Chapter 9
Geometric Minimizing Movements

We now examine some minimizing movements describing the motion of sets. Such a motion can be framed in the setting of the previous chapter after identification of a set A with its characteristic function $u = \chi_A$. The energies we are going to consider are of perimeter type; i.e., with

$$F(A) = \mathscr{H}^{n-1}(\partial A) \tag{9.1}$$

as a prototype in the notation of the previous chapter.

9.1 Motion by Mean Curvature

The prototype of a geometric motion is *motion by mean curvature*; i.e., a family of sets $A(t)$ whose boundary moves in the normal direction with velocity proportional to its curvature (inwards in convex regions and outwards in concave regions). In the simplest case when the initial datum is a ball in $\mathbb{R}^2$, $A(0) = A_0 = B_{R_0}(0)$, the motion is given by concentric balls with radii satisfying

$$\begin{cases} R' = -\dfrac{c}{R} \\ R(0) = R_0; \end{cases} \tag{9.2}$$

i.e., $R(t) = \sqrt{R_0^2 - 2ct}$, valid until the *extinction time* $t = R_0^2/2c$, when the radius vanishes.

A heuristic argument suggests that the variation of the perimeter be linked to the notion of curvature; hence, we expect to be able to obtain motion by mean curvature as a minimizing movement for the perimeter functional. We will see that, in order to obtain geometric motions as minimizing movements, we will have to modify the procedure described in the previous chapter.

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Example 9.1 (Pinning for the perimeter motion). Let $n = 2$. We apply the minimizing-movement procedure to the perimeter functional (9.1) and the initial datum $A_0 = B_{R_0}(0)$ in $\mathbb{R}^2$.

With fixed τ, since

$$\int_{\mathbb{R}^2} |\chi_A - \chi_B|^2 \, dx = |A \triangle B|,$$

the minimization to determine A_1 is

$$\min\Big\{\mathscr{H}^1(\partial A) + \frac{1}{2\tau}|A \triangle A_0|\Big\}. \tag{9.3}$$

We note that we can restrict our attention to sets A contained in A_0, since otherwise taking $A \cap A_0$ as test sets in their place would decrease both terms in the minimization. Once this is observed, we also note that, given $A \subset A_0$, if $B_R(x) \subset A_0$ has the same measure as A then it decreases the perimeter part of the energy (strictly, if A itself is not a ball) while keeping the second term fixed. Hence, we can limit our analysis to balls $B_R(x) \subset A_0$, for which the energy depends only on R. The incremental problem is then given by

$$\min\Big\{2\pi R + \frac{\pi}{2\tau}(R_0^2 - R^2) : 0 \le R \le R_0\Big\}, \tag{9.4}$$

whose minimizer is either $R = 0$ (with value $\frac{\pi}{2\tau} R_0^2$) or $R = R_0$ (with value $2\pi R_0$), since in (9.4) we are minimizing a concave function of R. For τ small the minimizer is then R_0. This means that the motion is trivial: $A_k = A_0$ for all k, and hence also the resulting minimizing movement is trivial.

9.2 A First (Unsuccessful) Generalization

We may generalize the scheme of the minimizing movements by taking a more general distance term in the minimization; e.g., considering x_k as a minimizer of

$$\min\Big\{F(x) + \frac{1}{\tau}\Phi(\|x - x_{k-1}\|)\Big\}, \tag{9.5}$$

where Φ is a continuous increasing function with $\Phi(0) = 0$. As an example, we can consider

$$\Phi(z) = \frac{1}{p}|z|^p.$$

Note that in this case we obtain the estimate

$$\|x_k - x_{k-1}\|^p \le p\,\tau(F(x_{k-1}) - F(x_k))$$

for the minimizer x_k. Using Hölder's inequality as in the case $p = 2$, we end up with (for $j > h$)

$$\begin{aligned}\|x_j - x_h\| &\le (j-h)^{(p-1)/p}\Big(\sum_{k=h+1}^{j} \|x_k - x_{k-1}\|^p\Big)^{1/p}\\ &\le (p\,F(x_0))^{1/p}(\tau^{1/(p-1)}(j-h))^{(p-1)/p}.\end{aligned}$$

In order to obtain the $(1-\frac{1}{p})$-Hölder continuity for the interpolated function u^τ, we have to define it as

$$u^\tau(t) = u_{\lfloor t/\tau^{1/(p-1)}\rfloor}.$$

Note that we may use the previous definition $u^\tau(t) = u_{\lfloor t/\tau\rfloor}$ with time step τ for the interpolated function if we change the parameter τ in (9.5) and, to define x_k, consider the problem

$$\min\Big\{F(x) + \frac{1}{\tau^{p-1}}\Phi(\|x - x_{k-1}\|)\Big\} \tag{9.6}$$

instead.

Example 9.2 ((non-)geometric minimizing movements). We use the scheme above, with a slight variation in the exponents, since we will be interested in the description of the motion in terms of the radius of a ball in $\mathbb{R}^2$ (which is the square root of the L^2-norm and not the norm itself). As in the previous example, we take the initial datum $A_0 = B_{R_0} = B_{R_0}(0)$, and consider A_k defined recursively as a minimizer of

$$\min\Big\{\mathscr{H}^1(\partial A) + \frac{1}{p\tau^{p-1}}|A \triangle A_0|^p\Big\}, \tag{9.7}$$

with $p > 1$. As above, at each step the minimizer is given by balls

$$B_{R_k}(x_k) \subset B_{R_{k-1}}(x_{k-1}). \tag{9.8}$$

The value of R_k is determined by solving

$$\min\Big\{2\pi R + \frac{\pi^p}{p\tau^{p-1}}(R_{k-1}^2 - R^2)^p : 0 \le R \le R_{k-1}\Big\}, \tag{9.9}$$

which gives

$$\frac{R_k - R_{k-1}}{\tau} = -\frac{1}{\pi R_k^{1/(p-1)}(R_k + R_{k-1})}. \tag{9.10}$$

Note that, in this case, the minimum value is not taken at $R_k = R_{k-1}$ (this can be verified, e.g., by checking that the derivative of the function to be minimized in (9.9) is positive at R_{k-1}). By passing to the limit in (9.10) we deduce the equation

$$R' = -\frac{1}{2\pi R^{p/(p-1)}} \tag{9.11}$$

(valid until the extinction time).

Despite having obtained an equation for R, we notice that this approach is not satisfactory, since we have:

- **(non-geometric motion)** in (9.8) we have infinitely many solutions; namely, all balls centered in x_k with

$$|x_{k-1} - x_k| \le R_{k-1} - R_k.$$

  This implies that we may have moving centres $x(t)$ provided that $|x'| \le R'$ and $x(0) = 0$; in particular, we may choose $x(t) = (R_0 - R(t))z$ for any $z \in B_1(0)$ which converges to $R_0 z$; i.e., the point where the sets concentrate at the vanishing time may be any point in $\overline{B}_{R_0}$ at the extinction time. This implies that the motion is not a geometric one: sets do not move according to geometric quantities.
- **(failure to obtain mean-curvature motion)** even if we obtain an equation for R we never obtain the mean curvature flow since $p/(p-1) > 1$.

9.3 A Variational Approach to Curvature-Driven Motion

In order to obtain motion by curvature, Almgren, Taylor and Wang have introduced a variation of the implicit-time scheme described above, where the term $|A \triangle A_k|$ is substituted by an integral term which favours variations which are 'uniformly distant' to the boundary of A_k. The problem defining A_k is then

$$\min\Big\{\mathscr{H}^1(\partial A) + \frac{1}{\tau}\int_{A\triangle A_{k-1}} \mathrm{dist}(x, \partial A_{k-1})\, dx\Big\}. \tag{9.12}$$

Note that the integral term can be indeed interpreted as an L^2 distance between the boundaries of the sets.

We will not prove a general convergence result for an arbitrary initial datum A_0, but we will check the convergence to mean-curvature motion for $A = B_{R_0}$ in $\mathbb{R}^2$.

In this case we note that if A_{k-1} is a ball centered in 0 then we have:

- A_k is contained in A_{k-1}. To check this, note that, given a test set A, considering $A \cap A_{k-1}$ as a test set in its place decreases the energy in (9.12), strictly if $A \setminus A_{k-1} \neq \emptyset$.
- A_k is convex and with baricenter in 0. To check this, first, note that each connected component of A_k is convex. Otherwise, considering the convex envelopes decreases the energy (strictly, if one of the connected components if not convex). Then note that if 0 is not the baricenter of a connected component of A_k then a small translation towards 0 strictly decreases the energy (this follows by computing the derivative of the volume term along the translation). In particular, we only have one (convex) connected component.

From these properties we can conclude that A_k is indeed a ball centered in 0. Were it not so, there would be a line through 0 such that the boundary of A_k does not intersect perpendicularly this line. By a reflection argument, we then obtain a non-convex set $\tilde{A}_k$ with total energy not greater than the one of A_k (note that the line considered subdivides A_k into two subsets with equal total energy). Its convexification would then strictly decrease the energy. This shows that each A_k is of the form

$$A_k = B_{R_k} = B_{R_k}(0).$$

We can now compute the equation satisfied by R_k, by minimizing (after passing to polar coordinates)

$$\min\Big\{2\pi R + \frac{2\pi}{\tau}\int_R^{R_{k-1}} (R_{k-1} - \rho)\rho\, d\rho\Big\}, \tag{9.13}$$

which gives

$$\frac{R_k - R_{k-1}}{\tau} = -\frac{1}{R_k}. \tag{9.14}$$

Passing to the limit gives the desired mean curvature equation (9.2).

9.4 Homogenization of Flat Flows

We now consider geometric functionals with many local minimizers (introduced in Example 4.3) which give a more refined example of homogenization of minimizing movements. The functionals we consider are defined on (sufficiently regular) subsets of $\mathbb{R}^2$ by

$$F_\varepsilon(A) = \int_{\partial A} a\Big(\frac{x}{\varepsilon}\Big) d\mathcal{H}^1, \tag{9.15}$$

where

$$a(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z} \text{ or } x_2 \in \mathbb{Z} \\ 2 & \text{otherwise.} \end{cases}$$

The Γ-limit of the energies F_ε is the *crystalline perimeter energy*

$$F(A) = \int_{\partial A} \|\nu\|_1 \, d\mathscr{H}^1, \tag{9.16}$$

with $\|(\nu_1, \nu_2)\|_1 = |\nu_1| + |\nu_2|$. A minimizing movement for F is called a *flat flow*. We will first briefly describe it, and then compare it with the minimizing movements for F_ε.

9.4.1 Motion by Crystalline Curvature

The incremental problems for the minimizing-movement scheme for F in (9.16) are of the form

$$\min\Big\{F(A) + \frac{1}{\tau}\int_{A \triangle A_{k-1}} \mathrm{dist}_\infty(x, \partial A_{k-1})\, dx\Big\}, \tag{9.17}$$

where for technical reasons we consider the ∞-distance

$$\mathrm{dist}_\infty(x, B) = \inf\{\|x - y\|_\infty : y \in B\}.$$

However, in the simplified situation below this will not be relevant in our computations.

We only consider the case when the initial datum A_0 is a rectangle, which plays the role played by a ball for motion by mean curvature. Note that, as in Sect. 9.3, we can prove that if A_{k-1} is a rectangle, then we can limit the computation in (9.17) to

- A contained in A_{k-1} (otherwise $A \cap A_{k-1}$ strictly decreases the energy).
- A with each connected component a rectangle (otherwise taking the least rectangle containing a given component would decrease the energy, strictly if A is not a rectangle).
- A connected and with the same center as A_0 (since translating the center towards 0 decreases the energy).

Hence, we may suppose that

$$A_k = \Big[-\frac{L_{k,1}}{2}, \frac{L_{k,1}}{2}\Big] \times \Big[-\frac{L_{k,2}}{2}, \frac{L_{k,2}}{2}\Big]$$

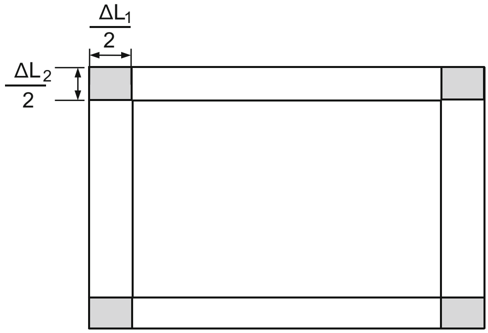

Fig. 9.1 Incremental crystalline minimization

for all k. In order to iteratively determine L_k, we have to minimize the energy

$$\min\Big\{2(L_{k,1}+\Delta L_1)+2(L_{k,2}+\Delta L_2)+\frac{1}{\tau}\int_{A\Delta A_{k-1}} \mathrm{dist}_\infty(x,\partial A_{k-1})\,dx\Big\}. \tag{9.18}$$

In this computation it is easily seen that for τ small the integral term can be substituted by

$$\frac{L_{k,1}}{4}(\Delta L_2)^2+\frac{L_{k,2}}{4}(\Delta L_1)^2.$$

This argument amounts to noticing that the contribution of the small rectangles at the corners highlighted in Fig. 9.1 is negligible as $\tau \to 0$. The optimal increments (more precisely, decrements) ΔL_j are then determined by the conditions

$$\begin{cases} 1+\dfrac{L_{k,2}}{4\tau}\Delta L_1 = 0 \\[2ex] 1+\dfrac{L_{k,1}}{4\tau}\Delta L_2 = 0. \end{cases} \tag{9.19}$$

Hence, we have the difference equations

$$\frac{\Delta L_1}{\tau} = -\frac{4}{L_{k,2}}, \qquad \frac{\Delta L_2}{\tau} = -\frac{4}{L_{k,1}}, \tag{9.20}$$

which finally gives the system of ODEs for the limit rectangles, with edges of length $L_1(t)$ and $L_2(t)$ respectively,

$$\begin{cases} L_1' = -\dfrac{4}{L_2} \\[2ex] L_2' = -\dfrac{4}{L_1}. \end{cases} \tag{9.21}$$

Geometrically, each edge of the rectangle moves inwards with velocity inversely proportional to its length; more precisely, equal to twice the inverse of its length (so that the other edge contracts with twice this velocity). Hence, in this context the inverse of the length of an edge plays the role of the curvature (crystalline curvature).

It is worth noticing that by (9.21) all rectangles are homothetic, since $\frac{d}{dt}\frac{L_1}{L_2} = 0$, and with area satisfying

$$\frac{d}{dt} L_1 L_2 = -8,$$

so that $L_1(t)L_2(t) = L_{0,1}L_{0,2} - 8t$, which gives the extinction time $t = L_{0,1}L_{0,2}/8$. In the case of an initial datum a square of side length L_0, the sets are squares whose side length at time t is given by $L(t) = \sqrt{L_0^2 - 8t}$, in analogy with the evolution of balls by mean curvature flow.

9.5 Homogenization of Oscillating Perimeters

We consider the sequence F_ε in (9.15). Note that, for any (sufficiently regular) initial datum A_0, we have that $A'_\varepsilon \subset A_0 \subset A''_\varepsilon$, where A'_ε and A''_ε are such that $F_\varepsilon(A'_\varepsilon) = \mathscr{H}^1(\partial A'_\varepsilon)$ and $F_\varepsilon(A''_\varepsilon) = \mathscr{H}^1(\partial A''_\varepsilon)$ and $|A''_\varepsilon \setminus A'_\varepsilon| = O(\varepsilon)$. Such sets are local minimizers for F_ε and hence the minimizing movement of F_ε from either of them is trivial. As a consequence, if $A_\varepsilon(t)$ is a minimizing movement for F_ε from A_0 we have

$$A'_\varepsilon \subset A_\varepsilon(t) \subset A''_\varepsilon.$$

This shows that for any set A_0 the only limit $\lim_{\varepsilon\to 0} A_\varepsilon(t)$ of minimizing movements for F_ε from A_0 is the trivial motion $A(t) = A_0$.

We now compute the minimizing movements along the sequence F_ε with initial datum a rectangle, and compare them with the flat flow described in the previous section.

For simplicity of computation we deal with a constrained case, when:

- For every ε the initial datum $A_0 = A_0^\varepsilon$ is a rectangle centered in 0 such that $F_\varepsilon(A) = \mathscr{H}^1(\partial A)$ (i.e., its edge lengths $L_{0,j}$ belong to $2\varepsilon\mathbb{Z}$). In analogy with x_0 in the example in Sect. 8.4, if this does not hold then either it does after one iteration or we have a pinned state $A_k = A_0$ for all k.
- All competing A are rectangles with $F_\varepsilon(A) = \mathscr{H}^1(\partial A)$ centered in 0. The fact that all competing sets are rectangles follows as for the flat flow in the previous section. The fact that $F_\varepsilon(A_k) \le F_\varepsilon(A_{k-1})$ then implies that the minimal rectangles satisfy $F_\varepsilon(A_k) = \mathscr{H}^1(\partial A_k)$. The only real assumption at this point is that they are centered in 0. This hypothesis can be removed, upon a slightly more complex computation, which would only make the arguments less clear.

After this simplifications, the incremental problem is exactly as in (9.17) since for competing sets we have $F_\varepsilon(A) = F(A)$, the only difference being that now $L_{k,1}$, $L_{k,2} \in 2\varepsilon\mathbb{Z}$. The problem in terms of $\triangle L_j$, using the same simplification for (9.18) as in the previous section, is then

$$\min\Big\{2(L_{k,1}+\Delta L_1)+2(L_{k,2}+\Delta L_2)+\frac{L_{k,1}}{4\tau}(\Delta L_2)^2+\frac{L_{k,2}}{4\tau}(\Delta L_1)^2 : \Delta L_j \in 2\varepsilon\mathbb{Z}\Big\}. \tag{9.22}$$

This is a minimization problem for a parabola as the ones in Sect. 8.4 that gives

$$\Delta L_1 = -\Big\lfloor \frac{4\tau}{\varepsilon L_{k,2}} + \frac{1}{2}\Big\rfloor \varepsilon \qquad \text{if } \frac{4\tau}{\varepsilon L_{k,2}} + \frac{1}{2} \notin \mathbb{Z} \tag{9.23}$$

(the other cases giving two solutions), and an analogous equation for ΔL_2. Passing to the limit, we have the system of ODEs governed by the parameter

$$w = \lim_{\varepsilon\to 0} \frac{\tau}{\varepsilon}$$

(which we may suppose exists, up to subsequences), which reads as

$$\begin{cases} L_1' = -\dfrac{1}{w}\Big\lfloor \dfrac{4w}{L_2} + \dfrac{1}{2}\Big\rfloor \\[2ex] L_2' = -\dfrac{1}{w}\Big\lfloor \dfrac{4w}{L_1} + \dfrac{1}{2}\Big\rfloor. \end{cases} \tag{9.24}$$

Note that the right-hand side is a discontinuous function of L_j, so some care must be taken at times t when $\frac{4w}{L_j(t)} + \frac{1}{2} \in \mathbb{Z}$. However, apart from some exceptional cases, this condition holds only for a countable number of t, and is therefore negligible.

We can compare the resulting minimizing movements with the crystalline curvature flow, related to F.

- **(total pinning)** if $\tau \ll \varepsilon$ ($w = 0$) then we have $A(t) = A_0$.
- **(crystalline curvature flow)** if $\varepsilon \ll \tau$ then we have the minimizing movements described in the previous section.
- **(partial pinning/asymmetric curvature flow)** if $0 < w < +\infty$ then we have
  - (i) (*total pinning*) if both $L_{0,j} > 8w$ then the motion is trivial $A(t) = A_0$.
  - (ii) (*partial pinning*) if $L_{0,1} > 8w$, $L_{0,2} < 8w$ and $\frac{4w}{L_{0,2}} + \frac{1}{2} \notin \mathbb{Z}$ then the horizontal edges do not move, but they contract with constant velocity until $L_1(t) = 8w$.
  - (iii) (*asymmetric curvature flow*) if $L_{0,1} \leq 8w$ and $L_{0,2} < 8w$ then we have a unique motion with $A(t) \subset\subset A(s)$ if $t > s$, up to a finite extinction time. Note, however, that the sets $A(s)$ are not homothetic, except for the trivial case when A_0 is a square.

Some cases are not considered above, namely those when we do not have uniqueness of minimizers in the incremental problem. This may lead to a multiplicity of minimizing movements, as remarked in Sect. 8.4.

It is worthwhile to highlight that we may rewrite the equations for L'_j as a variation of the crystalline curvature flow; e.g., for L'_1 we can write it as

$$L'_1 = -f\Big(\frac{L_2}{w}\Big)\frac{4}{L_2}, \qquad \text{with } f(z) = \frac{z}{4}\Big\lfloor \frac{4}{z} + \frac{1}{2}\Big\rfloor.$$

This suggests that the 'relevant' homogenized problem is the one obtained for $\frac{\tau}{\varepsilon} = 1$, as all the others can be obtained from this one by a scaling argument.

We note that the scheme can be applied to the evolution of more general sets, but the analysis of the rectangular case already highlights the new features deriving from the microscopic geometry.

9.6 Flat Flow with Oscillating Forcing Term

We now consider another minimizing-movement scheme linked to the functional F in (9.16). In this case, the oscillations are given by a lower-order forcing term. We consider, in $\mathbb{R}^2$,

$$G_\varepsilon(A) = \int_{\partial A} \|\nu\|_1 d\mathscr{H}^1 + \int_A g\Big(\frac{x_1}{\varepsilon}\Big)dx, \tag{9.25}$$

where g is 1-periodic and even, given by

$$g(s) = \begin{cases} \alpha & \text{if dist}\,(x, \mathbb{Z}) < \frac{1}{4} \\ \beta & \text{otherwise,} \end{cases}$$

with $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. Note that the additional term may be negative, so that this functional is not positive; however, the minimizing-movement scheme can be applied unchanged.

Since the additional term converges continuously in L^1 as $\varepsilon \to 0$, the Γ-limit is simply

$$G(A) = \int_{\partial A} \|\nu\|_1 d\mathscr{H}^1 + \frac{\alpha+\beta}{2}|A|. \tag{9.26}$$

9.6.1 Flat Flow with Forcing Term

We now consider minimizing movements for G. As in Sect. 9.4.1, we only deal with a constrained problem, when both the initial datum and the competing sets are rectangles centered in 0. With the notation of Sect. 9.4.1, we are led to the minimum problem

$$\min\Big\{2(L_{k,1}+\Delta L_1+L_{k,2}+\Delta L_2)+\frac{L_{k,1}}{4\tau}(\Delta L_2)^2+\frac{L_{k,2}}{4\tau}(\Delta L_1)^2 +\frac{\alpha+\beta}{2}(L_{k,1}+\Delta L_1)(L_{k,2}+\Delta L_2)\Big\}.$$

The minimizing pair $(\Delta L_1, \Delta L_2)$ satisfies

$$\frac{\Delta L_1}{\tau}=-\Big(\frac{4}{L_{k,2}}+(\alpha+\beta)\Big(1+\frac{\Delta L_2}{L_{k,2}}\Big)\Big) \tag{9.27}$$

and the analogous equation for $\frac{\Delta L_2}{\tau}$. Passing to the limit we have

$$\begin{cases} L_1'=-\Big(\dfrac{4}{L_2}+\alpha+\beta\Big)\\ L_2'=-\Big(\dfrac{4}{L_1}+\alpha+\beta\Big), \end{cases} \tag{9.28}$$

so that each edge moves with velocity $\frac{2}{L_2}+\frac{\alpha+\beta}{2}$, with the convention that it moves inwards if this number is positive, outwards if it is negative.

Note that if $\alpha+\beta\geq 0$ then L_1 and L_2 are always decreasing and we have finite-time extinction, while if $\alpha+\beta<0$ then there is an equilibrium for $L_j = 4/|\alpha+\beta|$, and we have expanding rectangles, with an asymptotic velocity of each side of $|\alpha+\beta|/2$ as the side length diverges.

9.6.2 Homogenization of Forcing Terms

We treat the case $\tau \ll \varepsilon$ only, in which we may highlight new homogenization phenomena. Again, we consider the constrained case when both the initial datum and the competing sets are rectangles centered in 0 and adopt the notation of Sect. 9.4.1. The geometry of the problem is pictured in Fig. 9.2, where the two colors in the background represent the two values of the forcing term.

Taking into account that $\tau \ll \varepsilon$, the incremental minimum problem can be approximated by

$$\min\Big\{2(L_{k,1}+\Delta L_1+L_{k,2}+\Delta L_2)+\frac{L_{k,1}}{4\tau}(\Delta L_2)^2+\frac{L_{k,2}}{4\tau}(\Delta L_1)^2 +\frac{\alpha+\beta}{2}L_{k,1}L_{k,2}+\frac{\alpha+\beta}{2}L_{k,1}\Delta L_2+g\Big(\frac{L_{k,1}}{2\varepsilon}\Big)L_{k,2}\Delta L_1\Big\}. \tag{9.29}$$

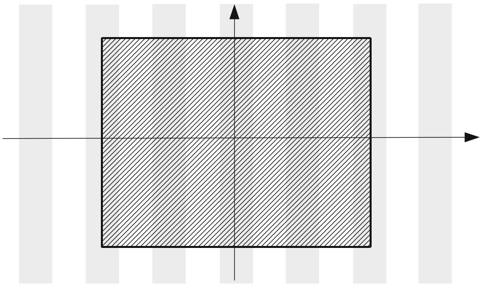

Fig. 9.2 Rectangle in a layered environment

In considering the term $g(L_{k,1}/2\varepsilon)$ we implicitly assume that τ is so small that both $L_{k,1}/2\varepsilon$ and $(L_{k,1}+\Delta L_1)/2\varepsilon$ belong to the same interval where g is constant. This can be assumed up to a number of k that is negligible as $\tau \to 0$.

For the minimizing pair of (9.29) we have

$$\begin{cases} 2 + \dfrac{L_{k,2}}{2\tau}\Delta L_1 + g\Big(\dfrac{L_{k,1}}{2\varepsilon}\Big) L_{k,2} = 0 \\ 2 + \dfrac{L_{k,1}}{2\tau}\Delta L_2 + \dfrac{\alpha+\beta}{2} L_{k,1} = 0; \end{cases} \tag{9.30}$$

that is,

$$\begin{cases} \dfrac{\Delta L_1}{\tau} = -\bigg(\dfrac{4}{L_{k,2}} + 2g\Big(\dfrac{L_{k,1}}{2\varepsilon}\Big)\bigg) \\ \dfrac{\Delta L_2}{\tau} = -\Big(\dfrac{4}{L_{k,1}} + (\alpha+\beta)\Big). \end{cases} \tag{9.31}$$

This systems shows that the horizontal edges move with velocity $\frac{2}{L_{k,1}} + \frac{\alpha+\beta}{2}$, while the velocity of the vertical edges depends on the location of the edge and is

$$\frac{2}{L_{k,2}} + g\Big(\frac{L_{k,1}}{2\varepsilon}\Big).$$

We then deduce that the limit velocity for the horizontal edges of length L_1 is

$$\frac{2}{L_1} + \frac{\alpha+\beta}{2}. \tag{9.32}$$

As for the vertical edges, we have:

- **(mesoscopic pinning)** if L_2 is such that

$$\Big(\frac{2}{L_2} + \alpha\Big)\Big(\frac{2}{L_2} + \beta\Big) < 0$$

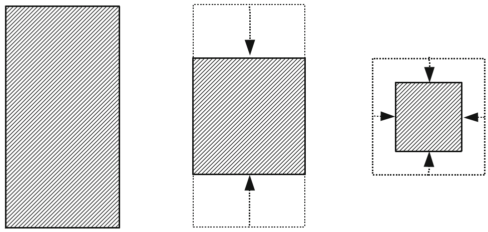

Fig. 9.3 Stages in the motion according to system (9.34)

then the vertical edge is eventually pinned in the minimizing-movement scheme. This pinning is not due to the equality $L_{k+1,1} = L_{k,1}$ in the incremental problem, but to the fact that the vertical edge moves in contrasting directions depending on the value of g.

- **(homogenized velocity)** if, on the contrary, the vertical edge length satisfies

$$\Big(\frac{2}{L_2} + \alpha\Big)\Big(\frac{2}{L_2} + \beta\Big) > 0$$

then we have a limit *effective velocity* of the vertical edge given by the harmonic mean of the two velocities $\frac{2}{L_2} + \alpha$ and $\frac{2}{L_2} + \beta$; namely,

$$\frac{(2 + \alpha L_2)(2 + \beta L_2)}{L_2\Big(2 + \frac{\alpha+\beta}{2} L_2\Big)}. \tag{9.33}$$

We examine some cases explicitly.

(i) Let $\alpha = -\beta$. Then we have

$$\begin{cases} L_2' = -\dfrac{4}{L_1} \\ L_1' = -2\dfrac{(2 - \beta L_2) \vee 0}{L_2}; \end{cases} \tag{9.34}$$

i.e., the vertical edges are pinned if their length is larger than $2/\beta$. In this case, the horizontal edges move inwards with constant velocity $\frac{2}{L_{0,1}}$. In this way the vertical edges shrink with rate $\frac{4}{L_{0,1}}$ until their length is $2/\beta$. After this, the whole rectangle shrinks in all directions. The stages of this evolution are pictured in Fig. 9.3.

(ii) Let $\alpha < \beta < 0$. Then for the vertical edges we have an interval of 'mesoscopic pinning' corresponding to

$$\frac{2}{|\beta|} \le L_2 \le \frac{2}{|\alpha|} \tag{9.35}$$

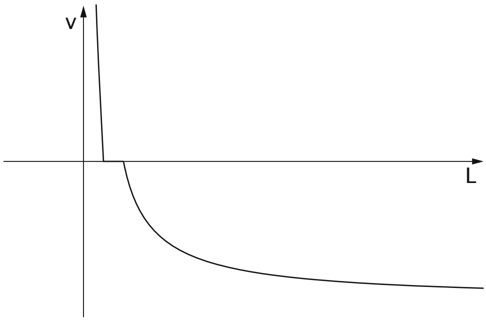

Fig. 9.4 Velocity of the vertical hedges with an interval of mesoscopic pinning

The velocity of the vertical edges in dependence of their length is then given by

$$v = \begin{cases} 0 & \text{if (9.35) holds} \\ \dfrac{(2+\alpha L_2)(2+\beta L_2)}{L_2\big(2+\frac{\alpha+\beta}{2}L_2\big)} & \text{otherwise} \end{cases}$$

and is pictured in Fig. 9.4. Instead, the velocity of the horizontal edges is given by (9.32), so that they move inwards if

$$L_1 < \frac{4}{|\alpha+\beta|},$$

and outwards if $L_1 > \frac{4}{|\alpha+\beta|}$.

In this case we can consider as initial datum a square of side length L_0. We have the following cases:

- If $L_0 \leq \frac{2}{|\beta|}$ then all edges move inwards until a finite extinction time.
- If $\frac{2}{|\beta|} < L_0 < \frac{4}{|\alpha+\beta|}$ then first only the horizontal edges move inwards until the vertical edge reaches the length $\frac{2}{|\beta|}$, after which all edges move inwards.
- If $\frac{4}{|\alpha+\beta|} < L_0 < \frac{2}{|\alpha|}$ then first only the horizontal edges move outwards until the vertical edge reaches the length $\frac{2}{|\alpha|}$, after which all edges move outwards.
- If $L_0 \geq \frac{2}{|\alpha|}$ then all edges move outwards, and the motion is defined for all times. The asymptotic velocity of the vertical edges as the length of the edges diverges is

$$\left|\frac{2\alpha\beta}{\alpha+\beta}\right|,$$

lower than $\left|\frac{\alpha+\beta}{2}\right|$ (the asymptotic velocity for the horizontal edges).

The critical case can be shown to be $\varepsilon \sim \tau$, so that for $\varepsilon \ll \tau$ we have the flat flow with averaged forcing term described in Sect. 9.6.1. The actual description in the case $\varepsilon \sim \tau$ would involve a complicated homogenization argument for the computation of the averaged velocity of vertical sides.

Appendix

The variational approach for motion by mean curvature is due to Almgren et al. [2]. The variational approach for crystalline curvature flow is contained in a paper by Almgren and Taylor [1].

The homogenization of the flat flow essentially follows the discrete analog contained in the paper by Braides et al. [3]. In that paper more effects of the microscopic geometry are described for more general initial sets. The homogenization with forcing term is part of ongoing work with A. Malusa and M. Novaga.

Geometric motions with a non-trivial homogenized velocity are described in the paper by Braides and Scilla [4], where example are shown of geometries which do not influence the crystalline perimeter obtained as Γ-limit, but do influence various features of the evolution.

References

1. Almgren, F., Taylor, J.E.: Flat flow is motion by crystalline curvature for curves with crystalline energies. J. Differ. Geom. **42**, 1–22 (1995)
2. Almgren, F., Taylor, J.E., Wang, L.: Curvature driven flows: a variational approach. SIAM J. Control Optim. **50**, 387–438 (1983)
3. Braides, A, Gelli, M.S., Novaga, M.: Motion and pinning of discrete interfaces. Arch. Ration. Mech. Anal. **95**, 469–498 (2010)
4. Braides, A., Scilla, G.: Motion of discrete interfaces in periodic media. Interfaces Free Bound. **15** (2013), to appear.