Chapter 3
Parameterized Motion Driven by Global Minimization

Energy-driven dynamic problems are in general associated with a local minimization procedure. Nevertheless, for 'slow movements' a meaningful notion of 'quasi-static' motion can be defined starting from a global-minimization criterion. The ingredients are:

- a parameter-dependent energy F,
- a dissipation D satisfying a non-decreasing constraint,
- a (time-)parameterized forcing condition.

Loosely speaking, a quasistatic motion is controlled by some parameterized forcing condition (applied forces, varying boundary conditions or other constraints); the motion is thought to be so slow so that the solution at a fixed value of the parameter (at fixed 'time') minimizes a total energy. This energy is obtained adding some 'dissipation' to some 'internal energy'. A further condition is that the total dissipation increases with time. An entire general theory (of rate-independent motion) can be developed starting from these ingredients.

An important feature of these rate-independent motions is that they can be characterized as the limit of a piecewise-constant (time-)parameterized family of functions, which are defined iteratively as solutions of minimum problems. Under suitable conditions, to such a characterization the Fundamental Theorem of Γ-convergence can be applied, so that this notion can be proved to be indeed compatible with Γ-convergence.

3.1 A Paradigmatical Example: Damage Models

In this section we deal with a simplified example and examine its stability with respect to perturbations. We will first define a damage process for a single material. Then we will consider the same definition for a mixture of two materials in the context of homogenization. A homogenized theory can be derived, with some care

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in the definition of the Γ-limit, that must take into account at the same time the energy and the dissipation.

3.1.1 Damage of a Homogeneous Material

We consider a one-dimensional setting. Our functions will be parameterized on a fixed interval $(0, 1)$. In this case we have:

- The parameter space will be that of all measurable subsets A of $(0, 1)$. The set A will be understood as the *damage set*.
- The energies depending on a set A will be

$$F_A(u) = \alpha \int_A |u'|^2\, dx + \beta \int_{(0,1)\setminus A} |u'|^2\, dx,$$

  where $0 < \alpha < \beta$. In an mechanical interpretation of the variables, u represents the deformation of a bar, whose elastic constant is β in the undamaged set and $\alpha < \beta$ in the damaged set.
- The dissipation is

$$D(A) = \gamma |A|,$$

  with $\gamma > 0$. The work done to damage a portion A of the material is proportional to the measure of A.
- The condition that forces the solution to be parameter dependent ('time-dependent') is a boundary condition

$$u(0) = 0, \qquad u(1) = g(t),$$

  where g is a continuous function with $g(0) = 0$. Here the parameter is $t \in [0, T]$.

Definition 3.1. A *solution* to the evolution related to the energy, dissipation and boundary conditions above is a pair (u^t, A^t) with $u^t \in H^1(0, 1)$, $A^t \subset (0, 1)$, and such that

- **(monotonicity)** we have $A^s \subset A^t$ for all $s < t$;
- **(minimization)** the pair (u^t, A^t) minimizes

$$\min\Big\{F_A(u) + D(A) : u(0) = 0, u(1) = g(t), A^t \subset A\Big\}; \tag{3.1}$$

- **(continuity)** the energy $\mathscr{E}(t) = F_{A^t}(u^t) + D(A^t)$ is continuous;
- **(homogeneous initial datum)** u^0 is the constant 0 and $A^0 = \emptyset$.

The continuity assumption allows to rule out trivial solutions as those with $A^t = (0,1)$ for all $t > 0$. It is usually replaced by a more physical condition of energy conservation. In our context this assumption is not relevant.

Note that t acts only as a parameter (the motion is 'rate independent'). Hence, for example if g is monotone increasing, it suffices to consider $g(t) = t$. We will construct by hand a solution in this simplified one-dimensional context.

Remark 3.1. Note that the value in the minimum problem

$$m(t) = \min\Big\{F_A(u) + D(A) : u(0) = 0, u(1) = t\Big\} \tag{3.2}$$

depends on A only through $\lambda = |A|$.

Indeed, given A, we can explicitly compute the minimum value

$$m(A,t) = \min\Big\{\int_A \alpha|u'|^2\,dx + \int_{(0,1)\setminus A} \beta|u'|^2\,dx : u(0) = 0, u(1) = t\Big\}.$$

In fact, for all test functions u we have, by Jensen's inequality

$$\int_A \alpha|u'|^2\,dx + \int_{(0,1)\setminus A} \beta|u'|^2\,dx \geq \alpha|A||z_1|^2 + \beta(1-|A|)|z_2|^2,$$

where

$$z_1 = \frac{1}{|A|}\int_A u'\,dx, \qquad z_2 = \frac{1}{1-|A|}\int_{(0,1)\setminus A} u'\,dx,$$

with a strict inequality if u' is not constant on A and $(0,1)\setminus A$. This shows that the unique minimizer satisfies

$$u' = z_1\chi_A + z_2(1-\chi_A), \qquad |A|z_1 + (1-|A|)z_2 = t,$$

where the second condition is given by the boundary data; hence,

$$m(A,t) = \min\{\alpha\lambda|z_1|^2 + \beta(1-\lambda)|z_2|^2 : \lambda z_1 + (1-\lambda)z_2 = t\} = \frac{\alpha\beta}{\lambda\beta + (1-\lambda)\alpha}t^2.$$

We conclude that the minimum value (3.2) is given by

$$\frac{\alpha\beta}{\lambda\beta + (1-\lambda)\alpha}t^2 + \gamma\lambda. \tag{3.3}$$

By minimizing over λ we obtain the optimal value of the measure of the damaged region

$$\lambda_{\min}(t) = \begin{cases} 0 & \text{if } |t| \le \sqrt{\frac{\alpha\gamma}{\beta(\beta-\alpha)}} \\ 1 & \text{if } |t| \ge \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}} \\ t\sqrt{\frac{\alpha\beta}{\gamma(\beta-\alpha)}} - \frac{\alpha}{\beta-\alpha} & \text{otherwise} \end{cases} \tag{3.4}$$

and the minimum value

$$m(t) = \begin{cases} \beta t^2 & \text{if } |t| \le \sqrt{\frac{\alpha\gamma}{\beta(\beta-\alpha)}} \\ \alpha t^2 + \gamma & \text{if } |t| \ge \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}} \\ 2t\sqrt{\frac{\alpha\beta\gamma}{\beta-\alpha}} - \frac{\gamma\alpha}{\beta-\alpha} & \text{otherwise.} \end{cases} \tag{3.5}$$

The interpretation of this formula is as follows. For small values of the total displacement t the material remains undamaged, until it reaches a critical value for the boundary datum. Then a portion of size $\lambda_{\min}(t)$ of the material damages, lowering the elastic constant of the material and the overall value of the sum of the internal energy and the dissipation, until all the material is damaged. Note that in this case $\mathscr{E}(t) = m(t)$, due to the increasing-load assumption.

The solutions for the evolution problem are given by any increasing family of sets A^t satisfying $|A^t| = \lambda_{\min}(t)$ and correspondingly functions u^t minimizing $m(A^t, t)$.

The value in (3.3) is obtained by first minimizing in u. Conversely, we may first minimize in A. We then have

$$\min\Big\{\int_0^1 \min_A\{\chi_A(\alpha|u'|^2+\gamma), \chi_{(0,1)\setminus A}\beta|u'|^2\}\, dx : u(0) = 0, u(1) = g(t)\Big\}. \tag{3.6}$$

The lower-semicontinuous envelope of the integral energy is given by the integral with energy function the convex envelope of

$$f(z) = \min\{\alpha z^2 + \gamma, \beta z^2\}, \tag{3.7}$$

which is exactly given by formula (3.5); i.e.,

$$m(t) = f^{**}(t)$$

(see Fig. 3.1).

Irreversibility. An important feature of the monotonicity condition for A^t is irreversibility of damage, which implies that for non-increasing g the values of

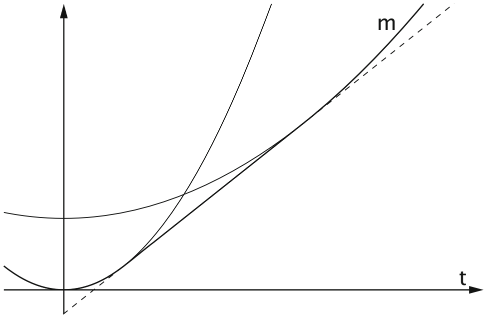


Fig. 3.1 Minimal value $m(t)$ for the damage problem

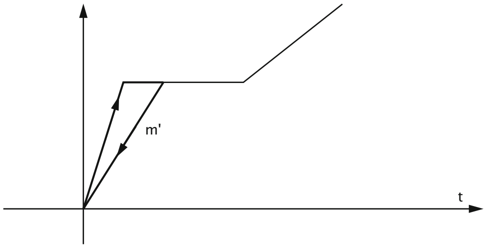


Fig. 3.2 Plot of $m'(t)$ along a cycle

$m(g(t))$ will depend on the highest value taken by $\lambda_{\min}(g(t))$ on $[0, t]$. In particular, for a 'loading–unloading' cycle with $g(t) = \frac{T}{2} - \left|t - \frac{T}{2}\right|$, the value of $\mathscr{E}(t)$ is given by

$$\mathscr{E}(t) = \begin{cases} m(t) & \text{for } 0 \le t \le \frac{T}{2} \\ \dfrac{\alpha\beta}{\lambda_{\min}(\frac{T}{2})\beta + (1-\lambda_{\min}(\frac{T}{2}))\alpha}(T-t)^2 + \gamma\,\lambda_{\min}(\dfrac{T}{2}) & \text{for } \frac{T}{2} \le t \le T. \end{cases}$$

This formula highlights that once the maximal value $\lambda_{\min}(T/2)$ is reached, then the damaged region A^t remains fixed, so that the problem becomes a quadratic minimization (plus the constant value of the dissipation). We plot $m'(t)$ and draw a cycle in Fig. 3.2.

Note that, in particular, if $\frac{T}{2} \ge \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}}$ then the material is completely damaged in the 'unloading' regime.

3.1.1.1 Threshold Formulation

Note that a solution u_t of (3.2) satisfies the Euler–Lagrange equation

$$((\alpha\chi_A + \beta(1-\chi_A))u')' = 0;$$

i.e.,

$$(\alpha\chi_A + \beta(1 - \chi_A))u' = \sigma_t, \tag{3.8}$$

where σ_t is a constant parameterized by t. Its plot as a function of $g = g(t)$ along a 'loading-unloading' cycle is the same as in Fig. 3.2.

The plateau for σ is obtained at the *threshold* value

$$\sigma = \sqrt{\frac{\alpha\beta\gamma}{\beta - \alpha}}.$$

We can interpret the g-σ graph as a *threshold principle*: the material does not damage until the *stress* σ reaches the threshold value. At this point, if the material is loaded further it damages so as to keep the value of σ below the threshold, until all the material is damaged. If the material is unloaded then σ follows a linear elastic behavior with the overall effective elastic constant corresponding to the total amount of damage produced.

3.1.2 Homogenization of Damage

We now examine the behaviour of the previous process with respect to Γ-convergence in the case of homogenization; i.e., when we have a fine mixture of two materials, each one of which can undergo a damage process as in the previous section. To that end we introduce the energies

$$F_{\varepsilon,A}(u) = \int_{(0,1)\setminus A} \beta\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx + \int_A \alpha\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx, \tag{3.9}$$

where α and β are 1-periodic functions with

$$\alpha(y) = \begin{cases} \alpha_1 & \text{for } 0 \le y < \frac{1}{2} \\ \alpha_2 & \text{for } \frac{1}{2} \le y < 1 \end{cases} \qquad \beta(y) = \begin{cases} \beta_1 & \text{for } 0 \le y < \frac{1}{2} \\ \beta_2 & \text{for } \frac{1}{2} \le y < 1 \end{cases}$$

with $0 < \alpha_j < \beta_j$. Note that, for fixed A, the functionals $F_{\varepsilon,A}$ Γ-converge to

$$F_{\mathrm{hom},A}(u) = \underline{\beta} \int_{(0,1)\setminus A} |u'|^2\,dx + \underline{\alpha} \int_A |u'|^2\,dx, \tag{3.10}$$

with

$$\underline{\alpha} = \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2} < \frac{2\beta_1\beta_2}{\beta_1 + \beta_2} = \underline{\beta}.$$

This can be easily checked if A is an interval (or a union of intervals), and then for a general A by approximation. Indeed if $A = (0, \lambda)$ then the liminf inequality trivially holds by separately applying the liminf inequality to the two energies

$$\int_0^\lambda \alpha\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx, \qquad \int_\lambda^1 \beta\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx. \tag{3.11}$$

Conversely, given a target function $u \in H^1(0,1)$, we can find recovery sequences (u^1_ε) and (u^2_ε) for u on $(0,\lambda)$ and $(\lambda,1)$, respectively, for the energies (3.11) with $u^1_\varepsilon(\lambda) = u^2_\varepsilon(\lambda)$, so that the corresponding u_ε defined as u^1_ε on $(0,\lambda)$ and as u^2_ε on $(\lambda,1)$ is a recovery sequence for $F_{\mathrm{hom},A}(u)$. Note that the Γ-limit is still of the form examined in Sect. 3.1.1 with constants $\underline{\alpha}$ and $\underline{\beta}$.

We now instead study the damage process at fixed ε. For simplicity of computation we suppose that $\frac{1}{\varepsilon} \in \mathbb{N}$. The general case can be always reduced to this assumption up to an error of order ε. The dissipation will be of the form

$$D_\varepsilon(A) = \int_A \gamma\Big(\frac{x}{\varepsilon}\Big)\,dx,$$

where again γ is a 1-periodic function with

$$\gamma(y) = \begin{cases} \gamma_1 & \text{for } 0 \le y < \frac12 \\ \gamma_2 & \text{for } \frac12 \le y < 1 \end{cases}$$

with $\gamma_j > 0$. In the case $\gamma_1 = \gamma_2$ we obtain the same dissipation as in Sect. 3.1.1, independent of ε.

In order to compute the minimum value

$$m^\varepsilon(t) = \min\Big\{F_{\varepsilon,A}(u) + D_\varepsilon(A) : u(0) = 0, u(1) = t, A \subset (0,1)\Big\} \tag{3.12}$$

we proceed as in Remark 3.1, noticing that the minimum value

$$m^\varepsilon(A,t) = \min\Big\{F_{\varepsilon,A}(u) : u(0) = 0, u(1) = t\Big\} \tag{3.13}$$

depends on A only through the volume fraction of each damaged component

$$\lambda_i = 2\Big|\Big\{x \in A : \alpha\Big(\frac{x}{\varepsilon}\Big) = \alpha_i\Big\}\Big|,$$

and its value is independent of ε and is given by

$$\min\Big\{\frac12\Big(\lambda_1\alpha_1 z_{11}^2 + (1-\lambda_1)\beta_1 z_{12}^2\Big) + \frac12\Big(\lambda_2\alpha_2 z_{21}^2 + (1-\lambda_2)\beta_1 z_{22}^2\Big) :$$

$$\frac{1}{2}(\lambda_1 z_{11} + (1-\lambda_1)z_{12}) + \frac{1}{2}(\lambda_2 z_{21} + (1-\lambda_2)z_{22}) = t\Big\}.$$

We conclude that $m^\varepsilon(t) = m_{\rm hom}(t)$ is independent of ε and satisfies

$$m_{\rm hom}(t) = \frac{1}{2}\min\Big\{m_1(t_1) + m_2(t_2) : \frac{t_1+t_2}{2} = t\Big\}, \tag{3.14}$$

where m_j is defined as m in (3.2) with α_j, β_j and γ_j in the place of α, β and γ (i.e., by the damage process in the i-th material). Hence, by (3.5)

$$m_j(t) = \begin{cases} \beta_j t^2 & \text{if } |t| \le \sqrt{\frac{\alpha_j\gamma_j}{\beta_j(\beta_j-\alpha_j)}} \\ \alpha_j t^2 + \gamma_j & \text{if } |t| \ge \sqrt{\frac{\beta_j\gamma_j}{\alpha_j(\beta_j-\alpha_j)}} \\ 2t\sqrt{\frac{\alpha_j\beta_j\gamma_j}{\beta_j-\alpha_j}} - \frac{\gamma_j\alpha_j}{\beta_j-\alpha_j} & \text{otherwise.} \end{cases} \tag{3.15}$$

We can therefore easily compute $m(t)$. In the hypothesis, e.g., that

$$p_2 := \sqrt{\frac{\alpha_2\beta_2\gamma_2}{\beta_2-\alpha_2}} < \sqrt{\frac{\alpha_1\beta_1\gamma_1}{\beta_1-\alpha_1}} =: p_1, \tag{3.16}$$

we can write $m'(t)$ as

$$m'_{\rm hom}(t) = \begin{cases} 2\underline{\beta}t & \text{if } |t| \le \frac{p_2}{\underline{\beta}} \\ 2p_2 & \text{if } \frac{p_2}{\underline{\beta}} < |t| < \frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \\ \frac{4\beta_1\alpha_2}{\beta_1+\alpha_2}t & \text{if } \frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \le |t| \le \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \\ 2p_1 & \text{if } \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2} < |t| < \frac{p_1}{\underline{\alpha}} \\ 2\underline{\alpha}t & \text{if } |t| \ge \frac{p_1}{\underline{\alpha}}. \end{cases}$$

The outcome is pictured in Fig. 3.3. It highlights that the behaviour is different from the one computed in Sect. 3.1.1: for small values of the total displacement t the overall response is the same as the one of the homogenized behaviour of the two 'strong' materials. At a first critical value for t one of the two materials (and only one except for the exceptional case $p_1 = p_2$) starts to damage (this corresponds

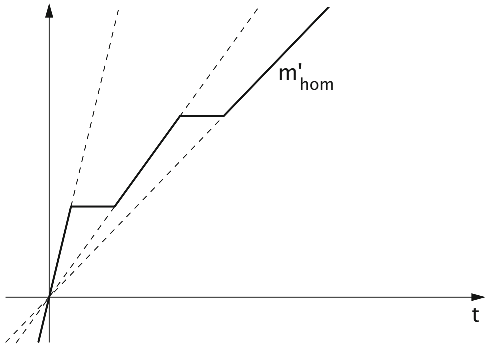


Fig. 3.3 The function $m'_{\mathrm{hom}}(t)$ describing the homogenized damage in a periodic microstructure

to the first constant value $2p_1$ of m') until it is completely damaged. Under condition (3.16) the first material to damage is material 2, and the corresponding damage volume fraction is

$$\lambda_{2,\min}(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{p_2}{\underline{\beta}} \\ \frac{2p_2}{\gamma_2}\left(t - \frac{p_2}{\underline{\beta}}\right) & \text{if } \frac{p_2}{\underline{\beta}} < t < \frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \\ 1 & \text{if } t > \frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2}. \end{cases} \tag{3.17}$$

Then the material behaves as a mixture of a strong material 1 and a damaged material 2. Subsequently, also material 1 starts to damage; the corresponding damage volume fraction is

$$\lambda_{1,\min}(t) = \begin{cases} 0 & \text{if } t \le \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \\ \frac{2p_1}{\gamma_1}\left(t - \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2}\right) & \text{if } \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2} < t < \frac{p_1}{\underline{\alpha}} \\ 1 & \text{if } t \ge \frac{p_1}{\underline{\alpha}}. \end{cases} \tag{3.18}$$

After also material 1 has completely damaged, the behaviour is that of the homogenized energy for two weak materials.

Note that at fixed ε we can define A^t_ε and u^t_ε by choosing increasing families of sets $A^t_{j,\varepsilon}$ describing the damage in the j-th material with $|A^t_{j,\varepsilon}| = \frac{1}{2}\lambda_{j,\min}(t)$, setting $A^t_\varepsilon = A^t_{1,\varepsilon} \cup A^t_{2,\varepsilon}$ and u^t_ε the corresponding solution of $m^\varepsilon(A^t, t)$. However the sets A^t_ε do not converge to sets as $\varepsilon \to 0$ except for the trivial cases $\emptyset$ and $(0, 1)$. In particular for $\frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \le t \le \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2}$ we have $\lambda_{2,\min}(t) = 1$ and $\lambda_{1,\min}(t) = 0$, so that $A^t_\varepsilon = \varepsilon(\mathbb{Z} + [\frac{1}{2}, 1])$, which do not converge as sets.

3.1.2.1 A Double-Damage-Set Formulation

The observation above highlights that a weaker notion of convergence of sets must be given in order to describe the behavior of (some solutions of) the sequence of damage problem. One way is to choose particular sequences of damaged sets $A^t_{j,\varepsilon}$, for examples intersections of intervals with the j-th material. For simplicity we consider intervals $[0,\lambda_{j,\varepsilon}(t)]$ with one endpoint in 0, so that

$$A^t_{1,\varepsilon} = [0,\lambda_{1,\varepsilon}(t)] \cap \varepsilon\Big(\mathbb{Z} + \Big[0,\frac{1}{2}\Big]\Big), \qquad A^t_{2,\varepsilon} = [0,\lambda_{2,\varepsilon}(t)] \cap \varepsilon\Big(\mathbb{Z} + \Big[\frac{1}{2},1\Big]\Big).$$

Note that under hypothesis (3.16) we have $\lambda_{2,\varepsilon}(t) \geq \lambda_{1,\varepsilon}(t)$ for all t. We have therefore to examine problems (3.13) rewritten in the form

$$m^\varepsilon(\lambda_{1,\varepsilon},\lambda_{2,\varepsilon},t) = \min\Big\{\int_0^{\lambda_{1,\varepsilon}} \alpha\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx + \int_{\lambda_{1,\varepsilon}}^{\lambda_{2,\varepsilon}} a\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx$$
$$+\int_{\lambda_{2,\varepsilon}}^1 \beta\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx : u(0)=0, u(1)=t\Big\}, \tag{3.19}$$

where a is the 1-periodic function with

$$a(y) = \begin{cases} \beta_1 & \text{for } 0 \leq y < \frac{1}{2} \\ \alpha_2 & \text{for } \frac{1}{2} \leq y < 1 \end{cases}.$$

If $\lambda_{j,\varepsilon} \to \lambda_j$ then these problems converge to

$$m_{\text{hom}}(\lambda_1,\lambda_2,t) = \min\Big\{\underline{\alpha}\int_0^{\lambda_1} |u'|^2\,dx + \underline{a}\int_{\lambda_1}^{\lambda_2} |u'|^2\,dx$$
$$+\underline{\beta}\int_{\lambda_2}^1 |u'|^2\,dx : u(0)=0, u(1)=t\Big\}. \tag{3.20}$$

Taking into account that in this case $\int_A \gamma(x/\varepsilon)\,dx \to \frac{1}{2}\gamma_2\lambda_2 + \frac{1}{2}\gamma_1\lambda_1$, the limit of $m^\varepsilon(t)$ can be written as

$$m_{\text{hom}}(t) = \min\Big\{\int_0^1 \Big(\chi_{[0,\lambda_1]}\Big(\underline{\alpha}|u'|^2 + \frac{\gamma_1+\gamma_2}{2}\Big) + \chi_{[\lambda_1,\lambda_2]}\Big(\underline{a}|u'|^2 + \frac{\gamma_2}{2}\Big)$$
$$+\chi_{[\lambda_2,1]}\underline{\beta}|u'|^2\Big)\,dx : u(0)=0,\ u(1)=t,\ 0\leq\lambda_1\leq\lambda_2\leq 1\Big\}. \tag{3.21}$$

Minimizing first in λ_1 and λ_2 we obtain

$$m_{\text{hom}}(t) = \min\Big\{\int_0^1 \min\Big\{\underline{\alpha}|u'|^2 + \frac{\gamma_1+\gamma_2}{2}, \underline{a}|u'|^2 + \frac{\gamma_2}{2}, \underline{\beta}|u'|^2\Big\}\,dx$$
$$: u(0)=0,\ u(1)=t\Big\}. \tag{3.22}$$

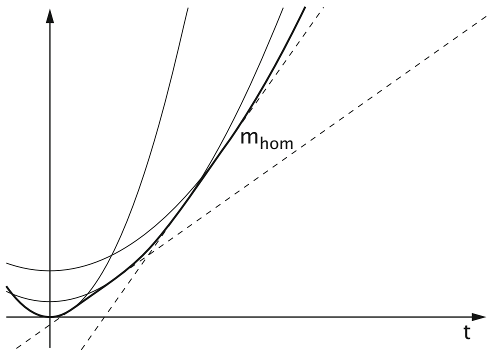


Fig. 3.4 The minimal energy $m_{\rm hom}(t)$ of Sect. 3.1.2

This observation highlights that the function $m_{\rm hom}(t)$ can be expressed as the convex envelope of

$$\min\Big\{\underline{\beta} z^2, \frac{2\alpha_2\beta_1}{\beta_1+\alpha_2\beta_2} z^2 + \frac{1}{2}\gamma_2, \underline{\alpha} + \frac{1}{2}(\gamma_1+\gamma_2)\Big\}, \tag{3.23}$$

(see Fig. 3.4) which are the three total energy densities corresponding to the mixtures of undamaged materials, equally damaged and undamaged materials (in the optimal way determined by condition (3.16)), and completely damaged materials.

The limit damage motion in this case is given in terms of the two sets $A^t_j = [0, \lambda_j(t)]$, where $\lambda_j(t)$ are the minimizers of problem (3.21), and of the corresponding u^t. Note that this is possible thanks to the particular choice of the damage sets $A^t_{j,\varepsilon}$, and does not give a description of the behavior of an arbitrary family of solutions $A^t_\varepsilon, u^t_\varepsilon$.

3.1.2.2 Double-Threshold Formulation

Also in this case we note that the damage process takes place when σ_t reaches some particular values. In this case the thresholds are two given by p_1 and p_2 (see Fig. 3.3 as compared with Fig. 3.2).

3.1.3 Dissipations Leading to a Commutability Result

We now slightly modify the dissipation in the example of the previous section. This will produce a 'commutatibility' result in the quasistatic motion outlined above; i.e., the process of homogenization and of damage can be interchanged. The first such modification is obtained by imposing that the domain of the dissipation be the set of intervals; i.e.,

$$D_\varepsilon(A) = +\infty \text{ if } A \text{ is not an interval,}$$

while D_ε remains unchanged otherwise. In this case, in the process described in Sect. 3.1.2, we may remark that, at fixed ε, the minimal A^t_ε will converge to some interval A^t for which we may pass to the limit obtaining the problem

$$\min\Big\{F_{\mathrm{hom},A^t}(u)+\overline{\gamma}|A^t| : u(0)=0, u(1)=t\Big\},$$

where

$$\overline{\gamma}=\frac{\gamma_1+\gamma_2}{2},$$

since

$$\lim_{\varepsilon\to 0} D_\varepsilon(A^t_\varepsilon)=\overline{\gamma}|A^t|.$$

Note that in the previous example this passage was not possible since the A^t_ε thus defined do not converge to a limit set.

We may conclude then that A^t minimizes the corresponding

$$\begin{aligned} m_{\mathrm{hom}}(t) &:= \min\Big\{F_{\mathrm{hom},A}(u)+\overline{\gamma}|A| : u(0)=0, u(1)=t, A \text{ subinterval of } (0,1)\Big\} \\ &= \min\Big\{F_{\mathrm{hom},A}(u)+\overline{\gamma}|A| : u(0)=0, u(1)=t, A\subset(0,1)\Big\} \\ &= f^{**}_{\mathrm{hom}}(t), \end{aligned} \tag{3.24}$$

where

$$f_{\mathrm{hom}}(z)=\min\{\underline{\alpha}z^2+\overline{\gamma}, \underline{\beta}z^2\}, \tag{3.25}$$

which describes the damage process corresponding to the limit homogenized functionals. Note that in the limit problem we may remove the constraint that A be an interval, since we have already remarked that solutions satisfying such a constraint exist.

3.1.3.1 Brutal Damage

We consider another dissipation, with

$$D_\varepsilon(A)=\int_A \gamma\Big(\frac{x}{\varepsilon}\Big)\,dx+\sigma\,\#(\partial A\cap[0,1]),$$

so that it is finite only on finite unions of intervals.

We may compute the limit of $m^\varepsilon(t)$ as in Sect. 3.1.2, noticing that, for a finite union of intervals, we may pass to the limit (taking possibly into account that the number of intervals may decrease in the limit process), and conclude that the limit damage process corresponds to the functionals $F_{\hom,A}$ and the homogenized dissipation

$$D_{\hom}(A) = \overline{\gamma}|A| + \sigma\#(\partial A \cap [0,1]).$$

Correspondingly, we can compute the minima

$$m_{\hom}(t) = \min\Big\{F_{\hom,A}(u) + D_{\hom}(A) : u(0) = 0, u(1) = t,$$
$$A \text{ union of subintervals of } (0,1)\Big\},$$

as

$$m_{\hom}(t) = \min\Big\{m^0_{\hom}(t), m^1_{\hom}(t)\Big\},$$

where $m^0_{\hom}$ corresponds to no damage,

$$m^0_{\hom}(t) = \min\Big\{F_{\hom,\emptyset}(u) : u(0) = 0, u(1) = t)\Big\} = \underline{\beta} t^2,$$

and $m^1_{\hom}$ corresponds to A a single interval (not being energetically convenient to have more than one interval),

$$\begin{aligned} m^1_{\hom}(t) &= \inf\Big\{F_{\hom,A}(u) + D_{\hom}(A) : u(0) = 0, u(1) = t, \\ &\qquad A \text{ subinterval of } (0,1), A \neq \emptyset\Big\} \\ &= \min\Big\{F_{\hom,A}(u) + \overline{\gamma}|A| : u(0) = 0, u(1) = t, \\ &\qquad A \text{ subinterval of } (0,1)\Big\} + 2\sigma \\ &= f^{**}_{\hom}(t) + 2\sigma, \end{aligned} \tag{3.26}$$

with $f_{\hom}$ as in (3.25).

The plot of $m_{\hom}$ is reproduced in Fig. 3.5. Note that we follow the curve corresponding to undamaged materials until we reach the graph of $m^1_{\hom}$, which corresponds to a positive value of the damage area; i.e., the damage is 'brutal' (once it is convenient to damage, we damage a large region). Correspondingly, in Fig. 3.6 we plot the value of $m'_{\hom}$ and the derivative of the homogenized energy $\mathscr{E}$ along a cycle in dependence of the boundary condition $g = g(t)$.

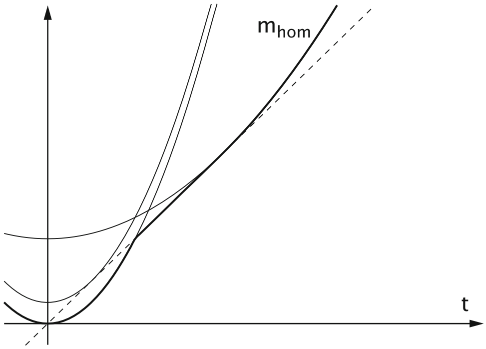


Fig. 3.5 The minimal energy $m_{\mathrm{hom}}(t)$

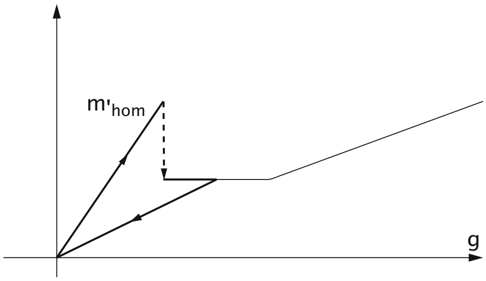


Fig. 3.6 Plot of $m'_{\mathrm{hom}}(t)$ and derivative of the energy along a cycle

3.1.4 Conditions for Commutability

Motivated by the examples above, we may derive a criterion of commutability of Γ-convergence and quasi-static motion, which we state in this particular case but is immediately generalized to more abstract situations. This easily follows from the remark that in order to pass to the limit we have to have the convergence of the minimum problems

$$\min\Big\{F_{\varepsilon,A}(u) + D_\varepsilon(A) : u(0) = 0, u(1) = g(t), B_\varepsilon \subset A\Big\} \tag{3.27}$$

with B_ε Borel sets converging to B (in (3.1) $B_\varepsilon = \bigcup\{A_s : s < t\}$) to

$$\min\Big\{F_{\mathrm{hom},A}(u) + D_{\mathrm{hom}}(A) : u(0) = 0, u(1) = g(t), B \subset A\Big\}. \tag{3.28}$$

Proposition 3.1 (A commutativity criterion). *Let $B_\varepsilon \to B$ and let*

$$G_\varepsilon(u, A) = \begin{cases} F_{\varepsilon,A}(u) + D_\varepsilon(A) & \text{if } B_\varepsilon \subset A \\ +\infty & \text{otherwise} \end{cases} \tag{3.29}$$

$$G_{\rm hom}(u,A)=\begin{cases}F_{{\rm hom},A}(u)+D_{\rm hom}(A) & \text{if } B\subset A\\ +\infty & \text{otherwise}.\end{cases} \tag{3.30}$$

Suppose that G_ε Γ-converges to $G_{\rm hom}$ with respect to the converge $L^2\times L^1$-convergence (the latter is understood as the convergence of the characteristic functions of sets). Then if a sequence of solutions $(u^t_\varepsilon, A^t_\varepsilon)$ to the evolutions related to the energies $F_{\varepsilon,A}$, dissipation D_ε and boundary conditions given by g is such that (up to subsequences) for all t u^t_ε converges to some u^t in L^2 and $B^t_\varepsilon=\bigcup\{A^s_\varepsilon : s<t\}$ converges to some B^t in L^1, then it converges (up to subsequences) to a solution to the evolution related to the energies $F_{{\rm hom},A}$, dissipation $D_{\rm hom}$ and boundary conditions given by g.

This criterion follows from the fundamental theorem of Γ-convergence, upon noting that the boundary conditions are compatible with the convergence of minima regardless to the constraint $B_\varepsilon\subset A$.

Remark 3.2. We may apply Proposition 3.1 to the two examples in Sect. 3.1.3. In fact, in both cases the boundedness of the dissipation implies that A^t_ε and hence B^t_ε are (increasing with t) intervals (or finite union of intervals in the second case), so that the pre-compactness of B^t_ε is guaranteed. The convergence for all t follows from an application of Helly's theorem.

We cannot apply Proposition 3.1 to the solutions in Sect. 3.1.2. Indeed, except for the trivial cases when $A^t_\varepsilon=\emptyset$ or $A^t_\varepsilon=(0,1)$, these do not converge strongly in L^1 but only weakly.

3.1.5 Relaxed Evolution

The criterion above suggests, in case it is not satisfied, to examine the behavior of the functionals (3.29) with respect to the $L^2\times L^1$-weak convergence. In this case, the limit of a sequence of characteristic functions may not be a characteristic function itself, so that the domain of the Γ-limit will be the space of pairs (u,θ), with $0\le\theta\le 1$. This formulation will necessarily be more complex, but will capture the behavior of all sequences $A^t_\varepsilon, u^t_\varepsilon$.

Proposition 3.2 (Relaxed total energies). *If hypothesis (3.16) holds, then the Γ-limit of the functionals (3.29) with respect to the $L^2\times L^1$-weak convergence is given by the functional (r stands for 'relaxed')*

$$G^r_{\rm hom}(u,\theta)=\int_{(0,1)} f_{\rm hom}(\theta,u')\,dx+\int_{(0,1)}\gamma_{\rm hom}(\theta)\,dx, \tag{3.31}$$

where

$$f_{\text{hom}}(\theta, z) = \begin{cases} \dfrac{2\alpha_2\beta_1\beta_2}{2\theta\beta_1\beta_2 + (1-2\theta)\alpha_2\beta_1 + \alpha_2\beta_2} z^2 & \text{if } 0 \le \theta \le \frac{1}{2} \\ \\ \dfrac{2\alpha_1\alpha_2\beta_1}{2(1-\theta)\alpha_1\alpha_2 + (2\theta-1)\alpha_2\beta_1 + \alpha_1\beta_1} z^2 & \text{if } \frac{1}{2} \le \theta \le 1 \end{cases} \tag{3.32}$$

and the dissipation energy density is

$$\gamma_{\text{hom}}(\theta) = \begin{cases} \gamma_2\theta & \text{if } 0 \le \theta \le \frac{1}{2} \\ \\ \frac{1}{2}\gamma_2 + \gamma_1\big(\theta - \frac{1}{2}\big) & \text{if } \frac{1}{2} \le \theta \le 1 \end{cases} \tag{3.33}$$

Proof. We do not dwell on this proof, since it is a variation of the usual homogenization theorem. A lower bound is obtained by minimizing on each periodicity cell. Upon scaling we are led to computing

$$\phi(z,\theta) := \min\Big\{ \int_A \alpha(y)|u'|^2\,dy + \int_{(0,1)\setminus A} \beta(y)|u'|^2\,dy + \int_A \gamma(y)\,dy \ : \ A \subset (0,1),\ |A| = \theta,\ u(0) = 0,\ u(1) = z \Big\}.$$

By a direct computation we get

$$\phi(z,\theta) = f_{\text{hom}}(\theta, z) + \gamma_{\text{hom}}(\theta).$$

Since ϕ is convex in the pair (z,θ), its integral is lower semicontinuous in $L^2 \times L^1$-weak, and hence is a candidate for the Γ-liminf. The proof of the limsup inequality is obtained by density, first dealing with the case when u is a piecewise-affine function and θ is a piecewise-constant function. □

Remark 3.3. The limit of problems (3.27) with B_ε converging weakly to some ϕ will be of the form

$$\min\Big\{ G^r_{\text{hom}}(u,\theta) : u(0) = 0, u(1) = g(t), \phi \le \theta \Big\}. \tag{3.34}$$

As in Sect. 3.1.2, we only consider the case $g(t) = t$, and the problem

$$m^r(t) = \min\Big\{ G^r_{\text{hom}}(u,\theta) : u(0) = 0, u(1) = t \Big\}. \tag{3.35}$$

We have

$$m^r(t) = \min\Big\{ \int_0^1 \min_{0\le\theta\le1} \Big\{ f_{\text{hom}}(\theta, u') + \gamma_{\text{hom}}(\theta) \Big\}\,dx : u(0) = 0, u(1) = t \Big\}. \tag{3.36}$$

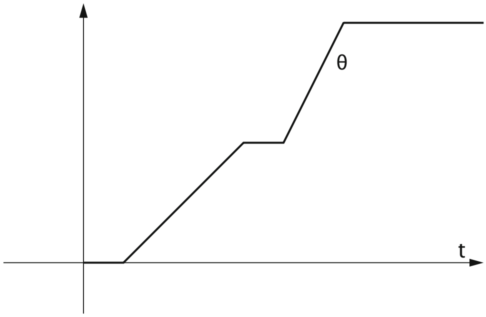


Fig. 3.7 Value of the damage $\theta(t)$

A direct computation shows that

$$\min_{0\leq\theta\leq 1}\Big\{f_{\rm hom}(\theta,z)+\gamma_{\rm hom}(\theta)\Big\}=m(z), \tag{3.37}$$

with m the one in Sect. 3.1.2; hence, by the convexity of m, we have $m^r(z)=m(z)$. Moreover, again using the convexity of m, a solution is simply given by $u_t(x)=tx$ and correspondingly $\theta=\theta(t)$ constant equal to the minimizer of (3.37) with $z=t$; namely,

$$\theta(t)=\begin{cases} 0 & \text{if } |t|\leq \frac{p_2}{\beta} \\ \frac{p_2^2}{\gamma_2}\left(\frac{t}{p_2}-\frac{1}{\beta}\right) & \text{if } \frac{p_2}{\beta}<|t|<\frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \\ \frac{1}{2} & \text{if } \frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2}\leq|t|\leq\frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \\ 1+\frac{p_1^2}{\gamma_1}\left(\frac{t}{p_1}-\frac{1}{\alpha}\right) & \text{if } \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2}<|t|<\frac{p_1}{\alpha} \\ 1 & \text{if } |t|\geq\frac{p_1}{\alpha} \end{cases}$$

(see Fig. 3.7). Note that we have

$$\theta(t)=\frac{\lambda_{1,\min}(t)+\lambda_{2,\min}(t)}{2}$$

with $\lambda_{j,\min}$ given by (3.17) and (3.18). The solution with θ constant corresponds to equi-distributed damage. Note that we have infinitely many solutions, among which the ones described above in terms of A_1^t and A_2^t.

3.2 Energetic Solutions for Rate-Independent Evolution

The examples in the previous theory can be framed in a general theory of rate-independent variational evolution. We introduce some of the concepts of the theory that are relevant to our presentation, without being precise in the hypotheses on spaces and topologies.

Definition 3.2. Let $\mathscr{F} = \mathscr{F}(t,\cdot)$ be a time-parameterized energy functional and $\mathscr{D}$ be a dissipation functional, which we assume to be positively-homogeneous of degree one; i.e. $\mathscr{D}(sU) = s\mathscr{D}(U)$ if $s > 0$. Then U is an *energetic solution* for the evolution inclusion

$$\partial\mathscr{D}(\dot{U}) + \partial_U \mathscr{F}(t,U) \ni 0$$

if the following two conditions hold:

(S) **global stability** for all t and $\hat{U}$ we have

$$\mathscr{F}(t,U(t)) \leq \mathscr{F}(t,\hat{U}) + \mathscr{D}(\hat{U} - U(t));$$

(E) **energy inequality** for all t

$$\mathscr{F}(t,U(t)) + \int_0^t D(\dot{U}) \leq \mathscr{F}(0,U(0)) + \int_0^t \partial_s \mathscr{F}(s,U(s))\,ds.$$

In this formula the integral $\int_0^t D(\dot{U})$ must be understood in the sense of measures, and can be equivalently defined as

$$\sup\Big\{\sum_{i=1}^n D(U(t_i) - U(t_{i-1})) : 0 = t_0 < t_1 < \cdots < t_n = t\Big\}. \tag{3.38}$$

If U is an absolutely continuous function then the integral reduces to $\int_0^t D(\dot{U}(s))\,ds$.

Remark 3.4 (Energy equality). Under mild assumptions, from (S) it can be deduced that in (E) equality holds, so that we have an *energy conservation* identity. This identity states that the difference of the energy at a final and an initial state equals the difference of the work of the applied actions and the total dissipation along the path.

Remark 3.5. In the case of damage we have $U = (u,v)$,

$$\mathscr{F}(t,u,v) = \begin{cases} \displaystyle\int_0^1 \Big(\alpha v|u'|^2 + \beta(1-v)|u'|^2\Big)\,dx & \text{if } v \in \{0,1\} \text{ a.e.,} \\ & u(0) = 0, u(1) = g(t) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\mathscr{D}(U) = \begin{cases} \gamma \displaystyle\int_0^1 v\,dx & \text{if } v \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

Condition (S) is meaningful only if $\hat{U} = (u, v)$ and $U(t) = (u^t, v^t)$ satisfy $v = \chi_A$ and $v^t = \chi_{A^t}$ with $A^t \subset A$, so that (S) implies that u^t and A^t are minimizers for (3.1). Conversely, it can be checked that the solutions to the damage evolution satisfy the energy inequality as an identity.

Remark 3.6 (Rate-independence). The requirement that $\mathscr{D}$ be positively homogeneous of degree one implies that the solution is *rate-independent*; i.e., that if we consider a re-parameterization of the energy $\tilde{\mathscr{F}}(t, U) = \mathscr{F}(\varphi(t), U)$ via an increasing diffeomorphism φ, then the energetic solutions $\tilde{U}$ of the corresponding evolution inclusion are exactly the $\tilde{U}(t) = U(\varphi(t))$ with U energetic solutions of the corresponding evolution inclusion for $\mathscr{F}$.

Example 3.1 (Mechanical play/hysteresis). The prototypical example of an evolution inclusion is by taking $U = x \in \mathbb{R}$ and

$$\mathscr{F}(t,x) = \frac{x^2}{2} - tx, \qquad \mathscr{D}(x) = |x|.$$

In this case we can write explicitly $\partial|\dot{x}| + x - t \ni 0$ as

$$\begin{cases} \dot{x} > 0 & \text{if } x = t-1 \\ \dot{x} < 0 & \text{if } x = t+1 \\ \dot{x} = 0 & \text{if } t-1 \le x \le t+1. \end{cases}$$

The solution with $x(0) = x_0 \in [-1, 1]$ is

$$x(t) = \begin{cases} x_0 & \text{if } t \le 1 + x_0 \\ t-1 & \text{if } t > 1 + x_0. \end{cases} \tag{3.39}$$

If we take a non-monotone load $g(t) = T - |t - T|$ with $T > 1 + x_0$ and

$$\mathscr{F}(t,x) = \frac{x^2}{2} - g(t)x, \qquad \mathscr{D}(x) = |x|.$$

then the solution x is as above for $t \le T$, and given solving $\partial|\dot{x}| + x - (2T - t) \ni 0$ by

$$x(t) = \begin{cases} T-1 & \text{if } T \le t \le T+2 \\ 2T - t + 1 = g(t) + 1 & \text{if } t \ge T+2. \end{cases}$$

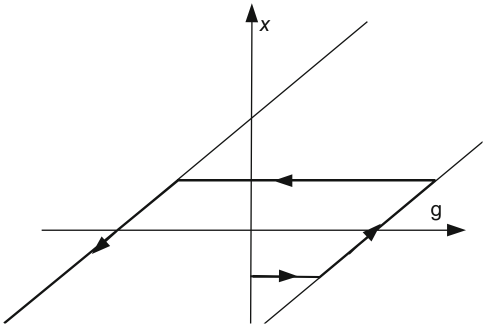


Fig. 3.8 Hysteretic trajectory $x(t)$ in dependence of $g(t)$

This solution shows a hysteretic behavior of this system, whose trajectory in the g-x plane is represented in Fig. 3.8.

3.2.1 Solutions Obtained by Time Discretization

Energetic solutions can be obtained as limits of discrete schemes as follows: fix $\tau > 0$ and define U_k^τ recursively by setting $U_0^\tau = U_0$, and choosing U_k^τ as a solution of the minimum problem

$$\min_{\hat{U}}\Big\{\mathscr{F}(\tau k, \hat{U}) + \mathscr{D}(\hat{U} - U_{k-1}^\tau)\Big\}.$$

Define the continuous trajectory $U^\tau(t) = U^\tau_{\lfloor t/\tau \rfloor}$. Under suitable assumptions, the limits of (subsequences of) U^τ are energetic solution of the variational inclusion for $\mathscr{F}$ and $\mathscr{D}$.

Example 3.2 (Mechanical play). It is easy to check that the solutions in Example 3.1 can be obtained by time-discretization, solving iteratively

$$\min\Big\{\frac{1}{2}x^2 - g(k\tau)x + |x - x_{k-1}^\tau|\Big\}.$$

In the case of $x_0 \in [-1, 1]$ and $g(t) = t$ the sequence $\{x_k^\tau\}$ is non-decreasing and hence x_k^τ solves

$$\min\Big\{\frac{1}{2}x^2 - (k\tau - 1)x - x_{k-1}^\tau : x \geq x_{k-1}^\tau\Big\};$$

i.e.,

$$x_k^\tau = \begin{cases} x_{k-1}^\tau & \text{if } k\tau - 1 \leq x_{k-1}^\tau \\ k\tau + 1 & \text{if } k\tau - 1 \geq x_{k-1}^\tau. \end{cases}$$

Passing to the limit as $\tau \to 0$ we then obtain

$$x(t) = \begin{cases} x_0 & \text{if } t - 1 \le x_0 \\ t - 1 & \text{if } t - 1 \ge x_0, \end{cases}$$

which corresponds to the solution in (3.39).

Example 3.3 (Nonconvex mechanical play). We can consider a double-well potential of the form

$$\mathscr{F}(t, x) = \frac{1}{2} \min\{(x-1)^2, (x+1)^2\} - tx, \qquad \mathscr{D}(x) = |x|$$

with $x_0 \in [-2, -1]$. Then the sequence x_k^τ is increasing and minimizes

$$\min\Big\{\frac{1}{2} \min\{(x-1)^2, (x+1)^2\} - (k\tau - 1)x - x_{k-1}^\tau : x \ge x_{k-1}^\tau\Big\}.$$

The solution satisfies

$$x_k^\tau = \begin{cases} x_{k-1}^\tau & \text{if } k\tau - 2 \le x_{k-1}^\tau \\ k\tau - 2 & \text{if } x_{k-1}^\tau \le k\tau - 2,\ k\tau \le 1 \\ k\tau & \text{if } k\tau \ge 1 \end{cases}$$

(with an ambiguity if $k\tau = 1$, in which case we may take equivalently $x_{k-1}^\tau = -1$ or $x_{k-1}^\tau = 1$). Passing to the limit we have either the solution

$$x(t) = \begin{cases} x_0 & \text{if } t \le x_0 + 2 \\ t - 2 & \text{if } x_0 \le t \le 1 \\ t & \text{if } t > 1 \end{cases}$$

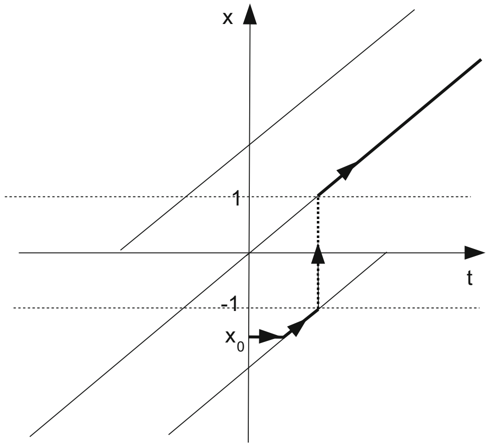
or the one equal to this except for 1 where $x(1) = 1$. The graph of the solution is pictured in Fig. 3.9.

Note that the solution is discontinuous at $t = 1$ and is not characterized completely by the differential inclusion. In this case the discontinuity exactly at $|x| = 1$ can be justified by the energy equality.

3.2.2 Stability

We can give a stability result with respect to Γ-convergence. As remarked in the case of damage, the separate Γ-convergence of $\mathscr{F}_\varepsilon$ and $\mathscr{D}_\varepsilon$ may not be sufficient to describe the limit of the corresponding variational motions.

Fig. 3.9 The trajectory $x(t)$ in Example 3.3



Theorem 3.1. *Suppose that $\mathscr{F}$ and $\mathscr{D}$ are lower bounds for $\mathscr{F}_\varepsilon$ and $\mathscr{D}_\varepsilon$, that U_ε are energetic solutions converging pointwise to some U as $\varepsilon \to 0$, that the initial data are well-prepared; i.e., that*

$$\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(0, U_\varepsilon(0)) = \mathscr{F}(0, U(0)),$$

that we have convergence of the external actions

$$\lim_{\varepsilon\to 0} \int_0^t \partial_s \mathscr{F}_\varepsilon(s, U_\varepsilon(s))\, ds = \int_0^t \partial_s \mathscr{F}(s, U(s))\, ds \qquad \text{for all } t,$$

and that the following **mutual recovery sequence** *existence condition holds: for all t and all $\hat{U}$ there exists a sequence $\hat{U}_\varepsilon$ such that*

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\Big(\mathscr{F}_\varepsilon(t, \hat{U}_\varepsilon) - \mathscr{F}_\varepsilon(t, U_\varepsilon(t)) + \mathscr{D}_\varepsilon(\hat{U}_\varepsilon - U_\varepsilon(t))\Big) \\ &\quad \le \mathscr{F}(t, \hat{U}) - \mathscr{F}(t, U(t)) + \mathscr{D}(\hat{U} - U(t)). \end{aligned} \tag{3.40}$$

Then U is an energetic solution for the limit energy and dissipation.

Proof. Let $0 = t_0 < t_1 < \cdots < t_n = t$; by the liminf inequality for D_ε and (3.38) we then have

$$\sum_{i=1}^n \mathscr{D}(U(t_i) - U(t_{i-1})) \le \liminf_{\varepsilon\to 0} \sum_{i=1}^n \mathscr{D}_\varepsilon(U_\varepsilon(t_i) - U_\varepsilon(t_{i-1})) \le \liminf_{\varepsilon\to 0} \int_0^t \mathscr{D}_\varepsilon(\dot{U}_\varepsilon).$$

Taking into account the liminf inequality for $\mathscr{F}_\varepsilon$ and the convergence hypotheses on initial data and external actions we then obtain

$$\mathscr{F}(t,U(t)) + \int_0^t D(\dot{U}(s))\,ds \le \liminf_{\varepsilon\to 0}\Big(\mathscr{F}_\varepsilon(0,U_\varepsilon(0)) + \int_0^t \partial_s\mathscr{F}_\varepsilon(s,U_\varepsilon(s))\,ds\Big)$$
$$= \mathscr{F}(0,U(0)) + \int_0^t \partial_s\mathscr{F}(s,U(s))\,ds$$

so that (E) holds.

Take any test $\hat{U}$ and use the mutual recovery sequence $\hat{U}_\varepsilon$ to obtain

$$\mathscr{F}(t,\hat{U}) - \mathscr{F}(t,U(t)) + \mathscr{D}(\hat{U} - U(t)) \ge 0;$$

i.e. the inequality in (S), from the same inequality for U_ε. □

Proposition 3.3 (Necessary and sufficient conditions).

(i) *Let $\mathscr{F}_\varepsilon$ Γ-converge to $\mathscr{F}$ and $\mathscr{D}_\varepsilon$ converge continuously to $\mathscr{D}$. Then the mutual recovery sequence condition is satisfied;.*

(ii) *Assume that $\mathscr{F}_\varepsilon$ and $\mathscr{D}_\varepsilon$ Γ-converge to $\mathscr{F}$ and $\mathscr{D}$, that $U_\varepsilon(t)$ is a recovery sequence for F_ε at $U(t)$ and that the mutual recovery sequence condition holds with $\hat{U}_\varepsilon \to \hat{U}$. Then $\mathscr{G}_\varepsilon(V) = \mathscr{F}_\varepsilon(t,V) + \mathscr{D}_\varepsilon(V - U_\varepsilon(t))$ Γ-converges to $\mathscr{G}(V) = \mathscr{F}(t,U(t)) + \mathscr{D}(V - U(t))$.*

Proof. (i) Follows by taking $\hat{U}_\varepsilon$ any recovery sequence for $\mathscr{F}_\varepsilon(t,\hat{U})$.

(ii) Is an immediate consequence of the fact that $\mathscr{F} + \mathscr{D}$ is a lower bound for $\mathscr{F}_\varepsilon + \mathscr{D}_\varepsilon$, while the mutual recovery sequence provides a recovery sequence for $\mathscr{F}(t,U(t)) + \mathscr{D}(V - U(t))$. □

Example 3.4 (An example with relaxed evolution). In $\mathbb{R}^2$ with $U = (x,y)$, consider the initial datum $u_\varepsilon(0) = (0,0)$ and the energy and dissipation

$$\mathscr{F}_\varepsilon(t,U) = \frac{1}{2}x^2 + \frac{1}{2\varepsilon^2}(y - \varepsilon x)^2 - tx, \qquad \mathscr{D}_\varepsilon(U) = |x| + \frac{1}{\varepsilon}|y|$$

with Γ-limits

$$\mathscr{F}(t,U) = \frac{1}{2}x^2 - tx, \qquad \mathscr{D}(U) = |x|$$

with domain $\{y = 0\}$.

The solution to the differential inclusion for $\mathscr{F}$ and $\mathscr{D}$ with initial datum $(0,0)$ is given by $x(t)$ as in (3.39) with $x_0 = 0$, and $y(t) = 0$. On the other hand, the solutions to the differential inclusions U_ε can be computed explicitly, and they tend to $U = (x,y)$ defined by $y(t) = 0$ and

$$x(t) = \begin{cases} 0 & \text{if } t \le 1 \\ \dfrac{t-1}{2} & \text{if } 1 \le t \le 3 \\ t-2 & \text{if } t \ge 3. \end{cases}$$

In this case we do not have convergence of the solutions. However, we can compute the Γ-limit of the sum $\mathscr{F}_\varepsilon + \mathscr{D}_\varepsilon$, whose domain is $\{y = 0\}$. Recovery sequences for $(x, 0)$ can be looked for of the form $(x, \varepsilon z)$. By minimizing in z we easily get that this Γ-limit is

$$\mathscr{G}(x) = \frac{1}{2}x^2 - tx + |x| + \psi(x),$$

where

$$\psi(x) = \min\Big\{\frac{1}{2}(z-x)^2 + |z|\Big\} = \min\Big\{\frac{x^2}{2}, \frac{1}{2} + ||x| - 1|\Big\},$$

whose derivative is

$$\psi'(x) = (x \wedge 1) \vee (-1).$$

It is easily seen that the function $x(t)$ above is the solution of

$$\partial|\dot{x}| + \mathscr{F}_0'(x) = \partial|\dot{x}| + x - t + \psi'(x) \ni 0,$$

where $\mathscr{F}_0(x) = \mathscr{G}(x) - |x| = \frac{1}{2}x^2 - tx + \psi(x)$. This energy $\mathscr{F}_0$ can then be regarded as the relaxed effective energy describing the limit behavior of the system.

3.3 Francfort and Marigo's Variational Theory of Fracture

A very interesting application of the theory outlined above is to variational models of Fracture following the formulation given by Griffith in the 1920s. In this case it is maybe clearer the definition via time-discrete motions (see Sect. 3.2.1) given as follows.

We consider the antiplane case where the variable u representing the displacement is scalar. By Ω we denote a bounded open subset of $\mathbb{R}^n$ which will be the reference configuration of a linearly elastic material subject to brittle fracture as a consequence of a varying boundary condition $u = g(t)$ on $\partial\Omega$. K will be a closed set representing the crack location in the reference configuration. We consider the case $g(0) = 0$, and we set $K_0 = \emptyset$.

With fixed $\tau > 0$ we define $u_0^\tau = 0$, $K_0^\tau = K_0$ and u_k^τ, K_k^τ recursively as minimizers of the problem

$$\min\Big\{\int_{\Omega\setminus K} |\nabla u|^2 dx + \mathscr{H}^{n-1}(K \setminus K_{k-1}^\tau) : K_{k-1}^\tau \subset K = \overline{K} \subset \overline{\Omega},$$
$$u \in H^1(\Omega \setminus K),\ u = g(t) \text{ on } \partial\Omega \setminus K\Big\}, \tag{3.41}$$

where $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. In this way K_k^τ is an increasing sequence of closed sets. Note that part of the crack may also lie on the boundary of Ω, in which case the boundary condition is satisfied only on $\partial\Omega \setminus K$.

In this formulation we have an elastic energy defined by

$$\mathscr{F}(t,u,K) = \begin{cases} \displaystyle\int_{\Omega\setminus K} |\nabla u|^2 dx & \text{if } u \in H^1(\Omega \setminus K) \text{ and } u = g(t) \text{ on } \partial\Omega \setminus K \\ +\infty & \text{otherwise,} \end{cases}$$

and a dissipation

$$\mathscr{D}(K) = \begin{cases} \mathscr{H}^{n-1}(K) & \text{if } K = \overline{K} \subset \overline{\Omega} \\ +\infty & \text{otherwise.} \end{cases}$$

The existence of minimizing pairs for (u, K) is not at all trivial. One way is by using the theory of SBV *functions*; i.e., functions of bounded variation u whose distributional derivative is a measure that can be written as a sum of a measure absolutely continuous with respect to the Lebesgue measure and a measure absolutely continuous with respect to the restriction of the $(n-1)$-dimensional Hausdorff measure to the complement of the Lebesgue points of u, the latter denoted by $S(u)$. For such functions the approximate gradient ∇u exists at almost all points. We can therefore define for all closed K the energy

$$\mathscr{E}_K(u) = \int_{\Omega\setminus K} |\nabla u|^2\, dx + \mathscr{H}^{n-1}(S(u) \cap (\Omega \setminus K)). \tag{3.42}$$

Such energies are L^1-lower semicontinuous and coercive, so that existence of weak solutions in $SBV(\Omega)$ are ensured from the direct methods of the Calculus of Variations. Regularity results give that $\mathscr{H}^{d-1}(\overline{S(u)} \setminus S(u)) = 0$ for minimizing u, so that to a minimizing $u \in SBV(\Omega \setminus K_{k-1}^\tau)$ of

$$\min\Big\{\mathscr{E}_{K_{k-1}^\tau}(u) : u \in SBV(\Omega \setminus K_{k-1}^\tau),\ u = g(t) \text{ on } \partial\Omega \setminus (S(u) \cup K_{k-1}^\tau)\Big\} \tag{3.43}$$

corresponds a minimizing pair $K_k^\tau = K_{k-1}^\tau \cup \overline{S(u)}$ and $u_k^\tau = u_{|\Omega\setminus K_k^\tau} \in H^1(\Omega \setminus K_k^\tau)$ for (3.41).

The passage from a discrete trajectory u^τ to a continuous one u for all t letting $\tau \to 0$ is possible thanks to some monotonicity arguments. The delicate step is the proof that such u still satisfies the global stability property, which is ensured by a *transfer lemma* (the Francfort–Larsen transfer lemma), which allows to approximate a test function $\hat{u}$ appearing in the limit stability estimate with a sequence $\hat{u}_\tau$ that can be used in the stability estimate holding for $u^\tau(t)$, which then carries to the limit.

Remark 3.7 (Existence of fractured solutions). Note that for large enough values of the boundary condition $g(t)$ we will always have a solution with $K^t \neq \emptyset$. Indeed, consider the case $g(t) = tg_0$ with $g_0 \neq 0$ on $\partial\Omega$. If $K^t = \emptyset$ then the corresponding u^t is a minimizer of

$$\min\Big\{\int_\Omega |\nabla u|^2\,dx : u = tg_0 \text{ on } \partial\Omega\Big\}$$
$$= t^2 \min\Big\{\int_\Omega |\nabla u|^2\,dx : u = g_0 \text{ on } \partial\Omega\Big\} =: t^2 C_0.$$

On the other hand, we can use as test function $u = 0$ and as test set $K = \partial\Omega$ in (3.41), for which the total energy is $C_1 = \mathscr{H}^{n-1}(\partial\Omega)$. This shows that for $t^2 C_0 > C_1$ we cannot have $K = \emptyset$.

Remark 3.8 (The one-dimensional case). In the one-dimensional case, the functional $\mathscr{E}$ reduces to the energy F obtained as a limit in Sect. 2.6 with the normalization $2J''(0) = J(\infty) = 1$, since $\mathscr{H}^0(K) = \#(K)$. Note that in this case the domain of $\mathscr{E}$ reduces to piecewise-H^1 functions. If $\Omega = (0,1)$ then the time-continuous solutions are of the form

$$(u^t(x), K^t) = \begin{cases} (g(t)x, \emptyset) & \text{for } t \leq t_c \\ \Big(g(t)\chi_{(x_0,1)}(x), \{x_0\}\Big) & \text{for } t > t_c, \end{cases}$$

or

$$(u^t(x), K^t) = \begin{cases} (g(t)x, \emptyset) & \text{for } < t_c \\ \Big(g(t)\chi_{(x_0,1)}(x), \{x_0\}\Big) & \text{for } t \geq t_c, \end{cases}$$

where $x_0 \in [0,1]$ and t_c is any value with $g(t_c) = 1$ and $g(s) \leq 1$ for $s < t_c$. This non-uniqueness is due to the fact that for $g(t) = 1$ we have two possible types of solutions $u(x) = x$ and $u(x) = \chi_{(x_0,1)}(x)$.

3.3.1 Homogenization of Fracture

The interpretation of fracture energies as functionals defined in *SBV* allows to consider the L^1-convergence in *SBV* along sequences with equibounded energy (3.42). With respect to such a convergence we can consider stability issues for energies and dissipations related to the oscillating total energy

$$\mathscr{E}_\varepsilon(u) = \int_{\Omega\setminus K} a_b\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2\,dx + \int_{S(u)\cap(\Omega\setminus K)} a_f\Big(\frac{x}{\varepsilon}\Big)d\mathscr{H}^{n-1}$$

(here the coefficients a_b and a_f, where b stands for bulk and f for fracture, are periodic functions). In this case the limit of the total energies $\mathscr{E}_\varepsilon$ is the sum of the energies obtained separately as limits of the energy and the dissipation parts (with respect to the same convergence), and has the form

$$\mathscr{E}_{\rm hom}(u) = \int_{\Omega\setminus K} \langle A_{\rm hom}\nabla u, \nabla u\rangle\, dx + \int_{S(u)\cap(\Omega\setminus K)} \varphi_{\rm hom}(\nu)\, d\mathscr{H}^{n-1},$$

where ν denotes the measure-theoretical normal to $S(u)$. Note that the homogenized $A_{\rm hom}$ is the same given by the homogenization process in H^1, while $\varphi_{\rm hom}$ is an effective fracture energies obtained by optimization on oscillating fractures, related to the homogenization of perimeter functionals. Thanks to this remark it is possible to show that the energetic solutions for $\mathscr{E}_\varepsilon$ converge to energetic solutions of $\mathscr{E}_{\rm hom}$. In terms of the construction of mutual recovery sequences this is possible since internal energy and dissipation can be optimized separately, contrary to what happens for the damage case, where both terms involve bulk integrals.

Appendix

Analyses of damage models linked to our presentation are contained in the work by Francfort and Marigo [5]. The higher-dimensional case is studied in a paper by Francfort and Garroni [3]. A threshold-based formulation is introduced by Garroni and Larsen [7]. The examples in Sect. 3.1.3 have been part of the course exam of B. Cassano and D. Sarrocco at Sapienza University in Rome.

An analysis of rate-independent processes is contained in the review article by Mielke [8]. The definitions given here can be traced back to the works by Mielke, Theil and Levitas [10] and [11]. The stability with respect to Γ-convergence is analyzed in the paper by Mielke, Roubiček, and Stefanelli [9]. Most of Sect. 3.2 is taken from a lecture given by Ulisse Stefanelli during the course at the University of Pavia. The homogenization examples in Sect. 3.1, framed in the theory of energetic solutions, are contained in the paper [2]

An account of the variational theory of fracture (introduced in [6]) is contained in the book by Bourdin et al. [1]. The fundamental transfer lemma is contained in the seminal paper by Francfort and Larsen [4].

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