Chapter 11
Dynamics of Planar Media

In previous chapters, we studied the equilibrium configurations of planar media — plates and membranes — governed by the two-dimensional Laplace and Poisson equations. In this chapter, we analyze their dynamics, modeled by the two-dimensional heat and wave equations. The heat equation describes diffusion of, say, heat energy in a thin metal plate, an animal population dispersing over a region, or a pollutant spreading out into a shallow lake. The wave equation models small vibrations of a two-dimensional membrane such as a drum. Since both equations fit into the general framework for dynamics that we established in Section 9.5, their solutions share many of the general qualitative and analytic properties possessed by their respective one-dimensional counterparts.

Although the increase in dimension may tax our analytical prowess, we have, in fact, already mastered the principal solution techniques: separation of variables, eigenfunction series, and fundamental solutions. When applied to partial differential equations in higher dimensions, separation of variables in curvilinear coordinates often leads to new linear, but non-constant-coefficient, ordinary differential equations, whose solutions are no longer elementary functions. Rather, they are expressed in terms of a variety of important *special functions*, which include the error and Airy functions we encountered earlier; the Bessel functions, which play a starring role in the present chapter; and the Legendre and Ferrers functions, spherical harmonics, and spherical Bessel functions arising in three-dimensional problems. Special functions are ubiquitous in more advanced applications in physics, chemistry, mechanics, and mathematics, and, over the last two hundred and fifty years, many prominent mathematicians have devoted significant effort to establishing their fundamental properties, to the extent that they are now, by and large, well understood, [**86**]. To acquire the requisite familiarity with special functions, in preparation for employing them to solve higher-dimensional partial differential equations, we must first learn basic series solution techniques for linear second-order ordinary differential equations.

11.1 Diffusion in Planar Media

As we learned in Chapter 4, the equilibrium temperature $u(x,y)$ of a thin, uniform, isotropic plate is governed by the two-dimensional Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Working by analogy, the dynamical diffusion of the plate's temperature should be modeled by the two-dimensional heat equation

$$u_t = \gamma\, \Delta u = \gamma\, (u_{xx} + u_{yy}). \tag{11.1}$$

P.J. Olver, *Introduction to Partial Differential Equations*, Undergraduate Texts in Mathematics,
DOI 10.1007/978-3-319-02099-0_11,

The coefficient $\gamma > 0$, assumed constant, measures the relative speed of diffusion of heat energy throughout the plate; its positivity is required on physical grounds, and also serves to avoid ill-posedness inherent in running diffusion processes backwards in time. In this model, we are assuming that the plate is uniform and isotropic, and experiences no loss of heat or external heat sources other than at its edge — which can be arranged by covering its top and bottom with insulation.

The solution $u(t, \mathbf{x}) = u(t, x, y)$ to the heat equation measures the temperature, at time t, at each point $\mathbf{x} = (x, y)$ in the (bounded) domain $\Omega \subset \mathbb{R}^2$ occupied by the plate. To uniquely specify the solution $u(t, x, y)$, we must impose suitable initial and boundary conditions. The initial data is the temperature of the plate

$$u(0, x, y) = f(x, y), \qquad (x, y) \in \Omega, \tag{11.2}$$

at an initial time, which for simplicity, we take to be $t_0 = 0$. The most important boundary conditions are as follows:

- *Dirichlet boundary conditions*: Specifying

$$u = h \qquad \text{on} \qquad \partial\Omega \tag{11.3}$$

  fixes the temperature along the edge of the plate.
- *Neumann boundary conditions*: Let $\mathbf{n}$ be the unit outwards normal on the boundary of the domain. Specifying the normal derivative of the temperature,

$$\frac{\partial u}{\partial \mathbf{n}} = k \qquad \text{on} \qquad \partial\Omega, \tag{11.4}$$

  effectively prescribes the heat flux along the boundary. Setting $k = 0$ corresponds to an insulated boundary.
- *Mixed boundary conditions*: More generally, we can impose Dirichlet conditions on part of the boundary $D \subsetneq \partial\Omega$ and Neumann conditions on its complement $N = \partial\Omega \setminus D$. For instance, homogeneous mixed boundary conditions

$$u = 0 \qquad \text{on} \qquad D, \qquad\qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \qquad \text{on} \qquad N, \tag{11.5}$$

  correspond to freezing a portion of the boundary and insulating the remainder.
- *Robin boundary conditions*:

$$\frac{\partial u}{\partial \mathbf{n}} + \beta\, u = \tau \qquad \text{on} \qquad \partial\Omega, \tag{11.6}$$

  where the edge of the plate sits in a heat bath at temperature τ.

Under reasonable assumptions on the domain, the initial data, and the boundary data, a general theorem, [**34**, **38**, **99**], guarantees the existence of a unique solution $u(t, x, y)$ to any of these initial-boundary value problems for all subsequent times $t > 0$. Our practical goal is to both compute and understand the behavior of the solution in specific situations.

Derivation of the Diffusion and Heat Equations

The physical derivation of the two-dimensional (and three-dimensional) heat equation relies on the same basic thermodynamic laws that were used, in Section 4.1, to establish the one-dimensional version. The first principle is that heat energy flows from hot to cold as rapidly as possible. According to multivariable calculus, [**8**, **108**], the negative temperature gradient $-\nabla u$ points in the direction of the steepest decrease in the temperature function u at a point, and so heat energy will flow in that direction. Therefore, the heat flux vector $\mathbf{w}$, which measures the magnitude and direction of the flow of heat energy, should be proportional to the temperature gradient:

$$\mathbf{w}(t,x,y) = -\,\kappa(x,y)\,\nabla u(t,x,y). \tag{11.7}$$

The scalar quantity $\kappa(x,y) > 0$ measures the *thermal conductivity* of the material, so (11.7) is the multi-dimensional form of *Fourier's Law of Cooling* (4.5). We are assuming that the thermal conductivity depends only on the position $(x,y) \in \Omega$, which means that the material in the plate

(a) is not changing in time;

(b) is *isotropic*, meaning that its thermal conductivity is the same in all directions;

(c) and, moreover, its thermal conductivity is not affected by any change in temperature.

Dropping either assumption (b) or (c) would result in a considerably more challenging nonlinear diffusion equation.

The second thermodynamic principle is that, in the absence of external heat sources, heat can enter any subregion $R \subset \Omega$ only through its boundary ∂R. (Keep in mind that the plate is insulated from above and below.) Let $\varepsilon(t,x,y)$ denote the heat energy density at each time and point in the domain, so that

$$H_R(t) = \iint_R \varepsilon(t,x,y)\,dx\,dy$$

represents the total heat energy contained within the subregion R at time t. The amount of additional heat energy entering R at a boundary point $\mathbf{x} \in \partial R$ is given by the normal component of the heat flux vector, namely $-\,\mathbf{w}\cdot\mathbf{n}$, where, as always, $\mathbf{n}$ denotes the *outward* unit normal to the boundary ∂R. Thus, the total heat flux entering the region R is obtained by integration along the boundary of R, resulting in the line integral $-\oint_{\partial R} \mathbf{w}\cdot\mathbf{n}\;ds$. Equating the rate of change of heat energy to the heat flux yields

$$\frac{dH_R}{dt} = \iint_R \frac{\partial \varepsilon}{\partial t}\,(t,x,y)\,dx\,dy = -\oint_{\partial R} \mathbf{w}\cdot\mathbf{n}\;ds = -\iint_R \nabla\cdot\mathbf{w}\;dx\,dy,$$

where we applied the divergence form of Green's Theorem, (6.80), to convert the flux line integral into a double integral. Thus,

$$\iint_R \left(\frac{\partial \varepsilon}{\partial t} + \nabla\cdot\mathbf{w} \right) dx\,dy = 0. \tag{11.8}$$

Keep in mind that this result must hold for *any* subdomain $R \subset \Omega$. Now, according to Exercise 11.1.13, the only way in which an integral of a continuous function can vanish for *all* subdomains is if the integrand is identically zero, and so

$$\frac{\partial \varepsilon}{\partial t} + \nabla\cdot\mathbf{w} = 0. \tag{11.9}$$

In this manner, we arrive at the basic *conservation law* relating the heat energy density ε and the heat flux vector $\mathbf{w}$.

As in our one-dimensional model, cf. (4.3), the heat energy density $\varepsilon(t,x,y)$ is proportional to the temperature, so

$$\varepsilon(t,x,y) = \sigma(x,y)\,u(t,x,y), \qquad \text{where} \qquad \sigma(x,y) = \rho(x,y)\,\chi(x,y) \tag{11.10}$$

is the product of the *density* ρ and the *specific heat capacity* χ of the material at the point $(x,y) \in \Omega$. Combining this with the Fourier Law (11.7) and the energy balance equation (11.10) leads to the general two-dimensional *diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{1}{\sigma}\,\nabla \cdot \big(\,\kappa\,\nabla u\,\big) \tag{11.11}$$

governing the thermodynamics of an isotropic medium in the absence of external heat sources or sinks. In full detail, this second-order partial differential equation is

$$\frac{\partial u}{\partial t} = \frac{1}{\sigma(x,y)}\left[\,\frac{\partial}{\partial x}\left(\kappa(x,y)\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa(x,y)\,\frac{\partial u}{\partial y}\right)\right]. \tag{11.12}$$

Such diffusion equations are also used to model movements of populations, e.g., bacteria in a petri dish or wolves in the Canadian Rockies, [**81**, **84**]. Here the solution $u(t,x,y)$ represents the population density at position (x,y) at time t, which diffuses over the domain due to random motions of the individuals. Similar diffusion processes model the mixing of solutes in liquids, with the diffusion induced by the random Brownian motion from molecular collisions. More generally, diffusion processes in the presence of chemical reactions and convection due to fluid motion are modeled by the more general class of *reaction-diffusion* and *convection-diffusion equations*, [**107**].

In particular, if the body (or the environment or the solvent) is uniform, then both σ and κ are constant, and so (11.11) reduces to the heat equation (11.1) with *thermal diffusivity*

$$\gamma = \frac{\kappa}{\sigma} = \frac{\kappa}{\rho\,\chi}\,. \tag{11.13}$$

Both the heat and more general diffusion equations are examples of *parabolic* partial differential equations, the terminology being adapted from Definition 4.12 to apply to partial differential equations in more than two variables. As we will see, all the basic qualitative features of solutions to the one-dimensional heat equation carry over to parabolic partial differential equations in higher dimensions.

Indeed, the general diffusion equation (11.12) can be readily fit into the self-adjoint dynamical framework of Section 9.5, taking the form

$$u_t = -\,\nabla^* \circ \nabla u. \tag{11.14}$$

The gradient operator ∇ maps scalar fields u to vector fields $\mathbf{v} = \nabla u$; its adjoint ∇^*, which goes in the reverse direction, is taken with respect to the weighted inner products

$$\langle\, u\,,\widetilde{u}\,\rangle = \iint_\Omega u(x,y)\,\widetilde{u}(x,y)\,\sigma(x,y)\,dx\,dy, \quad \langle\!\langle\, \mathbf{v}\,,\widetilde{\mathbf{v}}\,\rangle\!\rangle = \iint_\Omega \mathbf{v}(x,y)\cdot\widetilde{\mathbf{v}}(x,y)\;\kappa(x,y)\,dx\,dy, \tag{11.15}$$

between, respectively, scalar and vector fields. As in (9.33), a straightforward integration by parts tells us that

$$\nabla^*\mathbf{v} = -\,\frac{1}{\sigma}\,\nabla\cdot(\kappa\,\mathbf{v}) \;=\; -\,\frac{1}{\sigma}\left[\,\frac{\partial(\kappa\,v_1)}{\partial x} + \frac{\partial(\kappa\,v_2)}{\partial y}\,\right], \qquad \text{when} \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{11.16}$$

Therefore, the right-hand side of (11.14) equals

$$-\nabla^* \circ \nabla u = \frac{1}{\sigma}\, \nabla \cdot (\kappa\, \nabla u), \tag{11.17}$$

which thereby recovers the general diffusion equation (11.11). As always, the validity of the adjoint formula (11.16) rests on the imposition of suitable homogeneous boundary conditions: Dirichlet, Neumann, mixed, or Robin.

In particular, to obtain the heat equation, we take σ and κ to be constant, and so the inner products (11.15) reduce, up to a constant factor, to the usual L^2 inner products between scalar and vector fields. In this case, the adjoint of the gradient is, up to a scale factor, minus the divergence: $\nabla^* = -\gamma\, \nabla \cdot$, where $\gamma = \kappa/\sigma$. In this scenario, (11.14) reduces to the two-dimensional heat equation (11.1).

Separation of Variables

Let us now discuss analytical solution techniques. According to Section 9.5, the separable solutions to any linear evolution equation

$$u_t = -\, S[\,u\,] \tag{11.18}$$

are of exponential form

$$u(t,x,y) = e^{-\lambda t}\, v(x,y). \tag{11.19}$$

Since the linear operator S involves differentiation with respect to only the spatial variables x, y, we obtain

$$\frac{\partial u}{\partial t} = -\,\lambda\, e^{-\lambda t}\, v(x,y), \qquad \text{while} \qquad S[\,u\,] = e^{-\lambda t}\, S[\,v\,].$$

Substituting back into the diffusion equation (11.18) and canceling the exponentials, we conclude that

$$S[\,v\,] = \lambda\, v. \tag{11.20}$$

Thus, $v(x,y)$ must be an eigenfunction for the linear operator S, subject to the relevant homogeneous boundary conditions.

In the case of the heat equation (11.1),

$$S[\,u\,] = -\,\gamma\, \Delta u,$$

and hence, as in Example 9.40, the eigenvalue equation (11.20) is the two-dimensional *Helmholtz equation*

$$\gamma\, \Delta v + \lambda\, v = 0, \qquad \text{or, in detail,} \qquad \gamma \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \lambda\, v = 0. \tag{11.21}$$

According to Theorem 9.34, self-adjointness implies that the eigenvalues are all real and nonnegative: $\lambda \geq 0$. In the positive definite cases — Dirichlet and mixed boundary conditions — they are strictly positive, while the Neumann boundary value problem admits a zero eigenvalue $\lambda_0 = 0$ corresponding to the constant eigenfunction $v_0(x,y) \equiv 1$.

Let us index the eigenvalues in increasing order:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ \cdots \ , \tag{11.22}$$

repeated according to their multiplicities, where $\lambda_0 = 0$ is an eigenvalue only in the Neumann case, and $\lambda_k \to \infty$ as $k \to \infty$. For each eigenvalue λ_k, let $v_k(x,y)$ be an independent eigenfunction. The corresponding separable solution is

$$u_k(t,x,y) = e^{-\lambda_k t}\, v_k(x,y).$$

Those corresponding to positive eigenvalues are exponentially decaying in time, while a zero eigenvalue produces a constant solution $u_0(t,x,y) \equiv 1$. The general solution to the homogeneous boundary value problem can then be built up as an infinite series in these basic eigensolutions

$$u(t,x,y) = \sum_{k=1}^{\infty} c_k\, u_k(t,x,y) = \sum_{k=1}^{\infty} c_k\, e^{-\lambda_k t}\, v_k(x,y). \tag{11.23}$$

The coefficients c_k are prescribed by the initial conditions, which require

$$\sum_{k=1}^{\infty} c_k\, v_k(x,y) = f(x,y). \tag{11.24}$$

Since S is self-adjoint, Theorem 9.33 guarantees orthogonality[†] of the eigenfunctions under the L^2 inner product on the domain Ω:

$$\langle\, v_j \,, v_k \,\rangle = \iint_\Omega v_j(x,y)\, v_k(x,y)\, dx\, dy = 0, \qquad j \neq k. \tag{11.25}$$

As a consequence, the coefficients in (11.24) are given by the standard orthogonality formula (9.104), namely

$$c_k = \frac{\langle\, f \,, v_k \,\rangle}{\|\, v_k \,\|^2} = \frac{\displaystyle\iint_\Omega f(x,y)\, v_k(x,y)\, dx\, dy}{\displaystyle\iint_\Omega v_k(x,y)^2\, dx\, dy}\,. \tag{11.26}$$

(For the more general diffusion equation (11.11), one uses the appropriately weighted inner product.) The exponential decay of the eigenfunction coefficients implies that the resulting eigensolution series (11.23) converges and thus produces the solution to the initial-boundary value problem for the diffusion equation. See [**34**; p. 369] for a precise statement and proof of the general theorem.

Qualitative Properties

Before tackling examples in which we are able to construct explicit formulas for the eigenfunctions and eigenvalues, let us see what the eigenfunction series solution (11.23) can tell us about general diffusion processes. Based on our experience with the case of a one-dimensional bar, the final conclusions will not be especially surprising. Indeed, they also apply, word for word, to diffusion processes in three-dimensional solid bodies. A reader who is impatient to see the explicit formulas may wish to skip ahead to the following section, returning here as needed.

---

† As usual, in the case of a repeated eigenvalue, one chooses an orthogonal basis of the associated eigenspace to ensure orthogonality of all the basis eigenfunctions.

Keep in mind that we are still dealing with the solution to the homogeneous boundary value problem. The first observation is that all terms in the series solution (11.23), with the possible exception of a null eigenfunction term that appears in the semi-definite Neumann case, are tending to zero exponentially fast. Since most eigenvalues are large, all the higher-order terms in the series become almost instantaneously negligible, and hence the solution can be accurately approximated by a finite sum over the first few eigenfunction modes. As time goes on, more and more of the modes can be neglected, and the solution decays to thermal equilibrium at an exponentially fast rate. The rate of convergence to thermal equilibrium is, for most initial data, governed by the smallest positive eigenvalue $\lambda_1 > 0$ for the Helmholtz boundary value problem on the domain.

In the positive definite cases of homogeneous Dirichlet or mixed boundary conditions, thermal equilibrium is $u(t,x,y) \to u_\star(x,y) \equiv 0$. Here, the equilibrium temperature is equal to the zero boundary temperature — even if this temperature is fixed on only a small part of the boundary. The initial heat is eventually dissipated away through the uninsulated part of the boundary. In the semi-definite Neumann case, corresponding to a completely insulated plate, the general solution has the form

$$u(t,x,y) = c_0 + \sum_{k=1}^{\infty} c_k\, e^{-\lambda_k t}\, v_k(x,y), \tag{11.27}$$

where the sum is over the positive eigenmodes, $\lambda_k > 0$. Since all the summands are exponentially decaying, the final equilibrium temperature $u_\star = c_0$ is the same as the constant term in the eigenfunction expansion. We evaluate this term using the orthogonality formula (11.26), and so, as $t \to \infty$,

$$u(t,x,y) \;\longrightarrow\; c_0 = \frac{\langle\, f\,,1\,\rangle}{\|\,1\,\|^2} = \frac{\displaystyle\iint_\Omega f(x,y)\,dx\,dy}{\displaystyle\iint_\Omega dx\,dy} = \frac{1}{\text{area}\,\Omega}\iint_\Omega f(x,y)\,dx\,dy. \tag{11.28}$$

We conclude that the equilibrium temperature is equal to the average initial temperature distribution. Thus, when the plate is fully insulated, the heat energy cannot escape, and instead redistributes itself in a uniform manner over the domain.

Diffusion has a smoothing effect on the initial temperature distribution $f(x,y)$. Assume that the eigenfunction coefficients are uniformly bounded, so $|\,c_k\,| \le M$ for some constant M. This will certainly be the case if $f(x,y)$ is piecewise continuous or, more generally, belongs to L^2, since Bessel's inequality, (3.117), which holds for general orthogonal systems, implies that $c_k \to 0$ as $k \to \infty$. Many distributions, including delta functions, also have bounded Fourier coefficients. Then, at any time $t > 0$ after the initial instant, the coefficients $c_k\, e^{-\lambda_k t}$ in the eigenfunction series solution (11.23) are exponentially small as $k \to \infty$, which is enough to ensure smoothness of the solution $u(t,x,y)$ for each $t > 0$. Therefore, the diffusion process serves to immediately smooth out jumps, corners, and other discontinuities in the initial data. As time progresses, the local variations in the solution become less and less pronounced, as it asymptotically reaches a constant equilibrium state.

As a result, diffusion processes can be effectively applied to smooth and denoise planar images. The initial data $u(0,x,y) = f(x,y)$ represents the gray-scale value of the image at position (x,y), so that $0 \le f(x,y) \le 1$, with 0 representing black and 1 representing white. As time progresses, the solution $u(t,x,y)$ represents a more and more smoothed version

Figure 11.1. Smoothing a gray scale image.

of the image. Although this has the effect of removing unwanted high-frequency noise, there is also a gradual blurring of the actual features. Thus, the "time" or "multiscale" parameter t needs to be chosen to optimally balance between the two effects — the larger t is the more noise is removed, but the more noticeable the blurring. A representative illustration appears in Figure 11.1. The blurring affects small-scale features first, then, gradually, those at larger and larger scales, until eventually the entire image is blurred to a uniform gray. To further suppress undesirable blurring effects, modern image-processing filters are based on anisotropic (and thus *nonlinear*) diffusion equations; see [**100**] for a survey of recent progress in this active field.

Since the forward heat equation effectively blurs the features in an image, we might be tempted to reverse "time" in order to sharpen the image. However, the argument presented in Section 4.1 tells us that the backwards heat equation is ill-posed, and hence cannot be used directly for this purpose. Various "regularization" strategies have been devised to circumvent this mathematical barrier, and thereby design effective image enhancement algorithms, [**46**].

Inhomogeneous Boundary Conditions and Forcing

Let us next briefly discuss how to incorporate inhomogeneous boundary conditions and external heat sources into the general solution framework. Consider, as a specific example, the forced heat equation

$$u_t = \gamma\,\Delta u + F(x,y) \qquad \text{for} \qquad (x,y)\in\Omega, \tag{11.29}$$

where $F(x,y)$ represents an unvarying external heat source or sink, subject to inhomogeneous Dirichlet boundary conditions

$$u(x,y) = h(x,y) \qquad \text{for} \qquad (x,y)\in\partial\Omega, \tag{11.30}$$

that fixes the temperature of the plate on its boundary. When the external forcing does not vary in time, we expect the solution to eventually settle down to an equilibrium configuration: $u(t,x,y) \to u_\star(x,y)$ as $t\to\infty$. This will be justified below.

The time-independent equilibrium temperature $u_\star(x,y)$ satisfies the equation obtained by setting $u_t = 0$ in the evolution equation (11.29), which reduces it to the Poisson equation

$$-\gamma\,\Delta u_\star = F \qquad \text{for} \qquad (x,y)\in\Omega. \tag{11.31}$$

The equilibrium solution is subject to the same inhomogeneous Dirichlet boundary conditions (11.30). Positive definiteness of the Dirichlet boundary value problem implies that there is a unique equilibrium solution, which can be characterized as the sole minimizer of the associated Dirichlet principle; for details see Section 9.3.

With the equilibrium solution in hand, we let

$$v(t,x,y) = u(t,x,y) - u_\star(x,y)$$

measure the deviation of the dynamical solution u from its eventual equilibrium. By linearity $v(t,x,y)$ satisfies the unforced heat equation subject to homogeneous boundary conditions:

$$v_t = \gamma\,\Delta v, \qquad (x,y)\in\Omega, \qquad\qquad v = 0, \qquad (x,y)\in\partial\Omega. \tag{11.32}$$

Therefore, v can be expanded in an eigenfunction series (11.23), and will decay to zero, $v(t,x,y)\to 0$, at an exponentially fast rate prescribed by the smallest eigenvalue λ_1 of the associated homogeneous Helmholtz boundary value problem. (Special initial data can decay at a faster rate, prescribed by a larger eigenvalue.) Consequently, the solution to the forced inhomogeneous problem (11.29–30) will approach thermal equilibrium,

$$u(t,x,y) = v(t,x,y) + u_\star(x,y) \;\longrightarrow\; u_\star(x,y),$$

at exactly the same exponential rate as its homogeneous counterpart.

The Maximum Principle

Finally, let us state and prove the (Weak) Maximum Principle for the two-dimensional heat equation. As in the one-dimensional situation described in Section 8.3, it states that the maximum temperature in a body that is either insulated or having heat removed from its interior must occur either at the initial time or on its boundary. Observe that there are no conditions imposed on the boundary temperatures.

Theorem 11.1. *Suppose $u(t,x,y)$ is a solution to the forced heat equation*

$$u_t = \gamma\,\Delta u + F(t,x,y), \qquad \text{for} \qquad (x,y)\in\Omega, \qquad 0<t<c,$$

where Ω is a bounded domain, and $\gamma>0$. Suppose $F(t,x,y)\le 0$ for all $(x,y)\in\overline{\Omega}$ and $0\le t\le c$. Then the global maximum of u on the set $\{\,(t,x,y)\,|\,(x,y)\in\overline{\Omega},\ 0\le t\le c\,\}$ occurs either when $t=0$ or at a boundary point $(x,y)\in\partial\Omega$.

Proof: First, let us prove the result under the assumption that $F(t,x,y) < 0$ everywhere. At a local interior maximum, $u_t = 0$, and, since its Hessian matrix $\nabla^2 u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy}\end{pmatrix}$ must be negative semi-definite, both diagonal entries $u_{xx}, u_{yy}\le 0$ there. This would imply that $u_t - \gamma\,\Delta u \ge 0$, resulting in a contradiction. If the maximum were to occur when $t=c$, then $u_t\ge 0$ there, and also $u_{xx}, u_{yy}\le 0$, leading again to a contradiction.

To generalize to the case $F(t,x,y)\le 0$, which includes the heat equation when $F(t,x,y)\equiv 0$, set

$$v(t,x,y) = u(t,x,y) + \varepsilon\,(x^2+y^2), \qquad \text{where} \qquad \varepsilon>0.$$

Then,

$$\frac{\partial v}{\partial t} = \gamma\,\Delta v - 4\gamma\,\varepsilon + F(t,x,y) = \gamma\,\Delta v + \widetilde{F}(t,x,y),$$

where

$$\widetilde{F}(t,x,y) = F(t,x,y) - 4\gamma\,\varepsilon < 0.$$

Thus, by the previous paragraph, the maximum of v occurs either when $t = 0$ or at a boundary point $(x, y) \in \partial\Omega$. We then let $\varepsilon \to 0$ and conclude the same for u. More precisely, let $u(t, x, y) \le M$ on $t = 0$ or $(x, y) \in \partial\Omega$. Then

$$v(t,x,y) \le M + C\,\varepsilon, \qquad \text{where} \qquad C = \max\left\{\, x^2 + y^2 \;\middle|\; (x,y) \in \partial\Omega \,\right\} < \infty,$$

since Ω is a bounded domain. Thus,

$$u(t,x,y) \le v(t,x,y) \le M + C\,\varepsilon.$$

Letting $\varepsilon \to 0$ proves that $u(t, x, y) \le M$ at all $(x, y) \in \overline{\Omega}$, $0 \le t \le c$, which completes the proof. *Q.E.D.*

Remark: The preceding proof can be readily adapted to general diffusion equations (11.12) — assuming that the coefficients σ, κ remain strictly positive throughout the domain.

Exercises

11.1.1. A homogeneous, isotropic circular metal disk of radius 1 meter has its entire boundary insulated. The initial temperature at a point is equal to the distance of the point from the center. Formulate an initial-boundary value problem governing the disk's subsequent temperature dynamics. What is the eventual equilibrium temperature of the disk?

11.1.2. A homogeneous, isotropic, circular metal disk of radius 2 cm has half its boundary fixed at 100° and the other half insulated. Given a prescribed initial temperature distribution, set up the initial-boundary value problem governing its subsequent temperature profile. What is the eventual equilibrium temperature of the disk? Does your answer depend on the initial temperature?

11.1.3. Given the initial temperature distribution $f(x, y) = x\,y\,(1 - x)(1 - y)$ on the unit square $\Omega = \{0 \le x, y \le 1\}$, determine the equilibrium temperature when subject to homogeneous (*a*) Dirichlet boundary conditions; (*b*) Neumann boundary conditions.

11.1.4. A square plate with side lengths 1 meter has its right and left edges insulated, its top edge held at 100°, and its bottom edge held at 0°. Assuming that the plate is made out of a homogeneous, isotropic material, formulate an appropriate initial-boundary value problem describing the temperature dynamics of the plate. Then find its eventual equilibrium temperature.

11.1.5. A square plate with side lengths 1 meter has initial temperature 5° throughout, and evolves subject to the Neumann boundary conditions $\partial u/\partial \mathbf{n} = 1$ on its entire boundary. What is the eventual equilibrium temperature?

♡ 11.1.6. Let $u(t, x, y)$ be a solution to the heat equation on a bounded domain Ω subject to homogeneous Neumann conditions on its boundary $\partial\Omega$. (*a*) Prove that the total heat $H(t) = \iint_\Omega u(t, x, y)\, dx\, dy$ is conserved, i.e., is constant in time. (*b*) Use part (*a*) to prove that the eventual equilibrium solution is everywhere equal to the average of the initial temperature $u(0, x, y)$. (*c*) What can you say about the behavior of the total heat for the homogeneous Dirichlet boundary value problem? (*d*) What about an inhomogeneous Dirichlet boundary value problem?

11.1.7. Let $u(t,x,y)$ be a nonconstant solution to the heat equation on a connected, bounded domain Ω subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$. (*a*) Prove that its L^2 norm $N(t) = \sqrt{\iint_\Omega u(t,x,y)^2\,dx\,dy}$ is a strictly decreasing function of t. (*b*) Is this also true for mixed boundary conditions? (*c*) For Neumann boundary conditions?

11.1.8. Are the conclusions in Exercises 11.1.6 and 11.1.7 valid for the general diffusion equation (11.12)?

♢ 11.1.9. Write out the eigenvalue equation governing the separable solutions to the general diffusion equation (11.11), subject to appropriate boundary conditions. Given a complete system of eigenfunctions, write down the eigenfunction series solution to the initial value problem $u(0,x,y) = f(x,y)$, including the formulas for the coefficients.

11.1.10. *True or false*: The equilibrium temperature of a fully insulated nonuniform plate whose thermodynamics are governed by the general diffusion equation (11.12) equals the average initial temperature.

11.1.11. Let $\alpha > 0$, and consider the initial-boundary value problem $u_t = \Delta u - \alpha\, u$, $u(0,x,y) = f(x,y)$ on a bounded domain $\Omega \subset \mathbb{R}^2$, with boundary conditions $\partial u/\partial \mathbf{n} = 0$ on $\partial\Omega$.
(*a*) Write the equation in self-adjoint form (9.122). *Hint*: Look at Exercise 9.3.26.
(*b*) Prove that the problem has a unique equilibrium solution.

11.1.12. Write each of the following linear evolution equations in the self-adjoint form (9.122) by choosing suitable inner products and a suitable set of homogeneous boundary conditions. Is the operator you construct positive definite?
(*a*) $u_t = u_{xx} + u_{yy} - u$, (*b*) $u_t = y\,u_{xx} + x\,u_{yy}$, (*c*) $u_t = \Delta^2 u$.

♢ 11.1.13. Prove that if $f(x,y)$ is continuous and $\iint_R f(x,y)\,dx\,dy = 0$ for all $R \subset \Omega$, then $f(x,y) \equiv 0$ for $(x,y) \in \Omega$. *Hint*: Adapt the method in Exercise 6.1.23.

11.2 Explicit Solutions of the Heat Equation

Solving the two-dimensional heat equation in series form requires knowing the eigenfunctions for the associated Helmholtz boundary value problem. Unfortunately, as with the vast majority of partial differential equations, explicit solution formulas are few and far between. In this section, we discuss two specific cases in which the required eigenfunctions can be found in closed form. The calculations rely on a further separation of variables, which, as we know, works in only a very limited class of domains. Nevertheless, interesting solution features can be gleaned from these particular geometries.

The first example is a rectangular domain, and the eigensolutions can be expressed in terms of elementary functions — trigonometric functions and exponentials. We then study the heating of a circular disk. In this case, the eigenfunctions are no longer elementary functions, but, rather, are expressed in terms of Bessel functions. Understanding their basic properties will require us to take a detour to develop the fundamentals of power series solutions to ordinary differential equations.

Heating of a Rectangle

A homogeneous rectangular plate

$$R = \left\{\, 0 < x < a,\ 0 < y < b \,\right\}$$

is heated to a prescribed initial temperature,

$$u(0,x,y) = f(x,y), \qquad \text{for} \qquad (x,y) \in R. \tag{11.33}$$

Then its top and bottom are insulated, while its sides are held at zero temperature. Our task is to understand the thermodynamic evolution of the plate's temperature.

The temperature $u(t,x,y)$ evolves according to the two-dimensional heat equation

$$u_t = \gamma\,(u_{xx} + u_{yy}), \qquad \text{for} \qquad (x,y) \in R, \qquad t > 0, \tag{11.34}$$

where $\gamma > 0$ is the plate's thermal diffusivity, while subject to homogeneous Dirichlet conditions along the boundary of the rectangle at all subsequent times:

$$u(t,0,y) = u(t,a,y) = u(t,x,0) = u(t,x,b) = 0, \qquad 0 < x < a, \quad 0 < y < b, \qquad t > 0. \tag{11.35}$$

As in (11.19), the eigensolutions to the heat equation are obtained from the usual exponential ansatz $u(t,x,y) = e^{-\lambda t}\, v(x,y)$. Substituting this expression into the heat equation, we conclude that the function $v(x,y)$ solves the Helmholtz eigenvalue problem

$$\gamma\,(v_{xx} + v_{yy}) + \lambda\, v = 0, \qquad (x,y) \in R, \tag{11.36}$$

subject to the same homogeneous Dirichlet boundary conditions:

$$v(0,y) = v(a,y) = v(x,0) = v(x,b) = 0, \qquad 0 < x < a, \quad 0 < y < b. \tag{11.37}$$

To tackle the rectangular Helmholtz eigenvalue problem (11.36–37), we shall, as in (4.89), introduce a further separation of variables, writing the solution

$$v(x,y) = p(x)\, q(y)$$

as the product of functions depending on the individual Cartesian coordinates. Substituting this expression into the Helmholtz equation (11.36), we find

$$\gamma\, p''(x)\, q(y) + \gamma\, p(x)\, q''(y) + \lambda\, p(x)\, q(y) = 0.$$

To effect the variable separation, we collect all terms involving x on one side and all terms involving y on the other side of the equation, which is accomplished by dividing by $v = p\,q$ and rearranging the terms:

$$\gamma\,\frac{p''(x)}{p(x)} = -\,\gamma\,\frac{q''(y)}{q(y)} - \lambda \equiv -\,\mu.$$

The left-hand side of this equation depends only on x, whereas the middle term depends only on y. As before, this requires that the expressions equal a common *separation constant*, denoted by $-\,\mu$. (The minus sign is for later convenience.) In this manner, we reduce our partial differential equation to a pair of one-dimensional eigenvalue problems

$$\gamma\,\frac{d^2 p}{dx^2} + \mu\, p = 0, \qquad \gamma\,\frac{d^2 q}{dy^2} + (\lambda - \mu)\, q = 0, \tag{11.38}$$

each of which is subject to homogeneous Dirichlet boundary conditions

$$p(0) = p(a) = 0, \qquad q(0) = q(b) = 0, \tag{11.39}$$

stemming from the boundary conditions (11.37). To obtain a nontrivial separable solution to the Helmholtz equation, we seek nonzero solutions to these two supplementary eigenvalue problems.

We have already solved these particular two boundary value problems (11.38–39) many times; see, for instance, (4.21). The eigenfunctions are, respectively,

$$p_m(x) = \sin \frac{m\pi x}{a}, \qquad m = 1, 2, 3, \ldots, \qquad\qquad q_n(y) = \sin \frac{n\pi y}{b}, \qquad n = 1, 2, 3, \ldots,$$

with

$$\mu = \frac{m^2\pi^2\gamma}{a^2}, \qquad \lambda - \mu = \frac{n^2\pi^2\gamma}{b^2}, \qquad \text{so that} \qquad \lambda = \frac{m^2\pi^2\gamma}{a^2} + \frac{n^2\pi^2\gamma}{b^2}.$$

Therefore, the separable eigenfunction solutions to the Helmholtz boundary value problem (11.35–36) have the doubly trigonometric form

$$v_{m,n}(x,y) = \sin \frac{m\pi x}{a} \, \sin \frac{n\pi y}{b}, \qquad \text{for} \qquad m, n = 1, 2, 3, \ \ldots\ , \tag{11.40}$$

with associated eigenvalues

$$\lambda_{m,n} = \frac{m^2\pi^2\gamma}{a^2} + \frac{n^2\pi^2\gamma}{b^2} = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 \gamma. \tag{11.41}$$

Each of these corresponds to an exponentially decaying eigensolution

$$u_{m,n}(t,x,y) = e^{-\lambda_{m,n} t}\, v_{m,n}(x,y) = \exp\left[-\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 \gamma\, t \right] \sin \frac{m\pi x}{a} \, \sin \frac{n\pi y}{b} \tag{11.42}$$

to the original rectangular Dirichlet boundary value problem for the heat equation.

Using the fact that the univariate sine functions form a complete system, it is not hard to prove, [**120**], that the separable eigenfunction solutions (11.42) are complete, and so there are no non-separable eigenfunctions.† As a consequence, the general solution to the initial-boundary value problem can be expressed as a linear combination

$$u(t,x,y) = \sum_{m,n=1}^{\infty} c_{m,n}\, u_{m,n}(t,x,y) = \sum_{m,n=1}^{\infty} c_{m,n}\, e^{-\lambda_{m,n} t}\, v_{m,n}(x,y) \tag{11.43}$$

of the eigenmodes. The coefficients $c_{m,n}$ are prescribed by the initial conditions, which take the form of a double Fourier sine series

$$f(x,y) = u(0,x,y) = \sum_{m,n=1}^{\infty} c_{m,n} v_{m,n}(x,y) = \sum_{m,n=1}^{\infty} c_{m,n} \, \sin \frac{m\pi x}{a} \, \sin \frac{n\pi y}{b}.$$

Self-adjointness of the Laplacian operator coupled with the boundary conditions implies that‡ the eigenfunctions $v_{m,n}(x,y)$ are orthogonal with respect to the L^2 inner product

† This appears to be a general fact, true in all known examples, but I know of no general proof. Theorem 9.47 can be used to establish completeness of the eigenfunctions, but does not guarantee that they can all be constructed by separation of variables.

‡ Technically, orthogonality is guaranteed only when the eigenvalues are distinct: $\lambda_{m,n} \neq \lambda_{k,l}$. However, by a direct computation, one finds that orthogonality continues to hold even when the indicated eigenfunctions are associated with equal eigenvalues. See the final subsection of this chapter for a discussion of when such "accidental degeneracies" arise.

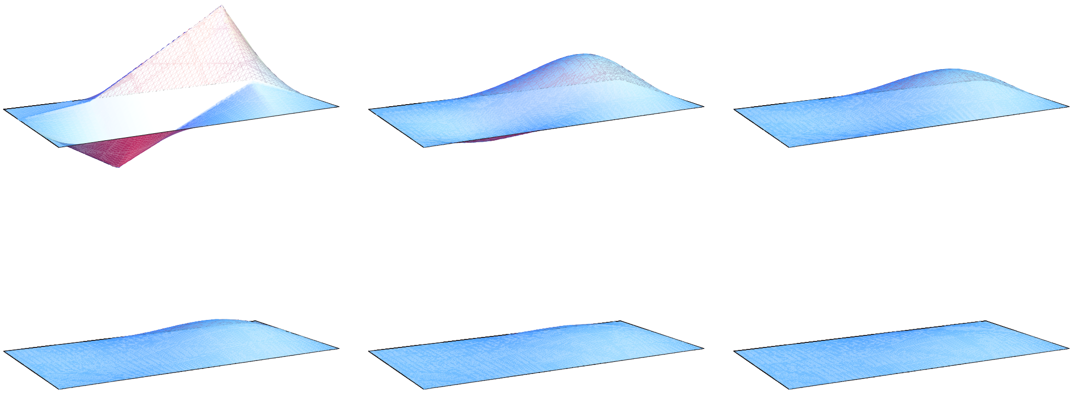


Figure 11.2. Heat diffusion in a rectangle.

on the rectangle:

$$\langle v_{k,l}, v_{m,n} \rangle = \int_0^b \int_0^a v_{k,l}(x,y)\, v_{m,n}(x,y)\, dx\, dy = 0 \qquad \text{unless} \qquad k = m \quad \text{and} \quad l = n.$$

(The skeptical reader can verify the orthogonality relations directly from the eigenfunction formulas (11.40).) Thus, we can appeal to our usual orthogonality formula (11.26) to evaluate the coefficients

$$c_{m,n} = \frac{\langle f, v_{m,n} \rangle}{\| v_{m,n} \|^2} = \frac{4}{a\, b} \int_0^b \int_0^a f(x,y)\ \sin \frac{m \pi x}{a}\ \sin \frac{n \pi y}{b}\, dx\, dy, \tag{11.44}$$

where the formula for the norms of the eigenfunctions

$$\| v_{m,n} \|^2 = \int_0^b \int_0^a v_{m,n}(x,y)^2\, dx\, dy = \int_0^b \int_0^a \sin^2 \frac{m \pi x}{a}\ \sin^2 \frac{n \pi y}{b}\, dx\, dy = \tfrac{1}{4}\, a\, b \tag{11.45}$$

follows from a direct evaluation of the double integral. Unfortunately, while orthogonality is (mostly) automatic, computation of the norms must inevitably be done "by hand".

For generic initial temperature distributions, the rectangle approaches thermal equilibrium at a rate equal to the smallest eigenvalue:

$$\lambda_{1,1} = \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \pi^2\, \gamma, \tag{11.46}$$

i.e., the sum of the reciprocals of the squared lengths of its sides multiplied by the diffusion coefficient. The larger the rectangle, or the smaller the diffusion coefficient, the smaller the value of $\lambda_{1,1}$, and hence the slower the return to thermal equilibrium. The exponentially fast decay rate of the Fourier series implies that the solution immediately smooths out any discontinuities in the initial temperature profile. Indeed, the higher modes, with m and n large, decay to zero almost instantaneously, and so the solution quickly behaves like a finite sum over a few low-order modes. Assuming that $c_{1,1} \neq 0$, the slowest-decaying mode in the Fourier series (11.43) is

$$c_{1,1}\, u_{1,1}(t,x,y) = c_{1,1} \exp\left[-\left(\frac{1}{a^2} + \frac{1}{b^2} \right) \pi^2 \gamma\, t \right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \tag{11.47}$$

Thus, in the long run, the temperature becomes entirely of one sign — either positive or negative depending on the sign of $c_{1,1}$ — throughout the rectangle. This observation is, in fact, indicative of the general phenomenon that an eigenfunction associated with the smallest positive eigenvalue of a self-adjoint elliptic operator is necessarily of one sign throughout the domain, [**34**]. A typical solution is plotted at several times in Figure 11.2. Non-generic initial conditions, with $c_{1,1} = 0$, decay more rapidly, and their asymptotic temperature profiles are not of one sign.

Exercises

11.2.1. A rectangle of size 2 cm by 1 cm has initial temperature $f(x,y) = \sin \pi x \sin \pi y$ for $0 \le x \le 2,\ 0 \le y \le 1$. All four sides of the rectangle are held at 0°. Assuming that the thermal diffusivity of the plate is $\gamma = 1$, write down a formula for its subsequent temperature $u(t,x,y)$. What is the rate of decay to thermal equilibrium?

11.2.2. Solve Exercise 11.2.1 when the initial temperature $f(x,y)$ is
(*a*) $x\,y$, (*b*) $\begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2; \end{cases}$ (*c*) $\bigl(1 - |\,1 - x\,|\bigr)\bigl(\frac{1}{2} - \bigl|\,\frac{1}{2} - y\,\bigr|\bigr)$.

11.2.3. Solve the initial-boundary value problem for the heat equation $u_t = 2\,\Delta u$ on the rectangle $-1 < x < 1,\ 0 < y < 1$ when the two short sides are kept at 0°, the two long sides are insulated, and the initial temperature distribution is $u(0,x,y) = \begin{cases} -1, & x < 0, \\ +1, & x > 0, \end{cases} \quad 0 < y < 1.$

11.2.4. Answer Exercise 11.2.3 when the two long sides are kept at 0° and the two short sides are insulated.

♡ 11.2.5. A rectangular plate of size 1 meter by 3 meters is made out a metal with unit diffusivity. The plate is taken from a 0° freezer, and, from then on, one of its long sides is heated to 100°, the other is held at 0°, while its top, bottom, and both of the short sides are fully insulated. (*a*) Set up the initial-boundary value problem governing the time-dependent temperature of the plate. (*b*) What is the equilibrium temperature? (*c*) Use your answer from part (*b*) to construct an eigenfunction series for the solution. (*d*) How long until the temperature of the plate is everywhere within 1° of its eventual equilibrium?
Hint: Once t is no longer small, you can approximate the series solution by its first term.

11.2.6. Among all rectangular plates of a prescribed area, which one returns to thermal equilibrium the slowest when subject to Dirichlet boundary conditions? The fastest? Use your physical intuition to explain your answer, but justify it mathematically.

11.2.7. Answer Exercise 11.2.6 for a fully insulated rectangular plate, i.e., subject to Neumann boundary conditions.

♡ 11.2.8. A square metal plate is taken from an oven, and then set out to cool, with its top and bottom insulated. Find the rate of cooling, in terms of the side length and the thermal diffusivity, if (*a*) all four sides are held at 0°; (*b*) one side is insulated and the other three sides are held at 0°; (*c*) two adjacent sides are insulated and the other two are held at 0°; (*d*) two opposite sides are insulated and the other two are held at 0°; (*e*) three sides are insulated and the remaining side is held at 0°. Order the cooling rates of the plates from fastest to slowest. Do your results confirm your intuition?

♡ 11.2.9. Two square plates are made out of the same homogeneous material, and both are initially heated to 100°. All four sides of the first plate are held at 0°, whereas one of the sides of the second plate is insulated while the other three sides are held at 0°. Which plate cools down the fastest? How much faster? Assuming the thermal diffusivity $\gamma = 1$, how long do you have to wait until every point on each plate is within 1° of its equilibrium temperature? *Hint*: Once t is no longer small, the series solution is well approximated by its first term.

♡ 11.2.10. *Multiple choice*: On a unit square that is subject to Dirichlet boundary conditions, the eigenvalues of the Laplace operator are
(*a*) all simple, (*b*) at most double, or (*c*) can have arbitrarily large multiplicity.

♡ 11.2.11. The thermodynamics of a thin circular cylindrical shell of radius a and height h, e.g., the side of a tin can after its top and bottom are removed, is modeled by the heat equation $\dfrac{\partial u}{\partial t} = \gamma\left(\dfrac{1}{a^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2}\right)$, in which $u(t,\theta,z)$ measures the temperature of the point on the cylinder at time $t > 0$, angle $-\pi < \theta \le \pi$, and height $0 < z < h$. Keep in mind that $u(t,\theta,z)$ must be a 2π–periodic function of the angular coordinate θ. Assume that the cylinder is everywhere insulated, while its two circular ends are held at 0°. Given an initial temperature distribution at time $t = 0$, write down a series formula for the cylinder's temperature at subsequent times. What is the eventual equilibrium temperature? How fast does the cylinder return to equilibrium?

♡ 11.2.12. Consider the initial-boundary value problem
$$u_t = u_{xx} + u_{yy}, \qquad u(0,x,y) = 0, \qquad 0 < x, y < \pi, \qquad t > 0,$$
for the heat equation in a square subject to the Dirichlet conditions
$$u(t,0,y) = u(t,\pi,y) = 0 = u(t,x,0), \quad u(t,x,\pi) = f(x), \quad 0 < x, y < \pi, \quad t > 0.$$
Write out an eigenfunction series formulas for
(*a*) the equilibrium solution $u_\star(x,y) = \lim\limits_{t\to\infty} u(t,x,y)$; (*b*) the solution $u(t,x,y)$.

11.2.13. Solve Exercise 11.2.1 when one long side of the plate is held at 100°.
Hint: See Exercise 11.2.12.

Heating of a Disk — Preliminaries

Let us perform a similar analysis of the thermodynamics of a circular disk. For simplicity (or by choice of suitable physical units), we will assume that the disk

$$D = \{x^2 + y^2 \le 1\} \subset \mathbb{R}^2$$

has unit radius and unit diffusivity $\gamma = 1$. We shall solve the heat equation on D subject to homogeneous Dirichlet boundary values of zero temperature at the circular edge

$$\partial D = C = \{x^2 + y^2 = 1\}.$$

Thus, the full initial-boundary value problem is

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, && x^2+y^2<1, && \\ &u(t,x,y) = 0, && x^2+y^2 = 1, && t > 0, \\ &u(0,x,y) = f(x,y), && x^2+y^2 \le 1. && \end{aligned} \tag{11.48}$$

We remark that a simple rescaling of space and time, as outlined in Exercise 11.4.7, can be used to recover the solution for an arbitrary diffusion coefficient and a disk of arbitrary radius from this particular case.

Since we are working in a circular domain, we instinctively pass to polar coordinates (r,θ). In view of the polar coordinate formula (4.105) for the Laplace operator, the heat equation and boundary and initial conditions assume the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2}\,, \qquad u(t,1,\theta) = 0, \qquad u(0,r,\theta) = f(r,\theta), \tag{11.49}$$

where the solution $u(t,r,\theta)$ is defined for all $0 \le r \le 1$ and $t \ge 0$. To ensure that the solution represents a single-valued function on the entire disk, it is required to be a 2π–periodic function of the angular variable:

$$u(t,r,\theta + 2\pi) = u(t,r,\theta).$$

To obtain the separable solutions

$$u(t,r,\theta) = e^{-\lambda t}\, v(r,\theta), \tag{11.50}$$

we need to solve the polar coordinate form of the Helmholtz equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\,\frac{\partial v}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 v}{\partial \theta^2} + \lambda\, v = 0, \qquad \begin{array}{c} 0 \le r < 1, \\ -\pi < \theta \le \pi, \end{array} \tag{11.51}$$

subject to the boundary conditions

$$v(1,\theta) = 0, \qquad v(r,\theta + 2\pi) = v(r,\theta). \tag{11.52}$$

To solve the polar Helmholtz boundary value problem (11.51–52), we invoke a further separation of variables by writing

$$v(r,\theta) = p(r)\, q(\theta). \tag{11.53}$$

Substituting this ansatz into (11.51), collecting all terms involving r and all terms involving θ, and then equating both to a common separation constant, we are led to the pair of ordinary differential equations

$$r^2\,\frac{d^2 p}{dr^2} + r\,\frac{dp}{dr} + (\lambda r^2 - \mu)\, p = 0, \qquad \frac{d^2 q}{d\theta^2} + \mu\, q = 0, \tag{11.54}$$

where λ is the Helmholtz eigenvalue, and μ the separation constant.

Let us start with the equation for $q(\theta)$. The second boundary condition in (11.52) requires that $q(\theta)$ be 2π–periodic. Therefore, the required solutions are the elementary trigonometric functions

$$q(\theta) = \cos m\,\theta \qquad \text{or} \qquad \sin m\,\theta, \qquad \text{where} \qquad \mu = m^2, \tag{11.55}$$

with $m = 0, 1, 2, \ldots$ a nonnegative integer.

Substituting the formula for the separation constant, $\mu = m^2$, the differential equation for $p(r)$ takes the form

$$r^2\,\frac{d^2 p}{dr^2} + r\,\frac{dp}{dr} + (\lambda r^2 - m^2)\, p = 0, \qquad 0 \le r \le 1. \tag{11.56}$$

Ordinarily, one imposes two boundary conditions in order to pin down a solution to such a second-order ordinary differential equation. But our Dirichlet condition, namely $p(1) = 0$, specifies its value at only one of the endpoints. The other endpoint is a *singular point* for the ordinary differential equation, because the coefficient of the highest-order derivative, namely r^2, vanishes at $r = 0$. This situation might remind you of our solution to the Euler differential equation (4.111) in the context of separable solutions to the Laplace equation on the disk. As there, we require that the solution be bounded at $r = 0$, and so seek eigensolutions that satisfy the boundary conditions

$$|\,p(0)\,| < \infty, \qquad\qquad p(1) = 0. \tag{11.57}$$

While (11.56) appears in a variety of applications, it is more challenging than any ordinary differential equation we have encountered so far. Indeed, most solutions cannot be written in terms of the elementary functions (rational functions, trigonometric functions, exponentials, logarithms, etc.) you see in first-year calculus. Nevertheless, owing to their ubiquity in physical applications, its solutions have been extensively studied and tabulated, and so are, in a sense, well known, [**85**, **86**, **119**].

To simplify the subsequent analysis, we make a preliminary rescaling of the independent variable, replacing r by

$$z = \sqrt{\lambda}\, r.$$

(We know the eigenvalue $\lambda > 0$, since we are dealing with a positive definite boundary value problem.) Note that, by the chain rule,

$$\frac{dp}{dr} = \sqrt{\lambda}\,\frac{dp}{dz}\,, \qquad\qquad \frac{d^2p}{dr^2} = \lambda\,\frac{d^2p}{dz^2}\,,$$

and hence

$$r\,\frac{dp}{dr} = z\,\frac{dp}{dz}\,, \qquad\qquad r^2\,\frac{d^2p}{dr^2} = z^2\,\frac{d^2p}{dz^2}\,.$$

The net effect is to eliminate the eigenvalue parameter λ (or, rather, hide it in the change of variables), so that (11.56) assumes the slightly simpler form

$$z^2\,\frac{d^2p}{dz^2} + z\,\frac{dp}{dz} + (z^2 - m^2)\,p = 0. \tag{11.58}$$

The resulting ordinary differential equation (11.58) is known as *Bessel's equation*, named after the early-nineteenth-century German astronomer Wilhelm Bessel, who first encountered its solutions, now known as *Bessel functions*, in his study of planetary orbits. Special cases had already appeared in the investigations of Daniel Bernoulli on vibrations of a hanging chain, and in those of Fourier on the thermodynamics of a cylindrical body. To make further progress, we need to take time out to study their basic properties, and this will require us to develop the method of power series solutions of ordinary differential equations. With this in hand, we can then return to complete our solution to the heat equation on a disk.

11.3 Series Solutions of Ordinary Differential Equations

When confronted with a novel ordinary differential equation, we have several available options for deriving and understanding its solutions. For instance, the "look-up" method relies on published handbooks. One of the most useful references that collects many solved differential equations is the classic German compendium by Kamke, [**62**]. Two more recent English-language handbooks are [**93**, **127**]. In addition, many symbolic computer algebra programs, including Mathematica and Maple, will produce solutions, when expressible in terms of both elementary and special functions, to a wide range of differential equations.

Of course, use of numerical integration to approximate solutions, [**24**, **60**, **80**], is always an option. Numerical methods do, however, have their limitations, and are best accompanied by some understanding of the underlying theory, coupled with qualitative or quantitative expectations of how the solutions should behave. Furthermore, numerical methods provide less than adequate insight into the nature of the special functions that regularly appear as solutions of the particular differential equations arising in separation of variables. A numerical approximation cannot, in itself, establish rigorous mathematical properties of the solutions of the differential equation.

A more classical means of constructing and approximating the solutions of differential equations is based on their power series expansions, a.k.a. Taylor series. The Taylor expansion of a solution at a point x_0 is found by substituting a general power series into the differential equation and equating coefficients of the various powers of $x - x_0$. The initial conditions at x_0 serve to uniquely determine the coefficients and hence all the derivatives of the solution at the initial point. The Taylor expansion of a special function is an effective tool for deducing some of its key properties, as well as providing a means of computing reasonable numerical approximations to its values within the radius of convergence of the series. (However, serious numerical computations more often rely on nonconvergent asymptotic expansions, [**85**].)

In this section, we provide a brief introduction to the basic series solution techniques for ordinary differential equations, concentrating on second-order linear differential equations, since these form by far the most important class of examples arising in applications. At a regular point, the method will produce a standard Taylor expansion for the solution, while so-called regular singular points require a slightly more general type of series expansion. Generalizations to irregular singular points, higher-order equations, nonlinear equations, and even linear and nonlinear systems are deferred to more advanced texts, including [**54**, **59**].

The Gamma Function

Before delving into the machinery of series solutions and special functions, we need to introduce the gamma function, which effectively generalizes the factorial operation to non-integers. Recall that the *factorial* of a nonnegative integer $n \geq 0$ is defined inductively by the iterative formula

$$n\,! = n \cdot (n-1)\,!, \qquad \text{starting with} \qquad 0\,! = 1. \tag{11.59}$$

When n is a positive integer, the iteration terminates, yielding the familiar expression

$$n\,! = n(n-1)(n-2)\ \cdots\ 3 \cdot 2 \cdot 1. \tag{11.60}$$

However, for more general values of n, the iteration never stops, and it cannot be used to compute its factorial. Our goal is to circumvent this difficulty, and introduce a function $f(x)$ that is defined for *all* values of x and will play the role of such a factorial. First, mimicking (11.59), the function should satisfy the functional equation

$$f(x) = x\, f(x-1) \tag{11.61}$$

where defined. If, in addition, $f(0) = 1$, then we know that $f(n) = n!$ whenever n is a nonnegative integer, and hence such a function will extend the definition of the factorial to more general real and complex numbers.

A moment's thought should convince the reader that there are many possible ways to construct such a function; see Exercise 11.3.6 for a nonstandard example. The most important version is due to Euler. The modern definition of Euler's gamma function relies on an integral formula discovered by the eighteenth-century French mathematician Adrien–Marie Legendre, who will play a starring role in Chapter 12.

Definition 11.2. The *gamma function* is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt. \tag{11.62}$$

The first fact is that, for real x, the gamma function integral converges only when $x > 0$; otherwise the singularity of t^{x-1} at $t = 0$ is too severe. The key property that turns the gamma function into a substitute for the factorial function relies on an elementary integration by parts:

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^x\, dt = -e^{-t}\, t^x \Big|_{t=0}^{\infty} + x \int_0^\infty e^{-t}\, t^{x-1}\, dt.$$

The boundary terms vanish whenever $x > 0$, while the final integral is merely $\Gamma(x)$. Therefore, the gamma function satisfies the *recurrence relation*

$$\Gamma(x+1) = x\, \Gamma(x). \tag{11.63}$$

If we set $f(x) = \Gamma(x+1)$, then (11.63) becomes (11.61). Moreover, by direct integration,

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1.$$

Combining this with the recurrence relation (11.63), we deduce that

$$\Gamma(n+1) = n! \tag{11.64}$$

whenever $n \geq 0$ is a nonnegative integer. Therefore, we can identify $x!$ with the value $\Gamma(x+1)$ whenever $x > -1$ is *any* real number.

Remark: The reader may legitimately ask why not replace t^{x-1} by t^x in the definition of $\Gamma(z)$, which would avoid the $n+1$ in (11.64). There is no good answer; we are merely following a well-established precedent set by Legendre and enshrined in all subsequent works.

Thus, at integer values of x, the gamma function agrees with the elementary factorial. A few other values can be computed exactly. One important case is at $x = \frac{1}{2}$. Using the substitution $t = s^2$, with $dt = 2\, s\, ds$, we obtain

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt = \int_0^\infty 2\, e^{-s^2}\, ds = \sqrt{\pi}, \tag{11.65}$$

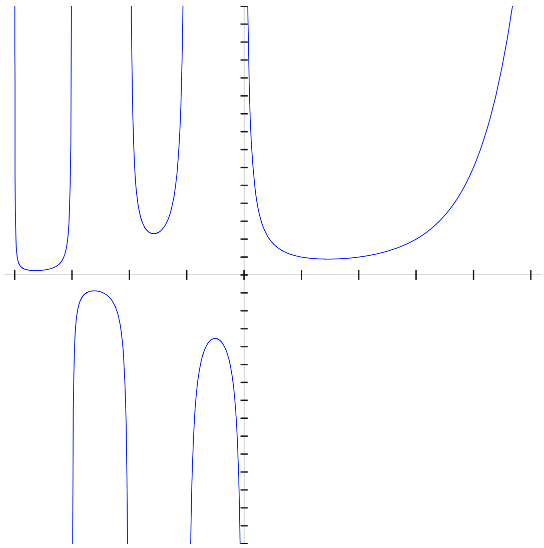


Figure 11.3. The gamma function.

where the final integral was evaluated in (2.100). Thus, using the identification with the factorial function, we identify this value with $\left(-\frac{1}{2}\right)! = \sqrt{\pi}$. The recurrence relation (11.63) will then produce the value of the gamma function at all half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. For example,

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \tag{11.66}$$

and hence $\frac{1}{2}\,! = \frac{1}{2}\sqrt{\pi}$. The recurrence relation can also be employed to extend the definition of $\Gamma(x)$ to (most) negative values of x. For example, setting $x = -\frac{1}{2}$ in (11.63), we have

$$\Gamma\left(\tfrac{1}{2}\right) = -\tfrac{1}{2}\,\Gamma\left(-\tfrac{1}{2}\right), \qquad \text{so} \qquad \Gamma\left(-\tfrac{1}{2}\right) = -2\,\Gamma\left(\tfrac{1}{2}\right) = -2\sqrt{\pi}\,.$$

The only points at which this device fails are the negative integers, and indeed, $\Gamma(x)$ has a singularity when $x = -1, -2, -3, \ldots$. A graph† of the gamma function is displayed in Figure 11.3.

Remark: Most special functions of importance for applications arise as solutions to fairly simple ordinary differential equations. The gamma function is a significant exception. Indeed, it can be proved, [**11**], that the gamma function *does not* satisfy *any* algebraic differential equation!

Regular Points

We are now ready to develop the method of series solutions to ordinary differential equations. Before we proceed to develop the general computational machinery, a naïve calculation in an elementary example will be enlightening.

† The axes are at different scales; the tick marks are at integer values.

Example 11.3. Consider the initial value problem

$$\frac{d^2u}{dx^2} + u = 0, \qquad u(0) = 1, \qquad u'(0) = 0. \tag{11.67}$$

Let us investigate whether we can construct an analytic solution in the form of a convergent power series

$$u(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots = \sum_{n=0}^{\infty} u_n x^n \tag{11.68}$$

that is based at the initial point $x_0 = 0$. Term-by-term differentiation yields the following series expansions† for its derivatives:

$$\begin{aligned} \frac{du}{dx} &= u_1 + 2u_2 x + 3u_3 x^2 + 4u_4 x^3 + \cdots &&= \sum_{n=0}^{\infty} (n+1)\, u_{n+1} x^n, \\ \frac{d^2u}{dx^2} &= 2u_2 + 6u_3 x + 12u_4 x^2 + 20u_5 x^3 + \cdots &&= \sum_{n=0}^{\infty} (n+1)(n+2)\, u_{n+2} x^n. \end{aligned} \tag{11.69}$$

The next step is to substitute the series (11.68–69) into the differential equation and collect common powers of x:

$$\frac{d^2u}{dx^2} + u = (2u_2 + u_0) + (6u_3 + u_1)\, x + (12u_4 + u_2)\, x^2 + (20u_5 + u_3)\, x^3 + \cdots = 0.$$

At this point, one focuses attention on the individual coefficients, appealing to the following basic observation:

Two convergent power series are equal if and only if all their coefficients are equal.

In particular, a power series represents the zero function‡ if and only if all its coefficients are 0. In this manner we obtain the following infinite sequence of algebraic *recurrence relations* among the coefficients:

$$\begin{array}{cr} 1 & 2u_2 + u_0 = 0, \\ x & 6u_3 + u_1 = 0, \\ x^2 & 12u_4 + u_2 = 0, \\ x^3 & 20u_5 + u_3 = 0, \\ x^4 & 30u_6 + u_4 = 0, \\ \vdots & \vdots \\ x^n & (n+1)(n+2)\, u_{n+2} + u_n = 0. \end{array} \tag{11.70}$$

Now, the initial conditions serve to prescribe the first two coefficients:

$$u(0) = u_0 = 1, \qquad u'(0) = u_1 = 0.$$

† When working with the series in summation form, it helps to re-index in order to display the term of degree n.

‡ Here it is essential that we work with analytic functions, since this result is *not* true for C^∞ functions! For example, the function e^{-1/x^2} has identically zero power series at $x_0 = 0$; see Exercise 11.3.21.

We then solve the recurrence relations in order: The first determines $u_2 = -\frac{1}{2}\,u_0 = -\frac{1}{2}$; the second, $u_3 = -\frac{1}{6}\,u_1 = 0$; next, $u_4 = -\frac{1}{12}\,u_2 = \frac{1}{24}$; then $u_5 = -\frac{1}{20}\,u_3 = 0$; then $u_6 = -\frac{1}{30}\,u_4 = -\frac{1}{720}$; and so on. In general, it is not hard to see that

$$u_{2k} = \frac{(-1)^k}{(2\,k)\,!}\,, \qquad u_{2k+1} = 0, \qquad k = 0, 1, 2, \; \ldots \; .$$

Hence, the required series solution is

$$u(x) = 1 - \tfrac{1}{2}\,x^2 + \tfrac{1}{24}\,x^3 - \tfrac{1}{720}\,x^6 + \;\cdots\; = \sum_{k=0}^{\infty} \frac{(-1)^k}{k\,!}\,x^{2k},$$

which, by the ratio test, converges for all x. We have thus recovered the well-known Taylor series for $\cos x$, which is indeed the solution to the initial value problem. Changing the initial conditions to $u(0) = u_0 = 0$, $u'(0) = u_1 = 1$, will similarly produce the usual Taylor expansion of $\sin x$. Note that the generation of the Taylor series does not rely on any a priori knowledge of trigonometric functions or the direct solution method for linear constant-coefficient ordinary differential equations.

Building on this experience, let us describe the general method. We shall concentrate on solving a second-order homogeneous linear differential equation

$$p(x)\,\frac{d^2u}{dx^2} + q(x)\,\frac{du}{dx} + r(x)\,u = 0. \tag{11.71}$$

The coefficients $p(x), q(x), r(x)$ are assumed to be analytic functions on some common domain. This means that, at a point x_0 within the domain, they admit convergent power series expansions

$$\begin{aligned} p(x) &= p_0 + p_1\,(x - x_0) + p_2\,(x - x_0)^2 + \;\cdots\;, \\ q(x) &= q_0 + q_1\,(x - x_0) + q_2\,(x - x_0)^2 + \;\cdots\;, \\ r(x) &= r_0 + r_1\,(x - x_0) + r_2\,(x - x_0)^2 + \;\cdots\;. \end{aligned} \tag{11.72}$$

We expect that solutions to the differential equation are also analytic. This expectation is justified, provided that the equation is *regular* at the point x_0, in the following sense.

Definition 11.4. A point $x = x_0$ is a *regular point* of a second-order linear ordinary differential equation (11.71) if the leading coefficient does not vanish there:

$$p_0 = p(x_0) \neq 0.$$

A point where $p(x_0) = 0$ is known as a *singular point*.

In short, at a regular point, the second-order derivative term does not disappear, and so the equation is "genuinely" of second order.

Remark: The definition of a singular point assumes that the other two coefficients do not also vanish there, so that either $q(x_0) \neq 0$ or $r(x_0) \neq 0$. If all three functions happen to vanish at x_0, we can cancel any common factor $(x - x_0)^k$, and hence, without loss of generality, assume that at least one of the coefficient functions is nonzero at x_0.

Proofs of the basic existence theorem for differential equations at regular points can be found in [**18**, **54**, **59**].

Theorem 11.5. *Let x_0 be a regular point for the second-order homogeneous linear ordinary differential equation* (11.71). *Then there exists a unique, analytic solution $u(x)$ to the initial value problem*

$$u(x_0) = a, \qquad u'(x_0) = b. \tag{11.73}$$

The radius of convergence of the power series for $u(x)$ is at least as large as the distance from the regular point x_0 to the nearest singular point of the differential equation in the complex plane.

Thus, every solution to an analytic differential equation at a regular point x_0 can be expanded in a convergent power series

$$u(x) = u_0 + u_1(x - x_0) + u_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} u_n (x - x_0)^n. \tag{11.74}$$

Since the power series necessarily coincides with the Taylor series for $u(x)$, its coefficients†

$$u_n = \frac{u^{(n)}(x_0)}{n!}$$

are multiples of the derivatives of the function at the point x_0. In particular, the first two coefficients,

$$u_0 = u(x_0) = a, \qquad u_1 = u'(x_0) = b, \tag{11.75}$$

are fixed by the initial conditions. The remaining coefficients will then be uniquely prescribed thanks to the uniqueness of solutions to initial value problems.

Near a regular point, the second-order differential equation (11.71) admits two linearly independent analytic solutions, which we denote by $\widehat{u}(x)$ and $\widetilde{u}(x)$. The general solution can be written as a linear combination of the two basis solutions:

$$u(x) = a\,\widehat{u}(x) + b\,\widetilde{u}(x). \tag{11.76}$$

A convenient choice is to have the first satisfy the initial conditions

$$\widehat{u}(x_0) = 1, \qquad \widehat{u}'(x_0) = 0, \tag{11.77}$$

and the second satisfy

$$\widetilde{u}(x_0) = 0, \qquad \widetilde{u}'(x_0) = 1, \tag{11.78}$$

although other conventions may be used depending on the circumstances. Given (11.77–78), the linear combination (11.76) automatically satisfies the initial conditions (11.73).

The basic computational strategy to construct the power series solution to the initial value problem is a straightforward adaptation of the method used in Example 11.3. One substitutes the known power series (11.72) for the coefficient functions and the unknown power series (11.74) for the solution into the differential equation (11.71). Multiplying out the formulas and collecting the common powers of $x - x_0$ will result in a (complicated) power series whose individual coefficients must be equated to zero. The lowest-order terms are multiples of $(x - x_0)^0 = 1$, i.e., the constant terms. They produce a linear relation

$$u_2 = R_2(u_0, u_1) = R_2(a, b)$$

† Some authors prefer to include the $n!$'s in the original power series; this is purely a matter of personal taste.

that prescribes the coefficient u_2 in terms of the initial data (11.75). The coefficient of $(x - x_0)$ leads to a relation

$$u_3 = R_3(u_0, u_1, u_2) = R_3(a, b, R_2(a, b))$$

that prescribes u_3 in terms of the initial data and the previously computed coefficient u_2. And so on. At the n^{th} stage of the procedure, the coefficient of $(x - x_0)^n$ produces the linear *recurrence relation*

$$u_{n+2} = R_n(u_0, u_1, \ldots, u_{n+1}), \qquad n = 0, 1, 2, \ldots \,, \tag{11.79}$$

that will prescribe the $(n+2)^{\text{nd}}$ order coefficient in terms of the previously computed coefficients. In this fashion, we will have constructed a formal power series solution to the differential equation at a regular point. The one remaining issue is whether the resulting power series actually converges. The full analysis can be found in [**54**, **59**], and will serve to complete the proof of the general Existence Theorem 11.5.

Rather than continue on in general, the best way to learn the method is to work through another, less trivial, example.

The Airy Equation

We will illustrate the procedure by constructing power series solutions to the *Airy equation*

$$\frac{d^2 u}{dx^2} = x\,u. \tag{11.80}$$

This second-order linear ordinary differential equation, which arises in applications to optics, rainbows, and dispersive waves, has solutions that cannot be expressed in terms of elementary functions.

For the Airy equation (11.80), the leading coefficient is constant, and so every point is a regular point. For simplicity, we will look only for power series based at the origin $x_0 = 0$, and therefore of the form (11.68). Equating the two series

$$\begin{aligned} u''(x) &= 2\,u_2 + 6\,u_3\,x + 12\,u_4\,x^2 + 20\,u_5 x^3 + \cdots &&= \sum_{n=0}^{\infty} (n+1)(n+2)\,u_{n+2}\,x^n, \\ x\,u(x) &= u_0\,x + u_1\,x^2 + u_2\,x^3 + \cdots &&= \sum_{n=1}^{\infty} u_{n-1}\,x^n, \end{aligned}$$

leads to the following recurrence relations relating the coefficients:

$$\begin{array}{cr} 1 & 2\,u_2 = 0, \\ x & 6\,u_3 = u_0, \\ x^2 & 12\,u_4 = u_1, \\ x^3 & 20\,u_5 = u_2, \\ x^4 & 30\,u_6 = u_3, \\ \vdots & \vdots \\ x^n & (n+1)(n+2)\,u_{n+2} = u_{n-1}. \end{array}$$

As before, we solve them in order: The first equation determines u_2. The second prescribes $u_3 = \frac{1}{6}\,u_0$ in terms of u_0. Next, we find $u_4 = \frac{1}{12}\,u_1$ in terms of u_1, followed by $u_5 = \frac{1}{20}u_2 = 0$; then $u_6 = \frac{1}{30}\,u_3 = \frac{1}{180}\,u_0$ is first given in terms of u_3, but we already know the latter in terms of u_0. And so on.

Let us now construct two basis solutions. The first has the initial conditions

$$u_0 = \widehat{u}(0) = 1, \qquad\qquad u_1 = \widehat{u}\,'(0) = 0.$$

The recurrence relations imply that the only nonzero coefficients c_n occur when $n = 3\,k$ is a multiple of 3. Moreover,

$$u_{3\,k} = \frac{u_{3\,k-3}}{3\,k\,(3\,k-1)}\,.$$

A straightforward induction proves that

$$u_{3\,k} = \frac{1}{3\,k\,(3\,k-1)(3\,k-3)(3\,k-4)\cdots 6\cdot 5\cdot 3\cdot 2}\,.$$

The resulting solution is

$$\widehat{u}(x) = 1 + \tfrac{1}{6}x^3 + \tfrac{1}{180}x^6 + \cdots = 1 + \sum_{k=1}^{\infty} \frac{x^{3\,k}}{3\,k\,(3\,k-1)(3\,k-3)(3\,k-4)\cdots 6\cdot 5\cdot 3\cdot 2}\,. \tag{11.81}$$

Note that the denominator is similar to a factorial, except every third term is omitted. A straightforward application of the ratio test confirms that the series converges for all (complex) x, in conformity with the general Theorem 11.5, which guarantees an infinite radius of convergence because the Airy equation has no singular points.

Similarly, starting with the initial conditions

$$u_0 = \widetilde{u}(0) = 0, \qquad\qquad u_1 = \widetilde{u}\,'(0) = 1,$$

we find that the only nonzero coefficients u_n occur when $n = 3\,k+1$. The recurrence relation

$$u_{3\,k+1} = \frac{u_{3\,k-2}}{(3\,k+1)(3\,k)} \qquad \text{yields} \qquad u_{3\,k+1} = \frac{1}{(3\,k+1)(3\,k)(3\,k-2)(3\,k-3)\cdots 7\cdot 6\cdot 4\cdot 3}\,.$$

The resulting solution is

$$\widetilde{u}(x) = x + \tfrac{1}{12}\,x^4 + \tfrac{1}{504}\,x^7 + \cdots = x + \sum_{k=1}^{\infty} \frac{x^{3\,k+1}}{(3\,k+1)(3\,k)(3\,k-2)(3\,k-3)\cdots 7\cdot 6\cdot 4\cdot 3}\,. \tag{11.82}$$

Again, the denominator skips every third term in the product. Every solution to the Airy equation can be written as a linear combination of these two basis power series solutions:

$$u(x) = a\,\widehat{u}(x) + b\,\widetilde{u}(x), \qquad \text{where} \qquad a = u(0), \qquad b = u'(0).$$

Both power series (11.81, 82), converge quite rapidly, and so the first few terms will provide a reasonable approximation to the solutions for moderate values of x.

We have, in fact, already encountered another solution to the Airy equation. According to formula (8.97), the integral

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\bigl(s\,x + \tfrac{1}{3}\,s^3\bigr)\,ds \tag{11.83}$$

defines the *Airy function of the first kind*. Let us prove that it satisfies the Airy differential equation (11.80):

$$\frac{d^2}{dx^2}\,\mathrm{Ai}(x) = x\,\mathrm{Ai}(x).$$

Before differentiating, we recall the integration by parts argument in (8.96) to re-express the Airy integral in absolutely convergent form:

$$\mathrm{Ai}(x) = \frac{2}{\pi}\int_0^\infty \frac{s\,\sin\bigl(s\,x + \frac{1}{3}\,s^3\bigr)}{(x+s^2)^2}\,ds.$$

We are now permitted to differentiate under the integral sign, producing (after some algebra)

$$\frac{d^2}{dx^2}\,\mathrm{Ai}(x) - x\,\mathrm{Ai}(x) = \frac{2}{\pi}\int_0^\infty \frac{d}{ds}\left[\frac{s\,(x+s^2)\cos\bigl(s\,x+\frac{1}{3}\,s^3\bigr) - \sin\bigl(s\,x+\frac{1}{3}\,s^3\bigr)}{(x+s^2)^3}\right] ds = 0.$$

Thus, the Airy function must be a certain linear combination of the two basic series solutions:

$$\mathrm{Ai}(x) = \mathrm{Ai}(0)\,\widehat{u}(x) + \mathrm{Ai}'(0)\,\widetilde{u}(x).$$

Its values at $x = 0$ are, in fact, given by

$$\begin{aligned}\mathrm{Ai}(0) &= \frac{1}{\pi}\int_0^\infty \cos\bigl(\tfrac{1}{3}\,s^3\bigr)\,ds = \frac{\Gamma\bigl(\frac{1}{3}\bigr)}{2\,\pi\,3^{1/6}} = \frac{1}{3^{2/3}\,\Gamma\bigl(\frac{2}{3}\bigr)} \approx .355028\,,\\ \mathrm{Ai}'(0) &= -\,\frac{1}{\pi}\int_0^\infty s\,\sin\bigl(\tfrac{1}{3}\,s^3\bigr)\,ds = -\,\frac{3^{1/6}\,\Gamma\bigl(\frac{2}{3}\bigr)}{2\,\pi} = -\,\frac{1}{3^{1/3}\,\Gamma\bigl(\frac{1}{3}\bigr)} \approx -\,.258819.\end{aligned} \tag{11.84}$$

The second and third expressions involve the gamma function (11.62); a proof, based on complex integration, can be found in [**85**; p. 54].

Exercises

11.3.1. Find (*a*) $\Gamma\left(\frac{5}{2}\right)$, (*b*) $\Gamma\left(\frac{7}{2}\right)$, (*c*) $\Gamma\left(-\frac{3}{2}\right)$, (*d*) $\Gamma\left(-\frac{5}{2}\right)$.

11.3.2. Prove that $\Gamma\bigl(n+\frac{1}{2}\bigr) = \dfrac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!}$ for every positive integer n.

11.3.3. Let $x \in \mathbb{C}$ be complex. (*a*) Prove that the gamma function integral (11.62) converges, provided $\mathrm{Re}\,x > 0$. (*b*) Is formula (11.63) valid when x is complex?

♢ 11.3.4. Prove that $\Gamma(x) = \displaystyle\int_0^1 (-\log s)^{x-1}\,ds$, and hence, for $0 \le n \in \mathbb{Z}$, we have $n! = \displaystyle\int_0^1 (-\log s)^n\,ds$. *Remark*: Euler first established the latter identity directly, and used it to define the gamma function.

11.3.5. Evaluate $\displaystyle\int_0^\infty \sqrt{x}\,e^{-x^3}\,dx$.

♢ 11.3.6. Can you construct a function $f(x)$ that satisfies the factorial functional equation (11.61) and has the values $f(x) = 1$ for $0 \le x \le 1$? If so, is $f(x) = \Gamma(x+1)$?

11.3.7. Explain how to construct the power series for $\sin x$ by solving the differential equation (11.67).

11.3.8. Construct two independent power series solutions to the Euler equation $x^2 u'' - 2u = 0$ based at the point $x_0 = 1$.

11.3.9. Construct two independent power series solutions to the equation $u'' + x^2 u = 0$ based at the point $x_0 = 0$.

11.3.10. Consider the ordinary differential equation $u'' + 2x u' + 2u = 0$. (*a*) Find two linearly independent power series solutions in powers of x. (*b*) What is the radius of convergence of your power series? (*c*) By inspection of your series, find one solution to the equation expressible in terms of elementary functions. (*d*) Find an explicit (non-series) formula for the second independent power series solution.

11.3.11. Answer Exercise 11.3.10 for the equation $u'' + \frac{1}{2} x u' - \frac{1}{2} u = 0$, which is a special case of equation (8.63).

11.3.12. Consider the ordinary differential equation $u'' + x u' + 2u = 0$. (*a*) Find two linearly independent power series solutions based at $x_0 = 0$. (*b*) Write down the power series for the solution to the initial value problem $u(0) = 1$, $u'(0) = -1$. (*c*) What is the radius of convergence of your power series solution in part (a)? Can you justify this by direct inspection of your power series?

♢ 11.3.13. The *Hermite equation* of order n is

$$\frac{d^2u}{dx^2} - 2x \frac{du}{dx} + 2n u = 0. \tag{11.85}$$

Assuming $n \in \mathbb{N}$ is a nonnegative integer: (a) Find two linearly independent power series solutions based at $x_0 = 0$, and then show that one of your solutions is a polynomial of degree n. (*b*) Prove that the Hermite polynomial $H_n(x)$ defined in (8.64) solves the Hermite equation (11.85) and hence is a multiple of the polynomial solution you found in part (*a*). What is the multiple? (*c*) Prove that the Hermite polynomials are orthogonal with respect to the inner product $\langle u , v \rangle = \int_{-\infty}^{\infty} u(x)\, v(x)\, e^{-x^2}\, dx$.

11.3.14. Use the ratio test to directly determine the radius of convergence of the series solutions (11.81, 82) to the Airy equation.

11.3.15. Write down the general solution to the following ordinary differential equations:
(*a*) $u'' + (x - c)\, u = 0$, where c is a fixed constant;
(*b*) $u'' = \lambda x u$, where $\lambda \neq 0$ is a fixed nonzero constant.

♢ 11.3.16. The *Airy function of the second kind* is defined by

$$\mathrm{Bi}(x) = \frac{1}{\pi} \int_0^\infty \left[\exp\left(s x - \tfrac{1}{3} s^3 \right) + \sin\left(s x + \tfrac{1}{3} s^3 \right) \right] ds. \tag{11.86}$$

(*a*) Prove that $\mathrm{Bi}(x)$ is well defined and a solution to the Airy equation. (*b*) Given that†

$$\mathrm{Bi}(0) = \frac{1}{3^{1/6}\, \Gamma\left(\frac{2}{3}\right)}, \qquad \mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)}, \tag{11.87}$$

explain why every solution to the Airy equation can be written as a linear combination of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. (*c*) Write the two series solutions (11.81, 82) in terms of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$.

11.3.17. Use the Fourier transform to construct an L^2 solution to the Airy equation. Can you identify your solution?

♢ 11.3.18. Apply separation of variables to the Tricomi equation (4.137), and write down all separable solutions. *Hint*: See Exercise 11.3.15(*b*) and Exercise 11.3.16.

---

† See [**85**; p. 54] for a proof.

♡ 11.3.19. (*a*) Show that $u(x) = \sum_{n=1}^{\infty} (n-1)!\, x^n$ is a power series solution to the first-order linear ordinary differential equation $x^2 u' - u + x = 0$. (*b*) For which x does the series converge? (*c*) Find an analytic formula for the general solution to the equation. (*d*) Find a second-order homogeneous linear ordinary differential equation that has this power series as a (formal) solution. *Remark*: The lesson of this exercise is that not all power series solutions to ordinary differential equations converge. Theorem 11.5 guarantees convergence at a regular point, but in this example the power series is based at the singular point $x_0 = 0$.

11.3.20. *True or false*: The only function $f(x)$ that has identically zero Taylor series is the zero function.

♢ 11.3.21. Define $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ (*a*) Prove that f is a C^∞ function for all $x \in \mathbb{R}$. (*b*) Prove that $f(x)$ is not analytic by showing that its Taylor series at $x_0 = 0$ does not converge to $f(x)$ when $x \neq 0$.

Regular Singular Points

As we have just seen, constructing power series solutions at regular points is a reasonably straightforward computational exercise: one writes down a power series with arbitrary coefficients, substitutes into the differential equation along with a pair of initial conditions, and recursively solves for the coefficients. Finding a general formula for the coefficients might be challenging, but producing their successive numerical values, degree by degree, is a mechanical exercise.

However, at a singular point, the solutions cannot be typically written as an ordinary power series, and one needs to be cleverer. Of course, you may object — why not just solve the equation away from the singular point and be done with it. But there are multiple reasons not to do this. First, one may be unable to discover a general formula for the power series coefficients at regular points. Second, the most informative and interesting behavior of solutions is typically found at the singular points, and so series solutions based at singular points are particularly enlightening. And finally, one of the boundary conditions required for us to complete our construction of separable solutions to partial differential equations often occurs at a singular point.

Singular points appear in two guises. The easier to handle, and, fortunately, the ones that arise in almost all applications, are known as "regular singular points". Irregular singular points are nastier, and we will not make any attempt to understand them in this text; the curious reader is referred to [**54**, **59**].

Definition 11.6. A second-order linear homogeneous ordinary differential equation that can be written the form

$$(x - x_0)^2\, a(x)\, \frac{d^2 u}{dx^2} + (x - x_0)\, b(x)\, \frac{du}{dx} + c(x)\, u = 0, \tag{11.88}$$

where $a(x), b(x)$, and $c(x)$ are analytic at $x = x_0$ and, moreover, $a(x_0) \neq 0$, is said to have a *regular singular point* at x_0.

The simplest example of a second-order equation with a regular singular point at $x_0 = 0$ is the *Euler equation*

$$a\, x^2 u'' + b\, x\, u' + c\, u = 0, \tag{11.89}$$

with a, b, c all constant and $a \neq 0$. Note that all other points are regular points. Euler equations can be readily solved by substituting the power ansatz $u(x) = x^r$. We find

$$a x^2 u'' + b x u' + c u = a r(r-1)x^r + b r x^r + c x^r = 0,$$

provided the exponent r satisfies the *indicial equation*

$$a r (r-1) + b r + c = 0.$$

If this quadratic equation has two distinct roots $r_1 \neq r_2$, we obtain two linearly independent (possibly complex) solutions $\widehat{u}(x) = x^{r_1}$ and $\widetilde{u}(x) = x^{r_2}$. The general solution $u(x) = c_1 x^{r_1} + c_2 x^{r_2}$ is a linear combination thereof. Note that unless r_1 or r_2 is a nonnegative integer, all nonzero solutions have a singularity at the singular point $x = 0$. A repeated root, $r_1 = r_2$, has only one power solution, $\widehat{u}(x) = x^{r_1}$, and requires an additional logarithmic term, $\widetilde{u}(x) = x^{r_1} \log x$, for the second independent solution. In this case, the general solution has the form $u(x) = c_1 x^{r_1} + c_2 x^{r_1} \log x$.

The series solution method at more general regular singular points is modeled on the simple example of the Euler equation. One now seeks a solution that has a series expansion of the form†

$$u(x) = (x-x_0)^r \sum_{n=0}^{\infty} u_n (x-x_0)^n = u_0(x-x_0)^r + u_1(x-x_0)^{r+1} + u_2(x-x_0)^{r+2} + \cdots . \qquad (11.90)$$

The exponent r is known as the *index*. If $r = 0$, or, more generally, if r is a positive integer, then (11.90) is an ordinary power series, but we allow the possibility of a non-integral, or even complex, index r. We can assume, without any loss of generality, that the leading coefficient $u_0 \neq 0$. Indeed, if $u_k \neq 0$ is the first nonzero coefficient, then the series begins with the term $u_k(x-x_0)^{r+k}$, and we merely replace r by $r + k$ to write it in the form (11.90). Since any scalar multiple of a solution is a solution, we can further assume that $u_0 = 1$, in which case we call (11.90) a *normalized Frobenius series* in honor of the German mathematician Georg Frobenius, who systematically established the calculus of series solutions at regular singular points in the late 1800s. The index r, and the higher-order coefficients $u_1, u_2, \ldots$, are then found by substituting the normalized Frobenius series into the differential equation (11.88) and equating the coefficients of the powers of $x - x_0$ to zero.

Warning: Unlike those in ordinary power series expansions, the coefficients $u_0 = 1$ and u_1 are *not* prescribed by the initial conditions at the point x_0.

Since

$$\begin{aligned} u(x) &= (x-x_0)^r + u_1(x-x_0)^{r+1} + \cdots , \\ (x-x_0)\, u'(x) &= r\,(x-x_0)^r + (r+1)u_1(x-x_0)^{r+1} + \cdots , \\ (x-x_0)^2\, u''(x) &= r\,(r-1)\,(x-x_0)^r + (r+1)\, r\, u_1(x-x_0)^{r+1} + \cdots , \end{aligned}$$

the terms of lowest order in equation (11.88) are multiples of $(x-x_0)^r$. Equating their coefficients to zero produces a quadratic equation of the form

$$a_0\, r\,(r-1) + b_0\, r + c_0 = 0, \qquad (11.91)$$

† If r is real but non-integral, and $x < x_0$, then one can replace $x - x_0$ by $x_0 - x$ or, alternatively, use absolute values throughout.

where, referring back to (11.71),

$$a_0 = a(x_0), \qquad b_0 = b(x_0), \qquad c_0 = c(x_0),$$

are the leading coefficients in the power series expansions of the individual coefficient functions. The quadratic equation (11.91) is known as the *indicial equation*, since it determines the possible indices r in the Frobenius expansion (11.90) of a solution.

As with the Euler equation, the quadratic indicial equation usually has two roots, say r_1 and r_2, which provide two allowable indices, and one thus expects to find two independent Frobenius expansions. Usually, this expectation is realized, but there is an important exception. The general result is summarized in the following list:

(*i*) If $r_1 - r_2$ is not an integer, then there are two linearly independent solutions $\widehat{u}(x)$ and $\widetilde{u}(x)$, each having convergent normalized Frobenius expansions of the form (11.90).

(*ii*) If $r_1 = r_2$, then there is only one solution $\widehat{u}(x)$ with a normalized Frobenius expansion (11.90). One can construct a second independent solution of the form

$$\widetilde{u}(x) = \log(x - x_0)\,\widehat{u}(x) + v(x), \qquad \text{where} \qquad v(x) = \sum_{n=1}^{\infty} v_n (x - x_0)^{n+r_1} \tag{11.92}$$

is a convergent Frobenius series.

(*iii*) Finally, if $r_1 = r_2 + k$, where $k > 0$ is a positive integer, then there is a nonzero solution $\widehat{u}(x)$ with a convergent Frobenius expansion corresponding to the larger index r_1. One can construct a second independent solution of the form

$$\widetilde{u}(x) = c\log(x - x_0)\,\widehat{u}(x) + v(x), \qquad \text{where} \qquad v(x) = (x - x_0)^{r_2} + \sum_{n=1}^{\infty} v_n (x - x_0)^{n+r_2} \tag{11.93}$$

is a convergent Frobenius series, and c is a constant, which may be 0, in which case the second solution $\widetilde{u}(x)$ is also of Frobenius form.

Thus, in every case, the differential equation has at least one nonzero solution with a convergent Frobenius expansion. If the second independent solution does not have a Frobenius expansion, then it requires an additional logarithmic term of a well-prescribed form. Rather than try to develop the general theory in any more detail here, we will content ourselves to work through a couple of particular examples.

Example 11.7. Consider the second-order ordinary differential equation

$$\frac{d^2u}{dx^2} + \left(\frac{1}{x} + \frac{x}{2}\right)\frac{du}{dx} + u = 0. \tag{11.94}$$

We look for series solutions based at $x = 0$. Note that, upon multiplying by x^2, the equation takes the form

$$x^2 u'' + x\left(1 + \tfrac{1}{2}x^2\right)u' + x^2 u = 0,$$

and hence $x_0 = 0$ is a regular singular point, with $a(x) = 1,\ b(x) = 1 + \frac{1}{2}x^2,\ c(x) = x^2$. We thus look for a solution that can be represented by a Frobenius expansion:

$$\begin{aligned}
u(x) &= x^r + u_1 x^{r+1} + \cdots + u_n x^{n+r} + \cdots, \\
x\,u'(x) &= r\,x^r + (r+1)u_1 x^{r+1} + \cdots + (n+r)u_n x^{n+r} + \cdots, \\
\tfrac{1}{2}x^3 u'(x) &= \tfrac{1}{2}r\,x^{r+2} + \tfrac{1}{2}(r+1)u_1 x^{r+3} + \cdots + \tfrac{1}{2}(n+r-2)\,u_{n-2}x^{n+r} + \cdots, \\
x^2 u''(x) &= r(r-1)\,x^r + (r+1)\,r\,u_1 x^{r+1} + \cdots + (n+r)(n+r-1)u_n x^{n+r} + \cdots.
\end{aligned} \tag{11.95}$$

Substituting into the differential equation, we find that the coefficient of x^r leads to the indicial equation

$$r^2 = 0.$$

There is only one root, $r = 0$, and hence, even though we are at a singular point, the Frobenius expansion reduces to an ordinary power series. The coefficient of $x^{r+1} = x$ tells us that $u_1 = 0$. The general recurrence relation, for $n \geq 2$, is

$$n^2 u_n + \tfrac{1}{2} n u_{n-2} = 0,$$

and hence

$$u_n = -\frac{u_{n-2}}{2n}.$$

Therefore, the odd coefficients $u_{2k+1} = 0$ are all zero, while the even ones are

$$u_{2k} = -\frac{u_{2k-2}}{4k} = \frac{u_{2k-4}}{4k(4k-4)} = -\frac{u_{2k-6}}{4k(4k-4)(4k-8)} = \cdots = \frac{(-1)^k}{4^k k!}, \qquad \text{since} \quad u_0 = 1.$$

The resulting power series assumes a recognizable form:

$$\widehat{u}(x) = \sum_{k=1}^{\infty} u_{2k} x^{2k} = \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{x^2}{4}\right)^k = e^{-x^2/4},$$

which is an explicit elementary solution to the ordinary differential equation (11.94).

Since there is only one root to the indicial equation, the second solution $\widetilde{u}(x)$ will require a logarithmic term. It can be constructed by a second application of the Frobenius method using the more complicated form (11.92). Alternatively, since the first solution is known, we can use a well-known reduction trick, [**23**]. Given one solution $\widehat{u}(x)$ to a second-order linear ordinary differential equation, the general solution can be found by substituting the ansatz

$$u(x) = v(x)\,\widehat{u}(x) = v(x)\, e^{-x^2/4} \tag{11.96}$$

into the equation. In this case,

$$\begin{aligned} u'' + \left(\frac{1}{x} + \frac{x}{2}\right) u' + u &= v \left[\widehat{u}'' + \left(\frac{1}{x} + \frac{x}{2}\right)\widehat{u}' + \widehat{u}\right] + v' \left[2\widehat{u}' + \left(\frac{1}{x} + \frac{x}{2}\right)\widehat{u}\right] + v''\,\widehat{u} \\ &= e^{-x^2/4} \left[v'' + \left(\frac{1}{x} - \frac{x}{2}\right) v'\right]. \end{aligned}$$

If u is to be a solution, v' must satisfy a linear first-order ordinary differential equation:

$$v'' + \left(\frac{1}{x} - \frac{x}{2}\right) v' = 0, \qquad \text{and hence} \qquad v' = \frac{c}{x}\, e^{x^2/4}, \qquad v = c \int \frac{e^{x^2/4}}{x}\, dx + b,$$

where c, b are arbitrary constants. We conclude that the general solution to the original differential equation is

$$\widetilde{u}(x) = v(x)\,\widehat{u}(x) = \left(c \int \frac{e^{x^2/4}}{x}\, dx + b \right) e^{-x^2/4}. \tag{11.97}$$

Bessel's Equation

Perhaps the most important "non-elementary" ordinary differential equation is

$$x^2 u'' + x\, u' + (x^2 - m^2)\, u = 0, \tag{11.98}$$

known as *Bessel's equation* of *order* m. We assume here that the order $m \geq 0$ is a nonnegative real number. (Exercise 11.3.30 investigates Bessel equations of imaginary order.) The Bessel equation arises from separation of variables in a variety of partial differential equations, including the Laplace, heat, and wave equations on a disk, a cylinder, and a spherical ball.

The Bessel equation cannot (except in a few particular instances) be solved in terms of elementary functions, and so the use of power series is essential. The leading coefficient, $p(x) = x^2$, is nonzero *except* when $x = 0$, and so all points except the origin are regular. Therefore, at any $x_0 \neq 0$, the standard power series construction can be used to produce the solutions of the Bessel equation. However, the recurrence relations for the coefficients are not particularly easy to solve in closed form. Moreover, applications tend to demand understanding the behavior of solutions at the singular point $x_0 = 0$.

Comparison with (11.88) immediately shows that $x_0 = 0$ is a regular singular point, and so we seek solutions in Frobenius form. We substitute the first, second, and fourth expressions in (11.95) into the Bessel equation and then equate the coefficients of the various powers of x to zero. The lowest power, x^r, provides the indicial equation

$$r(r-1) + r - m^2 = r^2 - m^2 = 0.$$

It has two solutions, $r = \pm\, m$, except when $m = 0$, for which $r = 0$ is the only index.

The higher powers of x lead to recurrence relations for the coefficients u_n in the Frobenius series. Replacing m^2 by r^2 produces

$$\begin{aligned} x^{r+1}: &\quad \big[\,(r+1)^2 - r^2\,\big]u_1 = (2r+1)u_1 = 0, &\quad u_1 &= 0,\\ x^{r+2}: &\quad \big[\,(r+2)^2 - r^2\,\big]u_2 + 1 = (4r+4)u_2 + 1 = 0, &\quad u_2 &= -\frac{1}{4r+4},\\ x^{r+3}: &\quad \big[\,(r+3)^2 - r^2\,\big]u_3 + u_1 = (6r+9)u_3 + u_1 = 0, &\quad u_3 &= -\frac{u_1}{6r+9} = 0, \end{aligned}$$

and, in general,

$$x^{r+n}: \quad \big[\,(r+n)^2 - r^2\,\big]u_n + u_{n-2} = n(2r+n)u_n + u_{n-2} = 0.$$

Thus, the general recurrence relation is

$$u_n = -\frac{1}{n(2r+n)}\,u_{n-2}, \qquad n = 2, 3, 4, \ldots\,. \tag{11.99}$$

Starting with $u_0 = 1$, $u_1 = 0$, it is easy to deduce that all $u_n = 0$ for all odd $n = 2k+1$, while for even $n = 2k$,

$$\begin{aligned} u_{2k} &= -\frac{u_{2k-2}}{4k(k+r)} = \frac{u_{2k-4}}{16\,k(k-1)(r+k)(r+k-1)} = \cdots \\ &= \frac{(-1)^k}{2^{2k}\,k(k-1)\cdots 3\cdot 2\,(r+k)(r+k-1)\cdots(r+2)(r+1)}\,. \end{aligned}$$

We have thus found the series solution

$$\widehat{u}(x) = \sum_{k=0}^{\infty} u_{2k}\,x^{r+2k} = \sum_{k=0}^{\infty} \frac{(-1)^k\,x^{r+2k}}{2^{2k}\,k!\,(r+k)(r+k-1)\cdots(r+2)(r+1)}\,. \tag{11.100}$$

So far, we have not paid attention to the precise values of the indices $r = \pm m$. In order to continue the recurrence, we need to ensure that the denominators in (11.99) are never 0. Since $n > 0$, a vanishing denominator will appear whenever $2r + n = 0$, and so $r = -\frac{1}{2}n$ is either a negative integer $-1, -2, -3, \ldots$ or half-integer $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. This will occur when the order $m = -r = \frac{1}{2}n$ is either an integer or a half-integer. Indeed, these are precisely the situations in which the two indices, namely $r_1 = -m$ and $r_2 = m$, differ by an integer, $r_2 - r_1 = n$, and so we are in the tricky case *(iii)* of the Frobenius method.

There is, in fact, a major difference between the integral and the half-integral cases. Recall that the odd coefficients $u_{2k+1} = 0$ in the Frobenius series automatically vanish, and so we only have to worry about the recurrence relation (11.99) for *even* values of n. When $n = 2k$, the factor $2r + n = 2(r + k) = 0$ vanishes only when $r = -k$ is a negative integer; the half-integral values do not, in fact cause problems. Therefore, if the order $m \geq 0$ is *not* an integer, then the Bessel equation of order m admits two linearly independent Frobenius solutions, given by the expansions (11.100) with exponents $r = +m$ and $r = -m$. On the other hand, if m is an integer, there is only one Frobenius solution, namely the expansion (11.100) for the positive index $r = +m$. The Frobenius recurrence with index $r = -m$ breaks down, and the second independent solution must include a logarithmic term; details appear below.

By convention, the standard *Bessel function* of order m is obtained by multiplying the Frobenius solution (11.100) with $r = m$ by

$$\frac{1}{2^m\, m!}, \qquad \text{or, more generally,} \qquad \frac{1}{2^m\, \Gamma(m+1)}, \tag{11.101}$$

where the first factorial form can be used if m is a nonnegative integer, while the more general gamma function expression must be employed for non-integral values of m. The result is

$$\begin{aligned} J_m(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{m+2k}}{2^{2k+m}\, k!\, (m+k)!} \\ &= \frac{1}{2^m\, m!}\left[x^m - \frac{x^{m+2}}{4(m+1)} + \frac{x^{m+4}}{32(m+1)(m+2)} - \frac{x^{m+6}}{384(m+1)(m+2)(m+3)} + \cdots \right]. \end{aligned} \tag{11.102}$$

When m is non-integral, the $(m + k)!$ should be replaced by $\Gamma(m + k + 1)$, and $m!$ by $\Gamma(m + 1)$. With this convention, the series is well defined for all real m except when $m = -1, -2, -3, \ldots$ is a negative integer. Actually, if m is a negative integer, the first m terms in the series vanish, because, at negative integer values, $\Gamma(-n) = \infty$. With this convention, one can prove that

$$J_{-m}(x) = (-1)^m J_m(x), \qquad m = 1, 2, 3, \ldots . \tag{11.103}$$

A simple application of the ratio test tells us that the power series converges for all (complex) values of x, and hence $J_m(x)$ is everywhere analytic. Indeed, the convergence is quite rapid when x is of moderate size, and so summing the series is a reasonably effective method for computing the Bessel function $J_m(x)$ — although in serious applications one adopts more sophisticated numerical techniques based on asymptotic expansions and integral formulas, [**85**, **86**]. In particular, we note that

$$J_0(0) = 1, \qquad J_m(0) = 0, \qquad m > 0. \tag{11.104}$$

Figure 11.4 displays graphs of the first four Bessel functions for $0 \leq x \leq 20$; the vertical axes range from $-.5$ to 1.0. Most software packages, both symbolic and numeric, include

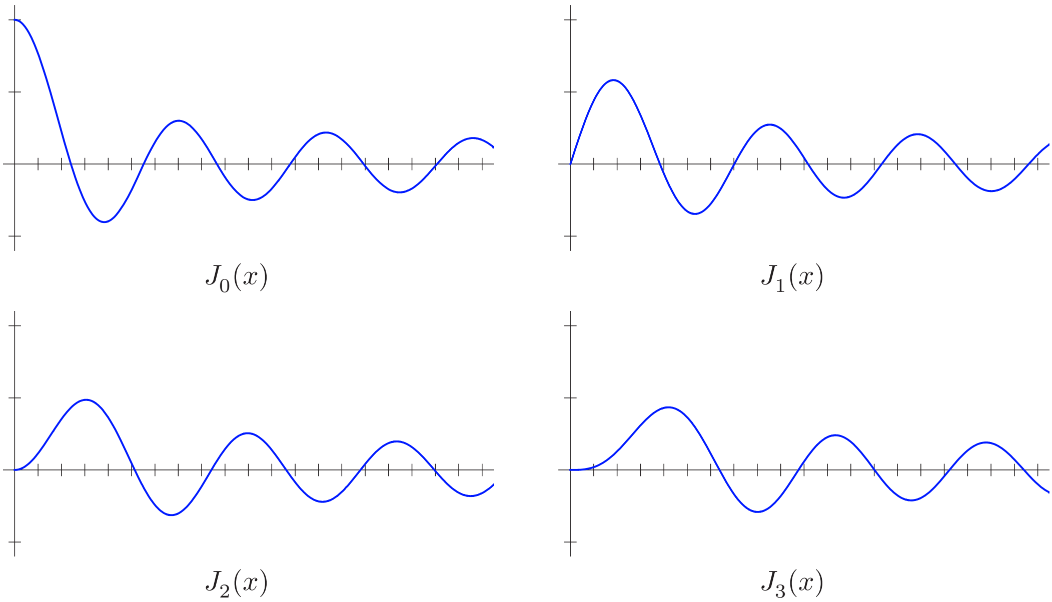


Figure 11.4. Bessel functions.

routines for accurately evaluating and graphing Bessel functions, and their properties can be regarded as well known.

Example 11.8. Consider the Bessel equation of order $m = \frac{1}{2}$. There are two indices, $r = \pm\frac{1}{2}$, and the Frobenius method yields two independent solutions: $J_{1/2}(x)$ and $J_{-1/2}(x)$. For the first, with $r = \frac{1}{2}$, the recurrence relation (11.99) takes the form

$$u_n = -\frac{u_{n-2}}{(n+1)\,n}\,.$$

Starting with $u_0 = 1$ and $u_1 = 0$, the general formula is easily found to be

$$u_n = \begin{cases} \dfrac{(-1)^k}{(n+1)\,!}\,, & n = 2\,k \text{ even}, \\ 0 & n = 2\,k+1 \text{ odd}. \end{cases}$$

Therefore, the resulting solution is

$$\widehat{u}(x) = \sqrt{x}\,\sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k+1)\,!}\,x^{2k} = \frac{1}{\sqrt{x}}\,\sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k+1)\,!}\,x^{2k+1} = \frac{\sin x}{\sqrt{x}}\,.$$

According to (11.101), the Bessel function of order $\frac{1}{2}$ is obtained by dividing this function by

$$\sqrt{2}\,\Gamma\left(\tfrac{3}{2}\right) = \sqrt{\frac{\pi}{2}}\,,$$

where we used (11.66) to evaluate the gamma function at $\frac{3}{2}$. Therefore,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x\,. \tag{11.105}$$

Similarly, for the other index $r = -\frac{1}{2}$, the recurrence relation

$$u_n = -\frac{u_{n-2}}{n\,(n-1)}$$

leads to the formula

$$u_n = \begin{cases} \dfrac{(-1)^k}{n!}, & n = 2k \text{ even}, \\ 0 & n = 2k+1 \text{ odd}, \end{cases}$$

for its coefficients, corresponding to the solution

$$\widetilde{u}(x) = x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} = \frac{\cos x}{\sqrt{x}}\,.$$

Therefore, in view of (11.101) and (11.65), the Bessel function of order $-\frac{1}{2}$ is

$$J_{-1/2}(x) = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\, \frac{\cos x}{\sqrt{x}} = \sqrt{\frac{2}{\pi x}}\, \cos x\,. \tag{11.106}$$

As we noted above, if m is not an integer, the two independent solutions to the Bessel equation of order m are $J_m(x)$ and $J_{-m}(x)$. However, when m is an integer, (11.103) implies that these two solutions are constant multiples of each other, and so one must look elsewhere for a second independent solution. One method is to use a generalized Frobenius expansion involving a logarithmic term, i.e., (11.92) when $m = 0$ (see Exercise 11.3.33) or (11.93) when $m > 0$. A second approach is to employ the reduction procedure used in Example 11.7. Yet another option relies on the following limiting procedure; see [**85**, **119**] for full details.

Theorem 11.9. *If $m > 0$ is* not *an integer, then the Bessel functions $J_m(x)$ and $J_{-m}(x)$ provide two linearly independent solutions to the Bessel equation of order m. On the other hand, if $m = 0, 1, 2, 3, \ldots$ is an integer, then a second independent solution, traditionally denoted by $Y_m(x)$ and called the* Bessel function of the second kind *of order m, can be found as a limiting case*

$$Y_m(x) = \lim_{\nu \to m} \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \tag{11.107}$$

of a certain linear combination of Bessel functions of non-integral order ν.

With some further analysis, it can be shown that the Bessel function of the second kind of order m has the logarithmic Frobenius expansion

$$Y_m(x) = \frac{2}{\pi}\left(\gamma + \log\frac{x}{2}\right) J_m(x) + \sum_{k=0}^{\infty} b_k x^{2k-m}, \qquad m = 0, 1, 2, \ldots\,, \tag{11.108}$$

with coefficients

$$b_k = \begin{cases} -\dfrac{(m-k-1)!}{\pi\, 2^{2k-m}\, k!}, & 0 \le k \le m-1, \\ \dfrac{(-1)^{k-m-1}(h_{k-m} + h_k)}{\pi\, 2^{2k-m}\, k!\,(k-m)!}, & k \ge m, \end{cases}$$

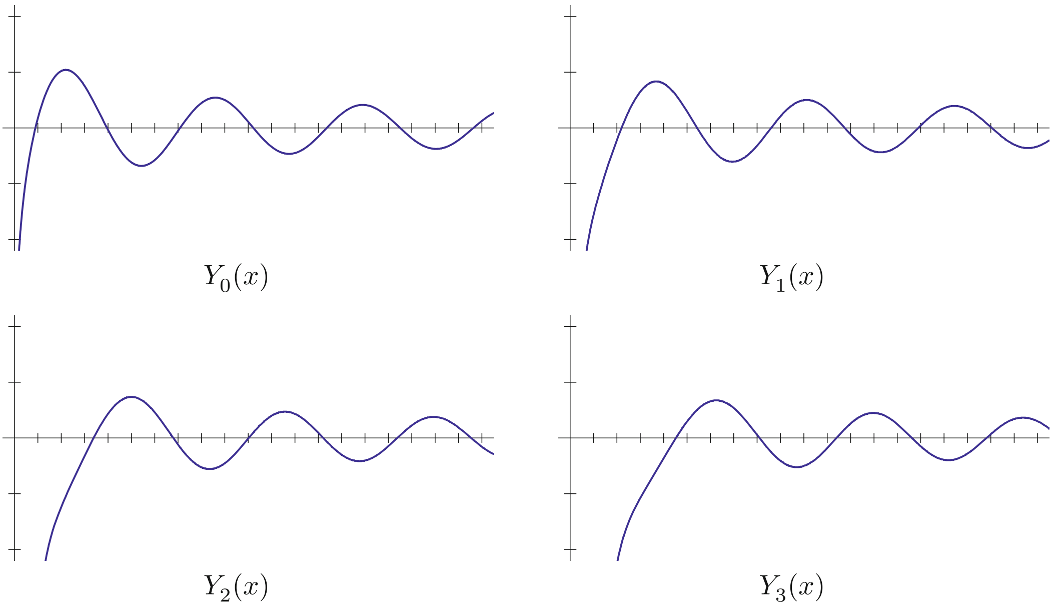


Figure 11.5. Bessel functions of the second kind.

where

$$h_0 = 0, \qquad h_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}, \qquad k > 0,$$

while

$$\gamma = \lim_{k \to \infty} \bigl(h_k - \log k \bigr) \approx .5772156649\ldots \tag{11.109}$$

is known as the *Euler* or *Euler–Mascheroni constant*. All Bessel functions of the second kind have a singularity at the origin $x = 0$; indeed, by inspection of (11.108), we find that the leading asymptotics as $x \to 0$ are

$$Y_0(x) \sim \frac{2}{\pi} \log x, \qquad Y_m(x) \sim -\frac{2^m (m-1)!}{\pi x^m}, \qquad m > 0. \tag{11.110}$$

Figure 11.5 contains graphs of the first four Bessel function of the second kind on the interval $0 < x \le 20$; the vertical axis ranges from -1 to 1.

Finally, we show how Bessel functions of different orders are interconnected by two important recurrence relations.

Proposition 11.10. *The Bessel functions are related by the following formulae*:

$$\frac{dJ_m}{dx} + \frac{m}{x} J_m(x) = J_{m-1}(x), \qquad -\frac{dJ_m}{dx} + \frac{m}{x} J_m(x) = J_{m+1}(x). \tag{11.111}$$

Proof: Differentiating the power series

$$x^m J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2m+2k}}{2^{2k+m}\, k!\, (m+k)!}$$

produces

$$\frac{d}{dx}\left[\,x^m J_m(x)\,\right] = \sum_{k=0}^{\infty} \frac{(-1)^k\, 2\,(m+k)x^{2m+2k-1}}{2^{2k+m}\, k!\,(m+k)!}$$
$$= x^m \sum_{k=0}^{\infty} \frac{(-1)^k x^{m-1+2k}}{2^{2k+m-1}\, k!\,(m-1+k)!} = x^m\, J_{m-1}(x). \tag{11.112}$$

Expansion of the left-hand side of this formula leads to

$$x^m \frac{dJ_m}{dx} + m\,x^{m-1} J_m(x) = \frac{d}{dx}\left[\,x^m J_m(x)\,\right] = x^m J_{m-1}(x),$$

which establishes the first recurrence formula (11.111). The second is proved by a similar manipulation involving differentiation of $x^{-m}\, J_m(x)$. *Q.E.D.*

For example, using the second recurrence formula (11.111) along with (11.105), we can write the Bessel function of order $\frac{3}{2}$ in elementary terms:

$$J_{3/2}(x) = -\,\frac{dJ_{1/2}(x)}{dx} + \frac{1}{2\,x}\, J_{1/2}(x)$$
$$= -\sqrt{\frac{2}{\pi}}\left(\frac{\cos x}{x^{1/2}} - \frac{\sin x}{2\,x^{3/2}}\right) + \sqrt{\frac{2}{\pi}}\,\frac{\sin x}{2\,x^{3/2}} = \sqrt{\frac{2}{\pi}}\,\frac{\sin x - x\cos x}{x^{3/2}}\,. \tag{11.113}$$

Iterating, one concludes that Bessel functions of half-integral order, $m = \pm\frac{1}{2}\,, \pm\frac{3}{2}\,, \pm\frac{5}{2}\,, \ldots,$ are all elementary functions, in that they can be written in terms of trigonometric functions and powers of $\sqrt{x}\,$. We will make use of these functions in our treatment of the three-dimensional heat and wave equations in spherical geometry. On the other hand, all of the other Bessel functions are non-elementary special functions.

With this, we conclude our brief introduction to the method of Frobenius and the basics of Bessel functions. The reader interested in delving further into either the general method or the host of additional properties of Bessel functions is encouraged to consult a more specialized text, e.g., [**59**, **85**, **119**].

Exercises

11.3.22. Consider the ordinary differential equation $2\,x\,u'' + u' + x\,u = 0$. (*a*) Prove that $x = 0$ is a regular singular point. (*b*) Find two independent series solutions in powers of x.

♡ 11.3.23. Consider the differential equation $\dfrac{u''}{2-x} = \dfrac{u}{x^2}\,.$ (*a*) Classify all $x_0 \in \mathbb{R}$ as either a (*i*) regular point; (*ii*) regular singular point; and/or (*iii*) irregular singular point. Explain your answers. (*b*) Find a series solution to the equation based at the point $x_0 = 0$, or explain why none exists. What is the radius of convergence of your series?

11.3.24. Consider the differential equation $u'' + \left(1 - \dfrac{1}{x}\right) u' + u = 0.$

(*a*) Classify all $x_0 \in \mathbb{R}$ as either (*i*) a regular point; (*ii*) a regular singular point; (*iii*) an irregular singular point; (*iv*) none of the above. Explain your answers.
(*b*) Write out the first five nonzero terms in a series solution.

11.3.25. Consider the differential equation $4x u'' + 2u' + u = 0$. (*a*) Classify the values of x for which the equation has regular points, regular singular points, and irregular singular points. (*b*) Find two independent series solutions, in powers of x. For what values of x do your series converge? (*c*) By inspection of your series, write the general solution to the equation in terms of elementary functions.

♡ 11.3.26. The *Chebyshev differential equation* is $(1 - x^2)u'' - x u' + m^2 u = 0$. (*a*) Find all (*i*) regular points; (*ii*) regular singular points; (*iii*) irregular singular points. (*b*) Show that if m is an integer, the equation has a polynomial solution of degree m, known as a *Chebyshev polynomial*. Write down the Chebyshev polynomials of degrees $1, 2$, and 3. (*c*) For $m = 1$, find two linearly independent series solutions based at the point $x_0 = 1$.

11.3.27. Write the following Bessel functions in terms of elementary functions:

$$\text{(a) } J_{5/2}(x), \qquad (b)\ J_{7/2}(x), \qquad (c)\ J_{-3/2}(x).$$

♢ 11.3.28. Prove the identity (11.103).

11.3.29. Suppose that $u(x)$ solves Bessel's equation. (a) Find a second order ordinary differential equation satisfied by the function $w(x) = \sqrt{x}\, u(x)$. (*b*) Use this result to rederive the formulas for $J_{1/2}(x)$ and $J_{-1/2}(x)$.

♢ 11.3.30. Let $m \geq 0$ be real, and consider the *modified Bessel equation* of order m:

$$x^2 u'' + x u' - (x^2 + m^2)\, u = 0. \tag{11.114}$$

(*a*) Explain why $x_0 = 0$ is a regular singular point.

(*b*) Use the method of Frobenius to construct a series solution based at $x_0 = 0$. Can you relate your solutions to the Bessel function $J_m(x)$?

♢ 11.3.31. (*a*) Let a, b, c be constants with $b, c \neq 0$. Show that the function $u(x) = x^a J_0(b x^c)$ solves the ordinary differential equation

$$x^2 \frac{d^2u}{dx^2} + (1 - 2a)\, x \frac{du}{dx} + (b^2 c^2 x^{2c} + a^2)\, u = 0.$$

What is the general solution to this equation?
(*b*) Find the general solution to the ordinary differential equation

$$x^2 \frac{d^2u}{dx^2} + \alpha x \frac{du}{dx} + (\beta x^{2c} + \gamma)\, u = 0,$$

for constants α, β, γ, c with $\beta, c \neq 0$.

♡ 11.3.32. Let $k > 0$ be a constant. The ordinary differential equation $\dfrac{d^2u}{dt^2} + e^{-2t} u = 0$ describes the vibrations of a weakening spring whose stiffness $k(t) = e^{-2t}$ is exponentially decaying in time. (*a*) Show that this equation can be solved in terms of Bessel functions of order 0. *Hint*: Perform a change of variables. (*b*) Does the solution tend to 0 as $t \to \infty$?

♡ 11.3.33. We know that $\widehat{u}(x) = J_0(x)$ is a solution to the Bessel equation of order 0, namely

$$x u'' + u' + x u = 0. \tag{11.115}$$

In accordance with the general Frobenius method, construct a second solution of the form

$$\widetilde{u}(x) = J_0(x) \log x + \sum_{n=1}^{\infty} v_n x^n.$$

11.3.34. Is it possible to have all solutions to an ordinary differential equation bounded at a regular singular point? If not, explain why not. If true, give an example where this happens.

11.4 The Heat Equation in a Disk, Continued

Now that we have acquired some familiarity with the solutions to Bessel's ordinary differential equation, we are ready to analyze the separable solutions to the heat equation in a polar geometry. At the end of Section 11.2, we were left with the task of solving the Bessel equation (11.58) of integer order m. As we now know, there are two independent solutions, namely the Bessel function of the first kind J_m, (11.102), and the more complicated Bessel function of the second kind Y_m, (11.107), and hence the general solution has the form

$$p(z) = c_1 J_m(z) + c_2 Y_m(z),$$

for constants c_1, c_2. Reverting to our original radial coordinate $r = z/\sqrt{\lambda}$, we conclude that every solution to the radial equation (11.56) has the form

$$p(r) = c_1 J_m\big(\sqrt{\lambda}\, r\big) + c_2 Y_m\big(\sqrt{\lambda}\, r\big).$$

Now, the singular point $r = 0$ represents the center of the disk, and the solutions must remain bounded there. While this is true for $J_m(z)$, the second Bessel function $Y_m(z)$ has, according to (11.110), a singularity at $z = 0$ and so is unsuitable for the present purposes. (On the other hand, it plays a role in other situations, e.g., the heat equation on an annular ring.) Thus, every separable solution that is bounded at $r = 0$ comes from the rescaled Bessel function of the first kind of order m:

$$p(r) = J_m\big(\sqrt{\lambda}\, r\big). \tag{11.116}$$

The Dirichlet boundary condition at the disk's rim $r = 1$ requires

$$p(1) = J_m\big(\sqrt{\lambda}\,\big) = 0.$$

Therefore, in order that λ be a bona fide eigenvalue, $\sqrt{\lambda}$ must be a *root* of the m^{th} order Bessel function J_m.

Remark: We already know, thanks to the positive definiteness of the Dirichlet boundary value problem, that the Helmholtz eigenvalues must all be positive, $\lambda > 0$, and so there will be no difficulty in taking its square root.

The graphs of $J_m(z)$ strongly indicate, and, indeed, it can be rigorously proved, [**85, 119**], that as z increases above 0, each Bessel function oscillates, with slowly decreasing amplitude, between positive and negative values. In fact, asymptotically,

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}}\, \cos\big[\, z - \big(\tfrac{1}{2} m + \tfrac{1}{4}\big)\pi\,\big] \qquad \text{as} \qquad z \longrightarrow \infty, \tag{11.117}$$

and so the oscillations become essentially the same as a (phase-shifted) cosine whose amplitude decreases like $z^{-1/2}$. As a consequence, there exists an infinite sequence of *Bessel roots*, which we number in increasing order:

$$\begin{gathered} J_m(\zeta_{m,n}) = 0, \qquad \text{where} \\ 0 < \zeta_{m,1} < \zeta_{m,2} < \zeta_{m,3} < \;\cdots \quad \text{with} \quad \zeta_{m,n} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty. \end{gathered} \tag{11.118}$$

It is worth emphasizing that the Bessel functions are *not* periodic, and so their roots are not evenly spaced. However, as a consequence of (11.117), the large Bessel roots are asymptotically close to the evenly spaced roots of the shifted cosine:

$$\zeta_{m,n} \sim \big(n + \tfrac{1}{2} m - \tfrac{1}{4}\big)\pi \qquad \text{as} \qquad n \longrightarrow \infty. \tag{11.119}$$

Owing to their physical importance in a wide range of problems, the Bessel roots have been extensively tabulated. The accompanying table displays all Bessel roots that are < 12 in magnitude. The columns of the table are indexed by m, the order of the Bessel function, and the rows by n, the root number.

Table of Bessel Roots $\zeta_{m,n}$

$n \setminus m$	0	1	2	3	4	5	6	7	$\ldots$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715	9.9361	11.0864	$\ldots$
2	5.5201	7.0156	8.4172	9.7610	11.0647	$\vdots$	$\vdots$	$\vdots$	
3	8.6537	10.1735	11.6198	$\vdots$	$\vdots$				
4	11.7915	$\vdots$	$\vdots$						
$\vdots$	$\vdots$								

Remark: According to (11.102),

$$J_m(0) = 0 \qquad \text{for} \qquad m > 0, \qquad \text{while} \qquad J_0(0) = 1.$$

However, we do not count 0 as a bona fide Bessel root, since it does not lead to a valid eigenfunction for the Helmholtz boundary value problem.

Summarizing our progress so far, the eigenvalues

$$\lambda_{m,n} = \zeta_{m,n}^2, \qquad n = 1, 2, 3, \ldots, \qquad m = 0, 1, 2, \ldots, \tag{11.120}$$

of the Bessel boundary value problem (11.56–57) are the squares of the roots of the Bessel function of order m. The corresponding eigenfunctions are

$$w_{m,n}(r) = J_m(\zeta_{m,n}\, r)\,, \qquad n = 1, 2, 3, \ldots, \qquad m = 0, 1, 2, \ldots, \tag{11.121}$$

defined for $0 \le r \le 1$. Combining (11.121) with the formula (11.55) for the angular components, we conclude that the separable solutions (11.53) to the polar Helmholtz boundary value problem (11.51) are

$$\begin{aligned} v_{0,n}(r) &= J_0(\zeta_{0,n}\, r), \\ v_{m,n}(r,\theta) &= J_m(\zeta_{m,n}\, r)\, \cos m\,\theta, \qquad \text{where} \qquad m, n = 1, 2, 3, \ldots . \\ \widehat{v}_{m,n}(r,\theta) &= J_m(\zeta_{m,n}\, r)\, \sin m\,\theta, \end{aligned} \tag{11.122}$$

These solutions define the *normal modes* for the unit disk; Figure 11.6 plots the first few of them. The eigenvalues $\lambda_{0,n}$ are simple, and contribute radially symmetric eigenfunctions, whereas the eigenvalues $\lambda_{m,n}$ for $m > 0$ are double, and produce two linearly independent separable eigenfunctions, with trigonometric dependence on the angular variable.

Recalling the original ansatz (11.50), we have at last produced the basic separable eigensolutions

$$\begin{aligned} u_{0,n}(t,r) &= e^{-\zeta_{0,n}^2 t}\, v_{0,n}(r) = e^{-\zeta_{0,n}^2 t}\, J_0(\zeta_{0,n}\, r), \\ u_{m,n}(t,r,\theta) &= e^{-\zeta_{m,n}^2 t}\, v_{m,n}(r,\theta) = e^{-\zeta_{m,n}^2 t}\, J_m(\zeta_{m,n}\, r)\, \cos m\,\theta, \\ \widehat{u}_{m,n}(t,r,\theta) &= e^{-\zeta_{m,n}^2 t}\, \widehat{v}_{m,n}(r,\theta) = e^{-\zeta_{m,n}^2 t}\, J_m(\zeta_{m,n}\, r)\, \sin m\,\theta, \qquad m, n = 1, 2, 3, \ldots , \end{aligned} \tag{11.123}$$

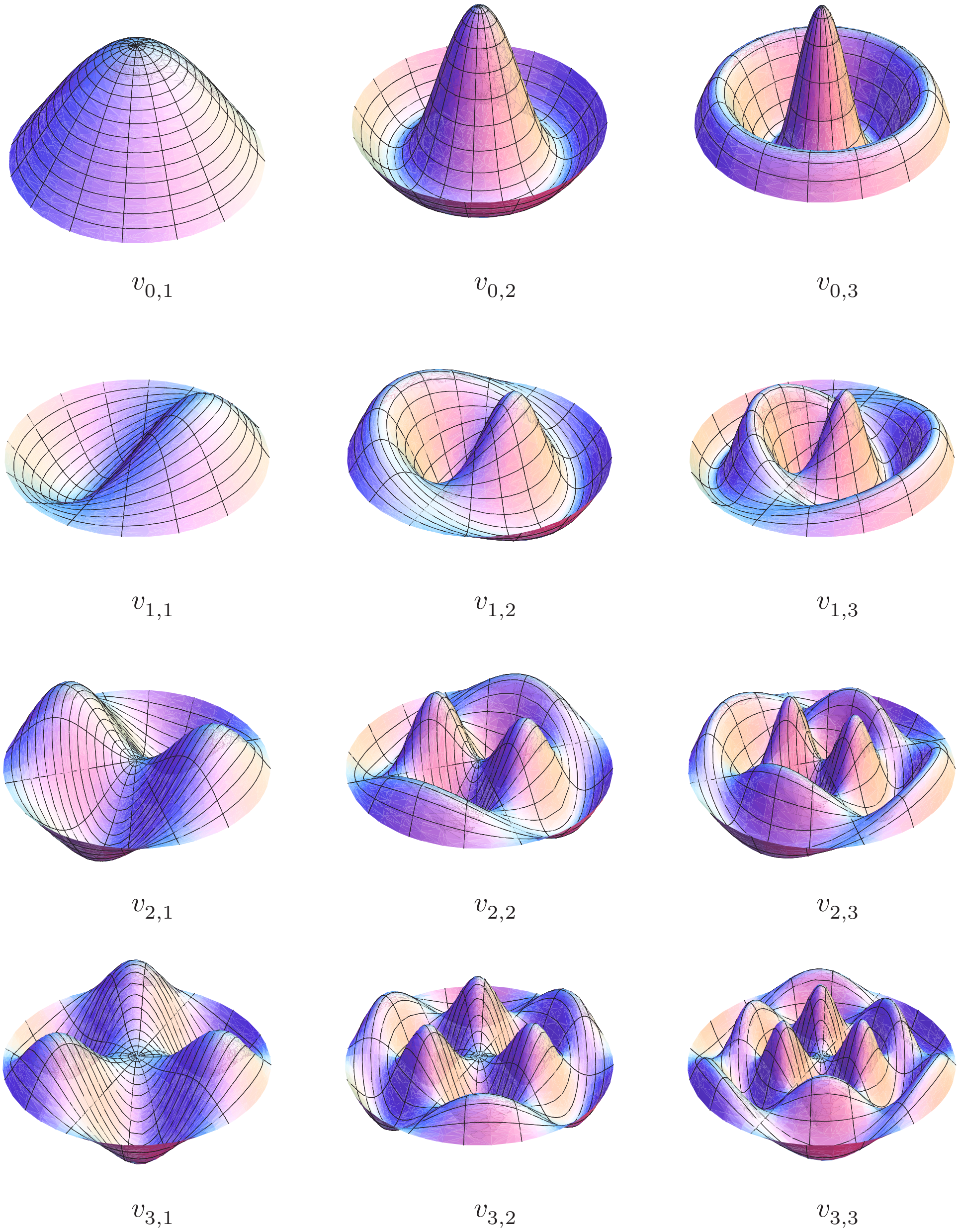


Figure 11.6. Normal modes for a disk.

to the homogeneous Dirichlet boundary value problem for the heat equation on the unit disk. The general solution is obtained by linear superposition, in the form of an infinite series

$$u(t,r,\theta) = \frac{1}{2}\sum_{n=1}^{\infty} a_{0,n}\, u_{0,n}(t,r) + \sum_{m,n=1}^{\infty} \big[\, a_{m,n}\, u_{m,n}(t,r,\theta) + b_{m,n}\, \widehat{u}_{m,n}(t,r,\theta) \,\big], \quad (11.124)$$

where the initial factor of $\frac{1}{2}$ is included, as with ordinary Fourier series, for later convenience. As usual, the coefficients $a_{m,n}, b_{m,n}$ are determined by the initial condition

$$u(0,r,\theta) = \frac{1}{2} \sum_{n=1}^{\infty} a_{0,n}\, v_{0,n}(r) + \sum_{m,n=1}^{\infty} \big[\, a_{m,n}\, v_{m,n}(r,\theta) + b_{m,n}\, \widehat{v}_{m,n}(r,\theta) \,\big] = f(r,\theta). \tag{11.125}$$

This requires that we expand the initial data into a *Fourier–Bessel series* in the eigenfunctions. As before, it is possible to prove, [**34**], that the separable eigenfunctions are *complete* — there are no other eigenfunctions — and hence every (reasonable) function defined on the unit disk can be written as a convergent series in the Bessel eigenfunctions.

Theorem 9.33 gurantees that the eigenfunctions are orthogonal† with respect to the standard L^2 inner product

$$\langle\, u\,, v \,\rangle = \iint_D u(x,y)\, v(x,y)\, dx\, dy = \int_0^1 \int_{-\pi}^{\pi} u(r,\theta)\, v(r,\theta)\; r\, d\theta\, dr$$

on the unit disk. (Note the extra factor of r coming from the polar coordinate form of the area element $dx\, dy = r\, dr\, d\theta$.) The L^2 norms of the Fourier–Bessel eigenfunctions are given by the interesting formulae

$$\|\, v_{0,n} \,\| = \sqrt{\pi}\; \big|\, J_1(\zeta_{0,n}) \,\big|\,, \qquad \|\, v_{m,n} \,\| = \|\, \widehat{v}_{m,n} \,\| = \sqrt{\frac{\pi}{2}}\; \big|\, J_{m+1}(\zeta_{m,n}) \,\big|\,, \tag{11.126}$$

which involve the value of the Bessel function of the next-higher order at the appropriate Bessel root. A proof of (11.126) can be found in Exercise 11.4.22, while numerical values are provided in the accompanying table.

Norms of the Fourier–Bessel Eigenfunctions $\|\, v_{m,n} \,\| = \|\, \widehat{v}_{m,n} \,\|$

$n \setminus m$	0	1	2	3	4	5	6	7
1	.9202	.5048	.4257	.3738	.3363	.3076	.2847	.2658
2	.6031	.3761	.3401	.3126	.2906	.2725	.2572	.2441
3	.4811	.3130	.2913	.2736	.2586	.2458	.2347	.2249
4	.4120	.2737	.2589	.2462	.2352	.2255	.2169	.2092
5	.3661	.2462	.2353	.2257	.2171	.2095	.2025	.1962

Orthogonality of the eigenfunctions implies that the coefficients in the Fourier–Bessel

† For the two independent eigenfunctions corresponding to one of the double eigenvalues, orthogonality must be verified by hand, but, in this case, it follows easily from the orthogonality of their trigonometric components.

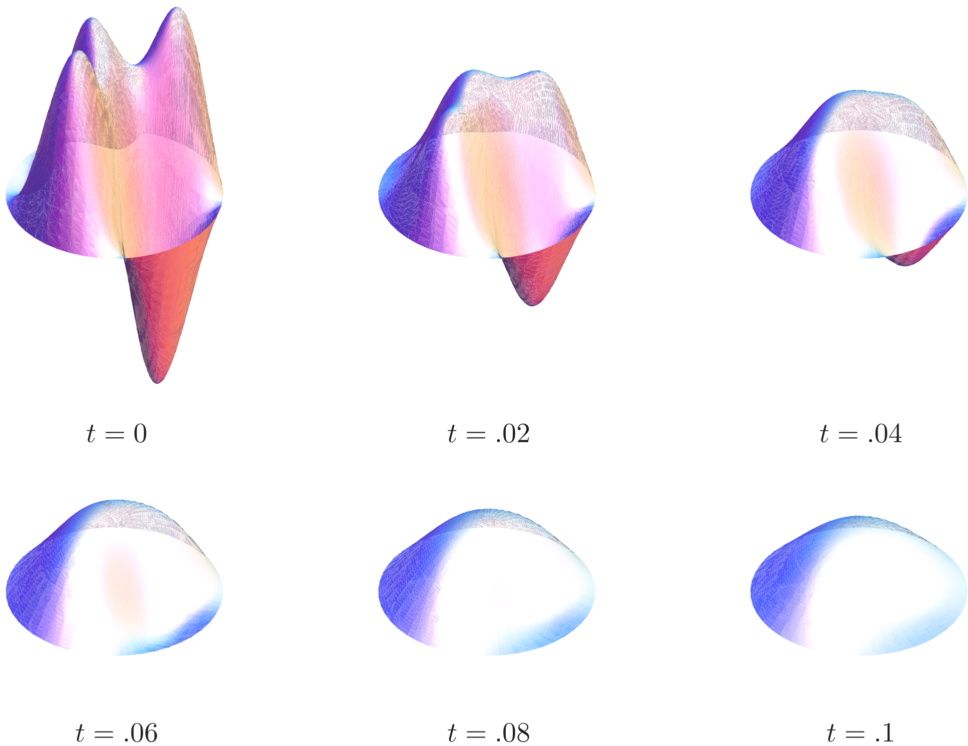


Figure 11.7. Heat diffusion in a disk.

series (11.125) are given by the inner product formulae

$$a_{0,n} = 2\,\frac{\langle\, f \,,v_{0,n}\,\rangle}{\|\, v_{0,n}\,\|^2} = \frac{2}{\pi\, J_1(\zeta_{0,n})^2}\int_0^1\!\!\int_{-\pi}^{\pi} f(r,\theta)\; J_0(\zeta_{0,n}\, r)\; r\,d\theta\,dr,$$
$$a_{m,n} = \frac{\langle\, f \,,v_{m,n}\,\rangle}{\|\, v_{m,n}\,\|^2} = \frac{2}{\pi\, J_{m+1}(\zeta_{m,n})^2}\int_0^1\!\!\int_{-\pi}^{\pi} f(r,\theta)\; J_m(\zeta_{m,n}\, r)\; r\cos m\,\theta\,d\theta\,dr, \qquad (11.127)$$
$$b_{m,n} = \frac{\langle\, f \,,\widehat{v}_{m,n}\,\rangle}{\|\, \widehat{v}_{m,n}\,\|^2} = \frac{2}{\pi\, J_{m+1}(\zeta_{m,n})^2}\int_0^1\!\!\int_{-\pi}^{\pi} f(r,\theta)\; J_m(\zeta_{m,n}\, r)\; r\sin m\,\theta\,d\theta\,dr.$$

In accordance with the general theory, each individual separable solution (11.123) to the heat equation decays exponentially fast, at a rate $\lambda_{m,n} = \zeta_{m,n}^2$ prescribed by the square of the corresponding Bessel root. In particular, the dominant mode, meaning the one that persists the longest, is

$$u_{0,1}(t,r,\theta) = e^{-\,\zeta_{0,1}^2\, t}\; J_0(\zeta_{0,1}\, r). \qquad (11.128)$$

Its decay rate is prescribed by the smallest positive eigenvalue:

$$\zeta_{0,1}^2 \;\approx\; 5.783, \qquad (11.129)$$

which is the square of the smallest root of the Bessel function $J_0(z)$. Since $J_0(z) > 0$ for $0 \le z < \zeta_{0,1}$, the dominant eigenfunction $v_{0,1}(r,\theta) = J_0(\zeta_{0,1}\, r) > 0$ is radially symmetric and strictly positive within the entire disk. Consequently, for most initial conditions (specifically those for which $a_{0,1} \neq 0$), the disk's temperature distribution eventually becomes entirely of one sign and radially symmetric, while decaying exponentially fast to zero at the rate given by (11.129). See Figure 11.7 for a plot of a typical solution. Note how, in accordance with the theory, the solution soon acquires a radial symmetry as it decays to thermal equilibrium.

Exercises

11.4.1. At the initial time $t_0 = 0$, a concentrated unit heat source is instantaneously applied at position $x = \frac{1}{2}$, $y = 0$, to a circular metal disk of unit radius and unit thermal diffusivity whose outside edge is held at 0°. Write down an eigenfunction series for the resulting temperature distribution at time $t > 0$. *Hint*: Be careful working with the delta function in polar coordinates; see Exercise 6.3.6.

11.4.2. Solve Exercise 11.4.1 when the concentrated unit heat source is instantaneously applied at the center of the disk.

♡ 11.4.3. (*a*) Write down the Fourier–Bessel series for the solution to the heat equation on a unit disk with $\gamma = 1$, whose circular edge is held at 0° and subject to the initial conditions $u(0, x, y) \equiv 1$ for $x^2 + y^2 \leq 1$. *Hint*: Use (11.112) to evaluate the integrals for the coefficients. (*b*) Approximate the time $t_\star \geq 0$ after which the temperature of the disk is everywhere $\leq .5^\circ$.

♣ 11.4.4. (*a*) Write down the first three nonzero terms in the Fourier–Bessel series for the solution to the heat equation on a unit disk with $\gamma = 1$ whose circular edge is held at 0° subject to the initial conditions $u(0, r, \theta) = 1 - r$ for $r \leq 1$. Use numerical integration to evaluate the coefficients. (*b*) Use your approximation to determine at which times $t \geq 0$ the temperature of the disk is everywhere $\leq .5^\circ$.

11.4.5. Prove that every separable eigenfunction of the Dirichlet boundary value problem for the Helmholtz equation in the unit disk can be written in the form

$$c\, J_m(\zeta_{m,n}\, r)\, \cos(m\,\theta - \alpha) \quad \text{for fixed } c \neq 0 \text{ and } -\pi < \alpha \leq \pi.$$

11.4.6. Suppose the initial data $f(r, \theta)$ in (11.49) satisfies $\displaystyle\int_0^1 \int_{-\pi}^{\pi} f(r, \theta)\, J_0(\zeta_{0,1}\, r)\, r\, d\theta\, dr = 0.$
(*a*) What is the decay rate to equilibrium of the resulting heat equation solution $u(t, r, \theta)$? (*b*) Prove that, generically, the asymptotic temperature distribution has half the disk above the equilibrium temperature and the other half below. Can you predict the diameter that separates the two halves? (*c*) If you know that $a_{0,1} = 0$, and also that the long-time temperature distribution is radially symmetric, what is the (generic) decay rate? What is the asymptotic temperature distribution?

♢ 11.4.7. Show how to use a scaling symmetry to solve the heat equation in a disk of radius R knowing the solution in a disk of radius 1.

11.4.8. Use rescaling, as in Exercise 11.4.7, to produce the solution to the Dirichlet initial-boundary value problem for a disk of radius 2 with diffusion coefficient $\gamma = 5$.

11.4.9. If it takes a disk of unit radius 3 minutes to reach (approximate) thermal equilibrium, how long will it take a disk of radius 2 made out of the same material and subject to the same homogeneous boundary conditions to reach equilibrium?

11.4.10. Assuming Dirichlet boundary conditions, does a square or a circular disk of the same area reach thermal equilibrium faster? Use your intuition first, and then check using the explicit formulas.

11.4.11. Answer Exercise 11.4.10 when the square and circle have the same perimeter.

11.4.12. Which reaches thermal equilibrium faster: a disk whose edge is held at 0° or a disk of the same radius that is fully insulated?

11.4.13. A circular metal disk is removed from an oven and then fully insulated. *True or false*: (*a*) The eventual equilibrium temperature is constant. (*b*) For large $t \gg 0$, the temperature $u(t, x, y)$ becomes more and more radially symmetric. If false, what can you say about the temperature profile at large times?

♡ 11.4.14. (*a*) Write down an eigenfunction series formula for the temperature dynamics of a disk of radius 1 that has an insulated boundary. (*b*) What is the eventual equilibrium temperature? (*c*) Is the rate of decay to thermal equilibrium (*i*) faster, (*ii*) slower, or (*iii*) the same as a disk with Dirichlet boundary conditions?

♡ 11.4.15. Write out a series solution for the temperature in a half-disk of radius 1, subject to (*a*) homogeneous Dirichlet boundary conditions on its entire boundary; (*b*) homogeneous Dirichlet conditions on the circular part of its boundary and homogeneous Neumann conditions on the straight part. (*c*) Which of the two boundary conditions results in a faster return to equilibrium temperature? How much faster?

11.4.16. A large sheet of metal is heated to 100°. A circular disk and a semi-circular half-disk of the same radius are cut out of it. Their edges are then held at 0°, while being fully insulated from above and below.
(*a*) *True or false*: The half-disk goes to thermal equilibrium twice as fast as the disk.
(*b*) If you need to wait 20 minutes for the circular disk to cool down enough to be picked up in your bare hands, how long do you need to wait to pick up the semi-circular disk?

♣ 11.4.17. Two identical plates have the shape of an annular ring $\{1 < r < 2\}$ with inner radius 1 and outer radius 2. The first has an insulated inner boundary and outer boundary held at 0°, while the second has an insulated outer boundary and inner boundary held at 0°. If both start out at the same temperature, which reaches thermal equilibrium faster? Quantify the rates of decay.

♡ 11.4.18. Let $m \geq 0$ be a nonnegative integer. In this exercise, we investigate the completeness of the eigenfunctions of the Bessel boundary value problem (11.56–57). To this end, define the Sturm–Liouville linear differential operator
$$S[u] = -\frac{1}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) + \frac{m^2}{x^2}\,u,$$
subject to the boundary conditions $|\,u'(0)\,| < \infty$, $u(1) = 0$, and either $|\,u(0)\,| < \infty$ when $m = 0$, or $u(0) = 0$ when $m > 0$.
(*a*) Show that S is self-adjoint relative to the inner product $\langle f, g \rangle = \int_0^1 f(x)\,g(x)\,x\,dx$.
(*b*) Prove that the eigenfunctions of S are the rescaled Bessel functions $J_m(\zeta_{m,n}x)$ for $n = 1, 2, 3, \ldots$. What are the orthogonality relations?
(*c*) Find the Green's function $G(x;\xi)$ and modified Green's function $\widehat{G}(x;\xi)$, cf. (9.59), associated with the boundary value problem $S[u] = 0$.
(*d*) Use the criterion of Theorem 9.47 to prove that the eigenfunctions are complete.

11.4.19. Determine the Bessel roots $\zeta_{1/2,n}$. Do they satisfy the asymptotic formula (11.119)?

♣ 11.4.20. Use a numerical root finder to compute the first 10 Bessel roots $\zeta_{3/2,n}$, $n = 1, \ldots, 10$. Compare your values with the asymptotic formula (11.119).

♢ 11.4.21. Prove that $J_{m-1}(\zeta_{m,n}) = -\,J_{m+1}(\zeta_{m,n})$.

♢ 11.4.22. In this exercise, we prove formula (11.126).
(*a*) First, use the recurrence formulae (11.111) to prove
$$\frac{d}{dx}\left[x^2\left(J_m(x)^2 - J_{m-1}(x)\,J_{m+1}(x)\right)\right] = 2x\,J_m(x)^2.$$
(*b*) Integrate both sides of the previous formula from 0 to the Bessel zero $\zeta_{m,n}$ and then

use Exercise 11.4.21 to show that

$$\int_0^{\zeta_{m,n}} x\, J_m(x)^2\, dx = -\frac{\zeta_{m,n}^2}{2}\, J_{m-1}(\zeta_{m,n})\, J_{m+1}(\zeta_{m,n}) = \frac{\zeta_{m,n}^2}{2}\, J_{m+1}(\zeta_{m,n})^2.$$

(*c*) Next, use a change of variables to establish the identity

$$\int_0^1 z\, J_m(\zeta_{m,n}\, z)^2\, dz = \tfrac{1}{2}\, J_{m+1}(\zeta_{m,n})^2.$$

(*d*) Finally, use the formulae for $v_{m,n}$ and $\widehat{v}_{m,n}$ to complete the proof of (11.126).

♢ 11.4.23. Prove directly that the eigenfunctions $v_{m,n}(r,\theta)$ and $\widehat{v}_{m,n}(r,\theta)$ in (11.122) are orthogonal with respect to the L^2 inner product on the unit disk.

11.4.24. Establish the following alternative formulae for the eigenfunction norms:

$$\|\, v_{0,n}\,\| = \sqrt{\pi}\, \left|\, J_0'(\zeta_{0,n}) \,\right|, \qquad \|\, v_{m,n}\,\| = \|\, \widehat{v}_{m,n}\,\| = \sqrt{\frac{\pi}{2}}\, \left|\, J_m'(\zeta_{m,n}) \,\right|.$$

11.5 The Fundamental Solution to the Planar Heat Equation

As we learned in Section 8.1, the fundamental solution to the heat equation measures the temperature distribution resulting from a concentrated initial heat source, e.g., a hot soldering iron applied instantaneously at a single point on a metal plate. The physical problem is modeled mathematically by taking a delta function as the initial data along with the relevant homogeneous boundary conditions. Once the fundamental solution is known, one is able to use linear superposition to recover the solution generated by any other initial data.

As in our one-dimensional analysis, we shall concentrate on the most tractable case, in which the domain is the entire plane: $\Omega = \mathbb{R}^2$. Thus, our first goal is to solve the initial value problem

$$u_t = \gamma\, \Delta u, \qquad u(0,x,y) = \delta(x-\xi)\,\delta(y-\eta), \tag{11.130}$$

for $t > 0$ and $(x,y) \in \mathbb{R}^2$. The solution $u = F(t,\mathbf{x};\boldsymbol{\xi}) = F(t,x,y;\xi,\eta)$ to this initial value problem is known as the *fundamental solution* for the heat equation on $\mathbb{R}^2$.

The quickest route to the desired formula relies on the following means of combining solutions of the one-dimensional heat equation to produce solutions of the two-dimensional version.

Lemma 11.11. *Let $v(t,x)$ and $w(t,x)$ be any two solutions to the one-dimensional heat equation $u_t = \gamma\, u_{xx}$. Then their product*

$$u(t,x,y) = v(t,x)\, w(t,y) \tag{11.131}$$

is a solution to the two-dimensional heat equation $u_t = \gamma\,(u_{xx} + u_{yy})$.

Proof: Our assumptions imply that $v_t = \gamma\, v_{xx}$, while $w_t = \gamma\, w_{yy}$ when we write $w(t,y)$ as a function of t and y. Therefore, differentiating (11.131), we find

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}\, w + v\, \frac{\partial w}{\partial t} = \gamma\, \frac{\partial^2 v}{\partial x^2}\, w + \gamma\, v\, \frac{\partial^2 w}{\partial y^2} = \gamma \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

and hence $u(t,x,y)$ solves the two-dimensional heat equation. *Q.E.D.*

For example, if

$$v(t,x) = e^{-\gamma \alpha^2 t} \sin \alpha x, \qquad w(t,y) = e^{-\gamma \beta^2 t} \sin \beta y,$$

are separable solutions of the one-dimensional heat equation, then

$$u(t,x,y) = e^{-\gamma (\alpha^2+\beta^2) t} \sin \alpha x \, \sin \beta y$$

are the separable solutions we used to solve the heat equation on a rectangle. A more interesting case is to choose

$$v(t,x) = \frac{1}{2\sqrt{\pi \gamma t}} \, e^{-(x-\xi)^2/(4\gamma t)}, \qquad w(t,y) = \frac{1}{2\sqrt{\pi \gamma t}} \, e^{-(y-\eta)^2/(4\gamma t)}, \tag{11.132}$$

to be the fundamental solutions (8.14) to the one-dimensional heat equation at respective locations $x = \xi$ and $y = \eta$. Multiplying these two solutions together produces the fundamental solution for the two-dimensional problem.

Theorem 11.12. *The* fundamental solution *to the heat equation* $u_t = \gamma \Delta u$ *corresponding to a unit delta function placed at position* $(\xi, \eta) \in \mathbb{R}^2$ *at the initial time* $t_0 = 0$ *is*

$$F(t,x,y;\xi,\eta) = \frac{1}{4\pi\gamma t} \, e^{-[(x-\xi)^2+(y-\eta)^2]/(4\gamma t)}. \tag{11.133}$$

Proof: Since we already know that both functions (11.132) are solutions to the one-dimensional heat equation, Lemma 11.11 guarantees that their product, which equals (11.133), solves the two-dimensional heat equation for $t > 0$. Moreover, at the initial time,

$$u(0,x,y) = v(0,x)\, w(0,y) = \delta(x-\xi)\, \delta(y-\eta)$$

is a product of delta functions, and hence the result follows. Indeed, the total heat

$$\iint u(t,x,y)\, dx\, dy = \int_{-\infty}^{\infty} v(t,x)\, dx \int_{-\infty}^{\infty} w(t,y)\, dy = 1, \qquad t \geq 0,$$

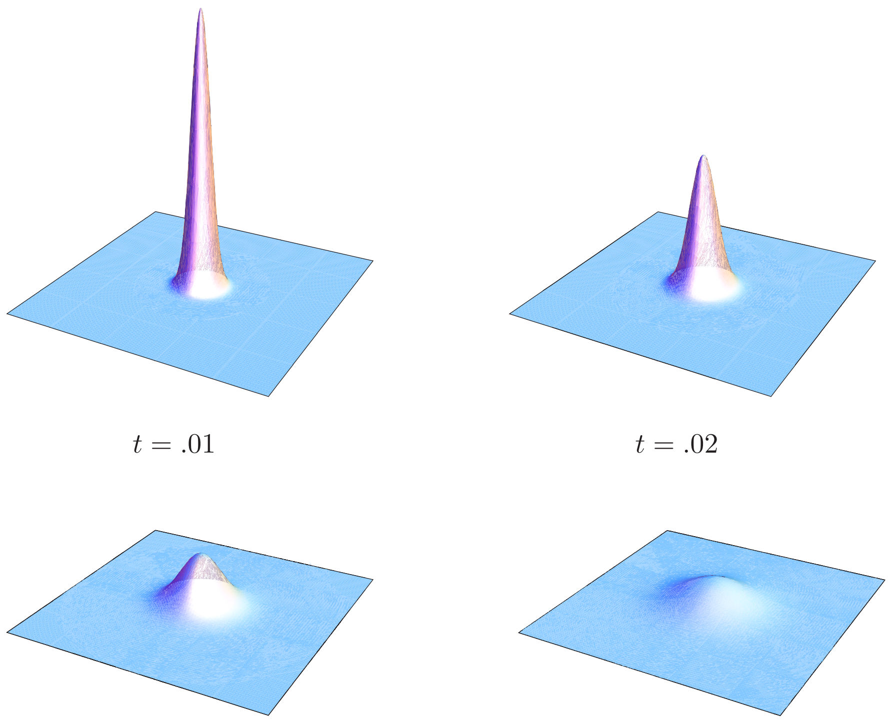
remains constant, while

$$\lim_{t \to 0^+} u(t,x,y) = \begin{cases} \infty, & (x,y) = (\xi,\eta), \\ 0, & \text{otherwise}, \end{cases}$$

has the standard delta function limit at the initial time instant. *Q.E.D.*

Figure 11.8 depicts the evolution of the fundamental solution when $\gamma = 1$ at the indicated times. Observe that the initially concentrated temperature spreads out in a radially symmetric manner, while the total amount of heat remains constant. At any individual point $(x,y) \neq (0,0)$, the initially zero temperature rises slightly at first, but then decays monotonically back to zero at a rate proportional to $1/t$. As in the one-dimensional case, since the fundamental solution is > 0 for all $t > 0$, the heat energy has an infinite speed of propagation.

Both the one- and two-dimensional fundamental solutions have bell-shaped profiles known as *Gaussian filters*. The most important difference is the initial factor. In a one-dimensional medium, the fundamental solution decays in proportion to $1/\sqrt{t}$, whereas in the plane the decay is more rapid, being proportional to $1/t$. The physical explanation is that the heat energy is able to spread out in two independent directions, and hence diffuses



$t = .01$ $t = .02$

$t = .05$ $t = .1$

Figure 11.8. The fundamental solution to the planar heat equation.

away from its initial source more rapidly. As we shall see, the decay in three-dimensional space is more rapid still, being proportional to $t^{-3/2}$ for similar reasons; see (12.120).

The principal use of the fundamental solution is for solving the general initial value problem. We express the initial temperature distribution as a superposition of delta function impulses,

$$u(0,x,y) = f(x,y) = \iint f(\xi,\eta)\,\delta(x-\xi,y-\eta)\,d\xi\,d\eta,$$

where, at the point $(\xi,\eta) \in \mathbb{R}^2$, the impulse has magnitude $f(\xi,\eta)$. Linearity implies that the solution is then given by the same superposition of fundamental solutions.

Theorem 11.13. *The solution to the initial value problem*

$$u_t = \gamma\,\Delta u, \qquad u(0,x,y) = f(x,y), \qquad (x,y) \in \mathbb{R}^2,$$

for the planar heat equation is given by the linear superposition formula

$$u(t,x,y) = \frac{1}{4\pi\gamma t}\iint f(\xi,\eta)\,e^{-[(x-\xi)^2+(y-\eta)^2]/(4\gamma t)}\,d\xi\,d\eta. \tag{11.134}$$

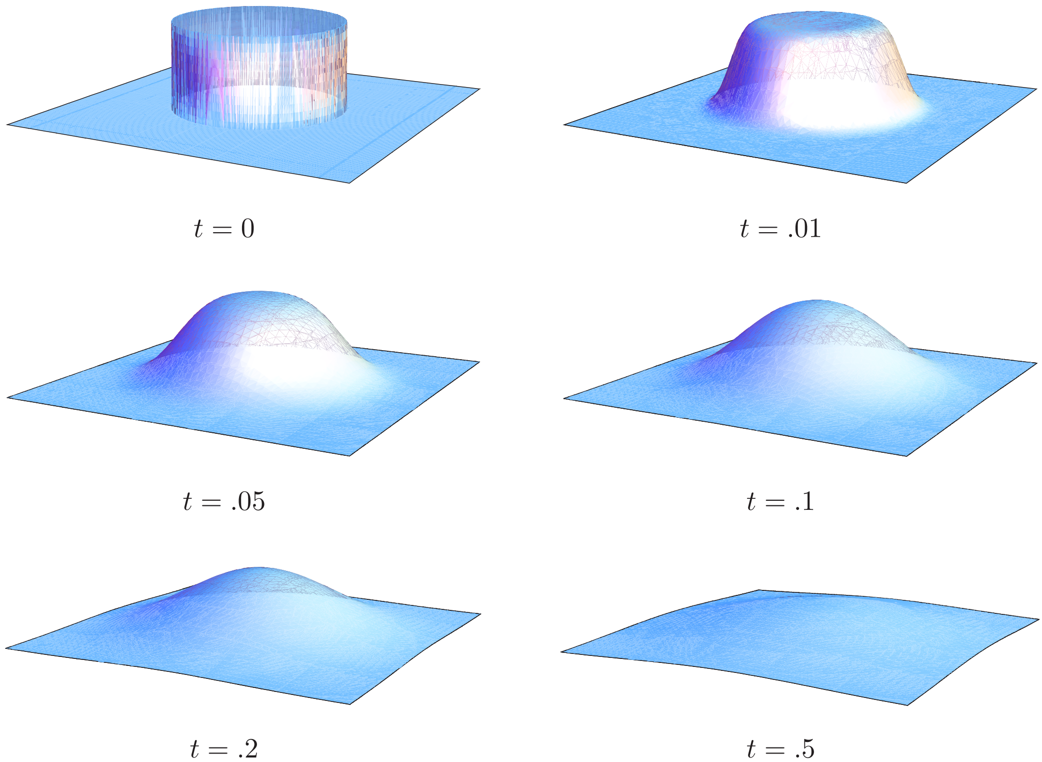


Figure 11.9. Diffusion of a disk.

We can interpret the solution formula (11.134) as a two-dimensional *convolution*

$$u(t,x,y) = F(t,x,y) * f(x,y) \tag{11.135}$$

of the initial data with a one-parameter family of progressively wider and shorter Gaussian filters

$$F(t,x,y) = F(t,x,y;0,0) = \frac{1}{4\pi\gamma t}\, e^{-(x^2+y^2)/(4\gamma t)}. \tag{11.136}$$

As in (7.54), such a convolution can be interpreted as a Gaussian weighted averaging of the function $f(x,y)$, which has the effect of smoothing out the initial data.

Example 11.14. If our initial temperature distribution is constant on a circular region, say

$$u(0,x,y) = \begin{cases} 1 & x^2+y^2<1, \\ 0, & \text{otherwise}, \end{cases}$$

then the solution can be evaluated using (11.134), as follows:

$$u(t,x,y) = \frac{1}{4\pi t} \iint_D e^{-[(x-\xi)^2+(y-\eta)^2]/(4t)}\, d\xi\, d\eta,$$

where the integral is over the unit disk $D = \{\xi^2+\eta^2 \le 1\}$. Unfortunately, the integral cannot be expressed in terms of elementary functions. On the other hand, numerical evaluation of the integral is straightforward. A plot of the resulting radially symmetric solution appears in Figure 11.9. One could also interpret this solution as the diffusion of an animal population in a uniform isotropic environment or bacteria in a similarly uniform large petri dish that are initially confined to a small circular region.

Exercises

11.5.1. Solve the following initial value problem: $u_t = 5\,(u_{xx} + u_{yy}), \quad u(0,x,y) = e^{-\,(x^2+y^2)}$.

11.5.2. Write down an integral formula for the solution to the following initial value problem:
$$u_t = 3(u_{xx} + u_{yy}), \qquad u(0,x,y) = (1 + x^2 + y^2)^{-2}.$$

11.5.3. At the initial time $t = 0$, a unit heat source is instantaneously applied at the origin of the (x,y)–plane. For $t > 0$, what is the maximum temperature experienced at a point $(x,y) \neq \mathbf{0}$? At what time is the maximum temperature achieved? Does the temperature approach an equilibrium value as $t \to \infty$? If so, how fast?

11.5.4. (*a*) Find an eigenfunction series representation of the fundamental solution for the heat equation $u_t = \Delta u$ on the unit square $\{\,0 \le x, y \le 1\,\}$ when subject to homogeneous Dirichlet boundary conditions. (*b*) Write the solution to the initial value problem $u(0,x,y) = f(x,y)$ in terms of the fundamental solution. (*c*) Discuss how your formula is related to the Fourier series solution (11.43).

11.5.5. Let $u(t,x,y)$ be a solution to the heat equation on all of $\mathbb{R}^2$ such that u and $\|\,\nabla u\,\| \to 0$ rapidly as $\|\,\mathbf{x}\,\| \to \infty$. (*a*) Prove that the total heat $H(t) = \iint u(t,x,y)\,dx\,dy$ is constant. (*b*) Explain how this can be reconciled with the statement that $u(t,x,y) \to 0$ as $t \to \infty$ at all points $(x,y) \in \mathbb{R}^2$.

♢ 11.5.6. Consider the initial value problem $u_t = \gamma\,\Delta u + H(t,x,y)$, $u(0,x,y) = 0$, for the inhomogeneous heat equation on the entire (x,y)–plane, where $H(t,x,y)$ represents a time-varying external heat source. Derive an integral formula for its solution. *Hint*: Mimic the solution method in Section 8.1.

11.5.7. A flat plate of infinite extent with unit thermal diffusivity starts off at 0°. From then on, a unit heat source is continually applied at the origin. Find the resulting temperature distribution. Does the temperature eventually reach a steady state?
Hint: Use Exercise 11.5.6.

♡ 11.5.8. Building on Example 11.14, we model the "diffusion" of a set $D \subset \mathbb{R}^2$ as the solution $u(t,x,y)$ to the heat equation $u_t = \Delta u$ subject to the initial condition $u(0,x,y) = \chi_D(x,y)$, where $\chi_D(x,y) = \begin{cases} 1, & (x,y) \in D, \\ 0, & (x,y) \notin D, \end{cases}$ is the *characteristic function* of the set D.
(*a*) Write down a formula for the diffusion of the set D.
(*b*) *True or false*: At each t, the diffusion $u(t,x,y)$ is the characteristic function of a set D_t.
(*c*) Prove that $0 < u(t,x,y) < 1$ for all (x,y) and $t > 0$. (*d*) What is $\displaystyle\lim_{t \to \infty} u(t,x,y)$?
(*e*) Write down a formula for the diffusion of a unit square $D = \{\,0 \le x, y \le 1\,\}$, and then plot the result at several times. Discuss what you observe.

11.5.9. (*a*) Explain why the delta function on $\mathbb{R}^2$ satisfies the scaling law $\delta(x,y) = \beta^2\,\delta(\beta x, \beta\,y)$, for $\beta \neq 0$. (*b*) Verify that the fundamental solution to the heat equation on $\mathbb{R}^2$ obeys the same scaling law: $F(t,x,y) = \beta^2\,F(\beta^2\,t, \beta\,x, \beta\,y)$. (*c*) Is the fundamental solution a similarity solution?

11.5.10. (*a*) Find the fundamental solution on $\mathbb{R}^2$ to the cable equation $u_t = \gamma\,\Delta u - \alpha\,u$, where $\alpha > 0$ is constant. (*b*) Use your solution to write down a formula for the solution to the general initial value problem $u(0,x,y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$.

11.5.11. (*a*) Prove that if $v(t,x)$ and $w(t,x)$ solve the dispersive wave equation (8.90), then their product $u(t,x,y) = v(t,x)\,w(t,y)$ solves the two-dimensional dispersive equation

$$u_t + u_{xxx} + u_{yyy} = 0.$$

(*b*) What is the fundamental solution on $\mathbb{R}^2$ of the latter equation? (*c*) Write down an integral formula for the solution to the initial value problem $u(0,x,y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$.

11.5.12. Define the two-dimensional convolution $f * g$ of functions $f(x,y)$ and $g(x,y)$ so that equation (11.135) is valid.

11.6 The Planar Wave Equation

Let us next consider the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{11.137}$$

which models the unforced transverse vibrations of a homogeneous membrane, e.g., a drum. Here, $u(t,x,y)$ represents the vertical displacement of the membrane at time t and position $(x,y) \in \Omega$, where the domain $\Omega \subset \mathbb{R}^2$, assumed bounded, represents the undeformed shape. The constant $c^2 > 0$ encapsulates the membrane's physical properties — density, tension, stiffness, etc.; its square root, c, is called, as in the one-dimensional case, the *wave speed*, since it represents the speed of propagation of localized signals.

Remark: In this simplified model, we are only allowing small, transverse (vertical) displacements of the membrane. Large elastic vibrations lead to the nonlinear partial differential equations of elastodynamics, [**7**]. In particular, the bending vibrations of a flexible elastic plate are governed by a more complicated fourth-order partial differential equation.

The solution $u(t,x,y)$ to the wave equation will be uniquely specified once we impose suitable boundary and initial conditions. The Dirichlet conditions

$$u(t,x,y) = h(x,y), \qquad (x,y) \in \partial\Omega, \tag{11.138}$$

correspond to gluing our membrane to a fixed boundary — a rim; more generally, we can also allow h to depend on t, modeling a membrane attached to a moving boundary. On the other hand, the homogeneous Neumann conditions

$$\frac{\partial u}{\partial \mathbf{n}}(t,x,y) = 0, \qquad (x,y) \in \partial\Omega, \tag{11.139}$$

represent a free boundary where the membrane is not attached to any support — although in this model, its edge is allowed to move only in a vertical direction. Mixed boundary conditions attach part of the boundary and leave the remaining portion free to vibrate:

$$u = h \quad \text{on} \quad D \subsetneq \partial\Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad N = \partial\Omega \setminus D. \tag{11.140}$$

Since the wave equation is of second order in time, to uniquely specify the solution we need to impose two initial conditions,

$$u(0,x,y) = f(x,y), \qquad \frac{\partial u}{\partial t}(0,x,y) = g(x,y), \qquad (x,y) \in \Omega. \tag{11.141}$$

The first specifies the membrane's initial displacement, while the second prescribes its initial velocity.

Separation of Variables

Unfortunately, the d'Alembert solution method does not apply to the two-dimensional wave equation in any obvious manner. The reason is that, unlike the one-dimensional version (2.69), one cannot factorize the planar wave operator $\square = \partial_t^2 - c^2\,\partial_x^2 - c^2\,\partial_y^2$, thus precluding any sort of reduction to a first-order partial differential equation. However, this is not the end of the story, and we will return to this issue at the end of Section 12.6.

We thus fall back on our universal solution tool for linear partial differential equations — separation of variables. According to the general framework established in Section 9.5, the separable solutions to the wave equation have the trigonometric form

$$u_k(t,x,y) = \cos(\omega_k t)\, v_k(x,y) \qquad \text{and} \qquad \widetilde{u}_k(t,x,y) = \sin(\omega_k t)\, v_k(x,y). \tag{11.142}$$

Substituting back into the wave equation, we find that $v_k(x,y)$ must be an eigenfunction of the associated Helmholtz boundary value problem

$$c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \lambda_k\, v = 0, \tag{11.143}$$

whose eigenvalue $\lambda_k = \omega_k^2$ equals the square of the vibrational frequency. According to Theorem 9.47, on a bounded domain, there is an infinite number of such *normal modes* with progressively faster vibrational frequencies: $\omega_k \to \infty$ as $k \to \infty$. In addition, in the positive semi-definite case — which occurs under homogeneous Neumann boundary conditions — there is a single constant null eigenfunction, leading to the additional separable solutions

$$u_0(t,x,y) = 1 \qquad \text{and} \qquad \widetilde{u}_0(t,x,y) = t. \tag{11.144}$$

The first represents a stationary membrane that has been displaced to a fixed height, while the second represents a membrane that is moving off in the vertical direction with constant unit speed. (Think of the membrane moving in outer space unaffected by any external gravitational force.) As in Section 9.5, the general solution can be written as an infinite series in the eigensolutions (11.142). Unfortunately, as we know, the Helmholtz boundary value problem can be explicitly solved only on a rather restricted class of domains. Here we will content ourselves with investigating the two most important cases: rectangular and circular membranes.

Remark: The vibrational frequencies represent the tones and overtones one hears when the drum membrane vibrates. An interesting question is whether two drums of different shapes can have identical sounds — the exact same vibrational frequencies. Or, more descriptively, can one "hear" the shape of a drum? It was not until 1992 that the answer was shown to be no, but for quite subtle reasons. See [**47**] for a discussion and some examples of differently shaped drums that have the same vibrational frequencies.

Vibration of a Rectangular Drum

Let us first consider the vibrations of a membrane in the shape of a rectangle

$$R = \{\, 0 < x < a,\ 0 < y < b \,\},$$

with side lengths a and b, whose edges are fixed to the (x, y)–plane. Thus, we seek to solve the wave equation

$$u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy}), \qquad 0 < x < a, \qquad 0 < y < b, \tag{11.145}$$

subject to the initial and boundary conditions

$$\begin{aligned} u(t,0,y) = u(t,a,y) = 0 = u(t,x,0) = u(t,x,b), \qquad & 0 < x < a, \\ u(0,x,y) = f(x,y), \qquad u_t(0,x,y) = g(x,y), \qquad & 0 < y < b. \end{aligned} \tag{11.146}$$

As we saw in Section 11.2, the eigenfunctions and eigenvalues for the associated Helmholtz equation on a rectangle,

$$c^2 (v_{xx} + v_{yy}) + \lambda\, v = 0, \qquad (x, y) \in R, \tag{11.147}$$

when subject to the homogeneous Dirichlet boundary conditions

$$v(0,y) = v(a,y) = 0 = v(x,0) = v(x,b), \qquad 0 < x < a, \qquad 0 < y < b, \tag{11.148}$$

are

$$v_{m,n}(x,y) = \sin \frac{m \pi x}{a}\, \sin \frac{n \pi y}{b}, \qquad \text{where} \qquad \lambda_{m,n} = \pi^2 c^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \tag{11.149}$$

with $m, n = 1, 2, \ldots$. The fundamental frequencies of vibration are the square roots of the eigenvalues, so

$$\omega_{m,n} = \sqrt{\lambda_{m,n}} = \pi\, c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\,, \qquad m, n = 1, 2, \ldots\ . \tag{11.150}$$

The frequencies will depend upon the underlying geometry — meaning the side lengths — of the rectangle, as well as the wave speed c, which, in turn, is a function of the membrane's density and stiffness. The higher the wave speed, or the smaller the rectangle, the faster the vibrations. In layman's terms, (11.150) quantifies the observation that smaller, stiffer drums made of less-dense material vibrate faster.

According to (11.142), the normal modes of vibration of our rectangle are

$$\begin{aligned} u_{m,n}(t,x,y) &= \cos\left(\pi\, c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\; t \right) \sin \frac{m \pi x}{a}\, \sin \frac{n \pi y}{b}, \\ \widetilde{u}_{m,n}(t,x,y) &= \sin\left(\pi\, c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\; t \right) \sin \frac{m \pi x}{a}\, \sin \frac{n \pi y}{b}. \end{aligned} \tag{11.151}$$

The general solution can then be written as a double Fourier series

$$u(t,x,y) = \sum_{m,n=1}^{\infty} \left[\, a_{m,n}\, u_{m,n}(t,x,y) + b_{m,n}\, \widetilde{u}_{m,n}(t,x,y) \,\right]$$

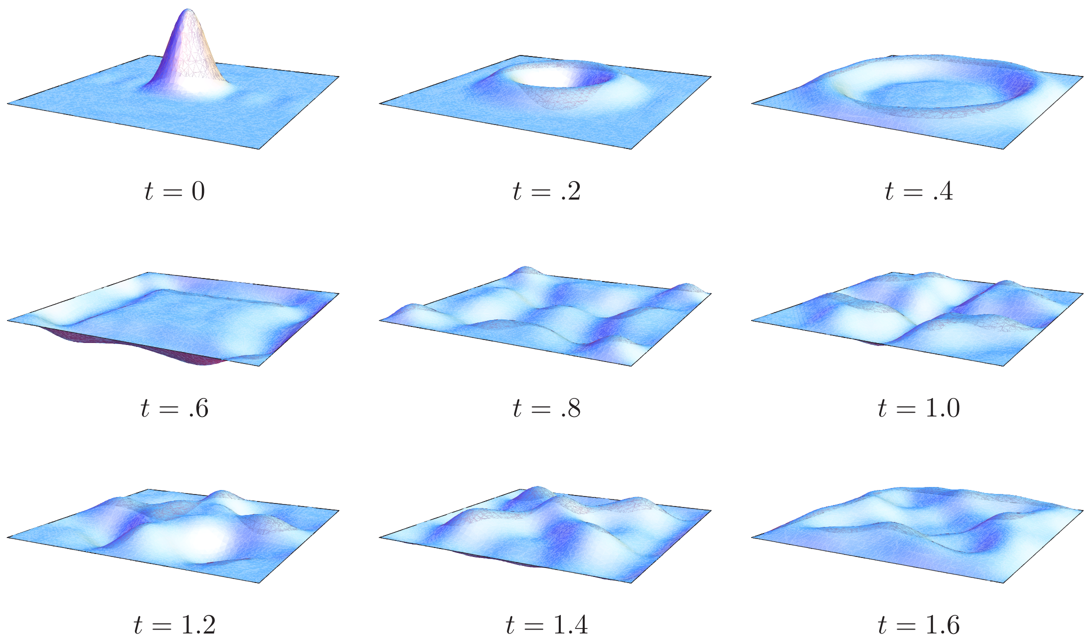


Figure 11.10. Vibrations of a square membrane.

in the normal modes. The coefficients $a_{m,n}, b_{m,n}$ are fixed by the initial displacement $u(0,x,y) = f(x,y)$ and the initial velocity $u_t(0,x,y) = g(x,y)$. Indeed, the usual orthogonality relations among the eigenfunctions imply

$$a_{m,n} = \frac{\langle\, v_{m,n}\,, f\,\rangle}{\|\, v_{m,n}\,\|^2} = \frac{4}{a\,b}\int_0^b\!\!\int_0^a f(x,y)\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,dx\,dy, \tag{11.152}$$

$$b_{m,n} = \frac{\langle\, v_{m,n}\,, g\,\rangle}{\omega_{m,n}\,\|\, v_{m,n}\,\|^2} = \frac{4}{\pi\,c\,\sqrt{m^2 b^2 + n^2 a^2}}\int_0^b\!\!\int_0^a g(x,y)\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,dx\,dy.$$

Since the fundamental frequencies are not rational multiples of each other, the general solution is a genuinely quasiperiodic superposition of the various normal modes.

In Figure 11.10, we plot the solution resulting from the initially concentrated displacement[†]

$$u(0,x,y) = f(x,y) = e^{-100\,[\,(x-.5)^2+(y-.5)^2\,]}$$

at the center of a unit square, so $a = b = 1$, with unit wave speed, $c = 1$. Note that, unlike a concentrated displacement of a one-dimensional string, which remains concentrated at all subsequent times and periodically repeats, the initial displacement here spreads out in a radially symmetric manner and propagates to the edges of the rectangle, where it reflects

[†] The alert reader may object that the initial displacement $f(x,y)$ does not exactly satisfy the Dirichlet boundary conditions on the edges of the rectangle. But this does not prevent the existence of a well-defined (weak) solution to the initial value problem, whose initial boundary discontinuities will subsequently propagate into the square. However, here these are so tiny as to be unnoticeable in the solution graphs.

and then interacts with itself. Moreover, due to the quasiperiodicity of the solution, the drum's motion never exactly repeats, and the initially concentrated displacement never quite reforms.

Vibration of a Circular Drum

Let us next analyze the vibrations of a circular membrane of unit radius. In polar coordinates, the planar wave equation (11.137) takes the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right). \tag{11.153}$$

We will again consider the homogeneous Dirichlet boundary value problem

$$u(t,1,\theta) = 0, \qquad t \geq 0, \qquad -\pi \leq \theta \leq \pi, \tag{11.154}$$

along with initial conditions

$$u(0,r,\theta) = f(r,\theta), \qquad \frac{\partial u}{\partial t}(0,r,\theta) = g(r,\theta), \tag{11.155}$$

representing the initial displacement and velocity of the membrane. As always, we build up the general solution as a quasiperiodic linear combination of the normal modes as specified by the eigenfunctions for the associated Helmholtz boundary value problem.

As we saw in Section 11.2, the eigenfunctions of the Helmholtz equation on a disk of radius 1, say, subject to homogeneous Dirichlet boundary conditions, are products of trigonometric and Bessel functions:

$$\begin{aligned} v_{0,n}(r,\theta) &= J_0(\zeta_{0,n} r), \\ v_{m,n}(r,\theta) &= J_m(\zeta_{m,n} r)\cos m\theta, \qquad m,n = 1,2,3,\ldots . \\ \widetilde{v}_{m,n}(r,\theta) &= J_m(\zeta_{m,n} r)\sin m\theta, \end{aligned} \tag{11.156}$$

Here r, θ are the usual polar coordinates, while $\zeta_{m,n} > 0$ denotes the n^{th} (positive) root of the m^{th} order Bessel function $J_m(z)$, cf. (11.118). The corresponding eigenvalue is its square, $\lambda_{m,n} = \zeta_{m,n}^2$, and hence the natural frequencies of vibration are equal to the Bessel roots scaled by the wave speed:

$$\omega_{m,n} = c\sqrt{\lambda_{m,n}} = c\,\zeta_{m,n}. \tag{11.157}$$

A table of their values (for the case $c = 1$) can be found in the preceding section. The Bessel roots do not follow any easily discernible pattern, and are not rational multiples of each other. This result, known as *Bourget's hypothesis*, [**119**; p. 484], was rigorously proved by the German mathematician Carl Ludwig Siegel in 1929, [**106**]. Thus, the vibrations of a circular drum are also truly quasiperiodic, thereby providing a mathematical explanation of why drums sound dissonant.

The frequencies $\omega_{0,n} = c\,\zeta_{0,n}$ correspond to simple eigenvalues, with a single radially symmetric eigenfunction $J_0(\zeta_{0,n} r)$, while the "angular modes" $\omega_{m,n}$, for $m > 0$, are double, each possessing two linearly independent eigenfunctions (11.156). According to the general

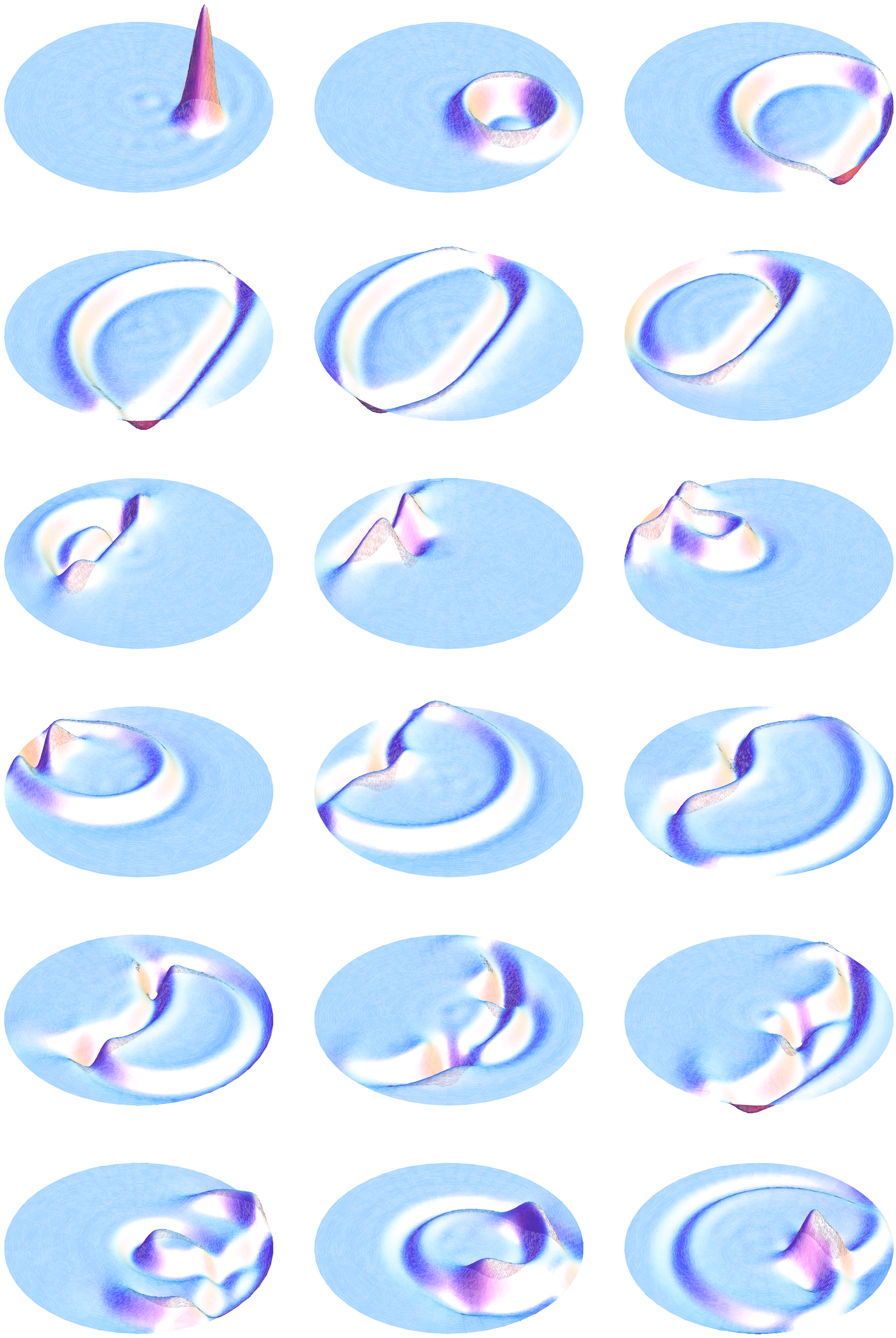


Figure 11.11. Vibration of a disk.

formula (11.142), each eigenfunction engenders two independent normal modes of vibration, having the explicit forms

$$\begin{aligned}
&\cos(c\,\zeta_{0,n}\,t)\; J_0(\zeta_{0,n}\,r), && \sin(c\,\zeta_{0,n}\,t)\; J_0(\zeta_{0,n}\,r),\\
&\cos(c\,\zeta_{m,n}\,t)\; J_m(\zeta_{m,n}\,r)\;\cos m\,\theta, && \sin(c\,\zeta_{m,n}\,t)\; J_m(\zeta_{m,n}\,r)\;\cos m\,\theta,\\
&\cos(c\,\zeta_{m,n}\,t)\; J_m(\zeta_{m,n}\,r)\;\sin m\,\theta, && \sin(c\,\zeta_{m,n}\,t)\; J_m(\zeta_{m,n}\,r)\;\sin m\,\theta.
\end{aligned} \tag{11.158}$$

The general solution to (11.153–154) is then expressed as a Fourier–Bessel series:

$$\begin{aligned}
u(t,r,\theta) = \frac{1}{2}\sum_{n=1}^{\infty}\big[\,a_{0,n}\cos(c\,\zeta_{0,n}\,t) + c_{0,n}\sin(c\,\zeta_{0,n}\,t)\,\big]\, J_0(\zeta_{0,n}\,r)\\
+ \sum_{m,n=1}^{\infty}\Big[\,\big(\,a_{m,n}\cos(c\,\zeta_{m,n}\,t) + c_{m,n}\sin(c\,\zeta_{m,n}\,t)\,\big)\cos m\,\theta\\
+ \big(\,b_{m,n}\cos(c\,\zeta_{m,n}\,t) + d_{m,n}\sin(c\,\zeta_{m,n}\,t)\,\big)\sin m\,\theta\,\Big]\, J_m(\zeta_{m,n}\,r),
\end{aligned} \tag{11.159}$$

whose coefficients $a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n}$ are determined, as usual, by the initial displacement and velocity of the membrane (11.155). In Figure 11.11, the vibrations due to an initially off-center concentrated displacement are displayed; the wave speed is $c = 1$, and the time interval between successive plots is $\Delta t = .3$. Again, the motion is only quasiperiodic and, no matter how long you wait, never quite returns to its original configuration.

Exercises

11.6.1. Use your physical intuition to decide whether the following statements are *true or false*. Then justify your answer.
(*a*) Increasing the stiffness of a membrane increases the wave speed.
(*b*) Increasing the density of a membrane increases the wave speed.
(*c*) Increasing the size of a membrane increases the wave speed

11.6.2. Two uniform membranes have the same shape, but are made out of different materials. Assuming that they are both subject to the same homogeneous boundary conditions, how are their vibrational frequencies related?

11.6.3. List the numerical values of the six lowest vibrational frequencies of a unit square with wave speed $c = 1$ when subject to homogeneous Dirichlet boundary conditions. How many linearly independent normal modes are associated with each of these frequencies?

♡ 11.6.4. The rectangular membrane $R = \{\,-1 < x < 1,\ 0 < y < 1\,\}$ has its two short sides attached to the (x, y)–plane, while its long sides are left free. The membrane is initially displaced so that its right half is one unit above, while its left half is one unit below the plane, and then released with zero initial velocity. (This discontinuous initial data serves to model a very sharp transition region.) Assume that the physical units are chosen so the wave speed $c = 1$. (*a*) Write down an initial-boundary value problem that governs the vibrations of the membrane. (*b*) What are the fundamental frequencies of vibration of the membrane? (*c*) Find the eigenfunction series solution that describes the subsequent motion of the membrane. (*d*) Is the motion (*i*) periodic? (*ii*) quasiperiodic? (*iii*) unstable? (*iv*) chaotic? Explain your answer.

11.6.5. Determine the solution to the following initial-boundary value problems for the wave equation on the rectangle $R = \{\,0 < x < 2,\ 0 < y < 1\,\}$:

(a) $\begin{cases} u_{tt} = u_{xx} + u_{yy}, & u(t,x,0) = u(t,x,1) = u(t,0,y) = u(t,2,y) = 0,\\ u(0,x,y) = \sin \pi y, & u_t(0,x,y) = \sin \pi y; \end{cases}$

(b) $\begin{cases} u_{tt} = u_{xx} + u_{yy}, & u(t,x,0) = u(t,x,1) = \dfrac{\partial u}{\partial x}(t,0,y) = \dfrac{\partial u}{\partial x}(t,2,y) = 0, \\ u(0,x,y) = \sin \pi y, & u_t(0,x,y) = \sin \pi y; \end{cases}$

(c) $\begin{cases} u_{tt} = u_{xx} + u_{yy}, & u(t,x,0) = u(t,x,1) = u(t,0,y) = u(t,2,y) = 0, \\ u(0,x,y) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2, \end{cases} & u_t(0,x,y) = 0; \end{cases}$

(d) $\begin{cases} u_{tt} = 2u_{xx} + 2u_{yy}, & u(t,x,0) = u(t,x,1) = u(t,0,y) = u(t,2,y) = 0, \\ u(0,x,y) = 0, & u_t(0,x,y) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2. \end{cases} \end{cases}$

11.6.6. *True or false*: The more sides of a rectangle that are tied down, the faster it vibrates.

11.6.7. Answer Exercise 11.6.3 when (a) two adjacent sides of the square are tied down and the other two are left free; (b) two opposite sides of the square are tied down and the other two are left free; (c) the membrane is freely floating in outer space.

11.6.8. A square drum has two sides fixed to a support and two sides left free. Does the drum vibrate faster if the fixed and free sides are adjacent to each other or on opposite sides?

11.6.9. Write down a periodic solution to the wave equation on a unit square, subject to homogeneous Dirichlet boundary conditions, that is *not* a normal mode. Does it vibrate at a fundamental frequency?

11.6.10. A rectangular drum with side lengths 1 cm by 2 cm and unit wave speed $c = 1$ has its boundary fixed to the (x,y)–plane while subject to a periodic external forcing of the form $F(t,x,y) = \cos(\omega t)\, h(x,y)$. (a) At which frequencies ω will the forcing incite resonance in the drum? (b) If ω is a resonant frequency, write down the condition(s) on $h(x,y)$ that ensure excitation of a resonant mode.

11.6.11. The right half of a rectangle of side lengths 1 by 2 is initially displaced, while the left half is quiescent. *True or false*: The ensuing vibrations are restricted to the right half of the membrane.

♡ 11.6.12. A torus (inner tube) can be obtained by gluing together each of the two pairs of opposite sides of a rubber rectangle. The (small) vibrations of the torus are described by the following periodic initial-boundary value problem for the wave equation, in which x, y represent angular variables:

$$\begin{array}{lll} u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy}), & u(0,x,y) = f(x,y), & u_t(0,x,y) = g(x,y), \\ u(t,-\pi,y) = u(t,\pi,y), & u_x(t,-\pi,y) = u_x(t,\pi,y), & -\pi < x < \pi, \\ u(t,x,-\pi) = u(t,x,\pi), & u_x(t,x,-\pi) = u_x(t,x,\pi), & -\pi < y < \pi. \end{array}$$

(a) Find the fundamental frequencies and normal modes of vibration. (b) Write down a series for the solution. (c) Discuss the stability of a vibrating torus. Is the motion (i) periodic; (ii) quasiperiodic; (iii) chaotic; (iv) none of these?

11.6.13. The *forced wave equation* $u_{tt} = c^2 \Delta u + F(x,y)$ on a bounded domain $\Omega \subset \mathbb{R}^2$ models a membrane subject to a constant external forcing function $F(x,y)$. Write down an eigenfunction series solution to the forced wave equation when the membrane is subject to homogeneous Dirichlet boundary conditions and initial conditions $u(0,x,y) = f(x,y)$, $u_t(0,x,y) = g(x,y)$. *Hint*: Expand the forcing function in an eigenfunction series.

11.6.14. A circular drum of radius $\zeta_{0,1} \approx 2.4048$ has initial displacement and velocity

$$u(0,x,y) = 0, \qquad \frac{\partial u}{\partial t}(0,x,y) = 2\, J_0\left(\sqrt{x^2+y^2}\,\right).$$

Assuming that the circular edge of the drum is fixed to the (x,y)–plane, describe, both qualitatively and quantitatively, its subsequent motion.

11.6.15. Write out the integral formulae for the coefficients in the Fourier–Bessel series solution (11.159) to the wave equation in a circular disk in terms of the initial data

$$u(0,r,\theta) = f(r,\theta), \ \ u_t(0,r,\theta) = g(r,\theta).$$

11.6.16. A circular drum at rest is struck with a concentrated blow at its center. Write down an eigenfunction series describing the resulting vibration.

♡ 11.6.17. (*a*) Set up and solve the initial-boundary value problem for the vibrations of a uniform circular drum of unit radius that is freely floating in space. (*b*) Discuss the stability of the drum's motion. (*c*) Are the vibrations slower or faster than when its edges are fixed to a plane?

11.6.18. A flat quarter-disk of radius 1 has its circular edge and one of its straight edges attached to the (x, y)–plane, while the other straight edge is left free. At time $t = 0$ the disk is struck with a hammer (unit delta function) at its midpoint, i.e., at radius $\frac{1}{2}$ and halfway between the straight edges. (a) Write down an initial-boundary value problem for the subsequent vibrations of the quarter-disk. *Hint*: Be careful with the form of the delta function in polar coordinates; see Exercise 6.3.6. (*b*) Assuming that the physical units are chosen so that the wave speed $c = 1$, determine the quarter-disk's vibrational frequencies. (*c*) Write down an eigenfunction series solution for the subsequent motion. (*d*) Is the motion unstable? periodic? If so, what is the period?

11.6.19. *True or false*: Assuming homogeneous Dirichlet boundary conditions, the fundamental frequencies of a vibrating half-disk are exactly twice those of the full disk of the same radius.

♡ 11.6.20. The edge of a circular drum is moved periodically up and down, so $u(t, 1, \theta) = \cos \omega t$. Assuming that the drum is initially at rest, discuss its response.

♣ 11.6.21. A drum is in the shape of a circular annulus with outer radius 1 meter and inner radius .5 meter. Find numerical values for its first three fundamental vibrational frequencies.

♡ 11.6.22. A homogeneous rope of length 1 and weight 1 is suspended from the ceiling. Taking x as the vertical coordinate, with $x = 1$ representing the fixed end and $x = 0$ the free end, the planar displacement $u(t, x)$ of the rope satisfies the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(x\, \frac{\partial u}{\partial x}\right), \qquad \begin{aligned} &|\, u(t,0)\, | < \infty, && u(t,1) = 0, \\ &u(0,x) = f(x), && \frac{\partial u}{\partial t}\,(0,x) = g(x), \end{aligned} \qquad t > 0, \quad 0 < x < 1.$$

(*a*) Find the solution. *Hint*: Let $y = \sqrt{x}$. (*b*) Are the vibrations periodic or quasiperiodic? (*c*) Describe the behavior of the rope when subject to uniform periodic external forcing $F(t, x) = a \cos \omega t$.

Scaling and Symmetry

Symmetry methods can also be effectively employed in the analysis of the wave equation. Let us consider the simultaneous rescaling

$$t \longmapsto \alpha\, t, \qquad x \longmapsto \beta\, x, \qquad y \longmapsto \beta\, y, \tag{11.160}$$

of time and space, whose effect is to change the function $u(t, x, y)$ into a rescaled version

$$U(t, x, y) = u(\alpha\, t, \beta\, x, \beta\, y). \tag{11.161}$$

The chain rule is employed to relate their derivatives:

$$\frac{\partial^2 U}{\partial t^2} = \alpha^2\, \frac{\partial^2 u}{\partial t^2}\,, \qquad \frac{\partial^2 U}{\partial x^2} = \beta^2\, \frac{\partial^2 u}{\partial x^2}\,, \qquad \frac{\partial^2 U}{\partial y^2} = \beta^2\, \frac{\partial^2 u}{\partial y^2}\,.$$

Therefore, if u satisfies the wave equation

$$u_{tt} = c^2\, \Delta u,$$

then U satisfies the rescaled wave equation

$$U_{tt} = \frac{\alpha^2 c^2}{\beta^2}\,\Delta U = C^2\,\Delta U, \quad \text{where the rescaled wave speed is} \quad C = \frac{\alpha\, c}{\beta}\,. \tag{11.162}$$

In particular, rescaling only time by setting $\alpha = 1/c$, $\beta = 1$, results in a unit wave speed $C = 1$. In other words, we are free to choose our unit of time measurement so as to fix the wave speed equal to 1.

If we set $\alpha = \beta$, scaling time and space in the same proportion, then the wave speed does not change, $C = c$, and so

$$t \longmapsto \beta\, t, \qquad x \longmapsto \beta\, x, \qquad y \longmapsto \beta\, y, \tag{11.163}$$

defines a *symmetry transformation* for the wave equation: If $u(t,x,y)$ is any solution to the wave equation, then so is its rescaled version

$$U(t,x,y) = u(\beta\, t, \beta\, x, \beta\, y) \tag{11.164}$$

for any choice of scale parameter $\beta \neq 0$. Observe that if $u(t,x,y)$ is defined on a domain Ω, then the rescaled solution $U(t,x,y)$ will be defined on the rescaled domain

$$\widetilde{\Omega} = \frac{1}{\beta}\,\Omega = \left\{ \left(\frac{x}{\beta}, \frac{y}{\beta} \right) \;\middle|\; (x,y) \in \Omega \right\} = \{\, (x,y) \mid (\beta\, x,\ \beta\, y) \in \Omega \,\}\,. \tag{11.165}$$

For instance, setting the scaling parameter $\beta = 2$ halves the size of the domain. The normal modes for the rescaled domain have the form

$$\begin{aligned} U_n(t,x,y) &= u_n(\beta\, t, \beta\, x, \beta\, y) = \cos(\beta\, \omega_n\, t)\ v_n(\beta\, x, \beta\, y), \\ \widetilde{U}_n(t,x,y) &= \widetilde{u}_n(\beta\, t, \beta\, x, \beta\, y) = \sin(\beta\, \omega_n\, t)\ v_n(\beta\, x, \beta\, y), \end{aligned}$$

and hence the rescaled vibrational frequencies are $\Omega_n = \beta\, \omega_n$. Thus, when $\beta < 1$, the rescaled membrane is larger by a factor $1/\beta$, and its vibrations are slowed down by the reciprocal factor β. For instance, a drum that is twice as large will vibrate twice as slowly, and hence have an octave lower overall tone. Musically, this means that all drums of a similar shape have the same pattern of overtones, differing only in their overall pitch, which is a function of their size, tautness, and density.

In particular, choosing $\beta = 1/R$ will rescale the unit disk into a disk of radius R. The fundamental frequencies of the rescaled disk are

$$\Omega_{m,n} = \beta\, \omega_{m,n} = \frac{c}{R}\, \zeta_{m,n}, \tag{11.166}$$

where c is the wave speed and $\zeta_{m,n}$ are the Bessel roots, defined in (11.118). Observe that the ratios $\omega_{m,n}/\omega_{m',n'}$ between vibrational frequencies remain the same, independent of the size of the disk R and the wave speed c. In general, we define the *relative vibrational frequencies* to be the ratios between the individual frequencies and the dominant, or smallest, one. Thus, the relative vibrational frequencies of a circular drum are

$$\rho_{m,n} = \frac{\omega_{m,n}}{\omega_{0,1}} = \frac{\zeta_{m,n}}{\zeta_{0,1}}\,, \qquad \text{where} \qquad \omega_{0,1} = \frac{c\, \zeta_{0,1}}{R} \approx 2.4\, \frac{c}{R}\,. \tag{11.167}$$

The relative frequencies (11.167) are independent of the size, stiffness or composition of the drum membrane. In the following table, we display a list of all relative vibrational frequencies (11.167) that are < 6. Once the lowest frequency $\omega_{0,1}$ has been determined

— either theoretically, numerically, or experimentally — all the higher overtones $\omega_{m,n} = \rho_{m,n}\,\omega_{0,1}$ are simply obtained by rescaling.

Relative Vibrational Frequencies of a Circular Disk

n \ m	0	1	2	3	4	5	6	7	8	9	...
1	1.000	1.593	2.136	2.653	3.155	3.647	4.132	4.610	5.084	5.553	...
2	2.295	2.917	3.500	4.059	4.601	5.131	5.651	⋮	⋮	⋮	
3	3.598	4.230	4.832	5.412	5.977	⋮	⋮				
4	4.903	5.540	⋮	⋮	⋮						
⋮	⋮	⋮									

Exercises

11.6.23. *True or false*: Two rectangular membranes, made out of the same material and both subject to Dirichlet boundary conditions, have the same relative vibrational frequencies if and only if they are have similar shapes.

11.6.24. *True or false*: (*a*) The vibrational frequencies of a square with side lengths $a = b = 2$ are four times as slow as those of a square with side lengths $a = b = 1$.
(*b*) The vibrational frequencies of a rectangle with side lengths $a = 2$, $b = 1$, are twice as slow as those of a square with side lengths $a = b = 1$.

11.6.25. A vibrating rectangle of unknown size has wave speed $c = 1$ and is subject to homogeneous Dirichlet boundary conditions. How many of its lowest vibrational frequencies do you need to know in order to determine the size of the rectangle?

11.6.26. Answer Exercise 11.6.25 when the rectangle is subject to homogeneous Neumann boundary conditions.

♣ 11.6.27. A circular drum has the A above middle C, which has a frequency of 440 Hertz, as its lowest tone. What notes are the first five overtones nearest? Try playing these on a piano or guitar. Or, if you have a synthesizer, try assembling notes of these frequencies to see how closely it reproduces the dissonant sound of a drum.

11.6.28. In an orchestra, a circular snare drum of radius 1 foot sits near a second circular drum made out of the same material. Vibrations of the first drum are observed to excite an undesired resonant vibration in its partner. What are the possible radii of the second drum?

11.6.29. *True or false*: The relative vibrational frequencies of a half-disk, subject to Dirichlet boundary conditions, are a subset of the relative vibrational frequencies of a full disk.

11.6.30. *True or false*: If $u(t,x,y) = \cos(\omega t)\, v(x,y)$ is a normal mode of vibration for a unit square subject to homogeneous Dirichlet boundary conditions, then the function $\widehat{u}(t,x,y) = \cos(\omega t)\, v\left(\frac{1}{2}x, \frac{1}{3}y\right)$ is a normal mode of vibration for a 2×3 rectangle that is subject to the same boundary conditions, but with a possibly different wave speed. If true, how are the wave speeds of the two rectangles related?

11.6.31. Prove that if $u(t,x,y)$ is a solution to the two-dimensional wave equation, so is the translated function $U(t,x,y) = u(t - t_0, x - x_0, y - y_0)$, for any constants t_0, x_0, y_0.

◇ 11.6.32. (a) Prove that if $u(t,x,y)$ solves the wave equation, so does $U(t,x,y) = u(-t,x,y)$. Thus, unlike the heat equation, the wave equation is time-reversible, and its solutions can be unambiguously followed backwards in time. (b) Suppose $u(t,x,y)$ solves the initial value problem (11.141). Write down the initial value problem satisfied by $U(t,x,y)$.

11.6.33. (a) Prove that, on $\mathbb{R}^2$, the solution to the pure displacement initial value problem $u_{tt} = c^2 \Delta u$, $u(0,x,y) = f(x,y)$, $u_t(0,x,y) = 0$, is an even function of t.
(b) Prove that the solution to the pure velocity initial value problem $u_{tt} = c^2 \Delta u$, $u(0,x,y) = 0$, $u_t(0,x,y) = g(x,y)$, is an odd function of t.
Hint: Use Exercise 11.6.32 and uniqueness of solutions to the initial value problem.

11.6.34. Suppose $v(t,x)$ is any solution to the one-dimensional wave equation $v_{tt} = v_{xx}$. Prove that $u(t,x,y) = v(t, ax+by)$, for any constants $(a,b) \neq (0,0)$, solves the two-dimensional wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$ for some choice of wave speed. Describe the behavior of such solutions.

11.6.35. A *traveling-wave solution* to the two-dimensional wave equation has the form $u(t,x,y) = v(x - at, y - at)$, where a is a constant. Find the partial differential equation satisfied by the function $v(\xi,\eta)$. Is the equation hyperbolic?

11.6.36. Is the counterpart of Lemma 11.11 valid for the wave equation? In other words, if $v(t,x)$ and $w(t,x)$ are any two solutions to the one-dimensional wave equation, is their product $u(t,x,y) = v(t,x)\,w(t,y)$ a solution to the two-dimensional wave equation?

11.6.37. (a) How would you solve an initial-boundary value problem for the wave equation on a rectangle that is not aligned with the coordinate axes? (b) Apply your method to set up and solve an initial-boundary value problem on the square $R = \{\,|\,x+y\,| < 1,\ |\,x-y\,| < 1\,\}$.

Chladni Figures and Nodal Curves

When a membrane vibrates, its individual atoms typically move up and down in a quasiperiodic manner. As such, there is little correlation between their motions at different locations. However, if the membrane is set to vibrate in a pure eigenmode, say

$$u_n(t,x,y) = \cos(\omega_n t)\, v_n(x,y), \tag{11.168}$$

then all points move up and down at a common frequency $\omega_n = \sqrt{\lambda_n}$, which is the square root of the eigenvalue corresponding to the eigenfunction $v_n(x,y)$. The exceptions are the points where the eigenfunction vanishes:

$$v_n(x,y) = 0, \tag{11.169}$$

which remain stationary. The set of all points $(x,y) \in \Omega$ that satisfy (11.169) is known as the n^{th} *Chladni figure* of the domain Ω, named in honor of the eighteenth-century German physicist and musician Ernst Chladni who first observed them experimentally by exciting a metal plate with his violin bow, [**43**]. The mathematical models governing such vibrating plates were formulated by the French mathematician Sophie Germain in the early 1800s. It can be shown that, in general, each Chladni figure consists of a finite system of *nodal curves*, [**34**, **43**], that partition the membrane into disjoint *nodal regions*. As the membrane vibrates, the nodal curves remain stationary, while each nodal region is entirely either above or below the equilibrium plane, except, momentarily, when the *entire* membrane has zero displacement. As Chladni discovered in his original experiments, scattering small

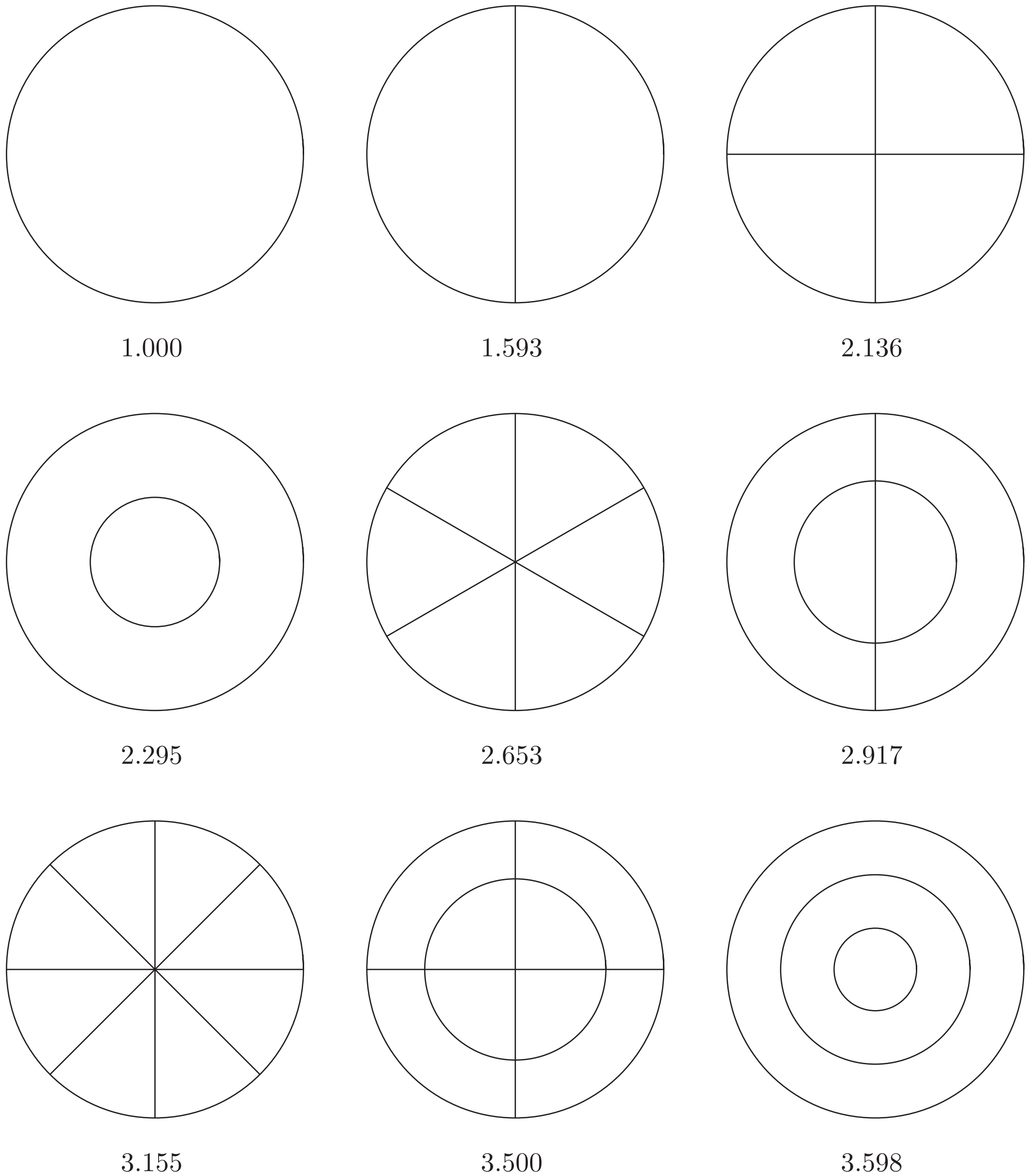


Figure 11.12. Nodal curves and relative vibrational frequencies of a circular membrane.

particles (e.g., fine sand) over a membrane or plate vibrating in an eigenmode will enable us to visualize the Chladni figure, because the particles will tend to accumulate along the stationary nodal curves. Adjacent nodal regions, lying on the opposite sides of a nodal curve, move in opposing directions — when one is up, its neighbors are down, and then they switch roles as the membrane becomes momentarily flat. Let us look at a couple of examples where the Chladni figures can be readily determined.

Example 11.15. *Circular Drums.* Since the eigenfunctions (11.156) for a disk are products of trigonometric functions in the angular variable and Bessel functions of the radius, the nodal curves for the normal modes of vibrations of a circular membrane are rays emanating from and circles centered at the origin. Consequently, the nodal regions are annular sectors. Chladni figures associated with the first nine normal modes, indexed by their relative frequencies, are plotted in Figure 11.12. Representative displacements of the membrane in each of the first twelve modes can be found earlier, in Figure 11.6. The dominant (lowest frequency) mode is the only one that has no nodal curves; it has the form of a radially symmetric bump where the entire membrane flexes up and down. The next lowest modes vibrate proportionally faster at a relative frequency $\rho_{1,1} \approx 1.593$. The most general solution with this vibrational frequency is a linear combination of the two eigensolutions: $\alpha\, u_{1,1} + \beta\, \widetilde{u}_{1,1}$. Each such combination has a single diameter as a nodal curve, whose angle with the horizontal depends on the ratio β/α. The two semicircular halves of the drum vibrate in opposing directions — when the top half is up, the bottom half is down and vice versa. The next set of modes have two perpendicular diameters as nodal curves; the four quadrants of the drum vibrate in tandem, with opposite quadrants moving in the same direction. Next in increasing order of vibrational frequency is a single mode, which has a circular nodal curve whose (relative) radius equals the ratio of the first two roots of the order zero Bessel function, $\zeta_{0,1}/\zeta_{0,2} \approx .43565$; see Exercise 11.6.39 for a justification. In this case, the inner disk and the outer annulus vibrate in opposing directions. And so on . . . .

Example 11.16. *Rectangular Drums.* For most rectangular drums, the Chladni figures are relatively uninteresting. Since the normal modes (11.151) are separable products of trigonometric functions in the coordinate variables x, y, the nodal curves are equally spaced straight lines parallel to the sides of the rectangle. The internodal regions are smaller rectangles, of identical size and shape, with adjacent rectangles vibrating in opposite directions.

More interesting figures appear when the rectangle admits multiple eigenvalues — so-called *accidental degeneracies*. Note that two of the eigenvalues (11.149) coincide, $\lambda_{m,n} = \lambda_{k,l}$, if and only if

$$\frac{m^2}{a^2} + \frac{n^2}{b^2} = \frac{k^2}{a^2} + \frac{l^2}{b^2}\,, \tag{11.170}$$

where $(m,n) \neq (k,l)$ are distinct pairs of positive integers. In such situations, the two eigenmodes happen to vibrate with a common frequency $\omega = \omega_{m,n} = \omega_{k,l}$. Consequently, any linear combination of the eigenmodes, e.g.,

$$\cos(\omega\, t) \left(\alpha \sin \frac{m\pi x}{a} \; \sin \frac{n\pi y}{b} + \beta \sin \frac{k\pi x}{a} \; \sin \frac{l\pi y}{b} \right), \qquad\qquad \alpha, \beta \in \mathbb{R},$$

is also a pure vibration, and hence qualifies as a normal mode. The associated nodal curves,

$$\alpha \sin \frac{m\pi x}{a} \; \sin \frac{n\pi y}{b} + \beta \sin \frac{k\pi x}{a} \; \sin \frac{l\pi y}{b} = 0, \qquad \begin{array}{l} 0 \le x \le a, \\ 0 \le y \le b, \end{array} \tag{11.171}$$

have a more intriguing geometry, which can change dramatically as the coefficients α, β vary.

For example, on the unit square $R = \{\, 0 < x, y < 1 \,\}$, an accidental degeneracy occurs whenever

$$m^2 + n^2 = k^2 + l^2 \tag{11.172}$$

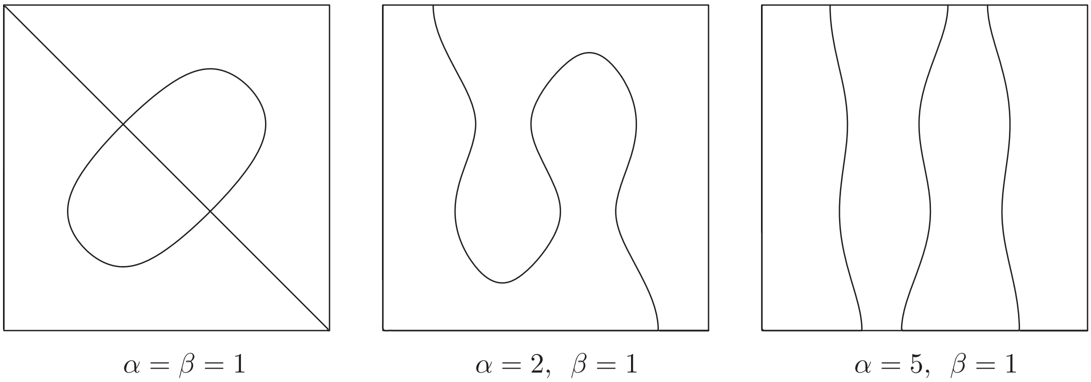


Figure 11.13. Some Chladni figures for a square membrane.

for distinct pairs of positive integers $(m, n) \neq (k, l)$. The simplest possibility arises whenever $m \neq n$, in which case we can merely reverse the order, setting $k = n$, $l = m$. In Figure 11.13 we plot three sample nodal curves

$$\alpha \sin 4\pi x \ \sin \pi y + \sin \pi x \ \sin 4\pi y = 0,$$

corresponding to three different linear combinations of the eigenfunctions with $m = l = 4$, $n = k = 1$. The associated vibrational frequency is, in all cases, $\omega_{4,1} = c\sqrt{17}\,\pi$, where c is the wave speed.

Classifying accidental degeneracies of rectangles takes us into the realm of number theory, [**9**, **29**]. In the case of a square, equation (11.172) is asking us to locate all integer points $(m, n) \in \mathbb{Z}^2$ that lie on a common circle.

Remark: Bourget's hypothesis, mentioned after (11.157), implies that $\zeta_{m,n} \neq \zeta_{k,l}$ whenever $(m, n) \neq (k, l)$. This implies that a disk has no accidental degeneracies, and hence all its nodal curves are concentric circles and diameters.

Exercises

♢ 11.6.38. Suppose that a membrane is vibrating in a normal mode. Prove that the membrane lies instantaneously completely flat at regular time intervals.

♢ 11.6.39. For a vibrating disk of unit radius, determine the radius of the circular nodal curve for the next-to-lowest circular mode.

11.6.40. Order the five nodal circles displayed in Figure 11.12 according to their size.

11.6.41. Sketch the Chladni figures in a unit disk corresponding to the following vibrational frequencies. Determine numerical values for the radii of any circular nodal curves.
(a) $\omega_{0,4}$, (b) $\omega_{2,4}$, (c) $\omega_{4,2}$, (d) $\omega_{3,3}$, (e) $\omega_{5,1}$.

11.6.42. *True or false*: Any diameter of a circular disk is a nodal curve for some normal mode.

11.6.43. *True or false*: The nodal curves on a semicircular disk are all semicircles and rays emanating from the center.

11.6.44. (*a*) Find the smallest distinct pair of positive integers $(k,l) \neq (m,n)$ satisfying (11.172) that are not obtained by simply reversing the order, i.e., $(k,l) \neq (n,m)$. (*b*) Find the next-smallest example. (*c*) Plot two or three Chladni figures arising from such degenerate eigenfunctions.

♡ 11.6.45. Let R be a rectangle all of whose sides are fixed to the (x,y)–plane. Suppose that all its nodal curves are straight lines. What can you say about its side lengths a, b?

11.6.46. *True or false*: The nodal regions of a vibrating rectangle are similarly shaped rectangles.

♢ 11.6.47. Prove that any point of intersection (x_0, y_0) of two nodal curves associated with the same normal mode is a critical point of the associated eigenfunction: $\nabla v(x_0, y_0) = \mathbf{0}$.

11.6.48. *True or false*: The nodal curves on a domain do not depend on the choice of boundary conditions.