BOLYAI SOCIETY
MATHEMATICAL STUDIES, 26

Contact and Symplectic Topology
pp. 173–244.

Higher Dimensional Contact Topology via Holomorphic Disks

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1. Introduction

In '85 Gromov published his article on pseudo-holomorphic curves [17] that made symplectic topology as we know it today only possible. Using these techniques, Gromov presented in his initial paper many spectacular results, and soon many other people started using these methods to settle questions that before had been out of reach [1, 9, 10, 18, 22, 23] and many others; for more recent results in this vein we refer to [31, 36].

While the references above rely on studying the topology of the moduli space itself, Gromov's J-holomorphic methods have also been used to develop powerful algebraic theories like Floer Homology, Gromov-Witten Theory, Symplectic Field Theory, Fukaya Theory etc. that basically rely on counting rigid holomorphic curves (that means holomorphic curves that are isolated). Note though that we will completely ignore such algebraic techniques in these notes.

Gromov's approach for studying a symplectic manifold (W, ω) consists in choosing an *auxiliary* almost complex structure J on W that is compatible with ω in a certain way. This auxiliary structure allows us to study so called J-holomorphic curves, that means, equivalence classes of maps

$$u\colon (\Sigma, j) \to (W, J)$$

F. Bourgeois et al. (eds.), *Contact and Symplectic Topology*,
Bolyai Society Mathematical Studies 26, DOI 10.1007/978-3-319-02036-5_5,

from a Riemann surface (Σ, j) to W whose differential at every point $x \in \Sigma$ is a (j,J)-complex map

$$Du_x : T_x\Sigma \to T_{u(x)}W.$$

Conceivable generalizations of such a theory based on studying J-holomorphic surfaces or even higher dimensional J-complex manifolds only work for *integrable* complex structures; otherwise generically such submanifolds do not exist. A different approach has been developed by Donaldson [7, 8], and consists in studying approximately holomorphic sections in a line bundle over W. This theory yields many important results, but has a very different flavor than the one discussed here by Gromov.

The J-holomorphic curves are relatively rare and usually come in finite dimensional families. Technical problems aside, one tries to understand the symplectic manifold (W,ω) by studying how these curves move through W.

Let us illustrate this strategy with the well-known example of $\mathbb{C}P^n$. We know that there is exactly one complex line through any two points of $\mathbb{C}P^n$. We fix a point $z_0 \in \mathbb{C}P^n$, and study the space of all holomorphic lines going through z_0. It follows directly that $\mathbb{C}P^n \setminus \{z_0\}$ is foliated by these holomorphic lines, and every line with z_0 removed is a disk. Using that the lines are parametrized by the corresponding complex line in $T_{z_0}\mathbb{C}P^n$ that is tangent to them, we see that the space of holomorphic lines is diffeomorphic to $\mathbb{C}P^{n-1}$, and that $\mathbb{C}P^n \setminus \{z_0\}$ will be a disk bundle over $\mathbb{C}P^{n-1}$.

In this example, we have used an ambient manifold that we understand rather well, $\mathbb{C}P^n$, to compute the topology of the space of complex lines. So far, it might seem unclear how one could obtain information about the topology of the space of complex lines in an ambient space that we do not understand equally well, to then extract in a second step missing information about the ambient manifold.

The common strategy is to assume that the almost complex manifold we want to study already contains a family of holomorphic curves. We then observe how this family evolves, hoping that it will eventually “fill up” the entire symplectic manifold (or produce other interesting effects).

To briefly sketch the type of arguments used in general, consider now a symplectic manifold W with a compatible almost complex structure, and suppose that it contains an open subset U diffeomorphic to a neighborhood of $\mathbb{C}P^1 \times \{0\}$ in $\mathbb{C}P^1 \times \mathbb{C}$ (see [22]). In this neighborhood we find a family of holomorphic spheres $\mathbb{C}P^1 \times \{z\}$ parametrized by the points z. We can explicitly write down the holomorphic spheres that lie completely inside U, but Gromov compactness tells us that as the holomorphic curves approach the boundary of U, they cannot just cease to exist but instead there is a well understood way in which they can degenerate, which is called *bubbling*. Bubbling means that a family of holomorphic curves decomposes in the limit into several smaller ones. Sometimes bubbling can be controlled or even excluded by imposing technical conditions, and in this case, the limit curve will just be a regular holomorphic curve.

In the example we were sketching above, this means that if no bubbling can happen, there will be regular holomorphic spheres (partially) outside U that are obtained by pushing the given ones towards the boundary of U. This limit curve is also part of the 2-parameter space of spheres, and thus it will be surrounded by other holomorphic spheres of the same family. As long as we do not have any bubbling, we can thus extend the family by pushing the spheres to the limit and then obtain a new regular sphere, which again is surrounded by other holomorphic spheres. This way, we can eventually show that the whole symplectic manifold is filled up by a 2-dimensional family of holomorphic spheres. Furthermore the holomorphic spheres do not intersect each other (in dimension 4), and this way we obtain a 2-sphere fibration of the symplectic manifold.

In conclusion, we obtain in this example just from the existence of the chart U, and the conditions that exclude bubbling that the symplectic manifold needs to be a 2-sphere bundle over a compact surface (the space of spheres).

Note that many arguments in the example above (in particular the idea that the moduli spaces foliate the ambient manifold) do not hold in general, that means for generic almost complex structures in manifolds of dimension more than 4. Either one needs to weaken the desired statements or find suitable work arounds. The principle that is universal is the use of a well understood local model in which we can detect a family of holomorphic curves. If bubbling can be excluded, this family extends into the unknown parts of the symplectic manifold, and can be used to understand certain topological properties of this manifold.

These notes are based on a course that took place at the Université de Nantes in June 2011 during the *Trimester on Contact and Symplectic Topology*. We will explain how holomorphic curves can be used to study symplectic fillings of a given contact manifold. Our main goal consists in showing that certain contact manifolds do not admit any symplectic filling at all. Since closed symplectic manifolds are usually studied using closed holomorphic curves, it is natural to study symplectic fillings by using holomorphic curves with boundary. We will explain how the existence of so called *Legendrian open books* (Lobs) and *bordered Legendrian open books* (bLobs) controls the behavior of holomorphic disks, and what properties we can deduce from families of such disks. The notions are direct generalizations of the overtwisted disk [9, 17] and standardly embedded 2-spheres in a contact 3-manifold [4, 17, 18].

For completeness, we would like to mention that symplectic fillings have also been studied successfully via punctured holomorphic curves whose behavior is linked to Reeb orbit dynamics, and via closed holomorphic curves by first capping off the symplectic filling to create a closed symplectic manifold.

1.1. Outline of the Notes

In the first part of these notes we will talk about Legendrian foliations, and in particular about Lobs and bLobs. We will not consider any holomorphic curves here, but the main aim will be instead to illustrate examples where these objects can be localized. In Section 3, we study the properties of holomorphic disks imposed by Legendrian foliations and convex boundaries. In the last section, we use this information to understand moduli spaces of holomorphic disks obtained from a Lob or a bLob, and we prove some basic results about symplectic fillings.

The content of these notes are based on an unfinished manuscript of [28].

1.2. Notation

We assume throughout a certain working knowledge on contact topology (for a reference see for example [24, Chapter 3.4] and [12]) and on holomorphic curves [3, 25]. The contact structures we consider in this text are always *cooriented*. Remember that by choice of a coorientation, (M, ξ) always obtains a natural orientation and its contact structure ξ carries a natural *conformal* symplectic structure. For both, it suffices to choose a **positive** contact form α, that means, a 1-form with $\xi = \ker \alpha$ that evaluates positively on vectors that are positively transverse to the contact structure. The orientation on M is then given by the volume form

$$\alpha \wedge d\alpha^n,$$

where $\dim M = 2n+1$, while the conformal symplectic structure is represented by $d\alpha|_\xi$.

One can easily check that these notions are well-defined by choosing any other positive contact form α' so that there exists a smooth function $f\colon M \to \mathbb{R}$ such that $\alpha' = e^f \alpha$.

Further Conventions. Note that $\mathbb{D}^2$ denotes in this text the *closed* unit disk.

I owe it to Patrick Massot to have been converted to the following *jargon*.

Definition. The term **regular equation** can refer in this text to any of the following objects:

(1) When Σ is a cooriented hypersurface in a manifold M, then we call a smooth function $h\colon M \to \mathbb{R}$ a **regular equation for** Σ, if 0 is a regular value of h and $h^{-1}(0) = \Sigma$.

(2) When $\mathcal{D} \le TM$ is a singular codimension 1 distribution, then we say that a 1-form β is a **regular equation for** $\mathcal{D}$, if $\mathcal{D} = \ker\beta$ and if $d\beta \neq 0$ at singular points of $\mathcal{D}$.

According to this definition, an equation of a contact structure is just a contact form.

2. Lobs & bLobs: Legendrian Open Books and Bordered Legendrian Open Books

2.1. Legendrian Foliations

2.1.1. General Facts about Legendrian Foliations. Let (M,ξ) be a contact manifold that contains a submanifold N. Generically, if we look at any point $p \in N$ the intersection between ξ_p and the tangent space T_pN will be a codimension 1 hyperplane. Globally though, the distribution $\mathcal{D} = \xi \cap TN$ may be singular, because there can be points $p \in N$ where $T_pN \subset \xi_p$, and equally important the distribution $\mathcal{D}$ will only be in very rare cases a foliation. In fact, if we choose a contact form α for ξ, then we obtain by the Frobenius theorem that $\mathcal{D}$ will only be a (singular) foliation if

$$(\alpha \wedge d\alpha)|_{TN} \equiv 0.$$

Another way to state this condition is to say that we have $d\alpha|_{\mathcal{D}_p} = 0$ at every regular point $p \in N$ of $\mathcal{D}$, so that $\mathcal{D}_p$ has to be an isotropic subspace of $(\xi_p, d\alpha_p)$. In particular, this shows that the induced distribution $\mathcal{D}$ can never be integrable if $\dim \mathcal{D} > \frac{1}{2} \dim \xi$.

We will usually denote the distribution $\xi \cap TN$ by $\mathcal{F}$ whenever it is a singular foliation. Furthermore, we will call such an $\mathcal{F}$ a **Legendrian foliation** if $\dim \mathcal{F} = \frac{1}{2} \dim \xi$, which implies that N has to be a submanifold of dimension $n + 1$ if the dimension of the ambient contact manifold is $2n + 1$. For reasons that we will briefly sketch below, but that will be treated extensively from Section 3 on, we will be mostly interested in submanifolds carrying such a Legendrian foliation. Note in particular that in a contact 3-manifold every hypersurface N carries automatically a Legendrian foliation.

Denote the set of points $p \in N$ where $\mathcal{F}$ is singular by $\mathrm{Sing}(\mathcal{F})$. One of the basic properties of a Legendrian foliation is that for any contact form α, the restriction $d\alpha|_{TN}$ does not vanish on $\mathrm{Sing}(\mathcal{F})$, because otherwise $T_pN \subset \xi_p$ would be an isotropic subspace of $(\xi_p, d\alpha_p)$ which is impossible for dimensional reasons. Since $d\alpha|_{TN}$ does not vanish on $\mathrm{Sing}(\mathcal{F})$, we deduce in particular that $N \setminus \mathrm{Sing}(\mathcal{F})$ is a dense and open subset of N.

The main reason, why we are interested in submanifolds that have a Legendrian foliation is that they often allow us to successfully use J-holomorphic curve techniques. On one side, such submanifolds will be automatically totally real for any suitable almost complex structure on a symplectic filling, thus posing a good boundary condition for the Cauchy-Riemann equation: The solution space of a Cauchy-Riemann equation with totally real boundary condition is often a finite dimensional smooth manifold, so that it follows that the moduli spaces of J-holomorphic curves whose boundaries lie in a submanifold with a Legendrian foliation will have a nice local structure. A second important property is that the topology of the Legendrian foliation controls the behavior of J-holomorphic curves, and will allow us to obtain many results in contact and symplectic topology. Elliptic codimension 2 singularities of the Legendrian foliation “emit” families of holomorphic disks; suitable codimension 1 singularities form “walls” that cannot be crossed by holomorphic disks.

In the rest of this section, we will state some general properties of Legendrian foliations. Theorem 2.2 shows that a manifold with a Legendrian foliation determines the germ of the contact structure on its neighborhood. This allows us to describe small deformations of the Legendrian foliation, and study almost complex structures more explicitly (see Section 3.2). Theo-

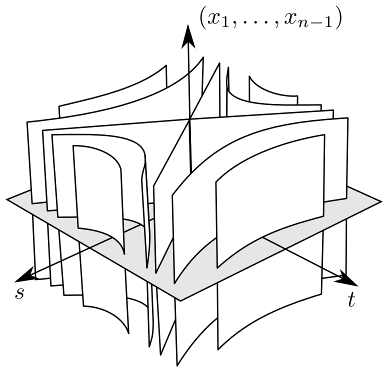

Fig. 1. The singularities of a Legendrian foliation look locally like the product of $\mathbb{R}^{n-1}$ with a foliation in the plane

rem 2.3 gives a precise characterization of the foliations that can be realized as Legendrian ones.

2.1.2. Singular Codimension 1 Foliations. The principal aim of this section will be to explain the following result due to Kupka [20] that tells us that the behavior of a Legendrian foliation close to a singular point can always be reduced to the 2-dimensional situation (see Figure 1).

Theorem 2.1. *Let N be a manifold with a singular foliation $\mathcal{F}$ that admits a regular equation β. Then we find around any $p \in \operatorname{Sing}(\mathcal{F})$ a chart with coordinates $(s, t, x_1, \ldots, x_{n-1})$, such that β is represented by the 1-form*

$$a(s,t)\, ds + b(s,t)\, dt$$

for smooth functions a and b.

We will call any chart of N of the form described in the theorem a **Kupka chart**. Note that the foliation in a Kupka chart restricts on every 2-dimensional slice $\{(x_1, \ldots, x_{n-1}) = \text{const}\}$ to one that does not have any isochore singularities (a term introduced in [15]).

Proof. From the Frobenius condition $\beta \wedge d\beta \equiv 0$, it follows that $d\beta^2 = 0$, so that if $\dim N > 2$, there is a non-vanishing vector field X on a neighborhood of p with $d\beta(X, \cdot) = 0$. We can also easily see that $X \in \ker \beta$ and $\mathcal{L}_X \beta = 0$, because

$$0 = \iota_X(\beta \wedge d\beta) = \beta(X)\, d\beta - \beta \wedge (\iota_X d\beta) = \beta(X)\, d\beta,$$

and $d\beta$ does not vanish on a neighborhood of p.

Let Φ_t^X be the flow of X, and choose a small hypersurface Σ transverse to X. Using the diffeomorphism

$$\Psi\colon \Sigma\times(-\varepsilon,\varepsilon)\to N, \quad (p,t)\mapsto \Phi_t^X(p)$$

we can pull back the 1-form β to $\Sigma\times(-\varepsilon,\varepsilon)$ and we see it reduces to $\beta|_{T\Sigma}$. By repeating this construction the necessary number of times we obtain the desired statement. ■

2.1.3. Local Behavior of Legendrian Foliations. We state the following two theorems without proof, and point the interested reader to [28] for more details. The situation in Section 2.2.2 is treated in these notes in full completeness to illustrate the flavor of the necessary methods. The first result tells us that a Legendrian foliation determines the germ of the contact structure in its neighborhood.

Theorem 2.2. *Let N be a compact manifold (possibly with boundary) and let (M_1,ξ_1) and (M_2,ξ_2) be contact manifolds. Assume that two embeddings $\iota_1\colon N\hookrightarrow M_1$ and $\iota_2\colon N\hookrightarrow M_2$ are given such that ξ_1 and ξ_2 induce on N the same cooriented Legendrian foliation $\mathcal{F}$. Then we find neighborhoods $U_1\subset M_1$ of $\iota_1(N)$ and $U_2\subset M_2$ of $\iota_2(N)$ together with a contactomorphism*

$$\Phi\colon (U_1,\xi_1)\to(U_2,\xi_2)$$

that preserves N, that means, $\Phi\circ\iota_1=\iota_2$.

Another useful fact is the following theorem that tells us that the singular foliations that can be realized as Legendrian ones are exactly those that admit a regular equation (using the convention from the introduction). This result generalizes the 3-dimensional situation [15], where this property was called a foliation without "*isochore singularities*".

Theorem 2.3. *Let N be a manifold with a singular codimension-1 foliation $\mathcal{F}$ given by a regular equation β. Then we can find an (open) cooriented contact manifold (M,ξ) that contains N as a submanifold such that ξ induces $\mathcal{F}$ as Legendrian foliation on N.*

2.2. Singularities of the Legendrian Foliation

The singular set of a Legendrian foliation $\mathcal{F}$ can be extremely complicated. We will only discuss briefly a few general properties of such points, before we specialize all considerations to two simple situations.

Let N have a singular foliation $\mathcal{F}$ given by a regular equation β, and let $p \in \operatorname{Sing}(\mathcal{F})$ be a singular point of $\mathcal{F}$. Choose a Kupka chart U with coordinates $(s, t, x_1, \dots, x_{n-1})$ centered at p. In this chart β is represented by

$$a(s,t)\,ds + b(s,t)\,dt$$

with two smooth functions $a, b\colon U \to \mathbb{R}$ that only depend on the s- and t-coordinates, and that vanish at the origin.

To understand the shape of the foliation depending on the functions a and b, we might study trajectories of the vector field

$$X = b(s,t)\frac{\partial}{\partial s} - a(s,t)\frac{\partial}{\partial t}$$

that spans the intersection of the foliation with the (s,t)-slices. Its divergence $\operatorname{div} X = \partial b/\partial s - \partial a/\partial t$ does not vanish, since $d\beta \neq 0$. Up to a genericity condition, we know by the Grobman-Hartman theorem that the flow of X is C^0-equivalent to the flow of its linearization (see [32]). In dimension 2, the Grobman-Hartman theorem even yields a C^1-equivalence, but this does not suffice for our purposes. For one, we would like to stick to a smooth model for all singularities, but in fact it even suffices for our goals to only look at singularities whose leaves are all radial, so we will use below a more hands-on approach.

2.2.1. Elliptic Singularities. The first type of singularities we allow for the foliation $\mathcal{F}$ on N are called **elliptic**: In this case, the point $p \in \operatorname{Sing}(\mathcal{F})$ admits a Kupka chart diffeomorphic to $\mathbb{R}^2 \times \mathbb{R}^n$ with coordinates $\{(s, t, x_1, \dots, x_n)\}$ in which the foliation is given as the kernel of the 1-form

$$s\,dt - t\,ds$$

that means, the leaves are just the radial rays in each (s,t)-slice.

We will always assume that the elliptic singularities of a foliation $\mathcal{F}$ are closed isolated codimension 2 submanifolds S in the interior of N with trivial normal bundle, so that the tubular neighborhood of S is diffeomorphic to $\mathbb{D}^2_\varepsilon \times S$. We assume additionally that the foliation $\mathcal{F}$ in this model

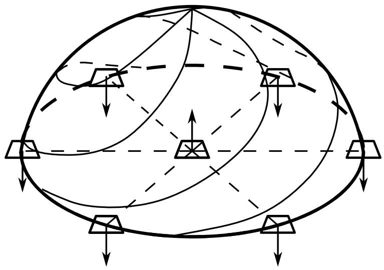

Fig. 2. In dimension 3 it is well-known that we can get rid of 1-dimensional singular sets of a Legendrian foliation by slightly tilting the surface along the singular set. The picture represents how to produce an overtwisted disk whose boundary is a regular compact leaf of the foliation

neighborhood is given by the points with constant angular coordinate on the $\mathbb{D}^2_\varepsilon$-factor.

2.2.2. Singularities of Codimension 1. Singular sets of codimension 1 are extremely ungeneric, but can be often found through explicit constructions (as in Example 2.7). We will show in this section that by slightly deforming the foliated submanifold one can sometimes modify the foliation in a controlled way so that the singular set turns into a regular compact leaf (see Figure 2).

We will treat this situation in detail to illustrate what type of methods are needed for the proofs in this section.

Lemma 2.4. *Let N be a compact manifold with a singular codimension 1 foliation $\mathcal{F}$ given by a regular equation β. Assume that the singular set $\operatorname{Sing}(\mathcal{F})$ of the foliation contains a closed codimension 1 submanifold $S \hookrightarrow N$ that is cooriented.*

Then we can find a tubular neighborhood of S diffeomorphic to $(-\varepsilon,\varepsilon) \times S$ such that β pulls back to

$$s \cdot \widetilde{\beta},$$

where s denotes the coordinate on $(-\varepsilon,\varepsilon)$, and $\widetilde{\beta}$ is a non-vanishing 1-form on S that defines a regular codimension 1 foliation on S.

Proof. Choose a coorientation for S. We first find a vector field X on a neighborhood of S that is transverse to S and lies in the kernel of β. Study the local situation in a Kupka chart U around a point $p \in S$ with coordinates

$(s, t, x_1, \ldots, x_{n-1})$. Assume that β restricts to

$$a(s,t)\,ds + b(s,t)\,dt,$$

such that $S \cap U$ corresponds to the subset $\{s = 0\}$, and such that s increases in direction of the chosen coorientation.

Since a and b vanish along $S \cap U$, we may write this form also as

$$s\,a_s(s,t)\,ds + s\,b_s(s,t)\,dt = s\big(a_s(s,t)\,ds + b_s(s,t)\,dt\big)$$

with smooth functions a_s and b_s that satisfy the conditions

$$a_s(0,t) = \frac{\partial a}{\partial s}(0,t) \quad \text{and} \quad b_s(0,t) = \frac{\partial b}{\partial s}(0,t).$$

The function b_s does not vanish in a small neighborhood of $S \cap U$, because $0 \neq d\beta = \partial_s b\, ds \wedge dt$. Choose then on the Kupka chart U the smooth vector field

$$X_U(s,t,x_1,\ldots,x_{n-1}) = \partial_s - \frac{a(s,t)}{b(s,t)}\partial_t = \partial_s - \frac{a_s(s,t)}{b_s(s,t)}\partial_t.$$

This field lies in $\mathcal{F}$, and is positively transverse to $S \cap U$.

Cover the singular set S with a finite number of Kupka charts $U_1, \ldots, U_N$, construct vector fields X_{U_j} according to the method described above, and glue them together to obtain the desired vector field X by using a partition of unity subordinate to the cover. We can use the flow of X to obtain a tubular neighborhood of S that is diffeomorphic to $(-\varepsilon, \varepsilon) \times S$, where $\{0\} \times S$ corresponds to the submanifold S, and X corresponds to the field ∂_s, where s is the coordinate on the interval $(-\varepsilon, \varepsilon)$, and since $\beta(X) \equiv 0$, it follows that β does not contain any ds-terms.

Let γ be the 1-form given by $\iota_X d\beta$. This form does not vanish on a neighborhood of the singular set S, because $d\beta \neq 0$ while $\beta|_{TS} \equiv 0$, and so we can write

$$0 \equiv \iota_X(\beta \wedge d\beta) = \beta(X)\,d\beta - \beta \wedge (\iota_X d\beta) = -\beta \wedge \gamma.$$

This means that there is a smooth function $F\colon (-\varepsilon, \varepsilon) \times S \to \mathbb{R}$ with $F|_S = 0$ such that $\beta = F\gamma$. Furthermore, we get that

$$\gamma = \iota_X d\beta = dF(X)\gamma + F\,\iota_X d\gamma$$

does not vanish along S, but F does, so we obtain on S that $dF(X) = 1$, and it follows that S is a regular zero level set of the function F. In fact, we can also easily see from

$$0 \equiv \beta \wedge d\beta = F^2 \gamma \wedge d\gamma$$

that $\gamma \wedge d\gamma$ vanishes everywhere so that $\ker \gamma$ defines a regular foliation $\widetilde{\mathcal{F}}$ that agrees with the initial foliation outside $\operatorname{Sing}(\mathcal{F})$.

Finally, we have $\iota_X \gamma \equiv 0$, and using a similar argument as before, we see

$$0 \equiv \iota_X(\gamma \wedge d\gamma) = -\gamma \wedge \iota_X d\gamma$$

so that there is a smooth function $f\colon (-\varepsilon, \varepsilon) \times S \to \mathbb{R}$ such that $\mathcal{L}_X \gamma = \iota_X d\gamma = f\gamma$. The flow in s-direction possibly rescales the 1-form γ, but it leaves its kernel invariant, thus the foliation $\widetilde{\mathcal{F}}$ is tangent to the s-direction and s-invariant. We can hence represent $\widetilde{\mathcal{F}}$ on $(-\varepsilon, \varepsilon) \times S$ as the kernel of the 1-form $\widetilde{\beta} = \gamma|_{TS}$ that does not depend on the s-coordinate, and does not have any ds-terms. It follows that γ is equal to $\widetilde{F}\gamma|_{TS}$ for a function $\widetilde{F}$ that restricts on S to 1.

For the initial 1-form β this means that $\beta = (F\widetilde{F})\widetilde{\beta}$, and $F\widetilde{F}$ is a smooth function and $\{0\} \times S$ is the (regular) level set of 0. We can redefine the model $(-\varepsilon, \varepsilon) \times S$ by using the flow of a vector field $G^{-1}\partial_s$ with $G = \partial_s(F\widetilde{F})$ to achieve that β reduces on this new model to $s\widetilde{\beta}$. ■

Suppose from now on that the singular foliation is of the form described in Lemma 2.4, that means, we have a closed manifold S with a regular codimension 1 foliation $\mathcal{F}_S$ given as the kernel of a 1-form $\widetilde{\beta}$, and N is diffeomorphic to $(-\varepsilon, \varepsilon) \times S$ with a singular foliation $\mathcal{F}$ given as the kernel of the 1-form $s\widetilde{\beta}$.

Remember that a 1-form σ on S defines a section in T^*S with the property that $\sigma^*\lambda_{\mathrm{can}} = \sigma$. We may realize $\mathcal{F}$ as a Legendrian foliation, by embedding $(-\varepsilon, \varepsilon) \times S$ into the 1-jet space $(\mathbb{R} \times T^*S, dz + \lambda_{\mathrm{can}})$ via the map

$$(s, p) \mapsto (0, s\widetilde{\beta}).$$

The foliations agree, and according to Theorem 2.2 this model describes a small neighborhood of $(N, s\widetilde{\beta})$ embedded into an arbitrary contact manifold.

Assume from now on additionally that $\widetilde{\beta}$ is a *closed* 1-form on S (by a result of Tischler, S fibers over the circle [34]). Choose a smooth odd function $f\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ with compact support such that the derivative $f'(0) = -1$. The section

$$(-\varepsilon, \varepsilon) \times S \hookrightarrow \mathbb{R} \times T^*S, \quad (s, p) \mapsto \big(\delta f(s), s\widetilde{\beta}\big)$$

describes for small $\delta > 0$ a C^∞-small deformation of N that agrees away from S with N. The perturbed submanifold N' also carries a Legendrian foliation induced by $\ker(ds + \lambda_{\mathrm{can}})$, because the pull-back form $\beta' = f'\,ds + s\widetilde{\beta}$ gives

$$\beta' \wedge d\beta' = \big(f'\,ds + s\widetilde{\beta}\big) \wedge (ds \wedge \widetilde{\beta} + s\,d\widetilde{\beta}) = s\big(f'\,ds + s\widetilde{\beta}\big) \wedge d\widetilde{\beta} = s f'\,ds \wedge d\widetilde{\beta},$$

which vanishes, so that β' satisfies the Frobenius condition. Furthermore, since β' itself does not vanish anywhere, it is easy to check that $\ker \beta'$ defines a regular foliation $\mathcal{F}'$, and that $\{0\} \times S$ is a closed leaf of $\mathcal{F}'$.

As a conclusion, we obtain

Corollary 2.5. *Let (M,ξ) be a contact manifold containing a submanifold N with an induced Legendrian foliation $\mathcal{F}$. Assume that the singular set of $\mathcal{F}$ contains a cooriented closed codimension 1-submanifold $S \subset N$, and that there is a regular foliation $\mathcal{F}$ that agrees outside N with $\mathcal{F}$, and that corresponds on S with a fibration over the circle. Using an arbitrary small C^∞ perturbation of N close to S, we obtain a new Legendrian foliation for which S has become a regular closed leaf.*

2.3. Examples of Legendrian Foliations

The following example relates Legendrian foliations to Lagrangian submanifolds. It is not important by itself, but it may help understanding the construction of the bLobs in blown down Giroux domains given in [21], and I believe that it might pave the way to other applications.

Example 2.6. Let P be a principal circle bundle over a base manifold B, and suppose that ξ is a contact structure on P that is transverse to the $\mathbb{S}^1$-fibers and invariant under the action. It is well-known that by averaging, we can choose an $\mathbb{S}^1$-invariant contact form α for ξ and that there exists a symplectic form ω on B such that $\pi^*\omega = d\alpha$, where π is the bundle projection $\pi\colon P \to B$. The symplectic form ω represents the image of the Euler class $e(P)$ in $H^2(B,\mathbb{R})$, and hence P cannot be a trivial bundle (see [5]). The manifold (P_L,α) is usually called the **pre-quantization of the symplectic manifold** (B,ω) (or the **Boothby-Wang manifold**).

Let L be a Lagrangian submanifold in (B,ω), and let $P_L := \pi^{-1}(L)$ be the fibration over L. Note first that in this situation, we have $\omega|_{TL} = 0$, so that $e(P_L) = e(P)|_L$ will automatically either vanish or be a torsion class. We assume that $e(P_L) = 0$, so that the fibration P_L will be trivial, and we can find a section $\sigma\colon L \to P_L$.

We have $(\alpha\wedge d\alpha)|_{TP_L} = (\alpha\wedge\pi^*\omega)|_{TP_L} \equiv 0$, so that ξ induces a Legendrian foliation $\mathcal{F}$ on P_L. Furthermore, since the infinitesimal generator X_φ of the circle action satisfies $\alpha(X_\varphi) \equiv 1$, it follows that $\mathcal{F}$ is everywhere regular. Using the section σ, we can identify P_L with $\mathbb{S}^1 \times L$, and write $\alpha|_{TP_L}$ as

$$d\varphi + \beta,$$

where φ is the coordinate on the circle and β is a closed 1-form on L. The leaves of the foliation $\mathcal{F}$ are local sections, but they need not be global ones, and usually these leaves will not even be compact. Instead the proper way to think of them is as the horizontal lift of the flat connection 1-form $\alpha|_{TP_L}$.

Choose any loop $\gamma \subset L$ based at a point $p_0 \in L$. We want to lift $\gamma(t)$ to a path $\widetilde{\gamma}(t) = (e^{i\varphi(t)}, \gamma(t))$ in $P_L \cong \mathbb{S}^1 \times L$ that is always tangent to a leaf of $\mathcal{F}$, so that

$$\widetilde{\gamma}'(t) = \big(-\beta\big(\gamma'(t)\big), \gamma'(t)\big).$$

In particular start and end point of $\widetilde{\gamma}$ are related by the monodromy

$$C_\gamma := -\int_\gamma \beta,$$

that means, if $\widetilde{\gamma}$ starts at $(e^{i\varphi_0}, p_0) \in \mathbb{S}^1 \times L$, then its end point will be $(e^{i(\varphi_0 + C_\gamma)}, p_0)$.

Note that since the connection is flat, that means, β is closed, two homologous paths from p_0 to p_1 will lift the end point in the same way. Thus we have a well-defined map

$$H_1(L, \mathbb{Z}) \to \mathbb{S}^1.$$

The leaves of the Legendrian foliation will only be compact, if the image of this map is discrete.

Note that the embedding of $H^1(L, \mathbb{Q}) \to H^1(L, \mathbb{R})$ is dense, and so we find a 1-form β' arbitrarily close to β such that the monodromy for every loop in L will be a rational number. Clearly, we can extend $\delta = \beta' - \beta$ to a 1-form defined on the whole bundle P, and suppose that δ is sufficiently small so that $\alpha' = \alpha + \delta$ determines a contact structure that is isotopic to the initial one. We may hence suppose that after a small perturbation of α that the Legendrian foliation on P_L is given by $d\phi + \beta'$.

In fact, since $H_1(L, \mathbb{Z})$ is finitely generated, we find a number $c \in \mathbb{Q}$ such that all possible values of the monodromy are a multiple of c, and by slightly perturbing α we obtain a regular Legendrian foliation on P_L, with compact leaves.

The second example gives a Legendrian foliation with a codimension 1 singular set.

Example 2.7. Let L be any smooth $(n+1)$-dimensional manifold with a Riemannian metric g. It is well-known that the unit cotangent bundle $\mathbb{S}(T^*L)$ carries a contact structure given as the kernel of the canonical 1-form λ_{can}. The fibers of this bundle are Legendrian spheres, hence if we choose any smooth regular loop $\gamma\colon \mathbb{S}^1 \to L$, and if we study the fibers lying over this path, we obtain the submanifold $N_\gamma := \pi^{-1}(\gamma)$ that has a singular Legendrian foliation.

In fact, we can naturally decompose $T^*L|_\gamma$ into the two subsets U_+ and U_- defined as

$$U_\pm = \left\{\nu \in N_\gamma \,\middle|\, \pm\nu\left(\gamma'\right) \geq 0\right\}.$$

These sets correspond in each fiber of N_γ to opposite hemispheres. The singular set of the Legendrian foliation on N_γ is $U_+ \cap U_-$, and that the regular leaves correspond to the intersection of each fiber of N_γ with the interior of U_+ and U_-. In particular, if N_γ is orientable, we obtain that it can be written as

$$\left(\mathbb{S}^1 \times \mathbb{S}^n, x_0\, d\varphi\right),$$

where φ is the coordinate on $\mathbb{S}^1$, and $(x_0, \dots, x_n)$ are the coordinates on $\mathbb{S}^n$.

Using the results of Section 2.2.2, we can perturb N_γ to a submanifold with a regular Legendrian foliation composed of two Reeb components.

2.4. Legendrian Open Books

Even though we discussed Legendrian foliations quite generally, we will only be interested in two special types: *Legendrian open books* introduced in [29] and *bordered Legendrian open books* introduced in [21]. Both objects were defined with the aim of generalizings results from 3-dimensional contact topology that hold for the 2-sphere with standard foliation and the overtwisted disk respectively [4, 9, 17, 18].

Definition. Let N be a closed manifold. An **open book** on N is a pair (B, ϑ) where:

- The **binding** B is a nonempty codimension 2 submanifold in the interior of N with trivial normal bundle.
- $\vartheta\colon N\setminus B\to\mathbb{S}^1$ is a fibration, which coincides in a neighborhood $B\times\mathbb{D}^2$ of $B=B\times\{0\}$ with the normal angular coordinate.

Definition. If N is a compact manifold with nonempty boundary, then a **relative open book** on N is a pair (B,ϑ) where:

- The **binding** B is a nonempty codimension 2 submanifold in the interior of N with trivial normal bundle.
- $\vartheta\colon N\setminus B\to\mathbb{S}^1$ is a fibration whose fibers are transverse to ∂N, and which coincides in a neighborhood $B\times\mathbb{D}^2$ of $B=B\times\{0\}$ with the normal angular coordinate.

We are interested in studying contact manifolds with submanifolds with a Legendrian foliation that either define an open book or a relative open book.

Definition. A closed submanifold N carrying a Legendrian foliation $\mathcal{F}$ in a contact manifold (M,ξ) is a **Legendrian open book** (abbreviated **Lob**), if N admits an open book (B,ϑ), whose fibers are the regular leaves of the Legendrian foliation (the binding is the singular set of $\mathcal{F}$).

Definition. A compact submanifold N with boundary in a contact manifold (M,ξ) is called a **bordered Legendrian open book** (abbreviated **bLob**), if N carries a Legendrian foliation $\mathcal{F}$ and if it has a relative open book (B,ϑ) such that:

(i) the regular leaves of $\mathcal{F}$ lie in the fibers of θ,

(ii) $\operatorname{Sing}(\mathcal{F})=\partial N\cup B$.

A contact manifold that contains a bLob is called PS**-overtwisted**.

Example 2.8.

(i) Every Lob in a contact 3-manifold is diffeomorphic to a 2-sphere with the binding consisting of the north and south poles, and the fibers being the longitudes. This special type of Lob has been studied extensively and has given several important applications, see for example [4, 9, 17, 18]. It is easy to find such Lobs locally, for example, the unit sphere in $\mathbb{R}^3$ with the standard contact structure $\xi = \ker(dz + x\,dy - y\,dx)$.

(ii) A bLob in a 3-dimensional contact manifold is an overtwisted disk (with singular boundary).

(iii) In higher dimensions, the plastikstufe had been introduced as a filling obstruction [27], but note that a plastikstufe is just a specific bLob that is diffeomorphic to $\mathbb{D}^2 \times B$, where the fibration is the one of an overtwisted disk (with singular boundary) on the $\mathbb{D}^2$-factor, extended by a product with a closed manifold B. Topologically a bLob might be *much* more general than the initial definition of the plastikstufe. For example, a plastikstufe in dimension 5 is always diffeomorphic to a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ while a 3-manifold admits a relative open book if and only if its boundary is a nonempty union of tori.

The importance of the previous definitions lie in the following two theorems, which will be proved in Section 4.

Theorem A ([21, 27])**.** *Let (M,ξ) be a contact manifold that contains a* bLob *N, then M does not admit any semi-positive weak symplectic filling (W,ω) for which $\omega|_{TN}$ is exact.*

The statement above is a generalization of the analogous statement found first for the overtwisted disk in [9, 17].

Remark 2.9. A bLob obstructs always (semi-positive) *strong* symplectic filling, because in that case the restriction of ω to N is exact.

Remark 2.10. In dimension 4 and 6, every symplectic manifold is automatically semi-positive.

Theorem B ([29])**.** *Let (M,ξ) be a contact manifold of dimension $(2n+1)$ that contains a* Lob *N. If M has a weak symplectic filling (W,ω) that is symplectically aspherical, and for which $\omega|_{TN}$ is exact, then it follows that N represents a trivial class in $H_{n+1}(W,\mathbb{Z}_2)$. If the first and second Stiefel-Whitney classes $w_1(N)$ and $w_2(N)$ vanish, then we obtain that N must be a trivial class in $H_{n+1}(W,\mathbb{Z})$.*

Remark 2.11. The methods from [18] can be generalized for Theorem A, see [2], and for Theorem B, see [29], to find closed contractible Reeb orbits.

2.5. Examples of bLobs

The most important result of these notes is the construction of non-fillable manifolds in higher dimensions. The first such manifolds were obtained by Presas in [33], and modifying his examples it was soon possible to show that every contact structure can be converted into one that is PS-overtwisted [35].

This result was reproved and generalized in [11], where it was shown that we may modify a contact structure into one that is PS-overtwisted without changing the homotopy class of the underlying almost contact structure.

A very nice explicit construction in dimension 5 that is similar to the 3-dimensional Lutz twist was given in [26]. In [21] the construction was extended and produced examples that are not PS-overtwisted but share many properties with 3-manifold that have positive Giroux torsion.

The following unpublished construction is due to Francisco Presas who explained it to me during a stay in Madrid. It is probably the easiest way to produce a closed PS-overtwisted manifolds of arbitrary dimensions.

Theorem 2.12 (Fran Presas). *Let (M_1, ξ_1) and (M_2, ξ_2) be contact manifolds of dimension $2n+1$ that both contain a PS-overtwisted submanifold (N, ξ_N) of codimension 2 with trivial normal bundle. The* ***fiber sum*** *of M_1 and M_2 along N is a PS-overtwisted $(2n+1)$-manifold.*

Proof. Let α_N be a contact form for ξ_N. The manifold N has neighborhoods $U_1 \subset M_1$ and $U_2 \subset M_2$ that are contactomorphic to

$$\mathbb{D}^2_{\sqrt{\varepsilon}} \times N$$

with contact structure given as the kernel of the 1-form $\alpha_N + r^2\,d\varphi$ [12, Theorem 2.5.15].

We can remove the submanifold $\{0\} \times N$ in this model, and do a reparametrization of the r-coordinate by $s = r^2$ to bring the neighborhood into the form

$$(0, \varepsilon) \times \mathbb{S}^1 \times N$$

with contact form $\alpha_N + s\,d\varphi$. We extend $M_1 \setminus N$ and $M_2 \setminus N$ by attaching the negative s-direction to the model collar, so that we obtain a neighborhood

$$\big((-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times N,\ \alpha_N + s\,d\varphi\big).$$

Denote these extended manifolds by $(\widetilde{M}_1, \widetilde{\xi}_1)$ and $(\widetilde{M}_2, \widetilde{\xi}_2)$, and glue them together using the contactomorphism

$$(-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times N \to (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times N$$
$$(s, \varphi, p) \mapsto (-s, -\varphi, p).$$

We call the contact manifold (M', ξ') that we have obtained this way the **fiber sum** of M_1 and M_2 along N.

If S is a bLob in N, then it is easy to see that $\{0\} \times \mathbb{S}^1 \times S$ is a bLob in the model neighborhood $(-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times N$. ■

With this proposition, we can now construct non-fillable contact manifolds of arbitrary dimension. Every oriented 3-manifold admits an overtwisted contact structure in every homotopy class of almost contact structures.

Let (M, ξ) be a compact manifold, let α_M be a contact form for ξ. A fundamental result due to Emmanuel Giroux gives the existence of a compatible open book decomposition for M [16]. Using this open book decomposition, it is easy to find functions $f, g\colon M \to \mathbb{R}$ such that

$$\big(M \times \mathbb{T}^2, \ker(\alpha_M + f\,dx + g\,dy)\big)$$

is a contact structure, see [6], where (x, y) denotes the coordinates on the 2-torus. The fibers $M \times \{z\}$ are contact submanifold with trivial normal bundle, so that in particular if (M, ξ) is PS-overtwisted, we can apply the construction above to glue two copies of $M \times \mathbb{T}^2$ along a fiber $M \times \{z\}$. This way, we obtain a PS-overtwisted contact structure on $M \times \Sigma_2$, where Σ_2 is a genus 2 surface.

Using this process inductively, we find closed PS-overtwisted contact manifolds of any dimension ≥ 3.

Note that in dimension 5, we can find more easily examples to which we can apply Theorem 2.12, so that it is not necessary to rely on [6]. Let (M, ξ) be an overtwisted 3-manifold with contact form α. After normalizing α with respect to a Riemannian metric, it describes a section

$$\sigma_\alpha\colon M \to \mathbb{S}\big(T^*M\big)$$

in the unit cotangent bundle. It satisfies the fundamental relation $\sigma_\alpha^* \lambda_{\mathrm{can}} = \alpha$, hence it gives a contact embedding of (M, ξ) into $(\mathbb{S}(T^*M), \ker \lambda_{\mathrm{can}})$.

For trivial normal bundle, this allows us to glue with Theorem 2.12 two copies together and obtain a PS-overtwisted 5-manifold.

3. Behavior of J-Holomorphic Disks Imposed by Convexity

The following section only fixes notation, and explains some well-known facts about J-convexity. With some basic knowledge on J-holomorphic curves, one can safely skip it and continue directly to Section 3.2, which describes the local models around the binding and the boundary of the Lobs and bLobs and the behavior of holomorphic disks that lie nearby. The next two sections include a description about moduli spaces and their basic properties, but most results are only explained in an intuitive way without giving any proofs. The fifth section deals with the Gromov compactness of the considered moduli spaces, and the chapter finishes proving the two applications that relate a Lob or a bLob to the topology of a symplectic filling.

3.1. Almost Complex Structures and Maximally Foliated Submanifolds

3.1.1. Preliminaries: J-Convexity.

The Maximum Principle. One of the basic ingredients in the theory of J-holomorphic curves with boundary is the maximum principle, which we will now briefly describe in the special case of Riemann surfaces. We assume in this section that (Σ, j) is a Riemann surface that does not need to be compact and may or may not have boundary. We define the differential operator d^j that associates to every smooth function $f\colon \Sigma \to \mathbb{R}$ a 1-form given by

$$\left(d^j f\right)(v) := -df(jv)$$

for $v \in T\Sigma$.

Definition. We say that a function $f\colon (\Sigma, j) \to \mathbb{R}$ is

(a) **harmonic** if the 2-form $dd^j f$ vanishes everywhere,

(b) it is **subharmonic** if the 2-form $dd^j f$ is a positive volume form with respect to the orientation defined by (v, jv) for any non-vanishing vector $v \in T\Sigma$.

(c) If f only satisfies

$$dd^j f(v, jv) \geq 0$$

then we call it **weakly subharmonic**.

In particular, if we choose a complex chart $(U \subset \mathbb{C}, \phi)$ for Σ with coordinate $z = x + iy$, we can represent f by $f_U := f \circ \phi^{-1} : U \to \mathbb{R}$. The 2-form $dd^j f$ simplifies on this chart to $dd^i f_U$, because ϕ is holomorphic with respect to j and i, and we can write $dd^i f_U$ in the form $(\triangle f_U)\, dx \wedge dy$, where the Laplacian is defined as

$$\triangle f_U = \frac{\partial^2 f_U}{\partial x^2} + \frac{\partial^2 f_U}{\partial y^2}.$$

Note that f_U is subharmonic, if and only if $dd^i f_U(\partial_x, \partial_y) > 0$, that means, $\triangle f_U > 0$.

For strictly subharmonic functions, it is obvious that they may not have any interior maxima, because the Hessian needs to be negative definite at any such point. We really need to consider both weakly subharmonic functions and the behavior at boundary points. To prove the maximum principle in this more general setup, we use the following technical result.

Lemma 3.1. *Let $f : \mathbb{D}^2 \subset \mathbb{C} \to \mathbb{R}$ be a function that is C^1 on the closed unit disk, and both C^2 and weakly subharmonic on the interior of the disk. Assume that f takes its maximum at a boundary point $z_0 \in \partial\mathbb{D}^2$ and is everywhere else strictly smaller than $f(z_0)$. Choose an arbitrary vector $X \in T_{z_0}\mathbb{C}$ at z_0 pointing transversely out of $\overline{\mathbb{D}}^2$.*

Then the derivative $\mathcal{L}_X f(z_0)$ in X-direction needs to be strictly *positive.*

Proof. We will perturb f to a *strictly* subharmonic function making use of the auxiliary function $g : \overline{\mathbb{D}}^2 \to \mathbb{R}$ defined by (see Figure 3)

$$g(r) = r^4 - \frac{9}{4} r^2 + \frac{5}{4}.$$

The function g vanishes along the boundary $\partial\mathbb{D}^2$, and its derivative in any direction v that is positively transverse to the boundary $\partial\mathbb{D}^2$ is strictly negative, because $\partial_\varphi g = 0$ and because

$$r\partial_r g = \frac{1}{2} r^2 \big(8r^2 - 9\big).$$

Finally, we also see that g is strictly subharmonic on the open annulus $\mathbb{A} = \{z \in \mathbb{C} \mid 3/4 < |z| < 1\}$ as

$$\triangle g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 16r^2 - 9.$$

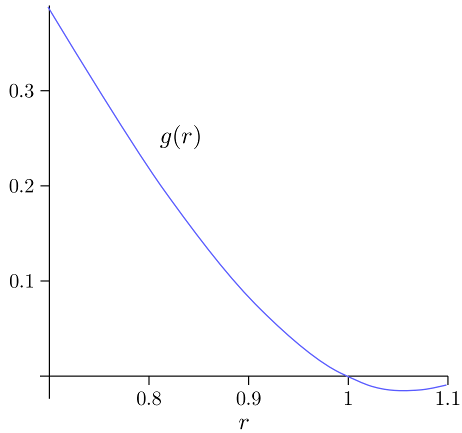

Fig. 3. The function $g(r)$ is subharmonic, vanishes on the boundary, and has negative radial derivative

We slightly perturb f by setting $f_\varepsilon = f + \varepsilon g$ for small $\varepsilon > 0$, and we additionally restrict f_ε to the closure of the annulus $\mathbb{A}$. Note in particular that f_ε must take its maximum on $\partial\mathbb{A}$, because f_ε is *strictly* subharmonic on the interior of $\mathbb{A}$ so that one of $\frac{\partial^2 f_\varepsilon}{\partial x^2}$ or $\frac{\partial^2 f_\varepsilon}{\partial y^2}$ must be strictly positive. This contradicts existence of possible interior maximum points. The functions f_ε are equal to f along the outer boundary of $\mathbb{A}$ so that the maximum of f_ε will either lie in z_0 or on the inner boundary of $\mathbb{A}$.

The initial function f is by assumption strictly smaller than $f(z_0)$ on the inner boundary of the annulus and by choosing ε sufficiently small, it follows that the perturbed function f_ε will still be strictly smaller than $f_\varepsilon(z_0) = f(z_0)$. Thus z_0 will also be the maximum of f_ε. Let X be a vector at z_0 that points transversely out of $\overline{\mathbb{D}}^2$. The derivative $\mathcal{L}_X f_\varepsilon$ at z_0 cannot be strictly negative, because z_0 is a maximum, and so since

$$0 \le \mathcal{L}_X f_\varepsilon = \mathcal{L}_X f + \varepsilon \mathcal{L}_X g,$$

the derivative of f in X-direction has to be *strictly* positive, yielding the desired result. ■

Now we are prepared to state and prove the maximum principle.

Theorem 3.2 (Weak maximum principle). *Let (Σ, j) be a connected compact Riemann surface. A weakly subharmonic function $f\colon \Sigma \to \mathbb{R}$ that attains its maximum at an interior point $z_0 \in \Sigma \setminus \partial\Sigma$ must be constant.*

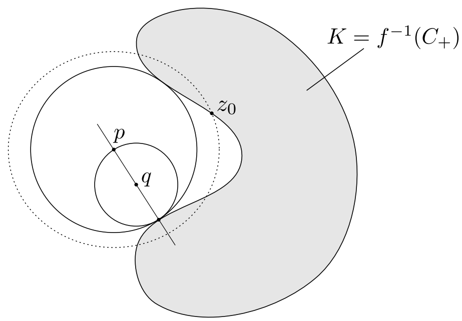

Fig. 4. Constructing a disk that has a single maximum on its boundary

Proof. The proof is classical and holds in much greater generality (see for example [14]). Nonetheless we will explain it in the special case needed by us to show that it only uses elementary techniques. The strategy is simply to find a closed disk in the interior of the Riemann surface with the properties required by Lemma 3.1. Then the function f increases in radial direction further, so that the maximum point was not really a maximum.

More precisely, assume f not to be constant, and to have a maximum at an interior point $z_+ \in \Sigma \setminus \partial\Sigma$ with $C_+ := f(z_+)$. The subset $K := f^{-1}(C_+) \cap \mathring{\Sigma}$ is closed in $\mathring{\Sigma}$. For every point $z \in K$, we find an $R_z > 0$ such that the open disk $D_{R_z}(z)$ is contained in some complex chart. There must be a point $z_0 \in K$ for which the half sized disk $D_{R_{z_0}/2}(z_0)$ intersects $\mathring{\Sigma} \setminus K$, for otherwise K would be open and hence as $\mathring{\Sigma}$ is connected, $K = \mathring{\Sigma}$.

Let p be a point in $D_{R_{z_0}/2}(z_0) \setminus K$ (see Figure 4). It lies so close to z_0 that the entire closed disk of radius $|p - z_0|$ lies in the chart U, and then we can choose first a disk $\overline{\mathbb{D}_R(p)}$ centered at p, where R is the largest number for which the *open* disk does not intersect $f^{-1}(C_+)$. We are interested in finding a closed disk that intersects $f^{-1}(C_+)$ at a *single* boundary point: For this let q be the mid point between p and one of the boundary points in $\partial\mathbb{D}^2_R(p) \cap f^{-1}(C_+)$. The disk $\mathbb{D}^2_{R/2}(q)$ touches $f^{-1}(C_+)$ at exactly one point.

This smaller disk satisfies the conditions of Lemma 3.1, and so it follows that the derivative of f at the maximum is strictly positive in radial direction. But since this point lies in the interior of Σ, it follows that f still increases in that direction and hence this point cannot be the maximum. Of course, the whole existence of the disk was based on the assumption that f was not constant, so we obtain the statement of the theorem. ■

If Σ has boundary, we also get the following refinement.

Theorem 3.3 (Boundary point lemma)**.** *Let $f\colon \Sigma \to \mathbb{R}$ be a weakly subharmonic function on a connected compact Riemann surface (Σ, j) with boundary. Assume f takes its maximum at a point $z_+ \in \partial\Sigma$, then f will either be constant or the derivative at z_+*

$$\mathcal{L}_X f(z_+) > 0$$

in any outward direction $X \in T_{z_+}\Sigma$ has to be strictly positive.

Proof. Denote the maximum $f(z_+)$ by C_+. By the maximum principle, Theorem 3.2, we know that f will be constant if there is a point $z \in \Sigma \setminus \partial\Sigma$ for which $f(z) = C_+$. We can thus assume that for all $z \notin \partial\Sigma$, we have $f < C_+$. Using a chart U around the point z_+, that represents an open set in $\mathbb{H} := \{z \in \mathbb{C} \,|\, \operatorname{Im} z \geq 0\}$, such that z_+ corresponds to the origin, we can easily find a small disk in $\mathbb{H}$ that touches $\partial\mathbb{H}$ only in 0, and hence allows us to directly apply Lemma 3.1 to complete the proof. ■

Plurisubharmonic Functions. We will now explain the connection between the previous section and contact topology.

Let (W, J) be an almost complex manifold, that means that J is a section of the endomorphism bundle $\operatorname{End}(TM)$ with $J^2 = -\mathbf{1}$. Define the differential $d^J f$ of a smooth function $f\colon W \to \mathbb{R}$ as before by

$$\bigl(d^J f\bigr)(v) := -df(J \cdot v)$$

for any vector $v \in TW$.

Definition. We say that a function $h\colon W \to \mathbb{R}$ is ***J*-plurisubharmonic**, if the 2-form

$$\omega_h := dd^J h$$

evaluates positively on J-complex lines, that means that $\omega_h(v, Jv)$ is strictly positive for every non-vanishing vector $v \in TW$.

If ω_h vanishes, then we say that h is ***J*-harmonic**.

Remark 3.4.

(1) If h is J-plurisubharmonic, then ω_h is an exact symplectic form that tames J.

(2) If ω_h is only non-negative, then we say that h is **weakly J-plurisubharmonic**. This notion might be for example interesting in the context of confoliations.

Let (Σ, j) be a Riemann surface that does not need to be compact, and may or may not have boundary. We say that a smooth map $u\colon \Sigma \to W$ is **J-holomorphic**, if its differential commutes with the pair (j, J), that means, at every $z \in \Sigma$ we have

$$J \cdot Du = Du \cdot j.$$

Using the commutation relation, we easily check for every J-holomorphic map u and every smooth function $f\colon U \to \mathbb{R}$ the formula

$$u^* d^J f = -df \cdot J \cdot Du = -df \cdot Du \cdot j = -d(f \circ u) \cdot j = d^j (f \circ u) = d^j u^* f. \tag{3.1}$$

Corollary 3.5. *If $u\colon (\Sigma, j) \to (W, J)$ is J-holomorphic and $h\colon W \to \mathbb{R}$ is a J-plurisubharmonic function, then $h \circ u$ will be weakly subharmonic, because*

$$dd^j (h \circ u) = d\, u^* d^J h = u^* dd^J h$$

and because the differential Du commutes with the complex structures, so that

$$dd^j (h \circ u)(v, jv) = dd^J h(Du \cdot v, J \cdot Du \cdot v) \geq 0$$

for every vector $v \in T\Sigma$. The function is strictly positive precisely at points $z \in U$, where Du_z does not vanish.

The maximum principle restricts severely the behavior of holomorphic maps:

Corollary 3.6. *Let $u\colon (\Sigma, j) \to (W, J)$ be a J-holomorphic map and $h\colon W \to \mathbb{R}$ be a J-plurisubharmonic function. If u is not a constant map then $h \circ u\colon \Sigma \to \mathbb{R}$ will never take its maximum on the interior of Σ.*

Proof. Since $h \circ u$ is weakly subharmonic, it follows immediately from the maximum principle (Theorem 3.2) that $h \circ u$ must be constant if it takes its maximum in the interior of Σ, and hence $d(h \circ u) = 0$. On the other hand, we know that if there were a point $z \in \Sigma$ with $D_z u \neq 0$, then $\omega_h(Du \cdot v, Du \cdot jv)$ would need to be strictly positive for non-vanishing vectors. This is not possible though, because $u^* \omega_h = dd^j (h \circ u) = 0$. ■

Corollary 3.7. *Let (Σ, j) be a Riemann surface with boundary, $u\colon (\Sigma, j) \to (W, J)$ a J-holomorphic map and $h\colon W \to \mathbb{R}$ be a J-plurisubharmonic function. If $h \circ u\colon \Sigma \to \mathbb{R}$ takes its maximum at $z_0 \in \partial\Sigma$ then it follows either that $d(h \circ u)(v) > 0$ for every vector $v \in T_{z_0}\Sigma$ pointing transversely out of the surface, or u will be constant.*

Proof. The proof is analogous to the previous one, but uses the boundary point lemma (Theorem 3.3) instead of the simple maximum principle. ■

Remark 3.8. Note that if h is only *weakly* plurisubharmonic, then we can only deduce in the two corollaries above that u has to lie in a level set of h, and not that u itself must be constant.

Contact Structures as Convex Boundaries. Now we will finally explain the relation between plurisubharmonic functions and contact manifolds.

Definition. Let (W, J) be an almost complex manifold with boundary. We say that W has ***J*-convex boundary**, if there exists a smooth function $h\colon W \to (-\infty, 0]$ with the properties

- h is J-plurisubharmonic on a *neighborhood* of ∂W,
- h is a regular equation for ∂W, that means, 0 is a regular value of h and $\partial W = h^{-1}(0)$.

Note that the function h in the definition takes its maximum on ∂W, so that it must be strictly increasing in outward direction.

We will show that the boundary of an almost complex manifold is J-convex if and only if it carries a natural cooriented contact structure (whose conformal symplectic structure tames J). Remember that we are always assuming our contact manifolds to be cooriented. Hence the manifold is oriented, and its contact structure will have a natural conformal symplectic structure.

Definition. Let M be a codimension 1 submanifold in an almost complex manifold (W, J). The **subbundle of complex tangencies** of M is the J-complex subbundle

$$\xi := TM \cap (J \cdot TM).$$

Proposition 3.9. *Let (W, J) be an almost complex manifold with boundary $M := \partial W$ and let ξ be the subbundle of complex tangencies of M. We have the following equivalence*:

(1) *The boundary M is J-convex.*

(2) *The subbundle ξ is a cooriented contact structure whose natural orientation is compatible with the boundary orientation of M, and whose natural conformal symplectic structure tames $J|_\xi$.*

Proof. To prove the direction "(1) ⇒ (2)", let h be the J-plurisubharmonic equation of M that exists by assumption. A straight forward calculation shows that the kernel of the 1-form $\alpha := d^J h|_{TM}$ is precisely ξ, and in particular that α does not vanish. Furthermore $d\alpha|_{TM} = \omega_h|_{TM}$ is a symplectic structure on ξ that tames $J|_\xi$, so that α is a contact form. To check that $\alpha \wedge d\alpha^{n-1}$ is a positive volume form with respect to the boundary orientation induced on M by (W, J), let R_α be the Reeb field of α, and define a vector field $Y = -JR_\alpha$. The field Y is positively transverse to ∂W, because $\mathcal{L}_Y h = dh(Y) = d^J h(R_\alpha) = \alpha(R_\alpha) = 1$ is positive. Choosing a basis $(v_1, \dots, v_{2n-2})$ for ξ at a point $p \in M$, we compute

$$\alpha \wedge d\alpha^{n-1}(R_\alpha, v_1, \dots, v_{2n-2}) = d\alpha^{n-1}(v_1, \dots, v_{2n-2}) = \omega_h^{n-1}(v_1, \dots, v_{2n-2}).$$

Similarly, we obtain

$$\begin{aligned}\omega_h^n(Y, R_\alpha, v_1, \dots, v_{2n-2}) &= n\omega_h(Y, R_\alpha) \cdot \omega_h^{n-1}(v_1, \dots, v_{2n-2}) \\ &= n\omega_h(R_\alpha, JR_\alpha) \cdot \omega_h^{n-1}(v_1, \dots, v_{2n-2}),\end{aligned}$$

where we have used that $\omega_h(R_\alpha, v_j) = d\alpha(R_\alpha, v_j) = 0$ for all $j \in \{1, \dots, n-1\}$. The first term $\omega_h(R_\alpha, JR_\alpha)$ is positive, and hence $\alpha \wedge d\alpha^{n-1}$ and $\iota_Y \omega_h^n$ induce identical orientations on M.

To prove the direction "(2) ⇒ (1)", choose any collar neighborhood $(-\varepsilon, 0] \times M$ for the boundary, and let t be the coordinate on $(-\varepsilon, 0]$. First note that $\alpha = d^J t|_{TM}$ is a non-vanishing 1-form with kernel ξ, so in particular it will be contact. Let R_α be the Reeb field of α, and set $Y := -JR_\alpha$. As before, the field Y is positively transverse to M, because of $\mathcal{L}_Y t = -dt(JR_\alpha) = \alpha(R_\alpha) = 1$.

Let C be a large constant, whose size will be determined below, and set $h(t, p) := e^{Ct} - 1$. Clearly, h is a regular equation for M, and we claim that for sufficiently large C, h will be a J-plurisubharmonic function.

Let $v \in T_pW$ be any non-vanishing vector at $p \in M$ and represent it as

$$v = aY + bR_\alpha + cZ,$$

where Y and R_α were defined above, and $Z \in \xi$ is a vector in the contact structure that has been normalized such that $d\alpha(Z, JZ) = \omega_t(Z, JZ) = 1$. Note that the 1-form $\alpha_C = d^J h|_{TM} = Ce^{Ct}\alpha$ is a contact form that represents the same coorientation as α.

We compute $\omega_h = dd^J h = Ce^{Ct}(\omega_t + C\,dt \wedge d^J t)$, which simplifies for $t = 0$ further to $\omega_h = C(\omega_t + C\,dt \wedge d^J t)$ and so we have

$$\omega_h(R_\alpha, \cdot) = C\big(\omega_t(R_\alpha, \cdot) - C\,dt\big) \quad \text{and} \quad \omega_h(Y, \cdot) = C\big(\omega_t(\cdot, JR_\alpha) + C\,d^J t\big).$$

This implies $\omega_h(R_\alpha, Z) = \omega_h(R_\alpha, JZ) = 0$ for all $Z \in \xi$, and $\omega_h(Y, R_\alpha) = C^2 + C\omega_t(R_\alpha, JR_\alpha)$ can be made arbitrarily large by increasing the size of C. With these relations we obtain

$$\begin{aligned}
&\omega_h(v, Jv) \\
&\quad = \omega_h(aY + bR_\alpha + cZ, aR_\alpha - bY + cJZ) \\
&\quad = \big(a^2 + b^2\big)\omega_h(Y, R_\alpha) + c^2\omega_h(Z, JZ) + ac\omega_h(Y, JZ) + bc\omega_h(Y, Z) \\
&\quad = \big(a^2 + b^2\big)\big(C^2 + O(C)\big) + C\big(c^2\omega_t(Z, JZ) + ac\omega_t(Y, JZ) + bc\omega_t(Y, Z)\big)
\end{aligned}$$

and setting $A_a = \omega_t(Y, JZ)$ and $A_b = \omega_t(Y, Z)$ and using that $\omega_t(Z, JZ) = 1$

$$\begin{aligned}
&= \big(a^2 + b^2\big)\big(C^2 + O(C)\big) + C\big(c^2 + A_a ac + A_b bc\big) \\
&= \big(a^2 + b^2\big)\big(C^2 + O(C)\big) + \frac{C}{2}\big((c + aA_a)^2 - a^2A_a^2 + (c + bA_b)^2 - b^2A_b^2\big) \\
&= a^2\big(C^2 + O(C)\big) + b^2\big(C^2 + O(C)\big) + \frac{C}{2}\big((c + aA_a)^2 + (c + bA_b)^2\big).
\end{aligned}$$

By choosing C large enough, we can ensure that the a^2- and b^2-coefficients are both positive. Then it is obvious from the computation above that ω_h tames J, and hence h is J-plurisubharmonic. ■

Legendrian Foliations in Convex Boundaries.

Definition. A **totally real submanifold** N of an almost complex manifold (W, J) is a submanifold of dimension $\dim N = \frac{1}{2}\dim W$ that is not tangent to any J-complex line, that means, $TN \cap (JTN) = \{0\}$, which is equivalent to requiring

$$TW|_N = TN \oplus (JTN).$$

Proposition 3.10. *Let (W, J) be an almost complex manifold with J-convex boundary (M, ξ). Assume N is a submanifold of M for which the complex tangencies ξ induce the Legendrian foliation $\mathcal{F} = TN \cap \xi$. Then it is easy to check that $N \setminus \operatorname{Sing}(\mathcal{F})$ is totally real.*

Proof. If $X \in TN$ is a non-vanishing vector with JX also in TN, then in particular

$$X \in TN \cap (JTN) \subset TM \cap (JTM) = \xi,$$

so that X and JX have to lie in $\mathcal{F}$. The 2-form $d\alpha$ tames $J|_\xi$ so that $d\alpha(X, JX) > 0$, but $d\alpha|_{\mathcal{F}}$ vanishes at regular points of the foliation, and hence X must be 0. ■

We will next study the restrictions imposed by a Legendrian foliation on J-holomorphic curves. Let (Σ, j) be a compact Riemann surface with boundary, and let A be a subset of an almost complex manifold (W, J). We introduce for J-holomorphic maps $u\colon \Sigma \to W$ with $u(\partial\Sigma) \subset A$ the notation

$$u\colon (\Sigma, \partial\Sigma, j) \to (W, A, J).$$

Note that we are always supposing that u is at least C^1 along the boundary.

Corollary 3.11. *Let (W, J) be an almost complex manifold with convex boundary (M, ξ). Let $N \hookrightarrow M$ be a submanifold with an induced Legendrian foliation $\mathcal{F}$, and let u be a J-holomorphic map*

$$u\colon (\Sigma, \partial\Sigma, j) \to \big(W, N \setminus \operatorname{Sing}(\mathcal{F}), J\big).$$

If there is an interior point $z_0 \in \Sigma \setminus \partial\Sigma$ at which u touches M, or if ∂u is not positively transverse to $\mathcal{F}$, then u is a constant map.

Proof. Choose a J-plurisubharmonic function $h\colon W \to \mathbb{R}$ that is a regular equation for M. The first implication follows directly from Corollary 3.6, because z_0 would be an interior maximum for $h \circ u$.

For the second implication note first that $h \circ u$ takes its maximum on $\partial\Sigma$ so that if u is not constant, we have by Corollary 3.7 that the derivative $\mathcal{L}_v(h \circ u)$ is strictly positive for every point $z_1 \in \partial\Sigma$ and every vector $v \in T_{z_1}\Sigma$ pointing out of Σ. Now if $w \in T\Sigma$ is a vector that is tangent to $\partial\Sigma$ such that jw points inward (so that w corresponds to the boundary orientation of $\partial\Sigma$, because $(-jw, w)$ is a positive basis of $T\Sigma$), we obtain

$$\alpha(Du \cdot w) = -dh(JDu \cdot w) = -dh(Du \cdot jw) = -d(h \circ u)(jw) > 0.$$

The boundary of ∂u has thus to be positively transverse to ξ, and so it is in particular positively transverse to the Legendrian foliation $\mathcal{F}$. ■

Note that the result above applies only for holomorphic maps that are C^1 along the boundary.

3.1.2. Preliminaries: ω-Convexity. Above we have explained the notion of J-convexity, and the relevant relationship between contact and almost complex structures. In this section, we want to discuss the notion of ω-convexity, that means the relationship between an (almost) symplectic and a contact structure.

In fact, we are not interested in studying almost complex manifolds for their own sake, but we would like to use the almost complex structure to understand instead a symplectic manifold (W,ω). As initiated by Gromov, we introduce an auxiliary almost complex structure to be able to study J-holomorphic curves in the hope that even though the J-holomorphic curves depend very strongly on the almost complex structure chosen, we'll be able to extract interesting information about the initial symplectic structure.

For this strategy to work, we need the almost complex structure to be **tamed** by ω, that means, we want

$$\omega(X, JX) > 0$$

for every non-vanishing vector $X \in TW$. This tameness condition is important, because it allows us to control the limit behavior of sequences of holomorphic curves (see Section 4.3).

As explained in the previous section, J-convexity is a property that greatly helps us in understanding holomorphic curves in ambient manifolds that have boundary. When (W,ω) is a symplectic manifold with boundary $M = \partial W$, we would thus like to chose an almost complex structure J that is

- tamed by ω, and
- that makes the boundary J-convex.

In particular, if such a J exists, we know that the boundary admits an induced contact structure

$$\xi = TM \cap (J \cdot TM).$$

From the symplectic or contact topological view point, the opposite setup would be more natural though: given a symplectic manifold (W,ω) with contact boundary (M,ξ), can we choose an almost complex structure J that is tamed by ω, and that makes the boundary J-convex such that ξ is the bundle of J-complex tangencies?

The general answer to that question was given in [21].

Definition. Let (M,ξ) be a cooriented contact manifold of dimension $2n-1$, and let (W,ω) be a symplectic manifold whose boundary is M. Let α be a positive contact form for ξ, and assume that the orientation induced by $\alpha \wedge d\alpha^{n-1}$ on M agrees with the boundary orientation of (W,ω). We call (W,ω) a **weak symplectic filling** of (M,ξ), if

$$\alpha \wedge (T\,d\alpha + \omega)^{n-1} > 0$$

for every $T \in [0,\infty)$.

The proofs of the following statements are very lengthy, hence we will omit the proofs referring instead to the Appendix of [21] for more details.

Theorem 3.12. *Let (M,ξ) be a cooriented contact manifold, and let (W,ω) be a symplectic manifold with boundary $M = \partial W$. The following two statements are equivalent*

- *(W,ω) is a weak symplectic filling of (M,ξ).*
- *There exists an almost complex structure J on W that is tamed by ω and that makes M a J-convex boundary whose J-complex tangencies are ξ.*

Furthermore the space of all almost complex structures that satisfy these conditions is contractible (if non-empty).

A weak filling is a notion that is relatively recent in higher dimensions; traditionally it is the concept of a strong symplectic filling that has been studied for a much longer time. Let (W,ω) be a symplectic manifold. A vector field X_L is called a **Liouville vector field**, if it satisfies the equation

$$\mathcal{L}_{X_L}\omega = \omega.$$

Definition. Let (M,ξ) be a cooriented contact manifold, and let (W,ω) be a symplectic manifold whose boundary is M. We call (W,ω) a **strong**

symplectic filling of (M, ξ), if there exists a Liouville vector field X_L on a neighborhood of M such that $\lambda := (\iota_{X_L}\omega)|_{TM}$ is a positive contact form for ξ.

It is easy to see that a strong filling is in particular a weak filling. Note that the symplectic form of a strong filling becomes always exact when restricted to the boundary, but that this needs not be true for a weak filling; if it is then it will usually still not be a strong symplectic filling, but by Corollary 3.15 it can deformed into one.

Lemma 3.13. *Let (W, ω) be a symplectic manifold and let M be a hypersurface (possibly a boundary component of W) together with a non-vanishing 1-form λ. Assume that the restriction of ω to $\ker\lambda$ is symplectic.*

Then there is a tubular neighborhood of M in W that is symplectomorphic to the model

$$\big((-\varepsilon, \varepsilon) \times M, d(t\lambda) + \omega|_{TM}\big),$$

where t is the coordinate on the interval $(-\varepsilon, \varepsilon)$. The 0-slice $\{0\} \times M$ corresponds in this identification to the hypersurface M. If M is a boundary component of W then of course we need to replace the model by $(-\varepsilon, 0] \times M$ or by $[0, \varepsilon) \times M$ depending on whether $\lambda \wedge \omega^{n-1}$ is oriented as the boundary of (W, ω) or not.

For the proof see [21, Lemma 2.6].

Proposition 3.14. *Let (W, ω) be a weak filling of a contact manifold (M, ξ), and let Ω be a 2-form on M that is cohomologous to $\omega|_{TM}$. Choose a positive contact form α for (M, ξ). Then if we allow $C > 0$ to be sufficiently large, we can attach a collar $[0, C] \times M$ to W with a symplectic form ω_C that agrees close to $\{C\} \times M$ with $d(t\alpha) + \Omega$, and such that the new manifold is a weak filling of $(\{t_0\} \times M, \xi)$ for every $t_0 \in [0, C]$.*

The proof can be found in [21, Lemma 2.10].

Corollary 3.15. *Let (W, ω) be a weak symplectic filling of (M, ξ) and assume that ω restricted to a neighborhood of M is an exact symplectic form. Then we may deform ω on a small neighborhood of M such it becomes a strong symplectic filling.*

Proof. Since $\omega|_{TM}$ is exact, we can apply the proposition above with $\Omega = 0$. Afterwards we can isotope the collar back into the neighborhood of the boundary of W. ■

Note that two contact structures that are strongly filled by the same symplectic manifold are isotopic, while a symplectic manifold may be a weak filling of two different contact manifolds. This is true even when the restriction of the symplectic structure to the boundary is exact, see [21, Remark 2.11].

3.2. Holomorphic Curves and Legendrian Foliations

Let (W, J) be an almost complex manifold with J-convex boundary (M, ξ), and let $N \subset M$ be a submanifold carrying a Legendrian foliation $\mathcal{F}$. The aim of this section will be to better understand the behavior of J-holomorphic maps

$$u\colon (\Sigma, \partial\Sigma, j) \to (W, N, J),$$

that lie close to a singular point $p \in \operatorname{Sing}(\mathcal{F})$ of the Legendrian foliation. For this we will assume that J is of a very specific form in a neighborhood of the point p.

3.2.1. Existence of J-Convex Functions Close to Totally Real Submanifolds. As a preliminary tool, we will need the following result.

Proposition 3.16. *Let (W, J) be an almost complex structure that contains a closed totally real submanifold L. Then there exists a smooth function $f\colon W \to [0, \infty)$ with $L = f^{-1}(0)$ that is J-plurisubharmonic on a neighborhood of L. In particular, it follows that $df_p = 0$ at every point $p \in L$.*

Proof. We will first show that we find around every point $p \in L$ a chart U with coordinates $\{(x_1, \dots, x_n; y_1, \dots, y_n)\} \subset \mathbb{R}^{2n}$ such that $L \cap U = \{y_1 = \dots = y_n = 0\}$ and

$$J\frac{\partial}{\partial x_j}\bigg|_{L\cap U} = \frac{\partial}{\partial y_j}\bigg|_{L\cap U}.$$

For this, start by choosing coordinates $\{(x_1, \dots, x_n)\} \subset \mathbb{R}^n$ for the submanifold L around the point p, and consider the associated vector fields

$$Y_1 = J\frac{\partial}{\partial x_1}, \dots, Y_n = J\frac{\partial}{\partial x_n}$$

along L. These vector fields are everywhere linearly independent and transverse to L, hence, we can define a smooth map from a small ball around 0 in $\mathbb{R}^{2n} = \{(x_1, \ldots, x_n; y_1, \ldots, y_n)\}$ to W by

$$y_1 Y_1(x_1, \ldots, x_n) + \cdots + y_n Y_1(x_1, \ldots, x_n) \mapsto \exp(y_1 Y_1 + \cdots + y_n Y_1),$$

where exp is the exponential map for an arbitrary Riemannian metric on W. If the ball is chosen sufficiently small, the map will be a chart with the desired properties.

For such a chart U, we will choose a function

$$f_U \colon U \to [0, \infty), \quad (x_1, \ldots, x_n; y_1, \ldots, y_n) \mapsto \frac{1}{2}\big(y_1^2 + \cdots + y_n^2\big).$$

It is obvious that both the function itself, and its differential vanish along $L \cap U$. Furthermore f is plurisubharmonic close to $L \cap U$, because

$$\begin{aligned} dd^J f_U &= d\big(y_1\, d^J y_1 + \cdots + y_n\, d^J y_n\big) \\ &= dy_1 \wedge d^J y_1 + \cdots + dy_n \wedge d^J y_n + y_1\, dd^J y_1 + \cdots + y_n\, dd^J y_n \end{aligned}$$

simplifies at $L \cap U$ to

$$dd^J f_U\big|_{L\cap U} = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n,$$

where we have used that all y_j vanish, and that $J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$ and $J\frac{\partial}{\partial y_j} = J^2\frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x_j}$. It is easy to check that this 2-form evaluates positively on complex lines along $L \cap U$, and hence also in a small neighborhood of p.

Now to obtain a global plurisubharmonic function as stated in the proposition, cover L with finitely many charts $U_1, \ldots, U_N$, each with a function $f_1, \ldots, f_N$ according to the construction given above. Choose a subordinate partition of unity $\rho_1, \ldots, \rho_N$, and define

$$f = \sum_{j=1}^{N} \rho_j \cdot f_j.$$

The function f and its differential $df = \sum_{j=1}^{N}(\rho_j\, df_j + f_j\, d\rho_j)$ vanish along L so that the only term in

$$\begin{aligned} dd^J f &= d\sum_{j=1}^{N}\big(\rho_j\, d^J f_j + f_j\, d^J \rho_j\big) \\ &= \sum_{j=1}^{N}\big(\rho_j\, dd^J f_j + d\rho_j \wedge d^J f_j + f_j\, dd^J \rho_j + df_j \wedge d^J \rho_j\big) \end{aligned}$$

that survives along L is the first one, giving us along L

$$dd^J f = \sum_{j=1}^{N} \rho_j \, dd^J f_j .$$

This 2-form is positive on J-complex lines, and hence there is a small neighborhood of L on which f is plurisubharmonic. Finally, we modify f to be positive outside this small neighborhood so that we have $L = f^{-1}(0)$ as required. ■

Corollary 3.17. *Let (W, J) be an almost complex structure that contains a closed totally real submanifold L. Then we find a small neighborhood U of L for which every J-holomorphic map*

$$u \colon (\Sigma, \partial\Sigma, j) \to (W, L, J)$$

from a compact Riemann surface needs to be constant if $u(\Sigma) \subset U$.

Proof. Let $f \colon W \to [0, \infty)$ be the function constructed in Proposition 3.16, and let $U \subset (W, J)$ be the neighborhood of L, where f is J-plurisubharmonic. Because $u(\Sigma) \subset U$, we obtain from Corollary 3.6 that $f \circ u$ must take its maximum on the boundary of Σ, but because $f \circ u$ is zero on all of $\partial\Sigma$, it follows that $f \circ u$ will vanish on the whole surface Σ. The image $u(\Sigma)$ lies then in the totally real submanifold L, and this implies that the differential of u vanishes everywhere. Hence there is a $\mathbf{q}_0 \in L$ with $u(z) = \mathbf{q}_0$ for all $z \in \Sigma$. ■

3.2.2. *J*-Holomorphic Curves Close to Elliptic Singularities of a Legendrian Foliation. The aim of this section will be to show that for a suitable choice of an almost complex structure, elliptic singularities give birth to a family of holomorphic disks, and that apart from these disks and their branched covers, no other holomorphic disks may get close to the elliptic singularities.

Before studying the higher dimensional case, we will construct a model situation for a 4-dimensional almost complex manifold with convex boundary.

Dimension 4. Consider $\mathbb{C}^2$ with its standard complex structure i. Then it is easy to check that $h(z_1, z_2) = \frac{1}{2}(|z_1|^2 + |z_2|^2)$ is a plurisubharmonic function whose regular level sets are the concentric spheres around the origin. We choose the level set $M = h^{-1}(1/2)$, that is, the boundary of the closed unit ball $W := h^{-1}((-\infty, 1/2])$ that is i-convex and has the induced contact form

$$\alpha_0 = d^i h\big|_{TM} = x_1\, dy_1 - y_1\, dx_1 + x_2\, dy_2 - y_2\, dx_2.$$

We only want to study a neighborhood U of $(0,1)$ in W. Embed a small disk by the map

$$\Phi\colon\, z \mapsto \left(z, \sqrt{1-|z|^2}\right)$$

into $M \cap U$, and denote the image of Φ by N_0. This submanifold is the intersection of $M = \mathbb{S}^3$ with a hyperplane whose z_2-coordinate is purely real. The restriction of α_0 to N_0 reduces to

$$\alpha_0|_{TN_0} = \Phi^*\alpha_0 = x\, dy - y\, dx, \tag{3.2}$$

so that the Legendrian foliation has at the origin an elliptic singularity (of the type described in Section 2.2.1).

Let U be the subset

$$U = \left\{(z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \mathrm{Re}(z_2) > 1-\delta\right\} \cap h^{-1}\big((-\infty, 1/2]\big)$$

for small $\delta > 0$, that means, we take the unit ball and cut off all points under a certain x_2-height.

The following propositions explain that there is essentially a unique holomorphic disk with boundary in N_0 passing through a given point $(z_1, z_2) \in N_0 \cap U$. All other holomorphic curves with the same boundary condition will either be constant or will be (branched) covers of that disk.

Proposition 3.18. *Denote the intersection of U with the complex plane $\mathbb{C} \times \{x\}$ for $x \in (1-\delta, 1)$ by L_x. For every $x_2 \in (1-\delta, 1)$, there exists a unique* injective *holomorphic map*

$$u_{x_2}\colon \left(\mathbb{D}^2, \partial\mathbb{D}^2\right) \to \left(L_{x_2}, \partial L_{x_2}\right)$$

that satisfies $u_{x_2}(0) = (0, x_2)$ and $u_{x_2}(1) \in \{(x_1, x_2) \in U | x_1 > 0\}$.

The last two conditions only serve to fix a parametrization of a given geometric disk.

Proof. The desired map u_{x_2} can be explicitly written down as

$$u_{x_2}(z) = (Cz, x_2)$$

with $C = \sqrt{1-x_2^2}$.

To prove uniqueness assume that there were a second holomorphic map

$$\tilde{u}_{x_2}\colon \left(\mathbb{D}^2, \partial\mathbb{D}^2\right) \to \left(L_{x_2}, \partial L_{x_2}\right)$$

with the required properties. It is clear that $L_{x_2} = \{(x+iy, x_2) \in \mathbb{C}^2 \mid x^2 + y^2 \le 1 - x_2^2\}$ is a round disk.

By Corollary 3.11, the restriction $u_{x_2}|_{\partial\mathbb{D}^2}$ of the map to the boundary has non-vanishing derivative, and it is by assumption injective, hence it is a diffeomorphism onto ∂L_{x_2}. This proves that u_{x_2} has to be for topological reasons surjective on L_{x_2} (otherwise we could construct a retract of the disk onto its boundary). Note also that the germ of a holomorphic map around the origin in $\mathbb{C}$ is always biholomorphic to $z \mapsto z^k$ for some integer $k \in \mathbb{N}_0$, so that the differential of u_{x_2} may not vanish anywhere, because otherwise u_{x_2} could not be injective.

Together this allows us to define a biholomorphism

$$\varphi := u_{x_2}^{-1} \circ \tilde{u}_{x_2}\colon \left(\mathbb{D}^2, \partial\mathbb{D}^2\right) \to \left(\mathbb{D}^2, \partial\mathbb{D}^2\right)$$

with $\varphi(0) = 0$ and $\varphi(1) = 1$, but the only automorphism of the disk with these properties is the identity, thus showing that $u_{x_2} = \tilde{u}_{x_2}$. ■

Proposition 3.19. *Let*

$$u\colon (\Sigma, \partial\Sigma; j) \to (U, N_0; i)$$

be any holomorphic map from a connected compact Riemann surface (Σ, j) *to* U *with* $u(\partial\Sigma) \subset N_0$.

Either u *is constant or its image is one of the slices* $L_{x_2} = U \cap (\mathbb{C} \times \{x_2\})$. *If* u *is injective at one of its boundary points, then* Σ *will be a disk, and after a reparametrization by a Möbius transformation,* u *will be equal to the map* u_{x_2} *given in Proposition 3.18.*

Proof. Note that we are supposing that u is at least C^1 on the boundary so that by Corollary 3.11 the map u will be constant if it touches the elliptic singularity in N.

The proof of the proposition will be based on the harmonicity of the coordinate functions x_1, y_1, x_2, and y_2. Let $f\colon U \to \mathbb{R}$ be the function $(z_1, z_2) \mapsto y_2 = \operatorname{Im}(z_2)$. Since Σ is a compact domain, the function $f \circ u$ attains somewhere on Σ its maximum and its minimum, and applying the maximum principle, Corollary 3.6, to $f \circ u$ itself and also to $-f \circ u$, we obtain that both the maximum and the minimum have to lie on $\partial\Sigma$. But since $u(\partial\Sigma) \subset N_0$ has vanishing imaginary z_2-part, it follows that $f \circ u \equiv 0$ on the whole surface. Using now the Cauchy-Riemann equations, it immediately follows that the real part of the z_2-coordinate of u has to be constant everywhere. We can deduce that the image of u has to lie in one of the slices $L_{x_2} = \mathbb{C} \times \{x_2\}$, and in particular the boundary $u(\partial\Sigma)$ lies in the circle $\partial L_{x_2} = \{(x+iy, x_2) \in \mathbb{C}^2 \mid x^2 + y^2 = 1 - x_2^2\}$.

Assume that u is not constant. Since u lies in L_{x_2}, we can use the map u_{x_2} from Proposition 3.18, to define a holomorphic map

$$\varphi := u_{x_2}^{-1} \circ u \colon (\Sigma, \partial\Sigma) \to \big(\mathbb{D}^2, \partial\mathbb{D}^2\big).$$

If u were not surjective on L_{x_2}, we could suppose (after a Möbius transformation on the target space) that the image of φ does not contain 0. The function $h(z) = -\ln|z|$ on $\mathbb{D}^2 \setminus \{0\}$ is harmonic, because it is locally the real part of a holomorphic function, and because $h \circ \varphi$ would have its maximum on the interior of Σ, we obtain that $h \circ \varphi$ is constant, so that the image of φ lies in $\partial\mathbb{D}^2$. The image of a non-constant holomorphic map is open, and hence u must be constant.

Assume now that u is injective at one of its boundary points. As we have shown in Proposition 3.18 the restriction $u|_{\partial\Sigma} \colon \partial\Sigma \to \partial L_{x_2}$ will be a diffeomorphism for each component of $\partial\Sigma$ so that $\partial\Sigma$ must be connected. Furthermore, it follows that u will also be injective on a small neighborhood of ∂L_{x_2}, because if we find two sequences $(z_k)_k$ and $(\tilde{z}_k)_k$ coming arbitrarily close to $\partial\Sigma$ with $u(z_k) = u(\tilde{z}_k)$ for every k, then after assuming that they both converge (reducing if necessary to subsequences), we see by continuity that $\lim u(z_k) = \lim u(\tilde{z}_k)$ and $\lim z_k, \lim \tilde{z}_k \in \partial\Sigma$, so that we can conclude that $\lim z_k = \lim \tilde{z}_k$. Using that the differential of u in $\lim z_k$ is not singular, we obtain that for k sufficiently large, we will always have $z_k = \tilde{z}_k$ showing that u is indeed injective on a small neighborhood of $\partial\Sigma$.

Assume $z_0 \in \Sigma$ is a point at which the differential $D\varphi$ vanishes. Then we know that φ can be represented in suitable charts as $z \mapsto z^k$ for some $k \in \mathbb{N}$, but if $k > 1$ this yields a contradiction, because we know that φ is a biholomorphism on a neighborhood of $\partial\Sigma$, and hence its degree must be 1. Since φ is holomorphic, it preserves orientations, so that on the other hand, we would have that the degree would need to be *at least* k, if there were such a critical point.

We obtain that φ has nowhere vanishing differential, and hence it must be a regular cover, but since it is of degree 1, it is in fact a biholomorphism, and Σ must be a disk. ∎

The Higher Dimensional Situation. In this section, L will always be a closed manifold, and we will choose for T^*L an almost complex structure J_L for which the 0-section L is totally real, so that there is by Proposition 3.16 a function $f_L\colon T^*L \to [0,\infty)$ that vanishes on L (and only on L) and that is plurisubharmonic on a small neighborhood of L.

As before, we will first describe a very explicit manifold that will serve as a model for the neighborhood of an elliptic singularity. Let $\mathbb{C}^2 \times T^*L$ be the almost complex manifold with almost complex structure $J = i \oplus J_L$, where i is the standard complex structure on $\mathbb{C}^2$. We define a function $f\colon \mathbb{C}^2 \times T^*L \to [0,\infty)$ by

$$f(z_1, z_2, \mathbf{q}, \mathbf{p}) = \frac{1}{2}\big(|z_1|^2 + |z_2|^2\big) + f_L(\mathbf{q}, \mathbf{p}).$$

If we stay in a sufficiently small neighborhood of the 0-section of T^*L, this function is clearly J-plurisubharmonic and we denote its regular level set $f^{-1}(1/2)$ by M; its contact form is given by

$$\alpha := d^J f\big|_{TM} = \big(x_1\,dy_1 - y_1\,dx_1 + x_2\,dy_2 - y_2\,dx_2 + d^{J_L} f_L\big)\big|_{TM}.$$

Now we define a submanifold N in M as the image of the map

$$\varPhi\colon \mathbb{D}^2 \times L \hookrightarrow M \subset \mathbb{C}^2 \times T^*L$$

given by $\varPhi(z;\mathbf{q}) = (z, \sqrt{1-|z|^2}; \mathbf{q}, \mathbf{0})$, that means, the image of $\varPhi$ is the product of the 0-section in T^*L and the submanifold N_0 given in the previous section. The submanifold has a Legendrian foliation $\mathcal{F}$ induced by

$$\alpha|_{TN} = \varPhi^* d^J f = x\,dy - y\,dx.$$

In particular, the leaves of the foliation are parallel to the L-factor in $\mathbb{D}^2 \times L$ and $\mathcal{F}$ has an elliptic singularity in $\{0\} \times L$.

Note that both the almost complex structure as well as the submanifold N split as a product, thus if we consider a holomorphic map

$$u\colon (\varSigma, \partial\varSigma; j) \to \big(\mathbb{C}^2 \times T^*L, N; J\big),$$

we can decompose it into $u = (u_1, u_2)$ with

$$u_1\colon (\varSigma, \partial\varSigma; j) \to \big(\mathbb{C}^2, N_0; i\big)$$
$$u_2\colon (\varSigma, \partial\varSigma; j) \to \big(T^*L, L; J_L\big).$$

This allows us to treat each factor independently from the other one, and we will easily be able to obtain similar results as in the previous section.

Since we are interested in finding a local model, we will first restrict our situation to the following subset

$$U := \big\{(z_1, z_2; \mathbf{q}, \mathbf{p}) \mid \mathrm{Re}(z_2) \geq 1 - \delta\big\} \cap f^{-1}\big([0, 1/2]\big) \tag{3.3}$$

that is, for δ sufficiently small, a compact neighborhood of N in $f^{-1}([0, 1/2])$, because the points $(z_1, z_2; \mathbf{q}, \mathbf{p})$ in U satisfy

$$0 \leq \frac{1}{2}|z_1|^2 + f_L(\mathbf{q}, \mathbf{p}) \leq \frac{1}{2}\big(1 - |z_2|^2\big) \leq \delta - \frac{1}{2}\delta^2 \leq \delta$$

so that all coordinates are bounded. Note in particular, that this bound on the $\mathbf{p}$-coordinates guarantees that f will be J-plurisubharmonic on U.

The submanifold $N \cap U$ can also be written in the following easy form

$$\big\{(z, x_2; \mathbf{q}, \mathbf{0}) \mid x_2 \geq 1 - \delta \text{ and } |z|^2 = 1 - x_2^2\big\} \times L.$$

Corollary 3.20. *Let*

$$u\colon (\Sigma, \partial\Sigma, j) \to (U, N \cap U; J)$$

be any holomorphic map from a connected compact Riemann surface (Σ, j) to U with $u(\partial\Sigma) \subset N$.

Either u is constant or its image is one of the slices $L_{x_2,\mathbf{q}_0} = (\mathbb{C} \times \{x_2\} \times \{\mathbf{q}_0\}) \cap U$ with $x_2 \in [1 - \delta, 1)$ and $\mathbf{q}_0$ a point on the 0-section of T^*L. If u is injective at one of its boundary points, then Σ will be a disk, and u is equal to

$$u(z) = \big(u_{x_2} \circ \varphi(z); \mathbf{q}_0, \mathbf{0}\big),$$

where u_{x_2} is the map given in Proposition 3.18, and φ is a Möbius transformation of the unit disk.

Proof. Let u be a J-holomorphic map as in the statement. We will study u by decomposing it into $u = (u_{\mathbb{C}^2}, u_{T^*L})$ with

$$u_{\mathbb{C}^2}\colon (\Sigma, \partial\Sigma, j) \to \big(\mathbb{C}^2, N, i\big)$$

$$u_{T^*L}\colon (\Sigma, \partial\Sigma, j) \to \big(T^*L, L, J_L\big).$$

Using that f_L is J_L-plurisubharmonic on the considered neighborhood of the 0-section contained in U, it follows from Corollary 3.17 that u_{T^*L} is constant.

Once we know that u_{T^*L} is constant, the situation for $u_{\mathbb{C}^2}$ is identical to the one in Proposition 3.19, so that we obtain the desired result. ■

The results obtained so far only explain the behavior of holomorphic curves that are completely contained in the model neighborhood U. Next we will extend this result to show that a holomorphic curve is either disjoint from the subset U or is lies completely inside U.

Assume (W, J) is a compact almost complex manifold with convex boundary $M = \partial W$. Let N be a submanifold of M, and assume that there is a compact subset U in W such that U is diffeomorphic to the model above, with $M \cap U$, $N \cap U$ and $J|_U$ all being equal to the corresponding objects in our model neighborhood.

Proposition 3.21. *Let*

$$u\colon (\Sigma, \partial\Sigma; j) \to (W, N; J)$$

be a holomorphic map, and let U be a compact subset of W that agrees with the model described above.

If $u(\Sigma)$ intersects U, then it has to lie entirely in U, and it will be consequently of the form given by Corollary 3.20.

Proof. Assume u to be a holomorphic map whose image lies partially in U. The set U is a compact manifold with corners, and we write $\partial U = \partial_M U \cup \partial_W U$ (see Figure 5), where

$$\partial_M U = U \cap M$$

$$\partial_W U = \big\{(z_1, z_2; \mathbf{q}, \mathbf{p}) \,\big|\, \mathrm{Re}(z_2) \geq 1 - \delta\big\} \cap f^{-1}\big([0, 1/2]\big).$$

We will show that the real part of the z_2-coordinate of u needs to be constant. This then proves the proposition, because it prevents u from leaving U.

Thus assume instead that the real part of z_2 does vary on u. Slightly decreasing the cut-off level δ in (3.3) using Sard's theorem, the holomorphic map u will intersect $\partial_W U$ transversely, so that $u^{-1}(\partial_W U)$ will be a properly embedded submanifold of Σ. We will restrict u to the compact subset $G = u^{-1}(U)$, and denote the boundary components of this domain

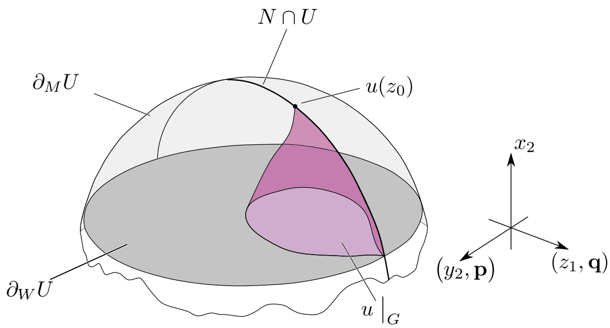

Fig. 5. Sketch of the symplectic model neighborhood of an elliptic singularity. A holomorphic curve lying only partially in this neighborhood would have two types of boundary, one part $u(\partial_M G)$ that lies in $N \cap U$, and a second one $u(\partial_W G)$ where the curve leaves the model neighborhood

by $\partial_M G = u^{-1}(N \cap U)$ and $\partial_W G = u^{-1}(\partial_W U)$. We thus have a holomorphic map

$$u|_G \colon (G, \partial G; j) \to (U, \partial U; J)$$

with $u(\partial_M G) \subset N \cap U$ and $u(\partial_W G) \subset \partial_W U$.

The coordinate maps $h_x \colon (z_1, z_2; \mathbf{q}, \mathbf{p}) \mapsto \mathrm{Re}(z_2)$ and $h_y \colon (z_1, z_2; \mathbf{q}, \mathbf{p}) \mapsto \mathrm{Im}(z_2)$ are harmonic, and it follows by the maximum principle that the maximum of $h_x \circ u|_G$ will lie for each component of G on the boundary of that component.

Furthermore the maximum of $h_x \circ u|_G$ cannot lie on $\partial_W G$, because by our assumption $u|_G$ is transverse to $\partial_W U$. It follows that the maximum of $h_x \circ u|_G$ will be a point $z_0 \in \partial_M G$; in particular z_0 does not lie on one of the edges of G. By the boundary point lemma, either $h_x \circ u|_G$ is constant or the outward derivative of this function at z_0 must be strictly positive. On the other hand, the function $h_y \circ u|_G$ is equal to 0 all along the boundary $\partial_M G$ so that the derivatives of $h_x \circ u|_G$ and $h_y \circ u|_G$ vanish at z_0 in directions that are tangent to the boundary. Using the Cauchy-Riemann equation we see that this implies that the derivatives of these two functions at z_0 vanish in *every direction*, in particular this implies that the function $h_x \circ u|_G$ needs to be constant.

In either case, we have proved that the image of u lies completely inside U. ■

The conclusion of the results in this section is that every curve that intersects a certain neighborhood of the elliptic singularities lies completely in this neighborhood and can be explicitly determined.

3.2.3. *J*-Holomorphic Curves Close to Codimension 1 Singularities. Let $(N, \mathcal{F})$ be a submanifold with Legendrian foliation and with non-empty boundary. We will show in this section that a boundary component of N lying in the singular set of $\mathcal{F}$ can sometimes exclude that any holomorphic curve gets close to this component. This way, the boundary may block any holomorphic disks from escaping the submanifold N.

The argument is similar to that of the previous section, where we constructed an almost complex manifold that served as a model for the neighborhood of the singular set.

Remark 3.22. We will only be dealing here with the easiest type of singular sets: Products of a closed manifold with $\mathbb{S}^1$. A more general situation has been considered in [21], where the singular set is allowed to be a fiber bundle over the circle.

Let T^*F be the cotangent bundle of a closed manifold F, choose an almost complex structure J_F on T^*F for which F is a totally real submanifold, and let $f_F \colon T^*F \to [0, \infty)$ be the function constructed in Proposition 3.16 that only vanishes along the 0-section of T^*F and that is J_F-plurisubharmonic close to the 0-section F.

Define (W, J) as

$$W := \mathbb{C} \times T^*\mathbb{S}^1 \times T^*F = \bigl\{(x + iy; \varphi, r; \mathbf{q}, \mathbf{p})\bigr\},$$

and let J be the almost complex structure $i \oplus i \oplus J_F$, where the complex structure on $T^*\mathbb{S}^1$ is the one induced from the identification of $T^*\mathbb{S}^1$ and $\mathbb{C}/(2\pi\mathbb{Z})$ with $\varphi + ir \sim \varphi + 2\pi + ir$. The function

$$f \colon W \to [0, \infty), (x + iy; \varphi, r; \mathbf{q}, \mathbf{p}) \mapsto \frac{1}{2}\bigl(x^2 + y^2\bigr) + \frac{1}{2}r^2 + f_F(\mathbf{q}, \mathbf{p})$$

is J-plurisubharmonic on a neighborhood where the values of $\mathbf{p}$ are small. We denote the level set $f^{-1}(1/2)$ by M, and note that for small values of $\mathbf{p}$, it is a smooth contact manifold with contact form

$$\alpha_M := \bigl(x\,dy - y\,dx - r\,d\varphi + d^{J_F} f_F\bigr)\big|_{TM}.$$

Let N be the submanifold of M given as the image of the map

$$\Phi\colon \mathbb{S}^1 \times [0,\varepsilon) \times F, (\varphi, r; \mathbf{q}) \mapsto \big(\sqrt{1-r^2}; \varphi, r; \mathbf{q}, \mathbf{0}\big).$$

It has a Legendrian foliation $\mathcal{F}$, because $\Phi^*\alpha_M = -r\,d\varphi$ that becomes singular exactly at the boundary $\partial N = \{1\} \times \mathbb{S}^1 \times F$.

Our local model will be the subset

$$U = \big\{(x+iy; \varphi, r; \mathbf{q}, \mathbf{p}) \mid x \geq 1-\delta\big\} \cap f^{-1}\big([0,1/2]\big)$$

for sufficiently small $\delta > 0$. Clearly U contains $\partial N = \operatorname{Sing}(\ker(-r\,d\varphi))$. Furthermore U is compact, because all coordinates are bounded: Points $(x+iy; \varphi, r; \mathbf{q}, \mathbf{p})$ in U satisfy

$$0 \leq \frac{1}{2}y^2 + \frac{1}{2}r^2 + f_F(\mathbf{q}, \mathbf{p}) = f(x+iy; \varphi, r; \mathbf{q}, \mathbf{p}) - \frac{1}{2}x^2 \leq 1/2\big(1-x^2\big) \leq \delta.$$

We also obtain that if δ has been chosen small enough, f is everywhere J-plurisubharmonic on U.

Remark 3.23. Note that the construction of the local model also applies in the case of contact 3-manifolds, because F may be just a point.

We will first exclude existence of holomorphic curves that are entirely contained in U.

Proposition 3.24. *A J-holomorphic map*

$$u\colon (\Sigma, \partial\Sigma, j) \to (U, N \cap U, J)$$

from a compact Riemann surface into U, whose boundary is mapped into $N \cap U$, must be constant.

Proof. As in the previous section, we can decompose u as $(u_{\mathbb{C}\times T^*\mathbb{S}^1}, u_{T^*F})$ with

$$u_{\mathbb{C}\times T^*\mathbb{S}^1}\colon (\Sigma, \partial\Sigma, j) \to \Big(\mathbb{C} \times T^*\mathbb{S}^1, \big\{\big(\sqrt{1-r^2}; \varphi, r\big) \;\big|\; \varphi \in \mathbb{S}^1, r \in [0,\varepsilon)\big\}, i \oplus i\Big)$$
$$u_{T^*F}\colon (\Sigma, \partial\Sigma, j) \to \big(T^*F, F, J_F\big).$$

Note in particular that the boundary conditions also split in this decomposition, so that we obtain two completely uncoupled problems. Furthermore, using Corollary 3.17, it follows that the second map is constant, because f_F is a J_F plurisubharmonic function on the considered neighborhood.

To show that $u_{\mathbb{C}\times T^*\mathbb{S}^1}$ is constant, use the harmonic function $g(z;\varphi,r) = \operatorname{Im}(z)$. Since $g\circ u_{\mathbb{C}\times T^*\mathbb{S}^1}$ vanishes along $\partial\Sigma$, it follows that $g\circ u_{\mathbb{C}\times T^*\mathbb{S}^1}$ has to be zero on the whole Riemann surface, and combining this with the Cauchy-Riemann equation, it follows that the real part of the z-coordinate of $u_{\mathbb{C}\times T^*\mathbb{S}^1}$ is equal to a constant $C\in[1-\delta,1]$. Now that we know that the first coordinate of $u_{\mathbb{C}\times T^*\mathbb{S}^1}$ is constant, we see that the boundary of $u_{\mathbb{C}\times T^*\mathbb{S}^1}$ has to lie in the circle $\{(C;\varphi,+\sqrt{1-C^2})\mid \varphi\in\mathbb{S}^1\}\subset\mathbb{C}\times T^*\mathbb{S}^1$.

This allows us to study only the second coordinate of $u_{\mathbb{C}\times T^*\mathbb{S}^1}$ reducing our map to the form

$$u_{T^*\mathbb{S}^1}\colon(\Sigma,\partial\Sigma,j)\to\big(T^*\mathbb{S}^1,S,i\big),$$

where $S=\{r=+\sqrt{1-C^2}\}$. Using that the map $(r,\varphi)\mapsto r$ is harmonic, and that it is constant along the boundary of Σ, we obtain that the whole image of the surface has to lie in the corresponding circle, implying with the Cauchy-Riemann equation that $u_{T^*\mathbb{S}^1}$ needs to be constant. ■

Next we will show that holomorphic curves may not enter the domain U even partially. Let (W,J) be now a compact almost complex manifold with convex boundary $M=\partial W$, and let N be a submanifold of M with $\partial N\neq\emptyset$. Assume that W contains a compact subset U that is identical to the model neighborhood constructed above such that $M\cap U$, $N\cap U$ and $J|_U$ all agree with the corresponding objects in the model.

Proposition 3.25. *If the image of a J-holomorphic map*

$$u\colon(\Sigma,\partial\Sigma,j)\to(W,N,J)$$

intersects the neighborhood U, then it will be constant.

Proof. It suffices to show that the image of u lies inside U, because we can then apply Proposition 3.24. Following the same line of arguments as in the proof of Proposition 3.21, one can show that the real part of the first coordinate of u needs to be constant. We recommend the reader to work out the details as an exercise. ■

Remark 3.26. Note that when the codimension 1 singular set lies in the interior of the maximally foliated submanifold, one can find under additional conditions a family of holomorphic annuli with one boundary component on each side of the singular set (see [30]). The reason why these curves do not appear in the results of this section are that we are assuming that all boundary components of the holomorphic curves lie locally on one side of the singular set.

4. Moduli Spaces of Disks and Filling Obstructions

4.1. The Moduli Space of Holomorphic Disks

Let us assume again that (W, J) is an almost complex manifold, and that $N \subset W$ is a totally real submanifold. We want to study the space of maps

$$u\colon \big(\mathbb{D}^2, \partial\mathbb{D}^2\big) \to (W, N; J)$$

that are J-holomorphic (strictly speaking they are (i, J)-holomorphic), meaning that we want the differential of u to be complex linear, so that it satisfies at every $z \in \Sigma$ the equation

$$Du_z \cdot i = J\big(u(z)\big) \cdot Du_z.$$

Note that J depends on the point $u(z)$!

A different way to state this equation is by introducing the Cauchy-Riemann operator

$$\bar\partial_J u = J(u) \cdot Du - Du \cdot i,$$

and writing $\bar\partial_J u = 0$, so that the space of J-holomorphic maps, we are interested in then becomes

$$\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big) = \big\{u\colon \mathbb{D}^2 \to W \;\big|\; \bar\partial_J u = 0 \text{ and } u\big(\partial\mathbb{D}^2\big) \subset N\big\}.$$

Remark 4.1. The situation of holomorphic disks is a bit special compared to the one of general holomorphic maps, because all complex structures on the disk are equivalent. If Σ were a smooth compact surface of higher genus, we would usually need to study the space of pairs (u, j), where j is a complex structure on Σ, and u is a map $u\colon (\Sigma, \partial\Sigma) \to (W, N)$ that should be (j, J)-holomorphic, that means, $J(u) \cdot Du - Du \cdot j = 0$.

To be a bit more precise, we do not choose pairs (u, j) with arbitrary complex structures j on Σ, but we only allow for j a single element in each equivalence class of complex structures: If $\varphi\colon \Sigma \to \Sigma$ is a diffeomorphism, and j is some complex structure, then of course $\varphi^* j$ will generally be a complex structure different from j, but we usually identify all complex structures up to isotopy, and use that the space of equivalence classes of complex structures can be represented as a smooth finite dimensional manifold (see [19] for a nice introduction to this theory).

Fortunately, these complications are not necessary for holomorphic disks (or spheres), and it is sufficient for us to work with the standard complex structure i on $\mathbb{D}^2$.

In this section, we want to explain the topological structure of the space $\widetilde{\mathcal{M}}(\mathbb{D}^2, N; J)$ without entering into too many technical details. Instead of starting directly with our particular case, we will try to argue on an intuitive level by considering a finite dimensional situation that has strong analogies with the problem we are dealing with.

Let us consider a vector bundle E of rank r over a smooth n-manifold B. Choose a section $\sigma\colon B \to E$, and let $M = \sigma^{-1}(0)$ be the set of points at which σ intersects the 0-section. We would "expect" M to be a smooth submanifold of dimension $\dim M = n - r$ (if $n - r < 0$, we could hope not to have any intersections at all); unfortunately, this intuitive expectation might very well be false. A sufficient condition under which it holds, is when σ is transverse to the 0-section, that means, for every $x \in M$, the tangent space to the 0-section T_xB in T_xE spans together with the image $D\sigma \cdot T_xB$ the whole tangent space T_xE. It is well-known that when the transversality condition is initially not true, it can be achieved by slightly perturbing the section σ.

Let us now come again to the Cauchy-Riemann problem. The role of B will be taken by the space of all maps $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, N)$, which we will denote by $\mathcal{B}(\mathbb{D}^2; N)$. We do not want to spend any time thinking about the regularity of the maps and point instead to [25] as reference. It is sufficient for us to observe that the space $\mathcal{B}(\mathbb{D}^2; N)$ is a Banach manifold, that means, an infinite dimensional manifold modeled on a Banach space.

The section σ will be replaced by the Cauchy-Riemann operator $\bar\partial_J$, and before pursuing this analogy further, we want first to specify the target space of this operator. In fact, $\bar\partial_J$ associates to every map $u \in \mathcal{B}(\mathbb{D}^2; N)$ a 1-form on Σ with values in TW. The formal way to state this is that we have for every map u a vector bundle u^*TW over $\mathbb{D}^2$, which allows us to construct

$$\operatorname{Hom}\bigl(T\mathbb{D}^2, u^*TW\bigr).$$

The sections in $\operatorname{Hom}(T\mathbb{D}^2, u^*TW)$ form a vector space, and if we look at all sections for *all* maps u, we obtain a vector bundle over $\mathcal{B}(\mathbb{D}^2; N)$, whose fiber over a point u are all sections in $\operatorname{Hom}(T\mathbb{D}^2, u^*TW)$. We denote this bundle by $\mathcal{E}(\mathbb{D}^2; N)$.

The operator $\bar\partial_J$ associates to every u, that means, to every point of $\mathcal{B}(\mathbb{D}^2; N)$ an element in $\mathcal{E}(\mathbb{D}^2; N)$ so that we can think of $\bar\partial_J$ as a section in the bundle $\mathcal{E}(\mathbb{D}^2; N)$. The J-holomorphic maps are the points of $\mathcal{B}(\mathbb{D}^2; N)$ where the section $\bar\partial_J$ intersects the 0-section. In fact, $\bar\partial_J u$ is always anti-holomorphic, because

$$J(u) \cdot \bar\partial_J u = -Du - J(u) \cdot Du \cdot i = \big(Du \cdot i - J(u) \cdot Du\big) \cdot i = -(\bar\partial_J u) \cdot i,$$

and for analytical reasons we will only consider sections in $\operatorname{Hom}(T\Sigma, u^*TW)$ taking values in the subbundle $\overline{\operatorname{Hom}_{\mathbb{C}}}(T\Sigma, u^*TW)$ of anti-holomorphic homomorphisms. We denote the subbundle of sections taking values in $\overline{\operatorname{Hom}_{\mathbb{C}}}(T\Sigma, u^*TW)$ by $\bar{\mathcal{E}}_{\mathbb{C}}(\mathbb{D}^2; N)$.

4.1.1. The Expected Dimension of $\widetilde{\mathcal{M}}(\mathbb{D}^2, N; J)$. The rank of $\bar{\mathcal{E}}_{\mathbb{C}}(\mathbb{D}^2; N)$ and the dimension of $\mathcal{B}(\mathbb{D}^2; N)$ are both infinite, hence we cannot compute the expected dimension of the solution space $\widetilde{\mathcal{M}}(\mathbb{D}^2, N; J)$ as in the finite dimensional case, where it was just the difference $\dim M - \operatorname{rank} E$. Nonetheless we can associate a so called Fredholm index to a Cauchy-Riemann problem. We will later give some more details about how the index is actually defined, for now we just note that it is an integer that determines the expected dimension of the space $\widetilde{\mathcal{M}}(\mathbb{D}^2, N; J)$.

For a Cauchy-Riemann problem with totally real boundary condition the index has an easy explicit formula (see for example [25, Theorem C.1.10]) that simplifies in our specific case of holomorphic disks to

$$\operatorname{ind}_u \bar\partial_J = \frac{1}{2} \dim W + \mu\big(u^*TW, u^*TN\big), \tag{4.1}$$

where we have used that the Euler characteristic of a disk is $\chi(\mathbb{D}^2) = 1$.

Remark 4.2. We would like to warn the reader that the dimension of a moduli space of holomorphic disks or holomorphic spheres tends to increase, if we increase the dimension of the symplectic ambient manifold. Unfortunately, the opposite is true for a higher genus curve Σ: The formula above becomes

$$\operatorname{ind}_u \bar\partial_J = \frac{1}{2}\chi(\Sigma) \dim W + \mu\big(u^*TW, u^*TN\big),$$

and since the Euler characteristic is negative, and it is harder to find curves with genus in high dimensional spaces than in lower dimensional ones.

The Maslov index μ is an integer that classifies loops of totally real subspaces up to homotopy:

Definition. Let $E_{\mathbb{C}}$ be a complex vector bundle over the closed 2-disk $\mathbb{D}^2$ and let $E_{\mathbb{R}}$ be a totally real subbundle of $E_{\mathbb{C}}|_{\partial\mathbb{D}^2}$ defined only over the boundary of the disk. The **Maslov index** $\mu(E_{\mathbb{C}}, E_{\mathbb{R}})$ is an integer that is computed by trivializing $E_{\mathbb{C}}$ over the disk, and choosing a continuous frame $A(e^{i\phi}) \in \mathrm{GL}(n,\mathbb{C})$ over the boundary $\partial\mathbb{D}^2$ representing $E_{\mathbb{R}}$. We then set

$$\mu(E_{\mathbb{C}}, E_{\mathbb{R}}) := \deg \frac{\det A^2}{\det(A^*A)},$$

where $\deg(f)$ is the degree of a continuous map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$.

In these notes, we will compute the Maslov index only once, in Section 4.1.3, but note that the index $\operatorname{ind}_u \bar{\partial}_J$ depends on the holomorphic disk u in $\widetilde{\mathcal{M}}(\mathbb{D}^2, N; J)$, we are considering; this should not confuse us however, because it only means that the space of disks may have different components and the expected dimensions of the different components do not need to agree.

We will now briefly explain how the index of $\bar{\partial}_J$ is defined. We have a map $\bar{\partial}_J\colon \mathcal{B}(\mathbb{D}^2; N) \to \bar{\mathcal{E}}_{\mathbb{C}}(\mathbb{D}^2; N)$, and we need to compute the linearization of $\bar{\partial}_J$ at a point of $u \in \mathcal{B}(\mathbb{D}^2; N)$, that means, we have to compute the differential

$$\bar{D}_J(u)\colon T_u\mathcal{B}\big(\mathbb{D}^2; N\big) \to T_{\bar{\partial}_J u}\bar{\mathcal{E}}_{\mathbb{C}}\big(\mathbb{D}^2; N\big).$$

To find $\bar{D}_J(u)$, choose a smooth path u_t of maps in $\mathcal{B}(\mathbb{D}^2; N)$ with $u_0 = u$, then we can regard the image $\bar{\partial}_J u_t$, and take its derivative with respect to t in $t = 0$. If we set $\dot{u}_0 = \frac{d}{dt}|_{t=0} u_t$, this allows us to obtain a linear operator $\bar{D}_J(u)$ by

$$\bar{D}_J(u) \cdot \dot{u}_0 = \frac{d}{dt}\bigg|_{t=0} \bar{\partial}_J u_t.$$

It is a good exercise to determine the domain and target space of this operator, and find a way to describe them.

The index of $\bar{\partial}_J$ at u is defined as

$$\operatorname{ind}_u \bar{\partial}_J := \dim\ker \bar{D}_J(u) - \dim\operatorname{coker} \bar{D}_J(u).$$

It is a remarkable fact that the index is finite and determined by formula (4.1) above. Also note that the index is constant on each connected component of $\mathcal{B}(\mathbb{D}^2; N)$.

4.1.2. Transversality of the Cauchy-Riemann Problem. Just as in the finite dimensional analogue, it may happen that the formal dimension we have computed does not correspond to the dimension we are observing in an actual situation. In fact, if the section σ (or in our infinite dimensional case, $\bar\partial_J$) are not transverse to the 0-section, there is no reason why M or $\widetilde{\mathcal{M}}(\mathbb{D}^2, N; J)$ would need to be smooth manifolds at all.

On the other hand, if σ is transverse to the 0-section, then $M = \sigma^{-1}(0)$ is a smooth submanifold of dimension $\dim M - \operatorname{rank} E$, and the analogue result is also true for the Cauchy-Riemann problem: If $\bar\partial_J$ is at every point of $\widetilde{\mathcal{M}}(\mathbb{D}^2, N; J)$ transverse to 0 (or said equivalently, if the cokernel of the linearized operator is trivial for every holomorphic disk), then $\widetilde{\mathcal{M}}(\mathbb{D}^2, N; J)$ will be a smooth manifold whose dimension is given by the index of $\bar\partial_J$.

In the finite dimensional situation, we can often achieve transversality by a small perturbation of σ, but of course, this might require a subtle analysis of the situation, when we want to perturb σ only within a space of sections satisfying certain prescribed properties.

Definition. Let $u\colon \Sigma \to W$ be a holomorphic map from a Riemann surface with or without boundary. We call u **somewhere injective**, if there exists a point $z \in \Sigma$ with $du_z \neq 0$, and such that z is the only point that is mapped by u to $u(z)$, that means,

$$u^{-1}\bigl(u(z)\bigr) = \{z\}.$$

We call a holomorphic curve that is not the multiple cover of any other holomorphic curve a **simple holomorphic curve**. Closed simple holomorphic curves are somewhere injective, [25, Proposition 2.5.1].

It is a non-trivial result that by perturbing the almost complex structure J, we can achieve transversality of the Cauchy-Riemann operator for every disk in W whose boundary is injective in a totally real submanifold N. We could hope that this theoretical result would be sufficient for us, because the considered disks are injective along their boundaries, but we have chosen a very specific almost complex structure in Section 3.2, and perturbing this J would destroy the results obtained in that section. Below, we will prove by hand that $\bar\partial_J$ is transverse to 0 for the holomorphic disks in our model neighborhood.

Remark 4.3. Note that often it is not possible to work only with somewhere injective holomorphic curves, and perturbing J will in that case not be sufficient to obtain transversality for holomorphic curves. Sometimes one can work around this problem by requiring that W is semi-positive, see Section 4.3. Unfortunately, there are many situations where this approach won't work either, as is the case of SFT, where transversality has been one of the most important outstanding technical problems.

4.1.3. The Bishop Family. In this section, we will show that the disks that we have found in Section 3.2.2, lying in the model neighborhood are regular solutions of the Cauchy-Riemann problem.

Before starting the actual proof of our claim, we will briefly recapitulate the situation described in Section 3.2.2. Let (W, J) be an almost complex manifold of dimension $2n$ with boundary that contains a model neighborhood U of the desired form. Remember that U was a subset of $\mathbb{C}^2 \times T^*L$ with almost complex structure $i \oplus J_L$, that we had a function $f\colon \mathbb{C}^2 \times T^*L \to [0, \infty)$ given by

$$f(z_1, z_2, \mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) + f_L(\mathbf{q}, \mathbf{p}),$$

and that the model neighborhood U was the subset

$$U := \left\{(z_1, z_2; \mathbf{q}, \mathbf{p}) \mid \operatorname{Re}(z_2) \geq 1 - \delta\right\} \cap f^{-1}\left([0, 1/2]\right).$$

The totally real manifold N is the image of the map

$$(z; \mathbf{q}) \in \mathbb{D}^2_\varepsilon \times L \mapsto \left(z, \sqrt{1 - |z|^2}; \mathbf{q}, \mathbf{0}\right) \subset \partial U.$$

For every pair $(s, \mathbf{q}) \in [1 - \delta, 1) \times L$, we find a holomorphic map of the form

$$\begin{aligned} u_{s,\mathbf{q}}\colon \left(\mathbb{D}^2, \partial\mathbb{D}^2\right) &\to U \\ z &\mapsto (C_s z, s; \mathbf{q}, \mathbf{0}) \end{aligned}$$

with $C_s = \sqrt{1 - s^2}$. We call this map a **(parametrized) Bishop disk**, and we call the collection of these disks, the **Bishop family**. Sometimes we will not be precise about whether the disks are parametrized or not, and whether we speak about disks with or without a marked point (see Section 4.2), but we hope that in each situation it will be clear what is meant.

To check that a given Bishop disk $u_{s,\mathbf{q}}$ is regular, we will first compute the index of the linearized Cauchy-Riemann operator that gives us the expected dimension for the space of holomorphic disks containing the Bishop family. Note that the observed dimension is $1 + \dim L + 3 = 1 + (n - 2) + 3 = n + 2$.

The first part, $1 + \dim L$ corresponds to the s- and $\mathbf{q}$-parameters of the family; the three corresponds to the dimension of the group of Möbius transformations acting on the complex unit disk: If $u_{s,\mathbf{q}}$ is a Bishop disk, and if $\varphi\colon \mathbb{D}^2 \to \mathbb{D}^2$ is a Möbius transformation, then of course $u_{s,\mathbf{q}} \circ \varphi$ will also be a holomorphic map with admissible boundary condition. On the other hand we showed in Corollary 3.20 that every holomorphic disk that lies in U is up to a Möbius transformation one of the Bishop disks.

For the index computations, it suffices by Section 4.1.1 to trivialize the bundle $E_{\mathbb{C}} := u_{s,\mathbf{q}}^* TW$ over $\mathbb{D}^2$, and study the topology of the totally real subbundle $E_{\mathbb{R}} = u_{s,\mathbf{q}}^* TN$ over $\partial\mathbb{D}^2$.

Before starting any concrete computations, we will significantly simplify the setup by choosing a particular chart: Note that the T^*L-part of a Bishop disk $u_{s,\mathbf{q}}$ is constant, we can hence choose a chart diffeomorphic to $\mathbb{R}^{2n-4} = \{(x_1, \dots, x_{n-2}; y_1, \dots, y_{n-2})\}$ for T^*L with the properties

- the point $(\mathbf{q}, \mathbf{0})$ corresponds to the origin,
- the almost complex structure J_L is represented at the origin by the standard i,
- the intersections of the 0-section L with the chart corresponds to the subspace $(x_1, \dots, x_{n-2}; 0, \dots, 0)$.

In the chosen chart, we write $u_{s,\mathbf{q}}$ as

$$u_{s,\mathbf{q}}(z) = (C_s z, s; 0, \dots, 0) \in \mathbb{C}^2 \times \mathbb{R}^{2n-4}$$

with $C_s = \sqrt{1 - s^2}$. By our assumption, the complex structure on the second factor is at the origin of $\mathbb{R}^{2n-4}$ equal to i, and there is then a direct identification of $u_{s,\mathbf{q}}^* TW$ with $\mathbb{C}^2 \times \mathbb{C}^{n-2}$. The submanifold N corresponds in the chart to

$$\left\{(z_1, z_2; x_1, \dots, x_{n-2}, 0, \dots, 0) \in \mathbb{C}^2 \times \mathbb{R}^{2n-4} \;\middle|\; \operatorname{Im} z_2 = 0, |z_1|^2 + |z_2|^2 = 1\right\}.$$

The boundary of $u_{s,\mathbf{q}}$ is given by $e^{i\varphi} \mapsto (\sqrt{1 - s^2} e^{i\varphi}, s; 0, \dots, 0)$, and the tangent space of TN over this loop is spanned over $\mathbb{R}$ by the vector fields

$$\big(ie^{i\varphi}, 0; 0, \dots, 0\big), \left(-\frac{s}{\sqrt{1 - s^2}} e^{i\varphi}, 1; 0, \dots, 0\right), (0, 0; 1, 0, \dots, 0), \dots,$$
$$(0, 0; 0, \dots, 0, 1, 0, \dots, 0).$$

We can now easily compute the Maslov index $\mu(E_{\mathbb{C}}, E_{\mathbb{R}})$ as

$$\deg \frac{\det A^2}{\det(A^* A)} = \deg \frac{-e^{2i\varphi}}{1} = 2,$$

where A is the matrix composed by the vector fields given above. Hence we obtain for the index

$$\operatorname{ind}_u \bar{\partial}_J = \frac{1}{2} \dim W + \mu\big(u_{s,\mathbf{q}}^* TW, u_{s.\mathbf{q}}^* TN\big) = n + 2,$$

which corresponds to the observed dimension computed above.

We will now show that the linearized operator $\bar{D}_J$ is surjective. We do not do this directly, but we compute instead the dimension of its kernel, and show that it is equal (and not larger than) the Fredholm index. From the definition of the index

$$\operatorname{ind}_u \bar{\partial}_J := \ker \bar{D}_J(u) - \operatorname{coker} \bar{D}_J(u),$$

we see that the cokernel needs to be trivial, and this way the surjectivity result follows.

We now compute the linearized Cauchy-Riemann operator at a Bishop disk $u_{s,\mathbf{q}}$. Let v_t be a smooth family of maps

$$v_t \colon \big(\mathbb{D}^2, \partial \mathbb{D}^2\big) \to (U, N)$$

with $v_0 = u_{s,\mathbf{q}}$ (think of each v_t as a smooth map, but for an analytically correct study, we would need to allow here for Sobolev maps).

In this chart, we can write the family v_t as

$$v_t(z) = \big(z_1(z,t), z_2(z,t); \mathbf{x}(z,t), \mathbf{y}(z,t)\big) \in \mathbb{C}^2 \times \mathbb{R}^{2n-4},$$

where we have set $\mathbf{x}(z,t) = (x_1(z,t), \dots, x_{n-2}(z,t))$ and $\mathbf{y}(z,t) = (y_1(z,t), \dots, y_{n-2}(z,t))$, and we require that the boundary of each of the v_t has to lie in N. When we now take the derivative of v_t with respect to t at $t = 0$, we obtain a vector in $T_{u_{s,\mathbf{q}}}\mathcal{B}$ that is represented by a map

$$\dot{v}_0 \colon \mathbb{D}^2 \to \mathbb{C}^2 \times \mathbb{R}^{2(n-2)}, \quad z \mapsto \big(\dot{z}_1(z), \dot{z}_2(z); \dot{\mathbf{x}}(z), \dot{\mathbf{y}}(z)\big)$$

with boundary conditions $\dot{\mathbf{y}}(z) = \mathbf{0}$ and $\operatorname{Im} \dot{z}_2(z) = 0$ for every $z \in \partial\mathbb{D}^2$. Furthermore taking the derivative of $|z_1(z,t)|^2 + |z_2(z,t)|^2 = 1$ for every $z \in \partial\mathbb{D}^2$ with respect to t, we obtain the condition

$$\bar{z}_1(z,0) \cdot \dot{z}_1(z) + z_1(z,0) \cdot \dot{\bar{z}}_1(z) + \bar{z}_2(z,0) \cdot \dot{z}_2(z) + z_2(z,0) \cdot \dot{\bar{z}}_2(z) = 0,$$

which simplifies by using the explicit form of $(z_1(z,0), z_2(z,0))$ to

$$C_s\bar{z}\cdot\dot{z}_1(z) + C_s z\cdot\dot{\bar{z}}_1(z) + s\dot{z}_2(z) + s\dot{\bar{z}}_2(z) = 0$$

for every $z\in\partial\mathbb{D}^2$.

The linearization of the Cauchy-Riemann operator $\bar\partial_J$ at $u_{s,\mathbf{q}}$ given by

$$\bar{D}_J\cdot\dot{v}_0 := \left.\frac{d}{dt}\right|_{t=0}\bar\partial_J v_s$$

decomposes into the $\mathbb{C}^2$-part

$$(id\dot{z}_1 - d\dot{z}_1 i, id\dot{z}_2 - d\dot{z}_2 i)$$

and the $\mathbb{R}^{2(n-2)}$-part

$$\left.\frac{d}{dt}\right|_{t=0}\Big(J_L\big(\mathbf{x}(z,t),\mathbf{y}(z,t)\big)\cdot\big(d\mathbf{x}(z,t), d\mathbf{y}(z,t)\big) - \big(d\mathbf{x}(z,t)\cdot i, d\mathbf{y}(z,t)\cdot i\big)\Big).$$

The second part can be significantly simplified by using first the product rule, and applying then that $\mathbf{x}(z,0)=\mathbf{0}$ and $\mathbf{y}(z,0)=\mathbf{0}$ are constant so that their differentials vanish. We obtain then

$$J_L(\mathbf{0},\mathbf{0})\cdot(d\dot{\mathbf{x}}, d\dot{\mathbf{y}}) - (d\dot{\mathbf{x}}\cdot i, d\dot{\mathbf{y}}\cdot i),$$

and using that $J_L(\mathbf{0},\mathbf{0}) = i$, it finally reduces to

$$(d\dot{\mathbf{y}} - d\dot{\mathbf{x}}\cdot i, -d\dot{\mathbf{x}} - d\dot{\mathbf{y}}\cdot i).$$

We have shown that linearized Cauchy-Riemann operator simplifies for all coordinates to the standard Cauchy-Riemann operator, so that if $\dot{v}_0(z) = (\dot{z}_1(z), \dot{z}_2(z); \dot{\mathbf{x}}(z), \dot{\mathbf{y}}(z))$ lies in the kernel of $\bar{D}_J$ then the coordinate functions $\dot{z}_1(z), \dot{z}_2(z)$ and $\dot{\mathbf{x}}(z) + i\dot{\mathbf{y}}(z)$ need all to be holomorphic in the classical sense.

Now using the boundary conditions, we easily deduce that $\dot{\mathbf{y}}(z)$ needs to vanish, because it is a harmonic function, and it takes both maximum and minimum on $\partial\mathbb{D}^2$. A direct consequence of $\dot{\mathbf{y}}\equiv\mathbf{0}$ and the Cauchy-Riemann equation is that $\dot{\mathbf{x}}(z)$ will be everywhere constant. We get the analogous result for the function $\dot{z}_2(z)$, so that we can write

$$\dot{v}_0(z) = \big(\dot{z}_1(z), \dot{s}; \dot{\mathbf{q}}_0, \mathbf{0}\big),$$

where $\dot{s}$ is a real constant, and $\dot{\mathbf{q}}_0$ is a fixed vector in $\mathbb{R}^{2(n-2)}$, and we only need to still understand the holomorphic function $\dot{z}_1(z)$.

The boundary condition for $\dot{z}_1(z)$ is $\bar{z}\cdot\dot{z}_1(z) + z\cdot\dot{\bar{z}}_1(z) = -\frac{2s\dot{s}}{C_s}$ for every $z \in \partial\mathbb{D}^2$. Using that the function $\dot{z}_1(z)$ is holomorphic, we can write it as power series in the form

$$\dot{z}_1(z) = \sum_{k=0}^{\infty} a_k z^k$$

and we get at $e^{i\varphi} \in \partial\mathbb{D}^2$

$$\dot{z}_1\big(e^{i\varphi}\big) = \sum_{k=0}^{\infty} a_k\, e^{ik\varphi}.$$

Plugging these series into the equation of the boundary condition, we find

$$e^{-i\varphi}\cdot\sum_{k=0}^{\infty} a_k\, e^{ik\varphi} + e^{i\varphi}\cdot\sum_{k=0}^{\infty} \bar{a}_k\, e^{-ik\varphi} = -\frac{2s\dot{s}}{C_s}$$

so that

$$\sum_{k=0}^{\infty}\big(a_k\, e^{(k-1)\,i\varphi} + \bar{a}_k\, e^{-(k-1)\,i\varphi}\big) = -\frac{2s\dot{s}}{C_s}$$

and by comparing coefficients we see that

$$a_1 + \bar{a}_1 = -\frac{2s\dot{s}}{C_s}, \qquad a_0 + \bar{a}_2 = 0, \qquad a_k = 0 \quad \text{for all } k \geq 3.$$

This means that the three (real) parameters we can choose freely are z_0 and $\operatorname{Im} z_1$.

Concluding, we have found that the dimension of the kernel of $\bar{D}_J$ is equal to $3+1+n-2 = n+2$ which corresponds to the Fredholm index of our problem. Thus there is no need to perturb J on the neighborhood of the Bishop family to obtain regularity.

Corollary 4.4. *Let (W,ω) be a compact symplectic manifold that is a weak symplectic filling of a contact manifold (M,ξ). Suppose that N is either a* Lob *or a* bLob *in M, then we can choose close to the binding and to the boundary of N the almost complex structure described in the previous sections, and extend it to an almost complex structure J that is tamed by ω, whose bundle of complex tangencies along M is ξ and that makes M J-convex. By a generic perturbation away from the binding and the boundary of N, we can achieve that all somewhere injective holomorphic curves become regular.*

*We call a J with these properties an **almost complex structure adapted to** N.*

The argument in the proof of the corollary above is that the Bishop disks are already regular, and that all other simple holomorphic curves have to lie outside the neighborhood where we require an explicit form for J. Thus it suffices to perturb outside these domains to obtain regularity for every other simple curve.

4.2. The Moduli Space of Holomorphic Disks with a Marked Point

Until now, we only have studied the space of certain J-holomorphic *maps*

$$\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; J\big) = \big\{u\colon \mathbb{D}^2 \to W \mid \bar\partial_J u = 0 \text{ and } u\big(\partial\mathbb{D}^2\big) \subset N\big\},$$

but many maps correspond to different parametrizations of the same geometric disk. To get rid of this ambiguity (and to obtain compactness), we quotient the space of maps by the biholomorphic reparametrizations of the unit disk, that means, by the Möbius transformations, but we will also add a marked point $z_0 \in \mathbb{D}^2$ to preserve the structure of the geometric disk. To simplify the notation, we will also omit the almost complex structure J in $\widetilde{\mathcal{M}}(\mathbb{D}^2, N)$.

From now on let

$$\begin{aligned}\widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; z_0\big) &= \big\{(u, z_0) \mid z_0 \in \mathbb{D}^2,\ \bar\partial_J u = 0 \text{ and } u\big(\partial\mathbb{D}^2\big) \subset N\big\} \\ &= \widetilde{\mathcal{M}}\big(\mathbb{D}^2, N\big) \times \mathbb{D}^2\end{aligned}$$

be the space of holomorphic maps together with a special point $z_0 \in \mathbb{D}^2$ that will be called the **marked point**. The **moduli space** we are interested in is the space of equivalence classes

$$\mathcal{M}\big(\mathbb{D}^2, N; z_0\big) = \widetilde{\mathcal{M}}\big(\mathbb{D}^2, N; z_0\big)/\sim$$

where we identify two elements (u, z_0) and (u', z_0'), if and only if there is a biholomorphism $\varphi\colon \mathbb{D}^2 \to \mathbb{D}^2$ such that $u = u' \circ \varphi^{-1}$ and $z_0 = \varphi(z_0')$. The map $(u, z) \mapsto u(z)$ descends to a well defined map

$$\begin{aligned}\mathrm{ev}\colon \mathcal{M}\big(\mathbb{D}^2, N; z_0\big) &\to W \\ [u, z_0] &\mapsto u(z_0)\end{aligned}$$

on the moduli space, which we call the **evaluation map**.

Let N be a Lob or a bLob, and assume that B_0 is one of the components of the binding of N. Since this is the only situation, we are really interested in these notes, we introduce the notation $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N)$ for the connected component in $\widetilde{\mathcal{M}}(\mathbb{D}^2, N)$ that contains the Bishop family around B_0. When adding a marked point, we write $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$ and $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ for the corresponding subspaces.

It is easy to see that $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ is a smooth (non-compact) manifold with boundary. Note first that $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$ is also a smooth and non-compact manifold with boundary: If J is regular, we know that $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N)$ is a smooth manifold, and so the boundary of the product manifold $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$ is

$$\partial\widetilde{\mathcal{M}}_0\bigl(\mathbb{D}^2, N; z_0\bigr) = \widetilde{\mathcal{M}}_0\bigl(\mathbb{D}^2, N\bigr) \times \partial\mathbb{D}^2.$$

Passing to the quotient preserves this structure, because the boundary of the maps in $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N)$ intersects each of the pages of the open book exactly once (this is a consequence of Corollary 3.11 and Section 3.2.2), and hence each of the disks is injective along its boundary. The only Möbius transformation that preserves the boundary pointwise is the identity, hence it follows that the group of Möbius transformations acts smoothly, freely and properly on $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$, and hence the quotient will be a smooth manifold of dimension

$$\dim \mathcal{M}_0\bigl(\mathbb{D}^2, N; z_0\bigr) = \dim \widetilde{\mathcal{M}}_0\bigl(\mathbb{D}^2, N; z_0\bigr) - 3 = \operatorname{ind}_u \bar{\partial}_J + 2 - 3 = n + 1.$$

As before the points on the boundary of $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ are the classes $[u, z]$ with $z \in \partial\mathbb{D}^2$. It is also clear that the evaluation map $\mathrm{ev}_{z_0} \colon \mathcal{M}_0(\mathbb{D}^2, N; z_0) \to W$ is smooth.

Remember that the Bishop disks contract to points as they approach the binding B_0. We will show that we incorporate B_0 into the moduli space $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ and that the resulting space carries a natural smooth structure that corresponds to the intuitive picture of disks collapsing to one point.

The neighborhood of the binding B_0 in W is diffeomorphic to the model

$$U = \bigl\{(z_1, z_2; \mathbf{q}, \mathbf{p}) \in \mathbb{C}^2 \times T^*B_0 \bigm| \operatorname{Re}(z_2) > 1 - \delta\bigr\} \cap h^{-1}\bigl((-\infty, 1/2]\bigr)$$

for small $\delta > 0$ with the function

$$h(z_1, z_2) = \frac{1}{2}\bigl(|z_1|^2 + |z_2|^2\bigr) + f_{B_0}(\mathbf{q}, \mathbf{p}),$$

see Section 3.2.2.

The content of Proposition 3.21 and of Corollary 3.20 is that for every point

$$(z, s; \mathbf{q}_0, \mathbf{0}) \in U$$

with $s \in (1-\delta, 1)$ and $\mathbf{q}_0$ in the 0-section of T^*B_0,

- there is up to a Möbius transformation a unique holomorphic map $u \in \widetilde{\mathcal{M}}_0(\mathbb{D}^2, N)$ containing that point in its image, and
- $\widetilde{\mathcal{M}}(\mathbb{D}^2, N)$ does not contain any holomorphic maps whose image is not entirely contained in $U \cap (\mathbb{C} \times \mathbb{R} \times B_0)$.

As a result, it follows that $V = \mathrm{ev}_{z_0}^{-1}(U)$ is an open subset of $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$, and that the restriction of the evaluation map

$$\mathrm{ev}_{z_0}|_V : V \to U$$

is a diffeomorphism onto $U \cap (\mathbb{C} \times (1-\delta, 1) \times B_0)$. The closure of this subset is the smooth submanifold

$$U \cap (\mathbb{C} \times \mathbb{R} \times B_0),$$

which we obtain by including the binding $\{0\} \times \{1\} \times B_0$ of N.

Using the evaluation map, we can identify V with its image in U, and this way glue B_0 to the moduli space $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$. The new space is also a smooth manifold with boundary, and the evaluation map extends to it, and is a diffeomorphism onto its image in U so that we can effectively identify U with a subset of the moduli space. In particular, it follows that B_0 is a submanifold that is of codimension 2 in the boundary of the moduli space.

The aim of the next section will consist in studying the Gromov compactification of $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$.

4.3. Compactness

Gromov compactness is a result that describes the possible limits of a sequence of holomorphic curves, and ensures under certain conditions that every such sequence contains a converging subsequence. In the limit, a given sequence of holomorphic curves may break into several components, called **bubbles**, each of which is again a holomorphic curve. We will not describe in detail what "convergence" in this sense really means, but we only sketch the idea: The holomorphic curves in a moduli space can be represented by holomorphic maps, and in the optimal case, one could hope that by choosing for each curve in the given sequence a suitable representative, we might have uniform convergence of the maps, and this way we would find the limit of the sequence as a proper holomorphic curve. Unfortunately, this is usually wrong, but it might be true that for the correct choice of parametrization we have convergence on subdomains. Choosing different reparametrizations, we then obtain convergence on different domains, and each such domain gives then rise to a bubble, that means, a holomorphic curve that represents one component of the Gromov limit.

Theorem 4.5 (Gromov compactness)**.** *Let (W, J) be a compact almost complex manifold (with or without boundary), and assume that J is tamed by a symplectic form ω. Let L be a compact totally real submanifold. Choose a sequence of J-holomorphic maps $u_k\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, L)$ whose ω**-energy***

$$E(u_k) := \int_{\mathbb{D}^2} u_k^*\omega$$

is bounded by a constant $C > 0$.

Then there is a subsequence of $(u_{k_l})_l$ that converges in the Gromov sense to a bubble tree composed of a finite family of non-constant holomorphic disks $u_\infty^{(1)}, \dots, u_\infty^{(K)}$ whose boundary lies in L, and a finite family of non-constant holomorphic spheres $v_\infty^{(1)}, \dots, v_\infty^{(K')}$. The total energy is preserved so that

$$\lim_{l\to\infty} E(u_{k_l}) = \sum_{j=1}^{K} E\big(u_\infty^{(j)}\big) + \sum_{j=1}^{K'} E\big(v_\infty^{(j)}\big).$$

If each of the disks u_k is equipped with a marked point $z_k \in \mathbb{D}^2$, then after possibly reducing to a another subsequence, there is a marked point z_∞ on one of the components of the bubble tree such that $\lim_k z_k = z_\infty$ in a suitable sense.

The ω-energy is fundamental in the proof of the compactness theorem to limit the number of possible bubbles: By [25, Proposition 4.1.4], there exists in the situation of Theorem 4.5 a constant $\hbar > 0$ that bounds the energy of every holomorphic sphere or every holomorphic disk $u_k\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, L)$ from below. Since every bubble needs to have at least an $\hbar$-quantum of energy, and since the total energy of the curves in the sequence is bounded by C, the limit curve will never break into more than $C/\hbar$ bubbles (the upper bound of the energy is also used to make sure that each bubble is a compact surface).

We will show in the rest of this section that we can apply Gromov compactness to sequences of holomorphic disks lying in the moduli space $\mathcal{M}_0(\mathbb{D}^2, N)$ studied in the previous section, and how we can incorporate these limits into $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ to construct the compactification $\overline{\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$.

Proposition 4.6. *Let N be a* Lob *or a* bLob *in the contact boundary (M, ξ) of a symplectic filling (W, ω), and assume that we find a contact form α for ξ such that $\omega|_{TN} = d\alpha|_{TN}$.*

There is a global energy bound $C > 0$ for all holomorphic disks in $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N)$.

Proof. There is a slight complication in our proof, because we may not assume that ω is globally exact, which would allow us to obtain the energy of a holomorphic disk by integrating over the boundary of the disk. To prove the desired statement, proceed as follows: Let $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, N)$ be any element in $\widetilde{\mathcal{M}}_0(\mathbb{D}^2, N)$. By our assumption, there exists a smooth path of maps u_t that starts at the constant map $u_0(z) \equiv b_0 \in B_0$ in the binding and ends at the chosen map $u_1 = u$. This family of disks may be interpreted as a map from the 3-ball into W. The boundary consists of the image of u_1, and the union of the boundary of all disks $u_t|_{\partial\mathbb{D}^2}$.

Using Stokes' theorem, we get

$$0 = \int_{[0,1]\times\mathbb{D}^2} u_t^* d\omega = \int_{\mathbb{D}^2} u_1^*\omega + \int_{[0,1]\times\partial\mathbb{D}^2} u_t^*\omega$$

so that $E(u) = -\int_{[0,1]\times\partial\mathbb{D}^2} u_t^*\omega$.

By our assumption, we have a contact form on the contact boundary M for which $\omega|_{TN} = d\alpha|_{TN}$, so that using Stokes' theorem a second time (and that $u_0(z) = b_0$) we get

$$E(u) = \int_{\partial\mathbb{D}^2} u^*\alpha.$$

The Legendrian foliation on N is an open book whose pages are fibers of a fibration $\vartheta\colon N \setminus B \to \mathbb{S}^1$. Hence the 1-form $d\vartheta$ and $\alpha|_{TN}$ have the same kernel, and it follows that there exists a smooth function $f\colon N \to [0, \infty)$ such that

$$\alpha|_{TN} = f\, d\vartheta.$$

The function f vanishes on the binding and on the boundary of a bLob, and f is hence bounded on N so that we define $C := 2\pi \max_{x\in N} |f(x)|$.

Using that the boundary of u intersects every leaf of the open book exactly once, we obtain for the energy of u the estimate

$$E(u) = \int_{\partial\mathbb{D}^2} u^*\alpha \le \max_{x\in N} \left|f(x)\right| \int_{\partial\mathbb{D}^2} u^* d\vartheta \le 2\pi \max_{x\in N} \left|f(x)\right| = C. \quad \blacksquare$$

With the given energy bound, we obtain now Gromov compactness in form of the following corollary.

Corollary 4.7. *Let N be a* Lob *or a* bLob *in the contact boundary (M,ξ) of a symplectic filling (W,ω), and assume that we find a contact form α for ξ such that $\omega|_{TN} = d\alpha|_{TN}$. Let $(u_k)_k$ be a sequence of holomorphic maps in $\widetilde{\mathcal{M}}_0(\mathbb{D}^2,N)$.*

There exists a subsequence $(u_{k_l})_l$ that converges either

- *uniformly up to reparametrizations of the domain to a J-holomorphic map $u_\infty \in \widetilde{\mathcal{M}}_0(\mathbb{D}^2,N)$,*
- *to a constant disk $u_\infty(z) \equiv b_0$ lying in the binding of N,*
- *or to a bubble tree composed of a single holomorphic disk $u_\infty : (\mathbb{D}^2,\partial\mathbb{D}^2) \to (W,N)$ and a finite family of non-constant holomorphic spheres $v_1,\dots,v_j$ with $j \ge 1$.*

Proof. We will apply Theorem 4.5. The submanifold N is not totally real along the binding B and ∂N, but we simply remove a small open neighborhood of both sets. By Proposition 3.24, none of the holomorphic disks u_k may get close to ∂N, and by Proposition 3.21 we know precisely how the curves look like that intersect a neighborhood of B. If we find disks in $(u_k)_k$ that get arbitrarily close to the binding of N, then using that B is compact, we may choose a subsequence that converges to a single point in the binding. If $(u_k)_k$ stays at finite distance from B, we may assume that the neighborhood, we have removed from N is so small that the holomorphic disks we are studying all lie inside.

If the sequence $(u_k)_k$ does not contain any subsequence that can be reparametrized in such a way that it converges to a single non-constant disk u_∞, we use Gromov compactness to obtain a subsequence that splits into a finite collection of holomorphic spheres and disks. But as a consequence from Corollary 3.11, we see that non-constant holomorphic disks attached to N need to intersect the pages of the open book transversely in positive direction. A sequence of holomorphic disks that intersects every page of the open book exactly once, cannot split into several disks intersecting pages several times. In particular possible bubble trees contain by this argument a single disk in its limit. ■

Above, we have obtained compactness for a sequence of disks, but we would like to understand how these limits can be incorporated into the moduli space. Adding the bubble trees to the space of parametrized maps does not give rise to a valid topology, because the bubbling phenomenon can only be understood by using different reparametrizations of the disk to recover all components of the bubble tree.

We will denote the compactification of $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ by $\overline{\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$. For us, it is not necessary to understand the topology of $\overline{\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$ in detail, but it will be sufficient to see that bubbling is a "codimension 2 phenomenon". In fact, it is not the topology of the moduli space itself we are interested in, but our aim is to obtain information about the symplectic manifold. For this we want to make sure that the image under the evaluation map of all bubble trees that appear in the limit, that means, of $\overline{\mathcal{M}}_0(\mathbb{D}^2, N; z_0) \setminus \mathcal{M}_0(\mathbb{D}^2, N; z_0)$ is contained in the image of a smooth map defined on a finite union of manifolds each of dimension at most

$$\dim \mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big) - 2.$$

For this to be true, we need to impose additional conditions for (W, ω).

Definition. A $2n$-dimensional symplectic manifold (M, ω) is called

- **symplectically aspherical**, if $\omega([A])$ vanishes for every $A \in \pi_2(M)$.
- It is called **semipositive** if every $A \in \pi_2(M)$ with $\omega([A]) > 0$ and $c_1(A) \geq 3 - n$ has non-negative Chern number.

Note that every symplectic 4- or 6-manifold is obviously semipositive.

In a symplectically aspherical manifold no J-holomorphic spheres exist, because their energy would be zero. So in particular they may not appear in any bubble tree and Corollary 4.7 implies in our situation that every sequence of holomorphic disks contains a subsequence that either collapses into the binding or that converges to a single disk in $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$. Using the results of Section 4.2, we obtain the following corollary.

Corollary 4.8. *Let (W, ω) be a compact symplectically aspherical manifold that is a weak filling of a contact manifold (M, ξ). Let N be a* Lob *or a* bLob *in M, and assume that we find a contact form for ξ such that $\omega|_{TN} = d\alpha|_{TN}$. Choose an almost complex structure J that is adapted to N (as in Corollary 4.4).*

Then the compactification of the moduli space $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ is a smooth compact manifold

$$\overline{\mathcal{M}}_0\big(\mathbb{D}^2, N; z_0\big) = \mathcal{M}_0\big(\mathbb{D}^2, N; z_0\big) \cup (\textit{binding of } N)$$

with boundary. The binding of N is a submanifold of codimension 2 in the boundary $\partial\overline{\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$.

The condition of asphericity is very strong, and we will obtain more general results by studying instead semipositive manifolds. The important point here is that a generic almost complex structure only ensure transversality for somewhere injective holomorphic curves, see Section 4.1.2. Even though the holomorphic disks in $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ are simple, it could happen that once the disks bubble, there appear spheres that are multiple covers. For these, we cannot guarantee transversality, and hence we cannot directly predict if the compactification of $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ consists of adding "codimension 2 strata" or if we will be forced to include too many bubble trees.

Still, we know that every sphere that is not simple is the multiple cover of a simple one (by the Riemann-Hurwitz formula a sphere can only multiply cover a sphere), we can hence compute the dimension of the moduli space of the underlying simple spheres, and use this information as an upper bound for the dimension of the spheres that appear in the bubble tree.

Let $v\colon \mathbb{S}^2 \to W$ be a holomorphic sphere that is a k-fold cover of a sphere $\widetilde{v}$ representing a homology class $[v]$ and $[\widetilde{v}] \in H_2(W, \mathbb{Z})$ respectively with $[v] = k[\widetilde{v}]$ and with $\omega([\widetilde{v}]) > 0$. The expected dimension of the space of maps containing v is by an index formula

$$\operatorname{ind}_v \bar{\partial}_J = 2n + 2c_1\big([v]\big) = 2n + 2kc_1\big([\widetilde{v}]\big).$$

The space of biholomorphisms of $\mathbb{S}^2$ has dimension 6, and hence the expected dimension of the moduli space of unparametrized spheres that contain $[v]$ is $\operatorname{ind}_v \bar{\partial}_J - 6 = 2(n-3) + 2kc_1([\widetilde{v}])$.

As we explained above and in Section 4.1.2, this expected dimension does not correspond in general to the observed dimension of the bubble trees, instead we study the expected dimension of the underlying simple spheres. The dimension of the space containing $\widetilde{v}$ is given by $\operatorname{ind}_{\widetilde{v}} \bar{\partial}_J - 6 =$

$2(n-3)+2c_1([\widetilde{v}])$. If $c_1([\widetilde{v}]) < n-3$, then the expected dimension will be negative, and since we obtain regularity of all simple holomorphic curves by choosing a generic almost complex structure, it follows that the moduli space containing $\widetilde{v}$ is generically empty. As a consequence bubble trees appearing as limits do not contain any component that is the k-fold cover of a simple sphere representing the homology class $[\widetilde{v}]$.

If $c_1([\widetilde{v}]) \geq n-3$, the definition of semipositivity implies that $c_1([\widetilde{v}]) \geq 0$. When we compare the expected dimension of the moduli space containing v with the one of the underlying disk $\widetilde{v}$, we observe that $\operatorname{ind}_v \bar{\partial}_J - 6 = 2(n-3) + 2kc_1([\widetilde{v}]) \geq 2(n-3) + 2c_1([\widetilde{v}]) = \operatorname{ind}_{\widetilde{v}} \bar{\partial}_J - 6$.

Consider now the image in W of all spheres in the moduli space of v that are k-fold multiple covers of some simple sphere. Their image is contained in the image of the simple spheres lying in the same moduli space as $\widetilde{v}$. The dimension of this second moduli space is smaller or equal than the expected dimension of the initial moduli space containing v, and even though we cannot ensure regularity for v, we have an estimate on the dimension of the subset containing all singular spheres.

The following result allows us to find the desired bound for the dimension of the image of complete bubble trees.

Proposition 4.9. *Assume that (W,ω) is semipositive. To compactify the moduli space $\mathcal{M}_0(W,N,z_0)$, one has to add bubbled curves. We find a finite set of manifolds $X_1,\dots,X_N$ with $\dim X_j \leq \dim \mathcal{M}_0(W,N,z_0) - 2$ and smooth maps $f_j\colon X_j \to W$ such that the image of the bubbled curves under the evaluation map ev_{z_0} is contained in*

$$\bigcup f_j(X_j).$$

When we consider instead the compactification of the boundary $\partial\mathcal{M}_0(W,N,z_0)$, that means the space of holomorphic disks with a marked point on the boundary of the disk only, then we obtain the analogue result, only that the manifolds $X_1,\dots,X_N$ have dimension $\dim X_j \leq \dim \partial\mathcal{M}_0(W,N,z_0) - 2 = \dim \mathcal{M}_0(W,N,z_0) - 3$.

Proof. The standard way to treat bubbled curves consists in considering them as elements in a bubble tree: Here such a tree is composed by a simple holomorphic disk $u_0\colon (\mathbb{D}^2,\mathbb{S}^1) \to (W,N)$ and holomorphic spheres $u_1,\dots,u_{k'}\colon \mathbb{S}^2 \to W$. These holomorphic curves are connected to each other in a certain way. We formalize this relation by saying that the holomorphic curves are vertices in a tree, i.e. in a connected graph without cycles. We denote the edges of this graph by $\{u_i, u_j\}$, $0 \le i < j \le k'$.

Now we assign to any edge two nodal points z_{ij} and z_{ji}, the first one in the domain of the bubble u_i, the other one in the domain of u_j, and we require that $\mathrm{ev}_{z_{ij}}(u_i) = \mathrm{ev}_{z_{ji}}(u_j)$. For technical reasons, we also require nodal points on each holomorphic curve to be pairwise distinct. To include into the theory, trees with more than one bubble connected at the same point to a holomorphic curve, we add "ghost bubbles". These are constant holomorphic spheres inserted at the point where several bubbles are joined to a single curve. Now all the links at that point are opened and reattached at the ghost bubble. Ghost bubbles are the only constant holomorphic spheres we allow in a bubble tree.

The aim is to give a manifold structure to these bubble trees, but unfortunately this is not always possible, when multiply covered spheres appear in the bubble tree.

Instead, we note that the image of every bubble tree is equal to the image of a simple bubble tree, that means, to a tree, where every holomorphic sphere is simple and any two spheres have different image. Since we are only interested in the image of the evaluation map on the bubble trees, it is for our purposes equivalent to consider the simple bubble tree instead of the original one. The disk u_0 is always simple, and does not need to be replaced by another simple curve.

Let $u_0, u_1, \ldots, u_{k'}$ be the holomorphic curves composing the original bubble tree, and let $A_i \in H_2(W)$ be the homology class represented by the holomorphic sphere u_i. The simple tree is composed by $u_0, v_1, \ldots, v_k$ such that for every u_j there is a bubble sphere v_{i_j} with equal image

$$u_j\big(\mathbb{S}^2\big) = v_{i_j}\big(\mathbb{S}^2\big)$$

and in particular $A_j = m_j B_{i_j}$, where $B_{i_j} = [v_{i_j}] \in H_2(W)$ and $m_j \ge 1$ is an integer. Write also $\mathbf{A}$ for the sum $\sum_{j=1}^{k'} A_j$ and $\mathbf{B}$ for the sum $\sum_{i=1}^{k} B_i$. Below we will compute the dimension of this simple bubble tree.

The initial bubble tree $u_0, u_1, \ldots, u_{k'}$ is the limit of a sequence in the moduli space $\mathcal{M}_0(W, N, z_0)$. Hence the connected sum $u_\infty := u_0 \# \cdots \# u_{k'}$ is, as element of $\pi_2(W, N)$, homotopic to a disk u in the bishop family, and the Maslov indeces

$$\mu(u) := \mu\big(u^*TW, u^*TN\big) \quad \text{and} \quad \mu(u_\infty) := \mu\big(u_\infty^*TW, u_\infty^*TN\big)$$

have to be equal. With the standard rules for the Maslov index (see for example [25, Appendix C.3]), we obtain

$$2 = \mu(u) = \mu(u_\infty) = \mu(u_0) + \sum_{j=1}^{k'} 2c_1\big([u_j]\big) = \mu(u_0) + 2c_1(\mathbf{A}).$$

The dimension of the unconnected set of holomorphic curves $\widetilde{\mathcal{M}}_{[u_0]}(W,N,z_0) \times \prod_{j=1}^{k} \widetilde{\mathcal{M}}_{B_j}(W)$ for the simple bubble tree is

$$\begin{aligned}\big(n+\mu(u_0)\big) + \sum_{j=1}^{k} 2\big(n + c_1(B_j)\big) &= n + 2 - 2c_1(A) + 2nk + \sum_{j=1}^{k} 2c_1(B_j)\\ &= n + 2 + 2nk + 2\big(c_1(\mathbf{B}) - c_1(\mathbf{A})\big).\end{aligned}$$

In the next step, we want to consider the subset of connected bubbles, i.e. we choose a total of k pairs of nodal points, which then have to be pairwise equal under the evaluation map. The nodal points span a manifold

$$Z(2k) \subset \big\{(1,\dots,2k) \to \mathbb{D}^2 \amalg \mathbb{S}^2 \amalg \cdots \amalg \mathbb{S}^2\big\}$$

of dimension $4k$. The dimension reduction comes from requiring that the evaluation map

$$\mathrm{ev}\colon \widetilde{\mathcal{M}}_{[u_0]}(W,N,z_0) \times \prod_{j=1}^{k} \widetilde{\mathcal{M}}_{B_j}(W) \times Z(2k) \to W^{2k}$$

sends pairs of nodal points to the same image in the symplectic manifold. By regularity and transversality of the evaluation map to the diagonal submanifold $\triangle(k) \hookrightarrow W^{2k}$, the dimension of the space of holomorphic curves is reduced by the codimension of $\triangle(k)$, which is $2nk$.

As a last step, we have to add the marked point z_0 used for the evaluation map ev_{z_0}, this way increasing the dimension by 2, and then we take the quotient by the automorphism group to obtain the moduli space. The dimension of the automorphism group is $6k+3$. Hence the dimension of the total moduli space is

$$\begin{aligned}&n + 2 + 2nk + 2\big(c_1(\mathbf{B}) - c_1(\mathbf{A})\big) + 4k - 2nk + 2 - (6k+3)\\ &\quad = n + 1 - 2k + 2\big(c_1(\mathbf{B}) - c_1(\mathbf{A})\big) \le n + 1 - 2k.\end{aligned}$$

The inequality holds because by the assumption of semipositivity, all the Chern classes are non-negative on holomorphic spheres, and all coefficients n_j

in the difference $c_1(\mathbf{B}) - c_1(\mathbf{A}) = \sum_j c_1(B_j) - \sum_i c_1(A_i) = \sum_j c_1(B_j) - \sum_i m_i c_1(B_{j_i}) = \sum_j n_j c_1(B_j)$ are non-positive integers.

The computations for the disks in $\partial\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ only differs by the requirement that the marked point needs to lie on the boundary of the disk u_0 instead of moving freely on the bubble tree. Instead of having two degrees of freedom for this choice, we thus only add one extra dimension. ■

4.4. Proof of the Non-fillability Theorem A

Theorem A. *Let (M, ξ) be a contact manifold that contains a* bLob *N, then M does not admit any semi-positive weak symplectic filling (W, ω) for which $\omega|_{TN}$ is exact.*

Assume there were a semi-positive symplectic filling (W, ω) for which $\omega|_{TN}$ is exact. Let α be a positive contact form for ξ. By Proposition 3.14, we can extend (W, ω) with a collar in such a way that we have $\omega|_{TN} = d\alpha|_{TN}$, which will allow us to use the energy estimates of the previous section. Now we choose an almost complex structure that is adapted to the bLob N as in Corollary 4.4, and we will study the moduli space $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ defined in Section 4.2 of holomorphic disks with one marked point lying in the same component as the Bishop family around a chosen component B_0 of the binding of N.

Trace a smooth path $\gamma\colon [0,1] \to N$ that starts at $\gamma(0) \in B_0$ and ends on the boundary ∂N. Assume further that γ is a regular curve, and that it intersects the binding and ∂N only on the endpoints of $[0,1]$. We want to select a 1-dimensional moduli space in $\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ by only considering

$$\mathcal{M}^\gamma := \mathrm{ev}_{z_0}^{-1}\big(\gamma(I)\big).$$

It will be important for us that $\gamma(I)$ does not intersect the image of any bubble trees in $\overline{\mathcal{M}}_0(\mathbb{D}^2, N; z_0) \setminus \mathcal{M}_0(\mathbb{D}^2, N; z_0)$.

By Proposition 4.9, we have that the bubble trees in $\overline{\partial\mathcal{M}}_0(\mathbb{D}^2, N; z_0)$ lie in the image of a finite union of smooth maps defined on manifolds of dimension $\dim \partial\mathcal{M}_0(\mathbb{D}^2, N; z_0) - 2 = \dim N - 2$. The subset $N \setminus \mathrm{ev}_{z_0}(\text{bubble trees})$ is connected and we can deform γ keeping the endpoints fixed so that it does not intersect any of the bubble trees.

For a small perturbation of J (away from the binding and the boundary of N), we can make sure that the evaluation map ev_{z_0} is transverse to the path $\gamma(I)$. If the perturbed J lies sufficiently close to the old one, then γ will also not intersect any bubble trees for this new J, for otherwise we could choose a sequence of almost complex structures J_k converging to the unperturbed J such that for everyone there existed a bubble tree v_k intersecting γ. We would find a converging subsequence of v_k yielding a bubble tree v_∞ for the unperturbed almost complex structure intersecting γ, which contradicts our assumption.

It follows that $\mathcal{M}^\gamma$ is a collection of compact 1-dimensional submanifolds of $\partial\mathcal{M}_0(\mathbb{D}^2, N; z_0)$. There is one component in $\mathcal{M}^\gamma$, which we will denote by $\mathcal{M}_0^\gamma$ that contains the Bishop disks that intersect $\gamma([0,\varepsilon))$. We know that the Bishop disks are the only disks close to the binding, and hence it follows that $\mathcal{M}_0^\gamma$ cannot be a loop that closes, but must be instead a closed interval.

The first endpoint of $\mathcal{M}_0^\gamma$ is the constant disk with image $\gamma(0) \in B_0$, and we will deduce a contradiction by showing that no holomorphic disk can be the second endpoint of $\mathcal{M}_0^\gamma$.

By Proposition 3.24, there is a small neighborhood of ∂N that cannot be entered by any holomorphic disk. By our construction the endpoint of $\mathcal{M}_0^\gamma$ cannot be any bubble tree either. It follows that the endpoint needs to be a regular disk $[u, z_0] \in \partial\mathcal{M}_0(\mathbb{D}^2, N; z_0)$ for which the boundary of u lies in $N \setminus (\partial N \cup B)$ and whose interior points cannot touch ∂W either, because we are assuming that the boundary of W is convex.

It follows that this regular disk cannot really be the endpoint of $\mathcal{M}_0^\gamma$, because the evaluation map ev_{z_0} will also be transverse to γ at $[u, z_0]$ so that we can extend $\mathcal{M}_0^\gamma$ further.

This leads to a contradiction that shows that the assumption that the boundary of W is everywhere convex cannot hold.

4.5. Proof of Theorem B

For the proof, we first recall the definition of the degree of a map.

Definition. Let X and Y be closed oriented n-manifolds. The **degree** of a map $f\colon X \to Y$ is the integer $d = \deg(f)$ such that

$$f_\#[X] = d \cdot [Y],$$

where $[X] \in H_n(X, \mathbb{Z})$ and $[Y] \in H_n(Y, \mathbb{Z})$ are the fundamental classes of the corresponding manifolds. When the manifolds X and Y are not orientable, we define the degree to be an element of $\mathbb{Z}_2$ using the same formula, where the fundamental classes are elements in $H_n(X,\mathbb{Z}_2)$ and $H_n(Y,\mathbb{Z}_2)$.

Note that we can easily compute the degree of a smooth map f between smooth manifolds by considering a regular value $y_0 \in Y$ of f (which by Sard's theorem exist in abundance), and adding

$$\deg f = \sum_{x\in f^{-1}(y_0)} \operatorname{sign} Df_x,$$

where the point x contributes to the sum with $+1$, whenever Df_x is orientation preserving, and contributes with -1 otherwise. In case the manifolds are not orientable, we can always add $+1$ in the above formula, but need to take sum over $\mathbb{Z}_2$.

Theorem B. *Let (M,ξ) be a contact manifold of dimension $(2n+1)$ that contains a* Lob *N. If M has a weak symplectic filling (W,ω) that is symplectically aspherical, and for which $\omega|_{TN}$ is exact, then it follows that N represents a trivial class in $H_{n+1}(W,\mathbb{Z}_2)$. If the first and second Stiefel-Whitney classes $w_1(N)$ and $w_2(N)$ vanish, then we obtain that $[N]$ must be a trivial class in $H_{n+1}(W,\mathbb{Z})$.*

Using Proposition 3.14 we can assume that $\omega|_{TN} = d\alpha|_{TN}$ for a chosen contact form α. Choose an almost complex structure J on W that is adapted to the Lob N, and let $\mathcal{M}_0(\mathbb{D}^2,N;z_0)$ be the moduli space of holomorphic disks with one marked point lying in the same component as the Bishop family around a chosen component of the binding of N.

Since W is symplectically aspherical, we obtain by Corollary 4.8 that $\overline{\mathcal{M}}_0(\mathbb{D}^2,N;z_0)$ is a compact smooth manifold with boundary. It was shown in [13] that $\mathcal{M}_0(\mathbb{D}^2,N;z_0)$ is orientable if the first and second Stiefel-Whitney classes of $N\setminus B$ vanish. With our assumptions this is the case, because $w_j(N\setminus B) = w_j(N)|_{(N\setminus B)}$. If $\mathcal{M}_0(\mathbb{D}^2,N;z_0)$ is orientable then $\overline{\mathcal{M}}_0(\mathbb{D}^2,N;z_0)$ will also be orientable: If there were an orientation reversing loop γ in the compactified moduli space (which is obtained from $\mathcal{M}_0(\mathbb{D}^2,N;z_0)$ by gluing in B as codimension 3 submanifold), then due to the large codimension we could easily push γ completely into the regular part of the moduli space, where it would still need to be orientation reversing.

It follows that the boundary $\partial\overline{\mathcal{M}}_0(\mathbb{D}^2,N;z_0)$ is also homologically a boundary (either with $\mathbb{Z}$- or $\mathbb{Z}_2$-coefficients depending on the orientability of the considered spaces).

Denote the restriction of the evaluation map

$$\mathrm{ev}_{z_0}|_{\partial\overline{\mathcal{M}}_0(\mathbb{D}^2,N;z_0)} \colon \partial\overline{\mathcal{M}}_0\big(\mathbb{D}^2,N;z_0\big) \to N,$$

by f. We know that close to the binding every point is covered by a unique Bishop disk, this implies by the remarks made above that the degree $\deg(f)$ needs to be ± 1.

We have the following obvious equation

$$\mathrm{ev}_{z_0} \circ \iota_{\partial\overline{\mathcal{M}}} = \iota_N \circ f,$$

where $\iota_{\partial\overline{\mathcal{M}}}$ denotes the embedding of $\partial\overline{\mathcal{M}}_0(\mathbb{D}^2,N;z_0)$ in $\overline{\mathcal{M}}_0(\mathbb{D}^2,N;z_0)$ and ι_N the embedding of N in W. The homomorphism induced by $\iota_{\partial\overline{\mathcal{M}}}$ is the trivial map on the $(n+1)$-st homology group, so that the left side of the equation gives rise to the 0-map

$$H_{n+1}\big(\partial\overline{\mathcal{M}}_0\big(\mathbb{D}^2,N;z_0\big),R\big) \to H_{n+1}(W,R)$$

with R being either $\mathbb{Z}$ or $\mathbb{Z}_2$. Since $f_\#$ is $\pm$ identity, it follows that ι_N has to induce the trivial map on homology, which implies that N is homologically trivial in W.

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