Contact Invariants in Floer Homology

GORDANA MATIĆ

1. Heegaard Floer Homology—A Very Quick Introduction

In a pair of seminal papers [24, 25] Peter Ozsváth and Zoltan Szabó defined a collection of homology groups they named Heegaard-Floer homologies. To define these groups, they first associate to a 3-manifold M a Heegaard diagram, use it to define a chain complex, and then show that the associated homology groups do not depend on choices made.

A Heegaard decomposition of a 3-manifold is a decomposition $M = H_1 \cup H_2$ into two handlebodies with $\Sigma_g = \partial H_1 = -\partial H_2 = H_1 \cap H_2$. We can define a genus g handlebody as a 3-manifold obtained by gluing g "handles" $D^2 \times [0,1]$ to B^3 by attaching $D^2 \times 0$ and $D^2 \times 1$ along disjoint pairs of discs in S^2 and smoothing. Every 3-manifold has a Heegaard decomposition. The easiest way to "visualize" one is to take a triangulation of M and take the neighborhood of the 1-skeleton as H_1 and the complement as H_2, It is clear that H_1 is a handlebody—if we take a maximal tree T in the 1-skeleton and cut the neighborhood along the discs perpendicular to the edges not in T, we have a ball. The complement is a neighborhood of the dual 1-skeleton, so is a handlebody by the same argument.

Another way to obtain a Heegaard decomposition is to look at Morse functions on M. Assume that a Morse function $f : M \to [0,3]$ is self-indexing, i.e. that it has critical points of index i at critical values $i = 0, \ldots, 3$. When there are g critical points of index 1 and 2, the mid-level surface $\Sigma = f^{-1}(3/2)$
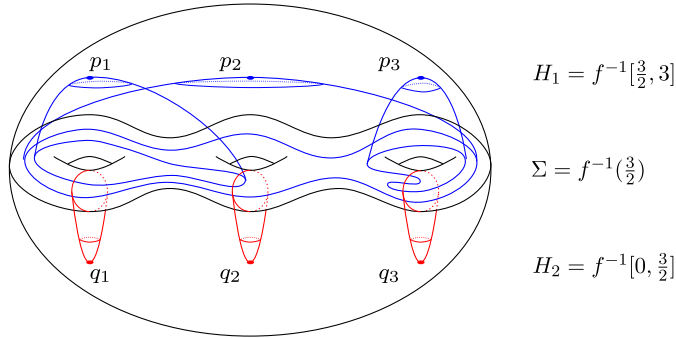
Fig. 1. Self-indexing Morse function $f : M \to [0,3]$ with critical points of index 1 and their ascending discs in red, and critical points of index 2 and their descending discs in blue tracing the Heegaard diagram on $\Sigma = f^{-1}(\frac{3}{2})$

has genus g. The descending discs D_{p_i} of the index 2 critical points $\{p_i | i = 1, \ldots, g\}$ and the ascending discs D_{q_i} of the index 1 critical points $\{q_i | i = 1, \ldots, g\}$ cut the handlebodies $H_1 = f^{-1}[3/2, 3]$ and $H_2 = f^{-1}[0, 3/2]$ into 3-balls (see Figure 1).

Collections of their boundary curves $\{\alpha_i | \alpha_i = \partial D_{p_i}\}$ (correspondingly $\{\beta_i | \beta_i = \partial D_{q_i}\}$) cut the Heegaard surface Σ_g into a $2g$-times punctured sphere. The *Heegaard diagram* $(\Sigma, \{\alpha_i\}, \{\beta_i\})$ determines M—it can be obtained from $\Sigma \times [-1, 1]$ by gluing compressing discs along $\alpha_i \times \{-1\}$ and along $\beta_i \times \{1\}$, thickening the discs and then finally gluing in two copies of B^3. It is a theorem of Reidemeister and Singer [28, 31] that any two Heegaard diagrams for the same 3-manifold can be related to each other by three moves: (1) stabilization, in which genus of Σ is increased by adding a one handle and a pair of dual curves α_0, β_0 is added to $\{\alpha_i\}$, $\{\beta_i\}$ (and destabilization), (2) isotopy of multicurves $\{\alpha_i\}$, $\{\beta_i\}$ and (3) handle slides, where $\{\alpha_1, \alpha_2, \ldots\}$ is replaced by $\{\alpha_1 + \alpha_2, \alpha_2 \ldots\}$.

Heegaard Floer homology is a variant of Lagrangian Floer homology applied to the two Lagrangian submanifolds $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ in the singular symplectic manifold $Sym^g(\Sigma_g)$, the symmetric product of g copies of Σ_g. In [24, 25] Ozsváth and Szabó define several versions of this invariant. To describe the simplest of them, $\widehat{HF(M)}$ with $\mathbb{Z}/2$ coefficients, we define the chain group $\widehat{CF}(\Sigma, \alpha, \beta) = \mathrm{Span}_{Z/2}\{\mathbf{x} | \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\}$ to be the free $\mathbb{Z}/2$-module generated by the points $\mathbf{x} = (x_1, \ldots, x_g)$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Note that such a g-tuple $\mathbf{x} = (x_1, \ldots, x_g)$ contains one point on each α_i and one point on each β_i curve.

Grading and the boundary maps are defined by considering pseudoholomorphic discs in $Sym^g(\Sigma_g)$ with boundary on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. More precisely, we look at maps ϕ from the unit disc $D^2 \subset \mathbb{C}$ to $Sym^g(\Sigma)$ that map $-i \mapsto \mathbf{x}$, $i \mapsto \mathbf{y}$, $S^1 \cap \{\text{Re } z \geq 0\}$ to $\mathbb{T}_\alpha$ and $S^1 \cap \{\text{Re } z \leq 0\}$ to $\mathbb{T}_\beta$. We denote the set of homotopy classes of such maps by $\pi_2(\mathbf{x}, \mathbf{y})$. With a generic choice of almost complex structure on $Sym^g(\Sigma_g)$, the space of pseudoholomorphic discs in the homotopy class of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is a smooth manifold $\mathcal{M}(\phi)$. These spaces carry a free $\mathbb{R}$ action by translation. To see this translation action we think of a biholomorphism between $D^2 \setminus \{i, -i\}$ and $[-1, 1] \times \mathbb{R}$ taking $\pm i$ to $\pm \infty$ and translate in the $\mathbb{R}$ direction. The dimension of the space $\mathcal{M}(\phi)$ of holomorphic maps in the homotopy class of ϕ is calculated via the Maslov index $\mu(\phi)$, which defines a relative grading on $\widehat{CF}(\Sigma, \alpha, \beta)$. When the dimension of $\mathcal{M}(\phi)$ is one, compactness arguments give finiteness of $\widehat{\mathcal{M}}_\phi = \mathcal{M}(\phi)/\mathbb{R}$, and we define the boundary operator on $\widehat{CF}(\Sigma, \alpha, \beta)$ by counting (mod 2) the number of points in the 0-dimensional space $\widehat{\mathcal{M}}_\phi$. If we fix a marked point z in the complement of the α and β curves, the function $n_z(\phi)$ on $\pi_2(\mathbf{x}, \mathbf{y})$ is given by the intersection number of $\phi(D^2)$ with $z \times Sym^{g-1}(\Sigma)$. If the image of ϕ misses $z \times Sym^{g-1}(\Sigma)$ then $n_z(\phi) = 0$. The differential

$$\partial \mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1, n_z(\phi)=0}} \#(\widehat{\mathcal{M}}_\phi) \mathbf{y}$$

has grading -1 and satisfies $\partial \circ \partial = 0$.

To understand better what we are summing over, note that the map $\phi : D^2 \to Sym^g(\Sigma)$ corresponds to a map $\widetilde{\phi} : \widetilde{D} \to \Sigma$ from a g-fold branched cover $\widetilde{D}$ of D^2 to Σ (branching is over the preimage of the intersection of $\phi(D^2)$ with the diagonal in $Sym^g(\Sigma)$). The two marked points i and $-i$ on the boundary of D^2 lift to $2g$ marked points on the boundary of $\widetilde{D}$, and under $\widetilde{\phi}$ these are mapped in alternating fashion to the coordinates x_i and y_i of $\mathbf{x}$ and $\mathbf{y}$. The segments on the boundary of $\widetilde{D}$ between these $2g$ points are mapped in alternating fashion to segments on α and β curves connecting x_i and y_j. The image of the map $\widetilde{\phi}$ is a union of domains D_i in the complement of the α and β curves in Σ, and the Maslov index can be computed from this picture according to a combinatorial formula of Lipshitz [20]. Analysis of the moduli spaces corresponding to Maslov index 2 maps shows that the boundary map satisfies $\partial^2 = 0$. The resulting homology is shown in [24] to be independent of the choice of a marked Heegaard diagram $(\Sigma, \alpha_i, \beta_i, z)$ for M, and therefore an invariant of the 3-manifold M.

Given a marked point z, each intersection point $\mathbf{x} = (x_1, \ldots, x_g)$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ defines a spin^c structure $\mathbf{s}(\mathbf{x})$ on the 3-manifold M determined by
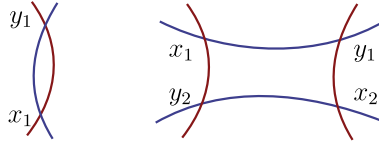
Fig. 2. A bigon and a rectangular domain

the Heegaard diagram. If there is a topological disc from $\mathbf{x} = (x_1, \ldots, x_g)$ to $\mathbf{y} = (y_1, \ldots, y_g)$ with boundary contained in the union of two tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ the two $spin^c$ structures $\mathbf{s}(\mathbf{x})$ and $\mathbf{s}(\mathbf{y})$ agree. Therefore $\widehat{CF}(\Sigma, \alpha, \beta, z)$ splits as the sum of $\widehat{CF}((\Sigma, \alpha, \beta, z), \mathbf{s})$ and, as is clear from the definition, the boundary map preserves this splitting. Hence the complex, and the homology of the complex as well, split according to $spin^c$ structures:

$$\widehat{HF}(M) = \sum_{\mathbf{s} \in Spin^c(M)} \widehat{HF}(M, \mathbf{s})$$

Other versions of Heegaard Floer homology, HF^∞ and HF^+, are defined by considering the free $\mathbb{Z}/2[U, U^{-1}]$ or $\mathbb{Z}/2[U]$ modules, respectively, generated by points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and counting also the pseudoholomorphic curves that cross the marked point z in the differential, by recording $n_z(\phi)$:

$$\partial \mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#(\widehat{\mathcal{M}}_\phi) U^{n_z(\phi)} \mathbf{y}$$

Defining orientations on the moduli spaces $\widehat{\mathcal{M}}_\phi$ makes it possible to count the number of points in the 0-dimensional moduli spaces $\widehat{\mathcal{M}}_\phi/\mathbb{R}$ with sign in order to work over $\mathbb{Z}$.

It is highly nontrivial to count the number of points in the moduli spaces $\widehat{\mathcal{M}}_\phi$ for a general ϕ, as that count depends on the choice of almost complex structure on $Sym^g(\Sigma)$. However, if the image of $\tilde{\phi}$ is a topological disc connecting two intersection points $\mathbf{x} = (x_1, x_2, x_3 \ldots x_g)$ and $\mathbf{y} = (y_1, x_2, x_3 \ldots x_g)$ that differ in just one coordinate, called a bigon domain (see Figure 2), or between two intersection points $\mathbf{x} = (x_1, x_2, x_3 \ldots x_g)$ and $\mathbf{y} = (y_1, y_2, x_3 \ldots x_g)$ that differ in exactly two coordinates, called a rectangular domain (and constant maps on other components of the cover), there is a unique holomorphic disc in that homotopy class.

This makes it possible to have a combinatorial calculation of the chain complex and the homology. Sarkar and Wang [30] described a method to
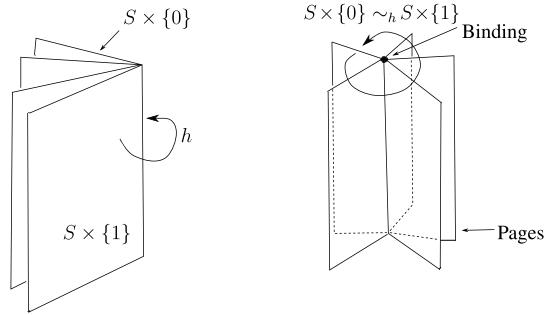
Fig. 3. A neighborhood of the binding in an open book decomposition

produce "nice Heegaard diagrams", i.e. diagrams for which all but one distinguished domain are bigons and rectangles and when the base point z is placed in that distinguished domain, the count involved in calculating $\widehat{HF}(M)$ becomes combinatorial. There are other combinatorial approaches to calculating the Heegaard Floer homology and a variant associated to a knot $K \subset M$, the knot Floer homology $\widetilde{HFK}(M, K)$, notably the gird diagrams and convenient diagrams ([22, 27] and many other sources).

2. Open Book Decompositions, Contact Structures and Convex Surfaces

Let S be a surface with boundary and let $Aut(S, \partial S) = \{h \in \mathrm{Diff}(S, \partial S) | h|_{\partial S} = id\}$. Moding $S \times [0, 1]$ out by the equivalence relation $(x, 1) \sim_h (h(x), 0)$ produces a manifold $M(S, h)$ with a torus boundary component for each component of ∂S. Moding out the boundary by further identifying $(x, t) \sim_h (x, t')$ for all $t, t' \in [0, 1]$ and $x \in \partial S$ reduces each torus boundary component to a knot in a closed 3 manifold $M_{(S,h)} = S \times [0, 1]/\sim_h$ (Figure 3). Denote by $B_{(S,h)} \subset M_{(S,h)}$ the image of $\partial S \times [0, 1]$ under the quotient projection. The complement of a small neighborhood of $B_{(S,h)}$ in $M_{(S,h)}$ is diffeomorphic to $M(S, h)$ and fibers over $S^1 = [0, 1]/0 \sim 1$. The link $B_{(S,h)}$ is called the *binding* of the open book and the image of each $S \times \{t\}$ under the quotient projection is a *page*. The projection $\pi : \mathrm{Int}(S) \times [0, 1]/\sim_h \to [0, 1]/_{0 \sim 1}$ induces a bundle structure on the complement of the binding with the preimage of a point in $t \in S^1 = [0, 1]/_{0 \sim 1}$ being the interior of the page S_t.

An open book decomposition for a 3-manifold M with page S, binding B and monodromy $h \in Aut(S, \partial S)$ is a projection $p : M \setminus B \to S^1$ and a homeomorphism $\Phi : (M, B) \to (M_{(S,h)}, B_{(S,h)})$ that commutes with projections p

and π. We say in this case that (S, h) is a *formal open book* decomposition for (M, B, p).

It is a theorem of Alexander that every 3-manifold has an open book decomposition. Different open book decompositions of the same manifold are related by sequences of positive/negative stabilizations, where stabilization changes S by adding a 1-handle to S and changes h by composing it with a positive/negative Dehn twist about a curve γ dual to the handle. More precisely, we attach a handle $H = [0, 1] \times [-\epsilon, \epsilon]$ to S along $\{0, 1\} \times [-\epsilon, \epsilon]$ to obtain S' and let $h \cup id$ be extension of h by identity on $[0, 1] \times [-\epsilon, \epsilon]$. Let γ be any simple closed curve in S' that intersects the co-core $1/2 \times [-\epsilon, \epsilon]$ of the handle H once, and let R_γ (R_γ^{-1}) be the right (left) handed Dehn twist. If $h' = R_\gamma^{\pm 1} \circ (h \cup id)$ we say that (S', h') is a positive (negative) stabilization of (S, h). It is not hard to see that if (S', h') is a positive (negative) stabilization of (S, h), and (S, h) is a formal open book decomposition of (M, B, p), then (S', h') gives a formal open book decomposition of (M, B', p') where the binding B' is obtained by a Murasugi sum along γ of the binding B with a positive (negative) Hopf link. For a nice description see [29].

Open book decompositions of 3-manifolds are intimately related to contact topology. A contact structure ξ on a 3-manifold M is a 2-plane filed which is the kernel of a nondegenerate 1-form α, the *contact form*. (Nondegeneracy means that $\alpha \wedge d\alpha = d\,vol$.) The contact structure ξ is said to be *supported by the open book decomposition* (M, B, p) if:

1. B is transverse to ξ,

2. there is a contact 1-form α for which $d\alpha$ induces a symplectic form on each fiber $S_t = p^{-1}(t)$ of $p : M \setminus B \to S^1$,

3. the orientation on B given by α is the same as the boundary orientation induced from S_t oriented by the symplectic structure induced by $d\alpha$.

Thurston and Winkelnkemper [32] showed that any open book decomposition (S, h) of M supports a contact structure ξ by constructing a contact form $\xi_{(S,h)}$ on $M_{(S,h)}$ with these properties. Torisu [33] and Giroux [10] proved that the converse is true, namely that every contact manifold is supported by (has a compatible) open book decomposition. Giroux established the following correspondence:

Theorem 2.1 (Giroux)**.** *Any contact structure (M, ξ) on a closed 3-manifold M is supported by an open book decomposition (S, h). Moreover, any two open book decompositions (S, h) and (S', h') which support the same*
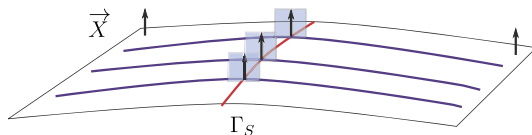
Fig. 4. Convex surface and its characteristic foliation and dividing set

contact structure (M, ξ) are equivalent under a sequence of positive stabilizations/destabilizations.

To give an indication of how a compatible open book decomposition can be found for a given a contact manifold (M, ξ), we need to first talk about convex decomposition methods in contact topology. In the early 90's Emanuel Giroux introduced an important tool in contact topology—convex surfaces. A vector field $\overrightarrow{v}$ in a contact manifold (M, ξ) is called *contact* if its flow preserves ξ. A surface $S \subset M$ is called *convex* if there exists a contact vector field $\overrightarrow{v}$ transverse to S. Note that a convex surface S can be transverse to many different contact vector fields. To a convex surface S and a contact vector field $\overrightarrow{v}$ transverse to S one can associate

$$\Gamma_S = \left\{ x \in S \,|\, \overrightarrow{v}(x) \in \xi(x) \right\}$$

its *dividing set* (see Figure 4). It is not hard to show that Γ_S is a smooth multicurve, and that the isotopy class of Γ_S is independent of $\overrightarrow{v}$.

If $g(\,,)$ is a Riemannian metric on M and we denote by $\overrightarrow{n}_\xi$ a normal vector field to ξ, the dividing set on a convex surface $S \subset M$ determined by a contact vector field $\overrightarrow{v}(p)$ can be described as $\Gamma_S = \{p \in S \,|\, g(\overrightarrow{v}(p), \overrightarrow{n}_\xi(p)) = 0\}$. The dividing set Γ clearly divides the convex surface S into two regions $R_+(S)$ and $R_-(S)$ where $R_+ = \{p \in S \,|\, g(\overrightarrow{v}(p), \overrightarrow{n}_\xi(p)) \geq 0\}$ and $R_- = \{p \in S \,|\, g(\overrightarrow{v}(p), \overrightarrow{n}_\xi(p)) \leq 0\}$. It is not hard to see that, if S is a closed convex surface in the contact manifold (M, ξ), the contact class of ξ evaluates on S as $\chi(\xi)[S] = \chi(R_+(S)) - \chi(R_-(S))$.

A knot is *Legendrian* if it is everywhere tangent to contact planes. Contact planes (cooriented by $\overrightarrow{n}_\xi$) induce, on a surface S oriented by $\overrightarrow{v}$, an oriented singular foliation $\mathcal{F}$ by Legendrian curves called the *characteristic foliation*. The regions $R_\pm$ correspond to "source" and "sink" regions in this picture. Legendrian curves of the foliation intersect the dividing set transversally as in Figure 4. If a convex surface S has Legendrian boundary, then the dividing set intersects the boundary in an even number of points $2n$, and n describes the number of twists that the contact plane makes relative to the framing for the Legendrian boundary curve coming from the convex surface.
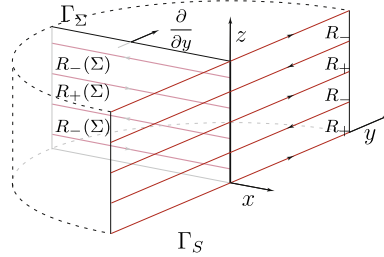
Fig. 5. Standard neighborhood of a Legendrian curve at the intersection of two perpendicular convex surfaces
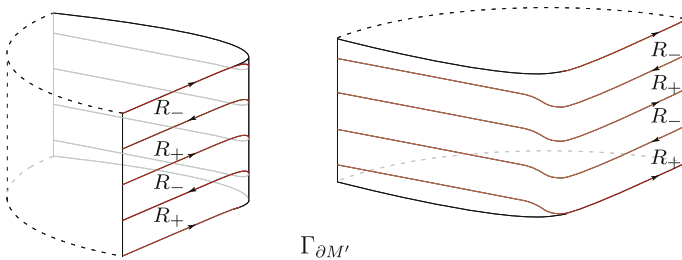

Fig. 6. Smoothing of the dividing set after cutting along convex surface Σ and rounding corners

We can decompose a contact manifold (M, ξ) by cutting it along properly embedded convex surfaces Σ. For a gentler introduction see [12]. When (M, ξ) is a contact manifold with boundary we assume that the boundary ∂M is convex, and that the convex cutting surface Σ has Legendrian boundary $L = \partial \Sigma \subset S$. A standard neighborhood theorem says that if L intersects the dividing set Γ_Σ geometrically $2n$ times, there is a neighborhood $N_\varepsilon(L)$ of L in M and local coordinates (x, y, z) on it so that $N_\varepsilon(L) = \{(x, y, z)|x^2 + y^2 < \varepsilon, x \leq 0\}$ in $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ (see Figure 5).

Here L lies on the z axis, the set of points with $x = 0$ corresponds to an annular neighborhood of L in Σ, the set of points with $y = 0$ corresponds to a neighborhood of L in S, and the contact structure ξ is given as the kernel of $\alpha = \sin(2\pi n z)dx + \cos(2\pi n z)dy$, $n \neq 0$. If we choose the contact vector fields for S and Σ to be $v_S = \frac{\partial}{\partial x}$ and $v_\Sigma = \frac{\partial}{\partial y}$ it is not hard to calculate that the dividing sets are $\Gamma_S = \{(0, y, \frac{k}{2n})|0 \leq k < 2n\}$ and $\Gamma_\Sigma = \{(x, 0, \frac{1+2k}{4n})|0 \leq k < 2n\}$.

After cutting M along Σ and rounding corners to obtain M', transverse contact vector fields on the new boundary components of $\partial M' = S'$ can be chosen to be $v_S = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ so that they rotate between $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. A simple
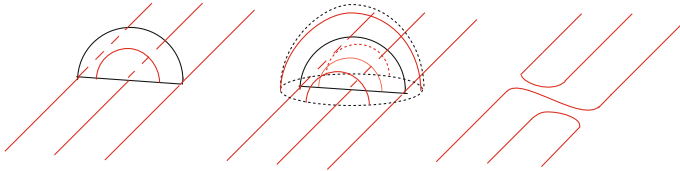
Fig. 7. Adding a bypass

calculation shows that the new dividing set on $\partial M'$ is obtained by the "turn to the right" method illustrated in Figure 6.

Giroux and Honda showed that isotoping a convex surface S through a contact manifold preserves the isotopy class of the dividing set except for at finitely many levels of the isotopy. Honda described how at these levels the product contact structure changes by adding a "bypass" to the product structure on $S \times I$, i.e. engulfing a "half overtwisted disc" B bounded by a Legendrian arc α in S connecting three dividing curves (possibly not globally different), and an Legendrian arc β along which the contact planes coincide with the tangent planes of B. Thickening the bypass disc to $B \times [-\varepsilon, \varepsilon]$ attached along $\alpha \times [-\varepsilon, \varepsilon]$, looking at the dividing set and rounding to obtain S' we see that the change in the dividing set after adding a bypass is as pictured in Figure 7.

To describe briefly how to find an open book compatible with a given contact structure ξ on M we start with a triangulation of M that is fine enough so that each 3-simplex is contained in a standard contact chart for ξ. The 1-skeleton can then be perturbed to be Legendrian in such a way that the relative twisting of the contact planes along the boundaries of the discs in the 2-skeleton is such that these discs can be made convex with the given Legendrian boundary. By adding extra pieces to the 1-skeleton to divide the discs in the 2-skeleton in such a way that each contains exactly one arc in its dividing set we can achieve the following:

(1) There is a handlebody decomposition of the 3-manifold into $H_1 = \nu(M^{(1)})$, the neighborhood of the 1-skeleton, and the complementary handlebody $H_2 = \nu(\bar{M}^{(1)})$ which is the nighbourhood of the dual 1-skeleton.

(2) The common boundary Σ is a convex surface.

(3) The contact handlebodies H_1 and H_2 with convex boundary are *disc decomposable*, i.e. there is a family of compressing discs with Legendrian boundary such that the boundary of each disc intersects the dividing set in exactly two points.
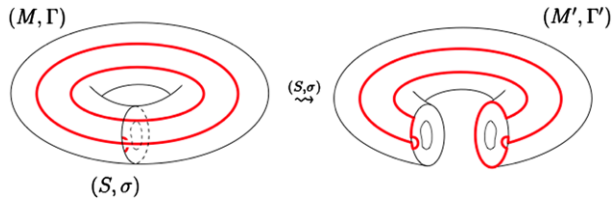
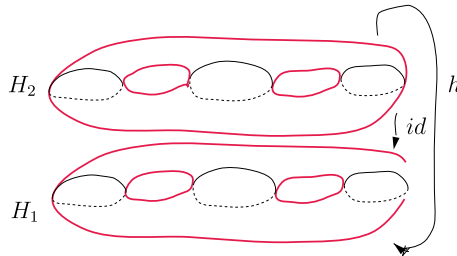Fig. 8. Cutting a genus one handlebody along a convex compressing disc

Fig. 9. Gluing disc decomposable contact structures on H_1 and H_2

Note that the convex discs in (3) are just small discs transverse to the 1-cells of the 1-skeleton in H_1 and the 2-cells of the subdivided complex for H_2. By cutting along the compressing discs until we obtain B^3, we can see that contact handlebodies H_i are contactomorphic to product contact manifolds $R \times I$ where R is homeomorphic to Σ_+ (and Σ_-), as we think of obtaining H_i by gluing contact 1-handles onto a standard ball with convex boundary S^2 and dividing set S^1. The simplest case, of a genus one surface with two parallel longitudinal dividing curves, is shown in Figure 8.

Note that when we consider Σ as the boundary of H_2 the roles of Σ_+ and Σ_- are reversed in comparison to what they are when we consider Σ as the boundary of H_1 (due to the change of orientation on Σ). We identify H_1 with $R \times [0, 1/2]$ in such a way that Σ_- corresponds to $R \times \{0\}$ and Σ_+ corresponds to $R \times \{1/2\}$, and identify H_2 with $R \times [1/2, 1]$ in such a way that Σ_- corresponds to $R \times \{1/2\}$ and Σ_+ to $R \times \{1\}$. Then the identification of $R \times \{0\}$ and $R \times \{1\}$ coming form the way H_1 and H_2 are glued inside M defines the monodromy map $h : R \to R$ which realizes M as corresponding to the open book (R, h), and the open book inside M is compatible with ξ (see Figure 9).

We say that a contact structure ξ on a 3-manifold M is *overtwisted* if there is an embedded D^2 in M such that the tangent plane $T_P D^2$ and the contact plane ξ_P agree at all points $P \in \partial D^2$. If ξ is not overtwisted we say it is *tight*. There are familiar examples of tight contact structures. It is a

theorem of Bennequin [4] that the standard contact structure on $\mathbb{R}^3$ is tight. Giroux showed that a product neighborhood of a convex surface is tight if and only if it has no contractible dividing curve, or is S^2 with $\Gamma = S^1$.

A theorem of Eliashberg [5] says that overtwisted contact structures are classified up to isotopy through contactomorphisms (contact isotopy) by the homotopy class of their plane fields, i.e. two overtwisted contact structures that are homotopic as plane fields are isotopic through a family of contact structures. This is not true in the case of tight contact structures, which are more closely related to the finer topology of the manifold. A central question in contact topology on 3-manifolds is to construct, recognize and classify tight contact structures on a given M^3.

There is a rich source of tight examples; the fillable contact structures. A contact manifold (M, ξ) is *Stein fillable* if there is a compact complex Stein manifold W with convex boundary such that $M = \partial W$ and contact planes are complex lines in TM. (M, ξ) is *strongly symplectically fillable* if M is the boundary of a symplectic manifold (W, ω) which looks like a Stein manifold near the boundary. Finally, (M, ξ) is *weakly symplectically fillable* if M is boundary of a symplectic manifold (W, ω) and $\omega|_\xi \geq 0$. It is a theorem of Gromov and Eliashberg that fillable contact structures are tight [7, 11].

To describe tightness in the framework of open book decompositions we use the notion of "right-veering" homeomorphisms [13]. We say that a homeomorphism $h \in Aut(S, \partial S)$ is right-veering if every properly embedded oriented arc α in S is mapped "to the right" of α as in Figure 10. For two properly embedded arcs α and β with the same initial point $\alpha(0) = \beta(0)$ which are isotoped rel boundary to intersect transversally in a minimal number of points, we say that β is to the right of α if the tangent vectors $\{\beta'(0), \alpha'(0)\}$ define the orientation of S.
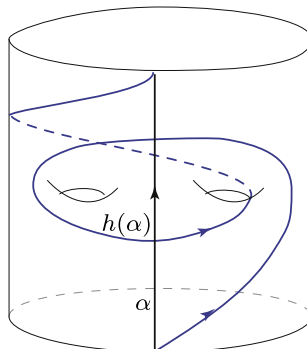

Fig. 10. The image arc $h(\alpha)$ is to the right of α

Theorem 2.2 (Honda-Kazez-Matić). *A contact 3-manifold (M,ξ) is tight if and only if all of its adapted open book decompositions have right-veering monodromy.*

Right-veering diffeomorphisms of S form a monoid $Veer(S,\partial S)$ in $Aut(S,\partial S)$. Monodromies in the submonoid $Dehn^+(S,\partial S) \subset Veer(S,\partial S)$ consisting of diffeomorphisms that are products of positive Dehn twists give rise to Stein fillable structures by the work of Eliashberg [6]. That every Stein fillable structure has a monodromy in $Dehn^+(S,\partial S)$ is a theorem by Loi-Piergallini and Akbulut-Ozbagci [1, 21]. John Baldwin [3] and Baker, Etnyre and Van-Horn Morris [2] have shown that there are open books for Stein fillable contact structures that are not in $Dehn^+(S,\partial S)$. They have also shown that the open book monodromies corresponding to Stein, strongly and weekly fillable contact structures form monoids. The natural inclusions $Dehn^+(S,\partial S) \subsetneq Stein(S,\partial S) \subsetneq Strong(S,\partial S) \subsetneq Week(S,\partial S) \subsetneq Tight(S,\partial S) \subsetneq Veer(S,\partial S)$ are all proper according to work of, in order, Baker-Etnyre-VanHorn Morris and independently Wand, Ghiggini, Eliashberg, Ghiggini and Honda-Kazez-Matić and Goodman. However, it is not clear that $Tight(S,\partial S)$ is a monoid, or how to describe the tightness in terms of one open book. It is easy to show that any open book can be stabilized to be right veering, hence there are right veering open books supporting overtwisted contact structures. Even if we know that an open book has right veering monodromy and that it is not the result of a stabilization, this does not guarantee tightness, as first shown by Lekili [19].

3. From Open Books to Contact Invariants

Ozsváth and Szabó [26] used the one-to-one correspondence between equivalence classes of open books for M under positive stabilization and isotopy classes of contact structures on M to define an invariant $c(\xi)$ of contact structures that lives in the Heegaard Floer homology of $-M$. To do this they associated to an open book decomposition (M, B, p) compatible with ξ the fibered manifold $M_0(B)$ obtained by performing 0-framed surgery with respect to the page framing on the binding B. Heegaard Floer homology for a fibered 3-manifold is special—it is one dimensional in the spin^c structure corresponding to the fibration. Ozsváth and Szabó defined $c(\xi)$ to be the image of the generator of this group under the map induced by the cobordism defined by the 0-handle attachment from $\widehat{HF}(-M_0(B))$ to $\widehat{HF}(-M)$. It has the property that $c(\xi) = 0$ if ξ is overtwisted, and that $c(\xi) \neq 0$ when ξ is fillable.
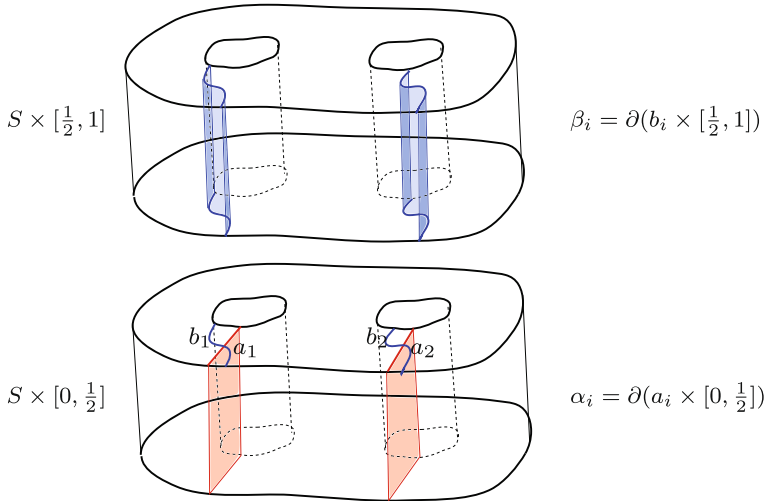
Fig. 11. The Heegaard decomposition and the diagram determined by a basis of arcs for the page of an open book decomposition

To see a concrete generator of $c(\xi)$ we will use the open book decomposition to construct a Heegaard diagram in which this generator sits in a distinguished way. An open book decomposition adapted to ξ gives rise to a Heegaard decomposition into two handlebodies $H_1 = S \times [0, 1/2]/_{\sim_h}$ and $H_2 = S \times [1/2, 1]/_{\sim_h}$.

A *basis of arcs* for the surface with boundary S is a collection of properly embedded arcs $\{a_i | i = 1, \ldots, r\}$ that cuts the surface into a disc, $S \setminus (\bigcup a_i) = D^2$. Given a basis of arcs we can construct the family of compressing discs $D_{a_i} = a_i \times [0, 1/2]$ for H_1 which cut H_1 into a ball. We perturb the a_i slightly to obtain a basis of arcs b_i such that each b_i intersects a_i transversely at exactly one point (positively) and consider discs $D_{b_i} = b_i \times [1/2, 1]$ which are a set of compressing discs that cut H_2 down to a ball.

Taking the boundary curves of these discs $\alpha_i = \partial(D_{a_i})$ and $\beta_i = \partial(D_{b_i})$ and looking at them as curves in the separating surface $\Sigma = S \times \{1/2\} \cup -S \times \{0\} = S \times \{1/2\} \cup -S \times \{1\}$ (here Σ is the common boundary of H_1 and H_2, and $S \times \{1\}$ is identified with $S \times \{0\}$ via the monodromy h), we obtain a Heegaard diagram $(\Sigma, \alpha_i, \beta_i, z)$ (see Figure 11). There is a distinguished generator $\mathbf{x} = (x_1, \ldots, x_i, \ldots)$ for $\widehat{CF}(\Sigma, \alpha_i, \beta_i, z)$ that is contained in $\Sigma = S \times \{1/2\}$ where $x_i = \alpha_i \cap \beta_i$. If we choose the marked point z to lie in $\Sigma = S \times \{1/2\}$ and outside the thin strips bounded by α_i and β_i curves, it is easy to see that $\mathbf{x}$ is a cycle, and in fact:
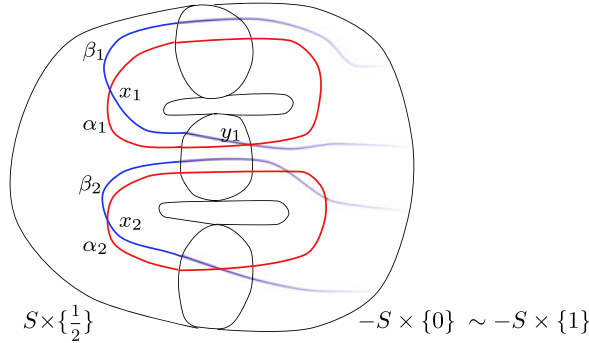
Fig. 12. The Heegaard diagram determined by a basis of arcs and the contact element
$$\mathbf{x} = (x_1, x_2)$$

Theorem 3.1 [14]. *The generator* $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ *is a cycle in* $\widehat{CF}(\Sigma, \beta_i, \alpha_1, z)$ *and its homology class is the Ozsváth-Szabó contact class,* $\mathbf{x} = c(\xi) \in \widehat{HF}(-M)$.

In particular, for a different open book decomposition compatible with the same contact structure ξ, we get the same class $c(\xi) \in \widehat{HF}(-M)$. Note that we needed to switch the role of α and β curves, hence the orientation of M in order for $\mathbf{x}$ to be a cycle.

From this description it is easy to see that for an overtwisted contact structure $c(\xi) = 0$. Specifically, an overtwisted contact structure has a compatible open book decomposition and a basis of arcs such that the holonomy takes at least one of the arcs, we can name it α_1, to the left of itself. Then in the corresponding Heegaard diagram, there is a bigon connecting $\mathbf{y} = (y_1, x_2, \ldots, x_r)$ to $\mathbf{x} = (x_1, x_2, \ldots, x_r)$, which gives $\partial \mathbf{y} = \mathbf{x}$, hence $\mathbf{x} = 0$ in homology. When looking at the Figure 12, it might seem that α_1 is in fact mapped to the right, but that is an artifact of reversal of orientation on $-S \times \{0\}$.

Given that $c(\xi) = 0$ for overtwisted contact structure, and that $c(\xi) \neq 0$ when ξ is fillable (by a theorem of Gromov and Eliashberg [7, 11]), it was a natural question to ask whether $c(\xi) \neq 0$ is a characterization of tightness. Ghiggini [8] showed that there are tight contact manifolds with $c(\xi) = 0$. The examples he found contain Giroux torsion, i.e. a contact embedding of $T^2 \times [0, 1] = \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1]$ with contact structure given by $\xi_{tor} = ker(\cos(2\pi z)dx - \sin(2\pi z)dy)$.

The question then was reformulated to ask if $c(\xi) = 0$ is equivalent to the requirement that ξ contains Giroux torsion. The answer was obtained with

the use of a contact invariant defined for contact manifolds with boundary, the sutured contact invariant, which we will discuss in the next section. On one hand, the invariant is used in [9] to prove that any contact manifold containing Giroux torsion has $c(\xi) = 0$. On the other hand, it is used in [16] to show that for a general surface with boundary S there are S^1-invariant contact structures on $S \times S^1$ generalizing Giroux torsion that have the property that any contact manifold that contains them has $c(\xi) = 0$, and is hence not fillable.

4. Sutured Manifolds and Partial Open Books

When a 3-manifold M has nonempty boundary, we will study contact structures on M for which the boundary ∂M is a convex surface with a prescribed dividing set $\Gamma \subset \partial M$ dividing ∂M into R_+ and R_- regions. A pair (M, Γ) of a manifold and a dividing set Γ on the boundary (which divides every boundary component) is called a *sutured manifold*. Sutured manifolds were first defined by Gabai for use in the study of foliations. When $\chi(R_+) = \chi(R_-)$ we call the sutured manifold (M, Γ) balanced (Figure 13).

We say that the balanced sutured manifold (M, Γ) carries a compatible contact structure ξ with convex boundary if the suture Γ agrees with the dividing set Γ_ξ on the boundary. Recall that on a closed convex surface Σ in a contact manifold (M, ξ) the contact class evaluates as $c(\xi)[\Sigma] = \chi(R_+(\Sigma)) - \chi(R_-(\Sigma))$. Since $\partial M = \Sigma$ is zero in homology, $c(\xi)[\Sigma] = 0$ and $\chi(R_+(\Sigma)) = \chi(R_-(\Sigma))$. Hence a sutured manifold that supports a contact structure with convex boundary is balanced.

We want to define an analogue of a Heegaard diagram in the case of a manifold with boundary, and sutured manifolds provide the right framework. A *sutured Heegaard diagram* consists of a surface with boundary Σ of genus g and two families of attaching curves $\{\alpha_i | i = 1, \ldots, k\}$ and $\{\beta_i | i = 1, \ldots, l\}$ with $k, l \leq g$. When we attach 2-handles to $\Sigma \times [0, 1]$ along those curves
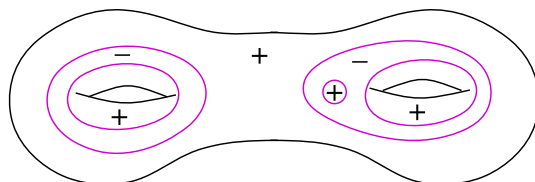

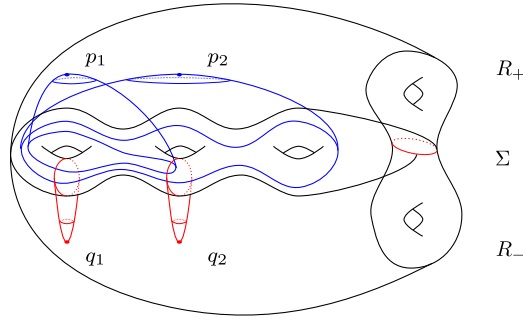*Fig. 13.* A balanced sutured manifold

Fig. 14. A Morse function picture of a balanced sutured manifold

we will build a 3-manifold with boundary. It will have a suture $\Gamma = \partial\Sigma \times \{\frac{1}{2}\}$ which divides the boundary into two regions: R_+ which is obtained by compressing $\Sigma \times \{1\}$ along α_i curves (cutting open along α and filling in by attaching pairs of discs), and R_- obtained by compressing $\Sigma \times \{0\}$ along β_i curves. On the 3-manifold with boundary we obtain this way there is clearly a Morse function picture, analogous to the closed case, that has the centers of the attached 2-handles as critical points (see Figure 14). Note that in the case of a closed manifold and a classical Heegaard diagram of genus g and α and β curves cutting it to a $2g$ times punctured sphere, adding 2-handles to the Heegaard surface along the α and β curves produces first a manifold with two S^2 boundary components to which, in the end, we add two 3-handles. A way to build a closed manifold from the sutured manifold is to attach enough handles until we obtain a union of sutured spheres on the boundary, one for each of the boundary components of the Heegaard surface, and then add 3-balls. It is easy to see that in the sutured Heegaard diagram where $k = l$, i.e. when we attach the same number of compressing discs to $\Sigma \times \{1\}$ as to $\Sigma \times \{0\}$, the sutured manifold we build is balanced.

Andras Juhasz [17] defined Sutured Floer Homology for a balanced sutured manifold in analogy to the Heegaard Floer Homology $\widehat{HF}$. As is done in the case of a closed manifold, he associated to a balanced Heegaard diagram $(\Sigma, \alpha_i, \beta_i)$ a chain complex generated by the intersection points of the two tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_k$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_k$ in $Sym^k(\Sigma)$ and defined the boundary operator by counting holomorphic disc. The role played by the base-point z in the closed case is played by the boundary $\partial\Sigma$, namely we consider only domains that do not go out to the boundary. The homology of this complex is denoted by $SFH(M, \Gamma)$ and Juhasz proved it does not depend on the choice of the sutured Heegaard diagram for (M, Γ).
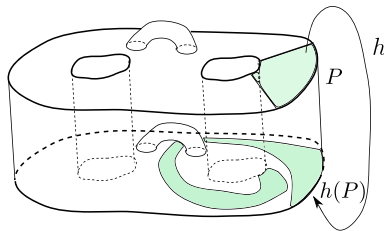
Fig. 15. Gluing to obtain a partial open book

We want to associate to a contact structure ξ with convex boundary supported by (M, Γ) an element in the Sutured Heegaard Floer homology $SFH(-M, -\Gamma)$. The change of orientation is parallel to the fact that contact invariant in the closed case lives in $\widehat{HF}(-M)$. In order to define $c(\xi) \in SFH(-M, -\Gamma)$, we first need to define an analogue of open book decompositions for manifolds with sutured boundary.

Definition 4.1. A *partial open book* (S, P, h) consists of the following data: a compact, oriented surface S with nonempty boundary, and a "partial" monodromy map $h : P \to S$ defined on a subset $P \subset S$ such that $\partial P \cap \partial S \neq 0$ and $h|_{\partial P \cap \partial S} = id$.

To obtain a sutured manifold associated to a partial open book decomposition (S, P, h) define an equivalence relation $\sim_h$ on $S \times [0, 1]$ by setting $(x, 1) \sim_h (h(x), 0)$ for $x \in P$, and $(x, t) \sim_h (x, t')$ for $t, t' \in [0, 1]$ $x \in \partial S$ (see Figure 15). It is not difficult to see that the glued-up space can be smoothed out to a manifold with boundary, where $R_+ = (S \setminus P) \times \{1\}$ and $R_- = (S \setminus h(P)) \times \{0\}$ and the dividing set is $\Gamma = \partial(S \setminus int\, P)$.

To motivate this definition let us think about the construction of the open book compatible with a contact structure ξ in the closed case that we described in Section 2. We will adapt this construction to the case of balanced sutured (M, Γ). We again take a cell decomposition of M by cells small enough so that the contact structure on them is standard, make the 1-skeleton Legendrian and 2-cells convex (keeping the boundary fixed), and this time consider a Legendrian 1-complex L that consists of the portion of the Legendrian 1-skeleton that is in the interior of the manifold together with enough of the Legendrian 1-simplices that come out and meet the boundary in points on the dividing set to meet every component of it in at least 2 points. If we take the cell decomposition to be fine enough, we can ensure that the neighborhood of L is a disc-decomposable contact handlebody H_2 with
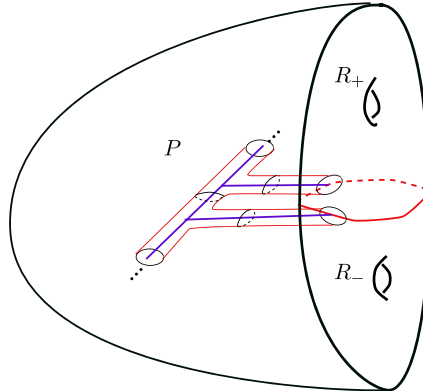
Fig. 16. Neighborhood of the Legendrian skeleton L near the boundary

convex boundary, and that its complement H_1 is also a disc-decomposable handlebody [15].

The boundary of $H_2 = \nu(L)$ consists of a "tube region" P and some discs that are part of the boundary and intersect the dividing set Γ in one segment each (see Figure 16). The tube P is divided by its dividing set into two regions $P_\pm$ and the boundary of the complement H_1 consists of $S_+ = R_+ \cup P_+$ and $S_- = R_- \cup P_-$ (we are being a bit sloppy and identifying $R_\pm$ with $R_\pm \setminus \partial(\nu(L))$. The choice of a fine enough decomposition ensures that the two contact handlebodies H_1 and H_2 are disc-decomposable and we can identify $H_1 = S \times [0, 1/2]$ and $H_2 = P \times [1/2, 1]$, where $S = R_+ \cup P_+ = R_- \cup P_-$ (see Figure 17). For simplicity of notation we ignore in this picture the fact that in the product handlebodies we mod out by $(x, t) \sim (x, t')$ to get to the real disc decomposable handlebody picture. By looking at the gluing of H_2 to H_1 inside M we see that we can think of M as obtained first by gluing along P_- to obtain the glued up $S \times [0, 1/2] \cup_{P \times \{1/2\}} P \times [1/2, 1]$ as homeomorphic to $S \times [0, 1]$, and of the final gluing along P_+ as gluing by the partial monodromy (after we identify $S \times [0, 1/2] \cup_{P \times \{1/2\}} P \times [1/2, 1]$ with $S \times [0, 1]$ in an obvious way).

We now want to associate a sutured Heegaard diagram to this decomposition. Define a *basis of arcs* in P to be a collection $\{a_i | i = 1, \ldots, k\}$ of disjoint properly embedded arcs in P with boundary on $\partial P \cap \partial S$ such that $S \setminus \bigcup \{a_i\}_{i=1,\ldots,k}$ deformation retracts onto $R_+ = S - P$. In our example in Figures 17 and 18, the basis consists of just one arc a_1. Let b_i, $i = 1, \ldots, k$, be pushoffs of a_i in the direction of ∂S so that a_i and b_i intersect exactly once at a point x_i. It is not hard to see that if we set $\Sigma = (S \times \{0\}) \cup (P \times \{\frac{1}{2}\})$,

$$\alpha_i = \partial(a_i \times [0, 1/2])$$
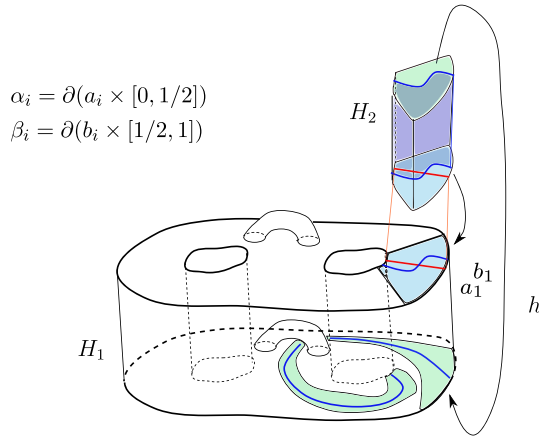$$\beta_i = \partial(b_i \times [1/2, 1])$$

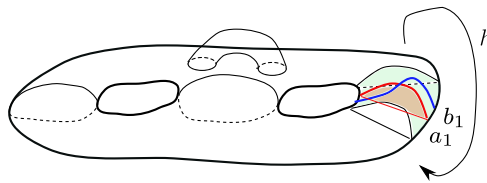Fig. 17. The partial open book and the handlebody decomposition

Fig. 18. The partial open book and the Heegaard diagram

$\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$ and $\beta_i = (b_i \times \{\frac{1}{2}\}) \cup (h(b_i) \times \{0\})$, then (Σ, β, α) is a Heegaard diagram for $(-M, -\Gamma)$. See Figure 19.

We can again look at the special generator $\mathbf{x} = (x_1, \ldots, x_i, \ldots)$ for $SFH(\Sigma, \beta, \alpha)$ and, as in the case of the closed manifold, $\mathbf{x} = [(x_1, \ldots, x_k)]$ is a cycle. It is shown in [15] that it defines a contact invariant.
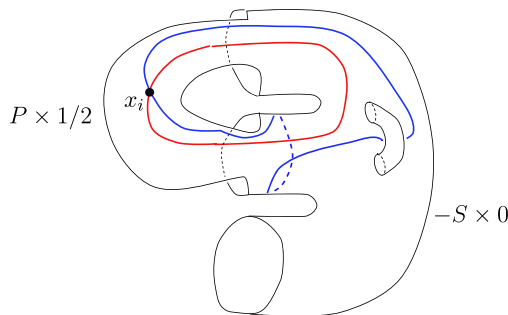

Fig. 19. The Heegaard diagram determined by a basis of arcs a_1 and the contact element
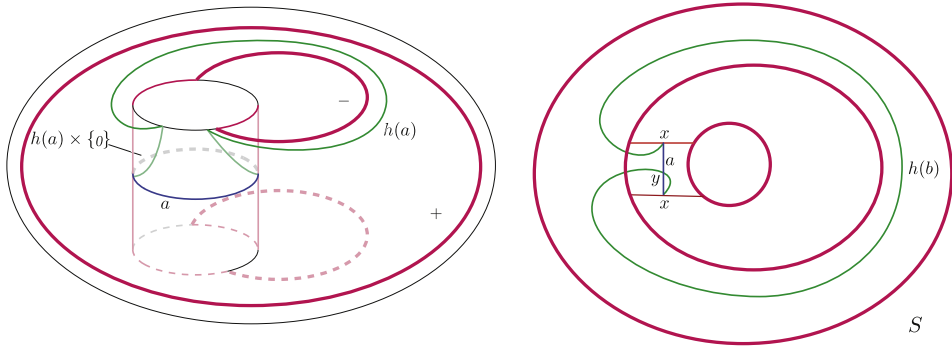$$\mathbf{x} = (x_1)$$

Fig. 20. A partial open book decomposition and a Heegaard diagram for a neighborhood of an overtwisted disc

Theorem 4.2. *The point* $\mathbf{x} = [(x_1, \dots, x_k)]$ *is independent of choices up to* ± 1 *and generates the sutured contact invariant* $c(\xi) \in SFH(-M, -\Gamma)$.

A concrete example we will look at here is a partial open book decomposition for a neighborhood of an overtwisted disc. The corresponding sutured manifold is (B^3, Γ) with the dividing set consisting of 3 parallel curves. It is enough to take just one segment connecting two nonadjacent components of the dividing set for the Legendrian 1-complex L, to get the complement to be a disc-decomposable handlebody. The segment comprising L is the core of the cylinder in Figure 20, while $P_\pm$ are two halves of the cylinder, and the basis of arcs consists of a single arc a on P. In the left-hand diagram of Figure 20 arc a is shown on $P \times \{1/2\} \subset S \times \{1/2\}$ (thus might more properly be denoted by $a \times \{1/2\}$). Isotoping a (rel endpoints) through $N(L)$ where we use the homeomorphism $N(L) = P \times [1/2, 1]$ produces, by the definition of the monodromy h, the arc $h(a) \times \{0\} \subset S \times \{0\}$. Finally pushing $h(a) \times \{0\}$ (rel endpoints) through the fibration $M - N(L)$ that identifies $M - N(L) = S \times [0, 1/2]$ results in $h(a) \times \{1/2\} \subset S \times \{1/2\}$, and this is denoted simply by $h(a)$. The right side of the figure shows a and $h(a)$ in $S = S \times \{1/2\}$.

5. Gluing Theorem for Sutured Manifolds

In [13] we work to understand the effect of cutting and gluing of contact manifolds along convex surfaces in sutured manifolds in the context of the contact invariant. We say that one balanced sutured manifold (M', Γ') is a *sutured submanifold* of another balanced sutured manifold (M, Γ) if M' is a

submanifold with boundary of M and $M' \subset int(M)$. A contact structure ξ defined on $M \subset int(M')$ is *compatible* with the sutured manifold structures of M and M' if the dividing set of ξ on the boundary of $M - int(M')$ is $\Gamma \cup -\Gamma'$. In this section we will define a map on sutured Floer homology induced by the inclusion of (M', Γ') into (M, Γ) in the presence of a compatible contact structure in the complement. We will see how triviality of the contact invariant on a sutured submanifold implies the triviality of the contact invariant of the manifold itself. In the next section we will use the gluing theorem to calculate the contact invariant in some examples, and obtain some interesting obstructions to fillability. For simplicity, we can think of all constructions as done over $\mathbb{Z}/2\mathbb{Z}$, so we do not have to worry about the sign ambiguity.

If a connected component N of $M \setminus int(M')$ contains no components of ∂M we say that N is *isolated*. When $M \setminus int(M')$ has no isolated components we have the following:

Theorem 5.1 [13]. *Let (M', Γ') be a sutured submanifold of (M, Γ), and let ξ be a compatible contact structure on $M \setminus int(M')$. Assume that $M - int(M')$ has no isolated components. Then ξ induces a natural map:*

$$\Phi_\xi : SFH\big(-M', -\Gamma'\big) \to SFH(-M, -\Gamma)$$

Moreover, if ξ' is any contact structure on M' compatible with Γ' then

$$\Phi_\xi\big(c\big(M', \Gamma', \xi'\big)\big) = c\big(M, \Gamma, \xi' \cup \xi\big)$$

where $\xi' \cup \xi$ is a contact structure on M that restricts to ξ on $M \setminus int(M')$ and to ξ' on M'.

There is a more complicated statement in the case of existence of isolated components which involves considering multi-pointed Heegaard diagrams and tensoring with $\widehat{HF}(S^1 \times S^2)$, see [13].

Brief description of Φ_ξ. To define this map we have to carefully extend a sutured Heegaard diagram for (M', Γ') to a diagram for (M, Γ). For details of this construction look at [13]. Here is just a very quick idea. We use, in an essential way, the contact structure ξ on $M \setminus int(M')$ compatible with the sutures Γ and Γ' to define the map. We start from a Heegaard surface for M'. If we are given ξ' on M' take $(\Sigma', \beta', \alpha')$ to be defined by a partial open book compatible with ξ'. If we are not given a ξ', let $(\Sigma', \beta', \alpha')$ be a Heegaard

diagram arising from a partial open book decomposition of some contact structure ζ which has dividing set Γ' on $\partial M'$. We would like to join this Heegaard diagram to one generated by ξ on $M \setminus int(M')$. However, that is in general not precise enough, as it does not provide enough compressing discs for the union. To connect the two sides, we need to start with N, a contact product neigbourhood of $\partial M'$ in $M \setminus int(M')$, and let $M'' = M \setminus int(M' \cup N)$. We carefully choose a Heegaard surface Σ_N which is compatible with the $[0,1]$-invariant contact structure $\xi|_N$, as well as a basis of arcs $\{a_i^N\}$ for it. We then extend Σ_N to a Heegaard surface $\Sigma_{M''}$ and denote a basis of arcs on the union extending $\{a_i^N\}$ in a way necessary to obtain sutured Heegaard diagram for $(M \setminus int\, M', \Gamma \cup \Gamma')$ by $\{\alpha_i''\}$. This needs to be done in a way compatible with $\xi \cup \zeta$, i.e. so that after $\{a_i''\}$ and their perturbations $\{b_i''\}$ are chosen, and we look at boundaries $\{\alpha_i''\}$ and $\{\beta_i''\}$ of corresponding compressing discs, the special point $\mathbf{x}'' = (\dots, \mathbf{x}_i'', \dots)$, consisting of $x_i'' = a_i'' \cap b_i''$, is the contact class of $\xi \cup \zeta$. After gluing we get a Heegaard diagram for (M, Γ) by taking $\alpha = \alpha' \cup \alpha''$ and $\beta' = \beta' \cup \beta''$. We then define

$$\Phi_\xi : CF\big(\Sigma', \beta', \alpha'\big) \to CF\big(\Sigma, \beta' \cup \beta'', \alpha' \cup \alpha''\big),$$

$$\mathbf{y} \mapsto \big(\mathbf{y}, \mathbf{x}''\big). \quad \blacksquare$$

Above theorem has as an immediate consequence:

Theorem 5.2 [13]**.** *Let* $i : (M', \Gamma', \xi') \to (M, \Gamma, \xi)$ *be an inclusion such that* $\xi|_{M'} = \xi'$. *If* $c(M, \Gamma, \xi) \neq 0$, *then* $c(M', \Gamma', \xi') \neq 0$.

Juhász [17] showed that we can recover the Heegaard Floer Homology of a closed manifold M by calculating the sutured Floer homology of the manifold with sutured boundary $(M \setminus B^3, \Gamma)$ obtained by removing a solid ball B^3 from M and letting the suture Γ be $S^1 \subset S^2$ on the boundary $S^2 = \partial(M \setminus B^3)$. This isomorphism is one to one on generators when we consider the sutured Heegaard diagram on $M \setminus B^3$ and the corresponding Heegaard diagram on M obtained by closing the Heegaard surface by adding a disc along its boundary and making its center a marked point. This isomorphism takes in a natural way the sutured contact invariant of $(M \setminus B^3, S^1)$ to the contact invariant of the closed manifold. Since fillable structures have nonzero contact invariants we have:

Theorem 5.3 [13]**.** *If* $c(M, \Gamma, \xi) = 0$, *then* (M, Γ, ξ) *does not embed into any fillable contact structure.*

It is clear from the construction, as we had remarked after defining the contact invariant generator in the closed case, that an arc that is taken to the left by the monodromy produces a holomorphic disc that kills the contact invariant. The partial open book we constructed in Figure 20 shows that $c(\xi) = 0$ for the sutured manifold which is the neighborhood of the overtwisted disc (there is a disc from y to x making $\partial y = x$). From this, coupled with the embedding theorem, we see another proof that $c(\xi) = 0$ for any overtwisted contact structure.

6. A TQFT Aspect of $c(\xi)$ and Fillability Obstructions

In this section we study S^1 invariant contact structures ξ on $\Sigma \times S^1$, such that $\Sigma \times \{t\}$ is convex surface with Legendrian boundary, and for all the components of $\partial \Sigma$ there is twisting of ξ with respect to the framing determined by Σ. These contact structures are classified by their dividing sets Γ_ξ which are properly embedded multicurves (disjoint union of curves and arcs) that intersect every component of the boundary of Σ in an even number of points, and divide Σ and hence $\partial \Sigma$ into positive and negative regions. We say that a properly embedded multicurve $K \subset \Sigma$ is *isolating* if $\Sigma \setminus K$ contains a component that does not intersect $\partial \Sigma$ (see Figure 21). Examples of contact structures with $c(M, \Gamma, \xi) = 0$ can now be obtained quite easily from:

Theorem 6.1 [13]**.** *Let ξ_K be the S^1 invariant contact structures on $\Sigma \times S^1$, such that $\Sigma \times \{t\}$ is convex with dividing set $\Gamma_\Sigma = K$. If K is isolating then $c(\Sigma \times S^1, \xi_K) = 0$.*

To prove this theorem we use some TQFT-like properties of contact invariants for S^1 invariant contact structures on $\Sigma \times S^1$. Consider a "bordered"
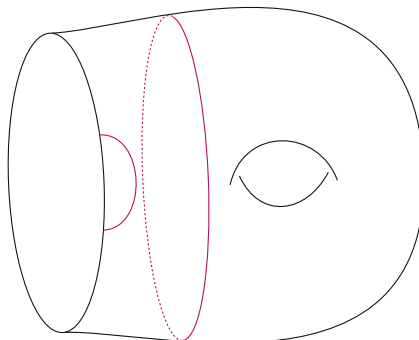

Fig. 21. An isolating dividing set on a punctured torus

surface with boundary (Σ, F), i.e. a surface Σ with a finite subset $F \subset \partial \Sigma$ consisting of 2n points that divide the boundary $\gamma = \partial \Sigma$ into alternating positive and negative regions $\gamma_\pm$, with $\gamma \setminus F = \gamma_- \sqcup \gamma_+$. We say that a union K of closed curves and properly embedded arcs in Σ with $\partial K = F$ is a dividing set on (Σ, F), or that K divides Σ, if $\Sigma \setminus K$ is a disjoint union of positive and negative regions $R_\pm$, and $\partial R_\pm = K \cup \gamma_\pm$. Denote by $\mathcal{D}(\Sigma, F)$ the family of all dividing sets for (Σ, F). We will use Sutured Floer Homology and sutured contact invariants to define a map that assigns a vector space $V(\Sigma, F)$ to each (Σ, F) and an element in that vector space to each $K \in \mathcal{D}(\Sigma, F)$ with following TQFT-like properties:

1. If Σ is connected, then

$$V(\Sigma, F) = \mathbb{F}^2 \otimes \cdots \otimes \mathbb{F}^2,$$

   where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, the number of copies of $\mathbb{F}^2$ is $r = n - \chi(\Sigma)$, and $\mathbb{F}^2 = \mathbb{F} \oplus \mathbb{F}$ is a graded $\mathbb{F}$-module whose first summand has grading 1 and the second summand has grading -1. If (Σ, F) is the disjoint union of (Σ_1, F_1) and (Σ_2, F_2), then

$$V(\Sigma_1 \sqcup \Sigma_2, F_1 \sqcup F_2) \simeq V(\Sigma_1, F_1) \otimes V(\Sigma_2, F_2).$$

2. To each $K \in \mathcal{D}(\Sigma, F)$ it assigns $c(K) \in V(\Sigma, F)$. If K has a homotopically trivial closed component, then $c(K) = 0$.

3. Given (Σ, F), possibly disconnected, let $\delta, \delta' \subset \partial \Sigma$ be mutually disjoint submanifolds of $\partial \Sigma$, such that their endpoints do not lie in F, and let τ be a diffeomorphism $\tau : \delta \xrightarrow{\sim} \delta'$ which identifies $\delta \cap F \xrightarrow{\sim} \delta' \cap F$ and preserves the $\pm$ labeling and reverses the orientation on δ, δ' inherited from $\partial \Sigma$. Denote by (Σ', F') the result of identifying γ and γ' via τ. For every $K \in \mathcal{D}(\Sigma, F)$ denote by $\overline{K}$ the dividing set obtained from K by gluing $K|_\gamma$ and $K|_{\gamma'}$. Then there exists a map

$$\Phi_\tau : V(\Sigma, F) \to V(\Sigma', F'),$$

   which satisfies

$$c(K) \mapsto c(\overline{K}).$$

Figure 22 shows the case when Σ is a union of disjoint surfaces Σ'' and Σ''', and hence Σ' is obtained by gluing Σ'' and Σ'''.

To define this assignment for every $(\Sigma, F, \gamma_\pm)$, we first perturb F by moving it slightly in the direction opposite to the one defined by the orientation
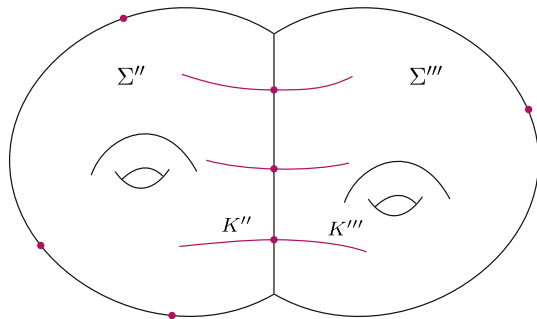
Fig. 22. Gluing (Σ'', K'') and (Σ''', K''')

on $\gamma = \partial \Sigma$ to obtain F^0, and shifting $\gamma_\pm$ in the same way to obtain $\gamma_\pm^0$. We then consider the sutured 3-manifold $(\Sigma \times S^1, \Gamma)$ where $\Gamma = F^0 \times S^1$ and $R_\pm = \gamma_\pm^0 \times S^1$. Denote by $V(\Sigma, F)$ the sutured Floer homology of $(-\Sigma \times S^1, -\Gamma)$. A dividing set $K \in \mathcal{D}(\Sigma, F)$ defines an S^1-invariant contact structure ξ_K on $(\Sigma \times S^1, \Gamma)$, and hence the corresponding contact invariant $c(K) = c(\xi) \in V(\Sigma, F)$. We use F^0 instead of F since the two convex surfaces $\Sigma \times \{t\}$ and $\partial(\Sigma \times S^1)$ are transverse, so the two dividing sets have to mark "interlocking points" along the Legendrian intersection curve, i.e. points in $F^0 = \Gamma \cap \partial \Sigma$ must lie between the endpoints F of K (see Figure 5).

To prove that $V(\Sigma, F) = \mathbb{F}^2 \otimes \cdots \otimes \mathbb{F}^2$ with appropriate number of factors, we first need to calculate $SFH(D^2 \times S^1, \Gamma_2)$ for a solid torus with dividing set Γ_2 made up of 4 longitudinal curves. It is shown in [13, Section 5, Example 3] that the result is $SFH(D^2 \times S^1, \Gamma_2) = \mathbb{F}^2 = \mathbb{F}_{(1)} \oplus \mathbb{F}_{(-1)}$. There are exactly two S^1 invariant contact structures compatible with these sutures, each determined by one of the two dividing sets on D^2 consisting of two arcs, and each generating one of the $\mathbb{F}_{(\pm 1)}$. We cut the surface Σ repeatedly by properly embedded arcs that connect $+$ and $-$ regions of $\partial \Sigma$ until we get a disc. Every time we do a cut, each of the two new arcs in the boundary that correspond to the cut gets one marked point in F, thus each cut adds two points to the boundary, increasing n by one. A tensor product formula by Juhász [18, Proposition 8.10] that holds for splitting sutured manifolds along product annuli applies. The annulus here is product of a cutting arc with S^1. The number of summands, $r = n - \chi(\Sigma)$ corresponds to the number of discs, each with 4 marked points on the boundary, that Σ is finally cut into, and the $\mathbb{F}^2$ factor corresponds to the contribution of each such disc to SFH.

If K has a homotopically trivial closed component, then ξ_K is overtwisted and hence $c(K) = 0$ since we defined $c(K) \in V(\Sigma, F)$ to be the value of the contact invariant for the S^1-invariant contact structure ξ_K determined by K.
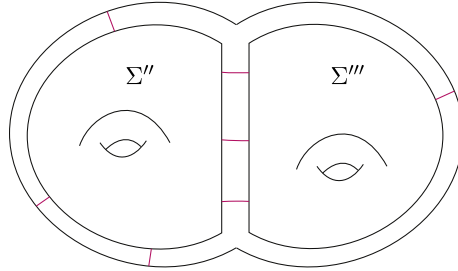
Fig. 23. The dividing set K_0 on $\Sigma' \setminus \Sigma$ is given in red
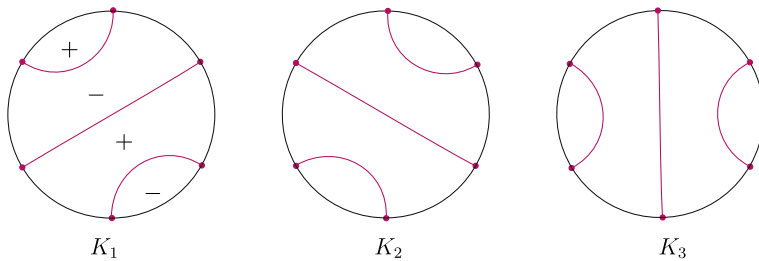

Fig. 24. Dividing sets on $\Sigma = D^2$ for $|F| = 6$

The Gluing Theorem 5.1 applied to $\Sigma \times S^1 \subset \Sigma' \times S^1$ gives us the map Φ_τ. We think of Σ as a subset of Σ' as in Figure 23, where $\Sigma = \Sigma'' \sqcup \Sigma'''$ and Σ'' and Σ''' are identified with slightly shrunk copies inside σ'.

The contact structure on $\Sigma' \times S^1 \setminus \Sigma \times S^1$ is the S^1 invariant structure determined by the dividing set K_τ on $\Sigma' \setminus \Sigma$ determined by F'', F''' and the identification τ.

By studying what we have in the case of $\Sigma = D^2$ and $|F| = 6$ we see in [16] that for dividing sets K_1, K_2, K_3 as in Figure 24 we have that the corresponding $c(K_1), c(K_2), c(K_3)$ are nonzero and distinct, and satisfy $c(K_1) = c(K_2) + c(K_3)$. Note that these three configurations are related by bypass addition. In fact, K_2 is obtained by adding a bypass to the front of K_1 along an arc connecting the three dividing curves, and K_3 is obtained by adding a bypass from the back of K_1 (digging a bypass).

When we combine this with the gluing theorem, we obtain the same relationship for any three dividing sets $\{K_i, \ i = 1, 2, 3\}$ related by bypass addition on a general Σ. We will quickly argue that $c(\xi_K) = 0$ for our example in Figure 21.

It is not hard to see that by adding and digging bypasses along the bypass arc δ given in blue in K_1 in Figure 25, we obtain dividing curves as in K_2
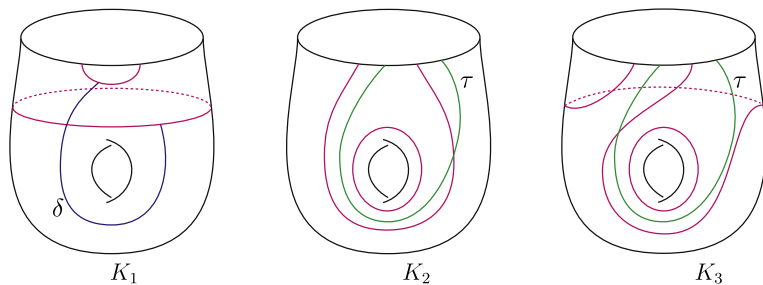
Fig. 25. Bypass adition in the proof of $c(\xi_K) = 0$ for an isolating dividin set

and K_3. Cutting Σ along the green curve τ and looking at the dividing sets K_2' and K_3' resulting from K_2 and K_3 on the resulting annulus, we see that $c(K_2') = c(K_3')$. Juházs' annulus theorem says gluing along τ induces an isomorphism, so that we get $c(K_2) = c(K_3)$. This completes the proof that $c(\xi_{K_1}) = 0$ since $c(K_1) = c(K_2) + c(K_3) = 0$ as we are working over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

More general isolating dividing sets on surfaces of higher genus can be dealt with by similar methods or reduced to this case, thus proving Theorem 6.1. For full explanation see [16]. The fact that the contact invariant vanishes for isolating dividing sets was proved over $\mathbb{Z}$ coefficients by Patrick Massot [23].

Theorem 6.1 together with Theorem 5.3 shows that $(\Sigma \times S^1, \xi_K)$ with contact structures corresponding to an isolating K form a vast family of universally tight contact structures that do not embed into fillable structures and are thus generalizing Giroux torsion. Similar results were obtained by looking at holomorphic curves and contact homology by Chris Wendl [34].

References

[1] S. Akbulut, B. Ozbagci, Lefschetz fibrations on compact Stein surfaces. Geom. Topol. **5**, 319–334 (2001)

[2] K. Baker, J.B. Etnyre, J. Van Horn-Morris, Cabling, contact structures and mapping class monoids. J. Differ. Geom. **90**, 1–80 (2012)

[3] J.A. Baldwin, Contact monoids and Stein cobordisms. Math. Res. Lett. **19**(1), 31–40 (2012)

[4] D. Bennequin, Entrelacements et equations de Pfaff. Astérisque **107–108**, 87–161 (1983)

[5] Y. Eliashberg, Classification of overtwisted contact structures on 3-manifolds. Invent. Math. **98**, 623–637 (1989)

[6] Y. Eliashberg, Topological characterization of Stein manifolds of dimension > 2. Int. J. Math. **1**, 29–46 (1990)

[7] Y. Eliashberg, Filling by holomorphic discs and its applications. Lond. Math. Soc. Lect. Note Ser. **151**, 45–67 (1991)

[8] P. Ghiggini, Infinitely many universally tight contact manifolds with trivial Ozsváth-Szabó contact invariants. Geom. Topol. **10**, 335–357 (2006)

[9] P. Ghiggini, K. Honda, J. Van Horn-Morris, The vanishing of the contact invariant in the presence of torsion. arXiv:0706.1602

[10] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, in *Proceedings of the International Congress of Mathematicians*, vol. II, Beijing, 2002 (Higher Ed. Press, Beijing, 2002), pp. 405–414

[11] M. Gromov, Pseudo holomorphic curves in symplectic manifolds. Invent. Math. **82**, 307–347 (1985)

[12] K. Honda, W. Kazez, G. Matić, Convex decomposition theory. Int. Math. Res. Not. **2002**, 55–88 (2002)

[13] K. Honda, W. Kazez, G. Matić, Right veering automorphisms of compact surfaces with boundary I. Invent. Math. **169** (2007)

[14] K. Honda, W. Kazez, G. Matić, On the contact class in Heegaard Floer homology. J. Differ. Geom. **83**(2), 289–311 (2009)

[15] K. Honda, W. Kazez, G. Matić, The contact invariant in sutured Floer homology. Invent. Math. **176**(3), 637–676 (2009)

[16] K. Honda, W. Kazez, G. Matić, Contact structures, sutured Floer homology and TQFT. Preprint, arXiv:0807.2431

[17] A. Juhász, Holomorphic discs and sutured manifolds. Algebr. Geom. Topol. **6**, 1429–1457 (2006) (electronic)

[18] A. Juhász, Floer homology and surface decompositions. Geom. Topol. **12**, 299–350 (2008)

[19] Y. Lekili, Planar open books with four binding components. Algebr. Geom. Topol. **11**, 909–928 (2011)

[20] R. Lipshitz, A cylindrical reformulation of Heegaard Floer homology. Geom. Topol. **10**, 955–1097 (2006) (electronic)

[21] A. Loi, R. Piergallini, Compact Stein surfaces with boundary as branched covers of B4. Invent. Math. **143**, 325–348 (2001)

[22] C. Manolescu, P. Ozsváth, Z. Szabó, D. Thurston, On combinatorial link Floer homology. Geom. Topol. **11**, 2339–2412 (2007)

[23] P. Massot, Infinitely many universally tight torsion free contact structures with vanishing Ozsváth–Szabó contact invariants. Math. Ann. **353**(4), 1351–1376 (2012)

[24] P. Ozsváth, Z. Szabó, Holomorphic disks and topological invariants for closed three-manifolds. Ann. Math. **159**, 1027–1158 (2004)

[25] P. Ozsváth, Z. Szabó, Holomorphic disks and three-manifold invariants: properties and applications. Ann. Math. **159**, 1159–1245 (2004)

[26] P. Ozsváth, Z. Szabó, Heegaard Floer homology and contact structures. Duke Math. J. **129**, 39–61 (2005)

[27] P. Ozsváth, A. Stipsicz, Z. Szabó, Combinatorial Heegaard Floer homology and nice Heegaard diagrams. Adv. Math. **231**, 102–171 (2012)

[28] K. Reidemeister, Zur dreidimensionalen Topologie. Abh. Math. Semin. Univ. Hamb. **9**, 189–194 (1933)

[29] D. Rolfsen, *Knots and Links* (Publish or Perish Inc., Houston, 1990). Corrected revision of the 1976 original

[30] S. Sarkar, J. Wang, An algorithm for computing some Heegaard Floer homologies. Ann. Math. **171**(2), 1213–1236 (2010)

[31] J. Singer, Three-dimensional manifolds and their Heegaard diagrams. Trans. Am. Math. Soc. **35**, 88–111 (1933)

[32] W. Thurston, H. Winkelnkemper, On the existence of contact forms. Proc. Am. Math. Soc. **52**, 345–347 (1975)

[33] I. Torisu, Convex contact structures and fibered links in 3-manifolds. Int. Math. Res. Not. **2000**, 441–454 (2000)

[34] C. Wendl, A hierarchy of local symplectic filling obstructions for contact 3-manifolds. Duke Math. J. **162**(12), 2197–2283 (2013)

G. Matić (✉)

University of Georgia
Athens, GA 30602
USA

e-mail: gordana@math.uga.edu

url: http://www.math.uga.edu/~gordana