution of*

$$\left(||q||\frac{d}{dt}||q||\right)^2 = -c^2||q||^4 + 2\mu||q||^3 + (I_1^2 - \mu^2)||q||^2 - I_2. \qquad (19)$$

Moreover, $(||q||^2\tau)^2 = I_2 = h^2$ *and* $\frac{dB}{dt}(t)$ *in the Serret-Frenet triad defined by*

$$\frac{dx}{dt} = T(t), \quad \frac{dT}{dt} = \kappa N, \quad \frac{dN}{dt} = -\kappa T + \tau B,$$

is contained in the linear span of $T(t), N(t), B(t)$*. Hence, the Serret-Frenet frame generated by an extremal curve is at most three dimensional.*

Proof. We have already remarked that $I_1 = ||p||$ and $I_2 = ||p||^2||q||^2 - (p \cdot q)^2 - (p \cdot y)^2||q||^2$ are constants of motion for (19). Then $\frac{d}{dt}||q|| = \frac{1}{||q||}(q \cdot \frac{dq}{dt}) = -\frac{1}{||q||}(p \cdot q)$ and therefore,

$$\begin{aligned}\left(||q||\frac{d}{dt}||q||\right)^2 &= (p \cdot q)^2 = ||p||^2||q||^2 - (p \cdot y)^2||q||^2 - I_2\\ &= I_1^2||q||^2 - (\mu - c||q||)^2||q||^2 - I_2\\ &= -c^2||q||^4 + 2\mu c||q||^3 + (I_1^2 - \mu^2)||q||^2 - I_2.\end{aligned}$$

Now $\frac{dT}{dt} = \frac{dy}{dt} = c\frac{q(t)}{||q(t)||}$ and so $N(t) = \frac{dq}{dt}$. The rest of the proof is the same as in the proof of Proposition 2. □

In the case $I_2 = 0$, Eq. (19) reduces to

$$\frac{d||q||}{dt}^2 = -c^2||q||^2 + 2\mu c||q|| + I_1^2 - \mu^2. \tag{20}$$

Recall that $I_2 = 0$ whenever $p, q(t), y(t)$ are linearly dependent for some time t. Apart from the stationary solutions $c||q|| = \mu \pm ||p||$, the solutions of (20) are of the form

$$c||q(t)|| = \mu - ||p|| \sin c((t - t_0)), \ ||q(t)|| \neq 0 \tag{21}$$

The stationary solutions result in helices since both the curvature and the torsion are constant (the case $\mu = 0$, $p = 0$ is ruled out by the Maximum Principle). Other cases are classified according to the size of $||p||$. For $||p|| \geq 1$ there are instances such that $||q|| = 0$. This phenomenon leads to

Definition 1. The hypersurface $S = \{(x, y, p, q) : q = 0\}$ is called the switching surface.

All extremal curves which cross the switching surface are confined to $I_2 = 0$. An extremal curve that does not originate on S may cross S either *tangentially or transversally*. If T is the time of crossing then the crossing is tangential if $-||q(T)||^2 + 2\mu||q(T)|| + ||p||^2 - \mu^2 = 0$, otherwise it is transversal. That is, the crossing depends on whether $\lim_{t\to T}\frac{d||q(t)||}{dt}$ is zero or not. The critical case $||p|| = 1$ is the only case in which the crossing is tangential. All other crossings are transversal and reside on $||p|| > 1$.

In the normal and non-geodesic case the time interval between two consecutive crossings is larger than π while in the abnormal case this time interval is equal to π. Both of these observations follow from (21).

Proposition 4. *Let $(x(t), y(t), p, q(t))$ be an extremal curve that crosses the switching surface transversally, i. e., $||p|| > 1$. Then $x(t)$ consists of concatenations of arcs of circles of radius $\frac{1}{\sqrt{c}}$ all contained in the plane defined by $x(0), y(0)$ and p.*

Proof. Let κ and τ be the curvature and torsion of $x(t)$. Then $\kappa(t) = c$ and $\tau = 0$ on every interval on which $||q|| > 0$. On these intervals $x(t)$ moves along an arc of a circle of radius $\frac{1}{\sqrt{c}}$ centered at some point a, and therefore, $x(t)$ can be represented as

$$x(t) - a = \frac{1}{\sqrt{c}}(A \cos \sqrt{c}t + B \sin \sqrt{c}t), \quad ||A||^2 = ||B||^2 = 1, \quad (A \cdot B) = 0.$$

Then, $\frac{dx}{dt} = y(t)$ and $\frac{dy}{dt} = -\sqrt{c}(x(t) - a) = \kappa N$. It follows that the normal vector $N(t)$ is equal to $\frac{1}{\sqrt{c}}(a - x))$.

Suppose now that T denotes the time when two adjacent circles meet, that is, suppose that $q(T) = 0$. Let $N_-(T) = \lim_{t<T, t \to T} N(t)$ and $N_+(T) = \lim_{t>T, t \to T} N(t)$. On any open interval I such that $q(t) \neq 0$, $N(t) = \frac{q(t)}{||q(t)||}$ and

$$\frac{dN(t)}{dt} = \frac{1}{||q(t)||}(-p + (p \cdot y - c||q(t)||)y(t)) - \frac{q(t)}{||q(t)||^2}\frac{d||q(t)||}{dt} = \kappa T = cy(t).$$

Therefore,

$$(-p + (p \cdot y)y(t) = N(t)\frac{d||q(t)||}{dt}.$$

It follows from Proposition 2 that $\lim_{t \to T} \frac{d||q(t)||}{dt} = \pm\sqrt{||p||^2 - 1}$ depending whether this limit is from the right or from the left. Since the Hamiltonian is equal to zero along the extremal curves, $p \cdot y(T) = \mu$ Hence,

$$N_- = -\frac{-p + \mu y(T)}{\sqrt{||p||^2 - 1}}, \quad N_+ = \frac{-p + \mu y(T)}{\sqrt{||p||^2 - 1}}.$$

Since these normals are colinear, the concatenated arcs of the circles are in the same plane. ◻

Proposition 5. *Suppose now that an extremal curve $(x(t), y(t), p, q(t))$ crosses the switching surface tangentially ($||p|| = 1$). Then either there is no switching at the time of the crossing and $x(t)$ is a circle of radius $\frac{1}{\sqrt{c}}$, or $x(t)$ is the concatenation of an arc of a circle of radius $\frac{1}{\sqrt{c}}$ with a straight line, possibly followed by another arc of a circle of radius $\frac{1}{\sqrt{c}}$.*

The proof is simple and will be omitted.

Let us now consider the extremals that reside on $I_2 \neq 0$. It will be convenient to rescale the time variable by $s = ct$. The rescaled variable $||q(s)||$ in Eq. (19) then satisfies

$$\left(||q||\frac{d}{ds}||q||\right)^2 = -||q||^4 + 2\frac{\mu}{c}||q||^3 + \frac{(I_1^2 - \mu^2)}{c^2}||q||^2 - \frac{I_2}{c^2}.$$

The preceding equation can be written in von Schwarz's form as

$$(u - \mu_0)^2 \frac{du}{ds}^2 = (u - \mu)^2(k^2 - u^2) - h_0^2$$

with $u = ||q|| - \frac{\mu}{c}$, $\mu_0 = -\frac{\mu}{c}$, $k = \frac{I_1}{c}$, $h_0 = \frac{\sqrt{I_2}}{c}$. The latter equation can be further rescaled to

$$(\zeta - l)^2 \left(\frac{d\zeta}{ds}\right)^2 = (\zeta - l)^2(1 - \zeta^2) - h^2 \tag{22}$$

with the rescaled variables

$$\zeta = \frac{u}{k}, \quad l = \frac{\mu}{k}, \quad h^2 = \frac{h_0^2}{k^4}.$$

Then ζ_1 and ζ_2, the roots of the equation $(\zeta - l)^2(1 - \zeta^2) - h^2 = 0$, are the stationary solutions of (27). Along them $||q(t)||$ is constant and hence the torsion of the corresponding extremal curve $x(t)$ is constant. Therefore, $x(t)$ is a helix.

Any other solution $\zeta(t)$ satisfies

$$\zeta_1 < \zeta(t) < \zeta_2$$

and can be expressed in terms of the elliptic functions by integrating

$$\int \frac{\zeta - l}{\sqrt{(\zeta - l)^2(1 - \zeta^2) - h^2}} d\zeta = \pm t. \tag{23}$$

Then $(||q||^2\tau)^2 = I_2$ gives $\tau(t)$ in terms of $\zeta(t)$ and the solutions are reduced to solving the Serret-Frenet system

$$\frac{dT}{dt} = cN(t), \quad \frac{dN}{dt} = -cT(t) + \tau(t)B(t), \quad \frac{dB}{dt} = -\tau(t)N(t).$$

To relate to Dubins' remarkable paper of 1958, consider the two dimensional case. Then p, q and y must be linearly dependent, since they lie in the same plane, hence $I_2 = 0$. Therefore, every solution is either an arc of a circle or a line segment, or a concatenation of arcs of circles and line segments.

An optimal solution that involves a line segment cannot have two consecutive circle switchings since such extremals reside on $||p|| = 1$ and the switching interval is 2π. Therefore, optimal solutions that involve a line segment must of the form CLC or any sub-path of these.

In the remaining case, optimal solutions are the concatenations of circles. We showed that such solutions reside on $||p|| > 1$ and the time interval between any two consecutive switchings are all equal and are greater than π. But then any path $C_t C_\alpha C_\alpha$ with $t > 0$ and $\alpha > \pi$ cannot be optimal (Monroy's Lemma, [16], pp. 141). Hence, optimal paths along arcs of circles must be of the form $C_\alpha C_\beta C_\gamma$ with $\beta > \pi$ or any sub-path of these arcs.

So, apart from the Four Circle Lemma of Monroy, the main contents of Dubins' paper can be read directly from our Hamiltonian setup. It would be of interest to investigate which of these optimal planar solutions remain optimal in higher dimensional spaces.

4 Non-Euclidean cases

We will continue with the notations introduced earlier, with M_ϵ equal to $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ together with the inner product $(v, w)_\epsilon = \sum_{i+1}^{n} v_i w_i + \epsilon v_0 w_0$ and the induced "norm" $||v||_\epsilon^2 = (v \cdot v)_\epsilon$. This quadratic form identifies T^*M_ϵ with TM_ϵ via the formula $l((\dot{x}, \dot{y})) = (p \cdot \dot{x})_\epsilon + \epsilon(q \cdot \dot{y})_\epsilon$. Hence, $(x, y, \dot{x}, \dot{y}) \in TM_\epsilon$ corresponds to $(x, y, p, \epsilon q) \in T^*M_\epsilon$. In these notations (x, y, p, q) denote the canonical coordinates of a point in T^*M_ϵ with the Poisson bracket given by

$$\{f, h\} = \left(\frac{\partial f}{\partial x} \cdot \frac{\partial h}{\partial p}\right)_\epsilon + \left(\frac{\partial f}{\partial y} \cdot \frac{\partial h}{\partial q}\right)_\epsilon - \left(\left(\frac{\partial f}{\partial p} \cdot \frac{\partial h}{\partial x}\right)_\epsilon + \left(\frac{\partial f}{\partial q} \cdot \frac{\partial h}{\partial y}\right)_\epsilon\right).$$

Then the cotangent bundle T^*N_ϵ is defined by the six constraints $G_1 = ||x||_\epsilon - \epsilon = 0$, $G_2 = ||y||_\epsilon^2 - 1 = 0$, $G_3 = (x \cdot y)_\epsilon = 0$, $G_4 = (x \cdot p)_\epsilon = 0$, $G_5 = (y \cdot q)_\epsilon = 0$, $G_6 = (y \cdot p)_\epsilon + \epsilon(x \cdot q)_\epsilon = 0$ which conform to the following Poisson bracket table

{ , }	G_1	G_2	G_3
G_1	0	0	0
G_2	0	0	0
G_3	0	0	0
G_4	$-2\|\|x\|\|_\epsilon^2$	0	$-(x \cdot y)_\epsilon$
G_5	0	$-2\|\|y\|\|_\epsilon^2$	$-(x \cdot y)_\epsilon$
G_6	$-2(x \cdot y)_\epsilon$	$-2\epsilon(x \cdot y)_\epsilon$	$-\epsilon\|\|x\|\|_\epsilon^2 - \|\|y\|\|_\epsilon^2$
{ , }	G_4	G_5	G_6
G_1	$2\|\|x\|\|_\epsilon^2$	0	$2(x \cdot y)_\epsilon$
G_2	0	$2\|\|y\|\|_\epsilon^2$	$2\epsilon(x \cdot y)_\epsilon$
G_3	$(x \cdot y)_\epsilon$	$(x \cdot y)_\epsilon$	$\epsilon\|\|x\|\|_\epsilon^2 + \|\|y\|\|_\epsilon^2$
G_4	0	0	$(y \cdot p)_\epsilon - (x \cdot q)_\epsilon$
G_5	0	0	$\epsilon(x \cdot q)_\epsilon - \epsilon(y \cdot p)_\epsilon$
G_6	$-(y \cdot p)_\epsilon + (x \cdot q)_\epsilon$	$\epsilon(y \cdot p)_\epsilon - \epsilon(x \cdot q)_\epsilon$	0

For the Euler-Griffiths problem the cost-extended Hamiltonian is given by

$$h_{u,\mu} = -\mu \frac{1}{2}||u||_\epsilon^2 + (y \cdot p)_\epsilon + (q \cdot (u - \epsilon x))_\epsilon + \sum_{i=1}^{6} \lambda_i G_i. \tag{24}$$

We leave it to the reader to show that the multipliers λ_i are given by

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{2}(q \cdot u)_\epsilon, \quad \lambda_3 = \frac{1}{2}((y \cdot q)_\epsilon + (x \cdot p)_\epsilon + (p \cdot u)_\epsilon),$$
$$\lambda_4 = \lambda_5 = \lambda_6 = 0.$$

The extremal controls maximize the restriction of $h_{u,\mu}$ to T^*N subject to the additional constraints $U_1 = (x \cdot u)_\epsilon = 0$, $U_2 = (y \cdot u)_\epsilon = 0$. According to the Lagrange multiplier rule, the maximum must be a critical point of $F = h_{u,\mu} + \lambda_{-1} U_1 + \lambda_{-2} U_2$. An easy calculation shows that $\lambda_{-1} = \epsilon(q \cdot x)_\epsilon$, $\lambda_{-2} = (q \cdot y)_\epsilon$. This implies that the extremal control, in the normal case, is of the form

$$u = q - \epsilon(q \cdot x)_\epsilon x - (q \cdot y)_\epsilon y. \tag{25}$$

The abnormal extremals will be ignored since they project onto the geodesics and the geodesics are also the projections of normal extremals.

The substitution of u in (25) into the Hamiltonian lift (24) gives

$$h = \frac{1}{2}(||q||^2 - (q \cdot x)_\epsilon^2 - (q \cdot y)_\epsilon^2) + (y \cdot p)_\epsilon - \epsilon(q \cdot x)_\epsilon + \lambda_2 G_2 + \lambda_3 G_3. \tag{26}$$

The Hamiltonian equations of $\vec{h}$ restricted to T^*N yield the Hamiltonian system for the Euler-Griffiths problem. They are as follows:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = q - \epsilon(q \cdot x)_\epsilon x - \epsilon x, \tag{27a}$$

$$\frac{dp}{dt} = \epsilon q + \epsilon(q \cdot x)_\epsilon q - \lambda_3 y, \quad \frac{dq}{dt} = -p - 2\lambda_2 y - \lambda_3 x. \tag{27b}$$

On T^*N, h reduces to

$$H = \frac{1}{2}(||q||^2 - (x \cdot q)_\epsilon^2) + 2(y \cdot p)_\epsilon = \frac{1}{2}\kappa^2 + 2(y \cdot p)_\epsilon, \tag{28}$$

where $\kappa = ||u||$ denotes the geodesic curvature of $x(t)$.

The Hamiltonian for the Delauney-Dubins problem is obtained in a similar manner. The reader may readily verify that in the case that $q(t)$ is neither equal to zero nor colinear with $x(t)$ on an open interval I, the extremal control must be of the form

$$u(t) = c \frac{q(t) - \epsilon(q \cdot x)_\epsilon x(t)}{||q(t) - \epsilon(q \cdot x)_\epsilon x(t)||} \tag{29}$$

and

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = c\frac{q(t) - \epsilon(q \cdot x)_\epsilon x(t)}{||q(t) - \epsilon(q \cdot x)_\epsilon x(t)||} - \epsilon x, \tag{30a}$$

$$\frac{dp}{dt} = c\frac{\epsilon(q \cdot x)_\epsilon q}{||q(t) - \epsilon(q \cdot x)_\epsilon x(t)||} - \lambda_3 y, \quad \frac{dq}{dt} = -p - 2\lambda_2 y - \lambda_3 x \tag{30b}$$

are the extremal equations for Dubins-Delauney problem corresponding to the Hamiltonian

$$H = -\mu + 2(y \cdot p)_\epsilon + c||q - \epsilon(q \cdot x)_\epsilon x||. \tag{31}$$

The reader may also verify that an extremal curve with $q(t)$ colinear with $x(t)$ projects onto a geodesic in the base space $\mathbb{S}^n_\epsilon$.

4.1 Integrability

We will use $v \otimes_\epsilon w$ to denote the matrix such that $(v \otimes_\epsilon w)x = (w \cdot x)_\epsilon v$ for all $x \in \mathbb{R}^{n+1}$ and use $v \wedge w_\epsilon$ to denote the matrix $v \wedge_\epsilon w = v \otimes w_\epsilon - w \otimes v_\epsilon$. Then SO_ϵ will denote the connected group that leaves the quadratic form $(\, , \,)_\epsilon$ invariant. This group is equal to SO_{n+1} when $\epsilon = 1$ and $SO(1, n)$ when $\epsilon = -1$.

Both the elastic and the Delauney-Dubins problems are invariant under the diagonal action of SO_ϵ and hence, the Hamiltonians generated by the infinitesimal generators are the symmetries for their Hamiltonian systems. The reader may readily verify that this means that the matrix

$$S = x \wedge_\epsilon p + y \wedge_\epsilon q$$

is constant along the solutions of either Hamiltonian system (27) or (30).

We will now show that the spectral invariants of M provide the appropriate integrals of motion in terms of which the extremal equations can be integrated. The vector space spanned by x, y, p, q is invariant under M. Then the restriction of M to this vector space is given by the matrix

$$M = \begin{pmatrix} 0 & (y \cdot p)_\epsilon & ||p||^2_\epsilon & (p \cdot q)_\epsilon \\ (q \cdot x)_\epsilon & 0 & (p \cdot q)_\epsilon & ||q||^2_\epsilon \\ -\epsilon & 0 & 0 & -(q \cdot x)_\epsilon \\ 0 & -1 & -(y \cdot p)_\epsilon & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix is of the form $-\lambda^4 + a\lambda^2 + b = 0$ with

$$a = ||q||^2_\epsilon - 2(y \cdot p)_\epsilon (x \cdot q)_\epsilon + \epsilon ||p||^2_\epsilon,$$
$$b = \epsilon(||p||^2_\epsilon ||q||^2_\epsilon - (q \cdot p)^2_\epsilon - ||q||^2_\epsilon (y \cdot p)^2_\epsilon) + (y \cdot p)^2_\epsilon (q \cdot x)^2_\epsilon - ||p||^2_\epsilon (q \cdot x)^2_\epsilon.$$

But $(y \cdot p)_\epsilon = -\epsilon(q \cdot x)_\epsilon$ on T^*N_ϵ and so

$$a = \epsilon(||p||^2_\epsilon + 2(y \cdot p)^2_\epsilon + \epsilon ||q||^2_\epsilon),$$
$$b = \epsilon((||p||^2_\epsilon - (y \cdot p)^2_\epsilon)\,(||q||^2_\epsilon - \epsilon(x \cdot q)^2_\epsilon) - (p \cdot q)^2_\epsilon).$$

It follows that

$$I_1 = (||p||_\epsilon^2 + 2(y \cdot p)_\epsilon^2 + \epsilon||q||_\epsilon^2),$$
$$I_2 = (||p||_\epsilon^2 - (y \cdot p)_\epsilon^2)(||q||_\epsilon^2 - \epsilon(x \cdot q)_\epsilon^2) - (p \cdot q)_\epsilon^2$$

are integrals of motion for each Hamiltonian system (27) and (30).

Our next propositions explain the relevance of these integrals of motion.

Proposition 6. *Let* $(x(t), y(t), p(t), q(t))$ *denote any solution of the Hamiltonian system associated with the Euer-Griffiths problem (Eqs.* (27)*). Let* $\kappa(t)$ *and* $\tau(t)$ *denote the curvature and the torsion of the projected curve* $x(t)$ *and let* $\xi(t) = \kappa^2(t)$*. Then:*

1) $$\frac{d\xi}{dt}^2 = -\xi^3 + 4(H - \epsilon)\xi^2 + 4(I_1 - H^2)\xi - 4I_2 = 0; \tag{32}$$

2) $(\kappa^2\tau)^2 = I_2$*;*

3) *if* $T(t), N(t), B(t)$ *denote the Serret-Frenet triad defined by*

$$\frac{dx}{dt} = T(t), \quad \frac{\mathcal{D}_x}{dt}T(t) = \kappa N(t), \quad \frac{\mathcal{D}_x}{dt}N(t) = -\kappa T(t) + \tau B(t),$$

then $\frac{\mathcal{D}_x}{dt}B(t)$ *is contained in the linear span of* $T(t), N(t), B(t)$*. Hence, the Serret-Frenet frame generated by an elastic curve is at most three dimensional.*

Proof. Since $\xi = \kappa^2 = ||q||^2 - \epsilon(q \cdot x)_\epsilon^2$,

$$\frac{d\xi}{dt} = 2((q \cdot \dot{q})_\epsilon - \epsilon((q \cdot x)_\epsilon((\dot{q} \cdot x_\epsilon) + (q \cdot \dot{x})_\epsilon) = -2(p \cdot q)_\epsilon.$$

Therefore,

$$\begin{aligned}\frac{1}{4}\left(\frac{d\xi}{dt}\right)^2 &= -I_2 + (||p||_\epsilon^2 - (y \cdot p)_\epsilon^2)\xi = -I_2 + (I_1 - 3(y \cdot p)_\epsilon^2 - \epsilon||q||_\epsilon^2)\xi \\ &= -I_2 + (I_1 - 4(y \cdot p)_\epsilon^2 - \epsilon\xi)\xi = -I_2 + \left(I_1 - (H - \frac{1}{2}\xi\right)^2 - \epsilon\xi)\xi \\ &= -\frac{1}{4}\xi^3 + (H - \epsilon)\xi^2 + (I_1 - H^2)\xi - I_2^2,\end{aligned}$$

and (1) holds. To prove the remaining parts, note that $\frac{\mathcal{D}_x}{dt}T(t) = \frac{dy}{dt} - \epsilon x(t) = u(t) = q(t) - \epsilon(x \cdot q)x(t)$ and that $\frac{\mathcal{D}_x}{dt}N = \frac{dN}{dt}$ and $\frac{\mathcal{D}_x}{dt}B = \frac{dB}{dt}$. Hence, $N = \frac{1}{||u||}u$. It follows from Eqs. (32) that

$$\frac{du}{dt} = -p - 2\lambda_2 y - (x \cdot q)_\epsilon y,$$

where $2\lambda_2 = (u \cdot q)_\epsilon = ||q||_\epsilon^2 - \epsilon(x \cdot q)_\epsilon^2 = ||u||_\epsilon^2$. Then

$$\frac{dN}{dt} = -\left(u \cdot \frac{du}{dt}\right)\frac{1}{||u||^3}u + \frac{1}{||q||}\frac{du}{dt}$$
$$= \frac{p \cdot q}{||u||^2}N + \frac{1}{||u||}(-p - ||u||_\epsilon^2 y - \epsilon(x \cdot q)_\epsilon y) = -\kappa T + \tau B.$$

The above yields

$$\kappa^2 \tau B = (p \cdot q)_\epsilon N - \kappa p - \epsilon(x \cdot q)_\epsilon y.$$

This implies that $T(t), N(t), B(t)$ are linearly dependent and hence $\frac{\mathcal{D}_x}{dt}B$ is contained in the linear span of T, N, B and moreover, it implies that

$$(\kappa^2\tau)^2 = ||(p \cdot q)_\epsilon N - \kappa p - \epsilon(x \cdot q)_\epsilon y||^2$$
$$= \kappa^2(||p||_\epsilon^2 - (p \cdot y)_\epsilon^2) - (p \cdot q)_\epsilon^2 = I_2.$$

The proof is now complete. □

Proposition 7. *Let $\kappa(t)$ and $\tau(t)$ denote the curvature and the torsion associated with an extremal curve $(x(t), y(t), p(t), q(t))$ for the Delauney-Dubins problem (Eqs.* (30)*). On any open interval that $\xi(t) = ||q(t) - \epsilon(x(t) \cdot q(t))_\epsilon x(t)||$ is not equal to zero, $\kappa(t) = c$, and $\xi(t)$ is a solution of*

$$\left(\xi\frac{d\xi}{dt}\right)^2 = -(c^2 + \epsilon)\xi^4 + 2c\mu\xi^3 + (I_1 - \mu^2)\xi^2 - I_2. \tag{33}$$

Moreover, $(\xi^2\tau)^2 = I_2$ and $\frac{dB}{dt}(t)$ in the Serret-Frenet triad is contained in the linear span of $T(t), N(t), B(t)$. Hence, the Serret-Frenet frame generated by an extremal curve is at most three dimensional.

Proof. Since $\xi^2 = ||q||_\epsilon^2 - \epsilon(x \cdot q)_\epsilon^2$, $I_2 = (||p||_\epsilon^2 - (y \cdot p)_\epsilon^2)\xi^2 - (p \cdot q)^2$ and $(y \cdot p)_\epsilon^2 = \frac{1}{4}(\mu - c\xi)^2$, because $0 = H = -\mu + 2(p \cdot y)_\epsilon + c\xi$. Moreover, $2\xi\frac{d\xi}{dt} = \frac{d}{dt}(||q||^2 - \epsilon(x \cdot q)_e^2 p) = -2(p \cdot q)_\epsilon$.

Then,

$$\left(\xi\frac{d\xi}{dt}\right)^2 = (p \cdot q)^2 = (||p||^2 - (p \cdot y)^2)\xi^2 - I_2$$
$$= (I_1 - 2(y \cdot p)_\epsilon^2 - \epsilon||q||_\epsilon^2 - (y \cdot p)_\epsilon^2)\xi^2 - I_2$$
$$= (I_1 - 3(y \cdot p)_\epsilon^2 - \epsilon||q||^2)\xi^2 - I_2$$
$$= (I_1 - 4(y \cdot p)_\epsilon^2 - \epsilon\xi^2)\xi^2 - I_2 = (I_1 - (\mu - c\xi)^2 - \epsilon\xi^2)\xi^2 - I_2$$
$$= -(c^2 + \epsilon)\xi^4 + 2c\mu\xi^3 + (I_1 - \mu^2)\xi^2 - I_2.$$

The rest of the proof consists of minor adaptations of the proof used in Proposition 6 and will be omitted. □

The content of this proposition is essentially the same as that found in ([18]) with the exception that in the hyperbolic case one should take $c > 1$. Otherwise, the control system is not controllable (([16])).

The hypersurface $S = \{(x, y, p, q) : \xi = ||q - \epsilon(x \cdot q_\epsilon)x|| = 0\}$ is the switching surface. If an extremal curve crosses the switching surface at some time T then either $q(T)$ equal to zero or it colinear with $x(T)$. In either case $p(T) \cdot q(T)_\epsilon$ is equal to zero and hence, $I_2 = 0$. So each extremal curve that crosses the switching surface resides on the hypersurface $I_2 = 0$.

It follows that the extremals on $I_2 = 0$ are the solutions of

$$\left(\frac{d\xi}{dt}\right)^2 = -(c^2 + \epsilon)\xi^2 + 2\mu c\,\xi + I_1 - \mu^2. \tag{34}$$

Analogous to the Euclidean case, the stationary solutions of this equation project onto non-Euclidean helices (curves having both the curvature and the torsion constant). Otherwise, the solutions are of the form

$$(c^2 + \epsilon)\xi(t) = a - b \sin\sqrt{c^2 + \epsilon}\,(t - t_0), \quad a = \mu c, \quad b = \sqrt{I_1(c^2 + \epsilon) - \mu^2\epsilon}.$$

The associated extremal curve does not cross the switching surface when $a > b$. It crosses the switching surface tangentially when $a = b$, that is, when $I_1 = \mu^2$. The crossing is transversal for all other values of a and b. In the normal and non-geodesic case, the time interval between two consecutive crossings is larger than $\frac{\pi}{\sqrt{c^2+\epsilon}}$ while in the abnormal case this time interval is equal to $\frac{\pi}{\sqrt{c^2+\epsilon}}$.

A curve $x(t)$ in $\mathbb{S}^n_\epsilon$ which has constant curvature and zero torsion will be called a circle. By this convention, a circle is a hyperbola in the ambient space $\mathbb{R}^{n+1}$ when $\epsilon = -1$. It follows that the projection $x(t)$ of an extremal curve that crosses the switching surface transversally, moves along a circle in $\mathbb{S}^n_\epsilon$ on a two dimensional "sphere" $S^2_\epsilon = \{(\alpha_1 a + \alpha_2 b + \alpha_3 d : (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3, \alpha_1^2 + \alpha_2^2 + \epsilon\alpha_3^2 = \epsilon\}$ where the vectors a, b, d are determined by $x(t_0)$, $y(t_0)$ and the normal $N(t_0) = \frac{1}{c}\frac{\mathfrak{D}_x}{dt}T(t)|_{t=t_0}$.

Proposition 8. *Let $(x(t), y(t), p, q(t))$ be an extremal curve that crosses the switching surface transversally. Then $x(t)$ consists of concatenations of arcs of circles all contained in the two dimensional sphere defined by $x(0), y(0)$ and $N(0) = \frac{1}{c}\frac{\mathfrak{D}_x}{dt}T(t)|_{t=0}$.*

Corollary 1. *A concatenation of four or more extremal circles can not be optimal.*

On any energy level $I_2 > 0$, the solutions of Eq. (33) can be expressed in terms of elliptic functions, much in the same manner as in the Euclidean case. They are generic and non-switching. It would be nice to know their optimality status. Some switching extremals also project onto optimal solutions. For instance, the concatenation of three circles is optimal in any dimensional Delauney-Dubins problem, but some extremals, which are optimal in two dimensional case, may loose their optimality in higher dimensions. It would be interesting to investigate this situation in more detail.

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On Local Approximation Theorem on Equiregular Carnot–Carathéodory Spaces

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Abstract We prove the Local Approximation Theorem on equiregular Carnot–Carathéodory spaces with C^1-smooth basis vector fields.

1 Introduction

In this paper we study a local geometry of equiregular Carnot–Carathéodory spaces (or simply Carnot manifolds) in the case of C^1-smooth vector fields. Our purpose in this paper is to compare the Carnot–Carathéodory metric d_{cc} on the initial space with the Carnot–Carathéodory metric d^u_{cc} on the local Carnot group at u and the metrics d^u_{cc}, and d^v_{cc} on two local Carnot groups in close points.

For vector fields smooth enough, the Local Approximation Theorem was stated by Gromov in the form $|d_{cc}(x, y) - d^u_{cc}(x, y)| = o(\varepsilon)$ as $\varepsilon \to 0$ for points x and y in a sub-Riemannian ball of radius ε centered at u [20]. Later, Bellaïche [4] and Jean [24] refined this result in the case of C^∞-smooth vector fields by obtaining the estimate $O(\varepsilon^{1+1/M})$ for the same difference, where M is the depth of the distribution.

The Local Approximation Theorem is a good alternative to the well-known in Riemannian geometry property that metrics in a manifold and in its tangent space are locally bi-Lipschitz equivalent (it is known that the last property does not hold in the Sub-Riemannian geometry). The Local Approximation Theorem plays crucial role

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in proofs of differentiability theorems for mappings of Carnot–Carathéodory spaces (see, e. g., [3, 31, 52–56, 59]).

Carnot–Carathéodory geometry is applied for studying hypoelliptic operators (see, e. g., [13, 23, 45]). Being an adequate means for describing properties of solutions to subelliptic equations, it is also extensively used in the theory of partial differential equations (see, e. g., [3, 8, 9, 16, 58]).

2 Basic Definitions and Results

Recall basic definitions.

Definition 1 ([3]; cf. [7, 31, 40]). Fix a connected Riemannian C^∞-manifold $\mathbb{M}$ of topological dimension N. The manifold $\mathbb{M}$ is called the *Carnot–Carathéodory space* if the tangent bundle $T\mathbb{M}$ has a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles such that every point $p \in \mathbb{M}$ has a neighborhood $U \subset \mathbb{M}$ equipped with a collection of C^1-smooth vector fields $X_1, \ldots, X_N$ enjoying the following two properties:

(1) at every point $v \in U$ we have a subspace

$$H_i\mathbb{M}(v) = H_i(v) = \operatorname{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\} \subset T_v\mathbb{M}$$

of the dimension $\dim H_i$ independent of v, $i = 1, \ldots, M$;

(2) the inclusion $[H_i, H_j] \subset H_{i+j}$, $i + j \leq M$, holds.

Moreover, if the third condition holds then the Carnot–Carathéodory space is called the *Carnot manifold*:

(3) $H_{j+1} = \operatorname{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \ldots, [H_k, H_{j+1-k}]\}$, where $k = \left[\frac{j+1}{2}\right]$, $H_0 = \{0\}$, $j = 1, \ldots, M-1$.

The subbundle $H\mathbb{M}$ is called *horizontal*.

The number M is called the *depth* of the manifold $\mathbb{M}$.

For specifying the situation, we emphasize that the tangent cone to a Carnot manifold is a Carnot group, and the tangent cone to a Carnot–Carathéodory space is a graded nilpotent group (i. e., a horizontal subbundle of its Lie algebra of vector fields may not generate the whole Lie algebra). Thus, the notions of a Carnot manifold and a Carnot–Carathéodory space are essentially different.

Remark 1. To this end, we assume that Carnot-Carathéodory spaces under consideration have the same collection of basis vector fields for all points.

Properties of Carnot-Carathéodory spaces and Carnot manifolds under assumptions of regularity mentioned in Definition 1 can be found in [3, 19, 26, 28, 30, 31, 60].

Many classical and modern results, development trends of the theory of Carnot-Carathéodory spaces and their applications can be found in [1,2,4–7,10,10,12,14,15,15,17,18,21,23,25,33–42,45,48–50,57,59].

Example 1. A Carnot group is an example of a Carnot manifold.

Example 2 (A Carnot–Carathéodory space with C^1-smooth basis vector fields [28]; *cf.* [27]*).* Consider arbitrary C^1-smooth functions $\psi, \varphi, \xi, \eta, \omega : \mathbb{R} \to \mathbb{R}$ with ψ, φ, ξ, η, $\omega \neq 0$, their derivatives are only continuous, and $\frac{d\varphi}{dy} \neq 0$ on a closed interval $W \subset \mathbb{R}$. Construct the vector fields X, Y, Z, T on $W \times W \times W \times W \Subset \mathbb{R}^4$ as follows:

$$\begin{aligned} X =& \psi(x)\partial_x + \psi(x)\varphi(y)\partial_y + \eta\Bigl(-\int_\varepsilon^y \widetilde{\xi}(x,s)\,ds + z\Bigr)\partial_z + \varphi(y)\omega(q)\partial_q, \\ Y =& \partial_y + \xi\Bigl(-\int_\varepsilon^y \frac{dt}{\varphi(t)} + x\Bigr)\partial_z, \\ Z =& -\psi(x)\partial_y + \omega(q)\partial_q, \\ T =& \partial_y. \end{aligned}$$

Here $\widetilde{\xi}(x,s) = \xi\Bigl(-\int_\varepsilon^s \frac{dt}{\varphi(t)} + x\Bigr)$, and ε is a number depending on W and the choice of all these functions. It is easy to see that $X \in C^1(x,y,z,q)$, $Y \in C^1(x,y)$, $Z \in C^1(x,q)$, and T is smooth. Moreover, $[X,Y] = \frac{d\varphi}{dy} \cdot Z$ and $[Y,Z] = -\frac{d\psi}{dx} \cdot T$. We put $H = H_1 = \operatorname{span}\{X,Y\}$, $H_2 = \operatorname{span}\{X,Y,Z\}$, and $H_3 = \operatorname{span}\{X,Y,Z,T\}$. The resulting system of vector fields is non-degenerate. Moreover, it cannot be reduced to a system with two smooth horizontal vector fields.

Thus, we obtain a Carnot–Carathéodory space with $M = 3$ and horizontal vector fields of class C^1 (but not of C^2) in the same collection of variables (thus, both X and Y are C^1-smooth with respect to x and y). Moreover, the vector fields $Z \in \operatorname{span}\{X,Y,[X,Y]\}$ and $T \in \operatorname{span}\{X,Y,[X,Y],[X,Z],[Y,Z]\}$ are C^1-smooth and smooth respectively.

Definition 2. Consider the initial value problem

$$\dot{\gamma}(t) = \sum_{i=1}^N y_i X_i(\gamma(t)),\ t \in [0,1], \quad \gamma(0) = x,$$

where the vector fields $X_1, \ldots, X_N$ are C^1-smooth. Then, for the point $y = \gamma(1)$, we write $y = \exp\Bigl(\sum_{i=1}^N y_i X_i\Bigr)(x)$.

Mappings $(y_1, \ldots, y_N) \mapsto \exp\Bigl(\sum_{i=1}^N y_i X_i\Bigr)(x)$ are called exponential mappings.

Definition 3. Consider $u \in \mathbb{M}$ and $(v_1, \dots, v_N) \in B_E(0,r)$, where $B_E(0,r)$ is a Euclidean ball in $\mathbb{R}^N$. Define a mapping $\theta_u : B_E(0,r) \to \mathbb{M}$ as follows:

$$\theta_u(v_1, \dots, v_N) = \exp\Bigl(\sum_{i=1}^{N} v_i X_i\Bigr)(u).$$

It is known that θ_u is a C^1-diffeomorphism if $0 < r \leq r_u$ for some $r_u > 0$. The collection $\{v_i\}_{i=1}^N$ is called *the normal coordinates* or *the coordinates of the* 1^{st} *kind* (*with respect to* $u \in \mathbb{M}$) of the point $v = \theta_u(v_1, \dots, v_N)$.

Proposition 1. *Given a point $p \in \mathbb{M}$, there exists a compactly embedded neighborhood $\mathcal{U} \Subset \mathbb{M}$ of p such that $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ for all $u \in \mathcal{U}$.*

Proof. This neighborhood exists due to theorems describing a size of a domain of existence of solution to ODE [22].

Definition 4. The *degree* $\deg X_k$ equals $\min\{m \mid X_k \in H_m\}$, $k = 1, \dots, N$.

Remark 2. The condition (2) of Definition 1 implies

$$[X_i, X_j](v) = \sum_{k:\, \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v), \tag{1}$$

$i, j = 1, \dots, N$, for all $v \in U$, where U is a neighborhood from Definition 1.

Theorem 1 ([31]). *Fix $u \in \mathbb{M}$. The collection*

$$\bar{c}_{ijk} = \begin{cases} c_{ijk}(u) \text{ of } (1) & \text{if } \deg X_i + \deg X_j = \deg X_k, \\ 0 & \text{otherwise} \end{cases}$$

constitutes a structure of a graded nilpotent Lie algebra.

We construct the Lie algebra $\mathfrak{g}^u$ of Theorem 1 as a graded nilpotent Lie algebra of vector fields $\{(\widehat{X}_i^u)'\}_{i=1}^N$ on $\mathbb{R}^N$ such that the exponential mapping $(x_1, \dots, x_N) \mapsto \exp\Bigl(\sum_{i=1}^N x_i (\widehat{X}_i^u)'\Bigr)(0)$ is the identity [5, 43]. As soon as this mapping is the identity, we have $x_i = \exp\bigl(x_i \bigl(\widehat{X}_i^u\bigr)'\bigr)(0)$. It follows that derivative at 0 of the left-hand side, equal to the vector e_i of the canonical basis in $\mathbb{R}^N$, coincides with the derivative of the right-hand side, equal to $(\widehat{X}_i^u)'(0)$. Thus the condition for the exponential mapping to be identical one implies the initial value

$$\bigl(\widehat{X}_i^u\bigr)'(0) = e_i \tag{2}$$

for the vector fields $\bigl(\widehat{X}_i^u\bigr)'$, $i = 1, \dots, N$.

By definition 3, we have $(\theta_u)_*\langle e_i\rangle = D\theta_u(0)\langle e_i\rangle = X_i(u)$. From here and (2) it follows

$$(\theta_u)_*\langle(\widehat{X}_i^u)'\rangle = X_i(u). \tag{3}$$

By the construction, the vector fields $\{(\widehat{X}_i^u)'\}_{i=1}^N$ satisfy

$$[(\widehat{X}_i^u)', (\widehat{X}_j^u)'] = \sum_{\deg X_k = \deg X_i + \deg X_j} c_{ijk}(u)(\widehat{X}_k^u)' \tag{4}$$

everywhere on $\mathbb{R}^N$.

Notation 1. We use the following standard notation: for each N-dimensional multi-index $\mu = (\mu_1, \dots, \mu_N)$, its *homogeneous norm* equals $|\mu|_h = \sum_{i=1}^N \mu_i \deg X_i$.

Definition 5. The graded nilpotent group $\mathbb{G}_u\mathbb{M}$ corresponding to the Lie algebra $\mathfrak{g}^u$ is said to be the *nilpotent tangent cone* of $\mathbb{M}$ at $u \in \mathbb{M}$. We construct $\mathbb{G}_u\mathbb{M}$ in $\mathbb{R}^N$ as a group algebra [43], that is, the exponential map from the Lie algebra $\mathfrak{g}^u$ to the graded nilpotent group $\mathbb{G}_u\mathbb{M}$ is identical:

$$\exp\left(\sum_{i=1}^N x_i(\widehat{X}_i^u)'\right)(0) = (x_1, \dots, x_N).$$

The group operation is defined by the Baker–Campbell–Hausdorff formula [43]: if

$$x = \exp\left(\sum_{i=1}^N x_i(\widehat{X}_i^u)'\right), \;\; y = \exp\left(\sum_{i=1}^N y_i(\widehat{X}_i^u)'\right)$$

then

$$x \cdot y = z = \exp\left(\sum_{i=1}^N z_i(\widehat{X}_i^u)'\right),$$

where

$$\begin{aligned}
z_i &= x_i + y_i, \quad \deg X_i = 1,\\
z_i &= x_i + y_i + \sum_{\substack{|e_l+e_j|_h=2,\\ l<j}} F^i_{e_l,e_j}(u)(x_l y_j - y_l x_j), \quad \deg X_i = 2,\\
z_i &= x_i + y_i + \sum_{\substack{|\mu+\beta|_h=k,\\ \mu>0,\, \beta>0}} F^i_{\mu,\beta}(u)x^\mu \cdot y^\beta\\
&= x_i + y_i + \sum_{\substack{|\mu+e_l+\beta+e_j|_h=k,\\ l<j}} G^i_{\mu,\beta,l,j}(u)x^\mu y^\beta(x_l y_j - y_l x_j), \quad \deg X_i = k.
\end{aligned} \tag{5}$$

With respect to this group operation, the basis vector fields $(\widehat{X}_i^u)' \in \mathfrak{g}^u$, $i = 1, \dots, N$, are left-invariant.

Theorem 2 ([15]). *If* $\{\frac{\partial}{\partial x_l}\}_{l=1}^N$ *is a standard basis in* $\mathbb{R}^N$ *then the* j*th coordinate of a vector field* $(\widehat{X}_i^u)'(x) = \sum_{j=1}^N z_i^j(u,x)\frac{\partial}{\partial x_j}$ *equals*

$$z_i^j(u,x) = \begin{cases} \delta_{ij} & \text{if} \quad j \leq \dim H_{\deg X_i}, \\ \sum_{\substack{|\mu+e_i|_h=\deg X_j, \\ \mu>0}} F^j_{\mu,e_i}(u)x^\mu & \text{if} \quad j > \dim H_{\deg X_i}, \end{cases}$$

$i = 1, \ldots, N.$

Using the exponential mapping θ_u, we can push forward the vector fields $(\widehat{X}_i^u)'$ onto $\mathcal{U} \subset \mathbb{M}$ as follows

$$[(\theta_u)_*\langle(\widehat{X}_i^u)'\rangle](\theta_u(x)) = D\theta_u(x)\langle(\widehat{X}_i^u)'(x)\rangle,$$

and obtain the vector fields $\widehat{X}_i^u = (\theta_u)_*(\widehat{X}_i^u)'$. Recall that $\widehat{X}_i^u(u) = X_i(u)$ by (3).

Definition 6. Associated to the Lie algebra $\{\widehat{X}_i^u\}_{i=1}^N$ at $u \in \mathbb{M}$, is a *local homogeneous group* $\mathcal{G}^u\mathbb{M}$. Define it so that the mapping θ_u is a *local group isomorphism* between some neighborhoods of the identity elements of the groups $\mathbb{G}_u\mathbb{M}$ and $\mathcal{G}^u\mathbb{M}$.

The canonical Riemannian structure on $\mathcal{G}^u\mathbb{M}$ is determined by the inner product at the identity element of $\mathcal{G}^u\mathbb{M}$ coinciding with that on $T_u\mathbb{M}$. The canonical Riemannian structure on the nilpotent tangent cone $\mathbb{G}_u\mathbb{M}$ is defined so that the local group isomorphism θ_u is an isometry.

Remark 3. If $\mathbb{M}$ is a Carnot manifold then its local tangent cone $\mathbb{G}_u\mathbb{M}$ is a stratified nilpotent graded Lie group.

Definition 7. If $\mathbb{M}$ is a Carnot manifold then its local homogeneous group $\mathcal{G}^u\mathbb{M}$ is called a *local Carnot group*.

Remark 4. Since $\{\widehat{X}_i^u\}_{i=1}^N$ are continuous [31] on $\mathcal{U}$ (not necessarily smooth) then formally the symbol $\exp\bigl(\sum_{i=1}^N x_i\widehat{X}_i^u\bigr)(v)$ is not well-defined. We define it to mean the point

$$\theta_u\left(\exp\left(\sum_{i=1}^N x_i(\widehat{X}_i^u)'\right)(\theta_u^{-1}(v))\right).$$

Proposition 2. *Given a point* $p \in \mathbb{M}$*, there exists a compactly embedded neighborhood* $\mathcal{U} \Subset \mathbb{M}$ *of* p *such that* $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ *for all* $u \in \mathcal{U}$.

Proof. By [31, Lemma 2.1.26] we have $\exp\bigl(\sum_{i=1}^N x_i\widehat{X}_i^u\bigr)(u) = \exp\bigl(\sum_{i=1}^N x_iX_i\bigr)(u)$. The proposition follows. □

Notation 2. Put $\widehat{\theta}_v^u(x_1,\ldots,x_N) = \exp\bigl(\sum_{i=1}^N x_i\widehat{X}_i^u\bigr)(v)$. For fixed $u, v \in \mathbb{M}$, it is a C^1-diffeomorphism of the ball $B_E(0, r_{u,v})$, $r_{u,v} > 0$, onto a neighborhood of v in $\mathbb{M}$.

Definition 8. Let $\mathbb{M}$ be a Carnot–Carathéodory space of a topological dimension N and a depth M, and $u \in \mathbb{M}$. For $x, v \in \mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ such that $x = \exp\bigl(\sum_{i=1}^{N} x_i \widehat{X}_i^u\bigr)(v)$, we define the quasimetric $d_\infty^u(x, v)$ as follows:

$$d_\infty^u(x, v) = \max_{i=1,\dots,N}\{|x_i|^{\frac{1}{\deg X_i}}\}.$$

Denote the ball $\{v \in \mathcal{G}^u\mathbb{M} : d_\infty^u(x, v) < r\}$ of radius r centered at x by $\mathrm{Box}^u(x, r)$.

Proposition 3. *Given a point $p \in \mathbb{M}$, there exists a compactly embedded neighborhood $\mathcal{U} \Subset \mathbb{M}$ of p satisfying*

$$\widehat{\theta}_v^u(B_E(0, r_{u,v})) \supset \mathcal{U} \quad \textit{for all} \quad u, v \in \mathcal{U}.$$

Thus, $d_\infty^u(v, w)$ is well-defined for all $u, v, w \in \mathcal{U}$.

Property 1. The quasimetric d_∞^u has the following properties:
1) $d_\infty^u(x, v) \geq 0$, and $d_\infty^u(x, v) = 0$ if and only if $x = v$;
2) $d_\infty^u(x, v) = d_\infty^u(v, x)$;
3) the value $d_\infty^u(x, v)$ is continuous with respect to each of its variables;
4) there exists a constant $Q = Q(\mathcal{U})$ such that the inequality

$$d_\infty^u(x, v) \leq Q(d_\infty^u(x, w) + d_\infty^u(w, v))$$

holds for every triple of points $x, w, v \in \mathcal{U}$. Here $\mathcal{U}$ is the same neighborhood as described in Propositions 1, 2 and 3.

Proof. The arguments explaining existence of this neighborhood are similar to those in the proof of Proposition 1. □

Definition 9. Given $u \in \mathcal{U}$ and $v \in \mathcal{U}$ such that $v = \exp\bigl(\sum_{i=1}^{N} v_i X_i\bigr)(u)$, define the mapping Δ_ε^u as

$$\Delta_\varepsilon^u(v) = \exp\Bigl(\sum_{i=1}^{N} v_i \varepsilon^{\deg X_i} X_i\Bigr)(u)$$

for $\varepsilon > 0$ such that the right-hand side of this relation is well-defined.

Definition 10. Let $\mathbb{M}$ be a Carnot–Carathéodory space of topological dimension N and depth M, and put $x = \exp\bigl(\sum_{i=1}^{N} x_i X_i\bigr)(u)$. Define the metric function $d_\infty(x, u)$:

$$d_\infty(x, u) = \max_{i=1,\dots,N}\{|x_i|^{\frac{1}{\deg X_i}}\}.$$

Denote the ball $\{x : d_\infty(x, u) < r\}$ of radius r centered at u by $\mathrm{Box}(x, r)$.

Theorem 3 ([29]). *Let $\mathbb{M}$ be a Carnot–Carathéodory space with $C^{1,\alpha}$-smooth basis vector fields, $\alpha \in [0, 1]$ (if $\alpha = 0$ then the vector fields are just C^1-smooth). For* *each* $w \in \mathbb{M}$*, there exists a neighborhood* $\mathcal{O} \ni w$*,* $\mathcal{O} \Subset \mathbb{M}$*, such that for* $u, x \in \mathcal{O}$ *the representations* $X_q(\Delta^u_\varepsilon x) = \sum_{p=1}^{N} a^u_{p,q}(\Delta^u_\varepsilon x)\widehat{X}^u_p(\Delta^u_\varepsilon x)$*, where*

$$a^u_{p,q}(\Delta^u_\varepsilon x) = \begin{cases} O(\varepsilon), & \deg X_p < \deg X_q, \\ \delta_{pq} + O(\varepsilon), & \deg X_p = \deg X_q, \\ O(\varepsilon^{\alpha+\deg X_p-\deg X_q}), & \deg X_p > \deg X_q \text{ and } \alpha > 0, \\ o(\varepsilon^{\deg X_p-\deg X_q}), & \deg X_p > \deg X_q \text{ and } \alpha = 0 \end{cases} \tag{6}$$

hold, $q = 1, \ldots, N$*, and the above estimates are uniform on* $\mathcal{O}$*.*

Theorem 3 implies immediately Gromov type Convergence Theorem [20] in the coordinates of the 1st kind.

Theorem 4 (see proof for C^1-case in [19]). *Let* $\mathbb{M}$ *be a Carnot–Carathéodory space with* C^1*-smooth basis vector fields. Given a point* $p \in \mathbb{M}$*, there exist a neighborhood* $\mathcal{O} \subset \mathbb{M}$ *of* p *and a positive number* $r > 0$ *such that the uniform convergence*

$$X^\varepsilon_i(x) = (\Delta^u_{\varepsilon^{-1}})_*\langle \varepsilon^{\deg X_i} X_i(\Delta^u_\varepsilon x)\rangle \to \widehat{X}^u_i(x)$$

as $\varepsilon \to 0$*,* $i = 1, \ldots, N$*, holds on* $\mathrm{Box}(u, r)$*,* $u \in \mathcal{O}$*, and this convergence is uniform in* $u \in \mathcal{O}$*.*

To make the understanding of the paper easier, we formulate all the assumptions on a neighborhood $\mathcal{U} \Subset \mathbb{M}$.

Assumption 1. To this end, we consider a compactly embedded neighborhood $\mathcal{U} \Subset \mathbb{M}$ such that
1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ for all $u \in \mathcal{U}$;
2) $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ for all $u \in \mathcal{U}$;
3) $\widehat{\theta}^u_v(B_E(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$;
4) $\mathcal{U} \Subset \mathcal{O}$, where $\mathcal{O}$ is a neighborhood from Theorem 3.

Remark 5. The existence of a neighborhood $\mathcal{U} \Subset \mathbb{M}$ is proved in Propositions 1, 2 and 3 and Theorem 3.

Theorem 3 has following corollaries.

Theorem 5 ([31]). *Let* $\mathbb{M}$ *be a Carnot–Carathéodory space with* C^1*-smooth basis vector fields. Assume that* $\mathcal{U} \Subset \mathbb{M}$ *is a compactly embedded neighborhood small enough such that*
1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ *for all* $u \in \mathcal{U}$*;*
2) $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ *for all* $u \in \mathcal{U}$*;*
3) $\widehat{\theta}^u_v(B_E(0, r_{u,v})) \supset \mathcal{U}$ *for all* $u, v \in \mathcal{U}$*;*
4) $\mathcal{U} \Subset \mathcal{O}$*, where* $\mathcal{O}$ *is a neighborhood from Theorem* 3.

The value d_∞ *is a quasimetric; i. e., for* $u, v, w \in \mathcal{U}$*, the generalized triangle inequality*

$$d_\infty(v, w) \le c(d_\infty(v, u) + d_\infty(u, w))$$

holds, where the constant $0 < c < \infty$ *depends only on* $\mathcal{U}$*.*

Proof. Note that by the choice of $\mathcal{U}$ its diameter $\operatorname{diam}\mathcal{U} = \sup\{d_\infty(u,v) \mid u,v \in \mathcal{U}\}$ is finite.

Our statement is equivalent to the following estimate of a diameter of a ball in d_∞: *in an arbitrary compact neighborhood* $\mathcal{W} \subset \mathbb{M}$, $\overline{\mathcal{W}} \subset \mathcal{U}$, *for each point* $u \in \mathcal{W}$ *and* $\varepsilon \le \operatorname{diam}\mathcal{W}$, *if* $\operatorname{Box}(u,\varepsilon) \subset \mathcal{W}$ *then we have* $\operatorname{diam}(\operatorname{Box}(u,\varepsilon)) \le L\varepsilon$, *where* L *depends only on* $\mathcal{U}$.

Assume the contrary: there exist a compact neighborhood $\mathcal{W} \subset \mathbb{M}$, $\overline{\mathcal{W}} \subset \mathcal{U}$, sequences $\{\varepsilon_k \in (0,\infty)\}_{k\in\mathbb{N}}$, $\{u_k \in \mathcal{W}\}_{k\in\mathbb{N}}$, $\{v_k \in \mathcal{W}\}_{k\in\mathbb{N}}$ and $\{w_k \in \mathcal{W}\}_{k\in\mathbb{N}}$ such that $d_\infty(u_k,v_k) = \varepsilon_k$ and $d_\infty(u_k,w_k) \le \varepsilon_k$ but $\operatorname{diam}\mathcal{W} \ge d_\infty(v_k,w_k) > k\varepsilon_k$. From the last inequality it follows immediately that $\varepsilon_k \to 0$ as $k \to \infty$.

Since $\overline{\mathcal{W}} \subset \mathcal{U} \subset \mathbb{M}$ is compact, we may assume without loss of generality that $u_k \to u_0 \in \overline{\mathcal{W}}$ as $k \to \infty$. Then $v_k \to u_0$ and $w_k \to u_0$ as $k \to \infty$.

By our assumptions on $\mathcal{U}$, $[\varepsilon^{\deg X_i} D\Delta^{u_k}_{\varepsilon^{-1}} X_i](x) - \widehat{X}^{u_k}_i(x) \to 0$ as $\varepsilon \to 0$ for $x \in \operatorname{Box}(u_0, Kr_0)$ uniformly in u_k, $i = 1,\ldots,N$, where $K = \max\{5, 5c^4\}$, c is such that $d^{u_k}_\infty(v,w) \le c(d^{u_k}_\infty(u,v) + d^{u_k}_\infty(u,w))$ for all $u,v,w \in \operatorname{Box}(u_0,Kr_0)$ and $k \in \mathbb{N}$ (see Theorem 4), $r_0 \le 1$ is such that $\operatorname{Box}(u_0,Kr_0) \subset \mathcal{U}$. Note that, $c < \infty$ since $c = c(u_k)$ depends continuously on values of $\{F^j_{\mu,\beta}(u_k)\}_{j,\mu,\beta}$, consequently, it depends continuously on u_k. Moreover, the choice of K implies the following:

1) For k big enough, we have that an integral curve of a vector field with constant coefficients connecting $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(w_k)$ and $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(v_k)$ in the local homogeneous group $\mathcal{G}^{u_k}\mathbb{M}$ lies in $\operatorname{Box}(u_0,Kr_0)$.

2) We may choose k as follows: $d_\infty(u_0,u_k) < r_0$ and the Riemannian distance between the integral curves corresponding to the collections $\{\widehat{X}^{u_k}_i\}_{i=1}^N$ and $\{(r_0{}^{-1}\varepsilon_k)^{\deg X_i} D\Delta^{u_k}_{r_0\varepsilon_k^{-1}}\langle X_i\rangle\}_{i=1}^N$ (with constant coefficients) that connect points $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(w_k)$ and $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(v_k)$, is less than r_0.

Fix $k \in \mathbb{N}$. Then

$$v_k = \exp\left(\sum_{i=1}^N \xi_i^k \varepsilon_k^{\deg X_i} X_i\right)(u_k), \quad w_k = \exp\left(\sum_{i=1}^N \eta_i^k \varepsilon_k^{\deg X_i} X_i\right)(u_k),$$

and $w_k = \exp\left(\sum_{i=1}^N \zeta_i(\varepsilon_k)\varepsilon_k^{\deg X_i} X_i\right)(v_k)$. Applying the mapping $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}$ to v_k and w_k we get

$$\begin{aligned}\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(w_k) &= \exp\left(\sum_{i=1}^N \zeta_i(\varepsilon_k)\varepsilon_k^{\deg X_i} D\Delta^{u_k}_{r_0\varepsilon_k^{-1}}\langle X_i\rangle\right)\left(\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(v_k)\right)\\ &= \exp\left(\sum_{i=1}^N \zeta_i(\varepsilon_k) r_0^{\deg X_i} X_i^{r_0^{-1}\varepsilon_k}\right)\left(\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(v_k)\right). \qquad (7)\end{aligned}$$

Note that, $d_\infty\big(u_k, \Delta^{u_k}_{r_0\varepsilon_k^{-1}}(v_k)\big) = r_0$ and $d_\infty\big(u_k, \Delta^{u_k}_{r_0\varepsilon_k^{-1}}(w_k)\big) \leq r_0$. In view of Theorem 3, we have

$$\big[(r_0{}^{-1}\varepsilon_k)^{\deg X_i} D\Delta^{u_k}_{r_0\varepsilon_k^{-1}}\langle X_i\rangle\big](x) = X_i^{r_0^{-1}\varepsilon_k} = \widehat{X}_i^{u_k}(x) + o(1), \quad i = 1, \dots, N,$$

where $o(1)$ is uniform in x and in u_k. Consequently, since $\dim\operatorname{span}\big\{\widehat{X}_i^{u_k}(x)\big\}_{i=1}^N = N$ at each $x \in \operatorname{Box}(u_0, r_0)$ and $\dim\operatorname{span}\big\{X_i^{r_0^{-1}\varepsilon_k}(x)\big\}_{i=1}^N = N$ for sufficiently small ε_k at each $x \in \operatorname{Box}(u_0, r_0)$, the Riemannian distance between $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(w_k)$ and $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(v_k)$ is bounded from above for all $k \in \mathbb{N}$ big enough (see (7)). Therefore, the coefficients $\zeta_i(\varepsilon_k)$, $i = 1, \dots, N$, are bounded from above for all $k \in \mathbb{N}$ big enough. The assumption $d_\infty(v_k, w_k) > k\varepsilon_k$ contradicts this conclusion. Thus there exists a constant $L = L(\mathcal{U})$ such that $\operatorname{diam}(\operatorname{Box}(u, \varepsilon)) \leq L\varepsilon$ for $u \in \mathcal{U}$. The statement follows. □

Theorem 6. *Let $\mathbb{M}$ be a Carnot–Carathéodory space with C^1-smooth basis vector fields. Assume that $\mathcal{U} \Subset \mathbb{M}$ is a compactly embedded neighborhood small enough such that*
1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ *for all* $u \in \mathcal{U}$;
2) $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ *for all* $u \in \mathcal{U}$;
3) $\widehat{\theta}^u_v(B_E(0, r_{u,v})) \supset \mathcal{U}$ *for all* $u, v \in \mathcal{U}$;
4) $\mathcal{U} \Subset \mathcal{O}$, *where* $\mathcal{O}$ *is a neighborhood from Theorem* 3.

Suppose that $\operatorname{Box}(u, \varepsilon) \subset \mathcal{U}$. *Then for any points* $v, w \in \operatorname{Box}(u, \varepsilon)$ *the following relation is valid:*

$$|d_\infty(v, w) - d^u_\infty(v, w)| = o(1) \cdot \varepsilon, \tag{8}$$

where $o(1) \to 0$ *as* $\varepsilon \to 0$, *and if* $u' \in \operatorname{Box}(u, \varepsilon)$ *then*

$$|d^{u'}_\infty(v, w) - d^u_\infty(v, w)| = o(1) \cdot \varepsilon,$$

where $o(1) \to 0$ *as* $\varepsilon \to 0$.

Moreover, $o(1)$ *is uniform in* $u \in \mathcal{W}$, *where* $\mathcal{W} \Subset \mathcal{U}$, *and in* $v, w \in \operatorname{Box}(u, \varepsilon) \Subset \mathcal{U}$.

Proof. Assume the opposite: (8) is not true. It means that there exist a neighborhood $\mathcal{W} \Subset \mathcal{U}$, a number $\eta > 0$, a sequence of positive numbers $\varepsilon_k \to 0$ as $k \to \infty$ and sequences of points $u_k \in \mathcal{W}$ and $v_k, w_k \in \operatorname{Box}(u_k, \varepsilon_k) \Subset \mathcal{U}$ such that

$$|d_\infty(v_k, w_k) - d^u_\infty(v_k, w_k)| \geq \eta \cdot \varepsilon_k, \tag{9}$$

By Theorem 5, $d_\infty(v_k, w_k) \leq S\varepsilon_k$, where S is the same for all $u_k, v_k, w_k \in \mathcal{U}$. By the choice, we consider points

$$v_k = \exp\Big(\sum_{i=1}^N p_i(k)\varepsilon_k^{\deg X_i} X_i\Big)(u_k) \quad \text{and} \quad w_k = \exp\Big(\sum_{i=1}^N q_i(k)\varepsilon_k^{\deg X_i} X_i\Big)(u_k)$$

in $\mathrm{Box}(u_k, \varepsilon_k)$. Then we have

$$w_k = \exp\left(\sum_{i=1}^{N} x_i(k)\varepsilon_k^{\deg X_i} X_i\right)(v_k) \quad\text{and}\quad w_k = \exp\left(\sum_{i=1}^{N} \hat{x}_i(k)\varepsilon_k^{\deg X_i} \widehat{X}_i^{u_k}\right)(v_k).$$

Similar to the proof of Theorem 5, we apply dilatations $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}$ for some fixed suitable $r_0 > 0$ to v_k and w_k. Note that, by the triangle inequality, the coefficients $\{x_i(k)\}_{i=1}^N$ and $\{\hat{x}_i(k)\}_{i=1}^N$ at scaled vector fields (in the expression similar to (7)) are totally bounded in k. Passing to subsequences, if necessary, we can assume that $u_k \to u_0 \in \mathcal{U}$, $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(v_k) \to v_0 \in \mathcal{U}$, $\Delta^{u_k}_{r_0\varepsilon_k^{-1}}(w_k) \to w_0 \in \mathcal{U}$ and $x_i(k) \to y_i$, $\hat{x}_i(k) \to \hat{y}_i$ as $k \to \infty$, $i = 1, \dots, N$.

By Theorem 3, the convergence $X_i^{r_0^{-1}\varepsilon_k} - \widehat{X}_i^{u_k}(x) \to 0$ as $k \to \infty, i = 1, \dots, N$, is uniform in $x \in \mathcal{U}$. By the continuous dependence of solutions to ODE on the right-hand side and initial data (see for instance [22]), there is a corresponding sequence of "scaled" integral lines converging as $k \to \infty$ to a curve which can be written as an integral line of the field $\sum_{i=1}^N y_i r_0^{\deg X_i} \widehat{X}_i^u$ in $\mathcal{G}^u\mathbb{M}$ with endpoints v_0, w_0. The last conclusion is justified by the property that in $\mathcal{G}^u\mathbb{M}$ the solution to ODE is unique since $\mathcal{G}^u\mathbb{M}$ is isomorphic to a neighborhood of the unity in $\mathbb{G}_u\mathbb{M}$ where the vector fields are smooth.

By the same reason, integral lines connecting points v_k and w_k in $\mathcal{G}^u\mathbb{M}$ converge to the integral line of the field $\sum_{i=1}^N \hat{y}_i r_0^{\deg X_i} \widehat{X}_i^u$ in $\mathcal{G}^u\mathbb{M}$ with endpoints v_0, w_0. Since this integral line is unique we have the equalities $y_i = \hat{y}_i$, $i = 1, \dots, N$.

It follows

$$\begin{aligned} &|d_\infty(v_k, w_k) - d_\infty^{u_k}(v_k, w_k)| \\ &\qquad = \varepsilon_k \cdot |\max_i\{|x_i(k)|^{\frac{1}{\deg X_i}}\} - \max_i\{|\hat{x}_i(k)|^{\frac{1}{\deg X_i}}\}| = o(1)\cdot\varepsilon_k, \end{aligned}$$

where $o(1) \to 0$ as $\varepsilon_k \to 0$. It contradicts (9).

The latter relation implies the second one: $|d_\infty^{u'}(v, w) - d_\infty^u(v, w)| = o(1)\cdot\varepsilon$, where $o(1) \to 0$ as $\varepsilon \to 0$, and $o(1)$ is uniform in $\mathcal{U}$. □

Definition 11. A curve $\gamma : [0, 1] \to \mathbb{M}$ which is absolutely continuous in the Riemannian sense is called *horizontal* if $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0, 1]$ with respect to the Lebesgue measure on $[0, 1]$.

Definition 12. Given $x, y \in \mathbb{M}$, the Carnot–Carathéodory distance $d_{cc}(x, y)$ is defined as

$$d_{cc}(x, y) = \inf\{\ell(\gamma) : \gamma : [0, 1] \to \mathbb{M},\ \dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}\},$$

where the length ℓ of each (horizontal) curve γ is calculated with respect to the Riemannian tensor on $\mathbb{M}$.

Definition 13. A horizontal curve in $\mathcal{G}^u\mathbb{M}$ is defined similarly: here we require that $\dot{\gamma}(t) \in \operatorname{span}\{\widehat{X}^u_1(\gamma(t)), \dots, \widehat{X}^u_{\dim H_1}(\gamma(t))\} = \widehat{H}^u_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0,1]$ with respect to the Lebesgue measure on $[0,1]$.

Definition 14. For $x, y \in \mathcal{G}^u\mathbb{M}$ the Carnot–Carathéodory distance $d^u_{cc}(x,y)$ is defined as

$$d^u_{cc}(x,y) = \inf\{\ell^u(\gamma) : \gamma : [0,1] \to \mathcal{G}^u\mathbb{M},\ \dot{\gamma}(t) \in \widehat{H}^u_{\gamma(t)}\mathbb{M}\},$$

where the length ℓ^u of each (horizontal) curve γ is calculated with respect to the Riemannian tensor on $\mathcal{G}^u\mathbb{M}$.

Theorem 7 ([3]). *Let $\mathbb{M}$ be a Carnot manifold with C^1-smooth basis vector fields. Fix the point $u_0 \in \mathbb{M}$. Let $X_1, \dots, X_{\dim H_1}$ be a basis in H_1. Then there is a neighborhood $U(u_0)$ such that for every point $u \in U(u_0)$ an element $v \in U(u_0)$ can be represented as follows*

$$v = \exp(a_L X_{j_L}) \circ \dots \circ \exp(a_2 X_{j_2}) \circ \exp(a_1 X_{j_1})(u), \tag{10}$$

where $1 \le j_i \le \dim H_1$, $i = 1, \dots, L$, $L \in \mathbb{N}$, $|a_i| \le c_2\, d_\infty(u,v)$, constants L and c_2 are independent of u and v.

Theorem 8 ([3]; see [11,44] for smooth case). 1) *Given a Carnot manifold $\mathbb{M}$ and $x \in \mathbb{M}$, there exists a neighborhood $\mathcal{W}$ of a point x such that every pair of points $u, v \in \mathcal{W}$ can be joined by a rectifiable absolutely continuous horizontal curve γ constituted of at most L segments of integral lines of given horizontal fields where L is independent of the choice of points $u, v \in \mathcal{W}$.*

2) *Every two points of $\mathbb{M}$ can be joined by a horizontal curve. Thus, Carnot–Carathéodory metric is well-defined on Carnot manifolds with C^1-smooth basis vector fields.*

Denote the ball of radius r in d_{cc} (d^u_{cc}) centered at x by $B_{cc}(x,r)$ ($B^u_{cc}(x,r)$).

3 Main Results

The goal of this section is to prove the following Local Approximation Theorem for Carnot–Carathéodory metrics.

Theorem 9. *Let $\mathbb{M}$ be a Carnot manifold with C^1-smooth basis vector fields. Assume that $\mathcal{U} \Subset \mathbb{M}$ is a compactly embedded neighborhood small enough such that*
1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ *for all* $u \in \mathcal{U}$;
2) $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ *for all* $u \in \mathcal{U}$;
3) $\widehat{\theta}^u_v(B_E(0, r_{u,v})) \supset \mathcal{U}$ *for all* $u, v \in \mathcal{U}$;
4) $\mathcal{U} \Subset \mathcal{O}$, *where $\mathcal{O}$ is a neighborhood from Theorem* 3;
5) $\mathcal{U} \subset U$, *where U is a neighborhood from Theorem* 7.

Then points $u, u', v \in \mathcal{U}$ possess following properties.

For any points $v, w \in B_{cc}(u,\varepsilon)$ *the inequality holds:*

$$|d_{cc}(v,w) - d^u_{cc}(v,w)| \le o(1)\cdot\varepsilon;$$

for any points $u', v, w \in B_{cc}(u,\varepsilon)$ *the inequality holds:*

$$|d^u_{cc}(v,w) - d^{u'}_{cc}(v,w)| \le o(1)\cdot\varepsilon.$$

Both estimates are uniform in all points $u \in \mathcal{U}$ *and* $u', v, w \in B_{cc}(u,\varepsilon)$.

Remark 6. Theorem 9 (6) implies Mitchell-Gromov type theorem [7,35] ([47]) concerning the tangent cones of Carnot manifolds (Carnot–Carathéodory spaces).

First, we prove an auxiliary assertion of independent interest.

Theorem 10. *Let* $\mathbb{M}$ *be a Carnot–Carathéodory space with* C^1*-smooth basis vector fields. Then for each point of* $\mathbb{M}$*, there exists a sufficiently small neighborhood* $\mathcal{U} \Subset \mathbb{M}$ *such that*
1) $\theta_u(B_E(0,r_u)) \supset \mathcal{U}$ *for all* $u \in \mathcal{U}$*;*
2) $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ *for all* $u \in \mathcal{U}$*;*
3) $\widehat{\theta}^u_v(B_E(0,r_{u,v})) \supset \mathcal{U}$ *for all* $u, v \in \mathcal{U}$*;*
4) $\mathcal{U} \Subset \mathcal{O}$*, where* $\mathcal{O}$ *is a neighborhood from Theorem* 3.

Moreover, this neighborhood $\mathcal{U}$ *possesses the following property: for* $u, v \in \mathcal{U}$ *and* $w = \gamma(1)$ *and* $\widehat{w} = \widehat{\gamma}(1)$*, where* $\gamma, \widehat{\gamma} : [0,1] \to \mathbb{M}$ *are absolutely continuous (in the classical sense) curves contained in* $\operatorname{Box}(u,\varepsilon)$ *such that*

$$\dot\gamma(t) = \sum_{i=1}^N b_i(t)X_i(\gamma(t)),\ \ \gamma(0) = v,\ \ \text{and}\ \ \dot{\widehat{\gamma}}(t) = \sum_{i=1}^N b_i(t)\widehat{X}^u_i(\gamma(t)),\ \ \widehat{\gamma}(0) = v,$$

and each measurable function $b_i(t)$ *meets the property*

$$\int_0^1 |b_i(t)|\,dt < S\varepsilon^{\deg X_i}, \tag{11}$$

$S < \infty$*,* $i = 1,\dots,N$*, we have*

$$\max\{d_\infty(w,\widehat{w}), d^u_\infty(w,\widehat{w})\} \le o(1)\cdot\varepsilon,$$

with $o(1)$ *to be uniform in* $u, v \in \mathcal{U}$ *and all collections of functions* $\{b_i(t)\}_{i=1}^N$ *with the property* (11).

Proof. The existence of $\mathcal{U}$ is provided by Propositions 1, 2 and 3 and Theorem 3. Consider the normal coordinates θ_u^{-1} with respect to the point u. To simplify notation, we set the field $D\theta_u^{-1}\langle X_i\rangle$ to be equal Y_i and denote $D\theta_u^{-1}\langle\widehat{X}^u_i\rangle$ by $\widehat{Y}^u_i$, $i = 1,\dots,N$. We also set $\gamma_u(t) = \theta_u^{-1}(\gamma(t))$ and $\widehat{\gamma}_u(t) = \theta_u^{-1}(\widehat{\gamma}(t))$. Let us rewrite the tangent vector to the curve γ_u at a point $\gamma_u(t)$ as

$$\sum_{i=1}^{N} b_i(t)Y_i(\gamma_u(t)) = \sum_{i=1}^{N} b_i(t)\widehat{Y}_i^u(\gamma_u(t)) + \sum_{i=1}^{N} b_i(t)\Big(\sum_{j=1}^{N}[a_{i,j}^u(\gamma_u(t)) - \delta_{ij}]\widehat{Y}_j^u(\gamma_u(t))\Big)$$

The tangent vector to the curve $\delta_u(t) = \widehat{\gamma}_u(1) + \gamma_u(t) - \widehat{\gamma}_u(t)$, which joines $\gamma_u(1)$ and $\widehat{\gamma}_u(1)$, can be written as

$$\begin{aligned}\dot{\delta}_u(t) &= \sum_{i=1}^{N} b_i(t)[\widehat{Y}_i^u(\gamma_u(t)) - \widehat{Y}_i^u(\widehat{\gamma}_u(t))] \\ &\quad + \sum_{i=1}^{N} b_i(t)\Big(\sum_{j=1}^{N}[a_{i,j}^u(\gamma_u(t)) - \delta_{ij}]\widehat{Y}_j^u(\gamma_u(t))\Big) \\ &= \sum_{i=1}^{N} b_i(t)[\widehat{Y}_i^u(\gamma_u(t)) - \widehat{Y}_i^u(\widehat{\gamma}_u(t))] \\ &\quad + \sum_{j=1}^{N}\Big(\sum_{i=1}^{N} b_i(t)[a_{i,j}^u(\gamma_u(t)) - \delta_{ij}]\Big)\widehat{Y}_j^u(\gamma_u(t)),\end{aligned}$$

where the coefficients $\{a_{i,j}^u\}_{i,j=1}^N$ coincide with those in (6). Taking into account the coordinate representations of the vector fields $\{\widehat{Y}_i^u\}_{i=1}^N$ (see Theorem 2) we obtain the following ODE system for $1 \leq k \leq \dim H_1$:

$$[\dot{\delta}_u]_k(t) = \sum_{j=1}^{\dim H_1} \sum_{i=1}^{N} b_i(t)[a_{i,j}^u(\gamma_u(t)) - \delta_{ij}]\delta_{kj}.$$

Applying said above and the estimate $|a_{i,j}^u(\gamma_u(t)) - \delta_{ij}| = o(1)$ for $j \leq \dim H_1$ (see Theorem 3), we get

$$|[\delta_u]_k(t) - [\delta_u]_k(0)| \leq \int_0^1 \sum_{j=1}^{\dim H_1} \sum_{i=1}^{N} |b_i(\tau)|[|a_{i,j}^u(\gamma_u(\tau)) - \delta_{ij}|]\,d\tau \leq o(1)\cdot P_1\varepsilon$$

$t \in [0,1]$, for some constant $P_1 < \infty$. Next, for $\dim H_1 < k \leq \dim H_2$ we have

$$\begin{aligned}[\dot{\delta}_u]_k(t) = &\sum_{i=1}^{\dim H_1} \sum_{|\mu|_h=1} b_i(t)F_{\mu,e_i}^k(u)[\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu] \\ &+ \sum_{j=1}^{\dim H_1}\Big(\sum_{i=1}^{N} b_i(t)[a_{i,j}^u(\gamma_u(t)) - \delta_{ij}]\Big)\sum_{|\mu|_h=1} F_{\mu,e_j}^k(u)\gamma_u(t)^\mu \\ &+ \sum_{j=\dim H_1+1}^{\dim H_2} \sum_{i=1}^{N} b_i(t)[a_{i,j}^u(\gamma_u(t)) - \delta_{ij}]\delta_{kj}.\end{aligned}$$

As in the preceding case, taking into account the estimate obtained above and fact that, by Theorem 3, in the last sum $|a^u_{i,j}(\gamma_u(t))| = o(\varepsilon)$ for $\deg X_i = 1$ and $|a^u_{i,j}(\gamma_u(t)) - \delta_{ij}| = o(1)$ for $\deg X_i \geq 2$, we obtain

$$|[\delta_u]_k(t) - [\delta_u]_k(0)| \leq o(1) \cdot P_2\varepsilon^2, \quad P_2 < \infty, \quad t \in [0,1].$$

Arguing by induction, we suppose that we have proved the inequality

$$|[\delta_u]_k(t) - [\delta_u]_k(0)| \leq o(1) \cdot P_l\varepsilon^l,$$

$P_l < \infty$, for $\deg X_k = l$, $l = 3, \dots, Q-1$, $t \in [0,1]$. Then, using this assumption and the preceding estimates, we obtain the following relation for $\dim H_{Q-1} < k \leq \dim H_Q$:

$$\begin{aligned}
[\dot{\delta_u}]_k(t) = &\sum_{i=1}^{\dim H_{Q-1}} \sum_{|\mu|_h = Q - \deg X_i} b_i(t) F^k_{\mu,e_i}(u)[\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu] \\
&+ \sum_{j=1}^{\dim H_{Q-1}} \Big(\sum_{i=1}^{N} b_i(t)[a^u_{i,j}(\gamma_u(t)) - \delta_{ij}]\Big) \sum_{|\mu|_h = Q - \deg X_j} F^k_{\mu,e_j}(u)\gamma_u(t)^\mu \\
&+ \sum_{j=\dim H_{Q-1}+1}^{\dim H_Q} \sum_{i=1}^{N} b_i(t)[a^u_{i,j}(\gamma_u(t)) - \delta_{ij}]. \qquad (12)
\end{aligned}$$

By our assumption, we deduce in the first sum for $\deg X_i \geq 2$ that

$$|\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu| \leq o(1) \cdot P_\mu \varepsilon^{|\mu|_h}.$$

Furthermore, we have for $\mu = e_q$, where $\deg X_q = Q - 1$, that

$$|\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu| \leq o(1) \cdot P_\mu \varepsilon^{|\mu|_h} = o(1) \cdot P_\mu \varepsilon^{Q-1}.$$

We represent an arbitrary multi-index μ with $|\mu|_h = Q - 1$ as $\mu = \mu_1 + \mu_2$, where $\mu_1 > 0$ and $\mu_2 > 0$. We infer

$$\begin{aligned}
&|\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu| = |\gamma_u(t)^{\mu_1}\gamma_u(t)^{\mu_2} - \widehat{\gamma}_u(t)^{\mu_1}\widehat{\gamma}_u(t)^{\mu_2}| \\
&= |\gamma_u(t)^{\mu_1}\gamma_u(t)^{\mu_2} - (\gamma_u(t)^{\mu_1} + o(1) \cdot p_{\mu_1}\varepsilon^{|\mu_1|_h})(\gamma_u(t)^{\mu_2} + o(1) \cdot p_{\mu_2}\varepsilon^{|\mu_2|_h})| \\
&\qquad = o(1) \cdot |p_{\mu_2}||\gamma_u(t)^{\mu_1}|\varepsilon^{|\mu_2|_h} + o(1) \cdot |p_{\mu_1}||\gamma_u(t)^{\mu_2}|\varepsilon^{|\mu_1|_h} \\
&\qquad\qquad + o(1) \cdot |p_{\mu_1}||p_{\mu_2}|\varepsilon^{|\mu|_h} \leq o(1) \cdot P_\mu \varepsilon^{|\mu|_h} = o(1) \cdot P_\mu \varepsilon^{Q-1}.
\end{aligned}$$

Next, in the last sum in (12) we have $|a_{i,j}(\gamma_u(t)) - \delta_{ij}| \leq o(1) \cdot C_{ij}\varepsilon^{Q - \deg X_i}$ for $\deg X_i < Q$. If $\deg X_i \geq Q$ then $|a_{i,j}(\gamma_u(t)) - \delta_{ij}| \leq o(1)$. Finally, in the middle sum in (12) for $\deg X_j > \deg X_i$ we obtain

$$|a_{i,j}(\gamma_u(t))\gamma_u(t)^\mu| \leq o(1) \cdot K_{ij\mu}\varepsilon^{Q - \deg X_i}.$$

For $\deg X_j \le \deg X_i$ we have $|[a_{i,j}(\gamma_u(t)) - \delta_{ij}]\gamma_u(t)^\mu| \le o(1)\cdot K_{ij\mu}\varepsilon^{Q-\deg X_j}$. This implies the following estimate for $|[\delta_u]_k(t) - [\delta_u]_k(0)|$:

$$\begin{aligned}
|[\delta_u]_k(t) - [\delta_u]_k(0)| &\le o(1)\cdot S_1\varepsilon^{Q-1}\int_0^1 \sum_{i=1}^{\dim H_{Q-1}} |b_i(\tau)|\,d\tau \\
&+ o(1)\cdot S_2 \sum_{i=\dim H_1+1}^{\dim H_{Q-1}} \varepsilon^{Q-\deg X_i}\int_0^1 |b_i(\tau)|\,d\tau \\
&+ o(1)\cdot S_3 \sum_{j=1}^{\dim H_{Q-1}} \varepsilon^{Q-\deg X_j}\int_0^1 \sum_{i:\deg X_i\ge \deg X_j} |b_i(\tau)|\,d\tau \\
&+ o(1)\cdot S_4 \sum_{j=1}^{\dim H_{Q-1}} \sum_{i:\deg X_i<\deg X_j} \varepsilon^{Q-\deg X_i}\int_0^1 |b_i(\tau)|\,d\tau \\
&+ o(1)\cdot S_5 \sum_{i:\deg X_i<Q} \varepsilon^{Q-\deg X_i}\int_0^1 |b_i(\tau)|\,d\tau \\
&+ S_6\cdot o(1)\cdot \int_0^1 \sum_{i:\deg X_i\ge Q} |b_i(\tau)|\,d\tau \le o(1)\cdot P_Q\varepsilon^Q.
\end{aligned}$$

Let us estimate $d_\infty^u(\gamma_u(1), \widehat{\gamma}_u(1))$. Recall that $\delta_u(t) = \widehat{\gamma}_u(1) + \gamma_u(t) - \widehat{\gamma}_u(t)$, $[\delta_u(1)]_k - [\delta_u(0)]_k = [\gamma_u(1)]_k - [\widehat{\gamma}_u(1)]_k$, $k = 1,\dots,N$, and the coordinates of $\{[\gamma_u(1)]_k\}_{k=1}^N$ and $\{[\widehat{\gamma}_u(1)]_l\}_{l=1}^N$ with respect to zero in the system $\{\widehat{Y}_i^u\}$ coincide with Cartesian ones. To obtain our estimate, we apply the group operation in $\mathbb{G}_u\mathbb{M}$: if $\gamma_u(1) = \exp\Bigl(\sum_{i=1}^N w_i\widehat{Y}_i^u\Bigr)(\widehat{\gamma}_u(1))$ then

$$\begin{aligned}
w_i &= [\delta_u(1)]_i \\
&+ \sum_{|\mu+\beta+e_k+e_l|_h=\deg X_i} G^i_{\mu,\beta,e_k,e_l}(u)\gamma_u(1)^\mu\widehat{\gamma}_u(1)^\beta([\gamma_u(1)]_k[\widehat{\gamma}_u(1)]_l - [\gamma_u(1)]_l[\widehat{\gamma}_u(1)]_k).
\end{aligned}$$

We have $|[\delta_u(1)]_i| \le o(1)\cdot P_{\deg X_i}\varepsilon^{\deg X_i}$ and

$$\begin{aligned}
&|[\gamma_u(1)]_k[\widehat{\gamma}_u(1)]_l - [\gamma_u(1)]_l[\widehat{\gamma}_u(1)]_k| \\
&\quad\le |[\gamma_u(1)]_k|\cdot|[\widehat{\gamma}_u(1)]_l - [\gamma_u(1)]_l| + |[\gamma_u(1)]_l|\cdot|[\gamma_u(1)]_k - [\widehat{\gamma}_u(1)]_k|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|w_i| &\le o(1)\cdot P_{\deg X_i}\varepsilon^{\deg X_i} + \sum_{|\mu+\beta+e_k+e_l|_h=\deg X_i} o(1)\cdot S_{i,\mu,\beta,e_k,e_l}\varepsilon^{|\mu+\beta|_h}\varepsilon^{\deg X_k+\deg X_l} \\
&= o(1)\cdot W_i\varepsilon^{\deg X_i}.
\end{aligned}$$

This implies $d_\infty^u(\gamma_u(1), \widehat{\gamma}_u(1)) \leq o(1) \cdot L\varepsilon$. Hence, the same estimate holds for $w = \gamma(1)$ and $\widehat{w} = \widehat{\gamma}(1)$. Thus, $d_\infty(\gamma(1), \widehat{\gamma}(1)) \leq L \cdot o(\varepsilon)$. Since all the coefficients at $o(1)$ and at $o(\varepsilon)$ are uniform on $\mathcal{U}$ then we can write

$$d_\infty(\gamma(1), \widehat{\gamma}(1)) \leq o(1) \cdot \varepsilon.$$

The theorem follows.

Corollary 1. *Let $\mathbb{M}$ be a Carnot–Carathéodory space with C^1-smooth basis vector fields. For each point of $\mathbb{M}$, there exists a sufficiently small neighborhood $\mathcal{U} \Subset \mathbb{M}$ such that*
1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ *for all* $u \in \mathcal{U}$;
2) $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ *for all* $u \in \mathcal{U}$;
3) $\widehat{\theta}^u_v(B_E(0, r_{u,v})) \supset \mathcal{U}$ *for all* $u, v \in \mathcal{U}$;
4) $\mathcal{U} \Subset \mathcal{O}$, *where* $\mathcal{O}$ *is a neighborhood from Theorem* 3.

Moreover, for $u, u', v \in \mathcal{U}$ such that $d_\infty(u, u') \leq C\varepsilon$ for some $0 < C < \infty$, and points $w = \gamma(1)$ and $\widehat{w} = \widehat{\gamma}(1)$, where $\gamma, \widehat{\gamma} : [0, 1] \to \mathbb{M}$ are absolutely continuous (in the classical sense) curves lying in $\mathrm{Box}(u, \varepsilon)$ *such that*

$$\dot{\gamma}(t) = \sum_{i=1}^{N} b_i(t)\widehat{X}_i^{u'}(\gamma(t)),\ \gamma(0) = v,\ \textit{and}\ \dot{\widehat{\gamma}}(t) = \sum_{i=1}^{N} b_i(t)\widehat{X}_i^{u}(\gamma(t)),\ \widehat{\gamma}(0) = v,$$

and (11) *holds, we have* $\max\{d_\infty^{u'}(w, \widehat{w}), d_\infty^u(w, \widehat{w})\} \leq o(1) \cdot \varepsilon$, *with* $o(1)$ *to be uniform in* $u, u', v \in \mathcal{U}$ *and all collections of functions* $\{b_i(t)\}_{i=1}^N$ *with the property* (11).

Theorem 11 (Ball–Box Theorem; see [6,7,36,40,49] for comparison). *Let $\mathbb{M}$ be a Carnot manifold with C^1-smooth basis vector fields. Assume that $\mathcal{U} \Subset \mathbb{M}$ is a compactly embedded neighborhood small enough such that*
1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ *for all* $u \in \mathcal{U}$;
2) $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ *for all* $u \in \mathcal{U}$;
3) $\widehat{\theta}^u_v(B_E(0, r_{u,v})) \supset \mathcal{U}$ *for all* $u, v \in \mathcal{U}$;
4) $\mathcal{U} \Subset \mathcal{O}$, *where* $\mathcal{O}$ *is a neighborhood from Theorem* 3;
5) $\mathcal{U} \Subset U$, *where* U *is a neighborhood from Theorem* 7.

Then the shape of a sufficiently small ball B_{cc} is comparable with a parallelepiped in the sense that, for a compact neighborhood $\mathcal{U} \Subset \mathbb{M}$, there exist constants $0 < C_1 \leq C_2 < \infty$ and $r_0 > 0$ independent of $x \in \mathcal{U}$ and such that

$$\mathrm{Box}(x, C_1 r) \subset B_{cc}(x, r) \subset \mathrm{Box}(x, C_2 r)$$

for any $r \in (0, r_0)$.

Proof. An estimate $d_{cc}(x, y) \leq C_1 d_\infty(x, y)$ for points x, y from a compact neighborhood $\mathcal{U} \Subset \mathbb{M}$ follows from Theorem 7.

Our next goal is to prove the converse estimate. Assuming the contrary we have a compact neighborhood $\mathcal{U} \Subset \mathbb{M}$ and sequences of points $x_l, y_l \in \mathcal{U}$

such that $d_\infty(x_l, y_l) \geq l d_{cc}(x_l, y_l)$. In this case we have $d_\infty(x_l, y_l) \to 0$ as $l \to \infty$ since otherwise, for some subsequences x_{l_n} and y_{l_n}, we have simultaneously $d_{cc}(x_{l_n}, y_{l_n}) \to 0$ as $n \to \infty$, and $d_\infty(x_{l_n}, y_{l_n}) \geq \alpha > 0$ for all $n \in \mathbb{N}$ what is impossible. We can assume also that $x_l \to x \in \overline{\mathcal{U}}$ as $l \to \infty$ and $x_l \neq y_l$. Setting $d_\infty(x_l, y_l) = \varepsilon_l$ we have $d_\infty\big(x_l, \Delta^{x_l}_{r\varepsilon_l^{-1}} y_l\big) = r$ where $r > 0$ is a normalizing factor. Let $\gamma : [0,1] \to \mathbb{M}$ be a Lipschitz horizontal path connecting x_l and y_l such that its length equals $d_{cc}(x_l, y_l)$ [7]. Then the length $\ell(\Gamma_l)$ of the curve Γ_l: $[0,1] \ni t \to \Delta^{x_l}_{r\varepsilon_l^{-1}}(\gamma(t))$ equals $\frac{r}{\varepsilon_l} d_{cc}(x_l, y_l)$ where the length $\ell(\Gamma_l)$ is measured with respect to the frame $\{X_i^{\varepsilon_l/r}\}$ with push-forward Riemannian tensor. Indeed, if $\dot\gamma(t) = \sum_{i=1}^{\dim H_1} \gamma_i(t) X_j(\gamma(t))$ a. e. in $t \in [0,1]$, then $\dot\Gamma_l(t) = \frac{r}{\varepsilon_l} \sum_{i=1}^{\dim H_1} \gamma_i(t) X^{\varepsilon_l/r}(\Gamma_l(t))$ a. e. in $t \in [0,1]$. This implies the equality $\ell(\Gamma_l) = \frac{r}{\varepsilon_l} d_{cc}(x_l, y_l)$. Theorem 3 impies that the vectors $X_i^{\varepsilon_l/r}$ are close to the corresponding nilpotentized vector fields $\widehat{X}_i^{x_l}$, $i = 1, \ldots, \dim H_1$, consequently, the Riemannian distance $\rho\big(x_l, \Delta^{x_l}_{r\varepsilon_l^{-1}} y_l\big) \to 0$ as $l \to \infty$:

$$\rho\big(x_l, \Delta^{x_l}_{r\varepsilon_l^{-1}} y_l\big) \leq \ell(\Gamma_l) = \frac{r}{\varepsilon_l} d_{cc}(x_l, y_l) \leq C r l^{-1} \frac{d_\infty(x_l, y_l)}{\varepsilon_l} = C r l^{-1}$$

where the constant C is independent of l.

It is in a contradiction with $d_\infty\big(x_l, \Delta^{x_l}_{r\varepsilon_l^{-1}} y_l\big) = r$ for all $l \in \mathbb{N}$. □

Remark 7. Theorem 11 implies Mitchell type theorem [35] on Hausdorff dimension of a Carnot manifold $\mathbb{M}$.

Now we start to prove the main result of the paper.

Proof (Proof of Theorem 9). We prove the first statement. The proof of the second one is similar.

Consider a neighborhood $\mathcal{U} \subset \mathbb{M}$ described in Theorem 11. By Theorems 11 and 5, the relation $d_\infty(v, w) \leq \mathcal{N}\varepsilon$ is valid for some $\mathcal{N} < \infty$. Consider a horizontal (with respect to $\mathbb{M}$) curve γ with the natural parameterization joining v and w (i. e., $\gamma(0) = v$ and $\gamma(1) = w$) such that $\ell(\gamma) = d_{cc}(v, w)$ (see [7] for the proof of existence). In view of local equivalence of d_{cc} and d_∞ (see Theorem 11), triangle inequality for d_{cc} and generalized triangle inequality for d_∞ (see Theorem 5), all points of γ belong to $\mathrm{Box}(u, Q\varepsilon)$, $Q < \infty$. Indeed it is easy to show that, for every point $\gamma(t)$, we have

$$d_{cc}(v, \gamma(t)) = \ell(\gamma_t) \leq d_{cc}(v, w) \leq P\varepsilon,$$

where γ_t is the part of the curve γ joining v and $\gamma(t)$. We write the tangent vector to γ at $\gamma(t)$ as

$$\sum_{i=1}^{\dim H_1} b_i(t) X_i(\gamma(t))$$

and consider a curve $\widehat{\gamma}$ such that

$$\dot{\widehat{\gamma}}(t) = \sum_{i=1}^{\dim H_1} b_i(t)\widehat{X}_i^u(\widehat{\gamma}(t)), \quad \widehat{\gamma}(0) = v.$$

Note that this ODE has a solution on the entire interval [0, 1] for sufficiently small $\varepsilon > 0$ since a horizontal path on a Carnot group is well defined by its horizontal components $b_1(t), \dots, b_{\dim H_1}$ (see for instance [51]). Note that

$$\int_0^1 \sum_{i=1}^{\dim H_1} |b_i(t)|\, dt \le C\varepsilon,$$

where $C < \infty$ depends only on $\mathcal{U} \Subset \mathbb{M}$. Therefore the lengths of the curves γ and $\widehat{\gamma}$ differ by a quantity comparable with $o(1) \cdot \varepsilon$ (since Theorem 3 implies that Riemann tensors on $\mathbb{M}$ and $\mathcal{G}^u\mathbb{M}$ differ by a quantity comparable with $o(1)$). Next, the curve $\widehat{\gamma}$ lies in the sub-Riemannian ball $\operatorname{Box}(u, \widehat{Q}\varepsilon)$, $\widehat{Q} < \infty$. Indeed, it suffices to note that since the distance $d_{cc}^u(v, \widehat{\gamma}(t))$ does not exceed a value comparable with ε, the distance $d_\infty^u(v, \widehat{\gamma}(t))$ also does not exceed a value comparable with ε for all $t \in [0, 1]$. Furthermore, $d_\infty^u(u, \widehat{\gamma}(t)) = d_\infty(u, \widehat{\gamma}(t))$, $t \in [0, 1]$. Thus, all assumptions of Theorem 10 hold.

Consequently, Theorem 10 implies

$$d_{cc}^u(v, w) \le \ell(\gamma) + L \cdot o(1) \cdot \varepsilon \le d_{cc}(v, w) + o(1) \cdot L\varepsilon.$$

Similarly, if we have a horizontal (with respect to $\mathcal{G}^u\mathbb{M}$) curve $\widehat{\gamma}$, joining v and w such that

$$\dot{\widehat{\gamma}}(t) = \sum_{i=1}^{\dim H_1} \widehat{b}_i(t)X_i(\widehat{\gamma}(t)),$$

$t \in [0, 1]$, and $\widehat{\ell}(\widehat{\gamma}) = d_{cc}^u(v, w)$, then there exists a curve γ meeting the equation

$$\dot{\gamma}(t) = \sum_{i=1}^{\dim H_1} \widehat{b}_i(t)X_i(\gamma(t)),$$

$t \in [0, 1]$ [46]. Since both curves enjoy conditions of Theorem 10 it follows that

$$d_{cc}(v, w) \le d_{cc}^u(v, w) + o(1) \cdot \mathcal{L}\varepsilon.$$

The applied arguments imply that $\mathcal{L}$ has the same properties as L in Theorem 10. Thus, $|d_{cc}(v, w) - d_{cc}^u(v, w)| = o(1) \cdot \varepsilon$ and the theorem follows. □

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On curvature-type invariants for natural mechanical systems on sub-Riemannian structures associated with a principle G-bundle

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Abstract The Jacobi curve of an extremal of an optimal control problem is a curve in a Lagrangian Grassmannian defined up to a symplectic transformation and containing all information about the solutions of the Jacobi equations along this extremal. For parametrized curves in Lagrange Grassmannians satisfying very general assumptions, the canonical bundle of moving frames and the complete system of symplectic invariants, called curvature maps, were constructed. The structural equation for a canonical moving frame of the Jacobi curve of an extremal can be interpreted as the normal form for the Jacobi equation along this extremal and the curvature maps can be seen as the "coefficients"of this normal form. In the present paper, we focus on the curvature maps for an optimal control problem of a natural mechanical system on a sub-Riemannian structure on a principle connection of a principle G-bundles with one dimensional fibers over a Riemannian manifold. We express the curvature maps in terms of the curvature tensor of the base Riemannian manifold and the curvature form and the potential.

1 Introduction

Let $\mathcal{D}$ be a vector distribution on a manifold M, i. e., a subbundle of the tangent bundle TM. Assume that an Euclidean structure $\langle\cdot,\cdot\rangle_q$ is given on each space $\mathcal{D}_q$ smoothly w.r.t. q. The triple $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$ defines a sub-Riemannian structure on M. Assume that M is connected and that $\mathcal{D}$ is completely nonholonomic. A Lipschitzian curve $\gamma : [0, T] \longrightarrow M$ is called *admissible* if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$, for a.e. t. It follows from the Rashevskii-Chow theorem that any two points in M can be connected by an admissible curve. One can define the length of an admissible curve $\gamma : [0, T] \longrightarrow M$ by $\int_0^T \|\dot{\gamma}(t)\| dt$, where $\|\dot{\gamma}(t)\| = \langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle^{\frac{1}{2}}$.

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Having an additional potential U, we consider the NMSR optimal control problems (optimal control problems on natural mechanical systems on a sub-Riemannian manifold):

$$A(\gamma(\cdot)) = \int_0^T (\frac{1}{2}\|\dot{\gamma}\|^2 - U(\gamma))dt \mapsto \min \tag{1}$$

$$\gamma(\cdot) \text{ is admissible}, \quad \gamma(0) = q_0, \quad \gamma(T) = q_1. \tag{2}$$

1.1 The extremals of the NMSR optimal control problems

The extremals of the NMSR optimal control problem can be described by the Pontryagin Maximum Principle of Optimal Control Theory ([7]). There are two different types of extremals: abnormal and normal, according to vanishing or nonvanishing of Lagrange multiplier near the functional, respectively. The extremals of the NMSR problem are the projections of either normal extremals or abnormal extremals.

In the present paper we will focus on normal extremals only. To describe them let us introduce some notations. Let T^*M be the cotangent bundle of M and σ be the canonical symplectic form on T^*M, i. e., $\sigma = -d\varsigma$, where ς is the tautological (Liouville) 1-form on T^*M. For each function $h : T^*M \to \mathbb{R}$, the Hamiltonian vector field $\overrightarrow{h}$ is defined by $i_{\overrightarrow{h}}\sigma = dh$. Given a vector $u \in T_qM$ and a covector $p \in T_q^*M$ we denote by $p \cdot u$ the value of p at u. For $\lambda = (p, q) \in T^*M,\ q \in M,\ p \in T_q^*M$, let

$$h(\lambda) \triangleq \max_{u \in \mathcal{D}}(p \cdot u - \frac{1}{2}\|u\|^2 + U(q)) = \frac{1}{2}\|p|_{\mathcal{D}_q}\|^2 + U(q), \tag{3}$$

where $p|_{\mathcal{D}_q}$ is the restriction of the linear functional p to $\mathcal{D}_q$ and the norm $\|p|_{\mathcal{D}_q}\|$ is defined w.r.t. the Euclidean structure on $\mathcal{D}_q$. The normal extremals are exactly the trajectories of $\dot{\lambda}(t) = \overrightarrow{h}(\lambda)$.

1.2 Jacobi curves along normal extremals

Let us fix the level set of the Hamiltonian function h:

$$\mathcal{H}_c \triangleq \{\lambda \in T^*M | h(\lambda) = c\}, c > 0.$$

Let Π_λ be the vertical subspace of $T_\lambda\mathcal{H}_c$, i. e.,

$$\Pi_\lambda = \{\xi \in T_\lambda\mathcal{H}_c : \pi_*(\xi) = 0\},$$

where $\pi : T^*M \longrightarrow M$ is the canonical projection. With any normal extremal $\lambda(\cdot)$ on $\mathcal{H}_c$, one can associate a curve in a Lagrange Grassmannian which describe the dynamics of the vertical subspaces Π_λ along this extremal w.r.t. the flow $e^{t\overrightarrow{h}}$, generated by $\overrightarrow{h}$. For this let

$$t \longmapsto \mathfrak{J}_\lambda(t) \triangleq e_*^{-t\overrightarrow{h}}(\Pi_{e^{t\overrightarrow{h}}\lambda})/\{\mathbb{R}\overrightarrow{h}(\lambda)\}. \tag{4}$$

The curve $\mathfrak{J}_\lambda(t)$ is the curve in the Lagrange Grassmannian of the linear symplectic space $W_\lambda = T_\lambda \mathcal{H}_c/\mathbb{R}\overrightarrow{h}(\lambda)$ (endowed with the symplectic form induced in the obvious way by the canonical symplectic form σ of T^*M). It is called the *Jacobi curve* of the extremal $e^{t\overrightarrow{h}}\lambda$ (attached at the point λ).

Using the method of Jacobi curves, one can construct feedback invariants of NMSR optimal control problems, namely, any symplectic invariant of Jacobi curve, i.e., invariant of the action of the linear symplectic group $Sp(W_\lambda)$ on the Lagrange Grassmannian $L(W_\lambda)$, produces a feedback invariant of the original NMSR optimal control problems.

1.3 Statement of the problem

In ([8, 9]), I. Zelenko and the author constructed the canonical bundle of moving frames and the complete system of symplectic invariants for parametrized curves in Lagrange Grassmannians satisfying very general assumptions. As a consequence, for any sub-Riemannian structure defined on any nonholonomic distribution on a manifold M one has the canonical (in general, non-linear) connection on an open subset of the cotangent bundle, the canonical splitting of the tangent spaces to the fibers of the cotangent bundle and the tuple of maps, called curvature maps, between the subspaces of the splitting intrinsically related to the sub-Riemannian structure. These constructions can be done in a similar way except that the potential appears as a "parameter ". We give a brief description of these constructions in Sect. 2. The structural equation for a canonical moving frame of the Jacobi curve of an extremal can be interpreted as the normal form for the Jacobi equation along this extremal and the curvature maps can be seen as the "coefficients"of this normal form. In the case of a Riemannian metric the canonical connection above coincides with the Levi-Civita connection and the splitting of the tangent spaces to the fibers is trivial. Moreover, there is only one curvature map and it is naturally related to the Riemannian sectional curvature tensor. If there is a potential on the Riemannian manifold, then in the formulation of the curvature map there appears one more term of the Hessian of the potential. Further, for the case of a sub-Riemannian structure associated on a principle connection of a principle G-bundle with one dimensional fibers over a Riemannian manifold, the curvature maps are expressed in terms of the Riemannian sectional curvature of the base manifold and the curvature form of the principle connection. In the present paper, we assume that there is an additional potential on the aforementioned sub-Riemannian structures and study the role of the potential in the expression of the curvature maps.

2 Differential geometry of curves in Lagrange Grassmannian

Denote by $L(W)$ the Lagrangian Grassmannian of an even dimensional linear symplectic space W endowed with a symplectic form ω. Given $\Lambda \in L(W)$, the tangent space $T_\Lambda L(W)$ of $L(W)$ at point Λ can be naturally identified with the space $Quad(\Lambda)$ of all quadratic forms on linear space $\Lambda \subset W$. A curve $\Lambda(\cdot)$ is called *monotonically nondecreasing* (*monotonically nonincreasing*) if the velocity is nonnegative definite (nonpositve definite) at any point.

2.1 Young diagrams

Denote by $C(\Lambda)$ the canonical bundle over Λ: the fiber of $C(\Lambda)$ over the point $\Lambda(t)$ is the linear space $\Lambda(t)$. Let $\Gamma(\Lambda)$ be the space of sections of $C(\Lambda)$. Define the ith extension of $\Lambda(\cdot)$ (or the i-th osculating space) by

$$\Lambda^{(i)}(t) = \text{span}\{\frac{d^j}{d\tau^j}\ell(\tau) : \ell(\tau) \in \mathcal{C}(\Lambda), 0 \le j \le i\}.$$

The flag $\Lambda(t) \subseteq \Lambda^{(1)}(t) \subseteq \Lambda^{(2)}(t) \subseteq ...$ is called *the associated flag of the curve* $\Lambda(\cdot)$ *at point* t. Assume that the following two conditions hold:

- $\dim \Lambda^{(i)}(t) - \dim \Lambda^{(i-1)}(t)$ is independent of t for any i;
- $\Lambda^{(p)}(t) = W$ for some $p \in \mathbb{N}$.

It follows from the first assumption above that

$$\dim \Lambda^{(i+1)}(t) - \dim \Lambda^{(i)}(t) \le \dim \Lambda^{(i)}(t) - \dim \Lambda^{(i-1)}(t).$$

Therefore, using the flag, to any $\Lambda(\cdot)$ we can assign the Young diagram in the following way: the number of boxes of the ith column is equal to $\dim\ \Lambda^{(i)}(t) - \dim\ \Lambda^{(i-1)}(t)$. Assume that the length of the rows of D be p_1 repeated r_1 times, p_2 repeated r_2 times, ..., p_d repeated r_d times with $p_1 > p_2 > \ldots > p_d$. In this case, the Young diagram D is the union of d rectangular diagrams of size $r_i \times p_i, 1 \le i \le d$. Denote them by $D_i, 1 \le i \le d$. The Young diagram Δ, consisting of d rows such that the ith row has p_i boxes, is called the reduced diagram or the reduction of the diagram D. The rows of Δ will be called levels. To the jth box a of the ith level of Δ one can assign the jth column of the rectangular subdiagram D_i of D and the integer number r_i (equal to the number of boxes of D in this subcolumn), called the size of the box a.

2.2 Normal moving frames

As usual, by $\Delta \times \Delta$ we will mean the set of pairs of boxes of Δ. Also denote by Mat the set of matrices of all sizes. The mapping $R : \Delta \times \Delta \longrightarrow$ Mat is called *compatible with the Young diagram* D, if to any pair (a, b) of boxes of sizes s_1 and s_2 respectively the matrix $R(a, b)$ is of the size $s_2 \times s_1$. The compatible mapping R

is called *symmetric* if for any pair (a, b) of boxes the following identity holds

$$R(b, a) = R(a, b)^T. \tag{5}$$

Denote by Υ_i the ith level of Δ. Also denote by a_i and σ_i the first and the last boxes of the ith level Υ_i respectively and by $r : \Delta \backslash \{\sigma_i\}_{i=1}^d \longrightarrow \Delta$ the right shift on the diagram Δ. The last box of any level will be called *special*. For any pair of integers (i, j) such that $1 \leq j < i \leq d$ consider the following tuple of pairs of boxes

$$\begin{aligned}&\big(a_j, a_i\big),\ \big(a_j, r(a_i)\big),\ \big(r(a_j), r(a_i)\big),\ \big(r(a_j), r^2(a_i)\big), \dots, \big(r^{p_i-1}(a_j), r^{p_i-1}(a_i)\big),\\ &\big(r^{p_i}(a_j), r^{p_i-1}(a_i)\big), \dots, \big(r^{p_j-1}(a_j), r^{p_i-1}(a_i)\big).\end{aligned} \tag{6}$$

Definition 1. A symmetric compatible mapping $R : \Delta \times \Delta \longrightarrow \text{Mat}$ is called normal if the following three conditions hold:

- for any $1 \leq j < i \leq d$, the matrices, corresponding to the first $(p_j - p_i - 1)$ pairs of the tuple (6), are equal to zero;
- among all matrices $\mathcal{R}(a, b)$, where the box b is not higher than the box a in the diagram Δ the only possible nonzero matrices are the following: the matrices $\mathcal{R}(a, a)$ for all $a \in \Delta$, the matrices $R\big(a, r(a)\big)$, $R\big(r(a), a\big)$ for all nonspecial boxes, and the matrices, corresponding to the pairs, which appear in the tuples (6), for all $1 \leq j < i \leq d$;
- the matrix $R\big(a, r(a)\big)$ is antisymmetric for any nonspecial box a.

Note that this notion depends only on the mutual locations of the boxes a and b in the diagram Δ. Now let us fix some terminology about the frames in W, indexed by the boxes of the Young diagram D. A frame $\big(\{e_\alpha\}_{\alpha \in D}, \{f_\alpha\}_{\alpha \in D}\big)$ of W is called *Darboux* or *symplectic*, if for any $\alpha, \beta \in D$ the following relations hold

$$\omega(e_\alpha, e_\beta) = 0, \quad \omega(f_\alpha, f_\beta) = 0, \quad \omega(e_\alpha, f_\beta) = \delta_{\alpha,\beta}, \tag{7}$$

where $\delta_{\alpha,\beta}$ is the analogue of the Kronecker index defined on $D \times D$. In the sequel it will be convenient to divide a moving frame $\big(\{e_\alpha(t)\}_{\alpha \in D}, \{f_\alpha(t)\}_{\alpha \in D}\big)$ of W indexed by the boxes of the Young diagram D into the tuples of vectors indexed by the boxes of the reduction Δ of D, according to the correspondence between the boxes of Δ and the subcolumns of D. More precisely, given a box a in Δ of size s, take all boxes $\alpha_1, \dots, \alpha_s$ of the corresponding subcolumn in D in the order from the top to the bottom and denote

$$E_a(t) = \big(e_{\alpha_1}(t), \dots, e_{\alpha_s}(t)\big), \quad F_a(t) = \big(f_{\alpha_1}(t), \dots, f_{\alpha_s}(t)\big). \tag{8}$$

Definition 2. The moving Darboux frame $(\{E_a(t)\}_{a \in \Delta}, \{F_a(t)\}_{a \in \Delta})$ is called the normal moving frame of a monotonically nondecreasing curve $\Lambda(t)$ with the Young diagram D, if

$$\Lambda(t) = \text{span}\{E_a(t)\}_{a \in \Delta}$$

for any t and there exists an one-parametric family of normal mappings R_t :

$\Delta \times \Delta \longrightarrow$ Mat such that the moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ satisfies the following structural equation:

$$\begin{cases} E_a'(t) = E_{l(a)}(t) & : \ a \in \Delta \backslash \mathcal{F}_1 \\ E_a'(t) = F_a(t) & : \ a \in \mathcal{F}_1 \\ F_a'(t) = -\sum_{b\in\Delta} E_b(t)R_t(a,b) - F_{r(a)}(t) & : \ a \in \Delta \backslash \mathcal{S} \\ F_a'(t) = -\sum_{b\in\Delta} E_b(t)R_t(a,b), & : \ a \in \mathcal{S} \end{cases}, \tag{9}$$

$\mathcal{F}_1$ is the first column of the diagram Δ, $\mathcal{S}$ is the set of all its special boxes, and $l : \Delta\backslash\mathcal{F}_1 \longrightarrow \Delta$, $r : \Delta\backslash \mathcal{S} \longrightarrow \Delta$ are the left and right shifts on the diagram Δ. The mapping R_t, appearing in (9), is called the normal mapping, associated with the normal moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$.

Theorem 1. *For any monotonically nondecreasing curve $\Lambda(t)$ with the Young diagram D in the Lagrange Grassmannian there exists a normal moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$. A moving frame*

$$(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$$

is a normal moving frame of the curve $\Lambda(\cdot)$ if and only if for any $1 \le i \le d$ there exists a constant orthogonal matrix U_i of size $r_i \times r_i$ such that for all t

$$\widetilde{E}_a(t) = E_a(t)U_i, \quad \widetilde{F}_a(t) = F_a(t)U_i, \quad \forall\, a \in \Upsilon_i. \tag{10}$$

As a matter of fact, normal moving frames define a principal $O(r_1) \times O(r_2) \times ... \times O(r_k)$-bundle of symplectic frame in W endowed with a canonical connection. The normal moving frames are horizontal curves of this connection.

Relations (10) imply that for any box $a \in \Delta$ of size s the following s-dimensional subspaces

$$V_a(t) = \text{span}\{E_a(t)\}, \ V_a^{\text{trans}}(t) = \text{span}\{F_a(t)\} \tag{11}$$

of $\Lambda(t)$ does not depend on the choice of the normal moving frame. In particular, there exists the *canonical splitting of the subspace* $\Lambda(t)$ defined by

$$\Lambda(t) = \bigoplus_{a\in\Delta} V_a(t), \ \dim V_a(t) = \text{size}(a) \tag{12}$$

and the *canonical complement* $\Lambda^{trans}(t)$ to $\Lambda(t)$ defined by

$$\Lambda^{\text{trans}}(t) = \bigoplus_{a\in\Delta} V_a^{\text{trans}}(t). \tag{13}$$

Moreover, each subspace $V_a(t)$(and $V_a^{\text{trans}}(t)$) is endowed with the *canonical Euclidean structure* such that the tuple of vectors E_a(and $F_a(t)$) constitute an orthonormal frame w.r.t. to it. Taking the canonical Euclidean structures on all $V_a(t)$ and assuming that subspaces $V_a(t)$ and $V_b(t)$ with different a and b are orthogonal, we get the canonical Euclidean structure on the whole $\Lambda(t)$.

The linear map from $V_a(t)$ to $V_b(t)$ with the matrix $R_t(a,b)$ from (9) in the basis $\{E_a(t)\}$ and $\{E_b(t)\}$ of $V_a(t)$ and $V_b(t)$ respectively, is independent of the choice of normal moving frames. It will be denoted by $\mathfrak{R}_t(a,b)$ and it is called the (a,b)-*curvature map of the curve* $\Lambda(\cdot)$ *at time* t.

2.3 Consequences for NMSR optimal control problems

Let $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$ be a sub-Riemannian structure. Note that the Jacobi curve associated with an extremal of NMSR optimal control problems in M is monotonically nondecreasing. A point $\lambda \in T^*M$ is called a D-regular point if the germ of the Jacobi curve $\mathfrak{J}_\lambda(t)$ at $t = 0$ has the Young diagram D. Assume that for some diagram D the set of D-regular point is open in $\mathcal{H}_{\frac{1}{2}}$ and let Δ be the reduced diagram of D. The structural Eq. (9) for the Jacobi curve $\mathfrak{J}_\lambda(t)$ can be seen as the intrinsic Jacobi equation along the extremal $e^{t\vec{h}}\lambda$ and the $(a,b)-$curvature maps are the coefficients of this Jacobi equation.

Since there is a canonical splitting of $\mathfrak{J}_\lambda(t)$ and taking into account that $\mathfrak{J}_\lambda(0)$ and Π_λ can be naturally identified, we have the canonical splitting of Π_λ:

$$\Pi_\lambda = \bigoplus_{a\in\Delta} \mathcal{V}_a(\lambda),\ \dim(\mathcal{V}_a(\lambda)) = \mathrm{size}(a),$$

where $\mathcal{V}_a(\lambda) = V_a(0)$.

Moreover, let $\mathfrak{R}_\lambda(a,b) : \mathcal{V}_a(\lambda) \to \mathcal{V}_b(\lambda)$ and the $\mathfrak{R}_\lambda : \Pi_\lambda \to \Pi_\lambda$ be the (a,b)-curvature map and the big curvature of the Jacobi curve $\mathfrak{J}_\lambda(\cdot)$ at $t = 0$. These maps are intrinsically related to the sub-Riemannian structure. They are called the *(a,b)-curvature* and the *big curvature* of NMSR optimal control problems at the point λ. Also, the canonical complement $\mathfrak{J}_\lambda^{\mathrm{trans}}(t)$ at $t = 0$ give rise a canonical complement of Π_λ in W_λ, where $W_\lambda = T_\lambda\mathcal{H}_{\frac{1}{2}}/\mathbb{R}\vec{h}$, as before. For any $a \in \Delta$, denote

$$\mathcal{V}_a^{\mathrm{trans}}(\lambda) = V_a^{\mathrm{trans}}(0). \tag{14}$$

Let $\lambda \in T^*M$ and let $\lambda(t) = e^{t\vec{h}}\lambda$. Assume that $(E_a^\lambda(t), F_a^\lambda(t))_{a\in\Delta}$ is a normal moving frame of the Jacobi curve $\mathfrak{J}_\lambda(t)$ attached at point λ.

Let $\mathfrak{E}$ be the Euler field on T^*M, i.e. the infinitesimal generator of the homotheties on its fibers. Clearly $T_\lambda(T^*M) = T_\lambda\mathcal{H}_{h^{-1}(\lambda)} \oplus \mathbb{R}\mathfrak{E}(\lambda)$. The flow $e^{t\vec{h}}$ on T^*M induces the push-forward maps $\left(e^{t\vec{h}}\right)_*$ between the corresponding tangent spaces $T_\lambda T^*M$ and $T_{e^{t\vec{h}}\lambda}T^*M$, which in turn induce naturally the maps between the spaces $T_\lambda(T^*M)/\mathbb{R}\vec{h}(\lambda)$ and $T_{e^{t\vec{h}}\lambda}T^*M/\mathbb{R}\vec{h}(e^{t\vec{h}}\lambda)$. The map $\mathcal{K}^t$ between $T_\lambda(T^*M)/\mathbb{R}\vec{h}(\lambda)$ and $T_{e^{t\vec{h}}\lambda}T^*M/\mathbb{R}\vec{h}(e^{t\vec{h}}\lambda)$, sending $E_a^\lambda(0)$ to $\left(e^{t\vec{h}}\right)_*E_a^\lambda(t)$, $F_a^\lambda(0)$ to $\left(e^{t\vec{h}}\right)_*F_a^\lambda(t)$ for any $a \in \Delta$, and the equivalence class of $\mathfrak{E}(\lambda)$ to the equivalence class of $\mathfrak{E}(e^{t\vec{h}}\lambda)$, is independent of the choice of normal moving frames.

The map $\mathcal{K}^t$ is called *the parallel transport* along the extremal $e^{t\vec{h}}\lambda$ at time t. For any $v \in T_\lambda(T^*M)/\mathbb{R}\vec{h}(\lambda)$, its image $v(t) = \mathcal{K}^t(v)$ is called *the parallel transport of v at time t*. Note that from the definition of the Jacobi curves and the construction of normal moving frame it follows that the restriction of the parallel transport $\mathcal{K}_t$ to the vertical subspace $T_\lambda(T^*_{\pi(\lambda)}M)$ of $T_\lambda(T^*M)$ can be considered as a map onto the vertical subspace $T_{e^{t\vec{h}}\lambda}(T^*_{\pi(e^{t\vec{h}}\lambda)}M)$ of $T_{e^{t\vec{h}}\lambda}(T^*M)$. A vertical vector field V is called *parallel* if $V(e^{t\vec{h}}\lambda) = \mathcal{K}^t\big(V(\lambda)\big)$.

In the *Riemannian case*, i.e., when $\mathcal{D} = TM$, the Young diagram of the Jacobi curve $\Lambda(\cdot)$ consists of only one column and the corresponding reduced diagram consists of only one box. Denote this box by a. The structure equation for a normal moving frame is of the form:

$$\begin{cases} E_a'(t) = F_a(t) \\ F_a'(t) = -E_a(t)\mathcal{R}_t(a,a). \end{cases} \tag{15}$$

In [2] and [1] it was shown that the canonical connection coincides with the Levi-Civita connection and the unique curvature map $\mathfrak{R}_\lambda(a,a) : \mathcal{V}_a(\lambda) \longrightarrow \mathcal{V}_a(\lambda)$ (where $\mathcal{V}_a(\lambda) = \Pi_\lambda$) was expressed by the Riemannian curvature tensor. In order to give this expression let R^∇ be the Riemannian curvature tensor. Below we will use the identification between the tangent vectors and the cotangent vectors of the Riemannian manifold M given by the Riemannian metric. More precisely, given $p \in T^*_qM$ let $p^h \in T_qM$ such that $p \cdot v = \langle p^h, v\rangle$ for any $v \in T_qM$. Since tangent spaces to a linear space at any point are naturally identified with the linear space itself we can also identify in the same way the space $T_\lambda(T^*_{\pi(\lambda)}M)$ with $T_{\pi(\lambda)}M$.

$$\begin{aligned} &\mathfrak{R}_\lambda(a,a)v = R^\nabla(p^h, v^h)p^h, \\ &\forall \lambda = (q,p) \in \mathcal{H}_{h^{-1}(\lambda)}, q \in M, p \in T^*_qM, \quad v \in \Pi_\lambda. \end{aligned} \tag{16}$$

Further, for the optimal control problems of natural mechanical systems on Riemannian manifolds, the curvature maps satisfies

$$\mathfrak{R}_\lambda(a,a)v = R^\nabla(p^h, v^h)p^h + (\nabla_{v^h}(\nabla U))(q), \quad v \in \Pi_\lambda, \tag{17}$$

where ∇U is the gradient of U.

For the nontrivial case of sub-Riemannian structures, i.e., when $\mathcal{D} \subsetneq TM$, let us consider the simplest case: the sub-Riemannian structure on a nonholonomic principle connection of a principle G-bundle with one dimensional fibers over a Riemannian manifold. Let $\mathrm{pr} : M \to \widetilde{M}$ be a principle G-bundle over a Riemannian manifold $(\widetilde{M}, g)$. A principle connection $\mathcal{D}$ is a G-equivariant horizontal distribution (complement to the vertical distribution tangent to the fibers). For any principle connection $\mathcal{D}$ one can introduce an inner product $\langle\cdot,\cdot\rangle = \mathrm{pr}^* g$ and hence we have a sub-Riemannian structure $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$. We assume that $\mathcal{D}$ is nonholonomic. In the paper we consider the simplest case that $G = \mathbb{R}$ or $\mathbb{S}$, i.e. the fibers of the principle bundle $\mathrm{pr} : M \to \widetilde{M}$ is one dimensional. For short, hereafter we call $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$ *a* *sub-Riemannian structure of 1-dim bundle type*. Let ω be the connection form of $\mathcal{D}$ and X_0 be the fundamental vector field. We actually have a sub-Riemannian structure on the contact distribution $\mathcal{D}$ with the contact form ω and the Reeb field X_0 which is a transversal symmetry.

The curvature form $d\omega$ gives a (1,1) tensor J on $\widetilde{M}$ by $g(JX, Y) = d\omega(X, Y)$. Define $u_0 : T^*M \longrightarrow \mathbb{R}$ by $u_0(p,q) \stackrel{\Delta}{=} p \cdot X_0(q)$. Since X_0 is a symmetry of the sub-Riemannian structure, the function u_0 is the first integral of the extremal flow, i.e., $\{h, u_0\} = 0$, where $\{\cdot,\cdot\}$ is the Poisson bracket. Actually, $d\omega$ can be seen as a magnetic field and J can be seen as a Lorenzian force on Riemannian manifold $\widetilde{M}$. The projection by pr of all sub-Riemannian geodesics describes all possible motion of a charged particle (with any possible charge u_0) given by the magnetic field $d\omega$ on the Riemannian manifold $\widetilde{M}$ (see e. g. [6, Chap. 12] and the references therein).

The curvature maps for such sub-Riemannian structures were expressed in terms of the Riemannian sectional curvature of the base manifold and tensor J of the principle bundle ([4]). See also [5] for an example. In the present paper, we consider the NMSR optimal control problems by imposing a potential U as an external force. We assume that the potential U is constant on the fibers of the principle bundle $\text{pr} : M \to \widetilde{M}$. We show how the potential U effects the curvature maps for the NMSR optimal control problems (see Theorems 2-4 below).

3 Algorithm for calculation of canonical splitting and (a,b)-curvature maps

Let

$$\mathcal{D}^{\perp} = \{(p,q) \in T^*M : p \cdot v = 0,\ \forall v \in \mathcal{D}_q\}, \quad \mathcal{D}_q^{\perp} = \mathcal{D}^{\perp} \cap T_q^*M.$$

It holds the following series of natural identifications:

$$T_q^*M/\mathcal{D}_q^{\perp} \sim \mathcal{D}_q^* \stackrel{\langle\cdot,\cdot\rangle}{\sim} \mathcal{D}_q, \tag{18}$$

where $\mathcal{D}_q^* \subseteq T_q^*M$ is the dual space of $\mathcal{D}_q$. Accordingly, $J_q = J|_q$ can be taken as the linear map from the fiber T_q^*M of T^*M to $T_q^*M/\mathcal{D}_q^{\perp}$ (in this case, $J_q|_{\mathcal{D}_q^{\perp}} = 0$).

Fix $\dim M = n$. Let D be the Young diagram consisting of two columns, with $(n-2)$ boxes in the first column and 1 box in the second column. Then the set of D-regular points coincides with $\{(p,q) \in \mathcal{H}_{\frac{1}{2}} : J_q p \neq 0\}$ (see step 1 of Sect. 3.3 Proposition 1 below for the proof in the particular case with symmetries) . In the case of $n > 3$, the reduced Young diagram consists of three boxes: two in the first column and one in the second. The box in the second column will be denoted by a, the upper box in the first column will be denoted by b and the lower box in the first column will be denoted by c. Note that $\text{size}(a) = \text{size}(b) = 1$ and $\text{size}(c) = n-3$. When $n = 3$, the reduced Young diagram consists of two boxes, a and b as above and the box c does not appear. All formulae for $n > 3$ will be true for $n = 3$ if one avoids the formulae containing the box c. In this case, the symmetric (Darboux) compatible mapping (with Young diagram D) is normal if and only if $R_t(a,b) = 0$ and the canonical splitting of Π_λ has the form: $\Pi_\lambda = \mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda)$, where $\mathcal{V}_a(\lambda), \mathcal{V}_b(\lambda)$ are of dimension 1 and $\mathcal{V}_c(\lambda)$ is of dimension $n-3$. These subspaces can be described as follows. As the tangent space of the fibers of T^*M can be naturally identified with the fibers themselves (the fibers are linear spaces), one can show that

$$\mathcal{V}_a(\lambda) = \mathcal{D}^{\perp}_{\pi(\lambda)}.$$

Using the fact that $\mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda) \oplus \mathbb{R}p$ is transversal to $\mathcal{D}_q^{\perp}$, one can get the following identification

$$\mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda) \oplus \mathbb{R}p \sim T_q^*M/\mathcal{D}_q^{\perp}, \tag{19}$$

Finally, combining (18) and (19), we have that

$$\mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda) \oplus \mathbb{R}p \sim \mathcal{D}_q^* \sim \mathcal{D}_q, \tag{20}$$

Under the identifications, one can show that (see step 1 in Sect. 3.2 below):

$$\mathcal{V}_b(\lambda) = \mathbb{R}J_q p, \quad \mathcal{V}_c(\lambda) = (\mathrm{span}\{p, Jp\})^{\perp}. \tag{21}$$

3.1 Algorithm of normalization

First let us describe the construction of the normal moving frames and the curvature maps for a monotonically nondecreasing curve $\Lambda(t)$ with the Young diagram D as in Sect. 2.3. The details can be found in [9]. In this case, the structural equation for the normal moving frame is of the form:

$$\begin{cases} E_a'(t) = E_b(t) \\ E_b'(t) = F_b(t) \\ E_c'(t) = F_c(t) \\ F_a'(t) = -E_c(t)R_t(a,c) - E_a(t)R_t(a,a) \\ F_b'(t) = -E_c(t)R_t(b,c) - E_b(t)R_t(b,b) - F_a(t) \\ F_c'(t) = -E_c(t)R_t(c,c) - E_b(t)R_t(c,b) - E_a(t)\mathcal{R}_t(c,a). \end{cases} \tag{22}$$

Assume that each element of the set $\{\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \mathcal{F}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)\}$ is either a vector field or a tuple of vector fields, depending on the size of the corresponding box in the Young diagram such that

$$\begin{aligned}(\mathcal{E}_a(e^{t\vec{h}}\lambda), \mathcal{E}_b(e^{t\vec{h}}\lambda), \mathcal{E}_c(e^{t\vec{h}}\lambda), \mathcal{F}_a(e^{t\vec{h}}\lambda), \mathcal{F}_b(e^{t\vec{h}}\lambda), \mathcal{F}_c(e^{t\vec{h}}\lambda))\\ = \mathcal{K}^t(\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \mathcal{F}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)),\end{aligned}$$

where $\mathcal{K}^t$ is the parallel transport, defined in Sect. 2.3. Recall that for any vector fields X, Y one has the following formula: $\frac{d}{dt}\big|_{t=0} e_*^{-tX} Y = \mathrm{ad}_X Y$. So, the derivative w.r.t. t on the level of curves can be substituted by taking the Lie bracket with $\vec{h}$ on the level of sub-Riemannian structure. The normalization procedure of [9] can be described in the following steps:

Step 1. The vector field $\mathcal{E}_a(\lambda)$ can be characterized , uniquely up to a sign, by the following conditions: $\mathcal{E}_a(\lambda) \in \Pi_\lambda$, $\mathrm{ad}\,\vec{h}\;\mathcal{E}_a(\lambda) \in \Pi_\lambda$, and

$$\sigma(\mathrm{ad}\,\vec{h}\;\mathcal{E}_a(\lambda), (\mathrm{ad}\,\vec{h})^2\mathcal{E}_a(\lambda)) = 1.$$

Then by the first two lines of (22) $\mathcal{E}_b(\lambda) = \mathrm{ad}\,\vec{h}\;\mathcal{E}_a(\lambda)$ and $\mathcal{F}_b(\lambda) = (\mathrm{ad}\,\vec{h})^2\mathcal{E}_a(\lambda)$.

Step 2. The subspace $\mathcal{V}_c$ is uniquely characterized by the following two conditions:

- $\mathcal{V}_c(\lambda)$ is the complement of $\mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda)$ in Π_λ;
- $\mathcal{V}_c(\lambda)$ lies in the skew symmetric complement of

$$\mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda) \oplus \mathbb{R}(\mathrm{ad}\,\vec{h})^2\mathcal{E}_a(\lambda) \oplus \mathbb{R}(\mathrm{ad}\,\vec{h})^3\mathcal{E}_a(\lambda).$$

It is endowed with the canonical Euclidean structure, which is the restriction of $\dot{\mathfrak{J}}_\lambda(0)$ on it.

Step 3. The restriction of the parallel transport $\mathcal{K}^t$ to $\mathcal{V}_c(\lambda)$ is characterized by the following two properties:

- $\mathcal{K}^t$ is an orthogonal transformation of spaces $\mathcal{V}_c(\lambda)$ and $\mathcal{V}_c\big(e^{t\vec{h}}\lambda\big)$;
- the space $\mathrm{span}\{\frac{d}{dt}\big((e^{-t\vec{h}})_*(\mathcal{K}^t v)\big)|_{t=0} : v \in \mathcal{V}_c(\lambda)\}$ is isotropic.

Step 4. To complete the construction of normal moving frame it remains to fix $\mathcal{F}_a(\lambda)$. The field $\mathcal{F}_a(\lambda)$ is uniquely characterized by the following two conditions (see line 4 of (22))):

- the tuple $\{\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \mathcal{F}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)\}$ constitutes a Darboux frame;
- $\sigma(\mathrm{ad}\,\vec{h}\,\mathcal{F}_a(\lambda), \mathcal{F}_b(\lambda)) = 0$.

In order to find $\mathcal{F}_a(\lambda)$, one can choose any $\widetilde{\mathcal{F}}_a(\lambda)$ such that $\{\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \widetilde{\mathcal{F}}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)\}$ constitutes a Darboux frame. Then

$$\mathcal{F}_a(\lambda) = \widetilde{\mathcal{F}}_a(\lambda) - \sigma(\mathrm{ad}\,\vec{h}\;\widetilde{\mathcal{F}}_a(\lambda), \mathcal{F}_b(\lambda))\mathcal{E}_a(\lambda). \tag{23}$$

3.2 Preliminary implementation of the algorithm

Now let us analyze the relation between T^*M and $T^*\widetilde{M}$ in more detail. Let Ξ be the 1-foliation such that its leaves are integral curves of $\vec{u}_0$ and $\mathrm{PR} : T^*P \to T^*P/\Xi$ be the canonical projection to the quotient manifold.

Fix a constant c. The quotient manifold $\{u_0 = c\}/\Xi$ can be naturally identified with $T^*\widetilde{M}$. Indeed, a point $\tilde{\lambda}$ in $\{u_0 = c\}/\Xi$ can be identified with a leaf

$\mathrm{PR}^{-1}(\tilde{\lambda})$ of Ξ which has a form $(e^{tX_0}q, (e^{-tX_0})^* p)$, where $\lambda = (p, q) \in \mathrm{PR}^{-1}(\tilde{\lambda})$, $q \in M$ and $p \in T_q^* M$. On the other hand, any element in T^*M can be identified with a one-parametric family of pairs $(e^{tX_0}q, (e^{-tX_0})^*(p|_{\mathcal{D}}))$. The mapping $I : \{u_0 = c\}/\Xi \to T^*\widetilde{M}$ sending $(e^{tX_0}q, (e^{-tX_0})^* p)$ to $(e^{tX_0}q, (e^{-tX_0})^*(p|_{\mathcal{D}}))$ is one-to-one (because $p(X_0) = u_0$ is already prescribed and equal to c) and it defines the required identification. Therefore, for any vector field X on $T^*\widetilde{M}$, we can assign the vector field $\underline{X}$ on T^*M s.t. $PR_*\underline{X} = (I^{-1})_* X$ and $\pi_*\underline{X} \in \mathcal{D}$.

Before going further, let us introduce some notations. Given $v \in T_\lambda T_q^* M$ ($\sim T_q^* M$), where $\lambda = (p, q)$, we can assign a unique vector $v^h \in T_{\mathrm{pr}(q)}\widetilde{M}$ to its equivalence class in $T^*M/\mathcal{V}_a(\lambda)$ by using the identifications (19) and (20). Conversely, to any $X \in T_{\pi(q)}\widetilde{M}$ one can assign an equivalence class of $T_\lambda(T_q^* M)/V_a(\lambda)$. Denote by $X^v \in T_\lambda(T_q^* M)$ the unique representative of this equivalence class such that $du_0(X^v) = 0$.

For later use, we present the following facts (see [4] and [3] for details).

Lemma 1. *For any vectors $X, V \in T_\lambda T^*M$ with $\pi_* V = 0$ we have $\sigma(X, v) = g(\pi_* X, V^h)$ and $\overrightarrow{U} = -(\nabla U)^v$, where ∇U is the gradient of U.*

Lemma 2. *The following formula holds on $\{u_0 = c\}$.*

$$\sigma = (I \circ PR)^* \tilde{\sigma} - u_0 \pi^* d\omega_0. \tag{24}$$

Lemma 3. *Let $\overrightarrow{h}$ be the Hamiltonian vector field on T^*M, as before, then*

$$\overrightarrow{h}(\lambda) = \underline{\nabla_{p^h}} - u_0 (Jp^h)^v - (\nabla U)^v, \tag{25}$$

where $\lambda = (p, q) \in T^*M$ and ∇_{p^h} is the lift of p^h to $T^*\widetilde{M}$ w.r.t. the Levi-Civita connection.

Now we give more precise descriptions of normal moving frames following the steps as in Sect. 3.1. Assume that $\mathcal{V}_a^{\mathrm{trans}}(\lambda)$, $\mathcal{V}_b^{\mathrm{trans}}(\lambda)$, $\mathcal{V}_c^{\mathrm{trans}}(\lambda)$ are defined by (14).

Step 1. First define the vector field $\widetilde{\mathcal{E}_a}$ on T^*M by

$$\widetilde{\mathcal{E}_a}(\lambda) \in \Pi_\lambda,\ \widetilde{\mathcal{E}_a}(\lambda) \in \mathcal{D}^\perp,\ du_0(\widetilde{\mathcal{E}_a}(\lambda)) = 1. \tag{26}$$

For further calculations it is convenient to denote $\widetilde{\mathcal{E}_a}$ by ∂_{u_0}, because to take the Lie brackets of $\widetilde{\mathcal{E}_a}$ with $\overrightarrow{h}$ is the same as to make "the partial derivatives w.r.t. u_0" in the left handside of (25). Indeed, by (25) $\mathrm{ad}\overrightarrow{h}\ \partial_{u_0} = (Jp^h)^v \in \Pi_\lambda$ and then $\pi_*\big((\mathrm{ad}\overrightarrow{h})^2\ \partial_{u_0}\big) = -Jp^h$. Then from Lemma 1 it follows immediately that

$$\sigma(\mathrm{ad}\overrightarrow{h}\ \partial_{u_0}, (\mathrm{ad}\overrightarrow{h})^2\ \partial_{u_0}) = \|Jp^h\|^2.$$

Since $\mathcal{D}$ is a contact distribution, J_q is nonsingular for each $q \in P$, hence as a direct consequence of the last identity we have

Proposition 1. *All points of T^*M out of the zero section are D-regular.*

Further from step 1 of Sect. 3.1, we have that

$$\mathcal{E}_a(\lambda) = \frac{\partial_{u_0}}{\|Jp^h\|}, \tag{27}$$

$$\mathcal{E}_b(\lambda) = \operatorname{ad}\vec{h}\ \mathcal{E}_a(\lambda) = \frac{(Jp^h)^v}{\|Jp^h\|} + \vec{h}\left(\frac{1}{\|Jp^h\|}\right)\partial_{u_0}, \tag{28}$$

$$\mathcal{F}_b(\lambda) = \operatorname{ad}\vec{h}\ \mathcal{E}_b(\lambda) = \frac{1}{\|Jp^h\|}[\vec{h}, (Jp^h)^v] + 2\vec{h}\left(\frac{1}{\|Jp^h\|}\right)(Jp^h)^v + (\vec{h})^2\left(\frac{1}{\|Jp^h\|}\right)\partial_{u_0}. \tag{29}$$

By direct computations,

$$\pi_*[\vec{h}, (Jp^h)^v] = -Jp^h. \tag{30}$$

Step 2. Let us characterize the space $\mathcal{V}_c(\lambda)$. For this let $\widetilde{\Pi}_\lambda = \{v \in \Pi_\lambda : du_0(v) = 0\}$ or, equivalently, $\widetilde{\Pi}_\lambda = \{(v^h)^v : v \in \Pi_\lambda\}$. Since $\mathcal{V}_c(\lambda) \in \Pi_\lambda$ and $\mathcal{V}_c(\lambda)$ lies in the skew symmetric complement of $(\operatorname{ad}\vec{h})^2\mathcal{E}_a(\lambda)$, we have, using (30) and Lemma 1, that

$$\mathcal{V}_c(\lambda) \equiv (\operatorname{span}\{p^h, Jp^h\}^\perp)^v \mod \mathbb{R}\mathcal{E}_a(\lambda). \tag{31}$$

Further, let $\widetilde{\mathcal{V}}_c(\lambda) = \mathcal{V}_c(\lambda) \cap \widetilde{\Pi}_\lambda = \{X^v : X \in \operatorname{span}\{p^h, Jp^h\}^\perp\}$. Using the condition that $\mathcal{V}_c(\lambda)$ is in the skew symmetric complement of $(\operatorname{ad}\vec{h})^3\mathcal{E}_a(\lambda)$, we have

$$\mathcal{V}_c(\lambda) = \{v + \mathcal{A}(\lambda, v)\mathcal{E}_a(\lambda) : v \in \widetilde{\mathcal{V}}_c(\lambda)\}. \tag{32}$$

where $\mathcal{A}(\lambda, v)$ is the linear functional on the Whitney sum $T^*M \oplus T^*M$ over M, given by

$$\mathcal{A}(\lambda, v) = \sigma\left(v, \frac{(\operatorname{ad}\vec{h})^2\,(Jp^h)^v}{\|Jp^h\|}\right). \tag{33}$$

Step 3. Since the normal moving frame is a Darboux frame, the space $\mathcal{V}_c^{\text{trans}}(\lambda)$ lies in the skew symmetric complement of $\mathcal{V}_b(\lambda)$. Besides, its image under π_* belongs to $\mathcal{D}\big(\pi(\lambda)\big)$. Then, using Lemma 1 we obtain that

$$\operatorname{pr}_* \circ \pi_*\big(\mathcal{V}_c^{\text{trans}}(\lambda)\big) \equiv \operatorname{span}\{p^h, Jp^h\}^\perp \mod \mathbb{R}\mathrm{p}^{\mathrm{h}}, \tag{34}$$

where, as before, $\operatorname{pr} : M \to \widetilde{M}$ is the canonical projection. Recall that $\mathcal{V}_c^{\text{trans}}(\lambda) \in T_\lambda(T^*M)/\mathbb{R}\vec{h}(\lambda)$. As a canonical representative of $\mathcal{V}_c^{\text{trans}}(\lambda)$ in $T_\lambda(T^*M)$ one can take the representative, which projects exactly to $\operatorname{span}\{p^h, Jp^h\}^\perp$ by π_*. In the sequel, this canonical representative will be denoted by $\mathcal{V}_c^{\text{trans}}(\lambda)$ as well.

Further, given any $X \in \operatorname{span}\{p^h, Jp^h\}^\perp$ denote by ∇^c_X the lift of X to $\mathcal{V}_c^{\text{trans}}(\lambda)$ i.e. the unique vector $\nabla^c_X \in V_c^{\text{trans}}(\lambda)$ such that $\text{pr}_* \circ \pi_* \nabla^c_X = X$. Then there exist the unique $B \in \operatorname{End}\big(\widetilde{\mathcal{V}}_c(\lambda)\big)$ and $\alpha, \beta, \gamma \in \mathcal{V}_c(\lambda)^*$ such that

$$\nabla^c_{v^h} = \underline{\nabla_{v^h}} + B\big(\pi_0(v)\big) + \alpha(v)\frac{(Jp^h)^v}{\|Jp^h\|^2} + \beta(v)\partial_{u_0} + \gamma(v)(p^h)^v, \quad \forall v \in \mathcal{V}_c(\lambda) \tag{35}$$

where, as before, ∇ stands for the lifts to the Levi-Civita connection on $T^*\widetilde{M}$. Let us describe the operator B and the functionals α, β, γ more precisely. For this, we introduce the following notation. Given a map $S : T^*M \oplus T^*M \longrightarrow \mathbb{R}$, define a map $S^{(1)} : T^*M \oplus T^*M \longrightarrow \mathbb{R}$ by

$$S^{(1)}(\lambda, v) = \frac{d}{dt} S(e^{t\vec{h}}\lambda, \mathcal{K}^t v)\Big|_{t=0}, \; \lambda, v \in T^*M, \tag{36}$$

where in the second argument we use again the natural identification of $T^*_{\pi(\lambda)}M$ with $T_\lambda(T^*_{\pi(\lambda)}M)$. Now we can state the following lemma.

Lemma 4. *For any* $v \in \mathcal{V}_c(\lambda)$*,*

- $B(v^h)^v = \frac{u_0}{2}\left(-(Jv^h)^v + g(Jv^h, Jp^h)\frac{(Jp^h)^v}{\|Jp^h\|^2}\right)$;
- $\alpha(v) = -\sigma(\underline{\nabla_{v^h}}, ad\vec{h}\ (Jp^h)^v)$*;*
- $\beta(v) = -\left(\frac{1}{\|Jp^h\|}\mathcal{A}\right)^{(1)}(\lambda, (v^h)^v)$
  $= -\frac{1}{\|Jp^h\|}\mathcal{A}^{(1)}(\lambda, (v^h)^v) - \vec{h}\left(\frac{1}{\|Jp^h\|}\right)\mathcal{A}(\lambda, (v^h)^v).$

Because of appearance of the potential U, the vector field $\underline{\nabla_{v^h}}$ is not necessarily tangent to h. More precisely,

Lemma 5. $\gamma(v) = \frac{1}{\|p^h\|^2} g\left(v^h, \nabla U\right).$

Step 4. According to the algorithm, described in Sect. 3.1, first find some vector field $\widetilde{\mathcal{F}}_a$ such that the tuple $\{\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c, \widetilde{\mathcal{F}}_a, \mathcal{F}_b, \mathcal{F}_c\}$ constitutes a Darboux frame. Let $\mathfrak{V}_0$ be a vector in $\mathcal{V}_c(\lambda)$ such that

$$\sigma(\mathfrak{V}_0, \nabla^c_{v^h}) = \beta(v), \quad \forall v \in \mathcal{V}_c(\lambda). \tag{37}$$

Also, let $\mathfrak{W}_0$ be a vector in $\mathcal{V}_c^{\text{trans}}(\lambda)$ such that

$$\sigma(v, \mathfrak{W}_0) = \mathcal{A}(\lambda, v), \quad \forall v \in \mathcal{V}_c(\lambda). \tag{38}$$

Note that by constructions the map $v \mapsto \nabla^c_{v^h}$ is an isomorphism between $\mathcal{V}_c(\lambda)$ and $\mathcal{V}_c^{\text{trans}}(\lambda)$. Let $\mathfrak{V}_1$ be a vector in $\mathcal{V}_c$ such that $\mathfrak{W}_0 = \nabla^c_{\mathfrak{V}_1^h}$. Then from (37) and (38) it follows that

$$\mathcal{A}(\lambda, \mathfrak{V}_0) = \beta(\mathfrak{V}_1). \tag{39}$$

The canonical $\mathcal{F}_a$ is obtained from $\widetilde{\mathcal{F}}_a$ and the following lemma by formula (23).

Lemma 6. *A vector field $\widetilde{\mathcal{F}}_a$ can be taken in the following form*

$$\widetilde{\mathcal{F}}_a(\lambda) = -\|Jp^h\|\overrightarrow{u}_0 + \|Jp^h\|\mathfrak{V}_0 - \mathfrak{W}_0 + \|Jp^h\|(\overrightarrow{h})^2\left(\frac{1}{\|Jp^h\|}\right)\mathcal{E}_b(\lambda)$$
$$-\|Jp^h\|\overrightarrow{h}\left(\frac{1}{\|Jp^h\|}\right)\mathcal{F}_b(\lambda) \tag{40}$$

Now as a direct consequence of structure Eq. (22), we get the following preliminary descriptions of (a, b)-curvature maps (under identification (21)).

Proposition 2. *Let V be a parallel vector field such that $V(\lambda) = v$. Then the curvature maps satisfy the following identities:*

$$g\big((\mathfrak{R}_\lambda(c,c)v)^h, w^h\big) = -\sigma(ad\overrightarrow{h}\ \nabla^c_{V^h}, \nabla^c_{w^h}), \quad \forall w \in \mathcal{V}_c(\lambda) \tag{41}$$

$$\mathfrak{R}_\lambda(c,b)v = \sigma(ad\overrightarrow{h}\ \nabla^c_{V^h}, \mathcal{F}_b(\lambda))\frac{(Jp^h)^v}{\|Jp^h\|} = \sigma(ad\overrightarrow{h}\ \mathcal{F}_b(\lambda), \nabla^c_{v^h})\frac{(Jp^h)^v}{\|Jp^h\|} \tag{42}$$

$$\mathfrak{R}_\lambda(c,a)v = \sigma(ad\overrightarrow{h}\ \nabla^c_{V^h}, \mathcal{F}_a(\lambda))\partial_{u_0} \tag{43}$$

$$\mathfrak{R}_\lambda(b,b)(\frac{(Jp^h)^v}{\|Jp^h\|}) = -\sigma(ad\overrightarrow{h}\ \mathcal{F}_b(\lambda), \mathcal{F}_b(\lambda))(\frac{(Jp^h)^v}{\|Jp^h\|}) \tag{44}$$

$$\mathfrak{R}_\lambda(a,a)\partial_{u_0} = -\sigma(ad\overrightarrow{h}\ \mathcal{F}_a(\lambda), \mathcal{F}_a(\lambda))\partial_{u_0} \tag{45}$$

4 Calculus and the canonical splitting

4.1 Some useful formulas

We need special calculus which will be given in Proposition 3 below. Let A be a tensor of type $(1, K)$ and B be a tensor of type $(1, N)$ on $\widetilde{M}$, $K, N \geq 0$. Define a new tensor $A \bullet B$ of type $(1, K+N-1)$ by

$$A \bullet B(X_1, ..., X_{K+N-1}) = \sum_{i=0}^{K-1} A(X_1, ..., X_i, B(X_{i+1}, \ldots X_{i+N}), X_{i+N+1}, ..., X_{K+N-1}).$$

Also define by induction $A^{i+1} = A \bullet A^i$. For simplicity, in this section, we denote

$$Ap^h = A(\underbrace{p^h, p^h, ..., p^h}_{K}),\ Ap = (Ap^h)^v. \tag{46}$$

Besides, we denote by ∇A the covariant derivative (w.r.t. the Levi-Civita connection) of the tensor A, i. e. ∇A is a tensor of type $(1, K+1)$ defined by

$$\nabla A(X_1, ..., X_K, X_{K+1}) = (\nabla_{X_{K+1}} A)(X_1, ..., X_K). \tag{47}$$

In particular, if $K = 0$, then $\nabla A(X) = \nabla_X A$. Also define by induction $\nabla^{i+1} A = \nabla(\nabla^i A)$.

Proposition 3. *The following identities hold ([4]):*

- $[Ap, Bp] = (B \bullet A)p - (A \bullet B)p;$
- $[\underline{\nabla_{Ap^h}}, Bp] = -\underline{\nabla_{(A\bullet B)p^h}} + ((\nabla_{Ap^h} B)p^h)^v;$
- $[\underline{\nabla_{Ap^h}}, \underline{\nabla_{Bp^h}}] = \underline{\nabla_{(\nabla_{Ap^h} B)p^h - (\nabla_{Bp^h} A)p^h}} + (R^\nabla(Ap^h, Bp^h)p^h)^v - d\omega(Ap^h, Bp^h)\vec{u}_0;$
- $\underline{\nabla_{p^h}}\big(g(Ap^h, Bp^h)\big) = g\big((\nabla A)p^h, Bp^h\big) + g\big(Ap^h, (\nabla B)p^h\big).$

Proposition 4. *Let V_1, V_2 be vector fields on T^*M with $\pi_* V_1 = \pi_* V_2 = 0$. Then*

- $\bar{\sigma}([\vec{U}, \underline{\nabla_{V_1^h}}], \underline{\nabla_{V_2^h}}] = -\text{Hess}\, U(V_1^h, V_2^h);$
- $\vec{U}\left(g(V_1^h, V_2^h)\right) - g\left(([\vec{U}, V_1])^h, V_2^h\right) - g\left(V_1^h, ([\vec{U}, V_2])^h\right) = 0.$

4.2 Calculations of the canonical splitting

Using formulas given by Proposition 3, we are ready to express the canonical splitting of W_λ $(= T_\lambda \mathcal{H}_{\frac{1}{2}}/\mathbb{R}\vec{h})$ in terms of the Riemannian structure and the tensor J on M. Note that by (27) the subspace $\mathcal{V}_a$ is already expressed in this way. To express the subspace $\mathcal{V}_b$ and $\mathcal{V}_b^{\text{trans}}$ we combine Propositions 3 and 4 and Lemma 4.1 in [4] to get

Lemma 7. *The following identities hold:*

- $\vec{h}\left(\frac{\|Jp^h\|^2}{2}\right) = g(Jp^h, \nabla J(p^h, p^h)) - g(J\nabla U, Jp^h);$
- $\vec{h}^2\left(\frac{\|Jp^h\|^2}{2}\right) = \|\nabla J(p^h, p^h)\|^2 + g(Jp^h, \nabla^2 J(p^h, p^h, p^h))$
$- g(\nabla_{p^h} J\nabla U, Jp^h) - 2g(J\nabla U, \nabla J(p^h, p^h))$
$- g(Jp^h, (\nabla J(\nabla U, p^h) + \nabla J(p^h, \nabla U))$
$+ \|J\nabla U\|^2 - u_0 g(J^2 p^h, \nabla J(p^h, p^h)) - u_0 g(Jp^h, \nabla J(p^h, Jp^h)$
$+ \nabla J(Jp^h, p^h)) + u_0 g(J\nabla U, J^2 p^h).$

Now substituting item (1) of Lemma 7 into (28) we get the expression for the subspace $\mathcal{V}_b$. Now let us find the expression for $\mathcal{V}_b^{\text{trans}}$. First by (25) and item (2) of Proposition 3 we have

$$\begin{aligned}[\vec{h}, (Jp^h)^v] &= [\underline{\nabla_{p^h}} - u_0(Jp^h)^v - (\nabla U)^v, (Jp^h)^v] \\ &= -\underline{\nabla_{Jp^h}} + (\nabla J(p^h, p^h))^v - (J\nabla U)^v\end{aligned} \tag{48}$$

Substituting the last formula and the items (1) and (2) of Lemma 7 into (29) we will get the required expression for $\mathcal{V}_b^{\text{trans}}$.

Further, according to (32) in order to find the expression for $\mathcal{V}_c$ we have to express $\mathcal{A}(\lambda, v)$.

Lemma 8. *Let* $v \in \Pi_\lambda$. *Then*

$$\mathcal{A}(\lambda, v) = \frac{2}{\|Jp^h\|} g(v^h, \nabla J(p^h, p^h) - J\nabla U) - \frac{u_0}{\|Jp^h\|} g(v^h, J^2 p^h). \tag{49}$$

In order to express $\mathcal{V}_c^{\text{trans}}(\lambda)$ it is sufficient to express the operator B and functionals α and β, defined by (35), as the operator B and the functional γ are already expressed in Lemmas 4 and 5, respectively. Firstly, from decomposition (24), Lemma 1, and the fact that the Levi-Civita connection is a Lagrangian distribution it follows that

$$\begin{aligned}\alpha(v) &= -\sigma(\underline{\nabla_{v^h}}, -\underline{\nabla_{Jp^h}} + (\nabla J(p^h, p^h))^v - (J\nabla U)^v) \\ &= -u_0 d\omega_0(v^h, Jp^h) - g(v^h, \nabla J(p^h, p^h)) + g(v^h, J\nabla U) \\ &= -u_0 g(Jv^h, Jp^h) - g(v^h, \nabla J(p^h, p^h)) + g(v^h, J\nabla U)\end{aligned} \tag{50}$$

Similar to Corollary 1 in [4], we have the following characterization of $\mathcal{A}^{(1)}$. For later use we will work in more general setting. Let $\mathfrak{S}$ be a tensor of type $(1, K)$ on M. This tensor induces a map $S : T^*M \oplus T^*M \longrightarrow \mathbb{R}$ by

$$S(\lambda, v) = g(\mathfrak{S} p^h, v^h),\ \lambda = (p, q) \in T^*M, q \in M, p \in T_q^*M. \tag{51}$$

where $\mathfrak{S} p^h$ is as in (46).

Proposition 5. *Let* $v \in \mathcal{V}_c(\lambda)$.

$$\begin{aligned}S^{(1)}(\lambda, v) = g(v^h, (\nabla\mathfrak{S})p^h - u_0(\mathfrak{S} \bullet J)p^h + \frac{1}{2}u_0(J \bullet \mathfrak{S})p^h) \\ + \frac{1}{\|p^h\|^2} g\left(v^h, \nabla U\right) g(p^h, \mathfrak{S} p^h) - g(v^h, (\mathfrak{S} \bullet \nabla U)p^h) \\ - \frac{1}{2} S\left(\lambda, \frac{(Jp^h)^v}{\|Jp^h\|}\right) \mathcal{A}(\lambda, v)\end{aligned}$$

Corollary 1.

$$\begin{aligned}\mathcal{A}^{(1)}(\lambda, v) = & -\mathcal{A}(\lambda, v)\mathcal{A}\left(\lambda, \frac{(Jp^h)^v}{\|Jp^h\|}\right) \\ & + \frac{1}{\|Jp^h\|} g\Big(v^h, 2\nabla^2 J(p^h, p^h, p^h) - 3u_0 \nabla J(Jp^h, p^h) \\ & \qquad - 2u_0 \nabla J(p^h, Jp^h) + \frac{1}{2} u_0^2 J^3 p^h\Big) \\ & + \frac{1}{\|p^h\|^2 \|Jp^h\|} g\big(v^h, \nabla U\big) g\big(p^h, 2\nabla J(p^h, p^h) - 2J\nabla U - u_0 J^2 p^h\big) \\ & - \frac{1}{\|Jp^h\|} g\left(v^h, 2\nabla J(\nabla U, p^h) + 2\nabla J(p^h, \nabla U) + 2\nabla_{p^h} J \nabla U\right).\end{aligned}$$

The function β can be expressed by substituting the last formula of $\mathcal{A}^{(1)}$ and item (1) of Lemma 7 into item (3) of Lemma 4. In this way one gets the required expression for the subspace $\mathcal{V}_c^{\text{trans}}(\lambda)$. To summarize, we have

$$\begin{aligned}\nabla^c_{v^h} = \underline{\nabla_{v^h}} - \frac{1}{2}\mathcal{A}(\lambda, v)\frac{(Jp^h)^v}{\|Jp^h\|} - \frac{u_0}{2}(Jv^h)^v \\ + \beta(v)\partial_{u_0} - \frac{1}{\|p^h\|^2} g\left(v^h, \nabla U\right)(p^h)^v. \quad (52)\end{aligned}$$

To finish the representation of the canonical splitting, we find more detailed expression for $\mathcal{V}_a^{\text{trans}}(\lambda) = \mathbb{R}\mathcal{F}_a(\lambda)$ on the base of Eqs. (23) and (40). For this we will describe the properties of vectors $\mathfrak{V}_0$, $\mathfrak{V}_1$, and $\mathfrak{W}_0$ from step 4 of Sect. 3.2 which will be used in the calculations of the curvature maps (Sect. 5). The proof is almost the same as that of Lemma 4.3 in [4].

Lemma 9. *Let $v \in \mathcal{V}_c(\lambda)$ and V be a parallel vector field such that $V(\lambda) = v$. Then the following identities hold:*

(1) $\mathfrak{V}_1^h = (\text{pr} \circ \pi)_* \mathfrak{W}_0 = -\frac{2}{\|Jp^h\|}\big(\nabla J(p^h, p^h) - J\nabla U\big) + \frac{u_0}{\|Jp^h\|} J^2 p^h$
$+ \left(\frac{2}{\|p^h\|^2 \|Jp^h\|} g(-J\nabla U, p^h) + u_0 \frac{\|Jp^h\|}{\|p^h\|^2}\right) p^h + \frac{2}{\|Jp^h\|^3} g(\nabla J(p^h, p^h)$
$- J\nabla U, Jp^h) Jp^h;$

(2) $\sigma\left(\mathfrak{W}_0, ad\,\overrightarrow{h}\,(\nabla^c_{V^h})\right) = g\left(\left(\mathfrak{R}_\lambda(c, c)v\right)^h, \mathfrak{V}_1^h\right),$

$\sigma(\mathfrak{W}_0, ad\,\overrightarrow{h}\,\mathcal{F}_b(\lambda)) = -g\left(\left(\mathfrak{R}_\lambda(c, b)\mathfrak{V}_1\right)^h, \frac{Jp^h}{\|Jp^h\|}\right).$

5 Curvature maps via the Riemannian curvature tensor and the tensor J on $\widetilde{M}$

Let $\lambda = (p, q),\ q \in M,\ p \in T_q^*M$ be the given D-regular point. Fix $v \in \mathcal{V}_c(\lambda)$. As before, denote by R^∇ the Riemannian curvature tensor. Note that the proofs in this section are modified based on that in [4] the appearance of the potential U.

Theorem 2. *The curvature map $\mathfrak{R}_\lambda(c, c)$ can be represented as follows*

$$\begin{aligned} g\Big(\big(\mathfrak{R}_\lambda(c,c)(v)\big)^h, v^h\Big) = {} & g(R^\nabla(p^h, v^h)p^h, v^h) + \operatorname{Hess} U(v^h, v^h) \\ & + u_0 g(v^h, \nabla J(p^h, v^h)) + \frac{u_0^2}{4}\|Jv^h\|^2 - \frac{1}{4}\mathcal{A}^2(\lambda, v) \\ & + \frac{3}{\|p^h\|^2} g^2(v^h, \nabla U), \end{aligned}$$

where $\mathcal{A}$ is as in (49)

Proof. Take $v \in \mathcal{V}_c(\lambda)$ and parallel vector fields V such that $V(\lambda) = v$. As in the proof of Lemma 3.4 in [4] we can take V such that

$$[(\nabla U)^v + u_0(Jp^h)^v, V](\bar\lambda) = 0, \quad \bar\lambda \in O \cap T_q^*M, \tag{53}$$

where O is a neighborhood of λ. For simplicity denote $\bar\sigma = (I \circ \mathrm{PR})^*\tilde\sigma$.

Recall that by Proposition 2, (relation (41) there)

$$g\big((\mathfrak{R}_\lambda(c,c)v)^h, w^h\big) = -\sigma(\mathrm{ad}\overrightarrow{h}\ \nabla^c_{V^h}, \nabla^c_{v^h}).$$

Let us simplify the right-hand side of the last identity. First, from the last line of the structural Eqs. (22) it follows that

$$\pi_*(\mathrm{ad}\overrightarrow{h}(\nabla^c_{V^h})) \in \mathbb{R}p^h. \tag{54}$$

Then from (52) it follows that

$$\begin{aligned} &\sigma(\mathrm{ad}\overrightarrow{h}(\nabla^c_{V^h}), \nabla^c_{v^h}) \\ &\quad = \sigma(\mathrm{ad}\overrightarrow{h}(\nabla^c_{V^h}), \underline{\nabla_{v^h}}) - \frac{1}{\|p^h\|^2} g\left(v^h, \nabla U\right)\sigma(\mathrm{ad}\overrightarrow{h}(\nabla^c_{V^h}), (p^h)^v) \end{aligned} \tag{55}$$

For the second term in (55), we carry out the following calculations:

$$\begin{aligned} 0 &= d\sigma(\overrightarrow{h}, \nabla^c_{V^h}, (p^h)^v) \\ &= -\nabla^c_{V^h}\left(\sigma(\overrightarrow{h}, (p^h)^v)\right) - \sigma([\overrightarrow{h}, \nabla^c_{V^h}], (p^h)^v) \\ &\quad + \bar\sigma([\overrightarrow{h}, (p^h)^v], \nabla^c_{V^h}) - \sigma([\nabla^c_{V^h}, (p^h)^v], \overrightarrow{h}) \end{aligned} \tag{56}$$

$$
\begin{aligned}
&= -\nabla^c_{V^h}(\|p^h\|^2) - \sigma([\overrightarrow{h}, \nabla^c_{V^h}], (p^h)^v) \\
&\quad + \sigma(-\underline{\nabla_{p^h}} - (\nabla U)^v, \nabla^c_{V^h}) + \nabla^c_{V^h}(\|p^h\|^2) \\
&= -\sigma([\overrightarrow{h}, \nabla^c_{V^h}], (p^h)^v) + \sigma(-\underline{\nabla_{p^h}} - (\nabla U)^v, \nabla^c_{V^h}) \\
&= -\sigma([\overrightarrow{h}, \nabla^c_{V^h}], (p^h)^v) + \sigma(-\overrightarrow{h} - 2(\nabla U)^v, \nabla^c_{V^h})) \qquad (57) \\
&= -\sigma([\overrightarrow{h}, \nabla^c_{V^h}], (p^h)^v) + 2g(\nabla U, v^h).
\end{aligned}
$$

For the first term in (55), from the decomposition (24) it follows that the form $u_0\pi^* d\omega_0 = \sigma - \bar{\sigma}$ is semi-basic (i. e. its interior product with any vertical vector field is zero). Besides, since $v \in \mathcal{V}_c(\lambda)$, from (31) it follows that $\pi^* d\omega_0(\overrightarrow{h}, \underline{\nabla_{v^h}}) = g(Jp^h, v^h) = 0$. Therefore,

$$
\sigma(\mathrm{ad}\,\overrightarrow{h}\,(\nabla^c_{V^h}), \underline{\nabla_{v^h}}) = \bar{\sigma}(\mathrm{ad}\,\overrightarrow{h}\;\nabla^c_{V^h}, \nabla_{v^h}). \qquad (58)
$$

Also, from relation (41) it follows that it is enough to consider $\mathrm{ad}\,\overrightarrow{h}\;\nabla^c_{V^h}$ modulo $\mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda)$. We also need the following

Lemma 10. *Let V, W be vector fields of T^*M such that $\pi_* V = \pi_* W = 0$. Let U be the potential, as before. Then*

1) $([(Jp^h)^v, (JV^h)^v])^h = J([(Jp^h)^v, (V^h)^v])^h$;
2) $\bar{\sigma}([(Jp^h)^v, \underline{\nabla_{V^h}}], \underline{\nabla_{W^h}}) = g(W^h, \nabla J(p^h, V^h))$;
3) $[\nabla U, J(V^h)^v] = J([\nabla U, (V^h)^v])$.

Proof. The first two terms were proved in Lemma 5.1 in [4]. And the third one can be shown in a similar way. Indeed, if item (3) holds for vector field V then also holds for vector field aV. Thus in order to prove item (3) it is sufficient to prove it when V is constant on the fibers of T^*M, i. e., when V^h is a vector field on $\widetilde{M}$. But in this case from item 1 of Proposition 3 for $K = 0, N = 0$ it follows that both sides of item (3) vanish.

Now we are ready to start our calculations:

$$
\begin{aligned}
\mathrm{ad}\,\overrightarrow{h}\,(\nabla^c_{V^h}) = {} & [\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}] + [-(\nabla U)^v - u_0(Jp^h)^v, \underline{\nabla_{V^h}}] \qquad (59) \\
& -\frac{\mathcal{A}(\lambda, v)}{2\|Jp^h\|}[\underline{\nabla_{p^h}}, (Jp^h)^v] + \frac{\mathcal{A}(\lambda, v)}{2\|Jp^h\|}[(\nabla U)^v, (Jp^h)^v] \\
& -\frac{u_0}{2}[\underline{\nabla_{p^h}}, (JV^h)^v] - \frac{u_0}{2}[-(\nabla U)^v - u_0(Jp^h)^v, (JV^h)^v] \\
& -\frac{1}{\|p^h\|^2} g\left(v^h, \nabla U\right)[\underline{\nabla_{p^h}} - u_0(Jp^h)^v - (\nabla U)^v, (p^h)^v], \\
& \mathrm{mod}\, \mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda) \oplus \mathbb{R}(p^h)^v \qquad (60)
\end{aligned}
$$

Now let us analyze each term in (59). First of all, as in the Riemannian case (see (2.14) in [4]),

$$\bar{\sigma}([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}], \underline{\nabla_{v^h}}) = -g(R^{\nabla}(p^h, v^h)p^h, v^h). \tag{61}$$

For the second term in (59), it follows from Proposition 4 and Lemma 10 that

$$\bar{\sigma}\left([(\nabla U)^v + u_0(Jp^h)^v, \underline{\nabla_{V^h}}], \underline{\nabla_{v^h}}\right) = \text{Hess } U(v^h, v^h) + u_0 g(v^h, \nabla J(p^h, v^h)). \tag{62}$$

For the third term, it follows from Proposition 3 that

$$\bar{\sigma}\left(-\frac{\mathcal{A}(\lambda, v)}{2\|Jp^h\|}[\underline{\nabla_{p^h}}, (Jp^h)^v], \underline{\nabla_{v^h}}\right) = \frac{\mathcal{A}(\lambda, v)}{2\|Jp^h\|} g(v^h, \nabla J(p^h, p^h)). \tag{63}$$

For the fourth term, again, from Proposition 3 we have

$$\bar{\sigma}\left(\frac{\mathcal{A}(\lambda, v)}{2\|Jp^h\|}[(\nabla U)^v, (Jp^h)^v], \underline{\nabla_{v^h}}\right) = \frac{\mathcal{A}(\lambda, v)}{2\|Jp^h\|} g(v^h, J\nabla U). \tag{64}$$

For the fifth term, we use the following fact (see Lemma 5.1 in [4]):

$$\pi_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]) = \frac{u_0}{2}(Jv^h)^v - \frac{1}{2}\mathcal{A}(\lambda, v)\frac{(Jp^h)^v}{\|Jp^h\|} \mod \mathbb{R}p^h. \tag{65}$$

Then it follows that

$$\begin{aligned} \bar{\sigma}(-\frac{u_0}{2}[\underline{\nabla_{p^h}}, (JV^h)^v], \underline{\nabla_{v^h}}) &= -\frac{u_0}{2}\bar{\sigma}(\pi_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]), (Jv^h)^v) \\ &= -\frac{1}{4}u_0^2\|Jv^h\|^2 + \frac{u_0\mathcal{A}(\lambda, v)}{4\|Jp^h\|} g(Jv^h, Jp^h). \end{aligned} \tag{66}$$

Note that the sixth term vanishes due to item (3) in Lemma 10 and (53). For the last term, using Proposition 3 we have

$$\begin{aligned} &\bar{\sigma}\left(\frac{1}{\|p^h\|^2} g\left(v^h, \nabla U\right)[\underline{\nabla_{p^h}} - u_0(Jp^h)^v - (\nabla U)^v, (p^h)^v], \underline{\nabla_{v^h}}\right) \\ &\qquad = \frac{1}{\|p^h\|^2} g^2(v^h, \nabla U). \end{aligned} \tag{67}$$

Summering up the above calculations, we conclude

$$\begin{aligned} \bar{\sigma}(\text{ad}\overrightarrow{h}\ \nabla^c_{V^h}, \nabla_{v^h}) = &-g(R^{\nabla}(p^h, v^h)p^h, v^h) - \text{Hess } U(v^h, v^h) \\ &- u_0 g(v^h, \nabla J(p^h, v^h)) \\ &+ \frac{\mathcal{A}(\lambda, v)}{2\|Jp^h\|} g(v^h, \nabla J(p^h, p^h)) - \frac{\mathcal{A}(\lambda, v)}{2\|Jp^h\|} g(v^h, J\nabla U) \\ &- \frac{1}{4}u_0^2\|Jv^h\|^2 + \frac{u_0\mathcal{A}(\lambda, v)}{4\|Jp^h\|} g(Jv^h, Jp^h) \\ &- \frac{1}{\|p^h\|^2} g^2(v^h, \nabla U). \end{aligned}$$

Put (56) and the last identity together, we complete the proof of the theorem. □

For the other curvature maps $\mathfrak{R}_\lambda(c,b), \mathfrak{R}_\lambda(b,b)$, we calculate them under the assumption that J is compatible with the metric g, i. e. $\nabla J = 0$ in order to avoid long formulas and calculations.

Theorem 3. *Assume* $\mathfrak{R}_\lambda(c,b)v = \rho_\lambda(c,b)(v)\mathcal{E}_b(\lambda)$, *where* $\rho_\lambda(c,b) \in \mathcal{V}_c(\lambda)^*$ *and* $\mathfrak{R}_\lambda(b,b)\mathcal{E}_b(\lambda) = \rho_\lambda(b,b)\mathcal{E}_b(\lambda)$, *then*

$$\begin{aligned}\rho_\lambda(c,b)(v) = &\frac{1}{\|Jp^h\|} g(R^\nabla(p^h, Jp^h)p^h, v^h)\\ &+ \frac{1}{\|Jp^h\|}\Big(- 3\text{Hess } U(Jv^h, p^h) + \text{Hess } U(Jp^h, v^h)\Big)\\ &- \frac{u_0}{\|Jp^h\|} g\big(Jv^h, -2J\nabla U - u_0 J^2 p^h\big)\end{aligned}$$

and

$$\begin{aligned}\rho_\lambda(b,b) = &\frac{1}{\|Jp^h\|^2} g(R^\nabla(p^h, Jp^h)p^h, Jp^h)\\ &+ \frac{1}{\|Jp^h\|^2}\text{Hess } U(Jp^h, Jp^h) + \frac{3}{\|Jp^h\|^2}\text{Hess } U(J^2p^h, p^h)\\ &- \frac{10}{\|Jp^h\|^4} g^2(J\nabla U, Jp^h) + \frac{6u_0}{\|Jp^h\|^2} g(J\nabla U, J^2p^h)\\ &+ \frac{6}{\|Jp^h\|^2}\|J\nabla U\|^2 + \frac{\|J^2p^h\|^2}{\|Jp^h\|^2} u_0^2.\end{aligned}$$

Theorem 4. *The curvature maps* $\mathfrak{R}_\lambda(c,a)$ *and* $\mathfrak{R}_\lambda(a,a)$ *can be represented as follows*

1) $\mathfrak{R}_\lambda(c,a)v = \rho_\lambda(c,a)(v)\dfrac{\partial_{u_0}}{\|Jp^h\|}$, *where* $\rho_\lambda(c,a) \in \mathcal{V}_c(\lambda)^*$ *and it satisfies*

$$\begin{aligned}\rho_\lambda(c,a)v = &\|Jp^h\| \left(\frac{1}{\|Jp^h\|}\mathcal{A}\right)^{(2)}(\lambda, v) - g\Big((\mathfrak{R}_\lambda(c,c)v)^h, \mathfrak{V}_1^h\Big)\\ &+ \|Jp^h\| \overrightarrow{h}\Big(\frac{1}{\|Jp^h\|}\Big)\rho_\lambda(c,b)v;\end{aligned}$$

2) $\mathfrak{R}_\lambda(a,a)\partial_{u_0} = \rho_\lambda(a,a)\partial_{u_0}$, *where* $\rho_\lambda(c,a) \in \mathcal{V}_c(\lambda)^*$ *and it satisfies*

$$\begin{aligned}\rho_\lambda(a,a) = &\overrightarrow{h}\left(\rho_\lambda(c,b)(\mathfrak{V}_1^h)\right) + \|Jp^h\|\overrightarrow{h}\left(\frac{1}{\|Jp^h\|}\right)\overrightarrow{h}(\rho_\lambda(b,b))\\ &+ \rho_\lambda(c,a)(\mathfrak{V}_1) - \|Jp^h\|\overrightarrow{h}\left(\frac{1}{\|Jp^h\|}\right)\rho_\lambda(c,b)(\mathfrak{V}_1)\\ &+ \|Jp^h\|\overrightarrow{h}^2\left(\frac{1}{\|Jp^h\|}\right)\rho_\lambda(b,b) + \|Jp^h\|\overrightarrow{h}^4\left(\frac{1}{\|Jp^h\|}\right)\end{aligned}$$

where $\rho_\lambda(c,b)$ *and* $\rho_\lambda(b,b)$ *are as in Theorem 3,* $\mathcal{A}$ *is expressed in* (49) *and* $\mathfrak{W}_1^h$ *is expressed by item (1) of Lemma 9.*

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On the Alexandrov Topology of sub-Lorentzian Manifolds

Irina Markina and Stephan Wojtowytsch

Abstract In the present work, we show that in contrast to sub-Riemannian geometry, in sub-Lorentzian geometry the manifold topology, the topology generated by an analogue of the Riemannian distance function and the Alexandrov topology based on causal relations, are not equivalent in general and may possess a variety of relations. We also show that 'opened causal relations' are more well-behaved in sub-Lorentzian settings.

1 Introduction

Recall that a SemiRiemannian (or PseudoRiemannian) manifold is a C^∞-smooth manifold M equipped with a non-degenerate symmetric tensor g. The tensor defines a scalar product on the tangent space at each point. The quadratic form corresponding to the scalar product can have different numbers of negative eigenvalues. If the quadratic form is positively definite everywhere, the manifold is usually called Riemannian. The special case of one negative eigenvalue received the name the Lorentzian manifold.

Let us assume that a smooth subbundle D of the tangent bundle TM is given, on which we shall later impose certain non-integrability conditions. Suppose also that a non-degenerate symmetric tensor g defines a scalar product g_D on planes $D_q \subset T_qM$. Then the triplet (M, D, g_D) is called a sub-SemiRiemannian manifold. If g_D is positive definite everywhere, then (M, D, g_D) is called a sub-Riemannian manifold. Those manifolds are an active area of research, see, for instance [2,4,15,21,37].

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In the case of exactly one negative eigenvalue the manifold is called sub-Lorentzian. This setting has been considered in [5, 7–10, 12–14].

Sub-SemiRiemannian manifolds are an abstract setting for mechanical systems with non-holonomic constraints, linear and affine control systems, the motion of particles in magnetic fields, Cauchy-Riemann geometry and other subjects from pure and applied mathematics.

The main goal of the present work is to study the relations between the given manifold topology and the Alexandrov and time-separation topologies defined by causality properties in the presence of a sub-Lorentzian metric. The main aim is to compare these topologies for Lorentzian and sub-Lorentzian manifolds.

Recall that on Riemannian manifolds the original topology of the manifold and metric topology defined by the Riemannian distance function are equivalent. Due to the Ball-Box theorem we observe that in sub-Riemannian geometry the metric topology induced by the sub-Riemannian distance function and the original manifold topology are equivalent, too.

Theorem 1 (Ball-Box Theorem). *Let* (M, D, g_D) *be a sub-Riemannian manifold. Then for every point* $p \in M$ *there exist coordinates* (U, x) *around* p *and constants* $c, C > 0$ *such that the sub-Riemannian distance function* d_{sR} *defined by*

$$d_{sR}(p,q) = \inf\Big\{ \text{length of absolutely continuous curves } \ \gamma\colon [0,1] \to M, \ \gamma(0) = p, \ \gamma(1) = q, \ \dot{\gamma}(t) \in D_{\gamma(t)} \ \text{for almost all } \ t\Big\},$$

can be estimated by $c \ \sum_{i=1}^{n} |x^i|^{\frac{1}{w_i}} \le d_{sR}(p,q) \le C \ \sum_{i=1}^{n} |x^i|^{\frac{1}{w_i}}, \quad x \in U,$ *where the constants* $w_i \in \mathbb{N}$ *are determined by the non-integrability properties of* D *at the point* p.

In the Lorentzian and sub-Lorentzian cases we cannot obtain a metric distance function from a given indefinite scalar product. The closest analogue is the time separation function, which behaves quite differently from metric distances in a number of aspects.

The other specific feature of Lorentzian manifolds is a causal structure. In a sense, causality theory is the natural replacement for the metric geometry of Riemannian manifolds in the Lorentzian case. From causal relations one obtains a new topology, called the *Alexandrov topology*. It is known that in Lorentzian manifolds the Alexandrov topology can be obtained also from the time separation function. We showed that for sub-Lorentzian manifolds this is not generally true anymore and that the time separation function defines a new *time separation topology*, that can be thought of as a different extension of the Alexandrov topology. Our main interest is the study these two topologies, their similarities and differences from Lorentzian Alexandrov topologies. We proved that the Alexandrov and time separation topologies on a sub-Lorentzian manifold do not generally coincide neither with each other nor with the manifold topology.

We introduce a third extension of the Alexandrov topology to sub-Lorentzian manifolds, in which we force the sets from the Alexandrov topology to be open with respect to the manifold topology. They will correspond to what we call *opened causal relations*. The opened causal relations provide a useful tool both for our purpose of comparison of topologies and for the generalization of the causal hierarchy of space-times to the sub-Lorentzian case.

The work is organized in the following way. In Sect. 2 we review the main definitions of Lorentzian geometry and introduce them for sub-Lorentzian manifolds. In Sect. 3 we study the reachable sets and the causal structure of sub-Lorentzian manifolds, which allows us to introduce the Alexandrov topology. Section 3 contains our main result, where we compare the Alexandrov topology with the manifold topology. We also present different ways of introducing the Alexandrov topology and study some special class of sub-Lorentzian manifolds, that we call *chronologically open*, in which the Alexandrov topology behaves similarly as in the classical Lorentzian case. The last Sect. 4 is devoted to the study of the Alexandrov topology and the time separation topology.

2 Basic Concepts

2.1 Lorentzian Geometry

Definition 1. Let M be a smooth manifold and g be a smoothly varying (0, 2)-tensor with one negative eigenvalue on M.[1] Then the pair (M, g) is called a Lorentzian manifold. Furthermore, we will assume that M is connected throughout the paper.

The symmetric torsion free Levi-Civita connection, normal neighbourhoods and exponential maps are defined as in Riemannian geometry. We refer the reader to [18] for notations and the main definitions and results.

Definition 2. Let (M, g) be a Lorentzian manifold. A vector $v \in T_pM$, $p \in M$ is called:

- spacelike, if $g(v, v) > 0$ or $v = 0$;
- null or lightlike, if $g(v, v) = 0$ and $v \neq 0$;
- timelike, if $g(v, v) < 0$;
- nonspacelike, if v is null or timelike.

A vector field V is called timelike, if V_p is timelike for all $p \in M$, and similarly for the other conditions.

Definition 3. Let (M, g) be a Lorentzian manifold. A globally defined timelike vector field T is called a time orientation of (M, g). The triplet (M, g, T) is known as space-time or time oriented manifold.

Every Lorentzian manifold is either time orientable or admits a twofold time orientable cover [3]. Curves on a space-time are distinguished according to their causal nature and time orientation as stated in the following definition.

[1] Note that we cannot define the eigenvalues of a quadratic form, but their sign due to Sylvester's Theorem of Inertia. We call the number of negative eigenvalues the index of the form.

Definition 4. Let (M, g, T) be a space-time. An absolutely continuous curve $\gamma\colon I \to M$ is:

- future directed, if $g(T, \dot\gamma) < 0$ almost everywhere;
- past directed, if $g(T, \dot\gamma) > 0$ almost everywhere.

We call an absolutely continuous curve $\gamma\colon I \to M$ null (timelike or nonspacelike) and future or past directed, if $g(\dot\gamma, \dot\gamma) = 0$ (< 0 or ≤ 0) almost everywhere and γ is future directed or past directed. We call γ simply null (timelike or nonspacelike), if it is null (timelike or nonspacelike) and either past or future directed.

We abbreviate timelike to t., nonspacelike to nspc., future directed to f.d., and past directed to p.d. from now on. Nspc.f.d. curves are also called *causal*.

Definition 5. Let M be a space-time and $p, q \in M$. We write:

- $p \leq q$ if $p = q$ or there exists an absolutely continuous nspc.f.d. curve from p to q;
- $p \ll q$ if there exists an absolutely continuous t.f.d. curve from p to q.

Define the chronological past I^-, future I^+, the causal past J^- and future J^+ of $p \in M$ by

$$I^+(p) = \{q \in M \mid p \ll q\}, \quad I^-(p) = \{q \in M \mid q \ll p\},$$
$$J^+(p) = \{q \in M \mid p \leq q\}, \quad J^-(p) = \{q \in M \mid q \leq p\}.$$

Let $U \subset M$ be an open set. Then we write $\ll_U$ for the causal relation $\ll$ taken in U, where U is considered as a manifold itself, and $I^+(p, U)$ for the future set obtained on it. Observe, that usually $I^+(p, U) \neq I^+(p) \cap U$ since U might lack a convexity.

In [18, Chap. 5, Proposition 34] it is shown that if U is a normal neighbourhood of p, then $I^+(p, U) = \exp_p(I^+(0) \cap V_p)$ where $I^+(0) \subset T_pM$ is the Minkowski light cone and V_p is the neighbourhood of the origin of T_pM on which $\exp_p$ is a diffeomorphism to U.

The definition of the order immediately gives that $\leq, \ll$ are transitive and $p \ll q \Rightarrow p \leq q$. We state the following result on stronger transitivity of these relations.

Proposition 1. *[18, 19] The following is true for space-times:*

1) *if either $p \leq r, r \ll q$ or $p \ll r, r \leq q$, then $p \ll q$;*
2) *let $\gamma\colon [0, 1] \to M$ be a nonspacelike curve, $p = \gamma(0)$, $q = \gamma(1)$. If γ is not a null geodesic (up to reparametrization), then there is a timelike curve $\sigma\colon [0, 1] \to M$ such that $\sigma(0) = p$ and $\sigma(1) = q$.*

Lorentzian manifolds can be categorized in different levels of a causal hierarchy.

Definition 6. We call a space-time (M, g, T):

1) chronological, if there is no $p \in M$ such that $p \ll p$;

2) causal, if there are no two points $p, q \in M$, $p \neq q$, such that $p \leq q \leq p$;
3) strongly causal, if for every point p and every open neighbourhood U of p there is a neighbourhood $V \subset U$ of p, such that no nspc. curve that leaves the neighbourhood V ever returns to it.

Neighbourhoods V in the definition of strongly causal manifolds are called causally convex. Each requirement in Definition 6 is stronger than the preceding one and no two requirements are equivalent, see [3]. An example of a strongly causal manifold is any convex neighbourhood U with compact closure of a point.

The time-separation function, or the Lorentzian distance function, is defined by

$$T^S(p,q) = \sup\left\{\int_0^1 \sqrt{-g(\dot\gamma,\dot\gamma)}\,dt \mid \gamma \in \Omega_{p,q}\right\}, \tag{1}$$

where the space $\Omega_{p,q}$ consists of future directed nonspacelike curves defined on the unit interval joining p with q. If $\Omega_{p,q} = \emptyset$ then we declare the supremum is equal to 0. Because the arc length $L(\gamma) = \int_0^1 \sqrt{-g(\dot\gamma,\dot\gamma)}\,dt$ of the curve γ is invariant under monotone reparametrization, the normalization to the unit interval is admissible. It follows immediately from the definition that T^S satisfies the inverse triangle inequality

$$T^S(p,q) \geq T^S(p,r) + T^S(r,q) \quad \text{for all points} \quad p \leq r \leq q \in M.$$

However, the distance function fails to be symmetric, possibly even to be finite, and it vanishes outside the causal future set.

2.2 Sub-Lorentzian Manifolds

The setting we are going to explore now is a generalization of Lorentzian geometry. On a sub-Lorentzian manifold a metric, that is a non-degenerate scalar product at each point, varying smoothly on the manifold, is defined only on a subspace of the tangent space, but not necessarily on the whole tangent space. If the subspace is proper, those manifolds may behave quite differently from Lorentzian ones.

Definition 7. A smooth distribution D on a manifold M is a smooth subbundle of TM.

If D is a smooth distribution, then for any point $p \in M$ there exists a neighbourhood U and smooth vector fields $X_1, \ldots, X_k$ satisfying $D_q = \operatorname{span}\{X_1(q), \ldots, X_k(q)\}$ for all $q \in U$. We write $\operatorname{rank}_p(D) = \dim(D_p)$. We assume that $2 \leq k \equiv \operatorname{rank}_p(D) < \dim(M) = n$ everywhere. An *admissible* or *horizontal* curve $\gamma\colon I \to M$ is an absolutely continuous curve such that $\dot\gamma(t) \in D_{\gamma(t)}$ almost everywhere and such that $\dot\gamma$ is locally square integrable with respect to an auxiliary Riemannian metric (see the discussion after Proposition 2). We are interested in whether two arbitrary points can be connected by an admissible curve. That need not be possible due to the Frobenius theorem.

Definition 8. Let $N \in \mathbb{N}$ and $I = (i_1, \ldots, i_N) \in \mathbb{N}^N$ be a multi-index. Let $X_1, \ldots, X_k$ be vector fields on M. We define $X_I = [X_{i_1}, [X_{i_2}, \ldots [X_{i_{N-1}}, X_{i_N}]]]$. A distribution D satisfies the bracket-generating hypothesis, if for every point $p \in M$ there are $N(p) \in \mathbb{N}$, a neighbourhood U of p such that $D_q = \text{span}\{X_1(q), \ldots, X_k(q)\}$ for all $q \in U$ and

$$T_p M = \text{span}\{X_I(p) \mid I = (i_1, ..., i_N),\ N \leq N(p)\}.$$

A sufficient condition of the connectivity by admissible curves is given by the Chow-Rashevskii theorem [6,20,37], stating that if M is a connected manifold with a bracket-generating distribution D, then any two points $p, q \in M$ can be connected by an admissible curve.

Definition 9. A sub-Lorentzian manifold is a triple (M, D, g) where M is a manifold with a smooth bracket generating distribution D and a non-degenerate symmetric bilinear form $g\colon D_p \times D_p \to \mathbb{R}$ of constant index 1 smoothly varying on M.

A sub-space-time is a quadruple (M, D, g, T), where (M, D, g) is a sub-Lorentzian manifold and T is a globally defined horizontal timelike vector field. Like in the Lorentzian case we call T a time orientation.

Similar to Lorentzian geometry, in a sub-Lorentzian manifold we have the order relations $\leq$, $\ll$, satisfying the following properties:

- $\leq$, $\ll$ are partial orders. Note that the first property in Proposition 1 does not hold anymore for sub-space-times, see Example 7. However, it still holds for a smaller class of sub-space-times, that we call chronologically open sub-space-times, see Definition 12;
- causality conditions are defined analogously to the Lorentzian case;
- the sets $J^{\pm}$, $I^{\pm}$ for sub-Lorentzian manifolds are defined as before for Lorentzian ones.

The following statement can be proved by the same arguments as in the sub-Riemannian case, see, for instance [16].

Proposition 2. *A sub-Lorentzian metric g on a sub-space-time (M, D, g, T) can always be extended to a Lorentzian metric $\tilde{g}$ over the whole manifold M. A sub-space-time (M, D, g, T) thus becomes a space-time $(M, \tilde{g}, T)$ with the same time orientation T.*

Proposition 2 allows us to use results from Lorentzian geometry, because every horizontal t.f.d. curve will be t.f.d. with respect to all extended metrics. Moreover, using extended metrics $\tilde{g}_\lambda = g \circ \pi_D + \lambda^2\, h \circ \pi_{D^\perp}$ for some Riemannian metric h, one can show by letting $\lambda \to \infty$, that a curve, which is t.f.d. with respect to all extended metrics $\tilde{g}_\lambda$, is actually t.f.d. horizontal.

Let (M, D, g) be a sub-Lorentzian manifold. An open subset $U \subset M$ is called convex, if U has compact closure $\overline{U}$ and there is an extension $\tilde{g}$ of g and an open set $V \supset \overline{U}$ such that both V and U are uniformly normal neighbourhoods of their points in $(M, \tilde{g})$ in the sense of Lorentzian geometry. Thus one can introduce coordinates and a Lorentzian orthonormal frame $\{T, X_1, ..., X_n\}$ on V. Having a Lorentzian metric one can construct an auxiliary Riemannian metric as in [3]. Then, using Proposition 2, we adapt the following statement for continuous causal curves, i. e. continuous curves γ satisfying $s < t \Rightarrow \gamma(s) \leq \gamma(t), \gamma(s) \neq \gamma(t)$:

Proposition 3. *[3] Let M be a sub-space-time and $\gamma\colon I \to M$ a continuous causal curve. Then γ is locally Lipschitz with respect to an auxiliary Riemannian metric.*

Using the Rademacher theorem we obtain the following corollary.

Corollary 1. *Let M be a sub-space-time. A continuous causal curve is absolutely continuous and its velocity vector almost everywhere is nspc.f.d. with respect to an extension of the sub-Lorentzian metric and square integrable with respect to an auxiliary Riemannian metric. It follows that the curve γ is horizontal.*

Lorentzian and sub-Lorentzian manifolds do not carry a natural metric distance function which would allow a natural topology on curves in the manifold. Since monotone reparametrization does not influence causal character, we define the C^0-topology in the following way.

Definition 10. Let U, V, W be open sets in topology τ of M such that $V, W \subset U$. Then we define the set

$$B_{U,V,W,0,1} = \{\gamma \in C([0,1], M) |\ \gamma(0) \in V,\ \gamma(1) \in W,\ \gamma([0,1]) \subset U\}$$

and to eliminate the need to fix parametrization we take the union over all possible parametrizations $B_{U,V,W} = \bigcup B_{U,V,W,0,1}$. The C^0-topology on curves is the topology generated by the basis

$$\mathscr{B} := \{B_{U,V,W} \mid U, V, W \in \tau, V, W \subset U\}.$$

The C^0-topology is constructed in such a way that curves $\gamma_n\colon [0,1] \to M$ converge to $\gamma\colon [0,1] \to M$ if and only if

$$\gamma_n(0) \to \gamma(0), \quad \gamma_n(1) \to \gamma(1),$$

and for any $U \in \tau$ and $\gamma([0,1]) \subset U$ there exists a positive integer N such that $\gamma_n([0,1]) \subset U$ for all $n \geq N$. For general space-times or sub-space-times this notion of convergence might not be too powerful, some information on that may be found in [3]. However, it becomes useful for strongly causal sub-space-times.

Theorem 2. *If nspc.f.d. horizontal curves $\gamma_n\colon [0,1] \to M$ converge to $\gamma\colon [0,1] \to M$ in the C^0-topology on curves in a strongly causal sub-space-time then γ is horizontal nspc.f.d.*

Proof. The result is standard in Lorentzian geometry. A proof, that γ is locally nspc.f.d. can be found in [11]. By the transitivity of $\leq$, it is also globaly nspc.f.d.. Using Proposition 2 and Corollary 1, we easily generalize the result to sub-Lorentzian manifolds. □

3 Reachable Sets, Causality and the Alexandrov Topology

3.1 Reachable Sets

As was mentioned above, the manifold topology and the metric topology of a Riemannain or a sub-Riemannian manifold are equivalent as follows from the Ball-Box theorem. The causal structure of Lorentzian manifolds allows us to introduce a new topology, called the Alexandrov topology that is (often strictly) coarser than the manifold topology. We are interested in comparing an analogue of the Alexandrov topology in sub-Lorentzian manifolds with the initial manifold topology. Let us begin by reviewing the background from Lorentzian manifolds.

It is well known that in a space-time (not a sub-space-time) (M, g, T) the sets $I^+(p)$ and $I^-(p)$ are open in the manifold topology for all points $p \in M$, see [3]. In sub-space-times this is not true anymore. We give two examples showing that sets $I^+(0)$, $I^-(0)$ may or may not be open in sub-space-times.

Example 1. [7] Let $M = \mathbb{R}^3 = \{(x, y, z)\}$, $D = \operatorname{span}\left\{T = \partial_y + x^2\partial_z, X = \partial_x\right\}$ and g be the sub-Lorentzian metric determined by $g(T, T) = -1$, $g(T, X) = 0$, $g(X, X) = 1$. As $\partial_z = \frac{1}{2}[X, [X, T]]$, the distribution D is bracket generating. We choose T as the time orientation and show that the set $I^+(0, U)$ is not open for any neighbourhood U of the origin. Precisely, we show that for small enough $\theta > 0$ the point $(0, \theta, 0)$ will be contained in $I^+(0, U)$, while the point $(0, \theta, -a)$ will not be in $I^+(0, U)$ for any $a > 0$.

The curve $\gamma(t) = (0, t, 0)$ is horizontal t.f.d. since $\dot\gamma(t) = T_{\gamma(t)}$, and for small enough times it runs in U, so $(0, \theta, 0) \in I^+(p, U)$.

Assume that there is a horizontal nspc.f.d. curve $\sigma\colon [0, \tau] \to M$, $\sigma = (x, y, z)$, from 0 to $(0, \theta, -a)$ for some $a > 0$. Then

$$\dot\sigma(t) = \alpha(t)T_{\sigma(t)} + \beta(t)X_{\sigma(t)} = (\beta(t), \alpha(t), x^2(t)\alpha(t)).$$

Since σ is future directed, we find that $\alpha(t) > 0$ almost everywhere. Then due to absolute continuity

$$-a = z(\tau) = \int_0^\tau \dot z(t)dt = \int_0^\tau \alpha(t)x^2(t)dt,$$

which is impossible since the integrand is non-negative almost everywhere. Hence, we obtain $(0, \theta, -a) \notin I^+(0, U)$ for any $a > 0$, which implies that $I^+(0, U)$ is not open in M.

The curve γ lies on the boundary of $I^+(0, U)$. It is an example of a rigid curve, or a curve that cannot be obtained by any variation with fixed endpoints, see [15,17,37]. The curve γ is the unique (up to reparametrization) horizontal nspc.f.d. curve from 0 to $(0, \theta, 0)$.

The next example shows that there are sub-space-times for which $I^{\pm}(p)$ are open.

Example 2. [7] Consider the sub-space-time with $M = \mathbb{R}^3$, $D = \text{span}\{X, Y\}$, where

$$X = \frac{\partial}{\partial x} + \frac{1}{2}\, y\, \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - \frac{1}{2}\, x\, \frac{\partial}{\partial z},$$

the metric $g(X, X) = -g(Y, Y) = -1$, $g(X, Y) = 0$, and the vector field X as time orientation. This sub-space-time is called the Lorentzian Heisenberg group. In this sub-space-time the sets $I^+(p)$, $I^-(p)$ are open for all $p \in M$. The details of the proof can be found in [7], where the chronological future set of the origin

$$I^+(0) = \big\{(x, y, z) \mid -x^2 + y^2 + 4|z| < 0,\ x > 0\big\}$$

is calculated. The set $I^+(0)$ is obviously open. We apply the Heisenberg group multiplication in order to translate the chronological future set $I^+(p_0)$ of an arbitrary point $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ to the set $I^+(0)$. The map

$$\Phi(x, y, z) = \left(x - x_0, y - y_0, z - z_0 + \frac{1}{2}(y\, x_0 - x\, y_0)\right)$$

maps p_0 to 0, preserves the vector fields X and Y and hence maps $I^+(p_0)$ to $I^+(0)$. Since Φ is also a diffeomorphism of $\mathbb{R}^3$, we conclude that $I^+(p_0)$ is open for all $p_0 \in \mathbb{R}^3$. Similarly $\Phi(x, y, z) = (-x, -y, z)$ exchanges X for $-X$ and Y for $-Y$, i. e. it preserves the distribution and the scalar product, but it reverses time orientation. Hence, it maps $I^-(p_1, p_2, p_3)$ to $I^+(-p_1, -p_2, p_3)$ which proves, that also all chronological past sets are open.

In the following definition we generalize the properties of the above mentioned map Φ.

Definition 11. Let (M, D, g, T) be a sub-space-time. A diffeomorphism $\Phi\colon M \to M$ is called a sub-Lorentzian isometry, if it preserves the sub-Lorentzian causal structure, i. e.

$$\Phi_* D = D, \quad \Phi^* g = g, \quad g(\Phi_* T, T) < 0.$$

Lemma 1. *Let G be a group with a properly discontinuous action by sub-Lorentzian isometries on a sub-space-time M. Then the quotient M/G carries a canonical sub-space-time structure, such that the projection to equivalent classes $\pi : M \to M/G$ is a covering map and a local sub-Lorentzian isometry. Furthermore $I^+_{M/G}\big(\pi(p)\big) = \pi\big(I^+_M(p)\big)$.*

Proof. The proof that M/G is a smooth manifold and π is a covering map can be found in any textbook on differential topology. Take $q \in M/G$ and $p \in \pi^{-1}(q)$. Then we define $D_q = \{v \in T_q(M/G) \mid \exists\, V \in D_p \subset T_pM : \pi_*V = v\}$. As the action of G preserves the distribution, D_q is defined independently of the choice of $p \in \pi^{-1}(q)$. As the commutators of π-related vector-fields are π-related and π is a local diffeomorphism, the distribution on M/G is bracket-generating.

Now take $q \in M/G$ and $v, w \in D_q \subset T_q(M/G)$. We define $g_q(v,w) := g_p(V,W)$, where $p \in \pi^{-1}(q)$, $V, W \in D_p \subset T_pM$ and $\pi_*V = v$, $\pi_*W = w$. As π is an isomorphism, V and W are uniquely defined, and as the action of G preserves the scalar product, the value we get is independent of the choice of p. This gives a well-defined sub-Lorentzian metric on M/G.

The construction of a time orientation on M/G is a little subtler. If all isometries $\Phi \in G$ satisfy $\Phi_*T = T$ (as is the case in our examples), we can set $T^{M/G} = \pi_*T$. If G is finite, we choose $T_q^{M/G} = \sum_{p\in\pi^{-1}(q)} \pi_*T_p$.

In the general case in order to define time orientation $T^{M/G}$ it is preferable to work with an alternative definition of a time-orientation. A future selection $\tilde{T}$ is a set-valued map $\tilde{T} : M \to \mathscr{P}(TM)$ to the power set $\mathscr{P}(TM)$ of TM such that:

1) $\tilde{T}(p) \subset T_pM$ is connected;
2) the set of timelike vectors in T_pM equals $\tilde{T}(p) \cup -\tilde{T}(p)$;
3) $\tilde{T}$ is continuous in the sense that if $U \subset M$ is open and connected and a vector field X is timelike on U, then either $X_p \in \tilde{T}(p)$ or $X_p \in -\tilde{T}(p)$ for all $p \in U$.

So a future selection is a continuous choice of a time cone in the tangent space as the future time cone. Clearly, a time orientation T by a vector field defines a future selection function by choosing $\tilde{T}(p) = \{v \in T_pM \mid g(v,v) < 0,\ g(v,T) < 0\}$, but also a time orientation map defines a time orienting vector field.

To see this, we cover the manifold M by a locally finite countable open cover $\{U^{(n)}\}$ such that on each $U^{(n)}$ we can define a timelike vector field $T^{(n)}$. By the first and the second properties, either $T_p^{(n)} \in \tilde{T}(p)$ or $T_p^{(n)} \in -\tilde{T}(p)$ for all $p \in U^{(n)}$. We choose all $T^{(n)}$ such that $T_p^{(n)} \in \tilde{T}(p)$ wherever it is defined. Now, if more than one $T^{(n)}$ is defined at $p \in U^{(n)}$, then $T_p^{(n_1)}, ..., T_p^{(n_N)} \in \tilde{T}(p)$. As future cones are convex, also any convex combination $\sum \lambda^{n_i} T_p^{(n_i)}$ will lie in $\tilde{T}(p)$. Consequently, for a partition of unity $\{\chi_n\}$ subordinate to $\{U^{(n)}\}$, the field $T := \sum \chi_n T^{(n)}$ is timelike everywhere and $T_p \in \tilde{T}(p)$. Thus from a future selection we obtain a time orientation and these two concepts hold equivalent information. For this reason, some authors define time orientations as what we called by future selection functions.

A future selection function on the sub-Lorentzian manifold $(M/G, D, g)$ is defined by

$$\tilde{T}(q) = \{v \in D_q \subset T_q(M/G) \mid g_q(v,v) < 0,\ g_p(V,T_p) < 0\}$$

for any $p \in \pi^{-1}(q)$ and $V \in D_p \subset T_pM$ such that $\pi_*V = v$. This is well defined by the property $g(\Phi_*T, T) < 0$ for all $\Phi \in G$. Of course, the future selection $\tilde{T}$ on M/G can be used to obtain a time orientation of M/G in the usual sense.

So there is a canonical well-defined sub-space-time structure on M/G and the quotient map π is a local isometry by construction. The properties of a curve of being timelike or future directed are entirely local, so under the map π t.f.d. curves lift and project to t.f.d. curves. This implies the identity of the chronological future sets. □

The Examples 1 and 2 lead to the consideration of a special type of sub-space-times.

Definition 12. A sub-space-time in which $I^{\pm}(p)$ are open for all $p \in M$ is called chronologically open.

Except of some special cases, such as the Minkowski space, in general $J^+(p)$ and $J^-(p)$ are not closed for Lorentzian manifolds. For example, if the point $(1,1)$ in two dimensional Minkowski space is removed, then $J^+(0,0)$ and $J^-(2,2)$ are not closed. The most what is known up to now is the following.

Proposition 4. *[8] Let (M, D, g, T) be a sub-space-time, $p \in M$ and U a convex neighbourhood of p. Then*

1. $\overline{\mathrm{int}(I^+(p,U))}^U = J^+(p,U)$. *In particular,* $\mathrm{int}(I^+(p,U)) \neq \emptyset$ *and* $J^{\pm}(p,U)$ *is closed in U. It holds globally that* $J^+(p) \subset \overline{\mathrm{int}(I^+(p))}$*;*
2. $\mathrm{int}(I^+(p,U)) = \mathrm{int}(J^+(p,U))$ *and* $\mathrm{int}(I^+(p)) = \mathrm{int}(J^+(p))$*;*
3. $\tilde{\partial} I^+(p,U) = \tilde{\partial} J^+(p,U)$ *and* $\partial I^+(p) = \partial J^+(p)$.

Here $\overline{A}^U$ is the closure of A relative to U and $\tilde{\partial} A$ is the boundary of A relative to U.

3.2 The Alexandrov Topology

Lemma 2. *In a chronologically open sub-space-time (M, D, g, T) the set*

$$\mathscr{B} = \{I^+(p) \cap I^-(q) \mid p, q \in M\}$$

is the basis of a topology. This is the first analogue of the Lorentzian Alexandrov topology that we wish to introduce and investigate in the sub-Lorentzian case.

Proof. We have to check the following two requirements:

1) for each $p \in M$ there is a set $B \in \mathscr{B}$ such that $p \in B$;
2) if $p \in B_1 \cap B_2$ then there is B_3 such that $p \in B_3 \subset B_1 \cap B_2$, where B_1, B_2, $B_3 \in \mathscr{B}$.

To show the first statement we take a t.f.d. horizontal curve $\gamma\colon (-1,1) \to M$ such that $\gamma(0) = p$. Then $p \in I^+(\gamma(-1)) \cap I^-(\gamma(1))$.

For the second statement we denote by $B_1 = (I^+(q) \cap I^-(r))$, $B_2 = (I^+(q') \cap I^-(r'))$ and chose a point $p \in B_1 \cap B_2$. The set $B_1 \cap B_2$ is an open neighbourhood of p in the manifold topology. Thus there is a t.f.d. curve $\gamma\colon(-2\epsilon,2\epsilon)\to B_1\cap B_2$ with $\gamma(0)=p$. Then we have

$$r,r'\ll\gamma(-\epsilon)\ll p\ll\gamma(\epsilon)\ll q,q',$$

or, in other words $p\in I^+(\gamma(-\epsilon))\cap I^-(\gamma(\epsilon)) =: B_3\subset B_1\cap B_2$.

Definition 13. Let (M,D,g,T) be a sub-space-time. The Alexandrov topology $\mathscr{A}$ on M is the topology generated by the subbasis $\mathscr{S}=\left\{I^+(p),I^-(p)\mid p\in M\right\}$.

Lemma 2 implies that for chronologically open sub-space-times the set

$$\mathscr{B}=\{I^+(p)\cap I^-(q)\mid p,q\in M\}$$

is a basis of the Alexandrov topology $\mathscr{A}$.

Remark 1. Note that, while we need a time orientation T to define the Alexandrov topology, $\mathscr{A}$ is independent of T. A choice of a different time-orientation can at most reverse future and past orientation of the manifold. Note also, that since $I^+(p)$, $I^-(p)$ are open for space-times, the manifold topology is always finer than the Alexandrov topology. This is only true for chronologically open sub-space-times in the sub-Lorentzian case.

3.3 Links to Causality

The definition of the Alexandrov topology suggests a link between the Alexandrov topology and causal structure of a sub-space-time. The following theorem generalizes a well known fact from the Lorentzian geometry.

Theorem 3. *Every sub-space-time compact in the Alexandrov topology (in particular every compact space-time) M fails to be chronological.*

Proof. Cover the manifold M by $\mathscr{U}=\{I^+(p)\mid p\in M\}$. Then we can extract a final subcover $I^+(p_1)$, ..., $I^+(p_n)$ of M. Since these sets cover M, for any index k with $1\leq k\leq n$ there exists $i(k)$, $1\leq i(k)\leq n$ such that $p_k\in I^+(p_{i(k)})$. It gives an infinite sequence $p_1, p_{i(1)}, p_{i(i(1))}, \ldots$ containing only finitely many elements. Thus there exists k, $1\leq k\leq n$, such that p_k appears more than once in the sequence. This means that $p_k\ll p_k$ by transitivity of the relation $\ll$ and M is not chronological.

Proposition 5. *[3, 19] For any space-time (M,g,T) the following are equivalent:*

1) *the Alexandrov topology on M is Hausdorff;*
2) *the space-time M is strongly causal;*
3) *the Alexandrov topology is finer than the manifold topology.*

The proof of Proposition 5 employs quite a few Lorentzian notions, specifically openness of the sets $I^\pm$ and strong transitivity of the causal relations $\leq$, $\ll$, that do not generally hold in the sub-Lorentzian case. We show that results of Proposition 5 still hold in chronologically open sub-space-times, see also Theorem 8.

Lemma 3. *Let (M, D, g, T) be a sub-space-time, $p \in M$ and $q \in \partial J^+(p) \cap J^+(p)$, $p \neq q$. Then any nspc.f.d. horizontal curve γ from p to q is totally contained in $\partial J^+(p)$. If γ is a nspc.f.d. curve with $\gamma(\theta) \in \operatorname{int}(J^+(p))$ then $\gamma(t) \in \operatorname{int}(J^+(p))$ for all $t \geq \theta$.*

Proof. Let $\gamma : I \to M$ be an absolutely continuous nspc.f.d. curve and $p = \gamma(0)$, $q = \gamma(\theta)$. Assume that $q \in \partial J^+(p)$. Let us choose a normal neighbourhood U of q and an orthonormal frame $T, X_1, ..., X_d$ on U. Then we can expand $\dot{\gamma} = T + \sum_{i=0}^{d} u^i X_i$ in that frame, where we fixed the parametrization. The theory of ordinary differential equations ensures that there is a neighbourhood $V \subset U$ of q and $\epsilon > 0$ such that a unique solution of the Cauchy system

$$\dot{\gamma}_r(t) = -\Big(T(\gamma_r(t)) + \sum_{i=0}^{d} u^i(t)\, X_i(\gamma_r(t))\Big), \quad t \in [0, \epsilon], \qquad \gamma_r(0) = r,$$

exists for continuous coefficients u^i, for any $r \in V$, and that $\gamma_r(\epsilon)$ depends continuously on r. The result can also be extended for $u^i \in L^2([0, \epsilon])$.

As $q \in \partial J^+(p)$, there is $r \in V \cap (J^+(p))^c$. Since γ_r is nspc.p.d., $\gamma_r(t) \notin J^+(p)$ for any $t \in [0, \epsilon]$ either, but as $\gamma_r(t)$ depends continuously on the initial data $r \in J^+(p)^c$, we can choose r such that $\gamma_r(t)$ is arbitrarily close to $\gamma(\theta - t)$. Thus, if $\gamma(\theta) \in \partial J^+(p) \cap J^+(p)$ for $\theta > 0$, then also $\gamma(t) \in \partial J^+(p) \cap J^+(p)$ for $t \in [\theta - \epsilon, \theta]$ for some $\epsilon > 0$. This implies $\gamma[0, \theta] \subset J^+(p) \cap \partial J^+(p)$. □

Theorem 4. *Let (M, D, g, T) be a chronologically open sub-space-time. Then $p \leq q \ll r$ or $p \ll q \leq r$ implies $p \ll r$.*

Proof. Assume $p \ll q \leq r$. Then $q \in I^+(p) = \operatorname{int}(J^+(p))$. Therefore, any causal curve γ connecting p to r and passing through q is contained in $\operatorname{int}(J^+(p))$ after crossing q due to Lemma 3. In particular $r \in \operatorname{int}(J^+(p)) = I^+(p)$.

Assume now that $p \leq q \ll r$. Reversing the time orientation does not change the Alexandrov topology, so the sub-space-time remains chronologically open. In $(M, D, g, -T)$ we have $r \ll q \leq p$, so by the first step $r \ll p$, and by reversing T again we have $p \ll r$ in (M, D, g, T). □

Note that the proof of Theorem 4 uses Proposition 4 instead of the calculus of variations as in the classical Lorentzian case. It shows that the strong transitivity of causal relations holds not because one can vary curves, but because the system of t.f.d. curves has nice properties. Combining the last two results, we state the following summary.

Corollary 2. *Let (M, D, g, T) be a sub-space-time. If M is strongly causal, its Alexandrov topology is finer than the manifold topology. If M is chronologically open, the equivalences formulated in Proposition 5 still hold.*

Proof. Let M be strongly causal, but not necessarily chronologically open. Let U be an open set in the manifold topology. Without loss of generality, we can also assume, that U is causally convex. Let $\gamma_q : (-2\epsilon, 2\epsilon) \to U$ be a t.f.d. curve such that

$\gamma_q(0) = q$, $\gamma_q(-\epsilon), \gamma_q(\epsilon) \in U$. Then $q \in I^+(\gamma_q(-\epsilon)) \cap I^-(\gamma_q(\epsilon)) =: A_q \subset U$. Thus $U = \bigcup_{q \in U} A_q$ and any set that is open in the manifold topology, is also open in the Alexandrov topology. The rest is just a special case of Theorem 8, which we state later. □

Example 3. For non-chronologically open sub-space-times it is possible to get an Alexandrov topology that is strictly finer than the manifold topology, but that are not strongly chronological. Consider

$$N = \left\{ \left((x, y, z) \in \mathbb{R}^3 \right) \setminus \left((0, 2n, z) \mid z \in \mathbb{R}, n \in \mathbb{Z} \right) \right\}.$$

Let the horizontal distribution D be spanned by $T = \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial z}$, $X = \frac{\partial}{\partial x}$ and equipped with the scalar product $g(T, T) = -g(X, X) = -1$, $g(T, X) = 0$. Now consider the action of the group $G = \{\Phi_n(x, y, z) = (x, y + 2n, z)\}$ by sub-Lorentzian isometries on N. Define $M = N/G$ to be the quotient space of the group action and let $\pi\colon N \to M$ denote the canonical projection. Then $(M, \pi_* D, \pi_* g, \pi_* T)$ is a sub-space-time by Lemma 1. To simplify the notation we write $D = \pi_* D$, $g = \pi_* g$ and $T = \pi_* T$.

Let us first show that (M, D, g, T) is not strongly causal. Let $p_0 = [0, y_0, z_0]$ be some point on M and U a neighbourhood of p_0. Let $0 < \delta < 1$ and consider the curve

$$\gamma_\delta(t) = \pi \left(\delta t, y_0 + t, z_0 + \frac{\delta^2 t^3}{3} \right).$$

By definition, the x-coordinate of γ_δ is strictly positive for positive times, and as $p_0 \in M$, the curve is well-defined on the quotient space. The curve satisfies $\gamma_\delta(0) = p_0$, $\dot{\gamma}_\delta(t) = T_{\gamma_\delta(t)} + \delta\, X_{\gamma_\delta(t)}$ and $\gamma_\delta(2) = \left[p_0 + (2\delta, 2, \frac{8\delta^2}{3}) \right]$. So for small U, any curve of the t.f.d. family $\{\gamma_\delta\}_{\delta \in (0,1)}$ leaves U before $t = 2$, but for small enough δ they return to U later. Hence p_0 does not have arbitrarily small neighbourhoods U in M such that no t.f.d. curve that once leaves U will never return, and strong causality fails at p_0.

Now we want to show that the Alexandrov topology on M is finer than the manifold topology. Take $p_0 = [x_0, y_0, z_0] \in M$ and a neighbourhood U of p_0. We want to show that there are points $p_1 = [x_1, y_1, z_1]$ and $p_2 = [x_2, y_2, z_2]$ such that $p_0 \in I^+(p_1) \cap I^-(p_2) \subset U$.

First assume that $x_0 = 0$. Then the curve $\gamma(t) = \pi(0, y_0 + t, z_0)$ is well-defined on some small parameter interval such that $y_0 + t \notin 2\mathbb{Z}$. Furthermore, it is t.f.d. as $\dot{\gamma} = T$, and clearly for some small $\epsilon > 0$: $\gamma(-\epsilon, \epsilon) \subset U$. As in Example 1 we find, that up to monotone reparametrization, γ is the only t.f.d. curve from $p_1 := [0, y_0 - \epsilon, z_0]$ to $p_2 := [0, y_0 + \epsilon, z_0]$. Then

$$p_0 \in I^+(p_1) \cap I^-(p_2) = \{\pi(0, y_0 + t, z_0) \mid t \in (-\epsilon, \epsilon)\} \subset U.$$

This also shows that there are sets which are open in the Alexandrov topology, but not in the manifold topology. Let us now suppose $x_0 \neq 0$ and choose $\epsilon < \frac{|x_0|}{16}$.

Consider the curve

$$\gamma\colon (-2\epsilon, 2\epsilon) \to M, \quad \gamma(t) = \pi\left(x_0, y_0 + t, z_0 + x_0^2 t\right)$$

with derivative $\dot\gamma(t) = T_{\gamma(t)}$ and set $p_1 = \gamma(-\epsilon)$, $p_2 = \gamma(\epsilon)$. Due to non-vanishing value of x_0 the curve is well-defined. Clearly $p_0 \in I^+(p_1) \cap I^-(p_2)$. Now take $\bar p = [\bar x, \bar y, \bar z] \in I^+(p_1) \cap I^-(p_2)$. The components $\bar x, \bar z$ are independent of the choice of a representative, and as z is non-decreasing along t.f.d. curves, we find

$$z_1 = z_0 - x_0^2\epsilon \le \bar z \le z_0 + x_0^2\epsilon = z_2.$$

Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ be a t.f.d. curve connecting $p_1 = \sigma(0)$ with $\bar p$. Fix parametrization of σ such that $\dot\sigma = T + \beta X$. Then we find a fixed time $\theta > 0$ such that $\bar p = \sigma(\theta)$ and

$$\begin{aligned}\sigma_z(t) &= z_1 + \int_0^t \dot\sigma_z(s)\,ds = z_1 + \int_0^t \sigma_x^2(s)\,ds \ge z_1 + \int_0^{\min\{t, \frac{|x_1|}{2}\}} \sigma_x^2(s)\,ds \\ &\ge z_1 + \int_0^{\min\{t, \frac{|x_1|}{2}\}} \left(\frac{x_1}{2}\right)^2 ds = z_1 + \min\left\{t, \frac{|x_1|}{2}\right\} \frac{x_1^2}{4}.\end{aligned}$$

As $x_1 = x_0$ we combine the above obtained inequalities and see

$$z_0 + x_0^2\epsilon \ge \bar z = \sigma_z(\theta) \ge z_0 - x_0^2\epsilon + \min\left\{\theta, \frac{|x_0|}{2}\right\} \frac{x_0^2}{4}$$

or equivalently $8\epsilon > \min\{\theta, |x_0|/2\}$, which in its turn implies $8\epsilon > \theta$ due to our choice of ϵ. This however leads to

$$|\bar y - y_0| \le |\bar y - y_1| + |y_1 - y_0| = \theta + \epsilon < 9\epsilon,$$

$$|\bar x - x_0| = |\bar x - x_1| = \left|\int_0^\theta \beta(s)\,ds\right| \le \theta < 8\epsilon,$$

$$|\bar z - z_0| = \left|\int_0^\theta \sigma_x^2(s)\,ds\right| < \int_0^\theta (|x_0| + 8\epsilon)^2 ds \le 8\epsilon\,(|x_0| + 8\epsilon)^2 \le 8\epsilon\left(\frac{3\,|x_0|}{2}\right)^2.$$

Hence, the set $I^+(p_1) \cap I^-(p_2)$ becomes arbitrarily small as we let ϵ tend to 0, but contains p_0 for any $\epsilon > 0$. So inside every neighbourhood U of p_0, that is open in the manifold topology, we can find $p_1, p_2 \in U$ such that $p_0 \in I^+(p_1) \cap I^-(p_2)$. This means also that at $x_0 \ne 0$ the Alexandrov topology is finer than the manifold topology: $\tau \subset \mathscr{A} \Rightarrow \tau \subsetneq \mathscr{A}$.

Theorem 5. *A sub-space-time (M, D, g, T) is chronological if and only if the pull-back of the Alexandrov topology along all t.f.d. curves to their parameter inverval $I \subset \mathbb{R}$ is finer than the standard topology on I as a subspace of $\mathbb{R}$.*

Proof. Assume that M fails to be chronological. Then there is a closed t.f.d. curve $\gamma\colon [0,1] \to M$, which means $\gamma(s) \ll \gamma(t)$ for all $t, s \in [0,1]$, hence the pullback of the Alexandrov topology along this specific γ is $\{\emptyset, [0,1]\} = \gamma^{-1}(\mathscr{A})$.

If, on the other hand, M is chronological, and γ is a t.f.d. curve on the open interval I, then $(s,t) = \gamma^{-1}\big(I^+(\gamma(s)) \cap I^-(\gamma(t))\big)$ for $s, t \in I$. If $I = [a,b]$ is closed, then $(s,b] = \gamma^{-1}(I^+(\gamma(s)) \cap I^-(p))$ for any $p \in I^+(\gamma(b))$ and similarly for the other cases. □

3.4 The Alexandrov and Manifold Topology in sub-Lorentzian Geometry

We now come to a result that is new in sub-Lorentzian geometry and at the core of our investigation, namely the relation of the Alexandrov topology $\mathscr{A}$ to the manifold topology τ in proper sub-space-times. In contrast to space-times, we have the following result.

Theorem 6. *For sub-space-times all inclusions between the Alexandrov topology $\mathscr{A}$ and the manifold topology τ are possible, i. e. there are sub-space-times such that*

$$(1)\ \tau = \mathscr{A}, \quad (2)\ \tau \supsetneq \mathscr{A}, \quad (3)\ \tau \subsetneq \mathscr{A}, \quad (4)\ \tau \not\subset \mathscr{A},\ \tau \not\supset \mathscr{A}.$$

Proof. The proof is contained in the following examples. The first case $\tau = \mathscr{A}$ in Theorem 6 is realized for strongly causal space-times and the second case $\tau \supsetneq \mathscr{A}$ is realised in space-times, which fail to be strongly causal. They are therefore in particular satisfied for certain sub-space-times. We show even more, namely that there are sub-space-times, with a smooth distribution $D \subsetneq TM$, which exhibit this type of behaviour.

Example 4. In the Lorentzian Heisenberg group $M = \mathbb{R}^3$, $D = \operatorname{span}\{T, Y\}$, where

$$T = \frac{\partial}{\partial x} + \frac{1}{2}\, y\, \frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} - \frac{1}{2}\, x\, \frac{\partial}{\partial z},$$

and $g(T,T) = -g(Y,Y) = -1$ the manifold topology τ and the Alexandrov topology $\mathscr{A}$ agree: $\tau = \mathscr{A}$.

We know from Example 2 that $\mathscr{A} \subset \tau$ since a subbasis of $\mathscr{A}$ is open in τ. To obtain the inverse inclusion $\tau \subset \mathscr{A}$ we show that the Lorentzian Heisenberg group is strongly causal. To that end we construct a family of neighbourhoods $B(p_0, \epsilon)$ of a point $p_0 = (x_0, y_0, z_0)$, that become arbitrarily small for $\epsilon \to 0$ and such that any nspc.f.d. horizontal curve leaving $B(p_0, \epsilon)$ will never return back. Define

$$B(p_0, \epsilon) = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{array}{l} |x - x_0| < \epsilon, \\ |y - y_0| < x - x_0 + \epsilon, \\ |z - z_0| < \frac{|x_0| + |y_0| + 7\epsilon}{2}(x - x_0 + \epsilon) \end{array} \right\}.$$

Obviously $B(p_0, \epsilon)$ is open in M for any choice of $p_0, \epsilon > 0$, and

$$B(p_0, \epsilon) \subset (x_0 - \epsilon, x_0 + \epsilon) \times (y_0 - 2\epsilon, x_0 + 2\epsilon) \times (z_0 - 2C\epsilon, z_0 + 2C\epsilon),$$

where $C := C(x_0, y_0) = \frac{|x_0|+|y_0|+7}{2}$ for $\epsilon < 1$, so $B(p_0, \epsilon)$ can be made arbitrarily small.

Let $\gamma \colon I \to M$, $0 \in I$, be a nspc.f.d. horizontal curve such that $\gamma(0) = p_1 = (x_1, y_1, z_1) \in B(p_0, \epsilon)$. Then we have $\dot{\gamma}(t) = \alpha(t)T_{\gamma(t)} + \beta(t)Y_{\gamma(t)}$ with $\alpha > 0$ and $|\beta| \leq \alpha$. Without loss of generality we assume that $\alpha \equiv 1$ by fixing the parametrization. Then

$$x(t) - x_1 = \int_0^t \alpha(s)\, ds = \int_0^t 1\, ds = t,$$
$$|y(t) - y_1| = \left| \int_0^t \beta(s)\, ds \right| \leq \int_0^t 1\, ds = x(t) - x_1.$$

If we take $\tau > 0$ such that $\gamma(\tau) \in B(p_0, \epsilon)$, then

$$\tau = x(\tau) - x_1 = x(\tau) - x_0 + x_0 - x_1 \leq |x(\tau) - x_0| + |x_0 - x_1| < 2\epsilon,$$

and it follows for $0 \leq t \leq \tau$ that

$$\begin{aligned}
|z(t) - z_1| &= \left| \int_0^t \frac{1}{2}(\alpha(s)y(s) - \beta(s)x(s))\, ds \right| \leq \frac{1}{2} \int_0^t \big(|y(s)| + |x(s)|\big)\, ds \\
&\leq \frac{1}{2} \int_0^t \Big(|y_1| + |y(s) - y_1| + |x_1| + |x(s) - x_1|\Big)\, ds \\
&\leq \frac{1}{2} \left(|y_1| + t + |x_1| + t\right) t \leq \frac{|y_1| + |x_1| + 4\epsilon}{2}(x(t) - x_0) \\
&\leq \frac{|y_1 - y_0| + |y_0| + |x_1 - x_0| + |x_0| + 4\epsilon}{2}(x(t) - x_0) \\
&\leq \frac{|x_0| + |y_0| + 7\epsilon}{2}(x(t) - x_1)
\end{aligned}$$

since by the construction of $B(p_0, \epsilon)$ we have $|x_1 - x_0| < \epsilon$, $|y_1 - y_0| < 2\epsilon$. Our aim is to show that $\gamma(t) \in B(p_0, \epsilon)$, and if γ leaves $B(p_0, \epsilon)$, then it can not return later. As $x(t)$ is increasing in t, we have $x(t) \in [x_1, x(\tau)]$ so $x(t) = \lambda x_1 + (1 - \lambda)x(\tau)$ for some $\lambda \in [0, 1]$ and

$$|x(t) - x_0| = |\lambda x_1 + (1 - \lambda)x(\tau) - x_0| \leq \lambda|x_1 - x_0| + (1 - \lambda)|x(\tau) - x_0| < \epsilon.$$

Finally, we get

$$\begin{aligned}
|y(t) - y_0| \leq |y(t) - y_1| + |y_1 - y_0| &< x(t) - x_1 + x_1 - x_0 + \epsilon \\
&= x(t) - x_0 + \epsilon,
\end{aligned}$$

$$|z(t) - z_0| \le |z(t) - z_1| + |z_1 - z_0| < \frac{|x_0| + |y_0| + 7\epsilon}{2}\big(x(t) - x_0 + \epsilon\big).$$

So $\gamma(\tau) \in B(p_0, \epsilon)$ implies $\gamma(t) \in B(p_0, \epsilon)$ for all $t \in [0, \tau]$, and no nspc.f.d. curve can leave $B(p_0, \epsilon)$ and return there.

Assume now that there is a nspc.p.d. curve γ leaving and returning to $B(p_0, \epsilon)$. By reversing the orientation of γ, we find a nspc.f.d. curve leaving and returning to $B(p_0, \epsilon)$, which leads to a contradiction. Hence, we have shown that the Lorentzian Heisenberg group is strongly causal, and by that $\tau \subset \mathscr{A}$ and $\tau = \mathscr{A}$.

For the proof of the second statement we give the following example.

Example 5. Let M be the Lorentzian Heisenberg group and $G = G_\varphi$ the group of isometries

$$G = \left\{ \Phi_n(x, y, z) = \begin{pmatrix} x - n\cosh(\varphi), \\ y - n\sinh(\varphi), \\ z + \frac{n}{2}\big(y\cosh(\varphi) - x\sinh(\varphi)\big) \end{pmatrix} \Bigg| \, n \in \mathbb{Z} \right\}$$

for some fixed $\varphi \in \mathbb{R}$. Then the manifold topology $\tau_{M/G}$ in M/G is strictly finer than the Alexandrov topology $\mathscr{A}_{M/G}$: $\mathscr{A}_{M/G} \subsetneq \tau_{M/G}$.

We write $[p]$ for points from M/G. Take a point $[q] \in I^+([p])$. We can pick up a pre-image $q \in I^+(p) \cap \pi^{-1}([q])$ by Lemma 1. Because $I^+(p)$ is open, a small neighbourhood $U \subset M$ of q is contained in $I^+(p)$. We choose U to be so small, that $\pi|_U$ is a homeomorphism. Using Lemma 1, we find $[q] \in \pi(U) \subset \pi(I^+(p)) = I^+([p])$, where $\pi(U)$ is open, since we choose U to be small enough. Therefore, $\mathscr{A}_{M/G} \subset \tau_{M/G}$.

On the other hand M/G contains the closed t.f.d. curve

$$\gamma\colon \mathbb{R} \to M/G, \quad \gamma(t) = \pi\big(t \cdot (\cosh(\varphi), \sinh(\varphi), 0)\big),$$

where $\gamma(0) = \gamma(1)$. This means, that for any $V \in \mathscr{A}$ we find $\gamma(s) \in V \Leftrightarrow \gamma(t) \in V$ for any $s, t \in \mathbb{R}$. Therefore, the Alexandrov topology cannot be Hausdorff, in particular, $\tau_{M/G} \not\subset \mathscr{A}_{M/G}$. Together with the previous inclusion, we deduce $\mathscr{A}_{M/G} \subsetneq \tau_{M/G}$.

For the third statement we refer the reader back to Example 3, see also Example 1. We now turn to the last example to finish the proof of Theorem 6.

Example 6. Take the sub-space-time from Example 1 and the group

$$G = \{\Phi_n(x, y, z) = (x, y + 2n, z) \,|\, n \in \mathbb{Z}\}.$$

Then the Alexandrov topology and the manifold topology of M/G are not comparable: $\tau \not\subset \mathscr{A}, \tau \not\supset \mathscr{A}$.

The set $I^+([0])$ is not open in M/G because on the one hand $[(0, \theta, 0)] \in I^+([0])$ for all $\theta > 0$ since $(0, \theta, 0) \in I^+(0)$, but on the other hand $[(0, \theta, -a)] \notin I^+([0])$. Indeed, if $[(0, \theta, -a)]$ were in $I^+([0])$ there would be a t.f.d. curve γ in M/G from

$[0]$ to $[(0, \theta, -a)]$. Then we could lift this curve to a t.f.d. curve $\tilde{\gamma}$ in M from 0 to some representative $(0, \theta + n, -a)$ in the pre-image of $[(0, \theta, -a)]$, $n \in \mathbb{Z}$. This is not possible as shown in Example 1. This proves $\mathscr{A}_{M/G} \not\subset \tau_{M/G}$.

On the other hand M/G contains the closed t.f.d. curve

$$\gamma \colon \mathbb{R} \to M/G, \quad \gamma = \pi\big(0, t, 0\big),$$

where $\gamma(0) = \gamma(1)$. This means as in the preceding example that $\tau_{M/G} \not\subset \mathscr{A}_{M/G}$.

3.5 The Open Causal Relations

We have seen that pathological cases occur, when we extend the Alexandrov topology $\mathscr{A}$ in the obvious way to sub-space-times. We now present a different extension of $\mathscr{A}$ with better behaviour, that unfortunately may seem less sensible from a physical point of view, because it simply ignores the pathological cases that can occur. Let us start by explaining what happens, if we just force our extension of the Alexandrov topology to be open in the manifold topology.

Lemma 4. *Let (M, D, g, T) be a sub-space-time. Then $q \in \text{int}(I^+(p)) \Leftrightarrow p \in \text{int}(I^-(q))$.*

Proof. Assume $q \in \text{int}(I^+(p))$. We have shown that also $q \in \overline{\text{int}(I^-(q))}$, which means

$$q \in \overline{\text{int}(I^+(p)) \cap \text{int}(I^-(q))}.$$

In particular, there is a point $r \in \text{int}(I^+(p)) \cap \text{int}(I^-(q))$ and a t.p.d. curve γ such that

$$\gamma(0) = q, \quad \gamma\left(\frac{1}{2}\right) = r, \quad \gamma(1) = p.$$

By reversing time and using Lemma 3, we see that $\gamma(t) \in \text{int}(I^-(q))$ for all $t \geq \frac{1}{2}$, so in particular $p \in \text{int}(I^-(q))$. The same holds with reversed roles.

We conclude, that for $p \in M$ there is some $q \in M$ such that $p \in \text{int}(I^+(q))$, and we can even choose q as close to p as we want. In the same way as for chronologically open sub-space-times we find a way to obtain the basis of a topology, that we call *open Alexandrov topology*.

Definition 14. Let (M, D, g, T) be a sub-space-time. We write $p \ll_o q$ if $q \in \text{int}(I^+(p))$. The open Alexandrov topology $\mathscr{A}_o$ is the topology generated by the basis

$$\mathscr{B}_o = \{\text{int}(I^+(p)) \cap \text{int}(I^-(q)) \mid p, q \in M\}.$$

Theorem 7. *The strong transitivity of order relations $\leq, \ll_o$ hold in the sense:*

1) $p \ll_o q \;\Rightarrow\; p \ll q \;\Rightarrow\; p \leq q$;
2) $p \ll_o q \leq r \;\Rightarrow\; p \ll_o r$;
3) $p \leq q \ll_o r \;\Rightarrow\; p \ll_o r$.

The Alexandrov topology is finer than the open Alexandrov topology and they are the same if and only if M is chronologically open.

Proof. Statement (1) follows trivially from the definition. To show (2) we choose $p, q, r \in M$ such that $p \ll_o q \leq r$. Then there is a horizontal nspc.f.d. curve from p to r passing through q. Due to Lemma 3, the curve is totally contained in $\text{int}(J^+(p)) = \text{int}(I^+(p))$ after passing q, in particular we find $p \ll_o r$. By reversing the time orientation and employing Lemma 4 we obtain (3).

To show that $\mathscr{A}_o \subset \mathscr{A}$ let now p, q, r be such that $p \ll_o q \ll_o r$. Then there are points p', r' on horizontal t.f.d. curves, connecting p to q and q to r respectively, such that $p \ll_o p' \ll q \ll r' \ll_o r$, hence

$$q \in I^+(p') \cap I^-(r') \subset \text{int}(I^+(p)) \cap \text{int}(I^-(r)),$$

or in other words, the Alexandrov topology $\mathscr{A}$ is finer than the open Alexandrov topology $\mathscr{A}_o$. Clearly, these two topologies agree if M is chronologically open, since then $I^+(p) = \text{int}(I^+(p))$ and bases agree. Nevertheless, they cannot coincide in the case when M is not chronologically open. □

This helps us to generalize Theorem 3.

Proposition 6. *Every compact sub-space-time fails to be chronological.*

Theorem 8. *Let (M, D, g, T) be a sub-space-time. Then the following are equivalent:*

$$(1)\ M \ \textit{is strongly causal}, \quad (2)\ \mathscr{A}_o = \tau, \quad (3)\ \mathscr{A}_o \ \textit{is Hausdorff}.$$

Proof. Corollary 2 implies that in strongly causal sub-space-times the Alexandrov topology is finer than the manifold topology. A combination of the same proof with Lemma 4 shows, that also the open Alexandrov topology is finer than the manifold topology, which is, in particular, Hausdorff.

It remains to show that if in a general sub-space-time the open Alexandrov topology is Hausdorff, the sub-space-time must be strongly causal. The proof essentially follows [19], only one needs to use the convergence of curves in the C^0-topology in a convex neighbourhood instead of geodesics. □

Since $\mathscr{A} = \mathscr{A}_o$ in chronologically open sub-space-times, we deduce Corollary 2 from Theorem 8. We actually find that the strongly causal sub-space-times are those, in which even the coarser topology $\mathscr{A}_o$ is Hausdorff, and not only $\mathscr{A}$.

3.6 Chronologically Open sub-Space-Times

We have seen in Theorem 8 how the open Alexandrov topology of a sub-space-time and the property of being strongly causal are linked. It is natural to ask, what other properties the topological space $(M, \mathscr{A}_o)$ possesses and whether they are also related to causality. Unfortunately, if $\mathscr{A}_o$ is Hausdorff, it is already the same as the manifold topology and thus is metrizable, so it does not make any sense to ask for stronger properties than Hausdorff. Neither is the condition that one-point sets be closed interesting, because it only means that M is chronological. The following result holds quite trivially, as $\mathscr{A}_o \subset \tau$.

Proposition 7. *Let* (M, D, g, T) *be a sub-space-time. Then the topological space* $(M, \mathscr{A}_o)$ *is second countable, path-connected and locally path-connected, i. e. also first countable, separable, connected and locally connected.*

Note that the same need not hold for the topological space $(M, \mathscr{A})$. In the sub-space-time from Example 1 we have $I^+(0, y_0, z_0) \cap I^-(0, y_0+\theta, z_0) = \{(0, y_0+t, z_0) | t \in (0, \theta)\}$, so the lines on which z is constant and $x = 0$ are open in the Alexandrov topology. As there are uncountably many such lines, the topology $\mathscr{A}$ is not second countable. Still, it is metrizable by the metric

$$d\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{cases} \min\{1, |x-a| + |y-b| + |z-c|\} & \text{if } x, a \neq 0, \\ 1 & \text{if } \begin{cases} x = 0, a \neq 0 \text{ or} \\ x \neq 0, a = 0, \end{cases} \\ 1 & \text{if } x = a = 0, z \neq c, \\ \min\{1, |y-b|\} & \text{if } x = a = 0, z = c, \end{cases}$$

and hence first countable. The topological space $(M, \mathscr{A})$ is not even connected as it can be written as a disjoint union of open sets $M = \{x < 0\} \cup \{x = 0\} \cup \{x > 0\}$. Since chronologically open sub-space-times are the most well-behaved, we are interested in finding criteria to see that sub-space-times are chronologically open.

Definition 15. [2, 15] Let M be a manifold with a smooth distribution D. A curve $\gamma : I \to M$ is called a Goh-curve, if there is a curve $\lambda : I \to T^*M$, such that

$$\pi \circ \lambda = \gamma, \quad \lambda(X) = \lambda[X, Y] = 0$$

for all horizontal vector fields X, Y and $\lambda \neq 0$ anywhere.

Proposition 8. *[8] Let* (M, D, g, T) *be a sub-space-time,* $p \in M$*, and* $\gamma : [0, 1] \to M$ *a horizontal t.f.d. curve such that* $\gamma([0, 1]) \subset \partial I^+(p) \cap I^+(p)$*. Then* γ *is a Goh-curve.*

Remember, that if $q \in \partial I^+(p) \cap I^+(p)$ then any horizontal t.f.d. curve γ from p to q must run entirely in $\partial I^+(p)$. The proof of the following proposition is trivial.

Proposition 9. *Two step generating sub-space-times have no Goh-curves and therefore they are chronologically open.*

Analogously to Theorem 5, we can test whether a sub-space-time is chronologically open using horizontal t.f.d. curves.

Theorem 9. *A sub-space-time* (M, D, g, T) *is chronologically open if and only if for all horizontal t.f.d. curves* $\gamma : I \to M$ *the pullback of the Alexandrov topology* *along γ to I is coarser than the standard topology on I, considered as a subspace of* $\mathbb{R}$.

Proof. It is clear that, if $\mathscr{A} \subset \tau_M$, then $\gamma^{-1}(\mathscr{A}) \subset \gamma^{-1}(\tau_M) \subset \tau_I$, as γ was assumed to be continuous to (M, τ_M). If on the other hand, $\mathscr{A} \not\subset \tau_M$, then there is a point $p \in M$, such that $I^+(p)$ or $I^-(p)$ is not open. Without loss of generality we may assume, that $I^+(p)$ is not open. Then there is a point $q \in I^+(p) \cap \partial I^+(p)$.

Now take $r \in \text{int}(I^-(q))$. If $r \in I^+(p)$, then $p \ll r \ll_o q \Rightarrow q \in \text{int}(I^+(p))$, which possesses a contradiction, so $r \notin I^+(p)$. In this way we can construct a sequence of points $r_1 \ll r_2 \ll \ldots \ll q$ converging to q, such that $r_i \notin I^+(p)$. By assumption there are horizontal t.f.d. curves

$$\gamma_n\colon\ [0, 2^{-n}] \to M, \quad \gamma_n(0) = r_n, \quad \gamma_n(2^{-n}) = r_{n+1}.$$

We can consider the continuous horizontal t.f.d. curve $\gamma\colon [0, 1) \to M$ that is obtained by

$$\gamma(t) = \gamma_n\left(t - \sum_{k=0}^{n-1} 2^{-k}\right), \quad t \in \left[\sum_{k=0}^{n-1} 2^{-k}, \sum_{k=0}^{n} 2^{-k}\right]$$

By Corollary 1, the curve γ is absolutely continuous with nspc.f.d. derivative almost everywhere. Moreover, since all segments of γ have t.f.d. derivatives almost everywhere, so does γ. Clearly γ can be continuously extended to $\gamma(1) = q$, and then $\gamma^{-1}(I^+(p)) = \{1\}$, which is not open in $[0, 1]$. □

4 The Time-Separation Topology

In Lorentzian geometry there is a relation between causality and the time separation function T^S defined in (1). To emphasise the analogy between the Lorentzian distance function and the Riemannian distance function define the outer balls

$$O^+(p, \epsilon) = \{q \in M \mid T^S(p, q) > \epsilon\}, \quad O^-(p, \epsilon) = \{q \in M \mid T^S(q, p) > \epsilon\}.$$

The outer balls suggest a way to introduce a topology related to the function T^S.

Definition 16. The topology τ_{T^S} created by the subbasis

$$\mathscr{S} = \left\{O^{\pm}(p, \epsilon) \mid p \in M,\ \epsilon > 0\right\}$$

is called time separation topology.

Proposition 10. *[3] Let (M, g, T) be a space-time. Then $\tau_{T^S} = \mathscr{A} = \mathscr{A}_o$.*

As a consequence of Proposition 1 in space-times we have $T^S(p, q) > 0 \Leftrightarrow q \in I^+(p)$. Clearly, the implication from right to left still holds in sub-space-times, but the following example shows that the converse direction fails for some sub-space-times, and that Proposition 10 can not be extended.

Example 7. Take the sub-space-time $M = \mathbb{R}^3$ with the bracket generating distribution generated by $X = \partial y + x^2 \partial z$, $Y = \partial x$. Define a metric $g(X, X) = a$, $g(X, Y) = b$, $g(Y, Y) = 1$, where functions $a, b \in C^\infty(\mathbb{R}^3)$ are such that

$$a \leq 0, \quad a - b^2 < 0, \quad a(0, y, 0) = \begin{cases} -1 & 0 \leq y \leq \frac{1}{3} \\ 0 & \frac{2}{3} \leq y \leq 1. \end{cases}$$

The metric g is sub-Lorentzian since the matrix $g = \begin{pmatrix} a & b \\ b & 1 \end{pmatrix}$ has one positive and one negative eigenvalue due to $\det(g) = a - b^2 < 0$. The vector field $T = X - bY$ will serve as time orientation. Indeed,

$$\begin{aligned} & g(T, X) = g(X - bY, X) = g(X, X) - b \cdot g(X, Y) = a - b^2 < 0, \\ & g(T, Y) = g(X, Y) - b\, g(Y, Y) = b - b = 0, \\ \Longrightarrow \qquad & g(T, T) = g(X, T) - b\, g(Y, T) = a - b^2 < 0. \end{aligned}$$

In this sub-space-time $T^S(0, (0, 1, 0)) > 0$ but $(0, 1, 0) \notin I^+(0)$. Also $0 \ll (0, \frac{1}{3}, 0)$ and $(0, \frac{1}{3}, 0) \leq (0, 1, 0)$ but $0 \not\ll (0, 1, 0)$. To prove it we consider $\gamma\colon [0, 1] \to M$, $\gamma(t) = (0, t, 0)$. We know that $\dot\gamma(t) = X\big(\gamma(t)\big)$, and it implies $g(\dot\gamma, \dot\gamma) = a \leq 0$ and $g(\dot\gamma, T) = a - b^2 < 0$. Thus, the curve γ is nspc.f.d. and has positive length $L(\gamma) \geq \frac{1}{3}$, and it is t.f.d. between $(0, \frac{2}{3}, 0)$ and $(0, 1, 0)$.

Like in Example 1 one sees that the curve γ is the only f.d. curve connecting 0 and $(0, 1, 0)$ up to monotone reparametrization. However, monotone reparametrization does not influence causal character.

Theorem 10. *There are sub-space-times where $\tau_{TS} \neq \mathscr{A}$.*

Proof. Consider the sub-space-time from Example 7. Here $p = (0, 1, 0) \in O^+\left(0, \frac{1}{4}\right)$. However, since $a \equiv 0$ around p and since the distribution is two-step bracket generating away from $\{x = 0\}$ the only possible nspc. Goh-curves around p are null. Therefore, if $p \in I^+(q)$, it cannot lie on the boundary, so $p \in \operatorname{int}(I^+(q))$ and the same for $I^-(q)$. Hence any neighbourhood U of p, that is open in the Alexandrov topology, contains a whole neighbourhood $V \subset U$ of p in the manifold topology. But $p \in \partial I^+(0) \subset \overline{I^+(0)}$ and therefore also $p \in \partial O^+(0, \frac{1}{4})$ which means $V \not\subset O^+(0, \frac{1}{4})$. Therefore the set $O^+(0, \frac{1}{4})$ is not open in the Alexandrov topology. □

Note that in the sub-space-time of Example 7, neither $\mathscr{A} \subset \tau$ nor $\tau_{TS} \subset \tau$, as $O^+(0, \frac{1}{4})$ and $I^+(0)$ are not open in the manifold topology τ. In this example, the three generalizations $\mathscr{A}$, $\mathscr{A}_0$ and τ_{TS} of the Alexandrov topology are all distinct. In [8] the existence of chronologically open sub-space-times where $\tau_{TS} \neq \tau$ is stated. We do not know either examples, where $\mathscr{A} \not\subset \tau_{TS}$, or a proof excluding this case.

Theorem 11. *Let (M, D, g, T) be a sub-space-time with manifold topology τ. Then always $\mathscr{A}_o \subset \tau_{TS}$ and $\tau_{TS} \subset \tau$ implies $\mathscr{A} \subset \tau$. If $\tau_{TS} = \tau$, then also $\tau = \mathscr{A}$. If $\tau_{TS} \subset \tau$ and $\mathscr{A} = \tau$, then also $\tau_{TS} = \tau$.*

Proof. Take a set $A \in \mathscr{A}_o$ and a point $p \in A$. Without loss of generality $A = \operatorname{int}(I^+(q)) \cap \operatorname{int}(I^-(r))$ for some points $q, r \in M$. Take a horizontal t.f.d. curve $\gamma\colon [0,1] \to M$ such that $\gamma(0) = q$, $\gamma(\frac{1}{2}) = p$, $\gamma(1) = r$ and define $p_1 = \gamma(\frac{1}{4})$, $p_2 = \gamma(\frac{3}{4})$. By the transitivity of $\leq$ and $\ll_o$ we obtain

$$p \in O^+\left(p_1, 2^{-1}T^S(p_1, p)\right) \cap O^-\left(p_2, 2^{-1}T^S(p, p_2)\right) \subset J^+(p_1) \cap J^-(p_2) \subset A.$$

Concerning the second statement, we have $O^+(p,\epsilon) \subset \operatorname{int}(J^+(p)) = \operatorname{int}(I^+(p))$ because $O^+(p,\epsilon) \subset J^+(p)$ and $O^+(p,\epsilon)$ is open. Therefore

$$\bigcup_{\epsilon>0} O^+(p,\epsilon) \subset I^+(p).$$

But for any $q \in I^+(p)$, the value $T^S(p,q)$ is positive or infinite, so $I^+(p) \subset \bigcup_{\epsilon>0} O^+(p,\epsilon)$. This implies that $I^+(p) = \bigcup_{\epsilon>0} O^+(p,\epsilon)$ and $I^+(p)$ is open in the manifold topology.

Now assume that $\tau_{T^S} = \tau$. Take $p \in M$, $\epsilon > 0$, and any $q \in O^+(p,\epsilon)$. Every horizontal t.p.d. curve γ starting at p will initially lie in $O^+(p,\epsilon)$, as the set is open. So there is a point $r \in I^-(q) \cap O^+(p,\epsilon)$. For all $s \in I^+(r)$ we find that $T^S(p,s) \geq T^S(p,r) + T^S(r,s) > T^S(p,r) > \epsilon$. We see that $s \in O^+(p,\epsilon)$, which means $q \in I^+(r) \subset O^+(p,\epsilon)$. The same arguments hold for past outer balls, so we find $\tau = \tau_{TS} \subset \mathscr{A}$. Combining this with the first statement, we obtain $\mathscr{A} = \tau$.

Now assume that $\tau_{TS} \subset \tau$ and $\mathscr{A} = \tau$. Take any point $p \in M$ and $q \in I^+(p)$. Clearly $T^S(p,q) > 0$, possibly infinite. Choose $\epsilon = \frac{1}{2}T^S(p,q)$, if the time separation is finite and choose any $\epsilon > 0$ otherwise. Then $q \in O^+(p,\epsilon) \subset J^+(p)$ clearly. Since $O^+(p,\epsilon)$ is open, it even holds that $O^+(p,\epsilon) = \operatorname{int}(O^+(p,\epsilon)) \subset \operatorname{int}(J^+(p)) = \operatorname{int}(I^+(p)) = I^+(p)$, hence $q \in O^+(p,\epsilon) \subset I^+(p)$, which means that $\tau = \mathscr{A} \subset \tau_{TS}$, and finally $\tau = \tau_{TS}$. □

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The regularity problem for sub-Riemannian geodesics

Roberto Monti

Abstract We review some recent results on the regularity problem of sub-Riemannian length minimizing curves. We also discuss a new nontrivial example of singular extremal that is not length minimizing near a point where its derivative is only Hölder continuous. In the final section, we list some open problems.

1 Introduction

One of the main open problems in sub-Riemannian geometry is the regularity of length minimizing curves, see [12, Problem 10.1]. All known examples of length minimizing curves are smooth. On the other hand, there is no regularity theory of a general character for sub-Riemannian geodesics.

It was originally claimed by Strichartz in [15] that length minimizing curves are smooth, all of them being normal extremals. The wrong argument relied upon an incorrect application of Pontryagin Maximum Principle, ingnoring the possibility of *abnormal* (also called *singular*) extremals. In 1994 Montgomery discovered the first example of a singular length minimizing curve [11]. In fact, manifolds with distributions of rank 2 are rich of abnormal geodesics: in [9], Liu and Sussmann introduced a class of abnormal extremals, called *regular abnormal extremals*, that are always locally length minimizing. On the other hand, when the rank is at least 3 the situation is different. In [4], Chitour, Jean, and Trélat showed that for a generic distribution of rank at least 3 every singular curve is of minimal order and of corank 1. As a corollary, they show that a generic distribution of rank at least 3 does not admit (nontrivial) minimizing singular curves.

The question about the regularity of length minimizing curves remains open. The point, of course, is the regularity of abnormal minimizers. Some partial results in

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this direction are obtained in [8] and [13]. In this survey, we describe these and other recent results. In Sect. 5.2, we present the classification of abnormal extremals in Carnot groups [6], that was announced at the meeting *Geometric control and sub-Riemannian geometry* held in Cortona in May 2012. The example of nonminimizing singular curve of Sect. 7 is new.

We refer the reader to the monograph [2] for an excellent introduction to Geometric Control Theory, see also the book in preparation [1].

2 Basic facts

Let M be an n-dimensional smooth manifold, $n \geq 3$, let $\mathcal{D}$ be a completely nonintegrable (i. e., bracket generating) distribution of r-planes on M, $r \geq 2$, called *horizontal distribution*, and let $g = g_x$ be a smooth quadratic form on $\mathcal{D}(x)$, varying smoothly with $x \in M$. The triple $(M, \mathcal{D}, g)$ is called *sub-Riemannian manifold*.

A Lipschitz curve $\gamma : [0,1] \to M$ is $\mathcal{D}$-*horizontal*, or simply *horizontal*, if $\dot{\gamma}(t) \in \mathcal{D}(\gamma(t))$ for a.e. $t \in [0,1]$. We can then define the length of γ

$$L(\gamma) = \Big(\int_0^1 g_{\gamma(t)}(\dot{\gamma}(t))\, dt \Big)^{1/2}.$$

For any couple of points $x, y \in M$, we define the function

$$d(x,y) = \inf \Big\{ L(\gamma) : \gamma \text{ is horizontal, } \gamma(0) = x \text{ and } \gamma(1) = y \Big\}. \tag{1}$$

If the above set is nonempty for any $x, y \in M$, then d is a distance on M, usually called Carnot-Carathéodory distance.

By construction, the metric space (M, d) is a *length space*. If this metric space is complete, then closed balls are compact, and by a standard application of Ascoli-Arzelà theorem, the infimum in (1) is attained. Namely, for any given pair of points $x, y \in M$ there exists at least one Lipschitz curve $\gamma : [0,1] \to M$ joining x to y and such that $L(\gamma) = d(x,y)$. This curve, which in general is not unique, is called a *length minimizing* curve. Its a priori regularity is the Lipschitz regularity. In particular, length minimizing curves are differentiable a.e. on $[0,1]$.

For our porpouses, we can assume that M is an open subset of $\mathbb{R}^n$ or the whole $\mathbb{R}^n$ itself, and that we have $\mathcal{D}(x) = \text{span}\{X_1(x), \dots, X_r(x)\}$, $x \in \mathbb{R}^n$, where $X_1, \dots, X_r$ are $r \geq 2$ linearly independent smooth vector fields in $\mathbb{R}^n$. With respect to the standard basis of vector fields in $\mathbb{R}^n$, we have, for any $j = 1, \dots, r$,

$$X_j = \sum_{i=1}^n X_{ji} \frac{\partial}{\partial x_i}, \tag{2}$$

where $X_{ji} : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. A Lipschitz curve $\gamma : [0,1] \to M$ is then horizontal if there exists a vector of functions $h = (h_1, \dots, h_r) \in$

$L^\infty([0,1];\mathbb{R}^r)$, called controls of γ, such that

$$\dot\gamma = \sum_{j=1}^r h_j X_j(\gamma), \quad \text{a.e. on } [0,1].$$

We fix on $\mathcal{D}(x)$ the quadratic form g_x that makes $X_1, \dots, X_r$ orthonormal. Any other choice of metric does not change the regularity problem. In this case, the length of γ is

$$L(\gamma) = \Big(\int_0^1 |h(t)|^2 dt \Big)^{1/2}.$$

Let $h = (h_1, \dots, h_r)$ be the controls of a horizontal curve γ. When γ is length minimizing we call the pair (γ, h) an *optimal pair*. Pontryagin Maximum Principle provides necessary conditions for a horizontal curve to be a minimizer.

Theorem 1. *Let (γ, h) be an optimal pair. Then there exist $\xi_0 \in \{0,1\}$ and a Lipschitz curve $\xi : [0,1] \to \mathbb{R}^n$ such that:*

i) $\xi_0 + |\xi| \neq 0$ *on* $[0,1]$*;*
ii) $\xi_0 h_j + \langle \xi, X_j(\gamma)\rangle = 0$ *on* $[0,1]$ *for all* $j = 1, \dots, r$*;*
iii) *the coordinates ξ_k, $k = 1, \dots, n$, of the curve ξ solve the system of differential equations*

$$\dot\xi_k = -\sum_{j=1}^r \sum_{i=1}^n \frac{\partial X_{ji}}{\partial x_k}(\gamma) h_j \xi_i, \quad \textit{a.e. on } [0,1]. \tag{3}$$

Above, $\langle \xi, X_j\rangle$ is the standard scalar product of ξ and X_j as vectors of $\mathbb{R}^n$. If we identify the curve ξ with the 1-form in $\mathbb{R}^n$ along γ

$$\xi = \xi_1 dx_1 + \ldots + \xi_n dx_n,$$

then $\langle \xi, X_j\rangle$ is the covector-vector duality.

The proof of Theorem 1 relies upon the open mapping theorem, see [2, Chap. 12]. For any $v \in L^2([0,1];\mathbb{R}^r)$, let γ^v be the solution of the problem

$$\dot\gamma^v = \sum_{j=1}^r v_j X_j(\gamma^v), \quad \gamma^v(0) = x_0.$$

The mapping $\mathcal{E} : L^2([0,1];\mathbb{R}^r) \to \mathbb{R}^n$, $\mathcal{E}(v) = \gamma^v(1)$, is called the *end-point mapping* with initial point x_0. The *extended end-point mapping* is the mapping $\mathcal{F} : L^2([0,1];\mathbb{R}^r) \to \mathbb{R}^{n+1}$

$$\mathcal{F}(v) = \Big(\int_0^1 |v|^2 dt, \mathcal{E}(v) \Big).$$

If (γ, h) is an optimal pair with $\gamma(0) = x_0$ then $\mathcal{F}$ is not open at $v = h$ and then its differential is not surjective. It follows that there exists a nonzero vector $(\lambda_0, \lambda) \in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}$ such that for all $v \in L^2([0,1];\mathbb{R}^r)$ there holds $\langle d\mathcal{F}(h)v, (\lambda_0, \lambda)\rangle =$

0. The case $\lambda_0 = 0$ is the case of abnormal extremals, that are precisely the critical points of the end-point mapping $\mathcal{E}$, i. e., points h where the differential $d\mathcal{E}(h)$ is not surjective. In particular, the notion of abnormal extremal is independent of the metric fixed on the horizontal distribution.

The curve ξ, sometimes called *dual curve* of γ, is obtained in the following way. Let h be the controls of an optimal trajectory γ starting from x_0. For $x \in \mathbb{R}^n$, let γ_x be the solution to the problem

$$\dot{\gamma}_x = \sum_{j=1}^{r} h_j X_j(\gamma_x) \quad \text{and} \quad \gamma_x(0) = x.$$

The *optimal flow* is the family of mappings $P_t : \mathbb{R}^n \to \mathbb{R}^n$, $P_t(x) = \gamma_x(t)$ with $t \in \mathbb{R}$. We are assuming that the flow is defined for any $t \in \mathbb{R}$. Let $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^n$ be a vector orthogonal to the image of $d\mathcal{F}(h)$. At the point x_0 we have the 1-form $\xi(0) = \lambda_1 dx_1 + \ldots + \lambda_n dx_n$, where $(\lambda_1, \ldots, \lambda_n)$ are the coordinates of λ. Then the curve $t \mapsto \xi(t)$ given by the pull-back of $\xi(0)$ along the optimal flow at time t, namely the curve

$$\xi(t) = P^*_{-t}(x_0)\xi(0), \tag{4}$$

satisfies the *adjoint Eq.* (3).

We can use i)–iii) in Theorem 1 to define the notion of *extremal*. We say that a horizontal curve $\gamma : [0, 1] \to \mathbb{R}^n$ is an *extremal* if there exist $\xi_0 \in \{0, 1\}$ and $\xi \in \mathrm{Lip}([0, 1]; \mathbb{R}^n)$ such that i), ii), and iii) in Theorem 1 hold. We say that γ is a *normal extremal* if there exists such a pair (ξ_0, ξ) with $\xi_0 \neq 0$. We say that γ is an *abnormal extremal* if there exists such a pair with $\xi_0 = 0$. We say that γ is a *strictly abnormal extremal* if γ is an abnormal extremal but not a normal one.

If γ is an abnormal extremal with dual curve ξ, then by ii) we have, for any $j = 1, \ldots, r$,

$$\langle \xi, X_j(\gamma) \rangle = 0 \quad \text{on } [0, 1]. \tag{5}$$

Further necessary conditions on abnormal extremals can be obtained differentiating identity (5). In fact, one gets for any $j = 1, \ldots, r$,

$$\sum_{i=1}^{r} h_i \langle \xi, [X_i, X_j](\gamma) \rangle = 0 \quad \text{a.e. on } [0, 1]. \tag{6}$$

When the rank is $r = 2$, from (6) along with the free assumption $|h| \neq 0$ a.e. on $[0, 1]$ we deduce that

$$\langle \xi, [X_1, X_2](\gamma) \rangle = 0 \quad \text{on } [0, 1]. \tag{7}$$

In the case of strictly abnormal minimizers, necessary conditions analogous to (7) can be obtained also for $r \geq 3$.

Theorem 2. *Let $\gamma : [0,1] \to \mathbb{R}^n$ be a strictly abnormal length minimizer. Then any dual curve $\xi \in \mathrm{Lip}([0,1],\mathbb{R}^n)$ of γ atisfies*

$$\langle \xi, [X_i, X_j](\gamma)\rangle = 0 \quad \textit{on } [0,1] \tag{8}$$

for any $i, j = 1, \dots, r$.

Condition (8) is known as *Goh condition*. Theorem 2 can be deduced from second order open mapping theorems. We refer to [2, Chap. 20] for a systematic treatment of the subject. See also the work [3].

The Goh condition naturally leads to the notion of Goh extremal. A horizontal curve $\gamma : [0,1] \to \mathbb{R}^n$ is a *Goh extremal* if there exists a Lipschitz curve $\xi : [0,1] \to \mathbb{R}^n$ such that $\xi \neq 0$, ξ solves the adjoint Eq. (3) and $\langle \xi, X_i(\gamma)\rangle = \langle \xi, [X_i, X_j](\gamma)\rangle = 0$ on $[0,1]$ for all $i, j = 1, \dots, r$.

3 Known regularity results

In this section, we collect some regularity results for extremal and length minimzing curves. Other results are discussed in Sect. 4. The case of normal extremal is clear and classical.

Theorem 3. *Let $(M, \mathcal{D}, g)$ be any sub-Riemannian manifold. Normal extremals are C^∞ curves that are locally length minimzing.*

In fact, with the notation of Sect. 2, if γ is a normal extremal with controls h and dual curve ξ, by condition ii) in Theorem 1 we have, for any $j = 1, \dots, r$,

$$h_j = -\langle \xi, X_j(\gamma)\rangle \quad \text{a.e. on } [0,1]. \tag{9}$$

This along with the adjoint Eq. (3) implies that the pair (γ, ξ) solves a.e. the system of Hamilton's equations

$$\dot\gamma = \frac{\partial H}{\partial \xi}(\gamma,\xi), \quad \dot\xi = -\frac{\partial H}{\partial x}(\gamma,\xi), \tag{10}$$

where H is the Hamiltonian function

$$H(x,\xi) = -\frac{1}{2}\sum_{j=1}^{r}\langle \xi, X_j(x)\rangle^2.$$

This implies that $\dot\gamma$ and $\dot\xi$ are Lipschitz continuous and thus $\gamma, \xi \in C^{1,1}$. By iteration, one deduces that $\gamma, \xi \in C^\infty$.

The fact that normal extremals are locally length minimizing follows by a calibration argument, see [9, Appendix C]. Indeed, using the Hamilton's Eq. (10), the

1-form ξ along γ can be locally extended to an *exact* 1-form ξ satisfying

$$\sum_{j=1}^{r} \langle \xi, X_j(x) \rangle^2 = 1.$$

This 1-form provides the calibration.

The distribution $\mathscr{D} = \mathrm{span}\{X_1, \dots, X_r\}$ on M is said to be *bracket-generating of step* 2 if for any $x \in M$ we have

$$\dim\big(\mathrm{span}\{X_j(x), [X_i, X_j](x) : i, j = 1, \dots, r\}\big) = n, \tag{11}$$

where $n = \dim(M)$. For distributions of step 2, Goh condition (8) implies the smoothness of any minimizer.

Theorem 4. *Let $(M, \mathscr{D}, g)$ be a sub-Riemannian manifold where $\mathscr{D}$ is a distribution that is bracket generating of step 2. Then any length minimizing curve in $(M, \mathscr{D}, g)$ is of class C^∞.*

In fact, if γ is a strictly abnormal length minimizing curve with dual curve ξ then by (5), (8), and (11) it follows that $\xi = 0$ and this is not possible. In other words, there are no strictly abnormal minimizers and this implies the claim made in Theorem 4.

When the step of the distribution is at least 3, then there can exist strictly abnormal extremals. When the step is precisely 3, the regularity question is clear within the setting of Carnot groups. Let $\mathfrak{g}$ be a stratified nilpotent n-dimensional real Lie algebra with

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s, \quad s \geq 2,$$

where $\mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i]$ for $i \leq s-1$ and $\mathfrak{g}_i = \{0\}$ for $i > s$.

The Lie algebra $\mathfrak{g}$ is the Lie algebra of a connected and simply-connected Lie group G that is diffeomorphic to $\mathbb{R}^n$. Such a Lie group is called *Carnot group*. The horizontal distribution $\mathscr{D}$ on G is induced by the first layer $\mathfrak{g}_1$ of the Lie algebra. In fact, $\mathscr{D}$ is spanned by a system of r linearly independent left-invariant vector fields. By nilpotency, the distribution is bracket-generating. So any quadratic form on $\mathfrak{g}_1$ induces a left-invariant sub-Riemannian metric on G. The number $r = \dim(\mathfrak{g}_1)$ is the *rank* of the group. The number $s \geq 2$ is the *step* of the group.

Theorem 5. *Let G be a Carnot group of step $s = 3$ with a smooth left-invariant quadratic form g on the horizontal distribution $\mathscr{D}$. Any length minimizing curve in $(G, \mathscr{D}, g)$ is of class C^∞.*

This theorem is proved in [16]. A short and alternative proof, given in [6, Theorem 6.1], relies upon the fact that a strictly abnormal length minimizing curve must be contained in (the lateral of) a proper Carnot subgroup. Then a reduction argument on the rank of the group reduces the analysis to the case $r = 2$, where abnormal extremals are easily shown to be integral curves of some horizontal left-invariant vector field.

When the step is $s = 4$, there is a regularity result only for Carnot groups of rank $r = 2$, see [8, Example 4.6].

Theorem 6. *Let G be a Carnot group of step $s = 4$ and rank $r = 2$ with a smooth left invariant quadratic form g on the horizontal distribution $\mathscr{D}$. Then any length minimizing curve in $(G, \mathscr{D}, g)$ is of class C^∞.*

The proof of this result relies upon two facts. First, one proves that the horizontal coordinates of any abnormal extremal are contained in the zero set of a quadratic polynomial in two variables. This shows that the only singularity that abnormal extremals can have is of corner type. Then using a general theorem proved in [8] (see Sect. 4) one concludes that extremal curves with corners are not length minimizing.

When the rank is $r = 2$ and the step s is larger than 4, the best regularity known for minimizers is the $C^{1,\delta}$ regularity.

Theorem 7. *Let G be a Carnot group of rank $r = 2$, step $s > 4$ and with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$ satisfying*

$$[\mathfrak{g}_i, \mathfrak{g}_j] = 0 \quad \textit{for all } i, j \geq 2 \textit{ such that } i + j > 4. \tag{12}$$

Then any length minimizing curve in $(G, \mathscr{D}, g)$, where g is a smooth left-invariant metric on the horizontal distribution $\mathscr{D}$, is of class $C^{1,\delta}$ for any

$$0 \leq \delta < \min\left\{\frac{2}{s-4}, \frac{1}{4}\right\}. \tag{13}$$

This theorem is proved in [37, Theorem 10.1]. It is a byproduct of a technique that is used to analyse the length minimality properties of extremals of class C^1 whose derivative is only δ-Hölder continuos for some $0 < \delta < 1$. We give an example of such techniques in Sect. 7. The restriction $\delta < 2/(s-4)$ is a technical one. The estimates developed in [13], however, show that the restriction $\delta < 1/4$ is deeper. We shall discuss (12) it in the next section.

4 Analysis of corner type singularities

Let M be a smooth manifold with dimension $n \geq 3$, and let $\mathscr{D}$ be a completely nonintegrable distribution on M. Let $\mathscr{D}_1 = \mathscr{D}$ and $\mathscr{D}_i = [\mathscr{D}_1, \mathscr{D}_{i-1}]$ for $i \geq 2$, i.e., $\mathscr{D}_i$ is the linear span of all commutators $[X, Y]$ with $X \in \mathscr{D}_1$ and $Y \in \mathscr{D}_{i-1}$. We also let $\mathscr{L}_0 = \{0\}$ and $\mathscr{L}_i = \mathscr{D}_1 + \ldots + \mathscr{D}_i$, $i \geq 1$. By the nonintegrability condition, for any $x \in M$ there exists $s \in \mathbb{N}$ such that $\mathscr{L}_s(x) = T_x M$, the tangent space of M at x. Assume that $\mathscr{D}$ is equiregular, i.e., assume that for each $i = 1, \ldots, s$

$$\dim\left(\mathscr{L}_i(x)/\mathscr{L}_{i-1}(x)\right) \text{ is constant for } x \in M. \tag{14}$$

In [8], Leonardi and the author proved the following theorem.

Theorem 8. *Let* $(M, \mathcal{D}, g)$ *be a sub-Riemannian manifold, where* g *is a metric on the horizontal distribution* $\mathcal{D}$. *Assume that* $\mathcal{D}$ *satisfies* (14) *and*

$$[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j-1}, \quad i, j \geq 2, \ i + j > 4. \tag{15}$$

Then any curve in M *with a corner is not length minimizing in* $(M, \mathcal{D}, g)$.

A "curve with a corner" is a $\mathcal{D}$-horizontal curve $\gamma : [0, 1] \to M$ such that at some point $t \in (0, 1)$ the left and right derivatives $\dot\gamma_L(t) \neq \dot\gamma_R(t)$ exist and are different. The proof of Theorem 8 is divided into several steps.

1) First one blows up the manifold M, the distribution $\mathcal{D}$, the metric g, and the curve γ at the corner point $x = \gamma(t)$. The blow-up is in the sense of the nilpotent approximation of Mitchell, Margulis and Mostow (see e. g. [10]). The limit structure is a Carnot group and the limit curve is the union of two half-lines forming a corner.

2) The limit curve is actually contained in a subgroup of rank 2, and after a suitable choice of coordinates one can assume that the manifold is $M = \mathbb{R}^n$ with a 2-dimensional distribution $\mathcal{D} = \text{span}\{X_1, X_2\}$ spanned by the vector fields in $\mathbb{R}^n$

$$X_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x_2} + \sum_{j=3}^{n} f_j(x) \frac{\partial}{\partial x_j}, \tag{16}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 3, \dots, n$, are polynomials with certain properties. The curve obtained after the blow-up is $\gamma : [-1, 1] \to \mathbb{R}^n$

$$\gamma(t) = \begin{cases} -t\mathrm{e}_2, \ t \in [-1, 0], \\ \ \ t\mathrm{e}_1, \ \ t \in [0, 1], \end{cases} \tag{17}$$

where $\mathrm{e}_1, \dots, \mathrm{e}_n$ is the standard basis of $\mathbb{R}^n$. If the limit curve is not length minimizing in the limit structure, then the original curve is not length minimizing in the original structure.

3) At this stage, one uses (15). If the original distribution satisfies (15), then the limit Lie algebra satisfies (12) and the polynomials f_j only depends on the variables x_1 and x_2. This makes possible an effective and computable way to prove that the curve γ in (17) is not length minimizing. One cuts the corner of γ in the x_1x_2 plane gaining some length. The new planar curve must be lifted to get a horizontal curve, changing in this way the end-point. One can use several different devices to bring the end-point back to its original position. To do this, we can use a total amount of length that is less than the length gained by the cut. This adjustment is in fact possible, and the entire construction is the main achievement of [8].

The restriction (15) has a technical character. The problem of dropping this restriction is adressed in [14] (see also Sect. 6.2). The cut-and-adjust technique introduced in [8] is extended in [13] to the analysis of curves having singularities of higher order. In Sect. 7, we study a nontrivial example of such a situation.

5 Classification of abnormal extremals

The notion of abnormal extremal is rather indirect or implicit. There is a differential equation, the differential Eq. (3), involving the dual curve and the controls of the extremal. Even though this equation can be translated into some better form (see Theorem 2.6 in [6]), nevertheless the carried information is not transparent. In this section, we present some attempts to describe abnormal extremals is a more geometric or algebraic way.

5.1 Rank 2 distributions

We consider first the case when $M = \mathbb{R}^n$ and $\mathcal{D}$ is a rank 2 distribution in M spanned by vector fields X_1 and X_2 as in (16), where $f_3, \dots, f_n \in C^\infty(\mathbb{R}^2)$ are functions depending on the variables x_1, x_2. We fix on $\mathcal{D}$ the quadratic form g making X_1 and X_2 orthonormal. Let $K : \mathbb{R}^{n-2} \times \mathbb{R}^2 \to \mathbb{R}$ be the function

$$K(\lambda, x) = \sum_{i=1}^{n-2} \lambda_i \frac{\partial f_{i+2}}{\partial x_1}(x), \tag{18}$$

where $\lambda = (\lambda_1, \dots, \lambda_{n-2}) \in \mathbb{R}^{n-2}$ and $x \in \mathbb{R}^2$.

In this special situation, Pontryagin Maximum Principle can be rephrased in the following way (see Propositions 4.2 and 4.3 in [8]).

Theorem 9. *Let $\gamma : [0, 1] \to M$ be a $\mathcal{D}$-horizontal curve that is length minimizing in $(M, \mathcal{D}, g)$. Let $\kappa = (\gamma_1, \gamma_2)$ and assume that $|\dot{\kappa}| = 1$ almost everywhere. Then one (or both) of the following two statements holds:*

1) *there exists $\lambda \in \mathbb{R}^{n-2}$, $\lambda \neq 0$, such that*

$$K(\lambda, \kappa(t)) = 0, \quad \textit{for all } t \in [0, 1]; \tag{19}$$

2) *the curve γ is smooth and there exists $\lambda \in \mathbb{R}^{n-2}$ such that κ solves the system of differential equations*

$$\ddot{\kappa} = K(\lambda, \kappa)\dot{\kappa}^\perp, \tag{20}$$

where $\kappa^\perp = (-\kappa_2, \kappa_1)$.

The geometric meaning of the curvature Eq. (20) was already noticed by Montgomery in [11].

The interesting case in Theorem 9 is the case 1): the curve κ, i. e., the horizontal coordinates of γ, is in the zero set of a nontrivial explicit function.

5.2 Stratified nilpotent Lie groups

In free stratified nilpotent Lie groups (free Carnot groups) there is an algebraic characterization of extremal curves in terms of an algebraic condition analogous to (19).

Let G be a *free nilpotent* Lie group with Lie algebra $\mathfrak{g}$. Fix a Hall basis $X_1, \dots, X_n$ of $\mathfrak{g}$ and assume that the Lie algebra is generated by the first r elements $X_1, \dots, X_r$.

We refer to [5] for a precise definition of the Hall basis. The basis determines a collection of *generalized structure constants* $c_{i\alpha}^k \in \mathbb{R}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index and $i, k \in \{1, \dots, n\}$. These constants are defined via the identity

$$[X_i, X_\alpha] = \sum_{k=1}^n c_{i\alpha}^k X_k, \tag{21}$$

where the iterated commutator X_α is defined via the relation

$$[X_i, X_\alpha] = [X_i, [\underbrace{X_1, \dots, [X_1}_{\alpha_1 \text{ times}}, \dots, \underbrace{[X_n, \dots, X_n]}_{\alpha_n \text{ times}} \dots] \dots]]. \tag{22}$$

Using the constants $c_{i\alpha}^k$, for any $i = 1, \dots, n$ and for any multi-index $\alpha \in \mathbb{N}^n$, we define the linear mappings $\phi_{i\alpha} : \mathbb{R}^n \to \mathbb{R}$

$$\phi_{i\alpha}(v) = \frac{(-1)^{|\alpha|}}{\alpha!} \sum_{k=1}^n c_{i\alpha}^k v_k, \quad v = (v_1, \dots, v_n) \in \mathbb{R}^n. \tag{23}$$

Finally, for each $i = 1, \dots, n$ and $v \in \mathbb{R}^n$, we introduce the polynomials $P_i^v : \mathbb{R}^n \to \mathbb{R}$

$$P_i^v(x) = \sum_{\alpha \in \mathbb{N}^n} \phi_{i\alpha}(v) x^\alpha, \quad x \in \mathbb{R}^n, \tag{24}$$

where we let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The group G can be identified with $\mathbb{R}^n$ via exponential coordinates of the second type induced by the basis $X_1, \dots, X_n$. For any $v \in \mathbb{R}^n$, $v \neq 0$, we call the set

$$Z_v = \{x \in \mathbb{R}^n : P_1^v(x) = \ldots = P_r^v(x) = 0\}$$

an *abnormal variety* of G *of corank* 1. For linearly independent vectors $v_1, \dots, v_m \in \mathbb{R}^n$, $m \geq 2$, we call the set $Z_{v_1} \cap \ldots \cap Z_{v_m}$ an *abnormal variety* of G *of corank* m. Recall that the property of having corank m for an abnormal extremal γ means that the range of the differential of the end-point map at the extremal curve is $n - m$ dimensional.

The main result of [6] is the following theorem.

Theorem 10. *Let $G = \mathbb{R}^n$ be a free nilpotent Lie group and let $\gamma : [0, 1] \to G$ be a horizontal curve with $\gamma(0) = 0$. The following statements are equivalent:*

A) *the curve γ is an abnormal extremal of corank $m \geq 1$;*
B) *there exist m linearly independent vectors $v_1, \dots, v_m \in \mathbb{R}^n$ such that $\gamma(t) \in Z_{v_1} \cap \ldots \cap Z_{v_m}$ for all $t \in [0, 1]$.*

A stronger version of Theorem 10 holds for Goh extremals. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$, we let $r_1 = \dim(\mathfrak{g}_1)$ and $r_2 = \dim(\mathfrak{g}_2)$. Then, for $v \in \mathbb{R}^n$ with $v \neq 0$ we define the zero set

$$\Gamma_v = \{x \in \mathbb{R}^n : P_i^v(x) = 0 \text{ for all } i = r_1 + 1, \dots, r_1 + r_2\}.$$

Theorem 11. *Let $G = \mathbb{R}^n$ be a free nilpotent Lie group and let $\gamma : [0, 1] \to G$ be a horizontal curve such that $\gamma(0) = 0$. The following statements are equivalent:*

A) *the curve γ is a Goh extremal;*
B) *there exists $v \in \mathbb{R}^n$, $v \neq 0$, such that $\gamma(t) \in \Gamma_v$ for all $t \in [0, 1]$.*

The zero set Γ_v is always nontrivial for $v \neq 0$ and, moreover, there holds $v_i = 0$ for all $i = 0, \dots, r_1 + r_2$. See Remark 4.12 in [6].

These results are obtained via an explicit integration of the adjoint Eq. (3). Some work in progress [7] shows that Theorems 10 and 11 also hold in *nonfree* stratified nilpotent Lie groups.

6 Some examples

In this section, we present two examples. In the first one, we exhibit a Goh extremal having no regularity beyond the Lipschitz regularity. In the second example, there are extremals with corner in a sub-Riemannian manifold violating (15).

6.1 Purely Lipschitz Goh extremals

Let G be the free nilpotent Lie group of rank $r = 3$ and step $s = 4$. This group is diffeomorphic to $\mathbb{R}^{32}$. By Theorem 11, Goh extremals of G starting from 0 are precisely the horizontal curves γ in G contained in the algebraic set

$$\Gamma_v = \big\{x \in \mathbb{R}^{32} : P_4^v(x) = P_5^v(x) = P_6^v(x) = 0\big\},$$

for some $v \in \mathbb{R}^{32}$ such that $v \neq 0$ and $v_1 = \ldots = v_6 = 0$. The structure constants $c_{i\alpha}^k$ are determined by the relations of the Lie algebra of G. Using (24), we can then compute the polynomials defining Γ_v (for details, see [6]). These are

$$\begin{aligned}
P_4^v(x) &= -x_1 v_7 - x_2 v_8 - x_3 v_9 + x_5 v_{30} + x_6 v_{31} \\
&\quad + \frac{x_1^2}{2} v_{15} + x_1 x_2 v_{16} + x_1 x_3 v_{17} + \frac{x_2^2}{2} v_{18} + x_2 x_3 v_{19} + \frac{x_3^2}{2} v_{20} \\
P_5^v(x) &= -x_1 v_{10} - x_2 v_{11} - x_3 v_{12} - x_4 v_{30} + x_6 v_{32} \\
&\quad + \frac{x_1^2}{2} v_{21} + x_1 x_2 v_{22} + x_1 x_3 v_{23} + \frac{x_2^2}{2} v_{24} + x_2 x_3 v_{25} + \frac{x_3^2}{2} v_{26} \\
P_6^v(x) &= x_1(v_9 - v_{11}) - x_2 v_{13} - x_3 v_{14} - x_4 v_{31} - x_5 v_{32} \\
&\quad + x_1^2\left(-\frac{1}{2} v_{17} + \frac{1}{2} v_{22} + v_{30}\right) \\
&\quad + x_1 x_2(-v_{19} + v_{24} + v_{31}) + x_1 x_3(-v_{20} + v_{25}) + \frac{x_3^2}{2} v_{29}.
\end{aligned}$$

Theorem 12. *For any Lipschitz function $\phi : [0, 1] \to \mathbb{R}$ with $\phi(0) = 0$, the horizontal curve $\gamma : [0, 1] \to G = \mathbb{R}^{32}$ such that $\gamma(0) = 0$, $\gamma_1(t) = t^2$, $\gamma_2(t) = t$, and $\gamma_3(t) = \phi(t)$ is a Goh extremal.*

With the choice $v_7 = 1$, $v_{18} = 2$, and $v_j = 0$ otherwise, the relevant polynomials are $P_4^v(x) = x_2^2 - x_1$, $P_5^v(x) = P_6^v(x) = 0$. Then, the curve γ is contained in the zero set Γ_v and, by Theorem 11, it is a Goh extremal. The Lipschitz function ϕ is arbitrary. It would be interesting to understand the length minimality properties of γ depending on the regularity of ϕ.

6.2 A family of abnormal curves

During the meeting *Geometric control and sub-Riemannian geometry*, A. Agrachev and J. P. Gauthier suggested the following situation, in order to find a *nonsmooth* length-minimizing curve.

In $M = \mathbb{R}^4$, consider the vector fields

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_3} + x_3^2\frac{\partial}{\partial x_4}, \quad X_2 = \frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{\partial x_3}, \tag{25}$$

and denote by $\mathcal{D}$ the distribution of 2-planes in $\mathbb{R}^4$ spanned pointwise by X_1 and X_2. Fix a parameter $\alpha > 0$ and consider the initial and final points $L = (-1, \alpha, 0, 0) \in \mathbb{R}^4$ and $R = (1, \alpha, 0, 0) \in \mathbb{R}^4$. Let $\gamma : [-1, 1] \to \mathbb{R}^4$ be the curve

$$\gamma_1(t) = t, \quad \gamma_2(t) = \alpha|t|, \quad \gamma_3(t) = 0, \quad \gamma_4(t) = 0, \quad t \in [-1, 1]. \tag{26}$$

The curve γ is horizontal and joins L to R. Moreover, it can be easily checked that γ is an abonormal extremal.

This situation is interesting because the distribution $\mathcal{D}$ violates condition (15) with $i = 2$ and $j = 3$. In fact, we have

$$[[X_2, X_1], [[X_2, X_1], X_1]] = 48\frac{\partial}{\partial x_4}.$$

That condition (15) is violated is also apparent from the fact that the nonhorizontal variable x_3 do appear in the coefficients of the vector field X_1 in (25). The fact that the distribution $\mathcal{D}$ is not equiregular, is not relevant.

Agrachev and Gauthier asked whether the curve γ is length minimizing or not, especially for small $\alpha > 0$. The results of [8] cannot be used, because of the failure of (15). In [14], we answered in the negative to the question, at least when $\alpha \neq 1$.

Theorem 13. *For any $\alpha > 0$ with $\alpha \neq 1$, the curve γ in* (26) *is not length minimizing in* $(\mathbb{R}^4, \mathcal{D}, g)$*, for any choice of metric g on $\mathcal{D}$.*

The proof is a lengthy adaptation of the cut-and-adjust technique of [8]. When $\alpha = 1$ the construction of [13] does not work and, in this case, the length minimality property of γ remains open.

7 An extremal curve with Hölder continuous first derivative

On the manifold $M = \mathbb{R}^5$, let $\mathcal{D}$ be the distribution spanned by the vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_1^5 \frac{\partial}{\partial x_4} + x_1 x_2^3 \frac{\partial}{\partial x_5}. \tag{27}$$

We look for abnormal curves passing through $0 \in \mathbb{R}^5$. In view of Theorem 9, case 1), we consider the function $K : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$, defined as in (18),

$$K(\lambda, x) = \lambda_1 + 5\lambda_2 x_1^4 + \lambda_3 x_2^3.$$

With the choice $\lambda_1 = 0$, $\lambda_2 = 1/5$, and $\lambda_3 = -1$, the equation $K(\lambda, x) = 0$ reads $x_1^4 - x_2^3 = 0$. Thus the curve $\kappa : [0, 1] \to \mathbb{R}^2$, $\kappa(t) = (t, t^{4/3})$, is in the zero set of K. It can be checked that the horizontal curve $\gamma : [0, 1] \to M$ such that $(\gamma_1, \gamma_2) = \kappa$ is an abnormal extremal with dual curve $\xi : [0, 1] \to \mathbb{R}^5$,

$$\xi(t) = \Big(0, \frac{4}{5}t^5, 0, \frac{1}{5}, -1\Big).$$

Notice that we have, for any $t \in [0, 1]$,

$$\begin{aligned} \langle \xi, [X_1, [X_1, X_2]](\gamma)\rangle &= 4t^3, \\ \langle \xi, [X_2, [X_1, X_2]](\gamma)\rangle &= -3t^{8/3}. \end{aligned}$$

Then, when $t > 0$ the curve γ is a *regular* abnormal extremal, in the sense of Definition 14 on page 36 of [9]. By Theorem 5 on page 59 of [9], the curve γ is therefore locally (uniquely) length minimizing on the set where $t > 0$.

The curve γ fails to be *regular* abnormal at $t = 0$. Moreover, there holds $\gamma \in C^{1,1/3}([0, 1]; \mathbb{R}^5)$ with no further regularity at $t = 0$. In this section, we show that γ is not length minimizing.

Theorem 14. *Let g be any metric on the distribution $\mathcal{D}$. The horizontal curve γ : $[0, 1] \to M$ defined above is not length minimizing in $(M, \mathcal{D}, g)$ at $t = 0$.*

Proof. For any $0 < \eta < 1$, let $T_\eta \subset \mathbb{R}^2$ be the set

$$T_\eta = \Big\{(x_1, x_2) \in \mathbb{R}^2 : x_1^{4/3} < x_2 < \eta^{1/3} x_1,\ 0 < x_1 < \eta\Big\}.$$

The boundary ∂T_η is oriented counterclockwise. Let $\kappa^\eta : [0, 1] \to \mathbb{R}^2$ be the curve $\kappa^\eta(t) = (t, \eta^{1/3} t)$ for $0 \le t \le \eta$ and $\kappa^\eta(t) = (t, t^{4/3})$ for $\eta \le t \le 1$, and let $\gamma^\eta : [0, 1] \to \mathbb{R}^5$ be the horizontal curve such that $(\gamma_1^\eta, \gamma_2^\eta) = \kappa^\eta$.

We assume without loss of generality that g is the quadratic form on $\mathcal{D}$ that makes X_1 and X_2 orthonormal. The *gain of length* in passing from γ to γ^η is

$$\Delta L(\eta) = \int_0^1 |\dot\kappa|\, dt - \int_0^1 |\dot\kappa^\eta|\, dt = \frac{1}{30}\eta^{5/3} + o(\eta^{5/3}). \tag{28}$$

On the generic monomial $x_1^{i+1}x_2^j$ with $i, j \in \mathbb{N}$, the cut T_η produces the error T_η^{ij} given by the formula

$$\begin{aligned} T_\eta^{ij} &= \int_{\partial T_\eta} x_1^{i+1}x_2^j dx_2 = (i+1)\int_{T_\eta} x_1^i x_2^j dx_1 dx_2 \\ &= \frac{i+1}{j+1}\Big[\frac{1}{i+j+2} - \frac{1}{i+\frac{4}{3}(j+1)+1}\Big]\eta^{i+\frac{4}{3}(j+1)+1}. \end{aligned} \tag{29}$$

We are interested in this formula when $i = j = 0$, when $i = 4$ and $j = 0$, when $i = 0$ and $j = 3$. The initial error produced by the cut T_η is the vector of $\mathbb{R}^3$

$$\begin{aligned} \mathcal{E}_0(\eta) &= \big(T_\eta^{0,0}, T_\eta^{4,0}, T_\eta^{0,4}\big) \\ &= \Big(\frac{1}{14}\eta^{7/3}, \frac{5}{114}\eta^{19/3}, \frac{1}{95}\eta^{19/3}\Big). \end{aligned} \tag{30}$$

Only the exponents $7/3$ and $19/3$ of η are relevant, not the coefficients.

Our first step is to correct the error of order $\eta^{7/3}$ on the third coordinate. For fixed parameters $b > 0$, $\lambda > 0$, and $\varepsilon > 0$, let us define the curvilinear rectangle

$$R_{b,\lambda}(\varepsilon) = \big\{(x_1, x_2) \in \mathbb{R}^2 : b < x_1 < b + |\varepsilon|^\lambda,\ x_1^{4/3} < x_2 < x_1^{4/3} + \varepsilon\big\}. \tag{31}$$

When $\varepsilon < 0$, we let

$$R_{b,\lambda}(\varepsilon) = \big\{(x_1, x_2) \in \mathbb{R}^2 : b < x_1 < b + |\varepsilon|^\lambda,\ x_1^{4/3} + \varepsilon < x_2 < x_1^{4/3}\big\}. \tag{32}$$

The boundary $\partial R_{b,\lambda}(\varepsilon)$ is oriented counterclockwise if $\varepsilon > 0$, while it is oriented clockwise when $\varepsilon < 0$. The curve κ^η is deviated along the boundary of this rectangle and then it is lifted to a horizontal curve. The effect of $R_{b,\lambda}(\varepsilon)$ on the generic monomial $x_1^{i+1}x_2^j$ is

$$\begin{aligned} R_{b,\lambda}^{ij}(\varepsilon) &= \int_{\partial R_{b,\lambda}^{ij}(\varepsilon)} x_1^{i+1}x_2^j dx_2 \\ &= \frac{i+1}{j+1}\sum_{k=0}^{j}\binom{j+1}{k}\frac{\varepsilon^{j+1-k}}{i+\frac{4}{3}k+1}\Big[(b+|\varepsilon|^\lambda)^{i+\frac{4}{3}k+1} - b^{i+\frac{4}{3}k+1}\Big]. \end{aligned} \tag{33}$$

The *cost of length* of $R_{b,\lambda}(\varepsilon)$ is

$$\Lambda\big(R_{b,\lambda}(\varepsilon)\big) = 2|\varepsilon|. \tag{34}$$

When $i = j = 0$, formula (33) reads $R_{b,\lambda}^{0,0}(\varepsilon) = \varepsilon|\varepsilon|^\lambda$, whereas

$$\begin{aligned} R_{b,\lambda}^{4,0}(\varepsilon) &= 5\varepsilon\big[(b+|\varepsilon|^\lambda)^5 - b^5\big] \\ &= 5\varepsilon\sum_{k=0}^{4}\binom{5}{k}|\varepsilon|^{\lambda(5-k)}b^\kappa, \end{aligned} \tag{35}$$

and

$$\begin{aligned} R^{0,3}_{b,\lambda}(\varepsilon) &= \frac{1}{4}\sum_{k=0}^{3}\binom{4}{k}\frac{\varepsilon^{4-k}}{\frac{4}{3}k+1}\Big[(b+|\varepsilon|^{\lambda})^{\frac{4}{3}k+1}-b^{\frac{4}{3}k+1}\Big] \\ &= \frac{\varepsilon}{5}\big[(b+|\varepsilon|^{\lambda})^{5}-b^{5}\big]+\frac{1}{4}\sum_{k=0}^{2}\binom{4}{k}\frac{\varepsilon^{4-k}}{\frac{4}{3}k+1}\Big[(b+|\varepsilon|^{\lambda})^{\frac{4}{3}k+1}-b^{\frac{4}{3}k+1}\Big] \\ &= \frac{1}{25}R^{4,0}_{b,\lambda}(\varepsilon)+\widehat{R}^{0,3}_{b,\lambda}(\varepsilon), \end{aligned} \tag{36}$$

where $\widehat{R}^{0,3}_{b,\lambda}(\varepsilon)$ is defined via the last identity.

We choose $b = \eta$. The parameter $0 < \lambda < 1$ will be fixed at the end of the argument. To correct the error on the third coordinate, we solve the equation

$$R^{0,0}_{\eta,\lambda}(\varepsilon) + T^{0,0}_{\eta} = 0,$$

in the unknown ε. In fact, this equation is $\varepsilon|\varepsilon|^{\lambda} + \frac{1}{14}\eta^{7/3} = 0$ and the solution is

$$\varepsilon = -c_0\eta^{\beta}, \quad \text{where } c_0 = \frac{1}{14^{1/(1+\lambda)}} \text{ and } \beta = \frac{7}{3(1+\lambda)}.$$

The choice $b = \eta$ is not relevant, here. By (28) and (34), the cost of length is admissible if $\eta > 0$ is small enough and we have

$$\frac{7}{3(1+\lambda)} > \frac{5}{3} \quad \Leftrightarrow \quad \lambda < \frac{2}{5}. \tag{37}$$

This is our first restriction on λ.

The rectangle $R_{\eta,\lambda}(\varepsilon)$ produces new errors on the fourth and fifth coordinates. Namely, by (35) we have

$$R^{4,0}_{\eta,\lambda}\big(-c_0\eta^{\beta}\big) = -5c_0\eta^{\beta}\sum_{k=0}^{4}\binom{5}{k}\big(c_0\eta^{\beta}\big)^{\lambda(5-k)}\eta^{k}. \tag{38}$$

When $\lambda < 3/4$, condition implied by (37), the leading term in η in the sum above is obtained for $k = 0$.

By (36), the error produced on the last coordinate is

$$R^{0,3}_{\eta,\lambda}\big(-c_0\eta^{\beta}\big) = \frac{1}{4}\sum_{k=0}^{3}\binom{4}{k}\big(-c_0\eta^{\beta}\big)^{4-k}\frac{3}{4k+3}\Big[\Big(\eta+\big(c_0\eta^{\beta}\big)^{\lambda}\Big)^{\frac{4}{3}k+1}-\eta^{\frac{4}{3}k+1}\Big]. \tag{39}$$

When $\lambda < 3/4$, the bracket $[\ldots]$ in the sum over k above is

$$\big(c_0\eta^{\beta}\big)^{\lambda(\frac{4}{3}k+1)}\Big[1+\Big(\frac{4}{3}k+1\Big)c_0^{-\lambda}\eta^{1-\frac{7\lambda}{3(1+\lambda)}}+\frac{2k}{3}\Big(\frac{4}{3}k+1\Big)c_0^{-2\lambda}\eta^{2-\frac{14\lambda}{3(1+\lambda)}}+\ldots\Big].$$

The leading term in the sum in (39) is obtained for $k = 3$, and the second leading term is obtained for $k = 2$.

We have the new vector of errors

$$\mathscr{E}_1(\eta) = \Big(0, R^{4,0}_{\eta,\lambda}\big(-c_0\eta^\beta\big) + T^{4,0}_\eta, R^{0,3}_{\eta,\lambda}\big(-c_0\eta^\beta\big) + T^{0,3}_\eta\Big). \tag{40}$$

When $\lambda < 3/4$, the errors on the fourth and fifth coordinates produced by the rectangle $R_{\eta,\lambda}$ dominate the errors produced by the cut, see (30). In fact, we have

$$\beta(1+5\lambda) < \frac{19}{3} \quad \Leftrightarrow \quad \lambda < \frac{3}{4}.$$

Also the second leading term in $R^{0,3}_{\eta,\lambda}\big(-c_0\eta^\beta\big)$ dominates $T^{0,3}_\eta$. In fact, we have

$$\beta\Big(2+\frac{11}{3}\lambda\Big) < \frac{19}{3} \quad \Leftrightarrow \quad \lambda < \frac{3}{4}.$$

Now we use a rectangle $R_{b,\mu}(\varepsilon)$ to correct the error on the fourth coordinate. Here, $\frac{1}{2} < b < 3/4$ is position parameter and $\mu > 0$ is small enough. Conceptually, we could take $\mu = 0$. The parameter $\mu > 0$ is only needed to confine the construction in a bounded region. We solve the equation

$$R^{4,0}_{b,\mu}(\varepsilon) + R^{4,0}_{\eta,\lambda}\big(-c_0\eta^\beta\big) = 0 \tag{41}$$

in the unknown ε. By the formulas computed above, we deduce that the solution $\varepsilon = \bar{\varepsilon}$ is

$$\bar{\varepsilon} = c_1\eta^{\beta\frac{1+5\lambda}{1+5\mu}} + \dots,$$

where $c_1 > 0$ is an explicit constant and the dots stand for lower order terms in η. The cost of length of the rectangle $R_{b,\mu}(\bar{\varepsilon})$ is admissible for any $\mu > 0$ close to 0, because $\beta(1+5\lambda) > 5/3$.

By (36) and (41), we have the identity

$$R^{0,3}_{b,\mu}(\bar{\varepsilon}) + R^{0,3}_{\eta,\lambda}\big(-c_0\eta^\beta\big) = \widehat{R}^{0,3}_{b,\mu}(\bar{\varepsilon}) + \widehat{R}^{0,3}_{\eta,\lambda}\big(-c_0\eta^\beta\big),$$

and, therefore, the new vector of errors is

$$\mathscr{E}_2(\eta) = \Big(R^{0,0}_{b,\mu}(\bar{\varepsilon}), T^{4,0}_\eta, \widehat{R}^{0,3}_{b,\mu}(\bar{\varepsilon}) + \widehat{R}^{0,3}_{\eta,\lambda}\big(-c_0\eta^\beta\big) + T^{0,3}_\eta\Big), \tag{42}$$

where we have

$$R^{0,0}_{b,\mu}(\bar{\varepsilon}) = \bar{\varepsilon}^{1+\mu} = c_2\eta^{\beta\frac{(1+5\lambda)(1+\mu)}{1+5\mu}} + \dots, \tag{43}$$

with the coefficient $c_2 = c_1^{1+\mu}$.

In the next step, we correct simultaneously the errors on the fourth and fifth coordinates. We need curvilinear squares. Let $0 < b < 1$ be a position parameter. For any $\varepsilon \in (-1,1)$, we let

$$Q_b(\varepsilon) = R_{b,1}(|\varepsilon|). \tag{44}$$

The parameter λ of the rectangle is set to $\lambda = 1$. Set-theoretically, the definition is the same for positive and negative ε. However, when $\varepsilon > 0$ the boundary $\partial Q_b(\varepsilon)$ of the square is oriented clockwise; when $\varepsilon < 0$ the boundary is oriented counterclockwise. The *cost of length* $\Lambda(Q_b(\varepsilon))$ of the square is the sum of the length of the four sides. For some constant $C > 0$ independent of b and ε we have

$$\Lambda(Q_b(\varepsilon)) \leq C|\varepsilon|. \tag{45}$$

By (33), when $\varepsilon > 0$ the effect $Q_b^{ij}(\varepsilon)$ of the square on the monomial $x_1^{i+1}x_2^j$ is

$$Q_b^{ij}(\varepsilon) = \frac{i+1}{j+1}\sum_{k=0}^{j}\binom{j+1}{k}\varepsilon^{j+1-k}\frac{1}{i+\frac{4}{3}k+1}\Big[(b+\varepsilon)^{i+\frac{4}{3}k+1} - b^{i+\frac{4}{3}k+1}\Big].$$

When $\varepsilon < 0$, we have $Q_b^{ij}(\varepsilon) = -Q_b^{ij}(|\varepsilon|)$.

Let $3/4 < b_1 < b_2 < 1$ be position parameters and let $\mu > 0$ be close to 0. We solve the system of equations

$$\begin{cases} Q_{b_1}^{4,0}(\varepsilon_1) + R_{b_2,\mu}^{4,0}(\varepsilon_2) + T_\eta^{4,0} = 0 \\ Q_{b_1}^{0,3}(\varepsilon_1) + R_{b_2,\mu}^{0,3}(\varepsilon_2) + \widehat{R}_{b,\mu}^{0,3}(\bar{\varepsilon}) + \widehat{R}_{\eta,\lambda}^{0,3}\big(-c_0\eta^\beta\big) + T_\eta^{0,3} = 0 \end{cases}$$

in the unknowns $\varepsilon_1, \varepsilon_2$. Subtracting the first equation from the second one and using (36), we get the equivalent system

$$\begin{cases} Q_{b_1}^{4,0}(\varepsilon_1) + R_{b_2,\mu}^{4,0}(\varepsilon_2) + T_\eta^{4,0} = 0 \\ \widehat{Q}_{b_1}^{0,3}(\varepsilon_1) + \widehat{R}_{b_2,\mu}^{0,3}(\varepsilon_2) + \mathcal{E}(\eta) = 0 \end{cases} \tag{46}$$

where

$$\begin{aligned} \mathcal{E}(\eta) &= \widehat{R}_{b,\mu}^{0,3}(\bar{\varepsilon}) + \widehat{R}_{\eta,\lambda}^{0,3}\big(-c_0\eta^\beta\big) + T_\eta^{0,3} - T_\eta^{4,0} \\ &= c_3\eta^{\beta(2+\frac{11}{3}\lambda)} + \dots \end{aligned}$$

for some $c_3 > 0$. The dots stand for lower order terms in η. In fact, the leading term in $R_{\eta,\lambda}^{0,3}\big(-c_0\eta^\beta\big)$ dominates the remaining terms. Using a notation consistent with (35), we also let

$$\begin{aligned} \widehat{Q}_{b_1}^{0,3}(\varepsilon_1) &= \operatorname{sgn}(\varepsilon_1)\frac{1}{4}\sum_{k=0}^{2}\binom{4}{k}\frac{|\varepsilon_1|^{4-k}}{\frac{4}{3}k+1}\Big[(b_1+|\varepsilon_1|)^{\frac{4}{3}k+1} - b_1^{\frac{4}{3}k+1}\Big] \\ &= c_4\varepsilon_1^3 + \dots. \end{aligned}$$

Above, $c_4 > 0$ is a constant and the dots stand for negligible terms. Notice that we have control on the sign of the leading term.

The system (46) can thus be approximated in the following way

$$\begin{cases} \operatorname{sgn}(\varepsilon_1)\varepsilon_1^2 + c_5\varepsilon_2|\varepsilon_2|^{5\mu} + c_6\eta^{\frac{19}{3}} + \ldots = 0 \\ \varepsilon_1^3 + c_7|\varepsilon_2|^{2+\frac{11}{3}\mu} + c_8\eta^{\beta(2+\frac{11}{3}\lambda)} + \ldots = 0, \end{cases} \tag{47}$$

where $c_5, \ldots, c_8 > 0$ are constants and the dots stand for negligible terms. We can compute ε_2 as a function of ε_1 from the first equation and replace this value into the second equation. This operation produces lower order terms. Thus the second equation reads

$$\varepsilon_1^3 + c_8\eta^{\beta(2+\frac{11}{3}\lambda)} + \ldots = 0,$$

and there is a solution $\varepsilon_1 < 0$ satisfying

$$\varepsilon_1 = -c_9\eta^{\beta(2+\frac{11}{3}\lambda)/3} + \ldots = -c_9\eta^{\frac{7(2+\frac{11}{3}\lambda)}{9(1+\lambda)}} + \ldots,$$

where $c_9 > 0$ and the dots stand for lower order terms in η. As a consequence, from the first equation in (47) we deduce that

$$\varepsilon_2 = -c_{10}\eta^{\frac{19}{3(1+5\mu)}} + \ldots.$$

The cost of length of the rectangle $R_{b_2,\mu}(\varepsilon_2)$ is $2|\varepsilon_2|$, and it is admissible because for $\mu > 0$ close to 0 we have

$$\frac{19}{3(1+5\mu)} > \frac{5}{3}.$$

By (45), the cost of lenght of the square $Q_{b_1}(\varepsilon_1)$ is at most $C|\varepsilon_1|$, and, for small η, it is admissible if and only if

$$\frac{7(2+\frac{11}{3}\lambda)}{9(1+\lambda)} > \frac{5}{3} \quad \Leftrightarrow \quad \lambda > \frac{3}{32}. \tag{48}$$

Here, we have a nontrivial restriction for λ. This restriction is compatible with (37). Now the parameter λ is fixed once for all in such a way that

$$\frac{3}{32} < \lambda < \frac{2}{5}. \tag{49}$$

The device $Q_{b_1}(\varepsilon_1)$ produces an error on the third coordinate of the order $|\varepsilon_1|^2$, that is of the order $\eta^{14(2+\frac{1}{3}\lambda)/9}$. The device $R_{b_2,\mu}(\varepsilon_2)$ produces an error on the third coordinate of the order $|\varepsilon_2|^{1+\mu}$, that is of the order $\eta^{19(1+\mu)/3}$. These errors are negligible with respect to the error $R^{0,0}_{b,\mu}(\bar{\varepsilon})$ appearing in (42)–(43). Eventually, after our last correction we have the vector of errors

$$\mathcal{E}_3(\eta) = \big(c_2\eta^{\frac{7}{3}\varrho} + \ldots, 0, 0\big), \quad \text{where } \varrho = \frac{(1+5\lambda)(1+\mu)}{(1+\lambda)(1+5\mu)}, \tag{50}$$

the dots stand for lower order terms and the number ϱ satisfies the key condition $\varrho > 1$, provided that $0 < \mu < \lambda$. Now also μ is fixed.

Comparing the initial error $\mathcal{E}_0(\eta)$ in (30) and the error $\mathcal{E}_3(\eta)$ in (50), we realize that the initial error $\eta^{7/3}$ on the third coordinate decreased by a geometric factor $\varrho > 1$. Now we can iterate the entire construction to set to zero all the three components of the error. Here, we omit the details of this standard part of the argument. This finishes the proof.

Remark 1. The curve γ studied in Theorem 14 is of class $C^{1,1/3}$. The curves considered in Theorem 7 are at most $C^{1,1/4}$. There is a gap between the two cases. In the proof of Theorem 14, the key step is the choice of λ made in (49). In particular, there is a very delicate bound from below for λ. In the proof of Theorem 7, there is no such a bound from below.

8 Final comments

Concerning the question about the regularity of length minimizing curves in sub-Riemannian manifolds, there are two possibilities. Either, in any sub-Riemannian manifold every length minimizing curve is C^∞ smooth (answer in the positive); or, there is some sub-Riemannian manifold with nonsmooth (non C^1, non C^2, etc.) length minimizing curves (answer in the negative). The author has no clear feeling on which of the two answers to bet.

Theorem 5 on step 3 Carnot groups suggests that, in sub-Riemannian manifolds of step 3, any length minimizing curve is C^∞ smooth. This seems to be the first question to investigate in view of an answer in the positive. In the same spirit, Theorem 6 suggests that in sub-Riemannian manifolds of rank 2 and step 4 any length minimizing curve is C^∞ smooth.

On the other hand, the first example to investigate in order to find a length minimizer with a corner type singularity is the one of Sect. 6.2 with the choice $\alpha = 1$. Moreover, Theorem 7 and the computations made in Sect. 7 suggest to look for nonsmooth length minimizing curves in the class of $C^{1,\delta}$ abnormal extremals with $0 < \delta < 1$ sufficiently close to 1. One interesting example could be the manifold $M = \mathbb{R}^5$ with the distribution spanned by the vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_1^{2m} \frac{\partial}{\partial x_4} + x_1 x_2^m \frac{\partial}{\partial x_5}, \tag{51}$$

for $m \in \mathbb{N}$ large.

Finally, the example of a purely Lipschitz Goh extremal of Sect. 6.1 proves that the first and second order necessary conditions for strictly abnormal extremals do not imply, in general, any further regularity beyond the given Lipschitz regularity. New and deeper techniques are needed in order to develop the regularity theory.

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A case study in strong optimality and structural stability of bang–singular extremals

Laura Poggiolini and Gianna Stefani

Abstract Motivated by the well known dodgem car problem, we give sufficient conditions for strong local optimality and structural stability of a bang–singular trajectory in a minimum time problem where the dynamics is single input, affine with respect to the control and depends on a finite–dimensional parameter, the initial point is fixed and the final one is constrained to an integral line of the controlled vector field.

On the nominal problem, we assume the coercivity of a suitable second variation along the singular arc and regularity both of the bang arc and of the junction point, thus obtaining sufficient conditions for strict strong local optimality for the given bang-singular extremal trajectory. Moreover, assuming the uniqueness of the adjoint covector along the singular arc, we prove that, for any sufficiently small perturbation of the parameter, there is a bang-singular extremal trajectory which is a strict strong local optimiser for the perturbed problem.
The results are proven via the Hamiltonian approach to optimal control and by taking advantage of previous results of the authors.

1 Introduction

The dodgem car problem has been widely studied and well understood in the framework of optimal control, see e. g. [4]. It consists in the minimum time problem for

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steering a car which moves with fixed speed and controlled-bounded angular velocity from a given position in the plane and a given orientation to another given position with free orientation. In the case when also the final orientation is prescribed, the problem is quoted as Dubins car problem, see e. g. [1].

It is well known that if the initial and final positions are far enough, then the minimum time trajectory is given by the concatenation of a bang and of a singular arc and such structure is stable under small perturbations of the boundary conditions.

The dodgem car problem is therefore a good test for the Hamiltonian approach to strong local optimality and structural stability of optimisers.

Motivated by these arguments we consider the minimum time problem $(\mathbf{P_r})$ for the following parameter dependent controlled dynamics:

$$\dot{\xi}^r(t) = f_0^r(\xi(t)) + u(t) f_1^r(\xi(t)) \tag{1}$$

$$u(t) \in [-1, 1] \tag{2}$$

with parameter dependent end-point constraints

$$\xi^r(0) = a^r , \quad \xi^r(T) \in N_f^r \tag{3}$$

where N_f^r is a given integral line of f_1^r, i. e.

$$N_f^r := \{\exp s f_1^r(y^r) \colon s \in \mathbb{R}\}$$

and the parameter r is finite dimensional, say $r \in \mathbb{R}^k$. The state space is a finite dimensional manifold M and all the data are assumed to be smooth with respect to all the variables.

The special structure of the end-point constraints (i. e. an integral line of the controlled vector field f_1^r) is inherited by the dodgem car problem and plays a crucial role in the obtained results.

When studying strong local optimality we consider localisation only with respect to trajectories without involving the associated controls. More precisely one considers the two following kinds of strong local optimality for an admissible triplet (T^r, ξ^r, u^r) for problem $(\mathbf{P_r})$.

Definition 1. The trajectory $\xi^r : [0, T^r] \to M$ is a *(time, state)–local minimiser* of $(\mathbf{P_r})$ if there exist a neighbourhood $\widetilde{\mathcal{U}}$ of its graph in $\mathbb{R} \times M$ and $\varepsilon > 0$ such that ξ^r is a minimiser among the admissible trajectories whose graphs are in $\widetilde{\mathcal{U}}$ and whose final time is greater than $T^r - \varepsilon$.

We point out that this kind of optimality is local both with respect to time and space. A *stronger version* of strong local optimality is the so–called *state–local optimality* which is defined as follows:

Definition 2. The trajectory ξ^r is a *state–local minimiser* of $(\mathbf{P_r})$ if there is a neighbourhood $\mathcal{U}$ of its range in M such that ξ^r is a minimiser among the admissible trajectories whose range is in $\mathcal{U}$.

The main point in the Hamiltonian approach to optimality sufficient conditions is studying the difference of the costs of two admissible trajectories by lifting them to the cotangent bundle according to the following paradigm:

- regularity conditions which allow to define a suitable over-maximised Hamiltonian flow;
- coercivity of a suitable second variation which allows to define a lift of trajectories to such flow. We point out that such second variation involves only the singular arc. For more details see Sect. 2.2.

Using this approach in [9] we considered the case of bang-singular-bang extremals for the fixed end-points constraint and gave sufficient conditions for both kinds of strong local optimality. State-local optimality of bang-singular trajectories is obtained there as a by product.

Here we prove that the same assumptions which ensure state-local optimality for the fixed end-points constraints are sufficient also if the final constraint is an integral line of the controlled vector field, see Theorem 1. The same result was obtained in [8] for (time, state)-local optimality.

The Hamiltonian approach to structural stability consists in applying the Implicit Function Theorem to the flow of a parameter dependent over-maximised Hamiltonian, thus it allows to obtain the smoothness with respect to the parameter of the switching and final times and of the singular control which is obtained as a feedback control on the cotangent bundle, i. e. it gives a hint on performing some sensitivity analysis.

As usual in the classical approach to structural stability the assumptions are the ones which ensure the strict state–local optimality of the given bang–singular extremal in the nominal problem, see [8] and [9], and add suitable controllability assumptions to obtain the structural stability result.

With this approach, in [11], we proved structural stability for the bang-singular-bang extremals studied in [9]. We point out explicitly that, differently from optimality, structural stability of bang-singular extremals for the fixed end-points constraint does not hold true.

Here we prove structural stability of bang-singular extremals for problem $(\mathbf{P_r})$, namely we prove that there exists $\epsilon > 0$ such that for any r, $\|r\| < \epsilon$, there is a bang–singular extremal, say ξ^r, which is a strict state–local optimiser for $(\mathbf{P_r})$.

To the authors knowledge state-local optimality for extremals containing singular arcs has not been studied but in [9]. For results on weaker kinds of optimality see [2] and the wide bibliography therein. Structural stability results for bang-singular-bang arcs in a Mayer problem with fixed–free end points constraint are in [5] where the author proves structural stability of extremals in a completely different framework. In [6] the author shows, with a counterexample, that if, under perturbations, the final constraint is no longer an integral line of the controlled vector field, then structural stability of bang-singular extremals is lost i. e. a new (small) final bang arc appears in the optimal trajectory.

1.1 Notation

In this paper we use some basic element of the theory of symplectic manifolds referred to the cotangent bundle T^*M.

Denote by $\pi\colon T^*M \to M$ the canonical projection, for $\ell \in T^*M$ the space $T^*_{\pi\ell}M$ is canonically embedded in $T_\ell T^*M$ as the space of tangent vectors to the fibers.

The canonical Liouville one–form ς on T^*M and the associated canonical symplectic two–form $\boldsymbol{\sigma} = \mathrm{d}\varsigma$ allow one to associate to any, possibly time-dependent, smooth Hamiltonian $H_t\colon T^*M \to \mathbb{R}$, a Hamiltonian vector field $\overrightarrow{H_t}$, by

$$\boldsymbol{\sigma}(v, \overrightarrow{H_t}(\ell)) = \langle \mathrm{d}H_t(\ell)\,,\, v\rangle, \quad \forall v \in T_\ell T^*M.$$

Finally recall that any vector field f on the manifold M defines, by lifting to the cotangent bundle, a Hamiltonian

$$F\colon \ell \in T^*M \mapsto \langle \ell\,,\, f(\pi\ell)\rangle \in \mathbb{R}.$$

We denote by F_0^r, F_1^r, the Hamiltonians associated to f_0^r, f_1^r, respectively, and by $F^r_{i_1 i_2 \ldots i_k}$, $i_1, \ldots, i_k \in \{0,\ 1\}$ the Hamiltonian associated to the vector field $f^r_{i_1 i_2 \ldots i_k} := [f^r_{i_1}, [\ldots[f^r_{i_{k-1}}, f^r_{i_k}]\ldots]$, where $[\cdot,\cdot]$ denotes the Lie brackets between vector fields.

Moreover we define $H^{\max,r}$ to be the continuous maximised Hamiltonian associated to the control system (1)–(2), i. e.

$$H^{\max,r}\colon \ell \mapsto \max_{u\in[-1,1]} \{F_0^r(\ell) + uF_1^r(\ell)\}.$$

For ease of reading, when $r = 0$ we omit the parameter, i. e. we write f_0 instead of f_0^0, f_1 instead of f_1^0, and so on.

We assume we are given a reference triple $(\widehat{T}, \widehat{\xi}, \widehat{u})$ which is a normal bang–singular Pontryagin extremal for the nominal problem $(\mathbf{P_0})$. By bang–singular trajectory (or triple or control) we mean that $\widehat{u}$ has the following structure

$$\begin{aligned} &\widehat{u}(t) \equiv u_1 \in \{-1, 1\} && \forall t \in [0, \widehat{\tau})\,, \\ &\widehat{u}(t) \in (-1, 1) && \forall t \in (\widehat{\tau}, \widehat{T}] \end{aligned} \tag{4}$$

We shall refer to $\widehat{\tau}$ as to the *switching time* of the reference control $\widehat{u}$ and to the time-dependent vector field $\widehat{f}_t := f_0 + \widehat{u}(t) f_1$ as to the reference vector field of the nominal problem $(\mathbf{P_0})$.

In particular we denote by $h_1 = f_0 + u_1 f_1$ the restriction of the reference vector field to the bang interval $[0, \widehat{\tau}]$ and by H_1 the associated Hamiltonian.

Since in this paper the switching time $\widehat{\tau}$ plays a special role, we consider all the flows as starting at time $\widehat{\tau}$. The flow from time $\widehat{\tau}$ of $\widehat{f}_t$ is a map defined in a neighbourhood of the point

$$\widehat{x} := \widehat{\xi}(\widehat{\tau})$$

which we denote as $\widehat{S}_t \colon M \to M$, $t \in [0, \widehat{T}]$ while

$$\widehat{F}_t = \begin{cases} H_1 & \text{if } t \in [0, \widehat{\tau}), \\ F_0 + \widehat{u}(t) F_1 & \text{if } t \in (\widehat{\tau}, \widehat{T}], \end{cases}$$

denotes the time–dependent reference Hamiltonian obtained lifting $\widehat{f}_t$.

We denote the flow of any Hamiltonian field $\overrightarrow{H_t}$ from time $\widehat{\tau}$ to time t, as

$$\mathcal{H} \colon (t, \ell) \mapsto \mathcal{H}(t, \ell) = \mathcal{H}_t(\ell).$$

and we call it *the flow of the Hamiltonian* H_t.

Notice that for $t < \widehat{\tau}$, the flow goes backwards in time.

We keep these notation throughout the paper, namely the overhead arrow denotes the vector field associated to a Hamiltonian and the script letter denotes its flow from time $\widehat{\tau}$, unless otherwise stated. Also we use the following notation from differential geometry: $f \cdot \alpha$ is the Lie derivative of a function α with respect to the vector field f. Moreover, if G is a C^1 map from a manifold X to a manifold Y, we denote its tangent map at a point $x \in X$, where the point x is clear from the context, as G_*.

2 Assumptions on the nominal problem

In this section we state the assumptions on the nominal extremal which allow to get the optimality and structural stability results.

2.1 Pontryagin Maximum Principle and Regularity Assumptions

In this section we recall the first order optimality condition which the reference triplet $(\widehat{T}, \widehat{\xi}, \widehat{u})$ must satisfy.

We call *extremal* of $(\mathbf{P_0})$ any curve in the cotangent bundle which satisfies Pontryagin Maximum Principle (PMP) and *state–extremal* of $(\mathbf{P_0})$ its projection on the state space.

In the minimum time problem PMP requires that the reference trajectory is a state extremal, here we ask for the reference trajectory to be a *normal* state extremal, i. e. we assume that the triplet $(\widehat{T}, \widehat{\xi}, \widehat{u})$ satisfies the following

Assumption 2 (Normal PMP). There exists a solution $\widehat{\lambda} \colon t \in [0, \widehat{T}] \to \widehat{\lambda}(t) \in T^*M$ of the Hamiltonian system

$$\dot{\lambda}(t) = \overrightarrow{\widehat{F}_t} \circ \lambda(t)$$

such that

$$\pi \circ \widehat{\lambda} = \widehat{\xi}, \quad \widehat{\lambda}(0) \neq 0 \tag{5}$$

and

$$\widehat{F}_t \circ \widehat{\lambda}(t) = H^{\max} \circ \widehat{\lambda}(t) = 1 \quad \text{a.e. } t \in [0, \widehat{T}]. \tag{6}$$

Remark 1. In this case the transversality condition at the final time $\widehat{T}$ is a consequence of (6).

$\widehat{\lambda}\colon [0, \widehat{T}] \to T^*M$ is called *adjoint covector*. We denote its end points and its junction point between the bang and the singular arc as

$$\widehat{\ell}_0 := \widehat{\lambda}(0), \quad \widehat{\ell}_f := \widehat{\lambda}(\widehat{T}), \quad \widehat{\ell} := \widehat{\lambda}(\widehat{\tau}),$$

respectively. Because of the structure of the reference control $\widehat{u}$, as defined by Eqs. (4), PMP implies

$$u_1 F_1 \circ \widehat{\lambda}(t) \geq 0 \;\; t \in [0, \widehat{\tau}), \quad F_1 \circ \widehat{\lambda}(t) = 0 \;\; t \in [\widehat{\tau}, \widehat{T}]. \tag{7}$$

As a consequence one gets

$$F_{01} \circ \widehat{\lambda}(t) \equiv 0 \qquad t \in [\widehat{\tau}, \widehat{T}], \tag{8}$$

$$(F_{001} + \widehat{u}(t) F_{101}) \circ \widehat{\lambda}(t) = 0 \quad t \in (\widehat{\tau}, \widehat{T}), \tag{9}$$

$$u_1 \left(F_{001} + u_1 F_{101}\right) (\widehat{\ell}_1) \geq 0, \tag{10}$$

see [9]. Moreover it is known that a necessary condition for the local optimality of a Pontryagin extremal is the *generalised Legendre condition* (GLC) along the singular arc:

$$F_{101} \circ \widehat{\lambda}(t) \geq 0 \quad t \in [\widehat{\tau}, \widehat{T}], \tag{11}$$

see for example [1], Corollary 20.18 page. 318; for a classical result see [7]. We assume that the inequality holds strictly.

Assumption 3 (Strong Generalised Legendre condition).

$$F_{101} \circ \widehat{\lambda}(t) > 0 \quad t \in [\widehat{\tau}, \widehat{T}], \tag{SGLC}$$

When (SGLC) holds, a singular extremal is called *of the first kind*, see e. g. [13].

PMP yields the mild inequalities in (7), (10) and (11), we assume the strict inequalities to hold, whenever possible.

Assumption 4 (Regularity along the bang arcs).

$$u_1 F_1 \circ \widehat{\lambda}(t) > 0 \quad \forall t \in [0, \widehat{\tau}).$$

Assumption 5 (Regularity at the junction point).

$$(u_1 F_{001} + F_{101}) (\widehat{\ell}) > 0.$$

Remark 2. (SGLC) implies that $\widehat{u} \in C^\infty((\widehat{\tau}, \widehat{T}])$ and that Assumption 5 is equivalent to the discontinuity of $\widehat{u}$ at times $\widehat{\tau}$, see [9].

Assumption 6 (Uniqueness of the adjoint covector). $\widehat{\lambda}\Big|_{[\widehat{\tau},\widehat{T}]}$ is the only extremal associated to $\widehat{\xi}\Big|_{[\widehat{\tau},\widehat{T}]}$ for the minimum time problem between its end points $\widehat{x} := \widehat{\xi}(\widehat{\tau})$ and $\widehat{x}_f := \widehat{\xi}(\widehat{T})$.

2.2 The extended second variation

System (1) is linear with respect to the control, therefore the standard second variation is completely degenerate. In order to obtain a second variation a typical procedure is to transform the minimum time problem into a Mayer one and, via a coordinate–free version of Goh's transformation, one obtains a suitable second order approximation on the singular arc, which we call *extended second variation*.

Such second order approximation takes into account variations of the singular control, variations of the lengths of the bang and of the singular intervals and of the final point on the constraint $\mathcal{N}_f$. In [10] we proved that the largest sub–space where the extended second variation can be coercive is the one relative to the minimum time problem with fixed end points $\xi(\widehat{\tau}) = \widehat{x}, \quad \xi(\widehat{T}) = \widehat{x}_f$. In other words, there is no need to take into account any variation of the switching time $\widehat{\tau}$.

We point out that the same assumption, together with Assumptions 4–5 is sufficient for $\widehat{\xi}$ to be a minimum time trajectory between $\widehat{x}_0$ and $\widehat{x}_f$, see [9].

For the sake of completeness we write here the reduced Mayer problem.

$$\text{Minimise} \quad \xi^0(\widehat{T}) \tag{12}$$

subject to

$$\dot{\xi}^0(s) = u_0(s) \quad s \in [\widehat{\tau}, \widehat{T}] \tag{13}$$

$$\dot{\xi}(s) = u_0(s)\, f_0(\xi(s)) + u_0(s)\, u(s)\, f_1(\xi(s)) \quad s \in [\widehat{\tau}, \widehat{T}] \tag{14}$$

$$\xi^0(\widehat{\tau}) = \widehat{\tau}, \quad \xi(\widehat{\tau}) = \widehat{x}, \quad \xi^0(\widehat{T}) \in \mathbb{R}, \quad \xi(\widehat{T}) = \widehat{x}_f \tag{15}$$

$$(u_0(s), u(s)) \in (0, +\infty) \times (-1, 1). \tag{16}$$

For a complete computation of the extended second variation see [9]. Here we give the final result. In particular we recall that in [9], f_0 and f_1 are proven to be linearly independent at $\widehat{x}$, so that we may choose local coordinates around $\widehat{x}$ which simplify computations. Namely, we choose coordinates such that

$$\begin{aligned} &1. \qquad f_1 \text{ is constant: } f_1 \equiv \frac{\partial}{\partial x_1}; \\ &2. \qquad f_0 = \frac{\partial}{\partial x_2} - x_1\,(f_{01}(\widehat{x}) + O(x)). \end{aligned} \tag{17}$$

In such coordinates choose β as

$$\beta(x) := -\sum_{i=2}^{n} \lambda_i x_i, \tag{18}$$

where $(0, \lambda_2, \dots, \lambda_n, 0, \dots, 0)$ are the coordinates of $\widehat{\ell}$. We get $\lambda_2 = 1$, $f_1 \cdot \beta \equiv 0$, and $f_0 \cdot f_0 \cdot \beta(\widehat{x}) = 0$. Finally, define the dragged vector fields at time $\widehat{\tau}$, along the reference flow, by setting

$$g_{i,t}(x) := \widehat{S}_{t*}^{-1} f_i \circ \widehat{S}_t(x)\,,\ i = 0, 1, \quad \widehat{g}_t := \widehat{S}_{t*}^{-1} \widehat{f}_t \circ \widehat{S}_t(x) = g_{0,t} + \widehat{u}(t) g_{1,t}. \tag{19}$$

and recall that

$$\dot{g}_{1,t}(x) = \widehat{S}_{t*}^{-1} f_{01} \circ \widehat{S}_t(x), \quad \dot{g}_{0,t}(x) = -\widehat{u}(t)\dot{g}_{1,t}(x). \tag{20}$$

Then the extended second variation is thus actually given by the quadratic form

$$J''_{\text{ext}}(\varepsilon_0, \varepsilon_1, w) = \frac{1}{2} \int_{\widehat{\tau}}^{\widehat{T}} \big(w^2(t)[\dot{g}_{1,t}, g_{1,t}] \cdot \beta(\widehat{x}) + 2w(t)\,\zeta(t) \cdot \dot{g}_{1,t} \cdot \beta(\widehat{x}) \big)\, \mathrm{d}t \tag{21}$$

defined on the linear sub–space $\mathcal{W}$ of $\mathbb{R}^2 \times L^2([\widehat{\tau}, \widehat{T}], \mathbb{R})$ of the triplets $(\varepsilon_0, \varepsilon_1, w)$ such that the linear system

$$\dot{\zeta}(t) = w(t)\dot{g}_{1,t}(\widehat{x}), \quad \zeta(\widehat{\tau}) = \varepsilon_0 f_0(\widehat{x}) + \varepsilon_1 f_1(\widehat{x}), \quad \zeta(\widehat{T}) = 0. \tag{22}$$

admits a solution ζ, see [9].

Assumption 7 (Coercivity). The extended second variation for the minimum time problem at fixed end points on the singular arc is coercive. Namely we require that the quadratic form (21) is coercive on the subspace $\mathcal{W}$ of $\mathbb{R}^2 \times L^2([\widehat{\tau}, \widehat{T}], \mathbb{R})$ given by the variations $\delta e = (\varepsilon_0, \varepsilon_1, w)$ such that system (22) admits a solution.

Remark 3. The quadratic form (21) is defined on the whole space $\mathbb{R}^2 \times L^2([\widehat{\tau}, \widehat{T}], \mathbb{R})$, but only its restriction to $\mathcal{W}$ is coordinate free and independent of the choice of β such that $\mathrm{d}\beta(\widehat{x}) = -\widehat{\ell}$

Remark 4. Notice that

$$R(t) := [\dot{g}_{1,t}, g_{1,t}] \cdot \beta(\widehat{x}) = F_{101}(\widehat{\lambda}(t)),$$

so that the coercivity of the extended second variation implies the *strong generalised Legendre condition* (SGLC).

2.3 Consequences of coercivity and controllability

In [11] it was proven that Assumption 6 is strongly related to the controllability space (see e. g. [3]) of system (22), i. e. the space

$$V := \operatorname{span}\left\{ f_0(\widehat{x}),\ f_1(\widehat{x}),\ \dot{g}_{1,t}(\widehat{x}),\ t \in [\widehat{\tau}_1, \widehat{T}] \right\}. \tag{23}$$

Namely, the following was proven in [11]:

Lemma 1. *Assumption 6 holds if and only if* $V = T_{\widehat{x}}M$.

In order to exploit the coercivity assumption we follow [12] and introduce the Lagrangian subspace and the Hamiltonian associated to the second variation (21)–(22), respectively given by

$$L'' := \{f_0(\widehat{x}),\ f_1(\widehat{x})\}^{\perp} \times \operatorname{span}\{f_0(\widehat{x}),\ f_1(\widehat{x})\} \subset T^*_{\widehat{x}}M \times T_{\widehat{x}}M, \tag{24}$$

$$H''_t\colon (\omega, \delta x) \in T^*_{\widehat{x}}M \times T_{\widehat{x}}M \mapsto \frac{-1}{2R(t)}\left(\langle \omega\,,\ \dot{g}_{1,t}(\widehat{x})\rangle + \delta x \cdot \dot{g}_{1,t} \cdot \beta(\widehat{x})\right)^2 \in \mathbb{R}. \tag{25}$$

Lemma 2. *Let* $\mathcal{H}''_t\colon T^*_{\widehat{x}}M \times T_{\widehat{x}}M \to T^*_{\widehat{x}}M \times T_{\widehat{x}}M$ *be the flow of the Hamiltonian* H''_t *defined in* (25). *Under Assumptions 6 and 7 the kernel of the linear mapping* $\pi_*\mathcal{H}''_{\widehat{T}}\Big|_{L''}$ *is trivial.*

For a proof, see Lemma 2.2. in [11].

3 Optimality in the nominal problem

Most of the results of this section are from [9]. Their proofs are quite technically involved so that here we only collect the results in a few lemmas.

3.1 Geometry near the singular arc

In this section we describe some properties of the Hamiltonians linked to our system near the singular arc of the reference extremal, for more details see [9]. Such properties depend only on the regularity assumptions 3–5.

By (7), (8) and (SGLC), any singular extremal of the first kind of $(\mathbf{P_0})$ belongs to the set

$$\mathcal{S} := \{\ell \in T^*M\colon F_1(\ell) = F_{01}(\ell) = 0, F_{101}(\ell) > 0\},$$

a subset of the set

$$\Sigma := \{\ell \in T^*M\colon F_1(\ell) = 0\},$$

where the maximised Hamiltonian of $(\mathbf{P_0})$, $H^{\max}$, coincides with every Hamiltonian $F_0 + uF_1, u \in \mathbb{R}$.

Notice that $\mathcal{S}$ and Σ are independent of the control constraints but, by (4) and (9), any singular extremal of problem $(\mathbf{P_0})$ is in

$$\mathcal{S} \cap \left\{\ell \in T^*M\colon \left|\frac{F_{001}}{F_{101}}(\ell)\right| < 1\right\}.$$

Taking advantage of (SGLC) it is easy to prove the following result.

Lemma 3. *There exists a neighbourhood* $\mathcal{V}$ *of* $\mathcal{S}$ *in* T^*M *in which the following statements hold true.*

1) *$\Sigma \cap \mathcal{V}$ is a hyper–surface and $\mathcal{S} \cap \mathcal{V}$ is a $(2n-2)$-dimensional symplectic manifold. Σ separates the regions defined by: $H^{\max} = F_0 + F_1$, $H^{\max} = F_0 - F_1$;*
2) *the Hamiltonian vector field $\overrightarrow{F_1}$ is tangent to Σ and transverse to $\mathcal{S}$, while $\overrightarrow{F_{01}}$ is transverse to Σ;*
3) *the maps $(s, \ell) \mapsto \exp s\overrightarrow{F_1}(\ell)$ and $(\tau, s, \ell) \mapsto \exp \tau\overrightarrow{F_{01}} \circ \exp s\overrightarrow{F_1}(\ell)$ are local diffeomorphisms from $\mathbb{R} \times \mathcal{S}$ to Σ and from $\mathbb{R} \times \mathbb{R} \times \mathcal{S}$ to T^*M respectively.*

Property *(3)* in Lemma 3 yields the possibility of defining a smooth function $v\colon \mathcal{V} \to \mathbb{R}$ as

$$v := \frac{-F_{001}}{F_{101}} \quad \text{on } \mathcal{S}$$

and extending it constant first on the integral lines of $\overrightarrow{F_1}$ and then on those of $\overrightarrow{F_{01}}$.

In this way we may define *the Hamiltonian of singular extremals of the first kind* as

$$F^S = F_0 + v\, F_1.$$

Indeed the associated vector field $\overrightarrow{F^S}$ is tangent to $\mathcal{S}$ and any singular extremal of the first kind of our problem is an integral curve of $\overrightarrow{F^S}$ contained in $\mathcal{S}$: consequently the singular arc of $\widehat{\lambda}$ is C^∞ and the same holds true for $\widehat{u}$. From now on we shall denote $\Sigma \cap \mathcal{V}$ and $\mathcal{S} \cap \mathcal{V}$ as Σ and $\mathcal{S}$, respectively.

The following lemma contains the main technical points which we need in order to prove our main result. Their proofs are quite technical and we refer the interested reader to [9].

Lemma 4. *Let Assumptions 2–5 and* (SGLC) *hold. Then there exists a neighbourhood of the range of $\widehat{\lambda}\big|_{[\widehat{\tau},\widehat{T}]}$ in T^*M where the followings hold.*

1) *There exists a non-negative smooth Hamiltonian χ such that the Hamiltonian vector fields associated to $K^S := F^S + \chi$, $\widehat{H}_t := \widehat{F}_t + \chi$ and $H_1 + \chi$ are tangent to Σ and*

$$\mathcal{K}_t^S\big|_{\mathcal{S}} = \mathcal{F}_t^S\big|_{\mathcal{S}} \quad \forall t \in [\widehat{\tau}, \widehat{T}]. \tag{26}$$

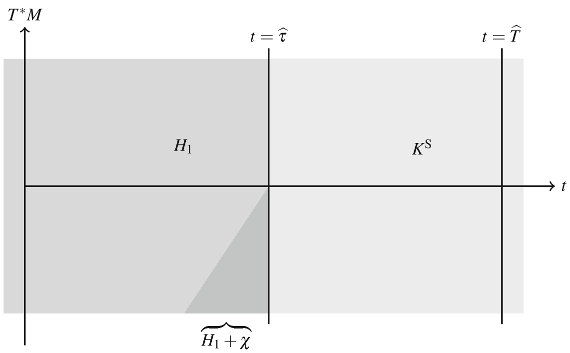

Fig. 1 The over–maximised Hamiltonian along the flow

2) *The following invariance properties hold:*

$$\widehat{\mathcal{H}}_{t\,*}\overrightarrow{F_0}(\widehat{\ell}) = \mathcal{K}^{\mathrm{S}}_{t\,*}\overrightarrow{F_0}(\widehat{\ell}) = \overrightarrow{F_0}\circ\widehat{\lambda}(t) \qquad \forall t\in[\widehat{\tau},\widehat{T}],$$

$$\widehat{\mathcal{H}}_{t\,*}\overrightarrow{F_1}(\ell) = \mathcal{K}^{\mathrm{S}}_{t\,*}\overrightarrow{F_1}(\ell) = \overrightarrow{F_1}\circ\mathcal{K}^{\mathrm{S}}_{t}(\ell) \quad \forall (t,\ell)\in[\widehat{\tau},\widehat{T}]\times\Sigma.$$

In particular

$$\left.\begin{aligned}\mathcal{K}^{\mathrm{S}}_t\circ\exp s\overrightarrow{F_1}(\ell) &= \exp s\overrightarrow{F_1}\circ\mathcal{K}^{\mathrm{S}}_t(\ell)\\ \widehat{\mathcal{H}}_t\circ\exp s\overrightarrow{F_1}(\ell) &= \exp s\overrightarrow{F_1}\circ\widehat{\mathcal{H}}_t(\ell)\end{aligned}\right. \quad \forall (s,t,\ell)\in\mathbb{R}\times[\widehat{\tau},\widehat{T}]\times\Sigma. \tag{27}$$

3) *There exist* $\varepsilon > 0$ *and a* $C^{1,1}$ *Hamiltonian function* K_1 *satisfying the following properties*

   1. $H_1 \le K_1 \le H_1 + \chi$*;*
   2. *the flow* $\mathcal{H}$ *of the Hamiltonian*

$$H_t(\ell) := \begin{cases} K_1(\ell) & \textit{if } t\in[-\varepsilon,\widehat{\tau}),\\ K^{\mathrm{S}}(\ell) & \textit{if } t\in[\widehat{\tau},\widehat{T}+\varepsilon]\end{cases} \tag{28}$$

emanating from a neighbourhood $\mathcal{U}$ *of* $\widehat{\ell}$ *in* Σ *is* C^1 *with respect to* ℓ *for any* t *and satisfies the following properties*

$$\mathcal{H}_t(\widehat{\ell}) = \widehat{\lambda}(t), \quad H_t\circ\widehat{\lambda}(t) = H^{\max}\circ\widehat{\lambda}(t), \quad t\in[-\varepsilon,\widehat{T}+\varepsilon], \tag{29}$$

$$(H_t - H^{\max})\circ\mathcal{H}_t(\ell)\ge 0 \qquad (t,\ell)\in[-\varepsilon,\widehat{T}+\varepsilon]\times\mathcal{U}. \tag{30}$$

Moreover $K_1\circ\mathcal{H}_t(\ell) = H_1\circ\mathcal{H}_t(\ell)$ *for any* $t\in[-\varepsilon,\widehat{\tau}-\varepsilon]$ *and, if* $t\in[\widehat{\tau}-\varepsilon,\widehat{\tau}]$ *then*

$$K_1\circ\mathcal{H}_t(\ell) = H_1\circ\mathcal{H}_t(\ell) \qquad \textit{if } u_1F_{01}\circ\mathcal{H}_t(\ell)\le 0,$$

$$K_1\circ\mathcal{H}_t(\ell) = (H_1+\chi)\circ\mathcal{H}_t(\ell) \quad \textit{if } u_1F_{01}\circ\mathcal{H}_t(\ell) < 0.$$

3.2 State–local optimality

Under the given assumptions we now prove state-local optimality of $\widehat{\xi}$ for the nominal problem $(\mathbf{P_0})$. Most of the proofs are in [9], here we adapt them to fit the final constraint $\mathcal{N}_f$. Namely, exploiting the Coercivity Assumption 7, we proved the results collected in the following lemma

Lemma 5. *There exist* C^2 *functions* $\alpha, \alpha^{\mathrm{b}}\colon M\to\mathbb{R}$ *such that locally around* $\widehat{\ell}$ *and locally around* $\widehat{x}$ *the followings hold:*

1) *the Lagrangian sub–manifold* $\Lambda^{\alpha} := \{\ell \in T^*M : \ell = \mathrm{d}\alpha(\pi\ell)\}$ *is transverse to the level set of* K^{S} *defined by* $K^{\mathrm{S}} = 1$*;*
2) $\widehat{\ell}$ *belongs to the isotropic sub–manifold* Λ^{s} *obtained intersecting* Λ^{α} *with the level set* $K^{\mathrm{S}} = 1$*;*
3) *the horizontal Lagrangian sub–manifold* $\Lambda^{\alpha^{\mathrm{b}}} := \{\ell \in T^*M : \ell = \mathrm{d}\alpha^{\mathrm{b}}(\pi\ell)\}$ *is contained in* Σ *and is transverse to the level set of* K_1 *defined by* $K_1 = 1$*;*
4) $\widehat{\ell}$ *belongs to the isotropic manifold* Λ^{b} *obtained intersecting* $\Lambda^{\alpha^{\mathrm{b}}}$ *with the level set* $K_1 = 1$*;*
5) *the projections* $\pi\Lambda^{\mathrm{s}}$ *and* $\pi\Lambda^{\mathrm{b}}$ *of the isotropic manifolds* Λ^{s} *and* Λ^{b} *on the state space* M *agree and are a* $(n-1)$*-dimensional sub-manifold* N *of the state space* M*. The intersection of* N *with the range* $\widehat{\xi}([0, \widehat{T}])$ *of the reference trajectory is the singleton* $\{\widehat{x}\}$*, and the functions* α^{b} *and* α *agree on* N*;*
6) *there exists a neighborhood* $\mathcal{U}$ *of the range of the reference trajectory such that* N *divides* $\mathcal{U}$ *in two parts,* $\mathcal{U}^{\mathrm{b}} \supset \widehat{\xi}\big|_{[0,\widehat{\tau}]}$ *and* $\mathcal{U}^{\mathrm{s}} \supset \widehat{\xi}\big|_{[\widehat{\tau},\widehat{T}]}$*;*
7) *for any admissible trajectory* $\xi\colon [0, T] \to M$ *whose range is in* $\mathcal{U}$*, there exists* $\overline{t} \in (0, T)$ *such that* $\xi([0, T]) \cap N = \xi(\overline{t})$*,* $\xi|_{[0,\overline{t}]} \subset \mathcal{U}^{\mathrm{b}}$ *and* $\xi|_{[\overline{t},T]} \subset \mathcal{U}^{\mathrm{s}}$*;*
8) *for small positive* ε*, we can consider the* C^1 *flows associated to* K_1 *and* K^{S}*:*

$$\mathcal{H}\colon (t, \ell) \in [-\varepsilon, \widehat{\tau}] \times \Lambda^{\mathrm{b}} \mapsto \mathcal{H}_t(\ell) = \mathcal{K}_{1,t}(\ell) \in T^*M,$$
$$\mathcal{H}\colon (t, \ell) \in [\widehat{\tau}, \widehat{T} + \varepsilon] \times \Lambda^{\mathrm{s}} \mapsto \mathcal{H}_t(\ell) = \mathcal{K}^{\mathrm{S}}_t(\ell) \in T^*M,$$

then $\pi\mathcal{H}\left([-\varepsilon, \widehat{\tau}] \times \Lambda^{\mathrm{b}}\right)$ *is bijective onto* $\mathcal{U}^{\mathrm{b}}$ *and* $\pi\mathcal{H}\left([\widehat{\tau}, \widehat{T} + \varepsilon] \times \Lambda^{\mathrm{s}}\right)$ *is bijective onto* $\mathcal{U}^{\mathrm{s}}$*.*

Using the previous lemma we now state and prove the optimality result

Theorem 1. *If* $\widehat{\xi}$ *is injective and Assumptions 2–5 and 7 hold then* $\widehat{\xi}$ *is a strict state–local optimal trajectory of the nominal problem* $(\mathbf{P_0})$*.*

Proof. Consider the one–forms

$$\boldsymbol{\omega}^{\mathrm{b}} := \mathcal{H}^*\varsigma \quad \text{on} \quad [-\varepsilon, \widehat{\tau}] \times \Lambda^{\mathrm{b}}$$
$$\boldsymbol{\omega}^{\mathrm{s}} := \mathcal{H}^*\varsigma \quad \text{on} \quad [\widehat{\tau}, \widehat{T} + \varepsilon] \times \Lambda^{\mathrm{s}}$$

It is well known that $\boldsymbol{\omega}^{\mathrm{b}}$ and $\boldsymbol{\omega}^{\mathrm{s}}$ are exact and that there exist functions φ^{b} and φ^{s},

$$\varphi^{\mathrm{b}}\colon [-\varepsilon, \widehat{\tau}] \times \Lambda^{\mathrm{b}} \to \mathbb{R}, \quad \varphi^{\mathrm{b}}\colon [\widehat{\tau}, \widehat{T} + \varepsilon] \times \Lambda^{\mathrm{s}} \to \mathbb{R}$$

such that

$$\boldsymbol{\omega}^{\mathrm{b}}(t, \ell) = \mathrm{d}\varphi^{\mathrm{b}}(t, \ell), \quad \varphi^{\mathrm{b}}(\widehat{\tau}, \ell) = \alpha^{\mathrm{b}}(\pi\ell) = \alpha(\pi\ell),$$
$$\boldsymbol{\omega}^{\mathrm{s}}(t, \ell) = \mathrm{d}\varphi^{\mathrm{s}}(t, \ell), \quad \varphi^{\mathrm{s}}(\widehat{\tau}, \ell) = \alpha(\pi\ell).$$

Let $\xi\colon [0, T] \to \mathcal{U}$ be an admissible trajectory and let $\overline{t} \in (0, T)$ such that $\xi(\overline{t}) \in N$ as stated in property 7 of Lemma 5. Thanks to Lemma 5 we can lift $\xi|_{[0,\overline{t}]}$ to

$[-\varepsilon, \widehat{\tau}] \times \Lambda^{\mathrm{b}}$ and $\xi|_{[\overline{t},T]}$ to $[\widehat{\tau}, \widehat{T} + \varepsilon] \times \Lambda^{\mathrm{s}}$. Let ψ be a curve on N joining $\xi(\overline{t})$ to $\widehat{\xi}(\widehat{\tau})$ and observe that

$$(\pi\mathcal{K}_1)^{-1}(x) = (\widehat{\tau}, \mathrm{d}\alpha^{\mathrm{b}}(x)), \quad \left(\pi\mathcal{K}^{\mathrm{S}}\right)^{-1}(x) = (\widehat{\tau}, \mathrm{d}\alpha(x)), \qquad \forall x \in N,$$
$$(\pi\mathcal{K}_1)^{-1}(\widehat{\xi}(s)) = (s, \widehat{\ell}), \qquad \forall s \in [0, \widehat{\tau}],$$
$$\left(\pi\mathcal{K}^{\mathrm{S}}\right)^{-1}(\widehat{\xi}(s)) = (s, \widehat{\ell}), \qquad \forall s \in [\widehat{\tau}, \widehat{T}].$$

We now obtain a closed path on $\mathcal{U}^{\mathrm{b}}$ by concatenating $\widehat{\xi}\Big|_{[0,\widehat{\tau}]}$ run backwards in time, $\xi|_{[0,\overline{t}]}$ and a curve ψ on N with initial and final points $\xi(\overline{t})$ and $\widehat{\xi}(\widehat{\tau})$, respectively. We then lift this path to $[-\varepsilon, \widehat{\tau}] \times \Lambda^{\mathrm{b}}$ by taking its preimage with respect to $\pi\mathcal{K}_1$. Integrating $\boldsymbol{\omega}^{\mathrm{b}}$ along this path we get

$$\begin{aligned} 0 = &- \int_0^{\widehat{\tau}} \langle \widehat{\lambda}(s)\,,\ \widehat{f}_s \circ \widehat{\xi}(s)\rangle\, \mathrm{d}s + \\ &+ \int_0^{\overline{t}} \langle \mathcal{K}_{1,\tau(s)} \circ \ell(s)\,,\ \dot{\xi}(s)\rangle\, \mathrm{d}s + \varphi^{\mathrm{b}}(\widehat{\tau}, \widehat{\ell}) - \varphi^{\mathrm{b}}(\widehat{\tau}, \ell(\overline{t})) \leq \\ &\leq -\widehat{\tau} + \overline{t} + \alpha(\widehat{x}) - \alpha(\xi(\overline{t})). \end{aligned} \tag{31}$$

In order to obtain a closed path on $\mathcal{U}^{\mathrm{s}}$ define a curve joining $\xi(T) := \exp \overline{s} f_1(\widehat{x}_f)$ to $\widehat{x}_f$:

$$\gamma : s \in [0, \overline{s}] \mapsto \exp(\overline{s} - s) f_1(\widehat{x}_f) \in M.$$

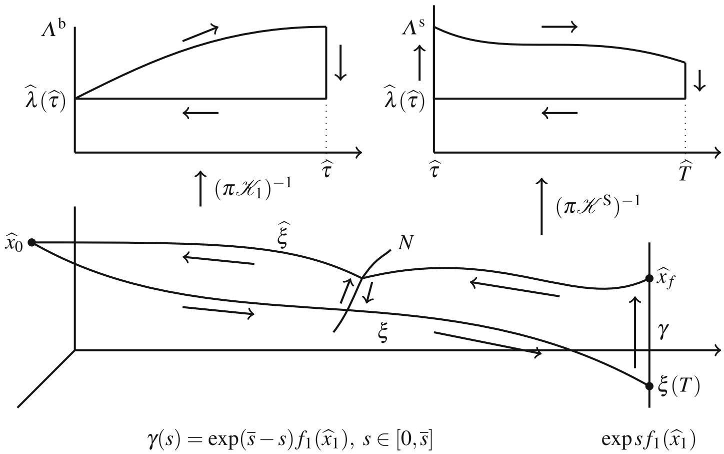

Fig. 2 Lifting trajectories

Concatenate ψ run backwards, $\xi|_{[\bar{t},T]}$, γ and then $\widehat{\xi}\Big|_{[\widehat{\tau},\widehat{T}]}$ run backwards. By taking the preimage of this path with respect to $\pi\mathcal{K}^{\mathrm{S}}$ we obtain a closed path in $[\widehat{\tau},\widehat{T}+\varepsilon]\times\Lambda^{\mathrm{s}}$ and, by property 2 of Lemma 4,

$$\left(\pi\mathcal{K}^{\mathrm{S}}\right)^{-1}\circ\gamma(s)=\left(\widehat{T},\exp(\bar{s}-s)\,\overrightarrow{F_1}(\widehat{\ell})\right)\qquad\forall s\in[0,\bar{s}]$$

so that

$$\int_{\left(\pi\mathcal{K}^{\mathrm{S}}\right)^{-1}\circ\gamma}\boldsymbol{\omega}^{\mathrm{s}}=0.$$

Integrating $\boldsymbol{\omega}^{\mathrm{s}}$ along the closed path we thus obtain

$$\begin{aligned}0=&\alpha(\xi(\bar{t}))-\alpha(\widehat{x})+\int_{\bar{t}}^{T}\langle\mathcal{K}^{\mathrm{S}}_{\tau(s)}\circ\ell(s)\,,\,\dot{\xi}(s)\rangle\,\mathrm{d}s-\int_{\widehat{\tau}}^{\widehat{T}}\langle\widehat{\lambda}(s)\,,\,\widehat{f}_s\circ\widehat{\xi}(s)\rangle\,\mathrm{d}s\leq\\ \leq&\alpha(\xi(\bar{t}))-\alpha(\widehat{x})+\left(T-\bar{t}\right)-\left(\widehat{T}-\widehat{\tau}\right).\end{aligned}\tag{32}$$

Summing inequalities (31)–(32) we get

$$0\leq T-\widehat{T}.\tag{33}$$

i.e. $T\geq\widehat{T}$ and $\widehat{\xi}$ is a state-optimal trajectory.

We omit the proof of strict state–local optimality since it is quite technically involved and requires the introduction of many other tools and properties. The proof is completely analougous to the one given in [9] for the bang–singular–bang case. □

Remark 5. Here we have considered only normal extremals, since normality is needed for stability. Nevertheless the same optimality result holds also in the abnormal case, see [9].

4 Structural stability

We now show that under all the given assumptions and for sufficiently small $\|r\|$, the perturbed problem $(\mathbf{P_r})$ has a bang–singular strict state–local optimal trajectory ξ^r. We start by extending the properties of $(\mathbf{P_0})$ due to the regularity assumptions.

Since for the nominal problem (SGLC) holds true in the neighbourhood $\mathcal{V}$ of $\widehat{\lambda}\Big|_{[\widehat{\tau},\widehat{T}]}$ defined in Lemma 3, then possibly restricting $\mathcal{V}$ and for small enough $\|r\|$, it holds also for the Hamiltonians F^r_{101}.

Therefore we can define, in $\mathcal{V}$, the Hamiltonians of singular extremals of $(\mathbf{P_r})$

$$F^{S,r}:=F^r_0-\frac{F^r_{001}}{F^r_{101}}F^r_1.\tag{34}$$

Moreover it is possible to define smooth Hamiltonians χ^r having analogous properties to those of χ and hence also

$$K^{S,r} := F^{S,r} + \chi^r.$$

Under the controllability and coercivity assumptions 6 and 7 we now prove the following crucial property:

Lemma 6. *Let*

$$L := \mathbb{R}\overrightarrow{F_0}(\widehat{\ell}) \oplus \mathbb{R}\overrightarrow{F_1}(\widehat{\ell}) \oplus \{f_0(\widehat{x}),\ f_1(\widehat{x})\}^{\perp}. \tag{35}$$

Under Assumption 6, the coercivity of J''_{ext} implies that the kernel of $\pi_* \mathcal{K}_{\widehat{T}*} \colon L \to T_{\widehat{x}_f} M$ is trivial.

Proof. In Corollary 4.3 of [11], Lemma 2 is proven to imply the following property:

$$\text{The linear map } \pi_* \widehat{\mathcal{H}}_{\widehat{T}*} \colon L \to T_{\widehat{x}_f} M \text{ is one–to–one.} \tag{36}$$

To prove the claim consider $\mathcal{G}_t := \widehat{\mathcal{H}}_t^{-1} \circ \mathcal{K}_t^{S} \colon \Sigma \to \Sigma$. $\mathcal{G}_t$ is the Hamiltonian flow associated to the Hamiltonian

$$G_t := \left(K^S{}_t - \widehat{H}_t\right) \circ \widehat{\mathcal{H}}_t = (v - \widehat{u}(t))\, \overrightarrow{F_1} \circ \widehat{\mathcal{H}}_t.$$

Since $\mathrm{D}G_t(\widehat{\ell}) = 0$, then $\mathcal{G}_{t*} \colon T_{\widehat{\ell}}\Sigma \to T_{\widehat{\ell}}\Sigma$ is the linear Hamiltonian flow associated to the quadratic Hamiltonian

$$\mathrm{D}^2 G_t(\widehat{\ell}) = (\mathrm{D}v \otimes \mathrm{D}F_1 + \mathrm{D}F_1 \otimes \mathrm{D}v)|_{\widehat{\lambda}(t)} \circ \widehat{\mathcal{H}}_{t*} \otimes \widehat{\mathcal{H}}_{t*}.$$

Namely, the restriction of $\mathcal{G}_{t*}$ to $T_{\widehat{\ell}}\Sigma$ is the flow associated to

$$\overrightarrow{G}''_t = \left(\mathrm{D}v(\widehat{\lambda}(t))\widehat{\mathcal{H}}_{t*}\right) \otimes \overrightarrow{F_1}(\widehat{\ell}).$$

By the invariance properties in Claim 2 of Lemma 4 we get

$$\mathcal{G}_{t*}\overrightarrow{F_0}(\widehat{\ell}) = \overrightarrow{F_0}(\widehat{\ell}), \qquad \mathcal{G}_{t*}\overrightarrow{F_1}(\widehat{\ell}) = \overrightarrow{F_1}(\widehat{\ell}).$$

Let $\omega \in \{f_0(\widehat{x}), f_1(\widehat{x})\}^{\perp}$, then $\mu(t) := \mathcal{G}_{t*}\omega$ satisfies the differential equation

$$\dot{\mu}(t) = \langle \mathrm{D}v(\widehat{\lambda}(t))\,,\ \widehat{\mathcal{H}}_{t*}\mu(t)\rangle \overrightarrow{F_1}(\widehat{\ell}), \qquad t \in [\widehat{\tau}, \widehat{T}]$$

so that

$$\mu(\widehat{T}) = \omega + \phi(\widehat{T}, \omega)\overrightarrow{F_1}(\widehat{\ell}), \qquad \phi(\widehat{T}, \omega) := \int_0^{\widehat{T}} \langle \mathrm{D}v(\widehat{\lambda}(t))\,,\ \widehat{\mathcal{H}}_{t*}\mu(t)\rangle\, \mathrm{d}t.$$

This proves that $\mathcal{G}_{\widehat{T}*}(L) \subseteq L$.

Assume that $\delta\ell \in L$, $\delta\ell = a_0\overrightarrow{F_0}(\widehat{\ell}) + a_1\overrightarrow{F_1}(\widehat{\ell}) + \omega$, $a_0, a_1 \in \mathbb{R}$, $\omega \in \left\{f_0(\widehat{x}), f_1(\widehat{x})\right\}^{\perp}$, is such that $\pi_*\mathcal{K}^{\mathrm{S}}_{\widehat{T}*}\,\delta\ell = 0$ i.e. $\pi_*\widehat{\mathcal{H}}_{\widehat{T}*}\mathcal{G}_{\widehat{T}*}\delta\ell = 0$. By (36) this is equivalent to $\mathcal{G}_{T*}\delta\ell = 0$, i.e.

$$a_0\overrightarrow{F_0}(\widehat{\ell}) + \left(a_1 + \phi(\widehat{T},\omega)\right)\overrightarrow{F_1}(\widehat{\ell}) + \omega = 0$$

so that $a_0 = 0$, $\omega = 0$, $a_1 = 0$, which yields the claim, i.e. $\delta\ell = 0$. □

We now show structural stability of extremals, i.e. we prove that if $\|r\|$ is sufficiently small, then $(\mathbf{P_r})$ has a bang–singular regular extremal. Later we show that such extremal is a local optimiser.

Lemma 7. *There exist $\rho > 0$, $\varepsilon > 0$ and a neighbourhood $\mathcal{O}$ of $\widehat{\ell_0}$ in T^*M such that for any r, $\|r\| < \rho$ there exists a normal bang–singular extremal λ^r of $(\mathbf{P_r})$. λ^r is the only bang–singular extremal of $(\mathbf{P_r})$ satisfying the following properties:*

1) $\lambda^r(0) \in \mathcal{O}$;
2) *the switching time $\tau(r)$ is in $[\widehat{\tau} - \varepsilon, \widehat{\tau} + \varepsilon]$;*
3) *the final time $T(r)$ is in $[\widehat{T} - \varepsilon, \widehat{T} + \varepsilon]$;*
4) $\xi^r(T(r)) = \exp(s(r)f_1^r)(y^r)$ *with* $s(r) \in [-\varepsilon, \varepsilon]$.

Moreover

1) *the bang arc of λ^r is regular:* $u_1F_1^r \circ \lambda^r(t) > 0 \quad \forall t \in [0, \tau(r))$;
2) *the singular arc of λ^r is of the first kind:* $F_{101}^r \circ \lambda^r(t) > 0 \quad \forall t \in [\tau(r), T(r)]$;
3) *the switching point of λ^r is regular:* $\left(F_{101}^r + u_1F_{001}^r\right) \circ \lambda^r(\tau(r)) > 0$.

Proof. Let us locally define the following map

$$\Phi\colon (r,\ell,\tau,T,s) \in \mathbb{R}^k \times T^*M \times \mathbb{R}^3 \mapsto \widehat{S}_{\widehat{T}}^{-1} \circ \exp(-sf_1^r) \circ \pi\exp(T-\tau)\overrightarrow{K^{\mathrm{S},r}} \circ \exp\tau\overrightarrow{H_1^r}(\ell) \in M \tag{37}$$

and let

$$\Psi(r,\ell,\tau,T,s) = \left(\pi\ell,\ \Phi(r,\ell,\tau,T,s),\ F_1^r \circ \exp\tau\overrightarrow{H_1^r}(\ell),\quad F_{01}^r \circ \exp\tau\overrightarrow{H_1^r}(\ell),\ F_0^r \circ \exp\tau\overrightarrow{H_1^r}(\ell)\right) \tag{38}$$

We claim that the implicit equation $\Psi(r,\ell,\tau,T,s) = (a^r, \widehat{S}_{\widehat{T}}^{-1}(y^r), 0, 0, 1)$ has rank $2n+3$ in $(0, \widehat{\ell_0}, \widehat{\tau}, \widehat{T}, 0)$ and it implicitly defines smooth functions

$$\ell = \ell(r),\ \tau = \tau(r),\ T = T(r),\ s = s(r),\ \|r\| < \rho$$

for some positive ρ.

Define

- Ψ_* as the tangent map to the map Ψ in $(0, \widehat{\ell}_0, \widehat{\tau}, \widehat{T}, 0)$;
- $c := u_1 + \frac{F_{001}}{F_{101}}(\widehat{\ell})$ (which is nonzero, see Remark 4);
- for any $\delta\ell_0$ in $T_{\widehat{\ell}_0} T^*M$, let $\delta\ell := \exp \widehat{\tau}\overrightarrow{H_1}_* \delta\ell_0$.

Since $\exp \widehat{\tau}\overrightarrow{H_1}_*$ is a linear diffeomorphism of $T_{\widehat{\ell}_0} T^*M$ onto $T_{\widehat{\ell}}T^*M$, then $\Psi_*(0, \delta\ell_0, \delta\tau, \delta T, \delta s)$ is null if and only if the followings are satisfied

$$\pi_*\delta\ell = 0 \tag{39}$$

$$\pi_*\mathcal{K}^{\mathrm{S}}_{\widehat{T}*}\left(\delta\ell + \delta\tau\, c\, \overrightarrow{F_1}(\widehat{\ell})\right) + \delta T \widehat{f}_{\widehat{T}}(\widehat{x}_f) - \delta s\, f_1(\widehat{x}_f) = 0 \tag{40}$$

$$\boldsymbol{\sigma}\left(\delta\ell, \overrightarrow{F_1}(\widehat{\ell})\right) = 0 \tag{41}$$

$$\boldsymbol{\sigma}\left(\delta\ell, \overrightarrow{F_{01}}(\widehat{\ell})\right) + \delta\tau\, c\, F_{101}(\widehat{\ell}) = 0 \tag{42}$$

$$\boldsymbol{\sigma}\left(\delta\ell, \overrightarrow{F_0}(\widehat{\ell})\right) = 0 \tag{43}$$

By Claim 2 of Lemma 4, Eq. (40) can be equivalently written as

$$\pi_*\mathcal{K}^{\mathrm{S}}_{\widehat{T}*}\left(\delta\ell + \delta T \overrightarrow{F_0}(\widehat{\ell}) + (\delta\tau\, c + \delta T - \delta s)\, \overrightarrow{F_1}(\widehat{\ell})\right) = 0. \tag{44}$$

and, by Eqs. (39), (41) and (43), $\delta\ell \in \{f_0(\widehat{x}),\ f_1(\widehat{x})\}^{\perp}$. Thus,

$$\delta\ell + \delta T \overrightarrow{F_0}(\widehat{\ell}) + (\delta\tau\, c + \delta T - \delta s)\, \overrightarrow{F_1}(\widehat{\ell}) \in L$$

so that, by Lemma 6,

$$\delta\ell = 0, \quad \delta T = 0, \quad \delta\tau\, c + \delta T - \delta s = 0. \tag{45}$$

Equations (42) and (45) yield $\delta\tau = 0$ and $\delta s = 0$, i. e. the kernel of Ψ_* is trivial.

Defining

$$u^r(t) := \begin{cases} u_1 & t \in [0, \tau(r)], \\ v^r(t) := \dfrac{-F^r_{001}}{F^r_{101}} \circ \lambda^r(t) & t \in (\tau(r), T(r)]. \end{cases} \tag{46}$$

we get that the associated trajectory ξ^r starting at a^r is a state extremal with associated adjoint covector

$$\lambda^r : t \in [0, T(r)] \mapsto \begin{cases} \exp t\overrightarrow{H^r_1}(\ell, a^r) & t \in [0, \tau(r)] \\ \exp(t - \tau(r))\overrightarrow{K^{\mathrm{S},r}} \circ \exp \tau(r)\overrightarrow{H^r_1}(\ell, a^r) & t \in (\tau(r), T(r)]. \end{cases} \tag{47}$$

The proof of the second part of the Lemma, i. e. of the regularity of λ^r is trivial and can be found in the final part of the proof of Lemma 4.4. in [11]. □

Theorem 2. *Let $\widehat{\xi}$ be injective and let Assumption 2–7 hold. Then there exist $\rho > 0$, $\varepsilon > 0$ and a neighbourhood $\mathcal{O}$ of $\widehat{\ell}_0$ in T^*M such that for any $r \in \mathbb{R}^k$, $\|r\| \le \rho$, there exists a bang–singular strict state–local optimal trajectory $\xi^r : [0, T(r)] \to M$ for problem $(\mathbf{P_r})$ such that:*

1) *the switching time $\tau(r)$ is in $[\widehat{\tau} - \varepsilon, \widehat{\tau} + \varepsilon]$;*
2) *the final time $T(r)$ is in $[\widehat{T} - \varepsilon, \widehat{T} + \varepsilon]$;*
3) $\xi^r(T(r)) = \exp(s(r) f_1^r)(y^r)$ *with $s(r) \in [-\varepsilon, \varepsilon]$.*

Moreover $\tau(r)$, $T(r)$ and $s(r)$ depend smoothly on r. The associated control $u^r(t)$ is such that

$$\sup\left\{|u^r(t) - \widehat{u}(t)| : t \in [\widehat{\tau}, \widehat{T}] \cap [\tau(r), T(r)]\right\} < \varepsilon. \tag{48}$$

Proof. Let u^r, ξ^r and λ^r be defined as in Lemma 7. In order to prove the strict state–local optimality of ξ^r it suffices prove its injectivity and that the coercivity assumption is satisfied. We omit the proofs of these facts. The interested reader can find them in Lemmata 4.5 and 4.6 of [11].

The smoothness of $\tau(r)$, $T(r)$ and $s(r)$ comes from the implicit function theorem. The estimate (48) is due to the feedback expression of u^r, (46). □

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Approximate controllability of the viscous Burgers equation on the real line

Armen Shirikyan

Abstract The paper is devoted to studying the 1D viscous Burgers equation controlled by an external force. It is assumed that the initial state is essentially bounded, with no decay condition at infinity, and the control is a trigonometric polynomial of low degree with respect to the space variable. We construct explicitly a control space of dimension 11 that enables one to steer the system to any neighbourhood of a given final state in local topologies. The proof of this result is based on an adaptation of the Agrachev–Sarychev approach to the case of an unbounded domain.

1 Introduction

Let us consider the following viscous Burgers equation on the real line:

$$\partial_t u - \mu \partial_x^2 u + u \partial_x u = f(t, x), \quad x \in \mathbb{R}. \tag{1}$$

Here $u = u(t, x)$ is an unknown function, $\mu > 0$ is a viscosity coefficient, and $f(t, x)$ is an external force which is assumed to be essentially bounded in x and integrable in t. Equation (1) is supplemented with the initial condition

$$u(0, x) = u_0(x), \tag{2}$$

where $u_0 \in L^\infty(\mathbb{R})$. Due to the maximum principle, one can easily prove the existence and uniqueness of a solution for (1), (2) in appropriate functional classes. Our aim is to study controllability properties of (1). Namely, we assume that f has the form

$$f(t, x) = h(t, x) + \eta(t, x), \tag{3}$$

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where h is a fixed regular function and η is a control, which is assumed to be a smooth function in time with range in a finite-dimensional subspace $E \subset L^\infty(\mathbb{R})$. We shall say that (1) is *approximately controllable* at a time $T > 0$ if for any initial state $u_0 \in L^\infty(\mathbb{R})$, any target $\hat{u} \in C(\mathbb{R})$, and any numbers $\varepsilon, r > 0$ there is a smooth function $\eta : [0, T] \to E$ such that the solution $u(t, x)$ of problem (1)–(3) satisfies the inequalities

$$\|u(T, \cdot)\|_{L^\infty(\mathbb{R})} \leq K, \quad \|u(T, \cdot) - \hat{u}\|_{L^\infty([-r,r])} < \varepsilon, \tag{4}$$

where $K > 0$ does not depend on r and ε. Given a finite subset $\Lambda \subset \mathbb{R}$, we denote by E_Λ the vector space spanned by the functions $\cos(\lambda x)$ and $\sin(\lambda x)$ with $\lambda \in \Lambda$. The following theorem is a weaker version of the main result of this paper.

Theorem 1. *Let* $\Lambda = \{0, \lambda_1, \lambda_2, 2\lambda_1, 2\lambda_2, \lambda_1 + \lambda_2\}$, *where* λ_1 *and* λ_2 *are incommensurable positive numbers, and let* $E = E_\Lambda$. *Then Eq.* (1) *is approximately controllable at any time* $T > 0$.

We refer the reader to Sect. 2 for a stronger result on approximate controllability and for an outline of its proof, which is based on an adaptation of a general approach introduced by Agrachev and Sarychev in [2] and further developed in [3]; see also [14–16] for some other extensions. Let us note that the Agrachev–Sarychev approach enables one to establish a much stronger property: given any initial and target states and any non-degenerate finite-dimensional functional, one can construct a control that steers the system to the given neighbourhood of the target so that the values of the functional on the solution and on the target coincide. However, to make the presentation simpler and shorter, we confine ourselves to the approximate controllability. The above-mentioned property of controllability will be analysed in [17] in the more difficult case of the 2D Navier–Stokes system.

The main theorem stated above proves the approximate controllability of the Burgers equation by a control whose Fourier transform is localised at 11 points. This result is in sharp contrast with the case of a control localised in the physical space, for which the approximate controllability does not hold even for the problem in a bounded interval. This fact was established by Fursikov and Imanuvilov; see Sect. I.6 of the book [9]. Other negative results on controllability of the Burgers equation via boundary were obtained by Diaz [7] and Guerrero and Imanuvilov [11]. On the other hand, Coron showed in [6] that any initial state can be driven to zero by a boundary control and Fernández-Cara and Guerrero [8] proved the exact controllability (with an estimate for the minimal time of control) for the problem with distributed control. Furthermore, Glass and Guerrero [10] established global controllability to non-zero constant states via boundary for small values of the viscosity and Chapouly [4] proved the global exact controllability to a given solution by two boundary and one distributed controls. Imanuvilov and Puel [12] proved the global boundary controllability of the 2D Burgers equation in a bounded domain under some geometric conditions. We refer the reader to the book [5] for a discussion of the methods used in the control theory for the Burgers equation on a bounded interval. To the best of our knowledge, the problem of controllability of the viscous Burgers equation was not studied in the case of an unbounded domain.

The paper is organised as follows. In Sect. 2, we formulate the main result and outline the scheme of its proof. Section 3 collects some facts about the Cauchy problem for Eq. (1) without decay condition at infinity. The proof of the main result of the paper is given in Sect. 4.

Notation. Let $J \subset \mathbb{R}$ be a bounded closed interval, let $D \subset \mathbb{R}^n$ be an open subset, and let X be a Banach space. We denote by $B_X(R)$ the closed ball in X of radius R centred at zero. We shall use the following functional spaces.

For $p \in [1, \infty]$, we denote by $L^p(J, X)$ the space of measurable functions $f : J \to X$ such that

$$\|f\|_{L^p(J,X)} := \left(\int_J \|f(t)\|_X^p \right)^{1/p} < \infty.$$

In the case $p = \infty$, this norm should be replaced by $\operatorname{ess\,sup}_{t \in J} \|f(t)\|_X$.

For an integer $k \in [0, +\infty]$, we write $C^k(J, X)$ for the space of k times continuously differentiable functions on J with range in X and endow it with natural norm. In the case $k = 0$, we omit the corresponding superscript.

For an integer $s \geq 0$, we denote by $H^s(D)$ the Sobolev space on D of order s with the standard norm $\|\cdot\|_s$. In the case $s = 0$, we write $L^2(D)$ and $\|\cdot\|$.

$L^\infty = L^\infty(\mathbb{R})$ is the space of bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$ with the natural norm $\|f\|_{L^\infty}$. The space $L^\infty(D)$ is defined in a similar way.

$W^{k,\infty}(\mathbb{R})$ is the space of functions $f \in L^\infty$ such that $\partial_x^j f \in L^\infty$ for $0 \leq j \leq k$.

$C_b^\infty = C_b^\infty(\mathbb{R})$ stands for the space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that are bounded together with all their derivatives.

$H_{\mathrm{ul}}^s = H_{\mathrm{ul}}^s(\mathbb{R})$ is the space of functions $f : \mathbb{R} \to \mathbb{R}$ whose restriction to any bounded interval $I \subset \mathbb{R}$ belongs $H^s(I)$ such that

$$\|f\|_{H_{\mathrm{ul}}^s} := \sup_{x \in \mathbb{R}} \|f(x + \cdot)\|_{H^s([0,1])} < \infty.$$

If $J = [a, b]$ and $X = H_{\mathrm{ul}}^s$ or $H_{\mathrm{ul}}^s \cap L^\infty$, then $C_*(J, X)$ stands for the space of functions $f : J \to X$ that are bounded and continuous on the interval $(a, b]$ and possess a limit in the space H_{loc}^s as $t \to a^+$.

We denote by C_i unessential positive constants.

2 Main result and scheme of its proof

We begin with the definition of the property of approximate controllability. As it will be proved in Sect. 3, the Cauchy problem (1), (2) is well posed. In particular, for any $T > 0$, any integer $s \geq 0$, and any functions $u_0 \in L^\infty(\mathbb{R})$ and $f \in L^1(J_T, H_{\mathrm{ul}}^s \cap L^\infty)$, there is a unique solution $u \in C_*(J_T, H_{\mathrm{ul}}^s \cap L^\infty)$ for (1), (2).

Definition 1. Let $T > 0$, let $h \in L^1(J_T, H^s_{\text{ul}})$ for any $s \ge 0$, and let $E \subset C_b^\infty$ be a finite-dimensional subspace. We shall say that problem (1), (3) is *approximately controllable at time T by an E-valued control* if for any integer $s \ge 0$, any numbers $\varepsilon, r > 0$, and any functions $u_0 \in L^\infty$ and $\hat{u} \in H^s_{\text{ul}}$ there is $\eta \in C^\infty(J_T, E)$ such that the solution $u(t,x)$ of (1)–(3) satisfies the inequalities[1]

$$\|u(T,\cdot)\|_{H^s_{\text{ul}} \cap L^\infty} \le K_s, \quad \|u(T,\cdot) - \hat{u}\|_{H^s([-r,r])} < \varepsilon, \tag{5}$$

where $K_s > 0$ is a constant depending only on $\|u_0\|_{L^\infty}$, $\|\hat{u}\|_{H^s_{\text{ul}}}$, T, and s (but not on r and ε).

Recall that, given a finite subset $\Lambda \subset \mathbb{R}$, we denote by $E_\Lambda \subset C_b^\infty$ the vector span of the functions $\cos(\lambda x)$ and $\sin(\lambda x)$ with $\lambda \in \Lambda$. The following theorem is the main result of this paper.

Theorem 2. *Let $T > 0$, $h \in L^2(J_T, H^s_{\text{ul}})$ for any $s \ge 0$, let λ_1 and λ_2 be incommensurable positive numbers, and let $\Lambda = \{0, \lambda_1, \lambda_2, 2\lambda_1, 2\lambda_2, \lambda_1 + \lambda_2\}$. Then problem* (1), (3) *is approximately controllable at time T by an E_Λ-valued control.*

A proof of this theorem is given in Sect. 4. Here we outline its scheme. Let us fix an integer $s \ge 0$ and functions $u_0 \in L^\infty$ and $\hat{u} \in H^s_{\text{loc}}$. In view of the regularising property of the resolving operator for (1) (see Proposition 5), there is no loss of generality in assuming that $u_0 \in C_b^\infty$, and by a density argument, we can also assume that $\hat{u} \in C_b^\infty$. Furthermore, as it is proved in Sect. 4.5, if inequalities (5) are established for $s = 0$, then simple interpolation and regularisation arguments show that it remains true for any $s \ge 1$. Thus, it suffices to prove (5) for $s = 0$.

Given a finite-dimensional subspace $G \subset C_b^\infty$, we consider the controlled equations

$$\partial_t u - \mu \partial_x^2 u + \mathcal{B}(u) = h(t,x) + \eta(t,x), \tag{6}$$

$$\partial_t u - \mu \partial_x^2 (u + \zeta(t,x)) + \mathcal{B}(u + \zeta(t,x)) = h(t,x) + \eta(t,x), \tag{7}$$

where η and ζ are G-valued controls, and we set $\mathcal{B}(u) = u\partial_x u$. We say that Eq. (6) is *(ε, r, G)-controllable at time T for the pair $(u_0, \hat{u})$* (or simply *G-controllable* if the other parameters are fixed) if one can find $\eta \in C^\infty(J_T, G)$ such that the solution u of (6), (2) satisfies inequalities (5) with $s = 0$. The concept of (ε, r, G)-controllability for (7) is defined in a similar way.

We need to prove that (6) is *E_Λ-controllable*. This fact will be proved in four steps. From now on, we assume that functions $u_0, \hat{u} \in C_b^\infty(\mathbb{R})$ and the positive numbers T, ε, and r are fixed and do not indicate explicitely the dependence of other quantities on them.

Step 1. Extension. Let us fix a finite-dimensional subspace $G \subset C_b^\infty$. Even though Eq. (7) contains more control functions than Eq. (6), the property of G-controllability is equivalent for them. Namely, we have the following result.

[1] Recall that the norm on the intersection of two Banach spaces is defined as the sum of the norms.

Proposition 1. *Equation* (6) *is G-controllable if and only if so is Eq.* (7)*.*

Step 2. Convexification. Let us fix a subset $N \subset C_b^\infty$ invariant under multiplication by real numbers such that

$$N \subset G, \quad \mathcal{B}(N) \subset G. \tag{8}$$

We denote by $\mathcal{F}(N, G) \subset C_b^\infty$ the vector span of functions of the form

$$\eta + \xi \partial_x \tilde{\xi} + \tilde{\xi} \partial_x \xi, \tag{9}$$

where $\eta, \xi \in G$ and $\tilde{\xi} \in N$. It is easy to see that $\mathcal{F}(N, G)$ is a finite-dimensional subspace contained in the convex envelope of G and $\mathcal{B}(G)$; cf. Lemma 1 in Sect. 4.2. The following proposition is an infinite-dimensional analogue of the well-known convexification principle for controlled ODE's (e. g., see [1, Theorem 8.7]).

Proposition 2. *Under the above hypotheses, Eq.* (7) *is G-controllable if and only if Eq.* (6) *is $\mathcal{F}(N, G)$-controllable.*

Step 3. Saturation. Propositions 1 and 2 (and their proof) imply the following result, which is a kind of "relaxation property" for the controlled Burgers equation.

Proposition 3. *Let $N, G \subset C_b^\infty$ be as in step 2. Then Eq.* (6) *is G-controllable if and only if it is $\mathcal{F}(N, G)$-controllable. Moreover, the constant K_0 of* (5) *corresponding to Eq.* (6) *with G-valued control can be made arbitrarily close to that for Eq.* (6) *with $\mathcal{F}(N, G)$-valued control.*

We now set $N = \{c \cos(\lambda_1 x), c \sin(\lambda_1 x), c \cos(\lambda_2 x), c \sin(\lambda_2 x), c \in \mathbb{R}\}$ and define $E_k = \mathcal{F}(N, E_{k-1})$ for $k \geq 1$, where $E_0 = E_\Lambda$. Note that $\mathcal{B}(N) \subset E_\Lambda$ (this inclusion will be important in the proof of Lemma 1). It follows from Proposition 2 that Eq. (6) is E_Λ-controllable if and only if it is E_k-controllable for some integer $k \geq 1$. We shall show that the latter property is true for a sufficiently large k. To this end, we first establish the following saturation property: there is a dense countable subset $\Lambda_\infty \subset \mathbb{R}_+$ such that

$$\bigcup_{k=1}^{\infty} E_k \text{ contains the functions } \sin(\lambda x) \text{ and } \cos(\lambda x) \text{ with } \lambda \in \Lambda_\infty. \tag{10}$$

Step 4. Large control space. Once (10) is proved, one can easily show that (6) is E_k-controllable for a sufficiently large k. To this end, it suffices to join u_0 and $\hat{u}$ by a smooth curve, to use Eq. (6) to define the corresponding control η, and to approximate it, in local topologies, by functions belonging to E_k. The fact that the corresponding solutions are close follows from continuity of the resolving operator for (6) in local norms (see Proposition 6). This will complete the proof of Theorem 2.

3 Cauchy problem

In this section, we discuss the existence and uniqueness of a solution for the Cauchy problem for the generalised Burgers equation

$$\partial_t u - \mu\partial_x^2\big(u + g(t,x)\big) + \mathcal{B}\big(u + g(t,x)\big) = f(t,x), \quad x \in \mathbb{R}, \tag{11}$$

where f and g are given functions. We also establish some a priori estimates for higher Sobolev norms and Lipschitz continuity of the resolving operator in local norms. The techniques of the maximum principle and of weighted energy estimates enabling one to derive this type of results are well known, and sometimes we confine ourselves to the formulation of a result and a sketch of its proof.

3.1 Existence, uniqueness, and regularity of a solution

Before studying the well-posedness of the Cauchy problem for Eq. (11), we recall some results for the linear equation

$$\partial_t v - \mu\partial_x^2 v + a(t,x)\partial_x v + b(t,x)v = c(t,x), \quad x \in \mathbb{R}, \tag{12}$$

supplement with the initial condition

$$v(0,x) = v_0(x), \tag{13}$$

where $v_0 \in L^\infty(\mathbb{R})$. The following proposition establishes the existence, uniqueness, and a priori estimates for a solution of problem (11), (12) in spaces with no decay condition at infinity.

Proposition 4. *Let $T > 0$ and let a, b, c, and f be some functions such that*

$$a \in L^2(J_T, L^\infty), \quad b, c \in L^1(J_T, L^\infty),$$

Then for any $v_0 \in L^\infty$ problem (12), (13) *has a unique solution $v(t,x)$ such that*

$$v \in L^\infty(J_T \times \mathbb{R}) \cap C_*(J_T, L^2_{\mathrm{ul}}), \quad \|\partial_x v(\cdot,x)\|_{L^2(J_T)} \in L^2_{\mathrm{ul}}.$$

Moreover, this solution satisfies the inequalities

$$\|v\|_{L^\infty(J_t\times\mathbb{R})} \le \exp\big(\|b\|_{L^1(J_t,L^\infty)}\big)\Big(\|v_0\|_{L^\infty} + \|c\|_{L^1(J_t,L^\infty)}\Big), \tag{14}$$

$$\|v(t)\|_{L^2_{\mathrm{ul}}} + \|\partial_x v\|_{L^2_{\mathrm{ul}}L^2(J_t)} \le C\, e^{C(\bar a(t)+\bar b(t))}\Big(\|v_0\|_{L^2_{\mathrm{ul}}} + \|c\|_{L^2_{\mathrm{ul}}L^2(J_t)}\Big), \tag{15}$$

where $0 \le t \le T$, $C > 0$ is an absolute constant, and

$$\bar b(t) = \|b\|^2_{L^2(J_t,L^2_{\mathrm{ul}})}, \quad \bar a(t) = \|a\|^2_{L^2(J_t,L^\infty)}, \quad \|c\|_{L^2_{\mathrm{ul}}L^2(J_t)} = \sup_{y\in\mathbb{R}} \|c\|_{L^2(J_t\times[y,y+1])}.$$

If, in addition, we have $a \in L^\infty(J_T \times \mathbb{R})$, then $u \in L^p(J_T, H^1_{\mathrm{ul}})$ for any $p \in [1, \frac{4}{3})$ and

$$\|v\|_{L^p(J_t,H^1_{\mathrm{ul}})} \le C_1\Big(\|v_0\|_{L^2_{\mathrm{ul}}} + \int_0^t \|c(r)\|_{L^2_{\mathrm{ul}}} dr\Big), \tag{16}$$

where $C_1 > 0$ depends only on p, $\|a\|_{L^\infty}$, and $\|b\|_{L^2(J_T,L^2_{\mathrm{ul}})}$.

Proof. Inequality (14) is nothing else but the maximum principle, while (15) can easily be obtained on multiplying (12) by $e^{-|x-y|}v$, integrating over $x \in \mathbb{R}$, and taking the supremum over $y \in \mathbb{R}$. Once these a priori estimates are established (by a formal computation), the existence and uniqueness of a solution in the required functional classes can be proved by usual arguments (e. g., see [13] for the more complicated case of the Navier–Stokes equations), and we omit them. The only non-standard point is inequality (16), and we now briefly outline its proof.

Let $K_t(x)$ be the heat kernel on the real line:

$$K_t(x) = \frac{1}{\sqrt{4\pi\mu t}} \exp\big(-\tfrac{x^2}{4\mu t}\big), \quad x \in \mathbb{R}, \quad t > 0. \tag{17}$$

The following estimates are easy to check:

$$\|K_t * g\|_{L^2_{\mathrm{ul}}} \le \|g\|_{L^2_{\mathrm{ul}}}, \quad \|\partial_x(K_t * g)\|_{L^2_{\mathrm{ul}}} \le C_1 t^{-\frac{3}{4}} \|g\|_{L^2_{\mathrm{ul}}}, \quad t > 0. \tag{18}$$

Here and henceforth, the constants C_i in various inequalities may depend on μ and T. We now use the Duhamel formula to write a solution of (12), (13) in the form

$$v(t,x) = (K_t * v_0)(x) + \int_0^t K_{t-r} * \big(c(r) - a\partial_x v(r) - bv(r)\big)\, dr.$$

It follows from (18) that

$$\begin{aligned}
\|v(t)\|_{H^1_{\mathrm{ul}}} &\le C_1 t^{-\frac{3}{4}} \|v_0\|_{L^2_{\mathrm{ul}}} \\
&\quad + C_2 \int_0^t (t-r)^{-\frac{3}{4}} \big(\|c\|_{L^2_{\mathrm{ul}}} + \|a\|_{L^\infty}\|v\|_{H^1_{\mathrm{ul}}} + \|b\|_{L^2_{\mathrm{ul}}}\|v\|_{L^\infty}\big)\, dr \\
&\le C_1 t^{-\frac{3}{4}} \|v_0\|_{L^2_{\mathrm{ul}}} + C_2 \int_0^t (t-r)^{-\frac{3}{4}} \big(\|c\|_{L^2_{\mathrm{ul}}} + \big(\|a\|_{L^\infty} + 1\big)\, \|v\|_{H^1_{\mathrm{ul}}}\big) dr \\
&\quad + C_3 \int_0^t (t-r)^{-\frac{3}{4}} \|b\|^2_{L^2_{\mathrm{ul}}} \|v\|_{L^2_{\mathrm{ul}}}\, dr\,,
\end{aligned}$$

where we used the interpolation inequality $\|v\|^2_{L^\infty} \le C\|v\|_{L^2_{\mathrm{ul}}}\|v\|_{H^1_{\mathrm{ul}}}$. Taking the left- and right-hand sides of this inequality to the p^{th} power, integrating in time, and using (15), after some simple transformations we obtain the following differential inequality for the increasing function $\varphi(t) = \int_0^t \|v(r)\|_{H^1_{\mathrm{ul}}}^p dr$:

$$\varphi(t) \le C_4 Q^p + C_4\Big(\int_0^t \|c(r)\|_{L^2_{\mathrm{ul}}} dr\Big)^p + C_4\big(\|a\|_{L^\infty(J_t\times\mathbb{R})}^p + 1\big)\int_0^t (t-r)^{-\frac{3}{4}}\varphi(r)\,dr,$$

where Q stands for the expression in the brackets on the right-hand side of (16), and C_4 depends on $\bar{a}(T)$, $\bar{b}(T)$, T, and μ. A Gronwall-type argument enables one to derive (16). □

Let us note that inequality (15) does not use the fact that $b, c \in L^1(J_T, L^\infty)$ and remains valid for any coefficient $b \in L^2(J_T, L^2_{\mathrm{ul}})$ and any right-hand side c for which $\|c\|_{L^2_{\mathrm{ul}} L^2(J_T)} < \infty$. This observation will be important in the proof of Theorem 3.

We now turn to the Burgers Eq. (11), supplemented with the initial condition (2). The proof of the following result is carried out by standard arguments, and we only sketch the main ideas.

Theorem 3. *Let $f \in L^1(J_T, L^\infty)$ and $g \in L^\infty(J_T \times \mathbb{R}) \cap L^2(J_T, W^{1,\infty}) \cap L^1(J_T, W^{2,\infty})$ for some $T > 0$ and let $u_0 \in L^\infty$. Then problem* (11), (2) *has a unique solution $u(t,x)$ such that*

$$u \in L^\infty(J_T \times \mathbb{R}) \cap C_*(J_T, L^2_{\mathrm{ul}}) \cap L^p(J_T, H^1_{\mathrm{ul}}), \quad \|\partial_x u(\cdot, x)\|_{L^2(J_T)} \in L^2_{\mathrm{ul}}, \tag{19}$$

where $p \in [1, \frac{4}{3})$ is arbitrary. Moreover, the mapping $(u_0, f, g) \mapsto u$ is uniformly Lipschitz continuous (in appropriate spaces) on every ball.

Proof. To prove the existence, we first derive some a priori estimates for a solution, assuming that it exists. Let us assume that the functions u_0, f, and g belong to the balls of radius R centred at zero in the corresponding spaces. If a function u satisfies (11), then it is a solution of the linear Eq. (12) with

$$a = u + g, \quad b = \partial_x g, \quad c = f + \mu\partial_x^2 g - g\partial_x g.$$

It follows from (14) that

$$\|u\|_{L^\infty(J_T\times\mathbb{R})} \le C_1(R). \tag{20}$$

Inequalities (15) and (16) now imply that

$$\|u\|_{L^\infty(J_T, L^2_{\mathrm{ul}})} + \|u\|_{L^p(J_T, H^1_{\mathrm{ul}})} + \|u\|_{H^1_{\mathrm{ul}} L^2(J_T)} \le C_2(R). \tag{21}$$

We have thus established some bounds for the norm of a solution in the spaces entering (19). The local existence of a solution can now be proved by a fixed point argument, whereas the absence of finite-time blowup follows from the above a priori estimates.

Let us prove a Lipschitz property for the resolving operator, which will imply, in particular, the uniqueness of a solution. Assume that u_i, $i = 1, 2$, are two solutions corresponding to some data (u_{0i}, f_i, g_i) that belong to balls of radius R centred at zero in the corresponding spaces. Setting $v = u_1 - u_2$, $f = f_1 - f_2$, $g = g_1 - g_2$, and $v_0 = u_{01} - u_{02}$, we see that v satisfies (12), (13) with

$$a = u_1 + g_1, \quad b = \partial_x(u_2 + g_2), \quad c = f + \mu\partial_x^2 g - (u_1 + g_1)\partial_x g - g\partial_x(u_2 + g_2).$$

Multiplying Eq. (12) by $e^{-|x-y|}v$, integrating in $x \in \mathbb{R}$, and using (20) and (21), after some transformations we obtain

$$\partial_t \|v\|_y^2 + \mu\|\partial_x v\|_y^2 \le C_3(R)\|v\|_y^2 + 2\|c\|_y\|v\|_y, \tag{22}$$

where we set

$$\|w\|_y^2 = \int_{\mathbb{R}} w^2(x) e^{-|x-y|}\, dx.$$

Application of a Gronwall-type argument implies that

$$\|v(t)\|_y^2 + \int_0^t \|\partial_x v\|_y^2\, ds \le C_4(R)\Big(\|v_0\|_y + \int_0^t \|c(s)\|_y\, ds\Big)^2. \tag{23}$$

Taking the square root and the supremum in $y \in \mathbb{R}$, we derive

$$\|v\|_{L^\infty(J_t, L^2_{\mathrm{ul}})} + \|\partial_x v\|_{L^2_{\mathrm{ul}} L^2(J_t)} \le C_5(R)\Big(\|v_0\|_{L^2_{\mathrm{ul}}} + \sup_{y\in\mathbb{R}} \int_0^t \|c(s)\|_y\, ds\Big). \tag{24}$$

Now note that

$$\|c\|_y \le \|f\|_y + \mu\|\partial_x^2 g\|_y + \|u_1 + g_1\|_{L^\infty}\|\partial_x g\|_y + \|g\|_{L^\infty}\|\partial_x u_2 + \partial_x g_2\|_y, \tag{25}$$

whence it follows that

$$\int_0^t \|c\|_y\, ds \le \|f\|_{L^2_{\mathrm{ul}} L^2(J_t)} + C_6(R)\big(\|\partial_x^2 g\|_{L^1(J_t, L^\infty)} + \|\partial_x g\|_{L^2(J_t, L^\infty)} + \|g\|_{L^2(J_t, L^\infty)}\big).$$

Substituting this inequality in (24), we obtain

$$\|v\|_{L^\infty(J_t, L^2_{\mathrm{ul}})} + \|\partial_x v\|_{L^2_{\mathrm{ul}} L^2(J_t)} \le C_8(R)\Big(\|v_0\|_{L^2_{\mathrm{ul}}} + \|f\|_{L^2_{\mathrm{ul}} L^2(J_t)} + |||g|||_t\Big), \tag{26}$$

where we set

$$|||g|||_t = \|g\|_{L^1(J_t, W^{2,\infty})} + \|g\|_{L^2(J_t, W^{1,\infty})}.$$

Inequality (26) establishes the required Lipschitz property of the resolving operator. □

Remark 1. An argument similar to that used in the proof of Theorem 3 enables one to estimate the H^1_{ul}-norm of the difference between two solutions. Namely, let $u_i(t, x)$,

$i = 1, 2$, be two solutions of (11), (2) corresponding to some data

$$(u_{0i}, f_i, g_i) \in H^1_{\mathrm{ul}} \times L^2(J_T, L^\infty) \times L^\infty(J_T, W^{2,\infty}), \quad i = 1, 2,$$

whose norms do not exceed R. Then the difference $v = u_1 - u_2$ satisfies the inequality

$$\|v\|_{L^\infty(J_T, H^1_{\mathrm{ul}})} \le C(R)\Big(\|v_0\|_{H^1_{\mathrm{ul}}} + \|f\|_{L^2(J_T, L^2_{\mathrm{ul}})} + \|g\|_{L^4(J_T, W^{2,\infty})}\Big), \tag{27}$$

where we retained the notation used in the proof of (26).

Finally, the following proposition establishes a higher regularity of solutions for (11) with $g \equiv 0$, provided that the right-hand side is sufficiently regular.

Proposition 5. *Under the hypotheses of Theorem 3, assume that $f \in L^2(J_T, H^s_{\mathrm{ul}})$ for an integer $s \ge 1$ and $g \equiv 0$. Then the solution $u(t,x)$ constructed in Theorem 3 belongs to $C([\tau, T], H^s_{\mathrm{ul}})$ for any $\tau > 0$ and satisfies the inequality*

$$\sup_{t \in J_T} \big(t^k \|\partial_x^k u(t)\|^2_{L^2_{\mathrm{ul}}}\big) + \sup_{y \in \mathbb{R}} \int_0^T t^k \|\partial_x^{k+1} u(t)\|^2_{L^2(I_y)} dt \\ \le Q_k\big(\|u_0\|_{L^\infty} + \|f\|_{L^2(J_T, H^k_{\mathrm{ul}} \cap L^\infty)}\big), \tag{28}$$

where $0 \le k \le s$, $I_y = [y, y+1]$, and Q_k is an increasing function. Furthermore, if $u_0 \in C_b^\infty$, then the solution belongs to $C(J_T, H^s_{\mathrm{ul}})$, and inequality (28) is valid without the factor of $t^{k/2}$ on the left-hand side and $\|u_0\|_{L^\infty}$ replaced by $\|u_0\|_{H^k_{\mathrm{ul}}}$ on the right-hand side.

Proof. We confine ourselves to the derivation of the a priori estimate (28) for $u_0 \in L^\infty$. Once it is proved, the regularity of a solution can be obtained by standard arguments. Furthermore, the case when $u_0 \in C_b^\infty$ can be treated by a similar, but simpler technique, and we omit it.

The proof of (28) is by induction on k. For $k = 0$, inequality (28) is a consequence of (21). We now assume that $l \in [1, s]$ and that (28) is established for all $k \le l - 1$. Let us set

$$\varphi_y(t) = t^l \int_{\mathbb{R}} e^{-\langle x-y \rangle} |\partial_x^l u|^2 dx = t^l \|\partial_x^l u\|_y^2, \quad y \in \mathbb{R},$$

where $\langle z \rangle = \sqrt{1 + z^2}$. In view of (11), the derivative of φ_y can be written as

$$\partial_t \varphi_y(t) = l\, t^{l-1} \|\partial_x^l u\|_y^2 + 2t^l \int_{\mathbb{R}} e^{-\langle x-y \rangle} \partial_x^l u\, \partial_x^l(\partial_x^2 u - u\partial_x u + f)\, dx. \tag{29}$$

Integrating by parts and using (20) and the Cauchy–Schwarz inequality, we derive

$$\int_{\mathbb{R}} e^{-\langle x-y\rangle}\partial_x^l u\,\partial_x^{l+2}u\,dx \le -\|\partial_x^{l+1}u\|_y^2 + \|\partial_x^{l+1}u\|_y\,\|\partial_x^l u\|_y,$$

$$\int_{\mathbb{R}} e^{-\langle x-y\rangle}\partial_x^l u\,\partial_x^l f\,dx \le \|\partial_x^l f\|_y\,\|\partial_x^l u\|_y,$$

$$\begin{aligned}\int_{\mathbb{R}} e^{-\langle x-y\rangle}\partial_x^l u\,\partial_x^l(u\partial_x u)\,dx &\le \frac{1}{2}\int_{\mathbb{R}} e^{-\langle x-y\rangle}\partial_x^l u\,\partial_x^{l+1}u^2\,dx\\ &\le \frac{1}{2}\big(\|\partial_x^{l+1}u\|_y + \|\partial_x^l u\|_y\big)\|\partial_x^l u^2\|_y.\end{aligned}$$

Substituting these inequalities into (29) and integrating in time, we obtain

$$\begin{aligned}\varphi_y(t) &+ \int_0^t t^l\,\|\partial_x^{l+1}u\|_y^2\,dt\\ &\le \int_0^t \big(s^{l-1}\|\partial_x^l u\|_y^2 + 4\varphi_y(s) + s^l\,\|\partial_x^l u^2\|_y^2 + s^l\,\|\partial_x^l f\|_y^2\big)\,ds.\end{aligned}$$

Taking the supremum over $y\in\mathbb{R}$ and using the induction hypothesis, we derive

$$\psi(t) \le Q_{l-1} + C_1\int_0^t \psi(s)\,ds + \sup_{y\in\mathbb{R}}\int_0^t s^l\,\|\partial_x^l u^2\|_y^2 ds + C_1\int_0^T \|f\|_{H^l_{\mathrm{ul}}}^2\,ds, \quad (30)$$

where Q_{l-1} is the function entering (28) with $k = l-1$, and

$$\psi(t) = t^l\,\|\partial_x^l u(t)\|_{L^2_{\mathrm{ul}}}^2 + \sup_{y\in\mathbb{R}}\int_0^t t^l\,\|\partial_x^{l+1}u\|_{L^2(I_y)}^2\,dt.$$

Now note that

$$\int_0^t s^l\,\|\partial_x^l u^2\|_y^2\,ds \le C_2\|u\|_{L^\infty}^2\sum_{k\in\mathbb{Z}} e^{-|k-y|}\int_0^t s^l\,\|u\|_{H^l(I_k)}^2\,ds.$$

Substituting this into (30) and using again the induction hypothesis and inequality (20), we obtain

$$\psi(t) \le C_3\int_0^t \psi(s)\,ds + Q\big(\|u_0\|_{L^\infty} + \|f\|_{L^2(J_T,H^l_{\mathrm{ul}}\cap L^\infty)}\big),$$

where Q is an increasing function. Application of the Gronwall inequality completes the proof. □

3.2 Uniform continuity of the resolving operator in local norms

Theorem 3 established, in particular, the Lipschitz continuity of the resolving operator for (11). The following proposition, which plays a crucial role in the next section, proves the uniform continuity of the resolving operator in local norms.

Proposition 6. *Under the hypotheses of Theorem 3, for any positive numbers T, R, r, and δ there are ρ and C such that, if triples (u_{0i}, f_i, g_i), $i = 1, 2$, satisfy the inclusions*

$$u_{0i} \in L^\infty, \quad f_i \in L^1(J_T, L^\infty), \quad g_i \in L^\infty(J_T \times \mathbb{R}) \cap L^2(J_T, W^{1,\infty}) \cap L^1(J_T, W^{2,\infty}),$$

and corresponding norms are bounded by R, then

$$\sup_{t \in J_T} \|u_1(t) - u_2(t)\|_{L^2([-r,r])} \le \delta + C\Big(\|u_{01} - u_{02}\|_{L^2(I_\rho)} + \|f_1 - f_2\|_{L^1(J_T, L^2(I_\rho))} + \|g_1 - g_2\|_{L^2(J_T, H^2(I_\rho))}\Big), \tag{31}$$

where $I_\rho = [-\rho, \rho]$, and $u_i(t)$ denotes the solution of (11) *issued from u_{0i}.*

Proof. We shall use the notation introduced in the proof of Theorem 3. It follows from inequality (23) with $y = 0$ that

$$e^{-r/2}\|v(t)\|_{L^2(I_r)} \le C_1(R)\Big(\|e^{-|\cdot|/2}v_0\|_{L^2} + \int_0^T \|e^{-|\cdot|/2}c(t,\cdot)\|_{L^2}\,dt\Big). \tag{32}$$

Now note that

$$\|e^{-|x|/2}v_0\|_{L^2}^2 = \int_{\mathbb{R}} |v_0|^2 e^{-|x|}dx \le \|v_0\|_{L^2(I_\rho)}^2 + 4e^{-\rho}\|v_0\|_{L^2_{\mathrm{ul}}}^2. \tag{33}$$

By a similar argument, we check that (cf. (25))

$$\|e^{-|\cdot|/2}c(t,\cdot)\|_{L^2} \le \|f\|_{L^2(I_\rho)} + \mu\|\partial_x^2 g\|_{L^2(I_\rho)} + C_2(R)\,\|\partial_x g\|_{L^2(I_\rho)} + C_3(R)e^{-\rho/2} + \big(\|g\|_{L^\infty(I_\rho)} + e^{-\rho/4}\|g\|_{L^\infty}\big)\,\|e^{-|\cdot|/4}(\partial_x u_2 + \partial_x g_2)\|_{L^2(\mathbb{R})}.$$

Integrating in time and using (21), we obtain

$$\int_0^T \|e^{-|\cdot|/2}c(t,\cdot)\|_{L^2}\,dt \le C_4(R)\Big\{\int_0^T \|f\|_{L^2(I_\rho)}dt + \Big(\int_0^T \|g\|_{H^2(I_\rho)}^2 dt\Big)^{1/2} + e^{-\rho/4}\Big\}. \tag{34}$$

Substituting (33) and (34) into (32) and taking $\rho > 0$ sufficiently large, we arrive at the required inequality (31). □

4 Proof of Theorem 2

4.1 Extension: proof of Proposition 1

We only need to prove that if Eq. (7) is G-controllable, then so is (6), since the converse implication is obvious. Let $\tilde{\eta}, \tilde{\zeta} \in C^\infty(J_T, G)$ be such that the solution $\tilde{u}$ of problem (7), (2) satisfies (5) with $s = 0$. In view of (26), replacing K_0 by a slightly larger constant, we can assume that $\tilde{\zeta}(0) = \tilde{\zeta}(T) = 0$. Let us set $u = \tilde{u} + \tilde{\zeta}$. Then u is a solution of (6), (2) with the control $\eta = \tilde{\eta} + \partial_t \tilde{\zeta}$, which takes values in G. Moreover, $u(T) = \tilde{u}(T)$ and, hence, u satisfies (5). This completes the proof of Proposition 1, showing in addition that the constants K_0 entering (5) and corresponding to Eqs. (6) and (7) can be chosen arbitrarily close to each other.

4.2 Convexification: proof of Proposition 2

We begin with a number of simple observations. Let us set $G_1 = \mathcal{F}(N, G)$. By Proposition 1, if Eq. (7) is G-controllability, then so is Eq. (6), and since $G \subset G_1$, we see that (6) is G_1-controllable. Thus, it suffices to prove that if (6) is G_1-controllable, then (7) is G-controllable. To establish this property, it suffices to prove that, for any $\eta_1 \in C^\infty(J_T, G_1)$ and any $\delta > 0$ there are $\eta, \zeta \in L^\infty(J_T, G)$ such that the solution $u(t, x)$ of (7), (2) satisfies the inequality

$$\|u(T) - u_1(T)\|_{H^1_{\mathrm{ul}}} < \delta, \tag{35}$$

where u_1 stands for the solution of (6), (2) with $\eta = \eta_1$. Indeed, if this property is established, then we take two sequences $\{\eta^n\}, \{\zeta^n\} \subset C^\infty(J_T, G)$ such that (cf. (27))

$$\|\eta^n - \eta\|_{L^2(J_T,G)} + \|\zeta^n - \zeta\|_{L^4(J_T,G)} \to 0 \quad \text{as } n \to \infty$$

and denote by $u^n(t, x)$ the solution of (7), (2) with $\eta = \eta^n$ and $\zeta = \zeta^n$. It follows from (27) that

$$\gamma_n := \|u^n(T) - u(T)\|_{H^1_{\mathrm{ul}}} \to 0 \quad \text{as } n \to \infty. \tag{36}$$

Combining (35) and (36) and using the continuous embedding $H^1_{\mathrm{ul}} \subset L^\infty$, we derive

$$\begin{aligned}
\|u^n(T)\|_{L^\infty} &\le \|u_1(T)\|_{L^\infty} + \|u(T) - u_1(T)\|_{L^\infty} + \|u^n(T) - u(T)\|_{L^\infty} \\
&\le K_0 + C_1(\delta + \gamma_n), \\
\|u^n(T) - \hat{u}\|_{L^2(I_r)} &\le \|u^n(T) - u(T)\|_{L^2(I_r)} + \|u(T) - u_1(T)\|_{L^2(I_r)} \\
&\quad + \|u_1(T) - \hat{u}\|_{L^2(I_r)} \\
&\le C_2(\gamma_n + \delta) + \|u_1(T) - \hat{u}\|_{L^2(I_r)},
\end{aligned}$$

where $I_r = [-r, r]$. Choosing $\delta > 0$ sufficiently small and n sufficiently large, we conclude that u^n satisfies inequalities (5), with a constant K_0 arbitrarily close to that for u_1. Finally, a similar approximation argument shows that, when proving (35), we can assume $\eta_1(t)$ to be piecewise constant, with finitely many intervals of constancy. The construction of controls $\eta, \zeta \in L^\infty(J_T, G)$ for which (35) holds is carried out in several steps.

Step 1. An auxiliary lemma. We shall need the following lemma, which establishes a relationship between G- and $\mathcal{F}(N, G)$-valued controls.

Lemma 1. *For any* $\eta_1 \in \mathcal{F}(N, G)$ *and any* $\nu > 0$ *there is an integer* $k \geq 1$*, numbers* $\alpha_j > 0$*, and vectors* $\eta, \zeta^j \in G$*,* $j = 1, \dots, k$*, such that*

$$\sum_{j=1}^{k} \alpha_j = 1, \tag{37}$$

$$\Big\| \eta_1 - \mathcal{B}(u) - \Big(\eta - \sum_{j=1}^{k} \alpha_j \big(\mathcal{B}(u + \zeta^j) - \mu \partial_x^2 \zeta^j\big)\Big) \Big\|_{H^1_{\mathrm{ul}}} \leq \nu \quad \text{for any } u \in H^1_{\mathrm{ul}}. \tag{38}$$

Proof. It suffices to find functions $\eta, \tilde{\zeta}^j \in G$, $j = 1, \dots, m$, such that

$$\Big\| \eta_1 - \eta + \sum_{j=1}^{k} \mathcal{B}(\tilde{\zeta}^j) \Big\|_{H^1_{\mathrm{ul}}} \leq \nu. \tag{39}$$

Indeed, if such vectors are constructed, then we can set $k = 2m$,

$$\alpha_j = \alpha_{j+m} = \frac{1}{2m}, \quad \zeta^j = -\zeta^{j+m} = \sqrt{m}\, \tilde{\zeta}^j \quad \text{for } j = 1, \dots, m,$$

and relations (37) and (38) are easily checked.

To construct $\eta, \tilde{\zeta}^j \in G$ satisfying (39), note that if $\eta_1 \in \mathcal{F}(N, G)$, then there are functions $\tilde{\eta}_j, \xi_j \in G$ and $\tilde{\xi}_j \in N$ such that

$$\eta_1 = \sum_{j=1}^{k} \big(\tilde{\eta}_j - \xi_j \partial_x \tilde{\xi}_j - \tilde{\xi}_j \partial_x \xi_j\big). \tag{40}$$

Now note that, for any $\varepsilon > 0$,

$$\xi_j \partial_x \tilde{\xi}_j + \tilde{\xi}_j \partial_x \xi_j = \mathcal{B}(\varepsilon \xi_j + \varepsilon^{-1} \tilde{\xi}_j) - \varepsilon^2 \mathcal{B}(\xi_j) - \varepsilon^{-2} \mathcal{B}(\tilde{\xi}_j).$$

Combining this with (40), we obtain

$$\eta_1 - \sum_{j=1}^{k} \big(\tilde{\eta}_j + \varepsilon^{-2} \mathcal{B}(\tilde{\xi}_j)\big) + \sum_{j=1}^{k} \mathcal{B}(\varepsilon \xi_j + \varepsilon^{-1} \tilde{\xi}_j) = \varepsilon^2 \sum_{j=1}^{k} \mathcal{B}(\xi_j).$$

Choosing $\varepsilon > 0$ sufficiently small and setting[2]

$$\eta = \sum_{j=1}^{k} \big(\tilde{\eta}_j + \varepsilon^{-2} \mathcal{B}(\tilde{\xi}_j)\big), \quad \tilde{\zeta}^j = \varepsilon \xi_j + \varepsilon^{-1} \tilde{\xi}_j, \tag{41}$$

we arrive at the required inequality (39). □

[2] Recall that $\mathcal{B}(N) \subset G$, so that the vector η defined in (41) belongs to G.

Step 2. Comparison with an auxiliary equation. Let $\eta_1 \in L^\infty(J_T, G_1)$ be a piecewise constant function and let u_1 be the solution of problem (6), (2) with $\eta = \eta_1$. To simplify notation, we assume that there are only two intervals of constancy for $\eta_1(t)$ and write

$$\eta_1(t,x) = I_{J_1}(t)\eta_1^1(x) + I_{J_2}(t)\eta_1^2(x),$$

where $\eta_1^1, \eta_1^2 \in G_1$ are some vectors and $J_1 = [0,a]$ and $J_2 = [a,T]$ with $a \in (0,T)$. We fix a small $\nu > 0$ and, for $i = 1,2$, choose numbers $\alpha_j^i > 0$, $j = 1,\dots,k_i$, and vectors $\eta^i, \zeta^{ji} \in G$ such that (37), (38) hold. Let us consider the following equation on J_T:

$$\partial_t u - \mu\partial_x^2 u + \sum_{j=1}^{k_i} \alpha_j^i \big(\mathcal{B}(u + \zeta^{ji}(x)) - \mu\partial_x^2\zeta^{ji}(x)\big) = h(t,x) + \eta^i(x), \quad t \in J_i. \tag{42}$$

This is a Burgers-type equation, and using the same arguments as in the proof of Theorem 3, it can be proved that problem (42), (2) has a unique solution $\tilde{u}(t,x)$ satisfying (19). Moreover, in view of the regularity of the data and an analogue of Proposition 5 for Eq. (42), we have

$$\tilde{u} \in C(J_T, H_{\mathrm{ul}}^k) \quad \text{for any } k \geq 0. \tag{43}$$

On the other hand, we can rewrite (42) in the form

$$\partial_t u - \mu\partial_x^2 u + u\partial_x u = h(t,x) + \eta_1^i(x) - c_\nu^i(t,x), \quad t \in J_i, \tag{44}$$

where $c_\nu^i(t,x)$ is defined for $t \in J_i$ by the function under sign of norm on the left-hand side of (38) in which $\eta_1 = \eta_1^i$, $\eta = \eta^i$, $\alpha_j = \alpha_j^i$, $\zeta^j = \zeta^{ji}$, and $u = \tilde{u}(t,x)$. Since the resolving operator for (44) is Lipschitz continuous on bounded subsets, there is a constant $C > 0$ depending only on the L^∞ norms of η_1^i such that (see Remark 1)

$$\|u_1(T) - \tilde{u}(T)\|_{H_{\mathrm{ul}}^1} \leq C\big(\|c_\nu^1\|_{L^2(J_1,L^\infty)} + \|c_\delta^2\|_{L^2(J_2,L^\infty)}\big) \leq C\sqrt{2T}\,\nu. \tag{45}$$

On the other hand, let us define $\eta \in L^\infty(J_T, G)$ by $\eta(t) = \eta^i$ for $t \in J_i$. We shall show in the next steps that there is a sequence $\{\zeta_m\} \subset L^\infty(J_T, G)$ such that

$$\|u^m(T) - \tilde{u}(T)\|_{H_{\mathrm{ul}}^1} \to 0 \quad \text{as } m \to \infty, \tag{46}$$

where $u^m(t,x)$ denotes the solution of problem (7), (2) in which $\zeta = \zeta_m$. Combining inequalities (45) and (46) with $\nu \ll 1$ and $m \gg 1$, we obtain the required estimate (35) for $u = u^m$.

Step 3. Fast oscillating controls. Following a classical idea in the control theory, we define functions $\zeta_m \in L^\infty(J_T, G)$ by the relation

$$\zeta_m(t) = \begin{cases} \zeta^{(1)}(mt/a) & \text{for } t \in J_1, \\ \zeta^{(2)}(m(t-a)/(T-a)) & \text{for } t \in J_2, \end{cases}$$

where $\zeta^{(i)}(t)$ is a 1-periodic G-valued function such that

$$\zeta^{(i)}(t) = \zeta^{ji} \quad \text{for } 0 \le t - (\alpha_1^i + \cdots + \alpha_{j-1}^i) < \alpha_j^i,\ j = 1, \ldots, k_i.$$

Let us rewrite (42) in the form

$$\partial_t u - \mu\partial_x^2(u + \zeta_m(t,x)) + \mathcal{B}(u + \zeta_m(t,x)) = h(t,x) + \eta(t,x) + f_m(t,x),$$

where we set $f_m = f_{m1} + f_{m2}$,

$$f_{m1}(t) = -\mu\partial_x^2\zeta_m + \mu\sum_{j=1}^{k_i}\alpha_j^i\partial_x^2\zeta^{ji}, \tag{47}$$

$$f_{m2}(t) = \mathcal{B}(\tilde{u} + \zeta_m) - \sum_{j=1}^{k_i}\alpha_j^i\mathcal{B}(\tilde{u} + \zeta^{ji}) \tag{48}$$

for $t \in J_i$. We now define an operator $\mathcal{K} : L^2(J_T, L^\infty) \to L^\infty(J_T \times \mathbb{R}) \cap C_*(J_T, L^2_{\rm ul})$ by the relation

$$(\mathcal{K}f)(t,x) = \int_0^t K_{t-s} * f(s)\,ds,$$

where the kernel K_t was introduced in (17). Setting $v_m = \tilde{u} - \mathcal{K}f_m$, we see that the function $v_m(t,x)$ satisfies the equation

$$\partial_t v - \mu\partial_x^2(v + \zeta_m) + \mathcal{B}(v + \zeta_m + \mathcal{K}f_m) = h + \eta. \tag{49}$$

Suppose we have shown that

$$\|\mathcal{K}f_m(T)\|_{H^1_{\rm ul}} + \|\mathcal{K}f_m\|_{L^4(J_T,W^{2,\infty})} \to 0 \quad \text{as } m \to \infty. \tag{50}$$

Then, by (27), we have

$$\|u^m(T) - \tilde{u}(T)\|_{H^1_{\rm ul}} \le \|u^m(T) - v_m(T)\|_{H^1_{\rm ul}} + \|\mathcal{K}f_m(T)\|_{H^1_{\rm ul}} \to 0 \quad \text{as } m \to \infty.$$

Thus, it remains to prove (50).

Step 4. Proof of (50). We first note that $\{f_m\}$ is a bounded sequence in $L^\infty(J_T, H^k_{\rm ul})$ for any $k \ge 0$. Integrating by parts, it follows that

$$\mathcal{K}f_m = F_m + \mu\,\mathcal{K}(\partial_x^2 F_m), \tag{51}$$

where we set

$$F_m(t) = \int_0^t f_m(s)\,ds.$$

In view of Proposition 4, the operator $\mathcal{K}$ is continuous from $L^1(J_T, H^k_{\rm ul})$ to $C(J_T, H^k_{\rm ul})$ for any integer $k \ge 0$. Therefore (50) will follow if we show that

$$\|F_m\|_{C(J_T,H^k_{\rm ul})} \to 0 \quad \text{as } m \to \infty.$$

This convergence is a straightforward consequence of relations (47) and (48); e. g., see [16, Sect. 3.3]. The proof of Proposition 2 is complete.

4.3 Saturation

We wish to prove (10). To this end, we shall need the following lemma describing explicitly some subspaces that are certainly included in E_k. Without loss of generality, we assume that $\lambda_1 > \lambda_2$.

Lemma 2. *Let us set* $\Lambda_k = \{n_1\lambda_1 + n_2\lambda_2 \geq 0 : n_1, n_2 \in \mathbb{Z}, |n_1| + |n_2| \leq k\}$. *Then* $E_{\Lambda_k} \subset E_k$ *for any integer* $k \geq 1$.

Proof. The proof is by induction on k. We confine ourselves to carrying out the induction step, since the base of induction can be checked by a similar argument.

Let us fix any integer $k \geq 2$ and assume that $E_{\Lambda_k} \subset E_k$. We need to show that that the functions $\sin(\lambda x)$ and $\cos(\lambda x)$ belong to E_{k+1} for $\lambda = n_1\lambda_1 + n_2\lambda_2 \in \Lambda_{k+1}$. We shall only consider the case when the coefficients n_1 and n_2 are non-negative, since the other situations can be treated by similar arguments. Assume first $n_1 \geq 2$ and $n_1 + n_2 \leq k + 1$. Then $\lambda' = \lambda - \lambda_1$ and $\lambda'' = \lambda - 2\lambda_1$ belong to Λ_k, and we have

$$\sin(\lambda x) = \frac{\lambda''}{\lambda} \sin(\lambda'' x) + \frac{2}{\lambda}\big(\sin(\lambda_1 x)\,\partial_x \sin(\lambda' x) + \sin(\lambda' x)\,\partial_x \sin(\lambda_1 x)\big), \tag{52}$$

$$\cos(\lambda x) = -\frac{\lambda}{\lambda''} \cos(\lambda'' x) + \frac{2}{\lambda''}\big(\cos(\lambda_1 x)\,\partial_x \sin(\lambda' x) + \sin(\lambda' x)\,\partial_x \cos(\lambda_1 x)\big), \tag{53}$$

whence we conclude that the functions on the left-hand side of these relations belong to E_{k+1}. If $\lambda = \lambda_1 + k\lambda_2 \in \Lambda_{k+1}$, then setting $\lambda' = \lambda - \lambda_2$ and $\lambda'' = \lambda - 2\lambda_2$, we see that relations (52) and (53) with λ_1 replaced by λ_2 remain valid, and we can conclude again that $\sin(\lambda x), \cos(\lambda x) \in E_{k+1}$. Finally, the same proof applies also in the case $\lambda = (k + 1)\lambda_2 \in \Lambda_{k+1}$. □

Lemma 2 shows that the union of E_k (which is a vector space) contains the trigonometric functions whose frequencies belong to the set $\Lambda_\infty := \cup_k \Lambda_k$. It is straightforward to check that Λ_∞ is dense in $\mathbb{R}_+$.

4.4 Large control space

Let us prove that (6) is E_{Λ_k}-controllable (and, hence, E_k-controllable) for a sufficiently large k. Indeed, let us set

$$u(t, x) = T^{-1}\big(t\hat{u}(x) + (T - t)u_0(x)\big), \quad (t, x) \in J_T \times \mathbb{R}. \tag{54}$$

This is an infinity smooth function in (t, x) all of whose derivatives are bounded. We now define

$$\eta(t, x) = \partial_t u - \mu\partial_x^2 u + u\partial_x u - h$$

and note that $\eta \in L^2(J_T, H^s_{ul})$ for any $s \geq 0$ and that the solution of problem (6), (2) is given by (54) and coincides with $\hat{u}$ for $t = T$. We have thus a control that steers a solution starting from u_0 to $\hat{u}$. To prove the required property, we approximate η, in local topologies, by an E_{Λ_k}-valued function and use the continuity of the resolving operator to show that the corresponding solutions are close.

More precisely, let $\chi \in C^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$, $\sup_{\mathbb{R}} |\chi'| \leq 2$, $\chi(x) = 0$ for $|x| \geq 2$, and $\chi(x) = 1$ for $|x| \leq 1$. Then the sequence $\eta_n(t, x) = \chi(x/n)\eta(t, x)$ possesses the following properties:

$$\eta_n(t, x) = 0 \quad \text{for } |x| \geq 2n \text{ and any } n \geq 1, \tag{55}$$

$$\|\eta_n\|_{L^2(J_T, H^1_{ul})} \leq 3\|\eta\|_{L^2(J_T, H^1_{ul})} \quad \text{for all } n \geq 1, \tag{56}$$

$$\|\eta_n - \eta\|_{L^2(J_T \times I_\rho)} \to 0 \quad \text{as } n \to \infty \text{ for any } \rho > 0, \tag{57}$$

where $I_\rho = [-\rho, \rho]$. Given a frequency $\omega > 0$ and an integer $N \geq 1$, we denote by $\mathsf{P}_{\omega,N} : L^2(I_{\pi/\omega}) \to L^\infty(\mathbb{R})$ a linear projection that takes a function g to its truncated Fourier series

$$(\mathsf{P}_{\omega,N} g)(x) = \sum_{|j| \leq N} g_j e^{\omega i j x}, \quad g_j = \frac{\omega}{2\pi} \int_{I_{\pi/\omega}} g(y) e^{-\omega i j y} dy.$$

The function $\mathsf{P}_{\omega,N} g$ is $2\pi/\omega$-periodic, and it follows from (55) and (56) that

$$\|\mathsf{P}_{\omega,N}\eta_n\|_{L^1(J_T, L^\infty)} \leq C_1 \|\mathsf{P}_{\omega,N}\eta_n\|_{L^2(J_T, H^1_{ul})} \leq C_2 \quad \text{for all } N, n \geq 1, \tag{58}$$

$$\|\mathsf{P}_{\omega,N}\eta_n - \eta_n\|_{L^2(J_T \times I_\rho)} \to 0 \quad \text{as } N \to \infty \text{ for any } n \geq 1. \tag{59}$$

Note that if $\omega \in \Lambda_\infty$, then for any $N \geq 1$ there is $k \geq 1$ such that the image of $\mathsf{P}_{\omega,N}$ is contained in E_{Λ_k}.

Let us denote by $u_{n,N}(t, x)$ the solution of problem (6), (2) with $\eta = \mathsf{P}_{\omega,N}\eta_n$. In view of inequality (31) with $\delta = \varepsilon/2$ and $R = \max\{\|u_0\|_{L^\infty}, \|\eta\|_{L^1(J_T, L^\infty)}, C_2\}$, we have

$$\begin{aligned} \|u_{n,N}(T) - \hat{u}\|_{L^2(I_r)} &= \|u_{n,N}(T) - u(T)\|_{L^2(I_r)} \\ &\leq \frac{\varepsilon}{2} + C\,\|\mathsf{P}_{\omega,N}\eta_n - \eta\|_{L^1(J_T, L^2(I_\rho))} \\ &\leq \frac{\varepsilon}{2} + C\sqrt{T}\Big(\|\mathsf{P}_{\omega,N}\eta_n - \eta_n\|_{L^1(J_T, L^2(I_\rho))} + \|\eta_n - \eta\|_{L^1(J_T, L^2(I_\rho))}\Big). \end{aligned} \tag{60}$$

We now choose $n \geq 1$ such that $C\sqrt{T}\,\|\eta_n - \eta\|_{L^1(J_T, L^2(I_\rho))} < \frac{\varepsilon}{4}$; see (57). We next find $\omega \in \Lambda_\infty$ so that $\frac{\pi}{\omega} > \max(2n, \rho)$ (this is possible since Λ_∞ is dense in $\mathbb{R}_+$) and choose $N \geq 1$ such that $C\sqrt{T}\|\mathsf{P}_{\omega,N}\eta_n - \eta_n\|_{L^1(J_T, L^2(I_\rho))} < \frac{\varepsilon}{4}$. Substituting these estimates into (60), we obtain

$$\|u_{n,N}(T) - \hat{u}\|_{L^2(I_r)} < \varepsilon,$$

which is the second inequality in (5) with $s = 0$. It remains to note that, in view of (20), (56), and (58), the first inequality in (5) is also satisfied.

4.5 Reduction to the case $s = 0$

We now prove that if inequalities (5) hold for $s = 0$ and arbitrary T, r, and ε, then they remain valid for any $s \geq 1$. Indeed, we fix an integer $s \geq 1$, positive numbers r and ε, and functions $u_0, \hat{u} \in C_b^\infty$. Let us define η by zero on the half-line $[T, +\infty)$ and denote by $\hat{u}(t)$ the solution of (1), (3) issued from $\hat{u}$ at $t = T$. Using interpolation, regularity of solutions (Proposition 5), and continuity of the resolving operator in local norms (Proposition 6), we can write

$$\begin{aligned}\|u(T+\tau)-\hat{u}(\tau)\|_{H^s(I_r)}^2 &\leq C_1\|u(T+\tau)-\hat{u}(\tau)\|_{L^2(I_r)}\|u(T+\tau)-\hat{u}(\tau)\|_{H^{2s}(I_r)} \\ &\leq C_2\tau^{-2s}\big(\delta + C\,\|u(T)-\hat{u}\|_{L^2(I_\rho)}\big)\,Q_{2s}\big(\|u(T)\|_{L^\infty}+K\big), \quad (61)\end{aligned}$$

where C_i are some constants depending on R and s, the quantities C and Q_{2s} are those entering (31) and (28), respectively, and $K = \|\hat{u}\|_{L^\infty} + \|h\|_{L^1(J_T, H_{\mathrm{ul}}^{2s})}$. Furthermore, in view of Proposition 5, we have

$$\|\hat{u}(\tau) - \hat{u}\|_{H_{\mathrm{ul}}^s} \to 0 \quad \text{as } \tau \to 0^+.$$

Let $\tau > 0$ be so small that the left-hand side of this relation is smaller than $\varepsilon^2/6$. We next choose $\delta > 0$ such that

$$C_2\tau^{-2s}Q_{2s}(K_0 + K)\delta < \varepsilon^2/6,$$

where K_0 is defined in (5) (and is independent of r and ε). Finally, we construct $\eta \in C^\infty(J_T, E_\Lambda)$ for which inequalities (5) hold with $r = \rho$ and $\varepsilon = \delta/C$. Comparing the above estimates with (61), we obtain

$$\|u(T+\tau) - \hat{u}\|_{H_{\mathrm{ul}}^s(I_r)} := \sup_{I\subset I_r} \|u(T+\tau) - \hat{u}\|_{H^s(I)} < \varepsilon,$$

where the supremum is taken oven all intervals $I \subset I_r$ of length ≤ 1. Furthermore, in view of (28), we have

$$\|u(T+\tau)\|_{H_{\mathrm{ul}}^s} \leq \tau^{-s}Q_s\big(K_0 + \|h\|_{L^1(J_T, H_{\mathrm{ul}}^s)}\big) =: K_s.$$

We have thus established inequalities (5) with T and $\|\cdot\|_{H^s(I_r)}$ replaced by $T + \tau$ and $\|\cdot\|_{H_{\mathrm{ul}}^s(I_r)}$, respectively. Since T is arbitrary and the positive numbers τ and ε can be chosen arbitrarily small, we conclude that inequalities (5) are true for any integer $s \geq 0$ and any numbers $T, r, \varepsilon > 0$. This completes the proof of Theorem 2.

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Homogeneous affine line fields and affine lines in Lie algebras

Michail Zhitomirskii

Abstract We prove that for $n = 2, 3$ any local homogeneous affine line field $L \subset T\mathbb{R}^n$ can be described by an affine line ℓ in an n-dimensional Lie algebra $\mathfrak{g}$, which means that L is diffeomorphic to the affine line field in a neighborhood of the identity of the Lie group of $\mathfrak{g}$ obtained by pushing ℓ along the flows of left-invariant vector fields. We show that this statement does not hold for $n = 4$, for one of several types of homogeneous line fields.

1 Introduction

1.1 Local homogeneous subsets of the tangent bundle

Let M^n be an analytic n-dimensional manifold. A subset Σ of the tangent bundle, $\Sigma = \{\Sigma_x \subset T_x M^n\}_{x \in M^n}$, is called homogeneous if for any points $p_1, p_2 \in M^n$ there exists a local analytic diffeomorphism $\Phi : (M^n, p_1) \to (M^n, p_2)$ such that $\Phi_{*,x}(\Sigma_x) = \Sigma_{\Phi(x)}$ for x close to p_1.

A local homogeneous subset of $T\mathbb{R}^n$ is the germ at $0 \in \mathbb{R}^n$ of a homogeneous subset of TU, where U is a neighborhood of 0. Here by germ we mean the germ with respect to $x \in R^n$ only so that a local homogeneous subset of $T\mathbb{R}^n$ is local with respect to a point of $\mathbb{R}^n$ and global with respect to a tangent vector.

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1.2 Symmetry algebra $sym(\Sigma)$

In what follows Σ is a local homogeneous subset of $T\mathbb{R}^n$.

A vector field germ V at $0 \in \mathbb{R}^n$ is called an infinitesimal symmetry of Σ if for the flow Φ^t of V one has $\Phi^t_*(\Sigma_x) = \Sigma_{\Phi^t(x)}$, for x close to $0 \in \mathbb{R}^n$ and t close to $0 \in \mathbb{R}$.

The set of all infinitesimal symmetries of Σ is a Lie algebra whose dimension might be finite or infinite. It is called the symmetry algebra of Σ. We will denote it by $sym(\Sigma)$.

Definition 1. A Lie algebra A of vector fields germs at $0 \in \mathbb{R}^n$ is called transitive if $\{V(0), V \in A\} = T_0\mathbb{R}^n$.

It is well-known that a local subset $\Sigma \subset T\mathbb{R}^n$ is homogeneous if and only if its symmetry algebra $sym(\Sigma)$ is transitive, see [1].

1.3 Construction of a local homogeneous subset of $T\mathbb{R}^n$ from an endowed n-dimensional Lie algebra

By an endowed n-dimensional Lie algebra we mean an n-dimensional Lie algebra $\mathfrak{g}$ endowed with a subset $\sigma \subset \mathfrak{g}$. Let (G, id) be a neighborhood of the identity of the Lie group G of $\mathfrak{g}$. We can push σ to $T(G, id)$ by the flows of left-invariant vector fields. We obtain a homogeneous subset of $T(G, id)$ for which we will use the notation $\widehat{(\mathfrak{g}, \sigma)}$.

Take any local diffeomorphism $(G, id) \to (\mathbb{R}^n, 0)$. It brings $\widehat{(\mathfrak{g}, \sigma)}$ to a local homogeneous subset of $T\mathbb{R}^n$, defined up to a local diffeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$.

Definition 2. We will say that a local homogeneous subset $\Sigma \subset T\mathbb{R}^n$ is induced by an endowed n-dimensional Lie algebra $(\mathfrak{g}, \sigma)$ if Σ can be obtained from the constructed homogeneous subset $\widehat{(\mathfrak{g}, \sigma)} \subset T(G, id)$ by a local diffeomorphism $(G, id) \to (\mathbb{R}^n, 0)$.

1.4 A general question on local homogeneous subsets of $T\mathbb{R}^n$

In the conference in Cortona in 2012, devoted to Andrei Agrachev's 60th Birthday, I formulated and discussed the following question.

Question. Are there local homogeneous subsets of $T\mathbb{R}^n$ which are *not* induced by any endowed n-dimensional Lie algebra, according to Definition 2? Is it possible to describe all of them, at least for small n?

At first observe the following almost obvious statement.

Proposition 1. *If Σ is a local homogeneous subset of $T\mathbb{R}^n$ such that $dim\, sym(\Sigma) = n$ then Σ is induced by an endowed n-dimensional Lie algebra.*

In fact, in this case Σ can be obtained from $\Sigma_0 = \Sigma \cap T_0\mathbb{R}^n$ by pushing Σ_0 along the flows of vector fields in $sym(\Sigma)$. Take an isomorphism $i : T_0\mathbb{R}^n \to sym(\Sigma)$ sending a tangent vector $v \in T_0\mathbb{R}^n$ to the unique $V \in sym(\Sigma)$ such that $V(0) = v$. Then Σ is diffeomorphic to $\widehat{(sym(\Sigma), \sigma)}$ with $\sigma = i(\Sigma_0)$.

On the other hand, there is a number of examples of local homogeneous subsets $\Sigma \subset T\mathbb{R}^n$ with the symmetry algebra of dimension ∞ or a finite dimension bigger than n which are also induced by an endowed n-dimensional Lie algebra.

Note that the symmetry algebra of $\widehat{(\mathfrak{g}, \sigma)}$ contains all left-invariant vector fields and therefore $\mathfrak{g} \subseteq sym\widehat{(\mathfrak{g}, \sigma)}$. It follows that the symmetry algebra of any local homogeneous subset $\Sigma \subset T\mathbb{R}^n$ induced by an endowed n-dimensional Lie algebra $(\mathfrak{g}, \sigma)$ contains a subalgebra isomorphic to $\mathfrak{g}$. But the whole symmetry algebra of Σ might be much bigger than $\mathfrak{g}$.

I have attacked the question above for $n = 2$ and $n = 3$. For $n = 2$ the *only* case that a local homogeneous subset $\Sigma \subset T\mathbb{R}^2$ is not induced by an endowed 2-dimensional Lie algebra $(\mathfrak{g}, \sigma)$, with Abelian or non-Abelian $\mathfrak{g}$, is the case that $sym(\Sigma) = \mathfrak{so}_3(\mathbb{R})$. The main example of such Σ is a field of ellipses in $T\mathbb{R}^2$ defining a Riemannian metrics with a constant positive curvature.

I have also proved that *any* local homogeneous subset of $T\mathbb{C}^2$ (the holomorphic part of the tangent bundle) is induced by an endowed 2-dimensional complex Lie algebra. In the definition of local homogeneous subset of the holomorphic part of $T\mathbb{C}^n$ and in the construction of Sect. 1.3 the analytic diffeomorphisms should be replaced by biholomorphisms.

As one can expect, the case $n = 3$ is much more difficult than $n = 2$. I conjectured that like for $n = 2$ any local homogeneous subset of the holomorphic part of $T\mathbb{C}^3$ is induced by some endowed 3-dimensional complex Lie algebras. Recently I found few counterexamples, with infinite-dimensional symmetry algebras.

The results answering the question formulated in the beginning of this subsection for $n = 2, 3$ will be published elsewhere. The present paper is only a small contribution to this question. It is devoted to the case that $\Sigma \subset T\mathbb{R}^n$ is an affine line field.

1.5 Local homogeneous affine line fields. Main theorems

Consider the case that $\Sigma \subset T\mathbb{R}^n$ is an *affine line field*, i. e. $\Sigma \cap T_x\mathbb{R}^n$ is a *straight line in* $T_x\mathbb{R}^n$ *which does* <u>*not*</u> *contain* $0 \in T_x\mathbb{R}^n$, for any $x \in \mathbb{R}^n$.

Theorem 1. *Let* $n = 2$ *or* $n = 3$. *Any local homogeneous affine line field in* $T\mathbb{R}^n$ *is induced by an* n*-dimensional Lie algebra* $\mathfrak{g}$ *endowed with an affine line* $\ell \subset \mathfrak{g}$.

Theorem 2. *There are homogeneous affine line fields in* $T\mathbb{R}^4$ *which are not induced by any endowed 4-dimensional Lie algebra* $(\mathfrak{g}, \ell)$ *where* $\ell \subset \mathfrak{g}$ *is an affine line.*

A local affine line field $L \subset T\mathbb{R}^n$ can be described by two non-vanishing vector field germs A and B, where B defines the direction of L and A is a "drift" vector field. In geometric control theory L is called a germ of a single input control affine system, defined up to feedback transformation. We will write $L = A + (B)$. The words "feedback transformation" correspond to the fact that the vector field germs A, B are defined by L up to transformations

$$\begin{gathered} B \to Q_1 B, \quad A \to \; A + Q_2 B, \\ Q_1, Q_2 \text{ function germs}, \;\; Q_1(0) \neq 0 \end{gathered} \tag{1}$$

Theorems 1 and 2 imply the following corollary.

Corollary 1. *Let $n = 2$ or $n = 3$ and let L be a local homogeneous affine line field in $T\mathbb{R}^n$. There are vector field germs A and B which generate an n-dimensional Lie algebra such that $L = A + (B)$. This statement does not hold for $n = 4$.*

1.6 Plan of the paper

In Sect. 2 we return to the general case, like in Sects. 1.1–1.4. We present general conceptual results which are the tools used in the proofs of Theorems 1 and 2. Illustrating the tools we prove a simple part of Theorem 1, the case $n = 2$. In Sect. 3 we prove Theorem 1 for $n = 3$. In Sect. 4 we use Theorem 1 to classify all local homogeneous affine line fields in $T\mathbb{R}^3$. In Sect. 5 we prove Theorem 2 and announce main results on the classification of local homogeneous affine line fields in $T\mathbb{R}^4$.

2 Tools

2.1 Splitting property of transitive Lie algebras

Any Lie algebra of vector fields germs at $0 \in \mathbb{R}^n$ has an important subalgebra, called the isotropy subalgebra.

Definition 3. The isotropy subalgebra of a Lie algebra A of vector fields germs at $0 \in \mathbb{R}^n$ is the subalgebra of A consisting of $V \in A$ such that $V(0) = 0$.

The question formulated in the beginning of Sect. 1.4 is tied with the following property of a transitive Lie algebra of vector field germs.

Definition 4. Let A be a transitive Lie algebra A of vector fields germs at $0 \in \mathbb{R}^n$ with the isotropy subalgebra I. We will say that A has the splitting property if there exists an n-dimensional Lie algebra $\mathfrak{g} \subset A$ such that $A = \mathfrak{g} + I$, meaning the direct sum of vector spaces, not necessarily of Lie algebras.

Proposition 2. *A local homogeneous subset $\Sigma \subset T\mathbb{R}^n$ is induced by an endowed n-dimensional Lie algebra if and only if the symmetry algebra of Σ has the splitting property.*

Proof. Assume that Σ is induced by an endowed n-dimensional Lie algebra $(\mathfrak{g}, \sigma)$. To prove that the Lie algebra $sym(\Sigma)$ has the splitting property it is sufficed to prove that the Lie algebra $sym\widehat{(\mathfrak{g},\sigma)}$, consisting of vector field germs at the identity of the Lie group of $\mathfrak{g}$, has the splitting property. The latter is clear because $\mathfrak{g}$, considered as the Lie algebra of left-invariant vector fields, belongs to $sym\widehat{(\mathfrak{g},\sigma)}$.

Assume now that $sym(\Sigma)$ has the splitting property: $sym(\Sigma) = \mathfrak{g} + I$, where $\mathfrak{g}$ is an n-dimensional Lie algebra spanned by non-vanishing vector field germs and I is the isotropy subalgebra of $sym(\Sigma)$. In this case we can construct Σ by pushing forward the set $\Sigma_0 = \Sigma \cap T_0\mathbb{R}^n$ by the flows of vector fields in $\mathfrak{g} \subset sym(\Sigma)$, "forgetting" about the isotropy subalgebra I. Take, as in the proof of Proposition 1, an isomorphism $i : T_0\mathbb{R}^n \to \mathfrak{g}$ sending a tangent vector $v \in T_0\mathbb{R}^n$ to the unique $V \in \mathfrak{g}$ such that $V(0) = v$. Then Σ is diffeomorphic to $\widehat{(\mathfrak{g},\sigma)}$ with $\sigma = i(\Sigma_0)$. □

2.2 Proof of Theorem 1 for n = 2

Let us illustrate Proposition 2 by proving Theorem 1 for $n = 2$.

Notation 1. *By* $\mathfrak{a}_2$ *we denote the Abelian 2-dimensional Lie algebra and by* $\mathfrak{b}_2$ *the non-Abelian 2-dimensional Lie algebra.*

A local affine line field L in $T\mathbb{R}^2$ can be described by equation $\alpha = 1$, where α is a differential 1-form: $L = \{v \in T_x\mathbb{R}^2 : \alpha(v) = 1\}$ where x is a point close to 0. For local homogeneous affine line fields there are the following and only the following possibilities:

(a) $d\alpha(0) \neq 0$;
(b) $d\alpha \equiv 0$ in a neighborhood of 0.

By the simplest part of Darboux theorem we have, in some local coordinates, $\alpha = (1 + x_2)dx_1$ in case (a) and $\alpha = dx_1$ in case (b).

In terms of the coordinates of these normal forms, in case (a) the symmetry algebra consists of vector fields of the form $f(x_1)\frac{\partial}{\partial x_1} - (1 + x_2)f'(x_1)\frac{\partial}{\partial x_2}$ and its isotropy subalgabra consists of vector fields of the same form with $f(x_1) = x_1^2 h(x_1)$. The isotropy subalgebra has a complement $\mathfrak{b}_2 = span\left\{\frac{\partial}{\partial x_1}, x_1\frac{\partial}{\partial x_1} - (1 + x_2)\frac{\partial}{\partial x_2}\right\}$. Therefore L is induced by $\mathfrak{b}_2$ endowed with a certain affine line.

In case (b) the the symmetry algebra consists of vector fields of the form $r\frac{\partial}{\partial x_1} + g(x_1, x_2)\frac{\partial}{\partial x_2}$, $r \in \mathbb{R}$. Its isotropy subalgebra consists of the vector fields of the same form with $r = 0$ and $g(0) = 0$. It has the Abelian complement $\mathfrak{a}_2 = span\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. Therefore L is induced by $\mathfrak{a}_2$ endowed with a certain affine line.

2.3 Classification of local homogeneous subsets of $T\mathbb{R}^n$ versus classification of endowed Lie algebras

Definition 5. Two endowed n-dimensional Lie algebras $(\mathfrak{g}_1, \sigma_1)$ and $(\mathfrak{g}_2, \sigma_2)$ are isomorphic if there exists an isomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ which sends σ_1 onto σ_2.

Our constructions imply the following statement.

Proposition 3. *Let Σ_1 and Σ_2 be local homogeneous subsets of $T\mathbb{R}^n$ induced by endowed n-dimensional Lie algebras $(\mathfrak{g}_1, \sigma_1)$ and $(\mathfrak{g}_2, \sigma_2)$:*

1) *if these endowed Lie algebras are isomorphic then Σ_1 and Σ_2 are diffeomorphic, i. e. there exists a local diffeomorphism $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\Phi_* \Sigma_1 = \Sigma_2$;*
2) *if $dim\,sym(\Sigma_1) = dim\,sym(\Sigma_2) = n$ and Σ_1 and Σ_2 are diffeomorphic then the endowed Lie algebras $(\mathfrak{g}_1, \sigma_1)$ and $(\mathfrak{g}_2, \sigma_2)$ are isomorphic.*

The second statement does not hold if $dim\,sym(\Sigma_1) = dim\,sym(\Sigma_2) > n$. The simplest example is the classification of homogeneous affine line fields in $T\mathbb{R}^2$, see Sect. 2.2. It is discrete (we have exactly two normal forms without parameters) whereas the classification of affine lines in $\mathfrak{b}_2$ is not discrete. In fact, an affine line in $\mathfrak{b}_2 = span\{x, y\}, [x, y] = x$ can be brought by an automorphism of $\mathfrak{b}_2$ to one of the normal forms $\ell = x + span\{y\}$ or $\ell = \lambda y + span\{x\}$ and in the second normal form λ is a modulus.

2.4 Nagano principle

Nagano gave the following definition of an *abstract* transitive Lie algebra.

Definition 6 (Nagano, [2]). An abstract transitive Lie algebra is a couple (A, I) where A is a Lie algebra and I is a subalgebra of A which does not contain non-trivial ideals of the whole A.

Proposition 4 (Nagano, [2]). *Any transitive Lie algebra of analytic vector field germs at $0 \in \mathbb{R}^n$ is an abstract transitive Lie algebra according to Definition 6.*

The following statement is a direct corollary of this proposition (it is not hard to prove it independently).

Proposition 5. *The isotropy subalgebra of a transitive Lie algebra of analytic vector field germs at $0 \in \mathbb{K}^n$ is trivial (consists of zero vector field only) if and only if it does not contain non-zero vector fields with zero linear approximation at 0.*

In fact, the absence of vector fields with the zero linear approximation in a transitive Lie algebra of vector field germs implies that the whole isotropy subalgebra is an ideal and therefore by Proposition 4 it must be trivial.

2.5 Finite dimensional transitive Lie algebras of vector fields

Proposition 6 below is a particular case of H. Sussmann's theorem, continuing the Nagano principle, on the relation between transitive Lie algebras of vector fields and abstract transitive Lie algebras according to Nagano's definition.

Proposition 6 (a particular case of H. Sussmann's theorem in [3]). *Two finite-dimensional transitive Lie algebras A_1 and A_2 of analytic vector field germs at $0 \in \mathbb{R}^n$ with isotropy subalgebras I_1 and I_2 are diffeomorphic, i. e. can be sent one to the other by a local diffeomorphism of $(\mathbb{R}^n, 0)$, if and only if (A_1, I_1) and (A_2, I_2) are isomorphic as abstract transitive Lie algebras, i. e. there exists an isomorphism from A_1 to A_2 which sends I_1 onto I_2.*

3 Proof of Theorem 1

We use the notations from Sect. 1.5. An affine line field $L = A + (B)$ will be called *bracket generating* if taking sufficiently many Lie brackets of A and B we obtain the whole tangent bundle.

At first consider a very simple case that a *homogeneous* affine line field in $T\mathbb{R}^3$ is *not* bracket generating. It is so if and only if L belongs to one of the following classes:
1. $[A, B](x) \in span\ \{B(x)\}$;
2. $[A, B](0) \notin span\ \{B(0)\},\ [A, B](x) \in span\ \{A(x), B(x)\}$

for *any* x close to 0. These classes are well defined, i. e. do not depend on the choice of vector fields describing an affine line field.

The cases **1** and **2** are very simple. By transformations (1) we can replace A and B by new vector fields such that $[A, B] = 0$ in case **1** and $[A, B] = -A$ in case **2**. By Proposition 6, we can take any vector fields satisfying these equations to get a normal form for L, for example

$$\text{case } \mathbf{1}: \ \frac{\partial}{\partial x_1} + \left(\frac{\partial}{\partial x_2}\right); \quad \text{case } \mathbf{2}: \ e^{x_2}\frac{\partial}{\partial x_1} + \left(\frac{\partial}{\partial x_2}\right). \tag{2}$$

The vector fields in these normal forms span a 2-dimensional Lie algebra. Therefore L is induced by an endowed Lie algebra $(\mathfrak{g}, \ell)$ where $\mathfrak{g} = \mathbb{R}^3$ in case **1**, $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{b}_2$ in case **2**, and ℓ is an affine line in $\mathfrak{g}$.

Now consider the case that L is bracket generating. In this case L belongs to one of the following classes:
A. $[A, B](0) \notin span\ \{A(0), B(0)\}$,
$[B, [A, B]](0) \notin \text{span}\ \{B(0), [A, B](0)\}$;
B. $[A, B](0) \notin span\ \{A(0), B(0)\}$,
$[B, [A, B]](x) \in \text{span}\ \{B(x), [A, B](x)\}$ for any x close to 0,
$[A, [A, B]](x) \in \text{span}\ \{B(x), [A, B]](x)\}$ for any x close to 0;

C. $[A, B](0) \notin span\ \{A(0), B(0)\}$,
$[B, [A, B]](x) \in \text{span}\ \{B(x), [A, B](x)\}$ for any x close to 0,
$[A, [A, B]](0) \notin \text{span}\ \{B(0), [A, B](0)\}$.

Again, these classes are well defined, i. e. the given conditions depend on L only. In case **A** we need the following statement.

Lemma 1. *If L belongs to class* **A** *then any infinitesimal symmetry of L which vanishes at* 0 *has zero linear approximation.*

Lemma 1 and Proposition 5 imply that $sym(L)$ is a 3-dimensional Lie algebra. By Proposition 3 the classification of class **A** is the same problem as classification with respect to isomorphisms of certain endowed 3-dimensional Lie algebras. In Sect. 4 we specify *which* ones.

In case **B** Theorem 1 follows from the following statement.

Lemma 2. *Any homogeneous affine line field of class* **B** *is diffeomorphic to*

$$\textit{class}\ \mathbf{B}: \quad \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + \left(\frac{\partial}{\partial x_3}\right). \tag{3}$$

The vector fields $A = \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}$ and $B = \frac{\partial}{\partial x_3}$ in this normal form generate the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_1$. Therefore any homogeneous L of class **B** is induced by an endowed Lie algebra $(\mathfrak{h}_1, \ell)$ where ℓ is an affine line in $\mathfrak{h}_1$.

Class **C** is harder for analysis. Within homogeneous affine line fields it also consists of one orbit, and we will use the following normal form.

Lemma 3. *Any homogeneous affine line field of class* **C** *is diffeomorphic to*

$$\textit{class}\ \mathbf{C}: \quad e^{x_2}\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + \left(\frac{\partial}{\partial x_3}\right). \tag{4}$$

In this case we cannot we cannot deduce Theorem 1 from normal form (4) in the same way as from normal form (3) because the vector fields $A = e^{x_2}\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}$ and $B = \frac{\partial}{\partial x_3}$ in (4) do not generate a 3-dimensional Lie algebra. One of the ways to prove Theorem 1 for case **C** is to find another normal form $\widetilde{A} + (\widetilde{B})$ with vector fields $\widetilde{A}, \widetilde{b}$ generating a 3-dimensional Lie algebra. Such normal form exists, but it is rather involved. In fact, to prove the existence of such normal form is the same task as to prove Theorem 1.

We will use another way: we will show that the symmetry algebra of normal form (4) has the splitting property given in Sect. 2.1. It is easy to compute that the symmetry algebra of (4) is

$$sym(L) = \left\{ f(x_1)\frac{\partial}{\partial x_1} + f'(x_1)\frac{\partial}{\partial x_2} + e^{x_2} f''(x_1)\frac{\partial}{\partial x_3} \right\}, \tag{5}$$

where $f(x_1)$ is an arbitrary function. The isotropy part of (5) consists of vector fields with $f(x_1)$ such that $f(0) = f'(0) = f''(0) = 0$. Taking $f(x_1) = 1,\ f(x_1) = x_1$, and $f(x_1) = x_1^2$ we obtain the complement

$$a_1 = \frac{\partial}{\partial x_1},\ a_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_1},$$
$$a_3 = 2e^{x_2}\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_2} + x_1^2\frac{\partial}{\partial x_1}. \tag{6}$$

We have $[a_1, a_2] = a_1,\ [a_1, a_3] = a_2,\ [a_2, a_3] = a_3$. We see that the vector fields a_1, a_2, a_3 span the $\mathfrak{sl}_2$ Lie algebra. Therefore any homogeneous affine line field L of class **C** is induced by an endowed Lie algebra $(\mathfrak{sl}_2, \ell)$ with some affine line $\ell \subset \mathfrak{sl}_2$.

Remark 1. In the next section we will show that the direction of ℓ is a special direction in $\mathfrak{sl}_2$ and we will explain what does it mean. We will also show that L is not induced by an endowed Lie algebra $(\mathfrak{g}, \ell)$ with $\mathfrak{g} \neq \mathfrak{sl}_2$.

Remark 2. The fact that vector fields (6) span $\mathfrak{sl}_2$ looks a bit mysterious. It is not so. The symmetry algebra (5) is isomorphic to the Lie algebra $Vect(1) = \left\{f(x)\frac{\partial}{\partial x}, x \in \mathbb{R}\right\}$. The isomorphism is simply the map sending a vector field (5) to the vector field $f(x)\frac{\partial}{\partial x}$. This isomorphism sends the isotropy subalgebra of (5) to the Lie algebra of vector fields $\left\{f(x)\frac{\partial}{\partial x},\ f(0) = f'(0) = f''(0) = 0\right\}$. Its complement $\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right\}$ in $Vect(1)$ is the classical realization of $\mathfrak{sl}_2$.

Now we will prove Lemmas 1 and 3. We will use techniques developed in [4]. As for Lemma 2, it is substantially simpler than Lemma 3 and can be proved by the same techniques; we omit its proof.

3.1 *Proof of Lemma 1*

The first condition describing class **A** implies that L can be described by vector fields

$$B = \frac{\partial}{\partial x_3}, \quad A = \left(1 + r_1x_3^2 + r_2x_1x_3 + f(x)\right)\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2},\ \ j^2 f = 0, \tag{7}$$

If $Z \in sym(L)$ and $Z(0) = 0$ then Z has the form

$$Z = \phi_1(x_1, x_2)\frac{\partial}{\partial x_1} + \phi_2(x_1, x_2)\frac{\partial}{\partial x_2} + \xi(x)\frac{\partial}{\partial x_3},\ \ \phi_1(0) = \phi_2(0) = \xi(0) = 0$$

and satisfies the equation $[Z, A] = 0 \mod \left(\frac{\partial}{\partial x_3}\right)$. What we need from this equation is its 2-jet:

$$j^2[Z, A] = 0\, mod\, \left(\frac{\partial}{\partial x_3}\right). \tag{8}$$

The second condition describing class **A** implies that $r_1 \neq 0$ in (7). It is easy to check that (8) and the condition $r_1 \neq 0$ imply $j^1\phi_1 = j^1\phi_2 = j^1\xi = 0$.

3.2 *Proof of Lemma 3*

We have the same preliminary normal form (7) following from the first condition describing class **C**. In fact, this condition implies a stronger normal form: we can reduce $f(x)$ to a function in the ideal (x_3^2, x_1x_3). The second condition describing class **C** implies $\frac{\partial^2 f}{\partial x_3^2} \equiv 0$. Therefore we obtain the normal form

$$B = \frac{\partial}{\partial x_3}, \quad A = (1 + r_2x_1x_3 + x_3g(x_1))\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}, \quad j^1g = 0. \tag{9}$$

Now we use the third condition describing class **C**. It implies $r_2 \neq 0$. We can scale r_2 to 1. Using the homotopy method (techniques can be found in [4]) one can prove that $g(x_1)$ can be reduced to 0 by a local diffeomorphism of the form $x_1 \to \Phi_1(x_1), x_2 \to \Phi_2(x_1), x_3 \to x_3R_1(x_1)+R_2(x_1)$. We obtain that all homogeneous affine line fields of class **C** are diffeomorphic to the affine line field

$$A + (B), \quad A = (1 + x_1x_3)\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}, \quad B = \frac{\partial}{\partial x_3}. \tag{10}$$

The fact that all L of class **C** are diffeomorphic allows us to replace A and B in this normal form by any other couple A, B satisfying the conditions describing class **C**. The vector fields $A = e^{x_2}\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}$ and $B = \frac{\partial}{\partial x_3}$ in normal form (4) satisfy these conditions.

Remark 3. Certainly we could work with normal form (10), but (4) is more convenient for computing the symmetry algebra. Note that neither in (10) nor in (4) the vector fields A and B generate a 3-dimensional Lie algebra. It is so for another normal form obtained by presentation of certain vectors in $\mathfrak{sl}_2$ as left-invariant vector fields in local coordinates of SL_2. It is not hard to display these vector fields, but there no need to do so.

4 Complete classification of homogeneous affine line fields in $T\mathbb{R}^3$

Now we can classify all local homogeneous affine line fields in $T\mathbb{R}^3$ in terms of normal forms for an affine line $\ell \subset \mathfrak{g}$ in certain 3-dimensional Lie algebras $\mathfrak{g}$, with respect to the group of automorphisms of $\mathfrak{g}$. We write an affine line in $\mathfrak{g}$ in the form $\ell = a + (b)$ where $a, b \in \mathfrak{g}$. We will say that ℓ is generating if $\mathfrak{g} = span\{a, b, [a, b]\}$.

The description of classes **A,B,C** in Sect. 3 and a simple work with 3-dimensional Lie algebras imply the following statement.

Table 1 Homogeneous affine lines in $T\mathbb{R}^3$ and endowed 3-dimensional Lie algebras

Class	All possible cases for $(\mathfrak{g}, \ell)$ with generating $\ell = a + (b0$
A	A1. $\mathfrak{g} = \mathfrak{so}_3$, any generating ℓ A2. $\mathfrak{g} = \mathfrak{sl}_2,\ b \notin [b, \mathfrak{g}]$ A3. $dim\, \mathfrak{g}^2 = 2,\ b \notin \mathfrak{g}^2$
B	B1. $dim\, \mathfrak{g}^2 = 1,\ [\mathfrak{g}, \mathfrak{g}^2] = 0$, any generating ℓ B2. $dim\, \mathfrak{g}^2 = 1,\ [\mathfrak{g}, \mathfrak{g}^2] \neq 0$, any generating ℓ B3. $dim\, \mathfrak{g}^2 = 2,\ b \in \mathfrak{g}^2$
C	$\mathfrak{g} = \mathfrak{sl}_2,\ b \in [b, \mathfrak{g}]$

Theorem 3. *Let $(\mathfrak{g}, \ell)$ be a 3-dimensional Lie algebra endowed with a generating affine line $\ell \subset \mathfrak{g}$, so that the induced homogeneous affine line field $L = \widehat{(\mathfrak{g}, \ell)}$ is bracket generating. Then L has type* **A**, **B** *or* **C** *given in Sect. 3 if and only if $(\mathfrak{g}, \ell)$ satisfies one of the conditions given in the corresponding row of Table 1.*

We see that when $\mathfrak{g}$ is fixed, the type of the endowed Lie algebra $(\mathfrak{g}, \ell)$ is defined by the direction (b) of ℓ only. If $\mathfrak{g} = \mathfrak{sl}_2$ then the direction is "special" if $b \in [b, \mathfrak{g}]$. There are no special directions in real $\mathfrak{so}_3$ (all directions are automorphic).

Remark 4. An equivalent definition of a special direction $(b) \subset \mathfrak{sl}_2$ is the condition that the operator $ad(b)$ is nilpotent.

The classification of real endowed Lie algebras of class **A** is given in Table 2.
Now we have a complete classification of all homogeneous affine line fields in $T\mathbb{R}^3$.
The results of Sects. 1–3 imply the following classification.

Theorem 4.

1) *Any local homogeneous affine line field in $T\mathbb{R}^3$ of class* **A** *is diffeomorphic to the affine line field $\widehat{(\mathfrak{g}, \ell)}$ where $(\mathfrak{g}, \ell)$ is one and only one of the normal forms*

Table 2 Classification, with respect to isomorphisms, of real 3-dimensional endowed Lie algebras $(\mathfrak{g}, \ell)$ of type **A**

Type of an endowed Lie algebra $(\mathfrak{g}, \ell),\ \ell = a + (b)$ of class **A**	Normal form with respect to isomorphisms. λ is a modulus
A1. $\mathfrak{g} = \mathfrak{so}_3$ any generating ℓ	$\mathfrak{g}$: $[x, y] = z,\ [y, z] = x,\ [z, x] = y$ $\ell = \lambda x + (y)$
A2. $\mathfrak{g} = \mathfrak{sl}_2,\ b \notin [b, \mathfrak{g}]$	$\mathfrak{g}$: $[x, y] = z,\ [z, x] = x,\ [z, y] = -y$ $\ell = \lambda x + (x \pm y)$ and $\ell = \lambda(x + y) + (z)$
A3. $dim\, \mathfrak{g}^2 = 2,\ b \notin \mathfrak{g}^2$	$\mathfrak{g}$: $[x, y] = 0,\ [z, x] = y,\ [z, y] = \pm x + \lambda y$ $\ell = x + (z)$

for endowed 3-dimensional Lie algebras given in the second column of Table 2. The parameter λ in these normal forms is a modulus of L with respect to diffeomorphisms.

2) *All local homogeneous affine line field in $T\mathbb{R}^3$ of class* **B** *are diffeomorphic. They can be described by the normal form $\widehat{(\mathfrak{g}, \ell)}$ where the endowed Lie algebra $(\mathfrak{g}, \ell)$ satisfies one of conditions B1, B2, B3 in Table 1 (all such normal forms are diffeomorphic). They also can be described by the normal form* (3).
3) *All local homogeneous affine line field in $T\mathbb{R}^3$ of class* **C** *are diffeomorphic. They can be described by the normal form $\widehat{(\mathfrak{sl}_2, \ell)}$ where $\ell = a + (b) \subset \mathfrak{sl}_2$ is an affine line whose direction (b) satisfies the condition that the operator $ad(b)$ is nilpotent. They also can be described by the normal form* (4).
4) *In the remaining case that a local homogeneous affine line field in $T\mathbb{R}^3$ is not bracket generating one has one of the normal forms* (2). *The first normal form is diffeomorphic to $\widehat{(\mathbb{R}^3, \ell)}$ where ℓ is any affine line. The second normal form is diffeomorphic to $\widehat{(\mathfrak{g}, \ell)}$ where $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{b}_2$ and the affine line $\ell = a + (b)$ satisfies the condition $b \notin \mathfrak{g}^2$.*

We see that the classification of the class **B** for endowed 3-dimensional Lie algebras is much more involved than the classification of class **B** for homogeneous affine line fields. On the other hand, it is not hard to prove that all endowed Lie algebras of class **C** are isomorphic.

5 Classification of homogeneous bracket generating affine line fields in $T\mathbb{R}^4$ and proof of Theorem 2

Theorem 2 follows from the following claim.

Lemma 4. *The affine line field*

$$L = e^{x_2}\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + \left(\frac{\partial}{\partial x_4}\right). \tag{11}$$

is homogeneous. Its symmetry algebra $sym(L)$ is ∞-dimensional and does **not** *have the splitting property given in Sect. 2.1.*

This lemma is proved below. Before that we announce some results on the classification of local homogeneous affine line fields in $T\mathbb{R}^4$.

Lemma 5. *Any local homogeneous bracket generating affine line field $L = A + (B)$ in $T\mathbb{R}^4$ satisfies one of the the following conditions (which are the properties of L only, i. e. do not depend on the choice of vector fields A and B):*

A. $[B, [A, B]](0) \notin span\,\{B(0), [A, B](0)\}$;

B. $[B, [A, B]](x) \in span\,\{B(x), [A, B](x)\}$,
$[A, [A, [A, B]]](x) \in span\,\{B(x), [A, B](x), [A, [A, B]](x)\}$
for any x close to 0;

C. $[B,[A,B]](x) \in span\,\{B(x),[A,B](x)\}$ *for any x close to* 0,
$[A,[A,[A,B]]](0) \notin span\,\{B(0),[A,B](0),[A,[A,B]](0)\}$,
$[B,[A,[A,[A,B]]]](x) \in span\,\{B(x),[A,B](x),[A,[A,B]](x)\}$
for any x close to 0.

The normal form (11) serves for the whole class **C** and it is the only case that L is not induced by a 4-dimensional Lie algebra endowed with an affine line field.

The class **B** also consists of one orbit which can be described by the normal form

$$\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + \left(\frac{\partial}{\partial x_4}\right).$$

The symmetry algebras of this affine line field is ∞-dimensional.

The class **A** can be decomposed onto certain classes **A1** and **A2**. Any L of class **A1** has 4-dimensional symmetry algebra and consequently the classification of homogeneous bracket generating line fields of this class is the same problem as classification of certain 4-dimensional Lie algebras endowed with an affine line. Any L of class **A2** is diffeomorphic to the normal form

$$\frac{\partial}{\partial x_1} + \frac{1}{2}\left(x_4^2 + \theta x_3^2\right)\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_3} + \left(\frac{\partial}{\partial x_4}\right)$$

where the parameter θ is a modulus. The symmetry algebra is the 5-dimensional solvable Lie algebra $span\{a_1,a_2,a_3,a_4,w\}$ with the isotropy 1-dimensional subalgebra $span\{w\}$ and the structure equations

$$\begin{gathered}[a_1,a_2]=0,\quad [a_1,a_3]=0,\quad [a_1,a_4]=0,\\ [a_2,a_3]=\theta a_4,\quad [a_2,a_4]=a_3,\quad [a_3,a_4]=a_4\\ [w,a_1]=2a_1,\quad [w,a_2]=0,\quad [w,a_3]=a_3,\quad [w,a_4]=a_4.\end{gathered}$$

The parameter θ can be reduced by an isomorphism to 1 unless $\theta=0$, but θ is a modulus of the *transitive* Lie algebra, i. e. with respect to isomorphims preserving the isotropy subalgebra $span\{w\}$.

Proof of Lemma 4. It is easy to compute that the symmetry algebra of the normal form (11) is as follows:

$$\Big\{f(x_1)\frac{\partial}{\partial x_1} + f'(x_1)\frac{\partial}{\partial x_2} + e^{x_3}f''(x_1)\frac{\partial}{\partial x_3} + h\frac{\partial}{\partial x_4},\\ h = e^{2x_2}f'''(x_1) + e^{x_2}f''(x_1)\Big\}. \tag{12}$$

It is isomorphic to the Lie algebra $Vect(1) = \{f(x)\frac{\partial}{\partial x},\ x \in \mathbb{R}\}$, the isomorphism is simply the map sending a vector field of form (12) to the vector field $f(x)\frac{\partial}{\partial x}$. This isomorphism sends the isotropy subalgebra of (12) to the Lie algebra $U = \{x^4 g(x)\frac{\partial}{\partial x}\} \subset Vect(1)$ where $g(x)$ is an arbitrary function (cf. Remark 2). Any vector-space-complement to U in $Vect(1)$ is spanned by vector fields of the form $a_i = \left(x^i + h_i(x)\right)\frac{\partial}{\partial x},\ \ j^3h(x) = 0,\ i = 0,1,2,3$. The vector field $[a_2, a_3]$ belongs to U. Therefore the vector space $span\{a_0, a_1, a_2, a_3\}$ is not a Lie algebra. It follows that the symmetry algebra (12) does not have the splitting property.

Remark 5. In fact it it easy to prove that $Vect(1)$ does not contain *any* 4-dimensional Lie subalgebra.

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