Chapter 15
Tomographic Represention of Quantum and Classical Cosmology

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Abstract In this paper we consider a tomographic representation of quantum cosmology, in which tomograms (i.e. a standard positive probability distribution function) describe the quantum state of universe in place of the the wave function or density matrices. This representation can be extended to classical cosmology and used to reconstruct the initial conditions of the universe by studying the evolution of a quantum tomogram to a classical one. To this end we give a definition of classical tomogram based on cosmological observations, and we give the criterion for the reconstruction of the states primordial universe.

15.1 Introduction

Quantum cosmology [1] is an application of quantum gravity in which the full field theory is reduced to a problem with a few degrees of freedom, by the restriction of the superspace, i.e. space of the spatial metrics, to the so-called minisuperspace, which is the space of the homogeneous metrics. In this case the quantum properties of homogeneous cosmological models can be described in terms of quantum mechanics. Quantum cosmology can be considered as a toy model designed to capture some of the fundamental properties of the complete field theory, even, if by fixing contemporarily most of the field modes and of their conjugate momenta to zero, it violates the uncertainty principle. There is a departure from the Copenhagen interpretation where the measurements are conceived by taking a quantum system embedded in a classical space, where some classical observer makes the measurements. By considering the whole universe as a quantum system, we necessarily have a non conventional approach to a quantum theory, because we are dealing with the entire universe as if it were a single particle. Unlike a microscopic system, in which generally are considered ensembles of many particles, the theory is not able to make predictions on the statistical evolution of the system. However, a phenomenological approach to the theory and how quantum effects affected the background radiation and on the galaxy distribution was recently proposed (see e.g. [2, 3]). Also our work

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proposes a phenomenological approach to quantum cosmology, on different lines, but not in contrast with those mentioned before.

In quantum cosmology one first introduces a canonical formalism and then defines a wavefunction of the universe which evolves according to the Wheeler-DeWitt equation. It is necessary to specify the initial conditions for the wave function in order to determine the evolution of the universe. They must be considered as a fundamental law of physics [4], because differently from any other physical system, where the initial conditions can change from one configuration to another, they determine the evolution of the whole universe, whose configuration is once and for all. Theoretical approaches to this law have been posed by Hartle-Hawking [5], Vilenkin [6] and Linde [7]. But as for many physical laws, we believe that a phenomenological approach to reconstruct the initial conditions of the universe is possible. A limiting condition to the reconstruction of the initial conditions could be the huge amount of entropy of the universe $S/k \sim 10^{80}$, which measures the lost of informations. But this number is significantly small when compared with the maximum entropy which, according to the holographic principle, is of the order of $S/k \approx 10^{120}$. This guarantees that the initial conditions of the universe can be reconstructed with a very high accuracy. But in order to do this we need to connect the cosmological observations available in the present time to the early state of the universe. This will be discussed in the following of the paper. Finally it is important to have a mechanism to describe the transition from quantum physics to classical physics by means of the quantum decoherence. Not all the solutions of the Wheeler-DeWitt lead to the decoherence, the so-called Hartle conditions are the necessary conditions needed to guarantee this transition.

In this paper we present an alternative representation of quantum cosmology by introducing the notion of standard positive probability distribution function or tomogram which has been used [8] to describe the quantum state of universe alternatively to the wave function or to the density matrix descriptions. Tomograms where introduced in quantum mechanics in analogy to the same notion used in quantum optics in [9]. The advantage of using the tomographic representation of quantum mechanics is that one can represent a quantum state by an observable function. Moreover even if we quantum tomograms instead of the wavefunction, it is also possible to define classical tomograms. Classical and quantum tomograms are both defined on the phase space and it is possible to show that quantum tomograms can evolve to classical ones. So one can think to reconstruct the initial conditions of the universe by first introducing a classical tomogram of the universe from observations and then one can reconstruct the initial state of the universe projecting back in time the this tomogram.

In this following sections we introduce first the tomographic representation of quantum mechanics and then apply it to quantum cosmology. We obtain the WheelerDeWitt equation for tomograms (TWDW equation), the probability transition function which defines the evolution of a tomogram and finally we give a prescription for deriving a classical tomogram for the present time based on a statistical function distribution of the distance and velocity of the galaxies, in doing so we slightly modify the paradigm of quantum cosmology, which traditionally assumes a rigorous homogeneity of the universe. Since the TWDW equation is not time dependent, we discuss the meaning of the time evolution of the tomogram as the evolution of the observations by a classical observer in the late stages of the cosmological evolution and by a quantum observer in the early universe.

15.2 Tomographic Representation of Quantum Mechanics

To understand better the role of the tomograms in reconstructing the state of a quantum system, we can refer to an experiment done by Kurtsiefer, Pfau and Mlynek [10], where a coherent beam of helium atoms in a double-slit experiment measurements of the quantum mechanical analogue of the classical phase space distribution function show that the motion of atoms behaves in a strongly non classical manner. The experiment was designed to reconstruct the Wigner function of the ensemble of the helium atoms by means of time-resolved diffraction patterns. These patterns are described by distribution functions $P(\tilde{x}, t_d)$ where t_d is the traveling time between the double-slit and the atom detector. It can be shown that this function is related to the marginal distribution $P_\Theta = P(\frac{\tilde{x}}{\cos\Theta}, \frac{mx_0^2}{\hbar}\tan\Theta)$, which is the corresponding tomogram. Finally the authors were able to reconstruct to the Wigner function, see Eq. (15.4). This paper therefore shows that tomograms are observable quantities which represent a quantum state. Therefore, according to [9] tomograms can be used in quantum mechanics alternatively to the wavefunctions.

Similarly, we introduce a tomogram in quantum cosmology instead of the wavefunction of the universe. We can also introduce the tomograms in classical cosmology and study directly the evolution of a quantum cosmological tomogram into a classical one or vice versa we can first construct from observations a classical tomogram and reconstruct the quantum tomogram of the early universe by using the transition probability functions, which are defined later.

To give the notion of tomogram let us first recall the definition of the Wigner function. It was introduced by Wigner to study quantum corrections to classical statistical mechanics and it can be assigned to represent a quantum state. It is expressed in terms of density matrix ($\hbar = 1$) by

$$W(q, p) = \int \rho\left(q + \frac{u}{2}, q - \frac{u}{2}\right) e^{-ipu} du \qquad (15.1)$$

The inverse transform reads

$$\rho(x, x') = \frac{1}{2\pi}\int W\left(\frac{x + x'}{2}, p\right) e^{ip(x-x')} dp. \qquad (15.2)$$

The Radon transform of the Wigner function in the modified form is the integral transform of the form

$$\mathcal{W}(X, \mu, \nu) = \int W(q, p) e^{ik(X-\mu q-\nu p)} \frac{dk\,dq\,dp}{(2\pi)^2} \qquad (15.3)$$

Here X, μ and ν are real numbers. The Wigner function can be found using the inverse Radon relation

$$W(q,p)=\frac{1}{2\pi}\int e^{i(X-\mu q-\nu p)}\mathcal{W}(X,\mu,\nu)dXd\mu d\nu. \tag{15.4}$$

The standard Radon transform is obtained from the two above by putting $\mu=\cos\Theta$, $\nu=\sin\Theta$.

One can see that the tomographic symbol of density matrix is given as a marginal distribution since

$$\mathcal{W}(X,\mu,\nu)=\int W(q,p)\delta(X-\mu q-\nu p)\frac{dqdp}{2\pi} \tag{15.5}$$

It is clear that

$$\int\mathcal{W}(X,\mu,\nu)dX=1, \tag{15.6}$$

since the Wigner function is normalized

$$\int W(q,p)\frac{dqdp}{2\pi}=1 \tag{15.7}$$

for normalized wave functions.

The formulae (15.5) – (15.7) are valid for arbitrary density matrices, both for pure and mixed states. For pure states of the universe, the tomographic symbol can be expressed directly in terms of the wave function of the universe. The relation between the wave function and the tomogram is given by

$$\mathcal{W}(X,\mu,\nu)=\frac{1}{2\pi|\nu|}\left|\int\psi(y)e^{\frac{i\mu}{2\nu}y^2-\frac{iX}{\nu}y}dy\right|^2. \tag{15.8}$$

this relation is invertible. Tomograms and wavefunctions are in a one-to-one relation (except for a phase factor). This is why tomograms represent the quantum states of a system.

Notice that the formula relating the tomographic symbol with the wave function contains the integral

$$I=\left|\int\psi(y)e^{\frac{i\mu}{2\nu}y^2-\frac{iX}{\nu}y}dy\right| \tag{15.9}$$

In case of $\mu=0$, $\nu=1$ this integral is a conventional Fourier transform of the wave function. For generic μ, ν the integral is identical to the modulus of Fractional Fourier transform of the wave function [11]. Thus, the Radon transform of the Wigner function in the case of pure states is related to the Fractional Fourier transform of the wave function.

Classical tomograms are defined by replacing in Eq. (15.3) the Wigner function in Eq. (15.3) with any classical distribution function $f(q, p)$ in the phase space,

$$\mathcal{W}(X, \mu, \nu) = \int f(q, p)\delta(X - \mu q - \nu p)\frac{dqdp}{2\pi}. \tag{15.10}$$

The main difference between the classical and quantum tomograms is that the quantum tomograms must satisfy the inequality

$$\left[\int \mathcal{W}(X, 1, 0)X^2 - \left\{\int \mathcal{W}(X, 1, 0)X\right\}^2\right] \times \left[\int \mathcal{W}(X, 0, 1)X^2 - \left\{\int \mathcal{W}(X, 0, 1)X\right\}^2\right] \geq \frac{1}{4} \tag{15.11}$$

derived from the uncertainty principle. In conclusion formulae (15.5) and (15.10) show that quantum and classical tomograms are the set of all the probability distribution functions defined on the straight lines $X = \mu q + \nu p$ which span the whole phase space by varying μ and ν.

15.3 The Phase Space in Cosmology

Many homogeneous cosmological models can be derived from a point particle Lagrangian. For example for a Friedman-Lemaitre-Robertson-Walker universe with metric

$$ds^2 = -c^2dt^2 + \frac{a^2}{1 - kr^2}\left(dr^2 + r^2d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{15.12}$$

the Lagrangian is obtained by substituting the metric (15.12) into the general relativistic action $\int \sqrt{-g}R$

$$\mathcal{L} = 3a\dot{a}^2 - 3\,ka - 8\pi G\rho_0 a_0^{3\gamma} a^{3(1-\gamma)}. \tag{15.13}$$

The material part is given by the potential term $\Phi(a) = 8\pi G\rho_0 a_0^{3\gamma} a^{3(1-\gamma)}$, when the matter source is a fluid with equation of state $P = (\gamma - 1)\rho$.

The tomogram has to be expressed in terms of the phase space coordinates which are cosmological observables. The expansion factor a is not an observable, instead it is the ratio $q = \frac{a}{a_0} = \frac{1}{z+1}$. Its conjugate momentum is $p_a = \frac{\partial L}{\partial \dot{a}} = 6a\dot{a}$. The observable is $p = 6\frac{a}{a_0^2}\dot{a} \equiv 6\frac{a^2}{a_0^2}H$

15.4 The Wheeler-DeWitt Equation and the Corresponding Equation for the Tomogram

The universe in a model of quantum cosmology is described by a wave functional which depends on the spatial metric. There exist several elaborated examples of minisuperspaces, here we consider, as an example, the model in which the metric dependence is reduced to dependence only on the expansion factor of the universe. This is a one dimensional Wheeler-DeWitt equation for a FLRW universe of the form

$$\frac{1}{2}\left\{\frac{1}{a^{\lambda}}\frac{d}{da}a^{\lambda}\frac{d}{da}-a^{2}+\Lambda a^{4}\right\}\psi(a)=0 \tag{15.14}$$

Here a, is the expansion term of the classical theory and λ is an index introduced to take into account the ambiguity of operator ordering. The Radon transform considered previously makes sense only for variables which take values from $-\infty$ to $+\infty$, then with the change of variables $a=\exp x$ and the Wheeler-DeWitt equation becomes

$$\frac{1}{2}\left\{\exp(-2x)\frac{d^{2}}{dx^{2}}+(\lambda-1)\exp(-2x)\frac{d}{dx}-2U(x)\right\}\Psi(x)=0 \tag{15.15}$$

where $U(x)=(\exp(2x)-\Lambda\exp(4x))/2$. This equation can be written also in the form [8]

$$\frac{1}{2}\left\{\exp(-2x')\frac{d^{2}}{dx'^{2}}+(\lambda-1)\exp(-2x')\frac{d}{dx'}-2U(x')\right\}\Psi^{*}(x')=0. \tag{15.16}$$

The TWDW equation corresponding to (15.15) is then

$$\begin{aligned}&\left\{\mathrm{Im}\left[\exp\left[2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}+i\nu\frac{\partial}{\partial X}\right]\left(\frac{1}{2}\mu\frac{\partial}{\partial X}-i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu}\right)^{2}\right]\right.\\&+(\lambda-1)\mathrm{Im}\left[\exp\left(2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}+i\nu\frac{\partial}{\partial X}\right)\left(\frac{1}{2}\mu\frac{\partial}{\partial X}-i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu}\right)\right]\\&-2\mathrm{Im}\left[\exp\left(-2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}+i\nu\frac{\partial}{\partial X}\right)-\Lambda\exp\left(-4\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}\right.\right.\\&\left.\left.\left.+2i\nu\frac{\partial}{\partial X}\right)\right]\right\}\mathcal{W}(X,\mu,\nu)=0.\end{aligned} \tag{15.17}$$

15.5 Evolution of a Tomogram

The tomographic map can be used not only for the description of the universe state by probability distributions, but also to describe the evolution of the universe (quantum transitions) by means of the standard real positive transition probabilities. The transition probability

$$\Pi(X, \mu, \nu, t, X', \mu', \nu', t_0)$$

is the propagator which gives the tomogram of the universe $\mathcal{W}(X, \mu, \nu, t)$, if the tomogram at the initial time t_0 is known, in the form

$$\mathcal{W}(X, \mu, \nu, t) = \int \Pi(X, \mu, \nu, t, X', \mu', \nu', t_0)\mathcal{W}(X', \mu', \nu', t_0)dX'd\mu'd\nu'. \quad (15.18)$$

The positive transition probability describing the evolution of the universe has the obvious nonlinear properties used in classical probability theory, namely

$$\Pi(X_3, \mu_3, \nu_3, t_3, X_1, \mu_1, \nu_1, t_1) = \int \Pi(X_3, \mu_3, \nu_3, t_3, X_2, \mu_2, \nu_2, t_2)$$
$$\times \Pi(X_2, \mu_2, \nu_2, t_2, X_1, \mu_1, \nu_1, t_1)\, dX_2\, d\mu_2\, d\nu_2. \quad (15.19)$$

They follow from the associativity property of the evolution maps.

We stress the importance of Eq. (15.18), because it shows that given a tomogram at a time t_0, one can reconstruct the tomogram, i.e. the state of the universe, at any other time by applying this equation. The probability functions are derived according to the cosmological equations at each stage of the universe (i.e. inflation epoch, radiation and matter epochs). The total transition function $\Pi(X, \mu, \nu, X', \mu', \nu', t)$ is obtained by repeatedly applying Eq. (15.19) for the transition functions of each cosmological epoch.

Therefore, in order to reconstruct the initial state of the universe, we need to find a phenomenological tomogram to insert in Eq. (15.18), defined on the phase space with coordinates (assuming $a_0 = 1$)

$$p = 6a\dot{a} = 6a^2 H \quad \text{and} \quad q = a. \quad (15.20)$$

15.6 The Classical Tomogram

The Wheeler-DeWitt equation is a Schrödinger-like equation in which time is not present. In the example of Sect. 15.4, the wave function depended only on a or on $x = \ln a$. So is for its tomographic version the TWDW equation. We assume to take the solutions of these equations on the light-cone, because all the information of the state of the universe come only from the cosmological observations along the light-cone. Let us suppose for the moment that we can observe the universe at all values of z.

The next point is how to construct a tomogram as a statistical function. If the universe were perfectly homogeneous and isotropic, since we are considering a constrained system, the tomogram should be reduced to a point for each value of a. Since on the other hand the universe is not totally homogeneous, generally many cosmological observables, like the Hubble constant H_0 are averages of measures taken along all directions. To construct a realistic tomogram we adopt the point of view of [14] where the variance of the Hubble flow is considered without making a priori cosmological assumptions and is viewed as the differential expansion of regions of different local densities, because during the evolution of the universe they may have had different expansion histories from an epoch when the density was close to uniform. Therefore taking measuring the observables along sectors of the sky defined by different directions and a particular angle around each direction, we calculate their averages on these sectors. These averages form a statistical sample to be inserted on the phase space defined above. The classical tomogram will be the distribution function of points on each straight line passing through the origin. Each straight line will correspond some definite redshift z. The classical tomogram so-defined is given by the state of the universe at all the epochs, in line of principle it should be extended to the farthest parts of the universe in causal connection with us. But at present the current observations show distances up to $z = 8, 9$ using the standard candles, and at about $z = 1000$ from observations of the CMB radiation, which is a limitation to the actual construction of the tomogram.

What is the meaning of the time dependence of this tomogram? As tomograms are related to the observations, so they *define* also the observer. By observer we mean any physical system which records the informations coming from the universe. A tomogram is classical when there is a very weak interaction between the observer and the rest of the universe and new version of the cosmological principle is that all the observers see the same tomogram. On the other side the resulting tomogram is quantum when the "observer" interacts with the entire universe and it should be subject to the uncertainty conditions (15.11).

15.7 Conclusions

In this paper we showed that the reconstruction of the initial conditions of the universe can be done first by constructing an observable functions which can be related to the early stages of the universe by means of probability transition functions. We defined such a function by weakening the assumption of homogeneity of the universe and considering the variance of Hubble flow in different regions of the universe. A classical tomogram was then defined by considering observations at any red-shift until the maximal distance where events can be observed. Even if the definition, due to the limitations of current observations, is not viable from a practical point of view, it is in principle interesting, because it relates the state of the universe with the observer. A better definition would be to define the tomogram on a spatial hypersurface at a time t_0 and consider the transitions from one hypersurface to another. This is the aim of a future work.

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