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Andrea Braides

Local Minimization, Variational Evolution and Γ -Convergence

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To Krys and Damian

Preface

These notes have been motivated by the interests of the author in variational problems depending on small parameters, for some of which a description based on a global minimization principle does not seem satisfactory. Such problems range from the derivation of physical theories from first principles to numerical problems involving energies with many local minima. Even though an asymptotic description of related global minimization problems can be given in terms of Γ -convergence, the Γ -limit often does not capture the behavior of local minimizers or of gradient flows. This failure is sometimes mentioned as the proof that Γ -convergence is ‘wrong’. It may well be so. The author’s standpoint is that it might nevertheless be a good starting point that may be systematically ‘corrected’.

The author’s program has been to examine the (few) results in the literature and try to connect them with his own work in homogenization and discrete systems, where often the local minimization issues are crucial due to the oscillations of the energies. The directions of research have been to:

- find criteria that ensure the convergence of local minimizers and critical points. In case this does not occur, then modify the Γ -limit into an equivalent Γ -expansion (as introduced by the author and L. Truskinovsky) in order to match this requirement. We note that in this way we ‘correct’ some limit theories, finding (or ‘validating’) other ones present in the literature.
- modify the concept of local minimizer, so that it may be more ‘compatible’ with the process of Γ -limit. One such concept is the small-scale stability of C. Larsen.
- treat evolution problems for energies with many local minima obtained by a time-discrete scheme introducing the notion of ‘minimizing movements along a sequence of functionals’. In this case the minimizing movement of the Γ -limit can always be obtained by a choice of the space- and time-scale, but more interesting behaviors can be obtained at a critical ratio between them. In many cases a ‘critical scale’ can be computed together with an effective motion, from which all other minimizing movements are obtained by scaling. Furthermore the choice of suitable Γ -converging sequences in the scheme above allows to address the issues of long-time behavior and backwards motion.

- examine the general variational evolution results that may be related to these minimizing movements, in particular recent theories of quasistatic motion and gradient flow in metric spaces.

The content of the present notes is taken from a series of lectures which formed a PhD course first given at Sapienza University of Rome from March to May 2012 and subsequently at the University of Pavia from November 2012 to January 2013. Those courses were addressed to an audience of students, some of which with an advanced background (meaning that they were already exposed to the main notions of the Calculus of Variations and of Γ -convergence), and researchers in the field of the Calculus of Variations and of Variational Evolution. This was an advanced course in that it was meant to address some current (or future) research issues rather than to discuss some subject systematically. Part of the notes has also been reworked during a 10-h course at the University of Narvik on October 25–30, 2012.

The reader should bear in mind that the scope of the notes has been to foster discussion on the problems presented rather than construct a general detailed theory (a worthy and very interesting objective, though). Hence, we have focused on highlighting the phenomena and issues linked to the interaction of scales, local minimization and variational evolution, rather than on the details of the Γ -convergence process, or the optimal hypotheses for the definition of gradient flows, and so on, for which we refer to the existing literature.

These notes would not have been written without the personal constant encouragement of Adriana Garroni, who is also responsible for the organization of the PhD course in Rome. I gratefully acknowledge the invitation of Enrico Vitali to give the PhD course in Pavia, his many interesting comments and his delightful hospitality. I greatly profited from the stimulating environments in both departments; special thanks go to all the students who interacted during the course and the final exams. A precious direct contribution has been given by Adriana Garroni for many ideas about the homogenization of damage in Sect. 3.1, by Ulisse Stefanelli, who provided the material for most of Sect. 3.2 by giving a beautiful lecture on the subject during the course at Pavia, and by Luigi Ambrosio for the proofs in Sect. 11.1.1. I also acknowledge the very fruitful discussions with Matteo Focardi, Chris Larsen, Alexander Mielke, Matteo Novaga, Andrey Piatnitski, Giuseppe Savaré and Lev Truskinovsky, which inspired many examples in these notes. I gratefully acknowledge a careful reading of the manuscript by Giovanni Scilla.

Rome, Italy
July 2013

Andrea Braides

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Chapter 1

Introduction

The theory of Γ -convergence was conceived by Ennio De Giorgi at the beginning of the 1970s. It originated from previous notions of convergence related mainly to elliptic operators, such as G -convergence or H -convergence, or to convex functionals, as Mosco convergence. The main issue in the definition of Γ -convergence is tracking the behaviour of global minimum problems (i.e., both of minimum values and minimizers) of a sequence F_ε by the computation of an ‘effective’ minimum problem involving the (suitably defined) Γ -limit of this sequence. Even though the definition of such a limit is local (in that in defining its value at a point x we only take into account sequences converging to x), its computation in general does not describe the behaviour of local minimizers of F_ε (i.e., points x_ε which are absolute minimizers of the restriction of F_ε to a small neighbourhood of the point x_ε itself). The possible situation, in a very simplified picture, is that in Fig. 1.1, where the original F_ε possess many local minimizers, some (or all) of which are ‘integrated out’ in the Γ -convergence process (note that this happens even when the depth of the oscillations does not vanish), or the one pictured in Fig. 1.2, where the limit F may present local minimizers that do not derive from local minimizers of F_ε . This over-simplification of the description given by the Γ -limit on some sets is not uncommon.

A notable exception is when we have an isolated local minimizer x_0 of the Γ -limit: in that case we may track the behaviour of local minimizers as absolute minimizers of F_ε restricted to a fixed neighbourhood of x_0 and conclude the existence of local minimizers for F_ε close to x_0 (this situation is pictured in Fig. 1.3). The possibility of the actual application of such a general principle has been envisaged by Kohn and Sternberg, who first used it to deduce the existence of local minimizers of the Allen–Cahn equation by exhibiting local area minimizing sets.

A recent different but related direction of research concerns the study of gradient flows. A general variational theory based on the solution of Euler schemes has been developed by Ambrosio, Gigli and Savaré. The stability of such schemes by Γ -perturbations is possible in the absence of local minimizers, which otherwise

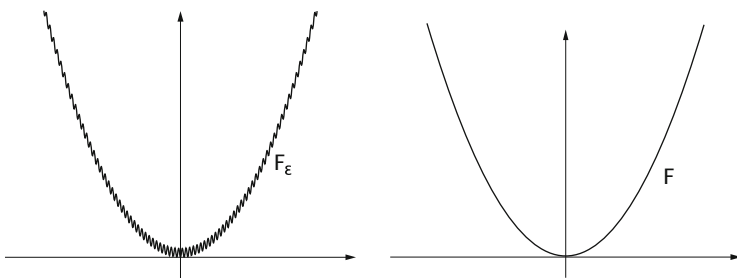


Fig. 1.1 Local minima may disappear in the limit

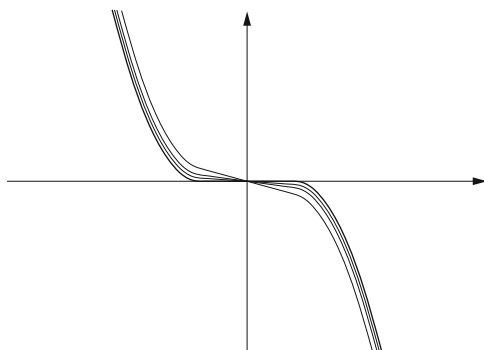


Fig. 1.2 Local minima may appear in the limit

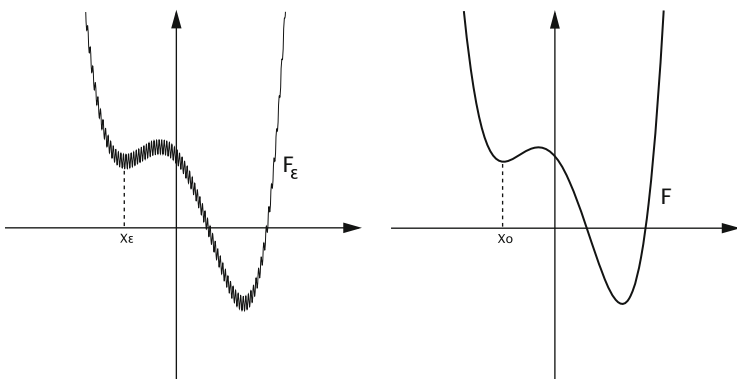


Fig. 1.3 Strict local minima are limits of local minima

could generate ‘pinned’ flows (i.e., stationary solutions or solutions ‘attracted’ by a local minimum) that are not detected by the Γ -limit. Conditions that guarantee such a stability are of convexity type on the energies. These conditions can be removed under other special assumptions on the gradient flows and for ‘well-prepared’ initial data following the scheme proposed for Ginzburg–Landau energies by Sandier

and Serfaty. Unfortunately, as remarked by those authors, the applicability of this scheme is often hard to verify.

Taking the above-mentioned results as a starting point we have explored some different directions. The standpoint of the analysis is that even though the Γ -limit may not give the correct description of the effect of local minimizers, it may nevertheless be ‘corrected’ in some systematic way.

After introducing the main notions of Γ -convergence in Chap. 2 and the general results on convergence of global minimum problems, in Chap. 3 we have examined another question related to global minimization, concerning the behavior of *quasistatic motions*, where the functions $u_\varepsilon = u_\varepsilon(t, x)$ minimize at each value of the parameter t (which is understood as a ‘slow time’) a total energy of the form $F_\varepsilon(u, v) + D_\varepsilon(v)$ subject to a t -depending forcing condition. Here, v is an additional parameter satisfying some monotonicity constraint and D_ε is a dissipation. We have examined conditions that imply that the separate Γ -convergences of F_ε to F and of D_ε to D guarantee the convergence of u_ε to an *energetic solution* of the quasistatic motion related to F and D , as envisaged by Mielke, Roubiček and Stefanelli. In general such conditions do not hold, and a relaxed formulation taking into account the interaction of F_ε and D_ε must be used. In this case, quasistatic motion turns out to be compatible with Γ -convergence provided that the latter is understood as the Γ -convergence of the functionals $G_\varepsilon(u, v) = F_\varepsilon(u, v) + D_\varepsilon(v)$ as depending on the variable (u, v) .

A first issue beyond global minimization is introduced in Chap. 4 and takes into account the notion of ‘equivalence by Γ -convergence’ as introduced and studied by Braides and Truskinovsky: in the case that a Γ -limit or a Γ -development may be insufficient to capture some desired feature of the minimum problems F_ε , we may introduce equivalent energies \tilde{F}_ε . These energies still integrate out the unimportant details of F_ε but maintain the particular desired feature and are equivalent to the original F_ε in that they have the same Γ -limit or Γ -development. In our context, one of the conditions that may be required to \tilde{F}_ε is that they have the same landscape of local minimizers as F_ε . As an example, we highlight that an argument using a Γ -development taking into account interactions between neighbouring transitions recovers the local minimizers of Allen–Cahn energies, that are integrated out by the usual sharp-interface models of phase transitions.

In Chap. 5 we have dealt with another case when global minimization arguments can be ‘localized’, which is the case of isolated local minimizers of the Γ -limit, as recalled above. We have noted that this principle can also be applied to scaled energies (e.g., to higher-order Γ -limits, or to ‘blown-up’ energies) obtaining existence of multiple local minimizers for the Γ -converging functionals.

Another issue is the problem of distinguishing ‘meaningful local minimizers’ from those that may ‘rightfully’ considered to disappear in the limit. Taking Fig. 1.1 as a pictorial example, local minimizers deriving from vanishing oscillations should be considered as different from those ‘trapped’ by energy barriers. To that end, in Chap. 6 we study the notion of small-scale stable state as recently introduced by C. Larsen, and the related notion of stable sequences of energies (in the original work of Larsen that notion is called ε -stability, a terminology which we abandon in

order to use the parameter ε as a label for energies). We show that Γ -convergence allows to exhibit classes of stable sequences. Furthermore, this notion allows to define parameterized motion driven by a stability criterion.

Linked to the study of local minimizers is the variational motion defined by the limit of Euler schemes at vanishing time step. This motion has been usually defined for a single functional F (and is sometimes referred to as a *minimizing movement*) by introducing a time step τ and define discrete trajectories (u_k^τ) iteratively as solutions of

$$\min \left\{ F(u) + \frac{1}{2\tau} \|u - u_{k-1}^\tau\|^2 \right\}$$

(for simplicity assume that the energies be defined on a Hilbert space, and $\|u\|$ be the related norm). A minimizing movement is a suitably defined continuum limit of such discrete trajectories. These concepts are introduced in Chap. 7 together with their analogues for time-dependent energies.

In Chap. 8 we examine a variation of the minimizing-movement scheme with two parameters: one is the time step τ , and the second one is the parameter ε (that we may often regard as a space scale) appearing in the Γ -converging sequence F_ε . The Euler scheme is then applied at fixed τ with $\varepsilon = \varepsilon(\tau)$ and F_ε in place of F , so that the resulting discrete trajectories (u_k^ε) , defined iteratively as solutions of

$$\min \left\{ F_\varepsilon(u) + \frac{1}{2\tau} \|u - u_{k-1}^\varepsilon\|^2 \right\},$$

may depend on the interaction between the two scales ε and τ . A general result, directly derived from the properties of Γ -convergence, allows to deduce the existence of a ‘fast’ space scale such that the limit of the ε - τ Euler scheme is just a minimizing movement for the Γ -limit. For ‘slow’ space scales the motion is often ‘pinned’ at local minimizers (e.g., for the F_ε pictured in Fig. 1.1). This observation highlights the existence of one or more critical ε - τ regimes which capture the most interesting features of the motion connected to these energies. We show examples where an ‘effective’ motion is obtained such that all minimizing movements are obtained by scaling of that motion.

An important class of minimizing movements are geometric minimizing movements treated in Chap. 9. The prototype of such a geometric motion is the motion by mean curvature obtained as a minimizing movement starting from the perimeter functional by Almgren, Taylor and Wang. In that case the scheme must be generalized to cover the case of an ‘asymmetric’ distance. This generalized scheme can be applied to treat sequences of perimeter energies.

Another possible phenomenon for these Euler schemes for Γ -converging energies is the existence of more superposed time scales, whose motions can themselves be interpreted as derived from Euler schemes for scaled functionals $F_\varepsilon/\lambda_\varepsilon$. Moreover, an appropriate choice of Γ -approximating sequences to a given F may be used to define a ‘backward’ motion. These issues are examined in Chap. 10.

Finally, in Chap. 11 we examine stability issues. The first one is linked to the theory of gradient flows elaborated by Ambrosio, Gigli and Savaré. Using their approach we can show that for convex equicoercive energies Γ -convergence always commutes with the minimizing-movement scheme. This is also true for some non-convex energies following the Sandier–Serfaty approach.

The notes are organized around some fundamental examples. We have analyzed a number of prototypical Γ -converging sequences $F_\varepsilon : X \rightarrow [-\infty, +\infty]$, highlighting different phenomena.

1. Elliptic homogenization:

$$F_\varepsilon(u) = \int_{\Omega} a\left(\frac{x}{\varepsilon}\right) |\nabla u|^2 dx,$$

with a 1-periodic. In this case X is (a subset of) $H^1(\Omega)$. The inhomogeneity a represents the fine properties of a composite medium. These energies are convex so that they do not possess local (non-global) minimizers. We may nevertheless introduce non-trivial perturbations for which we exhibit an isolated local minimizer for the Γ -limit, deducing existence of nontrivial local minimizers. Their convexity ensures that for the functionals F_ε Γ -convergence commutes with the minimizing movement schemes. As an application, we can deduce a parabolic homogenization theorem. Conversely, they may be ‘incompatible’ with simple types of dissipation, and lead to relaxed quasi static evolutions.

2. Oscillating metrics:

$$F_\varepsilon(u) = \int_0^1 a\left(\frac{u}{\varepsilon}\right) |u'| dt,$$

with a as above and X a subspace of $W^{1,\infty}([0, 1]; \mathbb{R}^n)$, and the analog oscillating perimeter energies

$$F_\varepsilon(A) = \int_{\partial A} a\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^{d-1}(x).$$

Here we are interested in the overall metric properties, or, in the case of perimeters, in the averaged interfacial energies of a composite medium. These energies have many local minimizers (which tend to be dense as $\varepsilon \rightarrow 0$), which are not stable as $\varepsilon \rightarrow 0$. As a consequence, we have a variety of limit minimizing movements that range from trivial (i.e., constant) motions to ‘crystalline’ motion, passing through interesting regimes where the scale ε interacts with the timescale τ . In the case of perimeter energies the limit motion is a kind of motion by curvature with a discontinuous dependence of the velocity on the curvature.

3. Van-der-Waals theory of phase transitions:

$$F_\varepsilon(u) = \int_{\Omega} \left(\frac{W(u)}{\varepsilon} + \varepsilon |\nabla u|^2 \right) dx,$$

with $X = H^1(\Omega)$. Here W is a double-well potential with minima in ± 1 . In one-dimension this is an example where the limit sharp-interface energy has many local minimizers (actually, all functions in the domain of the limit are local minimizers), so that a correction of the Γ -limit may be required to take into account the interaction of interfaces, through an exponentially decaying term. It must be noted that this corrected energy also provides an example of an exponentially scaled time-scale at which we have motion of interfaces (which otherwise stay pinned).

4. Atomistic theories: for a 1D chain of atoms

$$F_\varepsilon(u) = \sum_{i=1}^N J(u_i - u_{i-1}),$$

where J is an interatomic potential (e.g., the Lennard-Jones potential), and u_i represents the position of the i -th atom of a chain of N atoms, ordered with $u_i > u_{i-1}$. Here $\varepsilon = 1/N$. Starting from Lennard-Jones atomic interactions we show that the resulting Γ -limit in one-dimension (the Mumford–Shah functional or Griffith brittle fracture energy) must be modified to a Barenblatt cohesive fracture energy to maintain the features of local minimizers. Even though these energies are not convex, the Euler scheme commutes with Γ -convergence. Another feature of the variational motion is the appearance of a rescaled time scale, in which ‘multiple fractures’ (that correspond to ‘spurious’ local minimizers introduced by the limit process) tend to interact. A similar behavior is shown for Perona–Malik scaled energies, when $J(z)$ is of the form $\log(1 + z^2)$.

We believe that the examples and results highlighted provide a rather complex overview of the type of issues these notes want to be a stimulus to further analyze.

Chapter 2

Global Minimization

The issues related to the behavior of global minimization problems along a sequence of functionals F_ε are by now well understood, and mainly rely on the concept of Γ -limit. In this chapter we review this notion, which will be the starting point of our analysis. We will mainly be interested in the properties of Γ -limits as the convergence of minimization problems is concerned; further properties of Γ -limits will be recalled when necessary.

2.1 Upper and Lower Bounds

Here and afterwards F_ε will be functionals defined on a separable metric (or metrizable) space X , if not further specified.

Definition 2.1 (Lower bound). We say that F is a *lower bound* for the family (F_ε) if for all $u \in X$ we have

$$F(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \text{ for all } u_\varepsilon \rightarrow u, \quad (\text{LB})$$

or, equivalently, $F(u) \leq F_\varepsilon(u_\varepsilon) + o(1)$ as $\varepsilon \rightarrow 0$ for all $u_\varepsilon \rightarrow u$.

The inequality (LB) is usually referred to as the *liminf inequality*.

If F is a lower bound we obtain a lower bound also for minimum problems on *compact sets*.

Proposition 2.1. *Let F be a lower bound for F_ε and K be a compact subset of X . Then*

$$\inf_K F \leq \liminf_{\varepsilon \rightarrow 0} \inf_K F_\varepsilon. \quad (2.1)$$

Proof. Let $u_{\varepsilon_k} \in K$ be such that $u_{\varepsilon_k} \rightarrow \bar{u}$ and

$$\lim_k F_{\varepsilon_k}(u_{\varepsilon_k}) = \lim_{\varepsilon \rightarrow 0} \inf_K F_{\varepsilon}.$$

We set

$$\tilde{u}_{\varepsilon} = \begin{cases} u_{\varepsilon_k} & \text{if } \varepsilon = \varepsilon_k \\ \bar{u} & \text{otherwise.} \end{cases}$$

Then by (LB) we have

$$\inf_K F \leq F(\bar{u}) \leq \lim_{\varepsilon \rightarrow 0} \inf F_{\varepsilon}(\tilde{u}_{\varepsilon}) \leq \lim_k F_{\varepsilon_k}(u_{\varepsilon_k}) = \lim_{\varepsilon \rightarrow 0} \inf_K F_{\varepsilon}, \quad (2.2)$$

as desired. \square

Remark 2.1. Note that the hypothesis that K be compact cannot altogether be removed. A trivial example on the real line is:

$$F_{\varepsilon}(x) = \begin{cases} -1 & \text{if } x = 1/\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Then $F = 0$ is a lower bound according to Definition 2.1, but (2.1) fails if we take \mathbb{R} in the place of K .

Remark 2.2. The hypothesis that K be compact can be substituted by the hypothesis that K be closed and the sequence (F_{ε}) be *equi-coercive*; i.e., that

$$\text{if } \sup_{\varepsilon} F_{\varepsilon}(u_{\varepsilon}) < +\infty \text{ then } (u_{\varepsilon}) \text{ is precompact,} \quad (2.3)$$

the proof being the same.

Definition 2.2 (Upper bound). We say that F is an *upper bound* for the family (F_{ε}) if for all $u \in X$ we have

$$\text{there exists } u_{\varepsilon} \rightarrow u \text{ such that } F(u) \geq \limsup_{\varepsilon \rightarrow 0} F_{\varepsilon}(u_{\varepsilon}), \quad (\text{UB})$$

or, equivalently, $F(u) \geq F_{\varepsilon}(u_{\varepsilon}) + o(1)$ as $\varepsilon \rightarrow 0$.

The inequality (UB) is usually referred to as the *limsup inequality*.

If F is an upper bound for F_{ε} we obtain an upper bound also for the corresponding minimum problems on open sets.

Proposition 2.2. *Let F be an upper bound for F_{ε} and A be an open subset of X . Then*

$$\inf_A F \geq \limsup_{\varepsilon \rightarrow 0} \inf_A F_\varepsilon. \quad (2.4)$$

Proof. The proof is immediately derived from the definition after remarking that if $u \in A$ then we may suppose also that $u_\varepsilon \in A$, so that

$$F(u) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \limsup_{\varepsilon \rightarrow 0} \inf_A F_\varepsilon,$$

and (2.4) follows by the arbitrariness of $u \in A$. □

Remark 2.3. Again, note that the hypothesis that A be open cannot be removed. A trivial example on the real line is:

$$F_\varepsilon(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

(independent of ε). Then $F = 0$ is an upper bound according to Definition 2.2 (and also a lower bound!), but (2.4) fails taking $A = \{0\}$.

Note that in the remark above 0 is an upper bound for F_ε at 0 even though $F_\varepsilon(0) = 1$ for all ε , which trivially shows that an upper bound at a point can be actually (much) lower than any element of the family F_ε at that point.

2.2 Γ -Convergence

In this section we introduce the concept of Γ -limit.

Definition 2.3 (Γ -limit). We say that F is the Γ -limit of the sequence (F_ε) if it is both a lower and an upper bound according to Definitions 2.1 and 2.2.

If (LB) and (UB) hold at a point u then we say that F is the Γ -limit at u , and we write

$$F(u) = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u).$$

Note that this notation does not imply that u is in any of the domains of F_ε , even if $F(u)$ is finite.

Remark 2.4 (Alternate upper-bound inequalities). If F is a lower bound then requiring that (UB) holds is equivalent to any of the following

$$\text{there exists } u_\varepsilon \rightarrow u \text{ such that } F(u) = \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon); \quad (\text{RS})$$

$$\text{for all } \eta > 0 \text{ there exists } u_\varepsilon \rightarrow u \text{ such that } F(u) + \eta \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon). \quad (\text{AUB})$$

The latter is called the *approximate limsup inequality*, and is more handy in computations. A sequence satisfying (RS) is called a *recovery sequence*. The construction of a recovery sequence is linked to an *ansatz* on its form. The description of this *ansatz* gives an insight of the relevant features of the energies (oscillations, concentration, etc.) and is usually given on a subclass of u for which it is easier to prove its validity, while for general u one proceeds by a density argument.

Example 2.1. We analyze some simple examples on the real line.

1. From Remark 2.3 we see that the constant sequence

$$F_\varepsilon(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Γ -converges to the constant 0; in particular this is a constant sequence not converging to itself.

2. The sequence

$$F_\varepsilon(x) = \begin{cases} 1 & \text{if } x = \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

again Γ -converges to the constant 0. This is clearly a lower and an upper bound at all $x \neq 0$. At $x = 0$ any sequence $x_\varepsilon \neq \varepsilon$ is a recovery sequence.

3. The sequence

$$F_\varepsilon(x) = \begin{cases} -1 & \text{if } x = \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Γ -converges to

$$F(x) = \begin{cases} -1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Again, F is clearly a lower and an upper bound at all $x \neq 0$. At $x = 0$ the sequence $x_\varepsilon = \varepsilon$ is a recovery sequence.

4. Take the sum of the energies in Examples 2.2 and 2.3 above. This is identically 0, so is its limit, while the sum of the Γ -limits is the function F in Example 2.3. The same function F is obtained as the Γ -limit by taking the function $G_\varepsilon(x) = F_\varepsilon(x) + F_\varepsilon(-x)$ (F_ε in Example 2.3).
5. Let $F_\varepsilon(x) = \sin(x/\varepsilon)$. Then the Γ -limit is the constant -1 . This is clearly a lower bound. A recovery sequence for a fixed x is $x_\varepsilon = 2\pi\varepsilon \lfloor x/(2\pi\varepsilon) \rfloor - \varepsilon\pi/2$ ($\lfloor t \rfloor$ is the integer part of t).

The following fundamental property of Γ -convergence derives directly from its definition.

Proposition 2.3 (Stability under continuous perturbations). *Let F_ε Γ -converge to F and let G_ε converge continuously to G (i.e., $G_\varepsilon(u_\varepsilon) \rightarrow G(u)$ if $u_\varepsilon \rightarrow u$); then $F_\varepsilon + G_\varepsilon$ Γ -converges to $F + G$.*

Note that this proposition applies to $G_\varepsilon = G$ if G is continuous, but is in general false for $G_\varepsilon = G$ even if G is lower semicontinuous (take $G_\varepsilon = F$ as in Example 2.1(3), with $F_\varepsilon = -F$).

Example 2.2. The functions $\sin(x/\varepsilon) + x^2 + 1$ Γ -converge to x^2 . In this case we may apply the proposition above with $F_\varepsilon(x) = \sin(x/\varepsilon)$ (see Example 2.1(5)) and $G_\varepsilon(x) = x^2 + 1$. Note for future reference that F_ε has countably many local minimizers, which tend to be dense in the real line, while F has only one global minimizer.

It may be useful to define the lower and upper Γ -limits, so that the existence of a Γ -limit can be viewed as their equality.

Definition 2.4 (Lower and upper Γ -limits). We define

$$\Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \rightarrow u\} \quad (2.5)$$

$$\Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \rightarrow u\}. \quad (2.6)$$

Remark 2.5. 1. We immediately obtain that the Γ -limit exists at a point u if and only if

$$\Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u).$$

2. Comparing with the trivial sequence $u_\varepsilon = u$ we obtain

$$\Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u)$$

(and analogously for the Γ -lim sup). More in general, note that the Γ -limit depends on the topology on X . If we change topology, converging sequences are different and the value of the Γ -limit changes. A weaker topology will have more converging sequences and the value will decrease, a stronger topology will have less converging sequences and the value will increase. The pointwise limit corresponds to the Γ -limit with respect to the discrete topology.

3. From the formulas above it is immediate to check that a constant sequence $F_\varepsilon = F$ Γ -converges to itself if and only if F is lower semicontinuous; i.e., (LB) holds with $F_\varepsilon = F$. Indeed (LB) is equivalent to the validity of (2.5), while F is always an upper bound. More in general, a constant sequence $F_\varepsilon = F$ converges to the *lower-semicontinuous envelope* \overline{F} of F defined by

$$\overline{F} = \max\{G : G \leq F, G \text{ is lower semicontinuous}\};$$

in particular the Γ -limit is a lower-semicontinuous function.

4. It may be convenient to notice that the upper and lower Γ -limits are lower-semicontinuous functions and, with the notation just introduced, that

$$\Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} \overline{F_\varepsilon}(u) \quad (2.7)$$

$$\Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} \overline{F_\varepsilon}(u); \quad (2.8)$$

that is, Γ -limits are unchanged upon substitution of F_ε with its lower-semicontinuous envelope. These properties are an important observation for the actual computation of the Γ -limit, since in many cases lower-semicontinuous envelopes satisfy structural properties that make them easier to handle. As an example, we may consider (homogeneous) integral functionals of the form

$$F(u) = \int_{\Omega} f(u) \, dx,$$

defined on $L^1(\Omega)$ equipped with the weak topology. Under some growth conditions, the Γ -limits can be computed with respect to the weak topology on bounded sets of $L^1(\Omega)$, which is metrizable. In this case, the lower-semicontinuous envelope of F is

$$F(u) = \int_{\Omega} f^{**}(u) \, dx,$$

where f^{**} is the *convex and lower-semicontinuous envelope* of f ; i.e.,

$$f^{**} = \max\{g : g \leq f, g \text{ is lower-semicontinuous and convex}\}.$$

In particular, convexity is a necessary condition for a functional to be a Γ -limit of the integral form above.

2.3 Convergence of Minimum Problems

As we have already remarked, the Γ -convergence of F_ε will not imply the convergence of minimizers (or ‘almost minimizers’). It is necessary then to assume a *compactness* (or ‘mild coerciveness’) property as follows:

there exists a precompact sequence (u_ε) with $F_\varepsilon(u_\varepsilon) = \inf F_\varepsilon + o(1)$ as $\varepsilon \rightarrow 0$,

(2.9)

which is implied by the following stronger condition

$$\text{there exists a compact set } K \text{ such that } \inf F_\varepsilon = \inf_K F_\varepsilon \text{ for all } \varepsilon > 0. \quad (2.10)$$

This condition is implied by the *equi-coerciveness* hypothesis (2.3); i.e., if for all c there exists a compact set K such that the sublevel sets $\{F_\varepsilon \leq c\}$ are all contained in K . To check that (2.10) is stronger than (2.9) consider $F_\varepsilon(x) = \varepsilon e^x$ on the real line: any converging sequence satisfies (2.9) but (2.10) does not hold.

By arguing as for Propositions 2.1 and 2.2 we will deduce the convergence of minima. This result is made precise in the following theorem.

Theorem 2.1 (Fundamental Theorem of Γ -convergence). *Let (F_ε) satisfy the compactness property (2.9) and Γ -converge to F . Then*

- (i) F admits minimum, and $\min F = \liminf_{\varepsilon \rightarrow 0} \inf F_\varepsilon$.
- (ii) *If (u_{ε_k}) is a minimizing sequence for some subsequence (F_{ε_k}) (i.e., is such that $F_{\varepsilon_k}(u_{\varepsilon_k}) = \inf F_{\varepsilon_k} + o(1)$ as $\varepsilon \rightarrow 0$) which converges to some \bar{u} , then its limit point is a minimizer for F .*

Proof. By condition (2.9) we can argue as in the proof of Proposition 2.1 with $K = X$ and also apply Proposition 2.2 with $A = X$ to deduce that

$$\inf F \geq \limsup_{\varepsilon \rightarrow 0} \inf F_\varepsilon \geq \liminf_{\varepsilon \rightarrow 0} \inf F_\varepsilon \geq \inf F. \quad (2.11)$$

We then have that there exists the limit

$$\liminf_{\varepsilon \rightarrow 0} \inf F_\varepsilon = \inf F.$$

Since from (2.9) there exists a minimizing sequence (u_ε) from which we can extract a converging subsequence, it suffices to prove (ii). We can then follow the proof of Proposition 2.1 to deduce as in (2.2) that

$$\inf F \leq F(\bar{u}) \leq \lim_k F_{\varepsilon_k}(u_{\varepsilon_k}) = \liminf_{\varepsilon \rightarrow 0} \inf F_\varepsilon = \inf F;$$

i.e., $F(\bar{u}) = \inf F$ as desired. \square

Corollary 2.1. *In the hypotheses of Theorem 2.1 the minimizers of F are all the limits of converging minimizing sequences.*

Proof. If \bar{u} is a limit of a converging minimizing sequence then it is a minimizer of F by (ii) in Theorem 2.1. Conversely, if \bar{u} is a minimizer of F , then every its recovery sequence (u_ε) is a minimizing sequence. \square

Remark 2.6. Trivially, it is not true that all minimizers of F are limits of minimizers of F_ε , since this is not true even for (locally) uniformly converging sequences on the line. Take for example:

- (1) $F_\varepsilon(x) = \varepsilon x^2$ or $F_\varepsilon(x) = \varepsilon e^x$ and $F(x) = 0$. All points minimize the limit but only $x = 0$ minimizes F_ε in the first case, and we have no minimizer for the second case.
- (2) $F(x) = (x^2 - 1)^2$ and $F_\varepsilon(x) = F(x) + \varepsilon(x - 1)^2$. F is minimized by 1 and -1 , but the only minimum of F_ε is 1. Note, however, that -1 is the limit of strong local minimizers for F_ε .

2.4 An Example: Homogenization

The theory of homogenization of integral functionals is a very wide subject in itself. We will refer to monographs on the subject for details if needed. In this context, we want only to highlight some facts that will be used in the sequel and give a hint of the behaviour in the case of elliptic energies.

We consider $a : \mathbb{R}^n \rightarrow [\alpha, \beta]$, with $0 < \alpha < \beta < +\infty$ 1-periodic in the coordinate directions, and the integrals

$$F_\varepsilon(u) = \int_{\Omega} a\left(\frac{x}{\varepsilon}\right) |\nabla u|^2 dx$$

defined in $H^1(\Omega)$, where Ω is a bounded open subset of \mathbb{R}^n . The computation of the Γ -limit of F_ε is referred to as their *homogenization*, implying that a simpler ‘homogeneous’ (i.e., x -independent) functional can be used to capture the relevant features of F_ε . The limit can be computed both with respect to the L^1 -topology, but it can also be improved; e.g., in 1D it coincides with the limit in the L^∞ -topology. This means that the liminf inequality holds for u_ε converging in the L^1 -topology, while there exists a recovery sequence with u_ε tending to u in the L^∞ sense.

An upper bound is given by the pointwise limit of F_ε , whose computation in this case can be obtained by the following non-trivial but well-known result.

Proposition 2.4 (Riemann–Lebesgue lemma). *The functions $a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$ converge weakly* in L^∞ to their average*

$$\bar{a} = \int_{(0,1)^n} a(y) dy. \tag{2.12}$$

For fixed u the pointwise limit of $F_\varepsilon(u)$ is then simply $\bar{a} \int_{\Omega} |\nabla u|^2 dx$, which therefore gives an upper bound for the Γ -limit.

In a one-dimensional setting, the Γ -limit is completely described by a , and is given by

$$F_{\text{hom}}(u) = \underline{a} \int_{\Omega} |u'|^2 dx, \quad \text{where} \quad \underline{a} = \left(\int_0^1 \frac{1}{a(y)} dy \right)^{-1}$$

is the *harmonic mean* of a . We briefly sketch a proof which gives the *ansatz* for recovery sequences. We check the limit inequality as $u_\varepsilon \rightarrow u$. Suppose, for the sake of simplicity, that $N = 1/\varepsilon \in \mathbb{N}$, and write

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \sum_{i=1}^N \int_{\varepsilon(i-1)}^{\varepsilon i} a\left(\frac{x}{\varepsilon}\right) |u'_\varepsilon|^2 dx \\ &\geq \sum_{i=1}^N \varepsilon \min \left\{ \int_0^1 a(y) |v'|^2 dy : v(1) - v(0) = \frac{u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon(i-1))}{\varepsilon} \right\} \\ &= \underline{a} \sum_{i=1}^N \varepsilon \left| \frac{u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon(i-1))}{\varepsilon} \right|^2. \end{aligned}$$

The inequality in the second line is obtained by minimizing over all functions w with $w(\varepsilon(i-1)) = u_\varepsilon(\varepsilon(i-1))$ and $w(\varepsilon i) = u_\varepsilon(\varepsilon i)$; the minimum problem in the second line is obtained by scaling such w and using the periodicity of a , the third line is a direct computation of the previous minimum.

If we define \tilde{u}_ε as the piecewise-affine interpolation of u_ε on $\varepsilon\mathbb{Z}$ then the estimate above shows that

$$F_\varepsilon(u_\varepsilon) \geq F_{\text{hom}}(\tilde{u}_\varepsilon).$$

The functional on the right-hand side is independent of ε and with a convex integrand; hence, it is lower semicontinuous with respect to the weak H^1 -convergence. Since \tilde{u}_ε weakly converges to u in H^1 , we then deduce

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} F_{\text{hom}}(\tilde{u}_\varepsilon) \geq F_{\text{hom}}(u);$$

i.e., the liminf inequality. The *ansatz* for the upper bound is obtained by making the lower bound sharp: recovery sequences oscillate around the target function in an optimal way. If the target function $u(x) = zx$ is linear then a recovery sequence is obtained by taking the 1-periodic function v minimizing

$$\min \left\{ \int_0^1 a(y) |v' + 1|^2 dy : v(0) = v(1) = 0 \right\} = \underline{a},$$

and setting

$$u_\varepsilon(x) = z \left(x + \varepsilon v \left(\frac{x}{\varepsilon} \right) \right).$$

If u is affine then the construction is the same upon adding a constant. This construction can be repeated up to an error of order ε if u is piecewise affine, and then carries over to arbitrary u by density.

As a particular case, we can fix $\theta \in [0, 1]$ and consider the 1-periodic a given on $[0, 1)$ by

$$a(y) = \begin{cases} \alpha & \text{if } 0 \leq y < \theta \\ \beta & \text{if } \theta \leq y < 1. \end{cases} \quad (2.13)$$

In this case we have

$$\underline{a} = \frac{\alpha\beta}{\theta\beta + (1-\theta)\alpha}. \quad (2.14)$$

Note that if a is a 1-periodic function with $|\{y \in (0, 1) : a(y) = \alpha\}| = \theta$ and $|\{y \in (0, 1) : a(y) = \beta\}| = 1 - \theta$ then the Γ -limit is the same. Thus, in dimension one, the limit depends only on the volume fraction of α .

In the higher-dimensional case the limit can still be described by an elliptic integral, of the form

$$F_{\text{hom}}(u) = \int_{\Omega} \langle A \nabla u, \nabla u \rangle dx,$$

where A is a constant symmetric matrix with $\underline{a}I \leq A \leq \bar{a}I$ (I the identity matrix), with strict inequalities unless a is constant. If in two dimensions we take $a(y_1, y_2) = a(y_1)$ (this is called a *laminate* in the first direction), then A is the diagonal matrix $\text{diag}(\underline{a}, \bar{a})$. Of course, if $a(y_1, y_2) = a(y_2)$ then the two values are interchanged. If a takes only the values α and β in particular this shows that in the higher-dimensional case the results depends on the geometry of $\{y \in (0, 1) : a(y) = \alpha\}$ (often referred to as the *microgeometry* of the problem) and not only on the volume fraction.

A class of meaningful minimum problems is obtained by considering as F_ε the restriction of the previous functionals on the affine space $X = \varphi + H_0^1(\Omega)$ (i.e., we consider only functions with $u = \varphi$ on $\partial\Omega$). It can be proved that this boundary condition is ‘compatible’ with the Γ -limit; i.e., that the Γ -limit is the restriction to X of the previous one or, equivalently, that recovery sequences for the first Γ -limit can be taken satisfying the same boundary data as their limit. As a consequence of Theorem 2.1 we then conclude that oscillating minimum problems for F_ε with fixed boundary data are approximated by a simpler minimum problem with the same boundary data. Note, however, that all energies, both F_ε and F_{hom} , are strictly convex, which implies that they have no local non global minimizer.

Example 2.3. We can perturb the previous energies with some continuously converging perturbation to obtain some additional convergence result. For example, we can add perturbations of the form

$$G_\varepsilon(u) = \int_{\Omega} g\left(\frac{x}{\varepsilon}, u\right) dx.$$

On g we make the following hypothesis:

g is a Borel function 1-periodic in the first variable and uniformly Lipschitz in the second one; i.e.,

$$|g(y, z) - g(y, z')| \leq L|z - z'|.$$

We then have a perturbed homogenization result as follows.

Proposition 2.5. *The functionals $F_\varepsilon + G_\varepsilon$ Γ -converge in the L^1 -topology to the functional $F_{\text{hom}} + G$, where*

$$G(u) = \int_{\Omega} \bar{g}(u) \, dx, \quad \text{and} \quad \bar{g}(z) = \int_{(0,1)^n} g(y, z) \, dy$$

is simply the average of $g(\cdot, z)$.

Proof. By Proposition 2.3 it suffices to show that G_ε converge continuously with respect to the L^1 -convergence. If $u_\varepsilon \rightarrow u$ in L^1 then

$$\begin{aligned} |G_\varepsilon(u_\varepsilon) - G(u)| &\leq \int_{\Omega} \left| g\left(\frac{x}{\varepsilon}, u_\varepsilon\right) - g\left(\frac{x}{\varepsilon}, u\right) \right| dx + |G_\varepsilon(u) - G(u)| \\ &\leq L \int_{\Omega} |u_\varepsilon - u| \, dx + |G_\varepsilon(u) - G(u)|. \end{aligned}$$

It suffices then to show that G_ε converges pointwise to G . If u is piecewise constant then this follows immediately from the Riemann–Lebesgue Lemma. Noting that also $|\bar{g}(z) - \bar{g}(z')| \leq L|z - z'|$ we easily obtain the convergence for $u \in L^1(\Omega)$ by the density of piecewise-constant functions. \square

Note that, with a slightly more technical proof, we can improve the Lipschitz continuity condition to a local Lipschitz continuity of the form

$$|g(y, z) - g(y, z')| \leq L(1 + |z| + |z'|)|z - z'|.$$

In particular, in dimension one we can apply the result for $g(y, z) = a(y)|z|^2$ and we have that

$$\int_{\Omega} a\left(\frac{x}{\varepsilon}\right)(|u'|^2 + |u|^2) \, dx$$

Γ -converges to

$$\int_{\Omega} (\underline{a}|u'|^2 + \bar{a}|u|^2) \, dx.$$

As a consequence of Theorem 2.1, under the condition of coerciveness

$$\lim_{z \rightarrow \pm\infty} \inf g(\cdot, z) = +\infty,$$

we obtain a convergence result as follows.

Proposition 2.6. *The solutions to the minimum problems*

$$\min \left\{ F_\varepsilon(u) + G_\varepsilon(u) : u \in H^1(\Omega) \right\}$$

converge (up to subsequences) to a constant function \bar{u} , whose constant value minimizes \bar{g} .

Proof. The proof of the proposition follows immediately from Theorem 2.1, once we observe that, by the coerciveness and continuity of \bar{g} , a minimizer for that function exists, and the constant function \bar{u} defined above minimizes both F_{hom} and G . \square

If g is differentiable then by computing the Euler–Lagrange equations of $F_\varepsilon + G_\varepsilon$ we conclude that we obtain solutions of

$$-\sum_{ij} \frac{\partial}{\partial x_i} \left(a \left(\frac{x}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial x_i} \right) + \frac{\partial}{\partial u} g \left(\frac{x}{\varepsilon}, u_\varepsilon \right) = 0 \quad (2.15)$$

with Neumann boundary conditions, converging to the constant \bar{u} .

2.5 Higher-Order Γ -Limits and a Choice Criterion

We have noticed that if the hypotheses of Theorem 2.1 are satisfied then every minimum point of the Γ -limit F corresponds to a minimizing sequence for F_ε (see Corollary 2.1). However, not all points may be limits of minimizers for F_ε ; conversely, it may be interesting to discriminate between limits of minimizing sequences with different speeds of convergence. To this end, we may look at scaled Γ -limits. If we suppose that, e.g., u is a limit of a sequence (u_ε) with

$$F_\varepsilon(u_\varepsilon) = \min F + O(\varepsilon^\alpha) \quad (2.16)$$

for some $\alpha > 0$ (but, of course, the rate of convergence may also not be polynomial), then we may look at the Γ -limit of the scaled functionals

$$F_\varepsilon^\alpha(u) = \frac{F_\varepsilon(u_\varepsilon) - \min F}{\varepsilon^\alpha}. \quad (2.17)$$

Suppose that F_ε^α Γ -converges to some F^α not taking the value $-\infty$. Then:

- (i) The domain of F^α is contained in the set of minimizers of F (but may as well be empty).
- (ii) $F^\alpha(u) \neq +\infty$ if and only if there exists a recovery sequence for u satisfying (2.16).

Moreover, we can apply Theorem 2.1 to F_ε^α and obtain the following result, which gives a choice criterion among minimizers of F .

Theorem 2.2. *Let the hypotheses of Theorem 2.1 be satisfied and the functionals in (2.17) Γ -converge to some F^α not taking the value $-\infty$ and not identically $+\infty$. Then*

- (i) $\inf F_\varepsilon = \min F + \varepsilon^\alpha \min F^\alpha + o(\varepsilon^\alpha)$.
- (ii) *If $F_\varepsilon(u_\varepsilon) = \min F_\varepsilon + o(\varepsilon^\alpha)$ and $u_\varepsilon \rightarrow u$ then u minimizes both F and F^α .*

Proof. We can apply Theorem 2.1 to a (subsequence of a) converging minimizing sequence for F_ε^α ; i.e., a sequence satisfying hypothesis (ii). Its limit point u satisfies

$$F^\alpha(u) = \min F^\alpha = \lim_{\varepsilon \rightarrow 0} \min F_\varepsilon^\alpha = \lim_{\varepsilon \rightarrow 0} \frac{\min F_\varepsilon - \min F}{\varepsilon^\alpha},$$

which proves (i). Since, as already remarked u is also a minimizer of F , we also have (ii). \square

Example 2.4. Simple examples in the real line:

- (1) If $F_\varepsilon(x) = \varepsilon x^2$ then $F(x) = 0$. We have $F^\alpha(x) = 0$ if $0 < \alpha < 1$, $F^1(x) = x^2$ (if $\alpha = 1$), and

$$F^\alpha(x) = \begin{cases} 0 & x = 0 \\ +\infty & x \neq 0 \end{cases}$$

if $\alpha > 1$.

- (2) If $F_\varepsilon(x) = (x^2 - 1)^2 + \varepsilon(x - 1)^2$ then $F(x) = (x^2 - 1)^2$. We have

$$F^\alpha(x) = \begin{cases} 0 & |x| = 1 \\ +\infty & |x| \neq 1 \end{cases}$$

if $0 < \alpha < 1$,

$$F^1(x) = \begin{cases} 0 & x = 1 \\ 4 & x = -1 \\ +\infty & |x| \neq 1 \end{cases}$$

if $\alpha = 1$,

$$F^\alpha(x) = \begin{cases} 0 & x = 1 \\ +\infty & x \neq 1 \end{cases}$$

if $\alpha > 1$.

Remark 2.7. It must be observed that the functionals F_ε^α in Theorem 2.2 are often equicoercive with respect to a stronger topology than the original F_ε , so that we can improve the convergence in (ii), as in the following example.

Example 2.5 (Gradient theory of phase transitions). Let

$$F_\varepsilon(u) = \int_{\Omega} (W(u) + \varepsilon^2 |\nabla u|^2) dx \quad (2.18)$$

be defined in $L^1(\Omega)$ with domain in $H^1(\Omega)$. Here $W(u) = (u^2 - 1)^2$ (or a more general double-well potential; i.e., a non-negative function vanishing exactly at ± 1). Then (F_ε) is equicoercive with respect to the weak L^1 -convergence. Since this convergence is metrizable on bounded sets, we can consider $L^1(\Omega)$ equipped with this convergence. The Γ -limit is then simply

$$F^0(u) = \int_{\Omega} W^{**}(u) dx,$$

where W^{**} is the convex envelope of W ; i.e. $W^{**}(u) = ((u^2 - 1) \vee 0)^2$. All functions with $\|u\|_{\infty} \leq 1$ are minimizers of F^0 .

We take $\alpha = 1$ and consider

$$F_\varepsilon^1(u) = \int_{\Omega} \left(\frac{W(u)}{\varepsilon} + \varepsilon |\nabla u|^2 \right) dx. \quad (2.19)$$

Then (F_ε^1) is equicoercive with respect to the strong L^1 -convergence, and its Γ -limit is

$$F^1(u) = c_W \mathcal{H}^{n-1}(\partial\{u = 1\} \cap \Omega) \text{ for } u \in BV(\Omega; \{\pm 1\}), \quad (2.20)$$

and $+\infty$ otherwise, where $c_W = 8/3$ (in general $c_W = 2 \int_{-1}^1 \sqrt{W(s)} ds$). Here we denote by $BV(\Omega)$ the space of functions of bounded variation in Ω , and by $BV(\Omega; \{\pm 1\})$ the space of functions of bounded variation in Ω taking only the values ± 1 .

This results states that recovery sequences (u_ε) tend to sit in the bottom of the wells (i.e., $u \in \{\pm 1\}$) in order to make $\frac{W(u_\varepsilon)}{\varepsilon}$ finite; however, every ‘phase transition’ costs a positive amount, which is optimized by balancing the effects of the two terms in the integral. Indeed, by optimizing the interface between the phases $\{u = 1\}$ and $\{u = -1\}$ one obtains the optimal ‘surface tension’ c_W .

In one-dimension the ansatz on the recovery sequences around a jump point x_0 is that they are of the form

$$u_\varepsilon(x) = v\left(\frac{x - x_0}{\varepsilon}\right),$$

where v minimizes

$$\min\left\{\int_{-\infty}^{+\infty} (W(v) + |v'|^2) dx : v(\pm\infty) = \pm 1\right\} = 2 \int_{-1}^1 \sqrt{W(s)} ds.$$

In more than one-dimension the ansatz becomes

$$u_\varepsilon(x) = v\left(\frac{d(x, \partial\{u = 1\})}{\varepsilon}\right),$$

where $d(\cdot, A)$ is the signed distance from the set A . This means that around the interface $\partial\{u = 1\}$ the recovery sequence passes from -1 to $+1$ following the one-dimensional profile of v essentially on a $O(\varepsilon)$ -neighbourhood of $\partial\{u = 1\}$.

Note that:

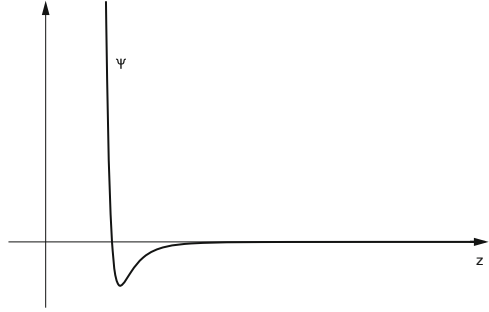
- (i) We have an improved convergence of recovery sequences from weak to strong L^1 -convergence.
- (ii) The domain of F^1 is almost disjoint from that of the F_ε^1 , the only two functions in common being the constants ± 1 .
- (iii) In order to make the Γ -limit properly defined we have to use the space of *functions of bounded variation* or, equivalently, (taking the constraint $u(x) = \pm 1$ a.e. into account) the family of *sets of finite perimeter* if we take as parameter the set $A = \{u = 1\}$. In this context the set $\partial\{u = 1\}$ is properly defined in a measure-theoretical way, as well as its $(n - 1)$ -dimensional Hausdorff measure. Even though ∂A may greatly differ from the topological boundary of A , we will use the same symbol in order not to overburden the notation. Note however that in our examples sets can be supposed to be smooth enough, so that the two notions coincide.

Example 2.6 (Linearized fracture mechanics from interatomic potentials). We now give an example in which the scaling of the variable, and not only of the energy, is part of the problem. We consider a systems of one-dimensional nearest-neighbour atomistic interactions through a Lennard-Jones type interaction. Note that, by the one-dimensional nature of the problem, we can parameterize the position of the atoms as an increasing function of the parameter.

Let ψ be a C^2 potential as in Fig. 2.1, with domain $(0, +\infty)$ (we set $\psi(z) = +\infty$ for $z \leq 0$), minimum in 1 with $\psi''(1) > 0$, convex in $(0, z_0)$, concave in $(z_0, +\infty)$ and tending to $\psi(\infty) < +\infty$ at $+\infty$. A possible choice is Lennard-Jones potential

$$\psi(z) = \frac{1}{z^{12}} - \frac{2}{z^6}.$$

Fig. 2.1 A Lennard-Jones potential



We consider the energy

$$\Psi_N(v) = \sum_{i=1}^N \psi(v_i - v_{i-1})$$

with $N \in \mathbb{N}$, defined on v_i with $v_i > v_{i-1}$. We introduce the small parameter $\varepsilon = 1/N$ and identify the vector (v_0, \dots, v_N) with a discrete function defined on $\varepsilon\mathbb{Z} \cap [0, 1]$ (i.e., $v_i = v(\varepsilon i)$). A non-trivial Γ -limit will be obtained by scaling and rewriting the energy in terms of a scaled variable

$$u = \sqrt{\varepsilon} \left(v - \frac{id}{\varepsilon} \right); \quad \text{i.e., } u_i = \sqrt{\varepsilon} (v_i - i).$$

This scaling can be justified noting that (up to additive constants) $v_i = i = id/\varepsilon$ is the absolute minimum of the energy. The scaled energies that we consider are

$$F_\varepsilon(u) = \Psi_N \left(\frac{id}{\varepsilon} + \frac{u}{\sqrt{\varepsilon}} \right) - \min \Psi_N = \sum_{i=1}^N J \left(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}} \right),$$

where

$$J(w) = \psi(1 + w) - \min \psi = \psi(1 + w) - \psi(1).$$

For convenience we extend the function to all \mathbb{R} setting $J(w) = +\infty$ if $w \leq -1$. Again, the vector (u_0, \dots, u_N) is identified with a discrete function defined on $\varepsilon\mathbb{Z} \cap [0, 1]$ or with its piecewise-affine interpolation. With this last identification, F_ε can be viewed as functionals in $L^1(0, 1)$, and their Γ -limit computed with respect to the strong $L^1(0, 1)$ -topology.

We denote by $w_0 = 1 + z_0$ the inflection point of J . It must be noted that for all $\bar{w} > 0$ we have

$$\#\left\{i : \frac{u_i - u_{i-1}}{\sqrt{\varepsilon}} > \bar{w}\right\} \leq \frac{1}{J(\bar{w})} F_\varepsilon(u),$$

so that this number of indices is equi-bounded along sequences with equibounded energy. We may therefore suppose that the corresponding points εi converge to a finite set $S \subset [0, 1]$. For fixed \bar{w} , we have $J(w) \geq \bar{c}|w|^2$ on $(-\infty, \bar{w}]$ for some $\bar{c} > 0$; this gives, if A is compactly contained in $(0, 1) \setminus S$, that

$$F_\varepsilon(u) \geq \bar{c} \sum_i \left(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\right)^2 = \bar{c} \sum_i \varepsilon \left(\frac{u_i - u_{i-1}}{\varepsilon}\right)^2 \geq \bar{c} \int_A |u'|^2 dt$$

(the sum is extended to those i such that $\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}} \leq \bar{w}$). By the arbitrariness of A in this estimate we then have that if $u_\varepsilon \rightarrow u$ and $F_\varepsilon(u_\varepsilon) \leq C < +\infty$ then u is piecewise- H^1 ; i.e., there exists a finite set $S \subset (0, 1)$ such that $u \in H^1((0, 1) \setminus S)$; we denote by $S(u)$ the minimal set such that $u \in H^1((0, 1) \setminus S(u))$. The reasoning above also shows that

$$\bar{c} \int_0^1 |u'|^2 dt + J(\bar{w})\#(S(u))$$

is a lower bound for the Γ -limit of F_ε . The Γ -limit on piecewise- $H^1(0, 1)$ functions can be computed by optimizing the choice of \bar{w} and \bar{c} , obtaining

$$F(u) = \frac{1}{2} J''(0) \int_0^1 |u'|^2 dt + J(\infty)\#(S(u)) \quad (2.21)$$

with the constraint that $u^+ > u^-$ on $S(u)$. This functional is the one-dimensional version of *Griffith's fracture energy* for brittle materials, and coincides with a functional introduced by *Mumford–Shah* in the framework of Image Reconstruction (without the constraint $u^+ > u^-$).

Note that the parameterization of v_i on $\varepsilon\mathbb{Z}$ would suggest to interpret $v_i - v_{i-1}$ as a different quotient and hence the change of variables $u_i = \varepsilon v_i - i d$. This would give an energy of the form

$$\tilde{F}_\varepsilon(u) = \sum_{i=1}^N J\left(\frac{u_i - u_{i-1}}{\varepsilon}\right);$$

it can be shown that \tilde{F}_ε converges to the energy with domain the set of piecewise-affine increasing u with $u' = 1$ a.e., and for such u

$$\tilde{F}(u) = J(\infty)\#(S(u)).$$

This different choice of the parameterization hence only captures the fracture part of the energy.

Appendix

For an introduction to Γ -convergence we refer to the ‘elementary’ book [1]. More examples, and an overview of the methods for the computation of Γ -limits can be found in [2]. More detailed information on topological properties of Γ -convergence are found in [6]. The use of higher-order Γ -limits is analyzed in [5].

Homogenization results are described in [3]. The reference to the original Γ -limit of the gradient theory of phase transitions is [7]. A more detailed treatment of Example 2.6 can be found in [4].

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Chapter 3

Parameterized Motion Driven by Global Minimization

Energy-driven dynamic problems are in general associated with a local minimization procedure. Nevertheless, for ‘slow movements’ a meaningful notion of ‘quasi-static’ motion can be defined starting from a global-minimization criterion. The ingredients are:

- a parameter-dependent energy F ,
- a dissipation D satisfying a non-decreasing constraint,
- a (time-)parameterized forcing condition.

Loosely speaking, a quasistatic motion is controlled by some parameterized forcing condition (applied forces, varying boundary conditions or other constraints); the motion is thought to be so slow so that the solution at a fixed value of the parameter (at fixed ‘time’) minimizes a total energy. This energy is obtained adding some ‘dissipation’ to some ‘internal energy’. A further condition is that the total dissipation increases with time. An entire general theory (of rate-independent motion) can be developed starting from these ingredients.

An important feature of these rate-independent motions is that they can be characterized as the limit of a piecewise-constant (time-)parameterized family of functions, which are defined iteratively as solutions of minimum problems. Under suitable conditions, to such a characterization the Fundamental Theorem of Γ -convergence can be applied, so that this notion can be proved to be indeed compatible with Γ -convergence.

3.1 A Paradigmatical Example: Damage Models

In this section we deal with a simplified example and examine its stability with respect to perturbations. We will first define a damage process for a single material. Then we will consider the same definition for a mixture of two materials in the context of homogenization. A homogenized theory can be derived, with some care

in the definition of the Γ -limit, that must take into account at the same time the energy and the dissipation.

3.1.1 Damage of a Homogeneous Material

We consider a one-dimensional setting. Our functions will be parameterized on a fixed interval $(0, 1)$. In this case we have:

- The parameter space will be that of all measurable subsets A of $(0, 1)$. The set A will be understood as the *damage set*.
- The energies depending on a set A will be

$$F_A(u) = \alpha \int_A |u'|^2 dx + \beta \int_{(0,1) \setminus A} |u'|^2 dx,$$

where $0 < \alpha < \beta$. In an mechanical interpretation of the variables, u represents the deformation of a bar, whose elastic constant is β in the undamaged set and $\alpha < \beta$ in the damaged set.

- The dissipation is

$$D(A) = \gamma |A|,$$

with $\gamma > 0$. The work done to damage a portion A of the material is proportional to the measure of A .

- The condition that forces the solution to be parameter dependent ('time-dependent') is a boundary condition

$$u(0) = 0, \quad u(1) = g(t),$$

where g is a continuous function with $g(0) = 0$. Here the parameter is $t \in [0, T]$.

Definition 3.1. A *solution* to the evolution related to the energy, dissipation and boundary conditions above is a pair (u^t, A^t) with $u^t \in H^1(0, 1)$, $A^t \subset (0, 1)$, and such that

- **(monotonicity)** we have $A^s \subset A^t$ for all $s < t$;
- **(minimization)** the pair (u^t, A^t) minimizes

$$\min \left\{ F_{A^t}(u) + D(A^t) : u(0) = 0, u(1) = g(t), A^t \subset A \right\}; \quad (3.1)$$

- **(continuity)** the energy $\mathcal{E}(t) = F_{A^t}(u^t) + D(A^t)$ is continuous;
- **(homogeneous initial datum)** u^0 is the constant 0 and $A^0 = \emptyset$.

The continuity assumption allows to rule out trivial solutions as those with $A^t = (0, 1)$ for all $t > 0$. It is usually replaced by a more physical condition of energy conservation. In our context this assumption is not relevant.

Note that t acts only as a parameter (the motion is ‘rate independent’). Hence, for example if g is monotone increasing, it suffices to consider $g(t) = t$. We will construct by hand a solution in this simplified one-dimensional context.

Remark 3.1. Note that the value in the minimum problem

$$m(t) = \min \left\{ F_A(u) + D(A) : u(0) = 0, u(1) = t \right\} \quad (3.2)$$

depends on A only through $\lambda = |A|$.

Indeed, given A , we can explicitly compute the minimum value

$$m(A, t) = \min \left\{ \int_A \alpha |u'|^2 dx + \int_{(0,1) \setminus A} \beta |u'|^2 dx : u(0) = 0, u(1) = t \right\}.$$

In fact, for all test functions u we have, by Jensen’s inequality

$$\int_A \alpha |u'|^2 dx + \int_{(0,1) \setminus A} \beta |u'|^2 dx \geq \alpha |A| |z_1|^2 + \beta (1 - |A|) |z_2|^2,$$

where

$$z_1 = \frac{1}{|A|} \int_A u' dx, \quad z_2 = \frac{1}{1 - |A|} \int_{(0,1) \setminus A} u' dx,$$

with a strict inequality if u' is not constant on A and $(0, 1) \setminus A$. This shows that the unique minimizer satisfies

$$u' = z_1 \chi_A + z_2 (1 - \chi_A), \quad |A| z_1 + (1 - |A|) z_2 = t,$$

where the second condition is given by the boundary data; hence,

$$m(A, t) = \min \{ \alpha \lambda |z_1|^2 + \beta (1 - \lambda) |z_2|^2 : \lambda z_1 + (1 - \lambda) z_2 = t \} = \frac{\alpha \beta}{\lambda \beta + (1 - \lambda) \alpha} t^2.$$

We conclude that the minimum value (3.2) is given by

$$\frac{\alpha \beta}{\lambda \beta + (1 - \lambda) \alpha} t^2 + \gamma \lambda. \quad (3.3)$$

By minimizing over λ we obtain the optimal value of the measure of the damaged region

$$\lambda_{\min}(t) = \begin{cases} 0 & \text{if } |t| \leq \sqrt{\frac{\alpha\gamma}{\beta(\beta-\alpha)}} \\ 1 & \text{if } |t| \geq \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}} \\ t \sqrt{\frac{\alpha\beta}{\gamma(\beta-\alpha)}} - \frac{\alpha}{\beta-\alpha} & \text{otherwise} \end{cases} \quad (3.4)$$

and the minimum value

$$m(t) = \begin{cases} \beta t^2 & \text{if } |t| \leq \sqrt{\frac{\alpha\gamma}{\beta(\beta-\alpha)}} \\ \alpha t^2 + \gamma & \text{if } |t| \geq \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}} \\ 2t \sqrt{\frac{\alpha\beta\gamma}{\beta-\alpha}} - \frac{\gamma\alpha}{\beta-\alpha} & \text{otherwise.} \end{cases} \quad (3.5)$$

The interpretation of this formula is as follows. For small values of the total displacement t the material remains undamaged, until it reaches a critical value for the boundary datum. Then a portion of size $\lambda_{\min}(t)$ of the material damages, lowering the elastic constant of the material and the overall value of the sum of the internal energy and the dissipation, until all the material is damaged. Note that in this case $\mathcal{E}(t) = m(t)$, due to the increasing-load assumption.

The solutions for the evolution problem are given by any increasing family of sets A^t satisfying $|A^t| = \lambda_{\min}(t)$ and correspondingly functions u^t minimizing $m(A^t, t)$.

The value in (3.3) is obtained by first minimizing in u . Conversely, we may first minimize in A . We then have

$$\min \left\{ \int_0^1 \min_A \{ \chi_A(\alpha|u'|^2 + \gamma), \chi_{(0,1) \setminus A} \beta |u'|^2 \} dx : u(0) = 0, u(1) = g(t) \right\}. \quad (3.6)$$

The lower-semicontinuous envelope of the integral energy is given by the integral with energy function the convex envelope of

$$f(z) = \min \{ \alpha z^2 + \gamma, \beta z^2 \}, \quad (3.7)$$

which is exactly given by formula (3.5); i.e.,

$$m(t) = f^{**}(t)$$

(see Fig. 3.1).

Irreversibility. An important feature of the monotonicity condition for A^t is irreversibility of damage, which implies that for non-increasing g the values of

Fig. 3.1 Minimal value $m(t)$ for the damage problem

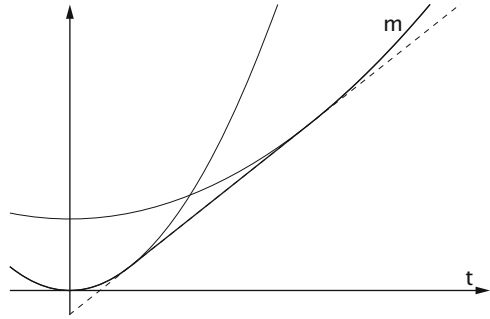
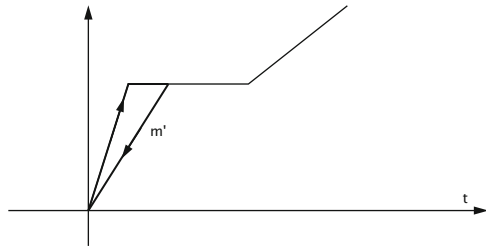


Fig. 3.2 Plot of $m'(t)$ along a cycle



$m(g(t))$ will depend on the highest value taken by $\lambda_{\min}(g(t))$ on $[0, t]$. In particular, for a ‘loading–unloading’ cycle with $g(t) = \frac{T}{2} - |t - \frac{T}{2}|$, the value of $\mathcal{E}(t)$ is given by

$$\mathcal{E}(t) = \begin{cases} m(t) & \text{for } 0 \leq t \leq \frac{T}{2} \\ \frac{\alpha\beta}{\lambda_{\min}(\frac{T}{2})\beta + (1 - \lambda_{\min}(\frac{T}{2}))\alpha} (T - t)^2 + \gamma \lambda_{\min}(\frac{T}{2}) & \text{for } \frac{T}{2} \leq t \leq T. \end{cases}$$

This formula highlights that once the maximal value $\lambda_{\min}(T/2)$ is reached, then the damaged region A^t remains fixed, so that the problem becomes a quadratic minimization (plus the constant value of the dissipation). We plot $m'(t)$ and draw a cycle in Fig. 3.2.

Note that, in particular, if $\frac{T}{2} \geq \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}}$ then the material is completely damaged in the ‘unloading’ regime.

3.1.1.1 Threshold Formulation

Note that a solution u_t of (3.2) satisfies the Euler–Lagrange equation

$$((\alpha\chi_A + \beta(1 - \chi_A))u')' = 0;$$

i.e.,

$$(\alpha\chi_A + \beta(1 - \chi_A))u' = \sigma_t, \quad (3.8)$$

where σ_t is a constant parameterized by t . Its plot as a function of $g = g(t)$ along a 'loading-unloading' cycle is the same as in Fig. 3.2.

The plateau for σ is obtained at the *threshold* value

$$\sigma = \sqrt{\frac{\alpha\beta\gamma}{\beta - \alpha}}.$$

We can interpret the g - σ graph as a *threshold principle*: the material does not damage until the *stress* σ reaches the threshold value. At this point, if the material is loaded further it damages so as to keep the value of σ below the threshold, until all the material is damaged. If the material is unloaded then σ follows a linear elastic behavior with the overall effective elastic constant corresponding to the total amount of damage produced.

3.1.2 Homogenization of Damage

We now examine the behaviour of the previous process with respect to Γ -convergence in the case of homogenization; i.e., when we have a fine mixture of two materials, each one of which can undergo a damage process as in the previous section. To that end we introduce the energies

$$F_{\varepsilon,A}(u) = \int_{(0,1)\setminus A} \beta\left(\frac{x}{\varepsilon}\right) |u'|^2 dx + \int_A \alpha\left(\frac{x}{\varepsilon}\right) |u'|^2 dx, \quad (3.9)$$

where α and β are 1-periodic functions with

$$\alpha(y) = \begin{cases} \alpha_1 & \text{for } 0 \leq y < \frac{1}{2} \\ \alpha_2 & \text{for } \frac{1}{2} \leq y < 1 \end{cases} \quad \beta(y) = \begin{cases} \beta_1 & \text{for } 0 \leq y < \frac{1}{2} \\ \beta_2 & \text{for } \frac{1}{2} \leq y < 1 \end{cases}$$

with $0 < \alpha_j < \beta_j$. Note that, for fixed A , the functionals $F_{\varepsilon,A}$ Γ -converge to

$$F_{\text{hom},A}(u) = \underline{\beta} \int_{(0,1)\setminus A} |u'|^2 dx + \underline{\alpha} \int_A |u'|^2 dx, \quad (3.10)$$

with

$$\underline{\alpha} = \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2} < \frac{2\beta_1\beta_2}{\beta_1 + \beta_2} = \underline{\beta}.$$

This can be easily checked if A is an interval (or a union of intervals), and then for a general A by approximation. Indeed if $A = (0, \lambda)$ then the liminf inequality trivially holds by separately applying the liminf inequality to the two energies

$$\int_0^\lambda \alpha \left(\frac{x}{\varepsilon} \right) |u'|^2 dx, \quad \int_\lambda^1 \beta \left(\frac{x}{\varepsilon} \right) |u'|^2 dx. \quad (3.11)$$

Conversely, given a target function $u \in H^1(0, 1)$, we can find recovery sequences (u_ε^1) and (u_ε^2) for u on $(0, \lambda)$ and $(\lambda, 1)$, respectively, for the energies (3.11) with $u_\varepsilon^1(\lambda) = u_\varepsilon^2(\lambda)$, so that the corresponding u_ε defined as u_ε^1 on $(0, \lambda)$ and as u_ε^2 on $(\lambda, 1)$ is a recovery sequence for $F_{\text{hom},A}(u)$. Note that the Γ -limit is still of the form examined in Sect. 3.1.1 with constants $\underline{\alpha}$ and $\underline{\beta}$.

We now instead study the damage process at fixed ε . For simplicity of computation we suppose that $\frac{1}{\varepsilon} \in \mathbb{N}$. The general case can be always reduced to this assumption up to an error of order ε . The dissipation will be of the form

$$D_\varepsilon(A) = \int_A \gamma \left(\frac{x}{\varepsilon} \right) dx,$$

where again γ is a 1-periodic function with

$$\gamma(y) = \begin{cases} \gamma_1 & \text{for } 0 \leq y < \frac{1}{2} \\ \gamma_2 & \text{for } \frac{1}{2} \leq y < 1 \end{cases}$$

with $\gamma_j > 0$. In the case $\gamma_1 = \gamma_2$ we obtain the same dissipation as in Sect. 3.1.1, independent of ε .

In order to compute the minimum value

$$m^\varepsilon(t) = \min \left\{ F_{\varepsilon,A}(u) + D_\varepsilon(A) : u(0) = 0, u(1) = t, A \subset (0, 1) \right\} \quad (3.12)$$

we proceed as in Remark 3.1, noticing that the minimum value

$$m^\varepsilon(A, t) = \min \left\{ F_{\varepsilon,A}(u) : u(0) = 0, u(1) = t \right\} \quad (3.13)$$

depends on A only through the volume fraction of each damaged component

$$\lambda_i = 2 \left| \left\{ x \in A : \alpha \left(\frac{x}{\varepsilon} \right) = \alpha_i \right\} \right|,$$

and its value is independent of ε and is given by

$$\min \left\{ \frac{1}{2} \left(\lambda_1 \alpha_1 z_{11}^2 + (1 - \lambda_1) \beta_1 z_{12}^2 \right) + \frac{1}{2} \left(\lambda_2 \alpha_2 z_{21}^2 + (1 - \lambda_2) \beta_1 z_{22}^2 \right) : \right.$$

$$\frac{1}{2}(\lambda_1 z_{11} + (1 - \lambda_1)z_{12}) + \frac{1}{2}(\lambda_2 z_{21} + (1 - \lambda_2)z_{22}) = t\}.$$

We conclude that $m^\varepsilon(t) = m_{\text{hom}}(t)$ is independent of ε and satisfies

$$m_{\text{hom}}(t) = \frac{1}{2} \min \left\{ m_1(t_1) + m_2(t_2) : \frac{t_1 + t_2}{2} = t \right\}, \quad (3.14)$$

where m_j is defined as m in (3.2) with α_j, β_j and γ_j in the place of α, β and γ (i.e., by the damage process in the i -th material). Hence, by (3.5)

$$m_j(t) = \begin{cases} \beta_j t^2 & \text{if } |t| \leq \sqrt{\frac{\alpha_j \gamma_j}{\beta_j(\beta_j - \alpha_j)}} \\ \alpha_j t^2 + \gamma_j & \text{if } |t| \geq \sqrt{\frac{\beta_j \gamma_j}{\alpha_j(\beta_j - \alpha_j)}} \\ 2t \sqrt{\frac{\alpha_j \beta_j \gamma_j}{\beta_j - \alpha_j}} - \frac{\gamma_j \alpha_j}{\beta_j - \alpha_j} & \text{otherwise.} \end{cases} \quad (3.15)$$

We can therefore easily compute $m(t)$. In the hypothesis, e.g., that

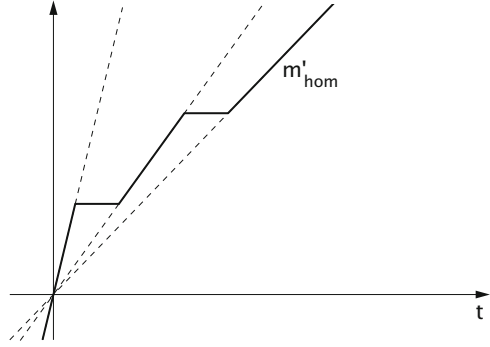
$$p_2 := \sqrt{\frac{\alpha_2 \beta_2 \gamma_2}{\beta_2 - \alpha_2}} < \sqrt{\frac{\alpha_1 \beta_1 \gamma_1}{\beta_1 - \alpha_1}} =: p_1, \quad (3.16)$$

we can write $m'(t)$ as

$$m'_{\text{hom}}(t) = \begin{cases} 2\underline{\beta}t & \text{if } |t| \leq \frac{p_2}{\underline{\beta}} \\ 2p_2 & \text{if } \frac{p_2}{\underline{\beta}} < |t| < \frac{p_2(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \\ \frac{4\beta_1\alpha_2}{\beta_1 + \alpha_2}t & \text{if } \frac{p_2(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \leq |t| \leq \frac{p_1(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \\ 2p_1 & \text{if } \frac{p_1(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} < |t| < \frac{p_1}{\underline{\alpha}} \\ 2\underline{\alpha}t & \text{if } |t| \geq \frac{p_1}{\underline{\alpha}}. \end{cases}$$

The outcome is pictured in Fig. 3.3. It highlights that the behaviour is different from the one computed in Sect. 3.1.1: for small values of the total displacement t the overall response is the same as the one of the homogenized behaviour of the two ‘strong’ materials. At a first critical value for t one of the two materials (and only one except for the exceptional case $p_1 = p_2$) starts to damage (this corresponds

Fig. 3.3 The function $m'_{\text{hom}}(t)$ describing the homogenized damage in a periodic microstructure



to the first constant value $2p_1$ of m') until it is completely damaged. Under condition (3.16) the first material to damage is material 2, and the corresponding damage volume fraction is

$$\lambda_{2,\min}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{p_2}{\underline{\beta}} \\ \frac{2p_2}{\gamma_2} \left(t - \frac{p_2}{\underline{\beta}} \right) & \text{if } \frac{p_2}{\underline{\beta}} < t < \frac{p_2(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \\ 1 & \text{if } t > \frac{p_2(\beta_1 + \alpha_2)}{2\beta_1\alpha_2}. \end{cases} \quad (3.17)$$

Then the material behaves as a mixture of a strong material 1 and a damaged material 2. Subsequently, also material 1 starts to damage; the corresponding damage volume fraction is

$$\lambda_{1,\min}(t) = \begin{cases} 0 & \text{if } t \leq \frac{p_1(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \\ \frac{2p_1}{\gamma_1} \left(t - \frac{p_1(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \right) & \text{if } \frac{p_1(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} < t < \frac{p_1}{\underline{\alpha}} \\ 1 & \text{if } t \geq \frac{p_1}{\underline{\alpha}}. \end{cases} \quad (3.18)$$

After also material 1 has completely damaged, the behaviour is that of the homogenized energy for two weak materials.

Note that at fixed ε we can define A_ε^t and u_ε^t by choosing increasing families of sets $A_{j,\varepsilon}^t$ describing the damage in the j -th material with $|A_{j,\varepsilon}^t| = \frac{1}{2}\lambda_{j,\min}(t)$, setting $A_\varepsilon^t = A_{1,\varepsilon}^t \cup A_{2,\varepsilon}^t$ and u_ε^t the corresponding solution of $m^\varepsilon(A^t, t)$. However the sets A_ε^t do not converge to sets as $\varepsilon \rightarrow 0$ except for the trivial cases \emptyset and $(0, 1)$. In particular for $\frac{p_2(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \leq t \leq \frac{p_1(\beta_1 + \alpha_2)}{2\beta_1\alpha_2}$ we have $\lambda_{2,\min}(t) = 1$ and $\lambda_{1,\min}(t) = 0$, so that $A_\varepsilon^t = \varepsilon(\mathbb{Z} + [\frac{1}{2}, 1])$, which do not converge as sets.

3.1.2.1 A Double-Damage-Set Formulation

The observation above highlights that a weaker notion of convergence of sets must be given in order to describe the behavior of (some solutions of) the sequence of

damage problem. One way is to choose particular sequences of damaged sets $A_{j,\varepsilon}^t$, for examples intersections of intervals with the j -th material. For simplicity we consider intervals $[0, \lambda_{j,\varepsilon}(t)]$ with one endpoint in 0, so that

$$A_{1,\varepsilon}^t = [0, \lambda_{1,\varepsilon}(t)] \cap \varepsilon \left(\mathbb{Z} + \left[0, \frac{1}{2}\right] \right), \quad A_{2,\varepsilon}^t = [0, \lambda_{2,\varepsilon}(t)] \cap \varepsilon \left(\mathbb{Z} + \left[\frac{1}{2}, 1\right] \right).$$

Note that under hypothesis (3.16) we have $\lambda_{2,\varepsilon}(t) \geq \lambda_{1,\varepsilon}(t)$ for all t . We have therefore to examine problems (3.13) rewritten in the form

$$\begin{aligned} m^\varepsilon(\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}, t) = \min \left\{ \int_0^{\lambda_{1,\varepsilon}} \alpha \left(\frac{x}{\varepsilon} \right) |u'|^2 dx + \int_{\lambda_{1,\varepsilon}}^{\lambda_{2,\varepsilon}} a \left(\frac{x}{\varepsilon} \right) |u'|^2 dx \right. \\ \left. + \int_{\lambda_{2,\varepsilon}}^1 \beta \left(\frac{x}{\varepsilon} \right) |u'|^2 dx : u(0) = 0, u(1) = t \right\}, \end{aligned} \quad (3.19)$$

where a is the 1-periodic function with

$$a(y) = \begin{cases} \beta_1 & \text{for } 0 \leq y < \frac{1}{2} \\ \alpha_2 & \text{for } \frac{1}{2} \leq y < 1 \end{cases}.$$

If $\lambda_{j,\varepsilon} \rightarrow \lambda_j$ then these problems converge to

$$\begin{aligned} m_{\text{hom}}(\lambda_1, \lambda_2, t) = \min \left\{ \underline{\alpha} \int_0^{\lambda_1} |u'|^2 dx + \underline{a} \int_{\lambda_1}^{\lambda_2} |u'|^2 dx \right. \\ \left. + \underline{\beta} \int_{\lambda_2}^1 |u'|^2 dx : u(0) = 0, u(1) = t \right\}. \end{aligned} \quad (3.20)$$

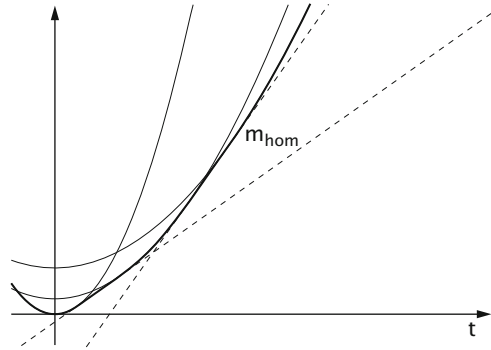
Taking into account that in this case $\int_A \gamma(x/\varepsilon) dx \rightarrow \frac{1}{2} \gamma_2 \lambda_2 + \frac{1}{2} \gamma_1 \lambda_1$, the limit of $m^\varepsilon(t)$ can be written as

$$\begin{aligned} m_{\text{hom}}(t) = \min \left\{ \int_0^1 \left(\chi_{[0,\lambda_1]} \left(\underline{\alpha} |u'|^2 + \frac{\gamma_1 + \gamma_2}{2} \right) + \chi_{[\lambda_1,\lambda_2]} \left(\underline{a} |u'|^2 + \frac{\gamma_2}{2} \right) \right. \right. \\ \left. \left. + \chi_{[\lambda_2,1]} \underline{\beta} |u'|^2 \right) dx : u(0) = 0, u(1) = t, 0 \leq \lambda_1 \leq \lambda_2 \leq 1 \right\}. \end{aligned} \quad (3.21)$$

Minimizing first in λ_1 and λ_2 we obtain

$$\begin{aligned} m_{\text{hom}}(t) = \min \left\{ \int_0^1 \min \left\{ \underline{\alpha} |u'|^2 + \frac{\gamma_1 + \gamma_2}{2}, \underline{a} |u'|^2 + \frac{\gamma_2}{2}, \underline{\beta} |u'|^2 \right\} dx \right. \\ \left. : u(0) = 0, u(1) = t \right\}. \end{aligned} \quad (3.22)$$

Fig. 3.4 The minimal energy $m_{\text{hom}}(t)$ of Sect. 3.1.2



This observation highlights that the function $m_{\text{hom}}(t)$ can be expressed as the convex envelope of

$$\min\left\{\underline{\beta}z^2, \frac{2\alpha_2\beta_1}{\beta_1 + \alpha_2\beta_2}z^2 + \frac{1}{2}\gamma_2, \underline{\alpha} + \frac{1}{2}(\gamma_1 + \gamma_2)\right\}, \quad (3.23)$$

(see Fig. 3.4) which are the three total energy densities corresponding to the mixtures of undamaged materials, equally damaged and undamaged materials (in the optimal way determined by condition (3.16)), and completely damaged materials.

The limit damage motion in this case is given in terms of the two sets $A_j^t = [0, \lambda_j(t)]$, where $\lambda_j(t)$ are the minimizers of problem (3.21), and of the corresponding u^t . Note that this is possible thanks to the particular choice of the damage sets $A_{j,\varepsilon}^t$, and does not give a description of the behavior of an arbitrary family of solutions $A_{\varepsilon}^t, u_{\varepsilon}^t$.

3.1.2.2 Double-Threshold Formulation

Also in this case we note that the damage process takes place when σ_t reaches some particular values. In this case the thresholds are two given by p_1 and p_2 (see Fig. 3.3 as compared with Fig. 3.2).

3.1.3 Dissipations Leading to a Commutability Result

We now slightly modify the dissipation in the example of the previous section. This will produce a ‘commutability’ result in the quasistatic motion outlined above; i.e., the process of homogenization and of damage can be interchanged. The first such modification is obtained by imposing that the domain of the dissipation be the set of intervals; i.e.,

$$D_{\varepsilon}(A) = +\infty \text{ if } A \text{ is not an interval,}$$

while D_ε remains unchanged otherwise. In this case, in the process described in Sect. 3.1.2, we may remark that, at fixed ε , the minimal A_ε^t will converge to some interval A^t for which we may pass to the limit obtaining the problem

$$\min \left\{ F_{\text{hom}, A^t}(u) + \bar{\gamma} |A^t| : u(0) = 0, u(1) = t \right\},$$

where

$$\bar{\gamma} = \frac{\gamma_1 + \gamma_2}{2},$$

since

$$\lim_{\varepsilon \rightarrow 0} D_\varepsilon(A_\varepsilon^t) = \bar{\gamma} |A^t|.$$

Note that in the previous example this passage was not possible since the A_ε^t thus defined do not converge to a limit set.

We may conclude then that A^t minimizes the corresponding

$$\begin{aligned} m_{\text{hom}}(t) &:= \min \left\{ F_{\text{hom}, A}(u) + \bar{\gamma} |A| : u(0) = 0, u(1) = t, A \text{ subinterval of } (0, 1) \right\} \\ &= \min \left\{ F_{\text{hom}, A}(u) + \bar{\gamma} |A| : u(0) = 0, u(1) = t, A \subset (0, 1) \right\} \\ &= f_{\text{hom}}^{**}(t), \end{aligned} \tag{3.24}$$

where

$$f_{\text{hom}}(z) = \min \{ \underline{\alpha} z^2 + \bar{\gamma}, \underline{\beta} z^2 \}, \tag{3.25}$$

which describes the damage process corresponding to the limit homogenized functionals. Note that in the limit problem we may remove the constraint that A be an interval, since we have already remarked that solutions satisfying such a constraint exist.

3.1.3.1 Brutal Damage

We consider another dissipation, with

$$D_\varepsilon(A) = \int_A \gamma \left(\frac{x}{\varepsilon} \right) dx + \sigma \#(\partial A \cap [0, 1]),$$

so that it is finite only on finite unions of intervals.

We may compute the limit of $m^\varepsilon(t)$ as in Sect. 3.1.2, noticing that, for a finite union of intervals, we may pass to the limit (taking possibly into account that

the number of intervals may decrease in the limit process), and conclude that the limit damage process corresponds to the functionals $F_{\text{hom},A}$ and the homogenized dissipation

$$D_{\text{hom}}(A) = \bar{\gamma}|A| + \sigma\#(\partial A \cap [0, 1]).$$

Correspondingly, we can compute the minima

$$m_{\text{hom}}(t) = \min \left\{ F_{\text{hom},A}(u) + D_{\text{hom}}(A) : u(0) = 0, u(1) = t, \right. \\ \left. A \text{ a union of subintervals of } (0, 1) \right\},$$

as

$$m_{\text{hom}}(t) = \min \left\{ m_{\text{hom}}^0(t), m_{\text{hom}}^1(t) \right\},$$

where m_{hom}^0 corresponds to no damage,

$$m_{\text{hom}}^0(t) = \min \left\{ F_{\text{hom},\emptyset}(u) : u(0) = 0, u(1) = t \right\} = \underline{\beta}t^2,$$

and m_{hom}^1 corresponds to A a single interval (not being energetically convenient to have more than one interval),

$$\begin{aligned} m_{\text{hom}}^1(t) &= \inf \left\{ F_{\text{hom},A}(u) + D_{\text{hom}}(A) : u(0) = 0, u(1) = t, \right. \\ &\quad \left. A \text{ subinterval of } (0, 1), A \neq \emptyset \right\} \\ &= \min \left\{ F_{\text{hom},A}(u) + \bar{\gamma}|A| : u(0) = 0, u(1) = t, \right. \\ &\quad \left. A \text{ subinterval of } (0, 1) \right\} + 2\sigma \\ &= f_{\text{hom}}^{**}(t) + 2\sigma, \end{aligned} \tag{3.26}$$

with f_{hom} as in (3.25).

The plot of m_{hom} is reproduced in Fig. 3.5. Note that we follow the curve corresponding to undamaged materials until we reach the graph of m_{hom}^1 , which corresponds to a positive value of the damage area; i.e., the damage is ‘brutal’ (once it is convenient to damage, we damage a large region). Correspondingly, in Fig. 3.6 we plot the value of m'_{hom} and the derivative of the homogenized energy \mathcal{E} along a cycle in dependence of the boundary condition $g = g(t)$.

Fig. 3.5 The minimal energy $m_{\text{hom}}(t)$

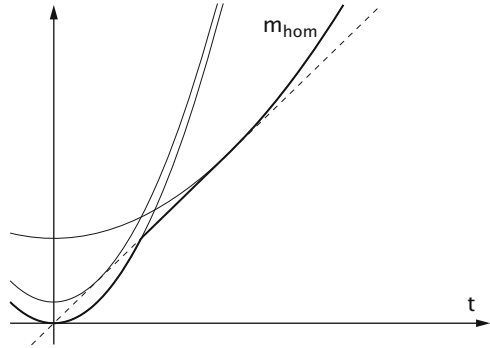
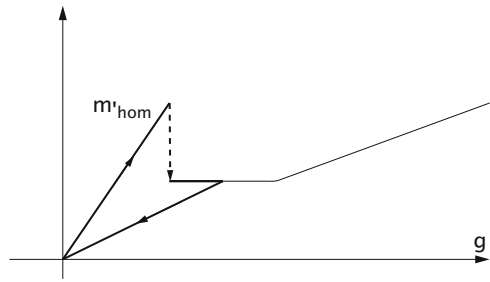


Fig. 3.6 Plot of $m'_{\text{hom}}(t)$ and derivative of the energy along a cycle



3.1.4 Conditions for Commutability

Motivated by the examples above, we may derive a criterion of commutability of Γ -convergence and quasi-static motion, which we state in this particular case but is immediately generalized to more abstract situations. This easily follows from the remark that in order to pass to the limit we have to have the convergence of the minimum problems

$$\min\{F_{\varepsilon,A}(u) + D_{\varepsilon}(A) : u(0) = 0, u(1) = g(t), B_{\varepsilon} \subset A\} \tag{3.27}$$

with B_{ε} Borel sets converging to B (in (3.1) $B_{\varepsilon} = \bigcup\{A_s : s < t\}$) to

$$\min\{F_{\text{hom},A}(u) + D_{\text{hom}}(A) : u(0) = 0, u(1) = g(t), B \subset A\}. \tag{3.28}$$

Proposition 3.1 (A commutativity criterion). *Let $B_{\varepsilon} \rightarrow B$ and let*

$$G_{\varepsilon}(u, A) = \begin{cases} F_{\varepsilon,A}(u) + D_{\varepsilon}(A) & \text{if } B_{\varepsilon} \subset A \\ +\infty & \text{otherwise} \end{cases} \tag{3.29}$$

$$G_{\text{hom}}(u, A) = \begin{cases} F_{\text{hom},A}(u) + D_{\text{hom}}(A) & \text{if } B \subset A \\ +\infty & \text{otherwise.} \end{cases} \quad (3.30)$$

Suppose that G_ε Γ -converges to G_{hom} with respect to the converge $L^2 \times L^1$ -convergence (the latter is understood as the convergence of the characteristic functions of sets). Then if a sequence of solutions $(u_\varepsilon^t, A_\varepsilon^t)$ to the evolutions related to the energies $F_{\varepsilon,A}$, dissipation D_ε and boundary conditions given by g is such that (up to subsequences) for all t u_ε^t converges to some u^t in L^2 and $B_\varepsilon^t = \bigcup\{A_\varepsilon^s : s < t\}$ converges to some B^t in L^1 , then it converges (up to subsequences) to a solution to the evolution related to the energies $F_{\text{hom},A}$, dissipation D_{hom} and boundary conditions given by g .

This criterion follows from the fundamental theorem of Γ -convergence, upon noting that the boundary conditions are compatible with the convergence of minima regardless to the constraint $B_\varepsilon \subset A$.

Remark 3.2. We may apply Proposition 3.1 to the two examples in Sect. 3.1.3. In fact, in both cases the boundedness of the dissipation implies that A_ε^t and hence B_ε^t are (increasing with t) intervals (or finite union of intervals in the second case), so that the pre-compactness of B_ε^t is guaranteed. The convergence for all t follows from an application of Helly's theorem.

We cannot apply Proposition 3.1 to the solutions in Sect. 3.1.2. Indeed, except for the trivial cases when $A_\varepsilon^t = \emptyset$ or $A_\varepsilon^t = (0, 1)$, these do not converge strongly in L^1 but only weakly.

3.1.5 Relaxed Evolution

The criterion above suggests, in case it is not satisfied, to examine the behavior of the functionals (3.29) with respect to the $L^2 \times L^1$ -weak convergence. In this case, the limit of a sequence of characteristic functions may not be a characteristic function itself, so that the domain of the Γ -limit will be the space of pairs (u, θ) , with $0 \leq \theta \leq 1$. This formulation will necessarily be more complex, but will capture the behavior of all sequences $A_\varepsilon^t, u_\varepsilon^t$.

Proposition 3.2 (Relaxed total energies). *If hypothesis (3.16) holds, then the Γ -limit of the functionals (3.29) with respect to the $L^2 \times L^1$ -weak convergence is given by the functional (r stands for 'relaxed')*

$$G_{\text{hom}}^r(u, \theta) = \int_{(0,1)} f_{\text{hom}}(\theta, u') dx + \int_{(0,1)} \gamma_{\text{hom}}(\theta) dx, \quad (3.31)$$

where

$$f_{\text{hom}}(\theta, z) = \begin{cases} \frac{2\alpha_2\beta_1\beta_2}{2\theta\beta_1\beta_2 + (1-2\theta)\alpha_2\beta_1 + \alpha_2\beta_2} z^2 & \text{if } 0 \leq \theta \leq \frac{1}{2} \\ \frac{2\alpha_1\alpha_2\beta_1}{2(1-\theta)\alpha_1\alpha_2 + (2\theta-1)\alpha_2\beta_1 + \alpha_1\beta_1} z^2 & \text{if } \frac{1}{2} \leq \theta \leq 1 \end{cases} \quad (3.32)$$

and the dissipation energy density is

$$\gamma_{\text{hom}}(\theta) = \begin{cases} \gamma_2\theta & \text{if } 0 \leq \theta \leq \frac{1}{2} \\ \frac{1}{2}\gamma_2 + \gamma_1(\theta - \frac{1}{2}) & \text{if } \frac{1}{2} \leq \theta \leq 1 \end{cases} \quad (3.33)$$

Proof. We do not dwell on this proof, since it is a variation of the usual homogenization theorem. A lower bound is obtained by minimizing on each periodicity cell. Upon scaling we are led to computing

$$\phi(z, \theta) := \min \left\{ \int_A \alpha(y) |u'|^2 dy + \int_{(0,1) \setminus A} \beta(y) |u'|^2 dy + \int_A \gamma(y) dy : \right. \\ \left. A \subset (0, 1), |A| = \theta, u(0) = 0, u(1) = z \right\}.$$

By a direct computation we get

$$\phi(z, \theta) = f_{\text{hom}}(\theta, z) + \gamma_{\text{hom}}(\theta).$$

Since ϕ is convex in the pair (z, θ) , its integral is lower semicontinuous in $L^2 \times L^1$ -weak, and hence is a candidate for the Γ -liminf. The proof of the limsup inequality is obtained by density, first dealing with the case when u is a piecewise-affine function and θ is a piecewise-constant function. \square

Remark 3.3. The limit of problems (3.27) with B_ε converging weakly to some ϕ will be of the form

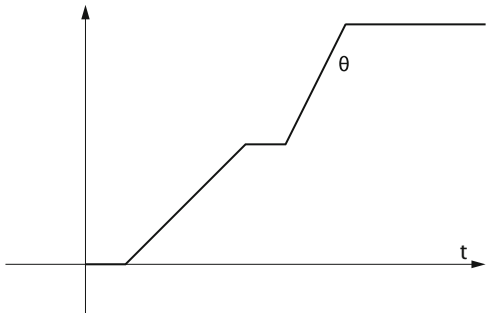
$$\min \left\{ G_{\text{hom}}^r(u, \theta) : u(0) = 0, u(1) = g(t), \phi \leq \theta \right\}. \quad (3.34)$$

As in Sect. 3.1.2, we only consider the case $g(t) = t$, and the problem

$$m^r(t) = \min \left\{ G_{\text{hom}}^r(u, \theta) : u(0) = 0, u(1) = t \right\}. \quad (3.35)$$

We have

$$m^r(t) = \min \left\{ \int_0^1 \min_{0 \leq \theta \leq 1} \left\{ f_{\text{hom}}(\theta, u') + \gamma_{\text{hom}}(\theta) \right\} dx : u(0) = 0, u(1) = t \right\}. \quad (3.36)$$

Fig. 3.7 Value of the damage $\theta(t)$ 

A direct computation shows that

$$\min_{0 \leq \theta \leq 1} \left\{ f_{\text{hom}}(\theta, z) + \gamma_{\text{hom}}(\theta) \right\} = m(z), \quad (3.37)$$

with m the one in Sect. 3.1.2; hence, by the convexity of m , we have $m^r(z) = m(z)$. Moreover, again using the convexity of m , a solution is simply given by $u_t(x) = tx$ and correspondingly $\theta = \theta(t)$ constant equal to the minimizer of (3.37) with $z = t$; namely,

$$\theta(t) = \begin{cases} 0 & \text{if } |t| \leq \frac{p_2}{\beta} \\ \frac{p_2^2}{\gamma_2} \left(\frac{t}{p_2} - \frac{1}{\beta} \right) & \text{if } \frac{p_2}{\beta} < |t| < \frac{p_2(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \\ \frac{1}{2} & \text{if } \frac{p_2(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \leq |t| \leq \frac{p_1(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} \\ 1 + \frac{p_1^2}{\gamma_1} \left(\frac{t}{p_1} - \frac{1}{\alpha} \right) & \text{if } \frac{p_1(\beta_1 + \alpha_2)}{2\beta_1\alpha_2} < |t| < \frac{p_1}{\alpha} \\ 1 & \text{if } |t| \geq \frac{p_1}{\alpha} \end{cases}$$

(see Fig. 3.7). Note that we have

$$\theta(t) = \frac{\lambda_{1,\min}(t) + \lambda_{2,\min}(t)}{2}$$

with $\lambda_{j,\min}$ given by (3.17) and (3.18). The solution with θ constant corresponds to equi-distributed damage. Note that we have infinitely many solutions, among which the ones described above in terms of A_1^t and A_2^t .

3.2 Energetic Solutions for Rate-Independent Evolution

The examples in the previous theory can be framed in a general theory of rate-independent variational evolution. We introduce some of the concepts of the theory that are relevant to our presentation, without being precise in the hypotheses on spaces and topologies.

Definition 3.2. Let $\mathcal{F} = \mathcal{F}(t, \cdot)$ be a time-parameterized energy functional and \mathcal{D} be a dissipation functional, which we assume to be positively-homogeneous of degree one; i.e. $\mathcal{D}(sU) = s\mathcal{D}(U)$ if $s > 0$. Then U is an *energetic solution* for the evolution inclusion

$$\partial\mathcal{D}(\dot{U}) + \partial_U\mathcal{F}(t, U) \ni 0$$

if the following two conditions hold:

(S) **global stability** for all t and \hat{U} we have

$$\mathcal{F}(t, U(t)) \leq \mathcal{F}(t, \hat{U}) + \mathcal{D}(\hat{U} - U(t));$$

(E) **energy inequality** for all t

$$\mathcal{F}(t, U(t)) + \int_0^t D(\dot{U}) \leq \mathcal{F}(0, U(0)) + \int_0^t \partial_s \mathcal{F}(s, U(s)) ds.$$

In this formula the integral $\int_0^t D(\dot{U})$ must be understood in the sense of measures, and can be equivalently defined as

$$\sup\left\{\sum_{i=1}^n D(U(t_i) - U(t_{i-1})) : 0 = t_0 < t_1 < \dots < t_n = t\right\}. \quad (3.38)$$

If U is an absolutely continuous function then the integral reduces to $\int_0^t D(\dot{U}(s)) ds$.

Remark 3.4 (Energy equality). Under mild assumptions, from (S) it can be deduced that in (E) equality holds, so that we have an *energy conservation* identity. This identity states that the difference of the energy at a final and an initial state equals the difference of the work of the applied actions and the total dissipation along the path.

Remark 3.5. In the case of damage we have $U = (u, v)$,

$$\mathcal{F}(t, u, v) = \begin{cases} \int_0^1 (\alpha v |u'|^2 + \beta(1-v) |u'|^2) dx & \text{if } v \in \{0, 1\} \text{ a.e.,} \\ & u(0) = 0, u(1) = g(t) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\mathcal{D}(U) = \begin{cases} \gamma \int_0^1 v \, dx & \text{if } v \in \{0, 1\} \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

Condition (S) is meaningful only if $\hat{U} = (u, v)$ and $U(t) = (u^t, v^t)$ satisfy $v = \chi_A$ and $v^t = \chi_{A^t}$ with $A^t \subset A$, so that (S) implies that u^t and A^t are minimizers for (3.1). Conversely, it can be checked that the solutions to the damage evolution satisfy the energy inequality as an identity.

Remark 3.6 (Rate-independence). The requirement that \mathcal{D} be positively homogeneous of degree one implies that the solution is *rate-independent*; i.e., that if we consider a re-parameterization of the energy $\tilde{\mathcal{F}}(t, U) = \mathcal{F}(\varphi(t), U)$ via an increasing diffeomorphism φ , then the energetic solutions \tilde{U} of the corresponding evolution inclusion are exactly the $\tilde{U}(t) = U(\varphi(t))$ with U energetic solutions of the corresponding evolution inclusion for \mathcal{F} .

Example 3.1 (Mechanical play/hysteresis). The prototypical example of an evolution inclusion is by taking $U = x \in \mathbb{R}$ and

$$\mathcal{F}(t, x) = \frac{x^2}{2} - tx, \quad \mathcal{D}(x) = |x|.$$

In this case we can write explicitly $\partial|\dot{x}| + x - t \ni 0$ as

$$\begin{cases} \dot{x} > 0 & \text{if } x = t - 1 \\ \dot{x} < 0 & \text{if } x = t + 1 \\ \dot{x} = 0 & \text{if } t - 1 \leq x \leq t + 1. \end{cases}$$

The solution with $x(0) = x_0 \in [-1, 1]$ is

$$x(t) = \begin{cases} x_0 & \text{if } t \leq 1 + x_0 \\ t - 1 & \text{if } t > 1 + x_0. \end{cases} \quad (3.39)$$

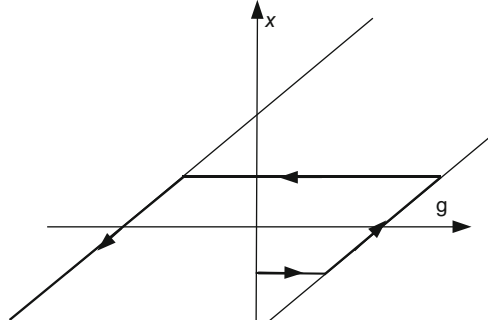
If we take a non-monotone load $g(t) = T - |t - T|$ with $T > 1 + x_0$ and

$$\mathcal{F}(t, x) = \frac{x^2}{2} - g(t)x, \quad \mathcal{D}(x) = |x|.$$

then the solution x is as above for $t \leq T$, and given solving $\partial|\dot{x}| + x - (2T - t) \ni 0$ by

$$x(t) = \begin{cases} T - 1 & \text{if } T \leq t \leq T + 2 \\ 2T - t + 1 = g(t) + 1 & \text{if } t \geq T + 2. \end{cases}$$

Fig. 3.8 Hysteretic trajectory $x(t)$ in dependence of $g(t)$



This solution shows a hysteretic behavior of this system, whose trajectory in the g - x plane is represented in Fig. 3.8.

3.2.1 Solutions Obtained by Time Discretization

Energetic solutions can be obtained as limits of discrete schemes as follows: fix $\tau > 0$ and define U_k^τ recursively by setting $U_0^\tau = U_0$, and choosing U_k^τ as a solution of the minimum problem

$$\min_{\hat{U}} \left\{ \mathcal{F}(\tau k, \hat{U}) + \mathcal{D}(\hat{U} - U_{k-1}^\tau) \right\}.$$

Define the continuous trajectory $U^\tau(t) = U_{\lfloor t/\tau \rfloor}^\tau$. Under suitable assumptions, the limits of (subsequences of) U^τ are energetic solution of the variational inclusion for \mathcal{F} and \mathcal{D} .

Example 3.2 (Mechanical play). It is easy to check that the solutions in Example 3.1 can be obtained by time-discretization, solving iteratively

$$\min \left\{ \frac{1}{2} x^2 - g(k\tau)x + |x - x_{k-1}^\tau| \right\}.$$

In the case of $x_0 \in [-1, 1]$ and $g(t) = t$ the sequence $\{x_k^\tau\}$ is non-decreasing and hence x_k^τ solves

$$\min \left\{ \frac{1}{2} x^2 - (k\tau - 1)x - x_{k-1}^\tau : x \geq x_{k-1}^\tau \right\};$$

i.e.,

$$x_k^\tau = \begin{cases} x_{k-1}^\tau & \text{if } k\tau - 1 \leq x_{k-1}^\tau \\ k\tau + 1 & \text{if } k\tau - 1 \geq x_{k-1}^\tau. \end{cases}$$

Passing to the limit as $\tau \rightarrow 0$ we then obtain

$$x(t) = \begin{cases} x_0 & \text{if } t - 1 \leq x_0 \\ t - 1 & \text{if } t - 1 \geq x_0, \end{cases}$$

which corresponds to the solution in (3.39).

Example 3.3 (Nonconvex mechanical play). We can consider a double-well potential of the form

$$\mathcal{F}(t, x) = \frac{1}{2} \min\{(x - 1)^2, (x + 1)^2\} - tx, \quad \mathcal{D}(x) = |x|$$

with $x_0 \in [-2, -1]$. Then the sequence x_k^τ is increasing and minimizes

$$\min \left\{ \frac{1}{2} \min\{(x - 1)^2, (x + 1)^2\} - (k\tau - 1)x - x_{k-1}^\tau : x \geq x_{k-1}^\tau \right\}.$$

The solution satisfies

$$x_k^\tau = \begin{cases} x_{k-1}^\tau & \text{if } k\tau - 2 \leq x_{k-1}^\tau \\ k\tau - 2 & \text{if } x_{k-1}^\tau \leq k\tau - 2, k\tau \leq 1 \\ k\tau & \text{if } k\tau \geq 1 \end{cases}$$

(with an ambiguity if $k\tau = 1$, in which case we may take equivalently $x_{k-1}^\tau = -1$ or $x_{k-1}^\tau = 1$). Passing to the limit we have either the solution

$$x(t) = \begin{cases} x_0 & \text{if } t \leq x_0 + 2 \\ t - 2 & \text{if } x_0 \leq t \leq 1 \\ t & \text{if } t > 1 \end{cases}$$

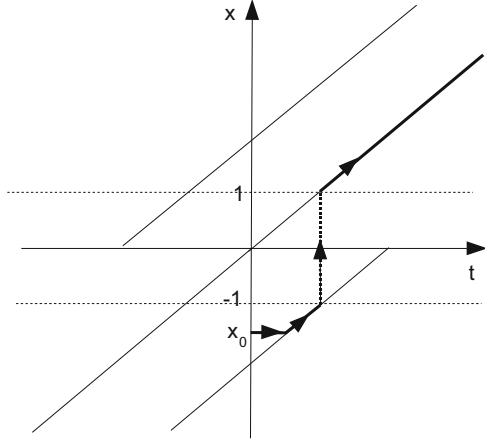
or the one equal to this except for 1 where $x(1) = 1$. The graph of the solution is pictured in Fig. 3.9.

Note that the solution is discontinuous at $t = 1$ and is not characterized completely by the differential inclusion. In this case the discontinuity exactly at $|x| = 1$ can be justified by the energy equality.

3.2.2 Stability

We can give a stability result with respect to Γ -convergence. As remarked in the case of damage, the separate Γ -convergence of \mathcal{F}_ε and \mathcal{D}_ε may not be sufficient to describe the limit of the corresponding variational motions.

Fig. 3.9 The trajectory $x(t)$ in Example 3.3



Theorem 3.1. *Suppose that \mathcal{F} and \mathcal{D} are lower bounds for \mathcal{F}_ε and \mathcal{D}_ε , that U_ε are energetic solutions converging pointwise to some U as $\varepsilon \rightarrow 0$, that the initial data are well-prepared; i.e., that*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(0, U_\varepsilon(0)) = \mathcal{F}(0, U(0)),$$

that we have convergence of the external actions

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \partial_s \mathcal{F}_\varepsilon(s, U_\varepsilon(s)) ds = \int_0^t \partial_s \mathcal{F}(s, U(s)) ds \quad \text{for all } t,$$

and that the following **mutual recovery sequence existence condition** holds: for all t and all \hat{U} there exists a sequence \hat{U}_ε such that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \left(\mathcal{F}_\varepsilon(t, \hat{U}_\varepsilon) - \mathcal{F}_\varepsilon(t, U_\varepsilon(t)) + \mathcal{D}_\varepsilon(\hat{U}_\varepsilon - U_\varepsilon(t)) \right) \\ \leq \mathcal{F}(t, \hat{U}) - \mathcal{F}(t, U(t)) + \mathcal{D}(\hat{U} - U(t)). \end{aligned} \quad (3.40)$$

Then U is an energetic solution for the limit energy and dissipation.

Proof. Let $0 = t_0 < t_1 < \dots < t_n = t$; by the liminf inequality for D_ε and (3.38) we then have

$$\sum_{i=1}^n \mathcal{D}(U(t_i) - U(t_{i-1})) \leq \liminf_{\varepsilon \rightarrow 0} \sum_{i=1}^n \mathcal{D}_\varepsilon(U_\varepsilon(t_i) - U_\varepsilon(t_{i-1})) \leq \liminf_{\varepsilon \rightarrow 0} \int_0^t \mathcal{D}_\varepsilon(\dot{U}_\varepsilon).$$

Taking into account the liminf inequality for \mathcal{F}_ε and the convergence hypotheses on initial data and external actions we then obtain

$$\begin{aligned} \mathcal{F}(t, U(t)) + \int_0^t D(\dot{U}(s)) ds &\leq \liminf_{\varepsilon \rightarrow 0} \left(\mathcal{F}_\varepsilon(0, U_\varepsilon(0)) + \int_0^t \partial_s \mathcal{F}_\varepsilon(s, U_\varepsilon(s)) ds \right) \\ &= \mathcal{F}(0, U(0)) + \int_0^t \partial_s \mathcal{F}(s, U(s)) ds \end{aligned}$$

so that (E) holds.

Take any test \hat{U} and use the mutual recovery sequence \hat{U}_ε to obtain

$$\mathcal{F}(t, \hat{U}) - \mathcal{F}(t, U(t)) + \mathcal{D}(\hat{U} - U(t)) \geq 0;$$

i.e. the inequality in (S), from the same inequality for U_ε . \square

Proposition 3.3 (Necessary and sufficient conditions).

- (i) Let \mathcal{F}_ε Γ -converge to \mathcal{F} and \mathcal{D}_ε converge continuously to \mathcal{D} . Then the mutual recovery sequence condition is satisfied;
- (ii) Assume that \mathcal{F}_ε and \mathcal{D}_ε Γ -converge to \mathcal{F} and \mathcal{D} , that $U_\varepsilon(t)$ is a recovery sequence for F_ε at $U(t)$ and that the mutual recovery sequence condition holds with $\hat{U}_\varepsilon \rightarrow \hat{U}$. Then $\mathcal{G}_\varepsilon(V) = \mathcal{F}_\varepsilon(t, V) + \mathcal{D}_\varepsilon(V - U_\varepsilon(t))$ Γ -converges to $\mathcal{G}(V) = \mathcal{F}(t, U(t)) + \mathcal{D}(V - U(t))$.

Proof. (i) Follows by taking \hat{U}_ε any recovery sequence for $\mathcal{F}_\varepsilon(t, \hat{U})$.

- (ii) Is an immediate consequence of the fact that $\mathcal{F} + \mathcal{D}$ is a lower bound for $\mathcal{F}_\varepsilon + \mathcal{D}_\varepsilon$, while the mutual recovery sequence provides a recovery sequence for $\mathcal{F}(t, U(t)) + \mathcal{D}(V - U(t))$. \square

Example 3.4 (An example with relaxed evolution). In \mathbb{R}^2 with $U = (x, y)$, consider the initial datum $u_\varepsilon(0) = (0, 0)$ and the energy and dissipation

$$\mathcal{F}_\varepsilon(t, U) = \frac{1}{2}x^2 + \frac{1}{2\varepsilon^2}(y - \varepsilon x)^2 - tx, \quad \mathcal{D}_\varepsilon(U) = |x| + \frac{1}{\varepsilon}|y|$$

with Γ -limits

$$\mathcal{F}(t, U) = \frac{1}{2}x^2 - tx, \quad \mathcal{D}(U) = |x|$$

with domain $\{y = 0\}$.

The solution to the differential inclusion for \mathcal{F} and \mathcal{D} with initial datum $(0, 0)$ is given by $x(t)$ as in (3.39) with $x_0 = 0$, and $y(t) = 0$. On the other hand, the solutions to the differential inclusions U_ε can be computed explicitly, and they tend to $U = (x, y)$ defined by $y(t) = 0$ and

$$x(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ \frac{t-1}{2} & \text{if } 1 \leq t \leq 3 \\ t-2 & \text{if } t \geq 3. \end{cases}$$

In this case we do not have convergence of the solutions. However, we can compute the Γ -limit of the sum $\mathcal{F}_\varepsilon + \mathcal{D}_\varepsilon$, whose domain is $\{y = 0\}$. Recovery sequences for $(x, 0)$ can be looked for of the form $(x, \varepsilon z)$. By minimizing in z we easily get that this Γ -limit is

$$\mathcal{G}(x) = \frac{1}{2}x^2 - tx + |x| + \psi(x),$$

where

$$\psi(x) = \min\left\{\frac{1}{2}(z-x)^2 + |z|\right\} = \min\left\{\frac{x^2}{2}, \frac{1}{2} + ||x| - 1|\right\},$$

whose derivative is

$$\psi'(x) = (x \wedge 1) \vee (-1).$$

It is easily seen that the function $x(t)$ above is the solution of

$$\partial|\dot{x}| + \mathcal{F}'_0(x) = \partial|\dot{x}| + x - t + \psi'(x) \ni 0,$$

where $\mathcal{F}_0(x) = \mathcal{G}(x) - |x| = \frac{1}{2}x^2 - tx + \psi(x)$. This energy \mathcal{F}_0 can then be regarded as the relaxed effective energy describing the limit behavior of the system.

3.3 Francfort and Marigo's Variational Theory of Fracture

A very interesting application of the theory outlined above is to variational models of Fracture following the formulation given by Griffith in the 1920s. In this case it is maybe clearer the definition via time-discrete motions (see Sect. 3.2.1) given as follows.

We consider the antiplane case where the variable u representing the displacement is scalar. By Ω we denote a bounded open subset of \mathbb{R}^n which will be the reference configuration of a linearly elastic material subject to brittle fracture as a consequence of a varying boundary condition $u = g(t)$ on $\partial\Omega$. K will be a closed set representing the crack location in the reference configuration. We consider the case $g(0) = 0$, and we set $K_0 = \emptyset$.

With fixed $\tau > 0$ we define $u_0^\tau = 0$, $K_0^\tau = K_0$ and u_k^τ , K_k^τ recursively as minimizers of the problem

$$\begin{aligned} \min \left\{ \int_{\Omega \setminus K} |\nabla u|^2 dx + \mathcal{H}^{n-1}(K \setminus K_{k-1}^\tau) : K_{k-1}^\tau \subset K = \overline{K} \subset \overline{\Omega}, \right. \\ \left. u \in H^1(\Omega \setminus K), u = g(t) \text{ on } \partial\Omega \setminus K \right\}, \end{aligned} \quad (3.41)$$

where \mathcal{H}^{n-1} denotes the $(n - 1)$ -dimensional Hausdorff measure. In this way K_k^τ is an increasing sequence of closed sets. Note that part of the crack may also lie on the boundary of Ω , in which case the boundary condition is satisfied only on $\partial\Omega \setminus K$.

In this formulation we have an elastic energy defined by

$$\mathcal{F}(t, u, K) = \begin{cases} \int_{\Omega \setminus K} |\nabla u|^2 dx & \text{if } u \in H^1(\Omega \setminus K) \text{ and } u = g(t) \text{ on } \partial\Omega \setminus K \\ +\infty & \text{otherwise,} \end{cases}$$

and a dissipation

$$\mathcal{D}(K) = \begin{cases} \mathcal{H}^{n-1}(K) & \text{if } K = \overline{K} \subset \overline{\Omega} \\ +\infty & \text{otherwise.} \end{cases}$$

The existence of minimizing pairs for (u, K) is not at all trivial. One way is by using the theory of *SBV functions*; i.e., functions of bounded variation u whose distributional derivative is a measure that can be written as a sum of a measure absolutely continuous with respect to the Lebesgue measure and a measure absolutely continuous with respect to the restriction of the $(n - 1)$ -dimensional Hausdorff measure to the complement of the Lebesgue points of u , the latter denoted by $S(u)$. For such functions the approximate gradient ∇u exists at almost all points. We can therefore define for all closed K the energy

$$\mathcal{E}_K(u) = \int_{\Omega \setminus K} |\nabla u|^2 dx + \mathcal{H}^{n-1}(S(u) \cap (\Omega \setminus K)). \quad (3.42)$$

Such energies are L^1 -lower semicontinuous and coercive, so that existence of weak solutions in $SBV(\Omega)$ are ensured from the direct methods of the Calculus of Variations. Regularity results give that $\mathcal{H}^{d-1}(\overline{S(u)} \setminus S(u)) = 0$ for minimizing u , so that to a minimizing $u \in SBV(\Omega \setminus K_{k-1}^\tau)$ of

$$\min \left\{ \mathcal{E}_{K_{k-1}^\tau}(u) : u \in SBV(\Omega \setminus K_{k-1}^\tau), u = g(t) \text{ on } \partial\Omega \setminus (S(u) \cup K_{k-1}^\tau) \right\} \quad (3.43)$$

corresponds a minimizing pair $K_k^\tau = K_{k-1}^\tau \cup \overline{S(u)}$ and $u_k^\tau = u|_{\Omega \setminus K_k^\tau} \in H^1(\Omega \setminus K_k^\tau)$ for (3.41).

The passage from a discrete trajectory u^τ to a continuous one u for all t letting $\tau \rightarrow 0$ is possible thanks to some monotonicity arguments. The delicate step is the proof that such u still satisfies the global stability property, which is ensured by a *transfer lemma* (the Francfort–Larsen transfer lemma), which allows to approximate a test function \hat{u} appearing in the limit stability estimate with a sequence \hat{u}_τ that can be used in the stability estimate holding for $u^\tau(t)$, which then carries to the limit.

Remark 3.7 (Existence of fractured solutions). Note that for large enough values of the boundary condition $g(t)$ we will always have a solution with $K^t \neq \emptyset$. Indeed, consider the case $g(t) = tg_0$ with $g_0 \neq 0$ on $\partial\Omega$. If $K^t = \emptyset$ then the corresponding u^t is a minimizer of

$$\begin{aligned} & \min \left\{ \int_{\Omega} |\nabla u|^2 dx : u = tg_0 \text{ on } \partial\Omega \right\} \\ & = t^2 \min \left\{ \int_{\Omega} |\nabla u|^2 dx : u = g_0 \text{ on } \partial\Omega \right\} =: t^2 C_0. \end{aligned}$$

On the other hand, we can use as test function $u = 0$ and as test set $K = \partial\Omega$ in (3.41), for which the total energy is $C_1 = \mathcal{H}^{n-1}(\partial\Omega)$. This shows that for $t^2 C_0 > C_1$ we cannot have $K = \emptyset$.

Remark 3.8 (The one-dimensional case). In the one-dimensional case, the functional \mathcal{E} reduces to the energy F obtained as a limit in Sect. 2.6 with the normalization $2J''(0) = J(\infty) = 1$, since $\mathcal{H}^0(K) = \#(K)$. Note that in this case the domain of \mathcal{E} reduces to piecewise- H^1 functions. If $\Omega = (0, 1)$ then the time-continuous solutions are of the form

$$(u^t(x), K^t) = \begin{cases} (g(t)x, \emptyset) & \text{for } t \leq t_c \\ \left(g(t)\chi_{(x_0,1)}(x), \{x_0\} \right) & \text{for } t > t_c, \end{cases}$$

or

$$(u^t(x), K^t) = \begin{cases} (g(t)x, \emptyset) & \text{for } t < t_c \\ \left(g(t)\chi_{(x_0,1)}(x), \{x_0\} \right) & \text{for } t \geq t_c, \end{cases}$$

where $x_0 \in [0, 1]$ and t_c is any value with $g(t_c) = 1$ and $g(s) \leq 1$ for $s < t_c$. This non-uniqueness is due to the fact that for $g(t) = 1$ we have two possible types of solutions $u(x) = x$ and $u(x) = \chi_{(x_0,1)}(x)$.

3.3.1 Homogenization of Fracture

The interpretation of fracture energies as functionals defined in *SBV* allows to consider the L^1 -convergence in *SBV* along sequences with equibounded energy (3.42). With respect to such a convergence we can consider stability issues for energies and dissipations related to the oscillating total energy

$$\mathcal{E}_\varepsilon(u) = \int_{\Omega \setminus K} a_b\left(\frac{x}{\varepsilon}\right) |\nabla u|^2 dx + \int_{S(u) \cap (\Omega \setminus K)} a_f\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^{n-1}$$

(here the coefficients a_b and a_f , where b stands for bulk and f for fracture, are periodic functions). In this case the limit of the total energies \mathcal{E}_ε is the sum of the energies obtained separately as limits of the energy and the dissipation parts (with respect to the same convergence), and has the form

$$\mathcal{E}_{\text{hom}}(u) = \int_{\Omega \setminus K} \langle A_{\text{hom}} \nabla u, \nabla u \rangle dx + \int_{S(u) \cap (\Omega \setminus K)} \varphi_{\text{hom}}(v) d\mathcal{H}^{n-1},$$

where ν denotes the measure-theoretical normal to $S(u)$. Note that the homogenized A_{hom} is the same given by the homogenization process in H^1 , while φ_{hom} is an effective fracture energies obtained by optimization on oscillating fractures, related to the homogenization of perimeter functionals. Thanks to this remark it is possible to show that the energetic solutions for \mathcal{E}_ε converge to energetic solutions of \mathcal{E}_{hom} . In terms of the construction of mutual recovery sequences this is possible since internal energy and dissipation can be optimized separately, contrary to what happens for the damage case, where both terms involve bulk integrals.

Appendix

Analyses of damage models linked to our presentation are contained in the work by Francfort and Marigo [5]. The higher-dimensional case is studied in a paper by Francfort and Garroni [3]. A threshold-based formulation is introduced by Garroni and Larsen [7]. The examples in Sect. 3.1.3 have been part of the course exam of B. Cassano and D. Sarrocco at Sapienza University in Rome.

An analysis of rate-independent processes is contained in the review article by Mielke [8]. The definitions given here can be traced back to the works by Mielke, Theil and Levitas [10] and [11]. The stability with respect to Γ -convergence is analyzed in the paper by Mielke, Roubiřek, and Stefanelli [9]. Most of Sect. 3.2 is taken from a lecture given by Ulisse Stefanelli during the course at the University of Pavia. The homogenization examples in Sect. 3.1, framed in the theory of energetic solutions, are contained in the paper [2].

An account of the variational theory of fracture (introduced in [6]) is contained in the book by Bourdin et al. [1]. The fundamental transfer lemma is contained in the seminal paper by Francfort and Larsen [4].

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Chapter 4

Local Minimization as a Selection Criterion

In this chapter we will leave the global minimization framework, and consider some issues linked to local minimization. By a *local minimizer* of F we mean a point u_0 such that there exists $\delta > 0$ such that

$$F(u_0) \leq F(u) \quad \text{if} \quad d(u, u_0) \leq \delta, \tag{4.1}$$

or, equivalently, is a minimum of the restriction of F to some ball centered in u_0 itself.

The Γ -limit F of a sequence F_ε is often interpreted as a simplified description of the energies F_ε , where unimportant details have been averaged out still keeping the relevant information about minimum problems. As far as global minimization problems are concerned this is ensured by the fundamental theorem of Γ -convergence, but this is in general false for local minimization problems. Nevertheless, if some information on the local minima is known, we may use the fidelity of the description of local minimizers as a way to ‘correct’ Γ -limits. In order to do that, we first introduce some notions of *equivalence by Γ -convergence*, and then show how to construct simpler equivalent theories as perturbations of the Γ -limit F in some relevant examples.

4.1 Equivalence by Γ -Convergence

Definition 4.1. Let (F_ε) and (G_ε) be sequences of functionals on a separable metric space X . We say that they are *equivalent by Γ -convergence* (or Γ -equivalent) if there exists a sequence (m_ε) of real numbers such that if $(F_{\varepsilon_j} - m_{\varepsilon_j})$ and $(G_{\varepsilon_j} - m_{\varepsilon_j})$ are Γ -converging sequences, then their Γ -limits coincide and are *proper* (i.e., not identically $+\infty$ and not taking the value $-\infty$).

Remark 4.1. (i) Since Γ -convergence is sequentially compact (i.e., every sequence has a Γ -converging subsequence), the condition in the definition is never

empty as long as the functional themselves are proper. On the set of proper lower-semicontinuous functionals the definition above is indeed an equivalence relation (in particular any sequence (F_ε) is equivalent to itself, even though it may not converge).

- (ii) Note that if F_ε Γ -converge to F and G_ε Γ -converge to G then equivalence amounts to $F = G$ and F proper, and (F_ε) is equivalent to the constant sequence F .
- (iii) The addition of the constants m_ε allows to consider and discriminate among diverging sequences (whose limit is otherwise not proper). For example, the sequences of constants $F_\varepsilon = 1/\varepsilon$ and $G_\varepsilon = 1/\varepsilon^2$ are not equivalent, even though they diverge to $+\infty$. Note instead that $F_\varepsilon(x) = x^2/\varepsilon$ and $G_\varepsilon(x) = x^2/\varepsilon^2$ are equivalent since their common limit is

$$F(x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

which is proper.

Definition 4.2 (Parameterized and uniform equivalence). For all $\lambda \in \Lambda$, let (F_ε^λ) and (G_ε^λ) be sequences of functionals on a separable metric space X . We say that they are equivalent by Γ -convergence if for all λ they are equivalent according to the definition above. If Λ is a metric space, we say that they are *uniformly Γ -equivalent* if there exist (m_ε^λ) such that

$$\Gamma\text{-}\lim_j (F_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}) = \Gamma\text{-}\lim_j (G_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j})$$

and are proper for all $\lambda_j \rightarrow \lambda$ and $\varepsilon_j \rightarrow 0$.

Remark 4.2. Suppose that F_ε^λ Γ -converge to F^λ and (F_ε^λ) and (F^λ) are uniformly Γ -equivalent as above, and that all functionals are equi-coercive and Λ is compact. Then we have

$$\sup_{\lambda \in \Lambda} |\inf F_\varepsilon^\lambda - \min F^\lambda| = o(1)$$

as $\varepsilon \rightarrow 0$, or, equivalently, that $f_\varepsilon(\lambda) = \inf F_\varepsilon^\lambda$ converges uniformly to $f(\lambda) = \min F^\lambda$ on Λ . This follows immediately from the fundamental theorem of Γ -convergence and the compactness of Λ .

Example 4.1. Take $\Lambda = [-1, 1]$, and

$$F_\varepsilon^\lambda(u) = \int_0^1 \left(\frac{W(u)}{\varepsilon} + \varepsilon |u'|^2 \right) dt, \quad \int_0^1 u dt = \lambda$$

with W as in Example 2.5. Then we have, for fixed λ , the Γ -limit

$$F^\lambda(u) = \begin{cases} 0 & \text{if } u(x) = \lambda \\ +\infty & \text{otherwise} \end{cases}$$

if $\lambda = \pm 1$ and

$$F^\lambda(u) = \begin{cases} c_W \#(S(u)) & \text{if } u \in BV((0, 1); \{\pm 1\}) \text{ and } \int_0^1 u \, dt = \lambda \\ +\infty & \text{otherwise} \end{cases}$$

if $|\lambda| \neq 1$. Note that $f_\varepsilon(\lambda) = \inf F_\varepsilon^\lambda$ is a continuous function, while

$$f(\lambda) = \min F^\lambda = \begin{cases} 0 & \text{if } |\lambda| = 1 \\ 1 & \text{otherwise} \end{cases}$$

is not continuous; hence, the convergence of f_ε to f is not uniform, which implies that (F_ε^λ) and (F^λ) are not uniformly Γ -equivalent.

4.2 A Selection Criterion

We will use the concept of equivalence introduced above to formalize a problem of the form: given F_ε find ‘simpler’ G_ε equivalent to F_ε , which capture the ‘relevant’ features of F_ε .

We will proceed as follows:

- Compute the Γ -limit F of F_ε . This suggests a limit domain and a class of energies (e.g., energies with sharp interfaces in place of diffuse ones, convex homogeneous functionals in place of oscillating integrals, etc.).
- If the description given by F is not ‘satisfactory,’ then ‘perturb’ F so as to obtain a family (G_ε) Γ -equivalent to (F_ε) , with G_ε in the limit class determined by the computation of the Γ -limit.

The same procedure may apply to parameterized families (F_ε^λ) .

Of course, the criteria of ‘relevance’ for the construction of G_ε as above may be of different types. In the following example we consider the parameterized family of Example 4.1, and the criterion of uniform equivalence.

Example 4.2. We consider the functionals F^λ in Example 4.1, which have been shown to be not uniformly equivalent to the sequence F_ε^λ . We wish to construct energies of the same form of F^λ ; i.e., with domain $u \in BV((0, 1); \{\pm 1\})$ with $\int_0^1 u \, dt = \lambda$, and uniformly Γ -equivalent to the sequence F_ε^λ . These energies must then depend on ε .

Suppose that $W \in C^2$. If we look for energies of the form

$$G_\varepsilon^\lambda(u) = \begin{cases} c_\varepsilon^\lambda \#(S(u)) & \text{if } u \in BV((0, 1); \{\pm 1\}) \text{ and } \int_0^1 u \, dx = \lambda \\ +\infty & \text{otherwise,} \end{cases}$$

then it is possible to show that the choice

$$c_\varepsilon^\lambda = \min \left\{ \frac{W(\lambda)}{\varepsilon}, c_W \right\}$$

gives G_ε^λ uniformly Γ -equivalent to F_ε^λ .

The form of c_ε^λ highlights that minimizers for F_ε^λ can either be close to a sharp interface (in which case their value is c_W), or close to the constant λ (which gives the energy value $W(\lambda)/\varepsilon$). When $\lambda = \pm 1 + O(\sqrt{\varepsilon})$ the second type of minimizers may have lower energy. Nevertheless since they converge to ± 1 this second type of minimizers are never detected by F^λ .

The choice of c_ε^λ is not unique, even within energies of the form prescribed; in fact we may also take the Taylor expansions of W at ± 1 in place of W

$$c_\varepsilon^\lambda = \min \left\{ \frac{W''(-1)}{2\varepsilon} (\lambda + 1)^2, \frac{W''(1)}{2\varepsilon} (\lambda - 1)^2, c_W \right\},$$

or any other function with the same Taylor expansion.

We may also take G_ε^λ belonging to a class of energies of a slightly more complex form, defined on piecewise-constant functions, setting

$$G_\varepsilon^\lambda(u) = \begin{cases} \int_0^1 \frac{W(u)}{\varepsilon} \, dx + c_W \#(S(u)) & \text{if } u \text{ piecewise constant and } \int_0^1 u \, dx = \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

This choice gives a better description of the minimizers of F_ε^λ .

4.3 A ‘Quantitative’ Example: Phase Transitions

In this section and the following the ‘unsatisfactory description’ of the original problems given by the Γ -limit, which we are trying to correct, will be a partial description of local minimizers. We will then try to perturb the Γ -limits as described above so as to satisfy this requirement.

We consider the same type of energies as in Examples 2.5 and 4.1

$$F_\varepsilon(u) = \int_0^1 \left(\frac{W(u)}{\varepsilon} + \varepsilon |u'|^2 \right) dx$$

with W a double-well potential with wells in ± 1 . For the sake of simplicity, in the present example the domain of F_ε is restricted to 1-periodic functions (i.e., u such that $u(1) = u(0)$). This constraint is compatible with the Γ -limit, which is then given by

$$F(u) = c_W \#(S(u) \cap [0, 1)) \quad \text{for } u \in BV((0, 1); \{\pm 1\}).$$

Again, u is extended to a periodic function, so that it may have a jump at 0 (and then by periodicity also at 1). By considering jumps in $[0, 1)$ such a jump is taken into account only once in the limit energy.

- Note that all functions in $BV((0, 1); \{\pm 1\})$ are L^1 -local minimizers (even though not isolated). This is a general fact when we have a lower-semicontinuous energy taking discrete values.
- We now show that, on the contrary, F_ε has no non-trivial L^1 -local minimizer. We consider the simplified case

$$W(u) = (|u| - 1)^2.$$

In this case $c_W = 2$. Suppose by contradiction that u is a local minimizer. If $u \geq 0$ (equivalently, $u \leq 0$) then

$$F_\varepsilon(u) = \int_0^1 \left(\frac{(u-1)^2}{\varepsilon} + \varepsilon |u'|^2 \right) dt.$$

Since this functional is convex, its only local minimizer is the global minimizer $u = 1$. Otherwise, we can suppose, up to a translation, that there exists $L \in (0, 1)$ such that $u(\pm L/2) = 0$ and $u(x) > 0$ for $|x| < L/2$. Again, using the convexity of

$$F_\varepsilon^L(u) = \int_{-L/2}^{L/2} \left(\frac{(u-1)^2}{\varepsilon} + \varepsilon |u'|^2 \right) dx$$

we conclude that u must be the global minimizer of F_L with zero boundary data; i.e., the solution of

$$\begin{cases} u'' = \frac{1}{\varepsilon^2}(u-1) \\ u(\pm \frac{L}{2}) = 0. \end{cases}$$

This gives

$$u(x) = 1 - \left(\cosh\left(\frac{L}{2\varepsilon}\right) \right)^{-1} \cosh\left(\frac{x}{\varepsilon}\right)$$

and

$$F_\varepsilon^L(u) = \frac{\sinh\left(\frac{L}{\varepsilon}\right)}{\left(\cosh\left(\frac{L}{2\varepsilon}\right)\right)^2}.$$

Note that

$$\frac{d^2}{dL^2} F_\varepsilon^L(u) = -\frac{1}{\varepsilon^2} \frac{\sinh\left(\frac{L}{2\varepsilon}\right)}{\left(\cosh\left(\frac{L}{2\varepsilon}\right)\right)^3},$$

i.e., this minimum value is a concave function of L . This immediately implies that no local minimizer may exist with changing sign; in fact, such a minimizer would be a local minimizer of the function

$$f(L_1, \dots, L_K) = \sum_{k=1}^K \frac{\sinh\left(\frac{L_k}{\varepsilon}\right)}{\left(\cosh\left(\frac{L_k}{2\varepsilon}\right)\right)^2}, \quad (4.2)$$

for some $K > 0$ under the constraint $L_k > 0$ and $\sum_k L_k = 1$, which is forbidden by the negative definiteness of its Hessian matrix. Note moreover that

$$F_\varepsilon^L(u) = 2 - 4e^{-\frac{L}{\varepsilon}} + O(e^{-\frac{2L}{\varepsilon}})$$

and that $-4e^{-\frac{L}{\varepsilon}}$ is still a concave function of L .

- We can now propose a ‘correction’ to F by considering in its place

$$G_\varepsilon(u) = c_W \#(S(u)) - \sum_{x \in S(u) \cap [0,1)} 4e^{-\frac{1}{\varepsilon}|x - \max(S(u) \cap (-\infty, x])|}$$

defined on periodic functions with $u \in BV((0, 1); \{\pm 1\})$. It is easily seen that G_ε Γ -converges to F , and, hence, is equivalent to F_ε ; thanks to the concavity of the second term, the same argument as above shows that we have no non-trivial local minimizers. As a side remark note that this approximation also maintains the stationary points of F_ε , which are functions with K jumps at distance $1/K$. This is easily seen after remarking that the distances between consecutive points must be a stationary point for (4.2). Moreover, the additional terms can also be computed as a development by Γ -convergence, which extends this equivalence to higher order.

Remark 4.3 (Interaction with the boundary). The case without periodicity conditions can be reduced to the case with periodic boundary conditions upon remarking that if \tilde{u} denotes the even extension of u to $(-1, 1)$ then

$$F_\varepsilon(u) = \frac{1}{2} \int_{-1}^1 \left(\frac{W(\tilde{u})}{\varepsilon} + \varepsilon |\tilde{u}'|^2 \right) dx$$

and \tilde{u} satisfies periodic boundary conditions. Then we can apply the arguments above to obtain that if $x_1 < \dots < x_n$ denote the location of points in $(0, 1)$ where u changes sign as in the computation above (where $L_k = x_k - x_{k-1}$), then we have the corresponding estimate

$$\begin{aligned} F_\varepsilon(u) &\geq \sum_{k=2}^n \frac{\sinh\left(\frac{x_k - x_{k-1}}{\varepsilon}\right)}{\left(\cosh\left(\frac{x_k - x_{k-1}}{2\varepsilon}\right)\right)^2} + \frac{\sinh\left(\frac{2x_1}{\varepsilon}\right)}{2\left(\cosh\left(\frac{2x_1}{2\varepsilon}\right)\right)^2} + \frac{\sinh\left(\frac{2-2x_n}{\varepsilon}\right)}{2\left(\cosh\left(\frac{2-2x_n}{2\varepsilon}\right)\right)^2} \\ &= \sum_{k=2}^n \frac{\sinh\left(\frac{x_k - x_{k-1}}{\varepsilon}\right)}{\left(\cosh\left(\frac{x_k - x_{k-1}}{2\varepsilon}\right)\right)^2} + \tanh\left(\frac{x_1}{\varepsilon}\right) + \tanh\left(\frac{1 - x_n}{\varepsilon}\right). \end{aligned}$$

The last two terms represent the interaction of the points x_0 and x_n with the boundary of $(0, 1)$. The ‘corrected’ Γ -converging sequence in this case can be written as

$$G_\varepsilon(u) = c_W \#(S(u)) - \sum_{k=2}^n 4e^{-\frac{1}{\varepsilon}|x_k - x_{k-1}|} - 2e^{-\frac{2}{\varepsilon}x_1} - 2e^{-2\varepsilon(1-x_n)}$$

if $S(u) = \{x_1, \dots, x_n\}$ with $x_k < x_{k-1}$.

4.4 A ‘Qualitative’ Example: Lennard-Jones Atomistic Systems

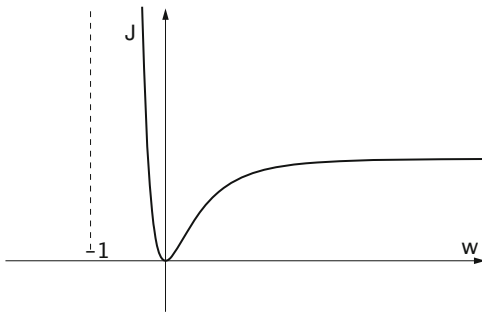
As in Example 2.6, we consider a scaled system of one-dimensional nearest-neighbour atomistic interactions through a Lennard-Jones type potential. Let J be a C^2 function as in Fig. 4.1, with domain $(-1, +\infty)$ (we set $J(w) = +\infty$ for $w \leq -1$), minimum in 0 with $J''(0) > 0$, convex in $(-1, w_0)$, concave in $(w_0, +\infty)$ and tending to $J(\infty) < +\infty$ at $+\infty$.

We consider the energy

$$F_\varepsilon^\lambda(u) = \sum_{i=1}^N J\left(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\right)$$

with the boundary conditions $u_0 = 0$ and $u_N = \lambda \geq 0$. Here $\varepsilon = 1/N$ with $N \in \mathbb{N}$. The vector (u_0, \dots, u_N) is identified with a discrete function defined on $\varepsilon\mathbb{Z} \cap [0, 1]$

Fig. 4.1 A (translation of a) Lennard-Jones potential J



or with its piecewise-affine interpolation. With this last identification, F_ε^λ can be viewed as functionals in $L^1(0, 1)$, and their Γ -limit computed with respect to that topology.

Taking into account the boundary conditions, we can extend all functions to $u(x) = 0$ for $x \leq 0$ and $u(x) = \lambda$ for $x \geq \lambda$, and denote by $S(u)$ (set of discontinuity points of u) the minimal set such that $u \in H^1((-s, 1 + s) \setminus S(u))$ for some $s > 0$. With this notation, the same arguments as in Example 2.6 give that the Γ -limit is defined on piecewise- $H^1(0, 1)$ functions by

$$F^\lambda(u) = 12J''(0) \int_0^1 |u'|^2 dt + J(\infty)\#(S(u) \cap [0, 1])$$

with the constraint that $u^+ > u^-$ on $S(u)$ and the boundary conditions $u^-(0) = 0$, $u^+(1) = \lambda$ (so that $S(u)$ is understood to contain also 0 or 1 if $u^+(0) > 0$ or $u^-(1) < \lambda$). For simplicity of notation we suppose

$$\frac{1}{2}J''(0) = J(\infty) = 1.$$

- *Local minimizers of F^λ .* If the average $z = \int_0^1 u' dt$ is given, by the strict convexity of $\int_0^1 |u'|^2 dt$ this part of the energy is minimized by the piecewise-constant gradient $u' = z$. From now on we tacitly assume that u' is constant; we then have two cases depending on the number of jumps:
 - (i) If $S(u) = \emptyset$ then $z = \lambda$, and this is a strict local minimizer since any L^1 perturbation with a jump of size w and (average) gradient z has energy $z^2 + 1$ independent of w , which is strictly larger than λ^2 if the perturbation is small.
 - (ii) If $\#S(u) \geq 1$ then L^1 local minimizers are all functions with $u' = 0$ (since otherwise we can strictly decrease the energy by taking a small perturbation v with the same set of discontinuity points and $v' = su'$ with $s < 1$).

The energy of the local minima in dependence of λ is pictured in Fig. 4.2.

Fig. 4.2 Energy of local minima for F^λ

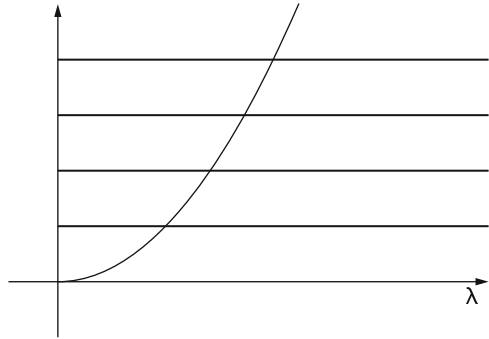
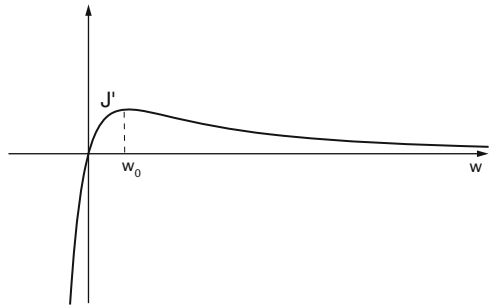


Fig. 4.3 Derivative of J



- *Local minimizers of F_ε^λ .* This is a finite-dimensional problem, whose stationarity condition is

$$J' \left(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}} \right) = \sigma \quad \text{for all } i,$$

for some $\sigma > 0$. The shape of J' is pictured in Fig. 4.3; its maximum is achieved for $w = w_0$. Note that for all $0 < \sigma < J'(w_0)$ we have two solutions of $J'(w) = \sigma$, while we have no solution for $\sigma > J'(w_0)$.

We have three cases:

- (i) We have

$$\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}} \leq w_0 \tag{4.3}$$

for all i . In this case the boundary condition gives

$$\frac{u_i - u_{i-1}}{\varepsilon} = \lambda$$

for all i , so that we have the constraint.

$$\lambda \leq \frac{w_0}{\sqrt{\varepsilon}}. \quad (4.4)$$

This solution is a local minimum. This is easily checked when $\lambda < \frac{w_0}{\sqrt{\varepsilon}}$ since small perturbations maintain condition (4.3). In the limit case $\lambda = \frac{w_0}{\sqrt{\varepsilon}}$ we may consider only perturbations where (4.3) is violated at exactly one index (see (ii) below), to which there corresponds an energy

$$J(w_0 + t) + (N - 1)J\left(w_0 - \frac{t}{N - 1}\right),$$

for $t \geq 0$, which has a local minimum at 0.

- (ii) Condition (4.3) is violated by two (or more) indices j and k . Let \bar{w} be such that

$$\frac{u_j - u_{j-1}}{\sqrt{\varepsilon}} = \frac{u_k - u_{k-1}}{\sqrt{\varepsilon}} = \bar{w} > w_0.$$

We may perturb $u_i - u_{i-1}$ only for $i = j, k$, so that the energy varies by

$$f(s) := J(\bar{w} + s) + J(\bar{w} - s) - 2J(\bar{w}). \quad (4.5)$$

We have $f'(0) = 0$ and $f''(0) = 2J''(\bar{w}) < 0$, which contradicts the minimality of u .

- (iii) Condition (4.3) is violated exactly by one index. The value of $w = \frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}$ for the $N - 1$ indices satisfying (4.3) is obtained by computing local minimizers of the energy on such functions, which is

$$f_\varepsilon^\lambda(w) := (N - 1)J(w) + J\left(\frac{\lambda}{\sqrt{\varepsilon}} - (N - 1)w\right)$$

defined for $0 \leq w \leq \min\left\{w_0, \frac{1}{N-1}\left(\frac{\lambda}{\sqrt{\varepsilon}} - w_0\right)\right\}$. We compute

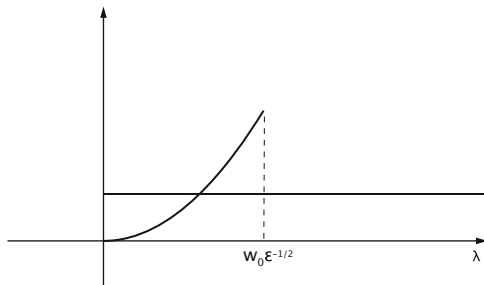
$$(f_\varepsilon^\lambda)'(w) := (N - 1)\left(J'(w) - J'\left(\frac{\lambda}{\sqrt{\varepsilon}} - (N - 1)w\right)\right).$$

Note that

$$f_\varepsilon^\lambda(0) = J\left(\frac{\lambda}{\sqrt{\varepsilon}}\right) = 1 - o(1)$$

and $(f_\varepsilon^\lambda)'(0) < 0$. If $\lambda > w_0/\sqrt{\varepsilon}$ then $(f_\varepsilon^\lambda)'(w) = 0$ has a unique solution, which is a local minimizer, while if $\lambda \leq w_0/\sqrt{\varepsilon}$ we have two solutions $w_1 < w_2$, of which the first one is a local minimizer. We then have a unique curve of local minimizers with one jump.

Fig. 4.4 Energy of local minima for F_ε^λ



The energy of the local minima in dependence of λ is schematically pictured in Fig. 4.4.

- *A qualitative comparison of local minimization.* First, the local minimizer for F_ε^λ which never exceeds the convexity threshold (corresponding to the minimizer with $S(u) = \emptyset$ for F^λ) exists only for $\lambda \leq w_0/\sqrt{\varepsilon}$; second, we only have one curve of local minimizers for F_ε^λ which exceed the convexity threshold for only one index (corresponding to the minimizers with $\#S(u) = 1$ for F^λ).
- *Γ -equivalent energies.* The class of energies suggested by the computation of the Γ -limits are those defined on piecewise- H^1 functions. We can further narrow their form by fixing the integral part, and look for energies defined on piecewise- H^1 functions of the form

$$G_\varepsilon^\lambda(u) = \int_0^1 |u'|^2 dt + \sum_{t \in S(u)} g\left(\frac{u^+ - u^-}{\sqrt{\varepsilon}}\right),$$

again with the constraint that $u^+ > u^-$ on $S(u)$ and the boundary conditions $u^-(0) = 0$, $u^+(1) = \lambda$. In order that local minimizers satisfy $\#(S(u)) \leq 1$ we require that $g : (0, +\infty) \rightarrow (0, +\infty)$ be strictly concave. In fact, with this condition the existence of two points in $S(u)$ is ruled out by noticing that given $w_1, w_2 > 0$ the function $t \mapsto g(w_1 + t) + g(w_2 - t)$ is concave. Moreover, we also require that g satisfy

$$\lim_{w \rightarrow +\infty} g(w) = 1.$$

With this condition it is easily seen that we have the Γ -convergence of G_ε^λ to F^λ .

In order to make a comparison with the local minimizers of F_ε^λ we first consider local minimizers with $S(u) = \emptyset$; i.e., $u(t) = \lambda t$. Such a function is a local minimizer if it is not energetically favourable to introduce a small jump of size w ; i.e., if 0 is a local minimizer for

$$g_\varepsilon^\lambda(w) := (\lambda - w)^2 + g\left(\frac{w}{\sqrt{\varepsilon}}\right),$$

where we have extended the definition of g by setting $g(0) = 0$. Note that if g is not continuous in 0 then 0 is a strict local minimizer for g_ε^λ for all λ . Otherwise, we can compute the derivative, and obtain that

$$\frac{d}{dw} g_\varepsilon^\lambda(0) = -2\lambda + \frac{1}{\sqrt{\varepsilon}} g'(0).$$

For ε small enough, 0 is a (isolated) local minimizer if and only if $\frac{d}{dw} g_\varepsilon^\lambda(0) > 0$; i.e.,

$$\lambda < \frac{1}{2\sqrt{\varepsilon}} g'(0).$$

If we choose

$$g'(0) = 2w_0$$

we obtain the desired constraint on this type of local minimizers. A possible simple choice of g is

$$g(w) = \frac{2w_0 w}{1 + 2w_0 w}.$$

We finally consider local minimizers with $\#(S(u)) = 1$. If w denotes the size of the jump then again computing the derivative of the energy, we conclude the existence of a single local minimizer w with

$$2(\lambda - w) = \frac{1}{\sqrt{\varepsilon}} g'\left(\frac{w}{\sqrt{\varepsilon}}\right),$$

and energy approaching 1 as $\varepsilon \rightarrow 0$.

- With the choice above, the pictures of the local minimizers for G_ε^λ and for F_ε^λ are of the same type, but may vary in quantitative details. We have not addressed the problem of the uniformity of this description, for which a refinement of the choice of g could be necessary.
- As a conclusion, we remark that this example has some modeling implications. The functional F^λ can be seen as a one-dimensional version of the energy of a brittle elastic medium according to Griffith's theory of Fracture ($S(u)$ represents the fracture site in the reference configuration), which is then interpreted as a continuum approximation of an atomistic model with Lennard Jones interactions. The requirement that also local minima may be reproduced by the limit theory has made us modify our functional F^λ obtaining another sequence of energies, which maintain an internal parameter ε . Energies of the form G_ε^λ are present in the literature, and are related to Barenblatt's theory of ductile fracture. Note that in all these considerations the parameter λ appears in the functionals only as a boundary condition, and does not influence the form of the energy.

4.5 A Negative Example: Oscillating Perimeters

The procedure described above cannot be always performed in a simple fashion. This may happen if the structure of the Γ -limit F cannot be easily modified to follow the pattern of the local minimizers of F_ε . We include an example where local minimizers of F_ε tend to be a dense set, while functionals with the structure of F have no local minimizers.

Example 4.3. We consider the function $a : \mathbb{Z}^2 \rightarrow \{1, 2\}$

$$a(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z} \text{ or } x_2 \in \mathbb{Z} \\ 2 & \text{otherwise,} \end{cases}$$

and the related scaled inhomogeneous-perimeter functionals

$$F_\varepsilon(A) = \int_{\partial A} a\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^1$$

defined on sets A with Lipschitz boundary. The energies F_ε Γ -converge, with respect to the convergence $A_\varepsilon \rightarrow A$, understood as the L^1 -convergence of the corresponding characteristic functions, to an energy of the form

$$F(A) = \int_{\partial A} g(\nu) d\mathcal{H}^1 \tag{4.6}$$

defined on all sets of finite perimeter (ν denotes the normal to ∂A). A direct computation shows that actually

$$g(\nu) = \|\nu\|_1 = |\nu_1| + |\nu_2|.$$

Furthermore, it is easily seen that the same F is equivalently the Γ -limit of

$$\tilde{F}_\varepsilon(A) = \mathcal{H}^1(\partial A),$$

defined on A which are the union of cubes $Q_{\varepsilon}^i := \varepsilon(i + (0, 1)^2)$ with $i \in \mathbb{Z}^2$. We denote by \mathcal{A}_ε the family of such A . Note that \tilde{F}_ε is the restriction of F_ε to \mathcal{A}_ε .

If $A \in \mathcal{A}_\varepsilon$ then A is trivially a L^1 -local minimizer for \tilde{F}_ε with $\delta < \varepsilon^2$, since any two distinct elements of \mathcal{A}_ε are at least at L^1 -distance ε^2 (the area of a single ε -square). It can be proved also that all $A \in \mathcal{A}_\varepsilon$ are L^1 -local minimizer for F_ε with $\delta = C\varepsilon^2$ for $C > 0$ sufficiently small.

Appendix

The notion of equivalence by Γ -convergence is introduced and analyzed in the paper by Braides and Truskinovsky [5].

Local minimizers for Lennard-Jones type potentials (also with external forces) are studied in the paper by Braides et al. [3]

More details on the derivation of fracture energies from interatomic potentials and the explanation of the $\sqrt{\varepsilon}$ -scaling can be found in the paper by Braides et al. [4] (see also the quoted paper by Braides and Truskinovsky for an explanation in terms of uniform Γ -equivalence).

For general reference on sets of finite perimeter and BV functions we refer to [1, 2, 6].

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Chapter 5

Convergence of Local Minimizers

In this chapter we consider a generalization of the fundamental theorem of Γ -convergence when we have strict local minimizers of the Γ -limit, in which case we are often able to deduce the existence and convergence of local minimizers of the converging sequence.

5.1 Convergence to Isolated Local Minimizers

The following theorem shows that we may extend (part of) the fundamental theorem of Γ -convergence to *isolated local minimizers* of the Γ -limit F ; i.e., (we suppose that F be defined on a metric space with distance d) to points u_0 such that there exists $\delta > 0$ such that

$$F(u_0) < F(u) \quad \text{if} \quad 0 < d(u, u_0) \leq \delta. \tag{5.1}$$

The proof of this theorem essentially consists in remarking that we may at the same time apply Proposition 2.1 (more precisely, Remark 2.2) to the closed ball of center u_0 and radius δ , and Proposition 2.2 to the open ball of center u_0 and radius δ .

Theorem 5.1. *Suppose that each F_ε is coercive and lower semicontinuous, and the sequence (F_ε) Γ -converges to F and is equicoercive. If u_0 is an isolated local minimizer of F then there exist a sequence (u_ε) converging to u_0 with u_ε a local minimizer of F_ε for ε small enough.*

Proof. Let $\delta > 0$ satisfy (5.1). Note that by the coerciveness and lower semicontinuity of F_ε there exists a minimizer u_ε of F_ε on $\overline{B_\delta(u_0)}$, the closure of $B_\delta(u_0) = \{u : d(u, u_0) \leq \delta\}$. By the equicoerciveness of (F_ε) , upon extracting a subsequence, we can suppose that $u_\varepsilon \rightarrow \bar{u}$. Since $\bar{u} \in \overline{B_\delta(u_0)}$ we then have

$$\begin{aligned}
F(u_0) &\leq F(\bar{u}) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \rightarrow 0} \min_{\overline{B_\delta(u_0)}} F_\varepsilon \\
&\leq \limsup_{\varepsilon \rightarrow 0} \inf_{B_\delta(u_0)} F_\varepsilon \leq \inf_{B_\delta(u_0)} F = F(u_0),
\end{aligned}$$

where we have used Proposition 2.2 in the last inequality. By (5.1) we have that $\bar{u} = u_0$ and $u_\varepsilon \in B_\delta(u_0)$ for ε small enough, which proves the thesis. \square

Remark 5.1. In the theorem above it is sufficient to require the coerciveness properties for F_ε only on bounded sets, since they are applied to minimization problems on $B_\delta(u_0)$.

Remark 5.2. Clearly, the existence of an isolated (local) minimizer in the limit does not imply that the converging (local) minimizers are isolated. It suffices to consider $F_\varepsilon(x) = ((x - \varepsilon) \vee 0)^2$ converging to $F(x) = x^2$.

Remark 5.3. In Sect. 4.4, we have noticed that the limit fracture energy F^λ possesses families of L^1 -local minimizers with an arbitrary number of jump points, while the approximating functionals F_ε^λ have local minimizers corresponding to limit functions with only one jump point. This cannot directly be deduced from the result above since those limit local minimizers are not isolated. Anyhow L^1 -local minimizers with one jump are strict local minimizers with respect to the distance

$$\begin{aligned}
d(u, v) &= \int_0^1 |u - v| dx + \sum_{x \in (0,1)} |(u^+ - u^-) - (v^+ - v^-)| \\
&= \int_0^1 |u - v| dx + \sum_{x \in S(u) \cap S(v)} |(u^+ - u^-) - (v^+ - v^-)| \\
&\quad + \sum_{x \in S(u) \setminus S(v)} |u^+ - u^-| + \sum_{x \in S(v) \setminus S(u)} |v^+ - v^-|,
\end{aligned}$$

which penalizes (large) jumps of a competitor v outside $S(u)$. Upon suitably defining interpolations of discrete functions in $SBV(0, 1)$ (where jumps correspond to difference quotients above the threshold $w_0/\sqrt{\varepsilon}$), it can be shown that the Γ -limit remains unchanged with this convergence, so that we may apply Theorem 5.1. Note that for discrete functions the notion of local minimizers is the same as for the L^1 -distance since we are in a finite-dimensional space. Note moreover that L^1 -local minimizers of F^λ with more than one jump are not strict local minimizers for the distance d above. Indeed, if $u' = 0$ and $S(u) = \{x_1, \dots, x_N\}$ with $0 \leq x_1 < \dots < x_N$ and $N \geq 2$, then any $u_s = u + s\chi_{(x_1, x_2)}$ is still a local minimizer for F^λ with $F^\lambda(u_s) = F^\lambda(u) = N$ and $d(u, u_s) = s(1 + |x_2 - x_1|)$.

5.2 Two Examples

We now use Theorem 5.1 to prove the existence of sequences of converging local minima in two examples.

Example 5.1 (Local minimizers for elliptic homogenization). Consider the functionals in Example 2.3. Suppose furthermore that \bar{g} has an isolated local minimum at z_0 . We will show that the constant function $u_0(x) = z_0$ is a L^1 -local minimizer of $F_{\text{hom}} + G$. Thanks to Theorem 5.1 we then deduce that there exists a sequence of local minimizers of $F_\varepsilon + G_\varepsilon$ (in particular, if g is differentiable with respect to u , of solutions of the Euler–Lagrange equation (2.15)) converging to u_0 .

We only prove the statement in the one-dimensional case, for which $\Omega = (0, L)$. We now consider $\delta > 0$ and u such that

$$\|u - u_0\|_{L^1(0,L)} \leq \delta.$$

Since z_0 is an isolated local minimum of \bar{g} there exists $h > 0$ such that $g(z_0) < g(z)$ if $0 < |z - z_0| \leq h$. If $\|u - u_0\|_\infty \leq h$ then $G(u) \geq G(u_0)$ with equality only if $u = u_0$ a.e., so that the thesis is verified. Suppose otherwise that there exists a set of positive measure A such that $|u - u_0| > h$ on A . We then have

$$h|A| \leq \int_A |u - u_0| dt \leq \delta,$$

so that $|A| \leq \delta/h$. We can then estimate

$$G(u) \geq \min \bar{g}|A| + (L - |A|)\bar{g}(z_0) \geq G(u_0) - \frac{\bar{g}(z_0) - \min \bar{g}}{h} \delta.$$

On the other hand, there exists a set of positive measure B such that

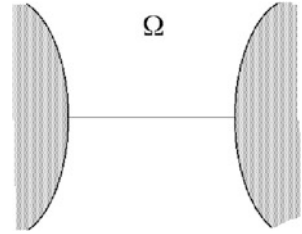
$$|u(x) - u_0| \leq \frac{\delta}{L}$$

for $x \in B$ (otherwise the L^1 estimate does not hold). Let $x_1 \in B$ and $x_2 \in A$; we can estimate (we can assume $x_1 < x_2$)

$$F_{\text{hom}}(u) \geq \alpha \int_{[x_1, x_2]} |u'|^2 dt \geq \alpha \frac{(u(x_2) - u(x_1))^2}{x_2 - x_1} \geq \alpha \frac{\left(h - \frac{\delta}{L}\right)^2}{L},$$

using Jensen's inequality. Summing up we have

Fig. 5.1 A neck in the open set Ω



$$\begin{aligned}
 F_{\text{hom}}(u) + G(u) &\geq F_{\text{hom}}(u_0) + G(u_0) + \alpha \frac{\left(h - \frac{\delta}{L}\right)^2}{L} - \frac{\bar{g}(z_0) - \min \bar{g}}{h} \delta \\
 &= F_{\text{hom}}(u_0) + G(u_0) + \alpha \frac{h^2}{L} + O(\delta) \\
 &> F_{\text{hom}}(u_0) + G(u_0)
 \end{aligned}$$

for δ small as desired.

Example 5.2 (Kohn–Sternberg). In order to prove the existence of L^1 local minimizers for the energies F_ε in (2.18), by Theorem 2.3 it suffices to prove the existence of isolated local minimizers for the minimal-interface problem related to the energy in (2.20). In order for this to hold we need some hypothesis on the set Ω (for example, it can be proved that no non-trivial local minimizer exists when Ω is convex).

We treat the two-dimensional case only. We suppose that Ω is bounded, regular, and has an ‘isolated neck’; i.e., it contains a straight segment whose endpoints meet $\partial\Omega$ perpendicularly, and $\partial\Omega$ is strictly concave at those endpoints (see Fig. 5.1).

We will show that the set with boundary that segment (we can suppose that the segment disconnects Ω) is an isolated local minimizer for the perimeter functional. We can set coordinates so that the segment is $(0, L) \times \{0\}$.

By the strict concavity of $\partial\Omega$, there exist $h > 0$ such that in a rectangular neighbourhood of the form $(a, b) \times (-2h, 2h)$ the lines $x = 0$ and $x = L$ meet $\partial\Omega$ only at $(0, 0)$ and $(L, 0)$, respectively. The candidate strict local minimizer is $A_0 = \{(x, y) \in \Omega : x > 0\}$, which we identify with the function $u_0 = -1 + 2\chi_{A_0}$, taking the value $+1$ in A_0 and -1 in $\Omega \setminus A_0$.

Take another test set A . The L^1 closeness condition for functions translates into

$$|A \Delta A_0| \leq \delta.$$

We may suppose that A is sufficiently regular (some minor extra care must be taken when A is just a set of finite perimeter, but the proof may be repeated essentially unchanged).

Fig. 5.2 Comparison with a uniformly close test set

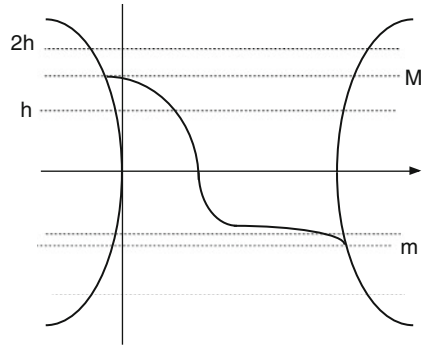
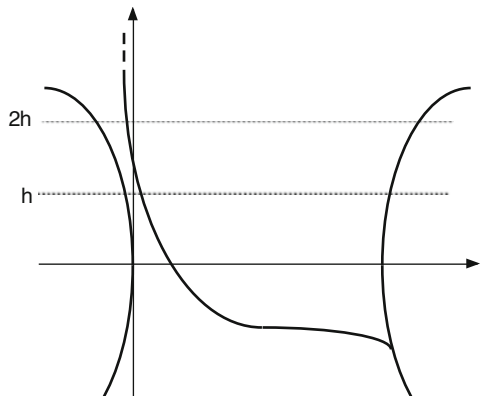


Fig. 5.3 Comparison with a L^1 -close test set



We first consider the case that A contains a horizontal segment $y = M$ with $M \in [h, 2h]$ and its complement contains a horizontal segment $y = m$ with $m \in [-2h, h]$. Then a portion of the boundary ∂A is contained in the part of Ω in the strip $|y| \leq 2h$, and its length is strictly greater than L , unless it is exactly the minimal segment (see Fig. 5.2).

If the condition above is not satisfied then we can suppose that the boundary of A does not contain, e.g., any horizontal segment $y = t$ with $t \in [h, 2h]$ (see Fig. 5.3). In particular, the length of the portion of ∂A contained with $h \leq y \leq 2h$ is not less than h . Consider now the one-dimensional set

$$B = \{t \in (0, L) : \partial A \cap (\{t\} \times (-h, h)) = \emptyset\}.$$

We have

$$\delta \geq |A \Delta A_0| \geq h|B|,$$

so that $|B| \leq \delta/h$, and the portion of ∂A with $h \leq y \leq 2h$ is not less than $L - \delta/h$. Summing up we have

$$\mathcal{H}^1(\partial A) \geq h + L - \frac{\delta}{h} = \mathcal{H}^1(\partial A_0) + h - \frac{\delta}{h},$$

and the desired strict inequality for δ small enough.

5.3 Generalizations

We can give some generalizations of Theorem 5.1 in terms of scaled energies.

Proposition 5.1. *Let F_ε satisfy the coerciveness and lower-semicontinuity assumptions of Theorem 5.1. Suppose furthermore that a bounded positive function $f : (0, +\infty) \rightarrow (0, +\infty)$ exists, together with constants m_ε , such that the scaled functionals*

$$\tilde{F}_\varepsilon(u) = \frac{F_\varepsilon(u) - m_\varepsilon}{f(\varepsilon)} \quad (5.2)$$

are equicoercive and Γ -converge on $\overline{B_\delta(u_0)}$ to \tilde{F}_0 given by

$$\tilde{F}_0(u) = \begin{cases} 0 & \text{if } u = u_0 \\ +\infty & \text{otherwise} \end{cases} \quad (5.3)$$

in $\overline{B_\delta(u_0)}$. Then there exists a sequence (u_ε) converging to u_0 of local minimizers of F_ε .

Remark 5.4. (i) First note that the functionals F_ε in Theorem 5.1 satisfy the hypotheses of the above proposition, taking, e.g., $f(\varepsilon) = \varepsilon$ and m_ε equal to the minimum of F_ε in $\overline{B_\delta(u_0)}$.

(ii) Note that the hypothesis above is satisfied if there exist constants m_ε such that:

$$(a) \quad \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} (F_\varepsilon(u_0) - m_\varepsilon) = 0.$$

$$(b) \quad \Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} (F_\varepsilon(u) - m_\varepsilon) > 0 \text{ on } \overline{B_\delta(u_0)} \setminus \{u_0\}.$$

Indeed condition (a) implies that we may change the constants m_ε so that the Γ -limit exists, is 0 at u_0 , and we have a recovery sequence with $F_\varepsilon(u_\varepsilon) = m_\varepsilon$, while (b) is kept unchanged. At this point it suffices to choose, e.g., $f(\varepsilon) = \varepsilon$.

Proof. The proof follows that of Theorem 5.1. Again, let u_ε be a minimizer of F_ε on $\overline{B_\delta(u_0)}$. We can suppose that $u_\varepsilon \rightarrow \bar{u} \in \overline{B_\delta(u_0)}$; we then have

$$\begin{aligned}
0 = \tilde{F}_0(u_0) &\leq \tilde{F}_0(\bar{u}) \leq \liminf_{\varepsilon \rightarrow 0} \tilde{F}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \rightarrow 0} \min_{\overline{B_\delta(u_0)}} \tilde{F}_\varepsilon \\
&\leq \limsup_{\varepsilon \rightarrow 0} \inf_{\overline{B_\delta(u_0)}} \tilde{F}_\varepsilon \leq \inf_{\overline{B_\delta(u_0)}} \tilde{F}_0 = 0,
\end{aligned} \tag{5.4}$$

so that $\bar{u} = u_0$ and $u_\varepsilon \in B_\delta(u_0)$ for ε small enough, which proves the thesis after remarking that (local) minimization of F_ε and \tilde{F}_ε are equivalent up to additive and multiplicative constants. \square

Proposition 5.2. *Let F_ε satisfy the coerciveness and lower-semicontinuity assumptions of Theorem 5.1. Suppose furthermore that there exist a bounded positive function $f : (0, +\infty) \rightarrow (0, +\infty)$, constants m_ε and ρ_ε with $\rho_\varepsilon > 0$ and $\rho_\varepsilon \rightarrow 0$, and $\tilde{u}_\varepsilon \rightarrow u_0$ such that the scaled functionals*

$$\tilde{F}_\varepsilon(v) = \frac{F_\varepsilon(\tilde{u}_\varepsilon + \rho_\varepsilon v) - m_\varepsilon}{f(\varepsilon)} \tag{5.5}$$

are equicoercive and Γ -converge on $\overline{B_\delta(v_0)}$ to \tilde{F}_0 , with v_0 an isolated local minimum. Then there exists a sequence (u_ε) converging to u_0 of local minimizers of F_ε .

Proof. We can apply Theorem 5.1 to the functionals $\tilde{F}_\varepsilon(v)$ concluding that there exist local minimizers v_ε of \tilde{F}_ε converging to v_0 . The corresponding $u_\varepsilon = \tilde{u}_\varepsilon + \rho_\varepsilon v_\varepsilon$ are local minimizers for F_ε converging to u_0 . \square

Example 5.3. We illustrate the proposition with the simple example

$$F_\varepsilon(x) = \sin\left(\frac{x}{\varepsilon}\right) + x,$$

whose Γ -limit $F(x) = x - 1$ has no local (or global) minimizers. Take any $x_0 \in \mathbb{R}$, $x_\varepsilon \rightarrow x_0$ any sequence with $\sin(x_\varepsilon/\varepsilon) = -1$, $m_\varepsilon = x_\varepsilon - 1$, $\rho_\varepsilon = \varepsilon^\beta$ with $\beta \geq 1$, and $f(\varepsilon) = \varepsilon^\alpha$ with $\alpha \geq 0$, so that

$$\begin{aligned}
\tilde{F}_\varepsilon(t) &= \frac{\sin\left(\frac{x_\varepsilon + \varepsilon^\beta t}{\varepsilon}\right) + 1}{\varepsilon^\alpha} + \varepsilon^{\beta-\alpha} t \\
&= \frac{\sin\left(\varepsilon^{\beta-1} t - \frac{\pi}{2}\right) + 1}{\varepsilon^\alpha} + \varepsilon^{\beta-\alpha} t = \frac{1 - \cos(\varepsilon^{\beta-1} t)}{\varepsilon^\alpha} + \varepsilon^{\beta-\alpha} t.
\end{aligned}$$

In this case the Γ -limit \tilde{F} coincides with the pointwise limit of \tilde{F}_ε . If $\beta = 1$ and $0 \leq \alpha \leq 1$ then we have (local) minimizers of \tilde{F} at all points of $2\pi\mathbb{Z}$; indeed if $\alpha = 0$ then the sequence converges to $\tilde{F}(x) = 1 - \cos x$, if $0 < \alpha < 1$ we have

$$\tilde{F}(x) = \begin{cases} 0 & \text{if } x \in 2\pi\mathbb{Z} \\ +\infty & \text{otherwise,} \end{cases}$$

and if $\alpha = 1$

$$\tilde{F}(x) = \begin{cases} x & \text{if } x \in 2\pi\mathbb{Z} \\ +\infty & \text{otherwise.} \end{cases}$$

In the case $2 > \beta > 1$ we have two possibilities: if $\alpha = 2\beta - 2$ then $\tilde{F}(x) = \frac{1}{2}t^2$; if $\beta \geq \alpha > 2\beta - 2$ then

$$\tilde{F}(x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

If $\alpha = \beta = 2$ then $\tilde{F}(x) = \frac{1}{2}t^2 + t$. In all these cases we have isolated local minimizers in the limit.

Note that in this computation x_ε are not themselves local minimizers of F_ε , but are chosen for computational convenience.

We now consider an infinite-dimensional example in the same spirit as the one above.

Example 5.4 (Existence of infinitely many local minima for oscillating metrics). Let the 1-periodic coefficient $a : \mathbb{R}^2 \rightarrow \{1, 2\}$ be defined on $[0, 1]^2$ as

$$a(v_1, v_2) = \begin{cases} 1 & \text{if } (v_1 - v_2)(v_1 + v_2 - 1) = 0 \\ 4 & \text{otherwise.} \end{cases} \quad (5.6)$$

Let

$$F_\varepsilon^0(u) = \int_0^1 a\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}\right)(1 + |u'|^2) dx$$

be defined on

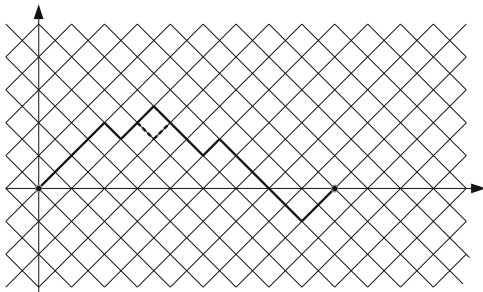
$$X = \{u \in W^{1,\infty}((0, 1); \mathbb{R}^2), u(0) = 0, u(1) = 1\}$$

equipped with the L^2 -convergence. It may be useful to remark that F_ε^0 can be rewritten in terms of the curve $\gamma(x) = (x, u(x))$ as the energy

$$\int_0^1 a\left(\frac{\gamma}{\varepsilon}\right) |\gamma'(x)|^2 dx,$$

of an inhomogeneous Riemannian metric which favors curves lying on the network where $a = 1$ (we will call that the 1-network), which is a sort of *opus reticulatum* as pictured in Fig. 5.4.

Fig. 5.4 A local minimizer and a competitor (*dashed line*)



The Γ -limit of F_ε^0 is of the form

$$F_{\text{hom}}^0(u) = \int_0^1 \varphi(u') \, dx,$$

with domain X . It can be shown that $\varphi(z) = \sqrt{2}$ if $|z| \leq 1$, and that for functions with $|u'| \leq 1$ recovery sequences for $F_{\text{hom}}^0(u)$ are functions with $a(x/\varepsilon, u_\varepsilon(x)/\varepsilon) = 1$ a.e. (i.e., that follow the lines of the 1-network). This will also follow from the computations below.

We consider the functionals

$$F_\varepsilon(u) = F_\varepsilon^0(u) + G(u), \quad \text{where} \quad G(u) = \int_0^1 |u|^2 \, dx$$

(perturbations more general than G can be added). Since G is a continuous perturbation, the Γ -limit of F_ε is simply $F = F_{\text{hom}}^0 + G$. Since G is strictly convex, then F is also strictly convex, and hence admits no local minimizers other than the absolute minimizer $u = 0$. We will show that F_ε admit infinitely many local minimizers. To that end we make some simplifying hypotheses: we suppose that ε are of the form 2^{-k} . In this way, both $(0, 0)$ and $(1, 0)$ (corresponding to the boundary conditions) belong to the 1-network for all ε , and 1-networks are decreasing (in the sense of inclusion) with ε . We consider any function $u_0 \in X$ such that $a(x2^{k_0}, u_0(x)2^{k_0}) = 1$ a.e. for some k_0 , and hence for all $k \geq k_0$; i.e., a function following the lines of the 1-network for all ε sufficiently small. We will prove that every such u_0 is a local minimum for F_ε if ε is small enough.

We consider the scaled functionals

$$\tilde{F}_\varepsilon(v) = \frac{F_\varepsilon(u_0 + \varepsilon^2 v) - F_\varepsilon(u_0)}{\varepsilon^2}.$$

We note that the term deriving from G still gives a continuously converging term, and can be dealt with separately, since

$$\frac{G(u_0 + \varepsilon^2 v) - G(u_0)}{\varepsilon^2} = 2 \int_0^1 u_0 v \, dx + \varepsilon^2 \int_0^1 |v|^2 \, dx.$$

We concentrate our analysis on the term of \tilde{F}_ε coming from F_ε^0 : let \tilde{v}_ε be such that

$$\|\tilde{v}_\varepsilon - u_0\|_{L^2} \leq \varepsilon^2 \delta; \quad (5.7)$$

i.e., that $\tilde{v}_\varepsilon = u_0 + \varepsilon^2 v_\varepsilon$ with $\|v_\varepsilon\|_{L^2} \leq \delta$, and $\tilde{F}_\varepsilon(\tilde{v}_\varepsilon) \leq C_1 < +\infty$.

We denote $\tilde{\gamma}_\varepsilon(x) = (x, \tilde{v}_\varepsilon(x))$ and $\gamma_0(x) = (x, u_0(x))$. Note that if we set $e^1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $e^2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ then $x \mapsto \langle \gamma_0(x), e^1 \rangle$ and $x \mapsto \langle \gamma_0(x), e^2 \rangle$ are both non decreasing. We may then suppose that the same holds for $\tilde{\gamma}_\varepsilon$. We also denote

$$\|z\|_1 = |\langle z, e^1 \rangle| + |\langle z, e^2 \rangle|.$$

For each ε fixed we consider points $0 = x_0 < x_1 < \dots < x_N = 1$ such that

$$a\left(\frac{\tilde{\gamma}_\varepsilon}{\varepsilon}\right) = 1 \quad \text{a.e. or} \quad a\left(\frac{\tilde{\gamma}_\varepsilon}{\varepsilon}\right) = 4 \quad \text{a.e. alternately on } [x_{k-1}, x_k];$$

we can suppose that the first case occurs for e.g. k odd and the second one for k even. In the first case, by convexity and taking into account that the image of $\tilde{\gamma}_\varepsilon$ is contained in the 1-network, we have

$$\int_{x_{k-1}}^{x_k} a\left(\frac{\tilde{\gamma}_\varepsilon}{\varepsilon}\right) |\tilde{\gamma}'_\varepsilon|^2 dx \geq (x_k - x_{k-1}) \left\| \frac{\tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}} \right\|_1^2$$

In the second case, again by convexity and by the inequality $\|z\|_1 \leq \sqrt{2}|z|$,

$$\begin{aligned} \int_{x_{k-1}}^{x_k} a\left(\frac{\tilde{\gamma}_\varepsilon}{\varepsilon}\right) |\tilde{\gamma}'_\varepsilon|^2 dx &\geq 4(x_k - x_{k-1}) \left| \frac{\tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}} \right|^2 \\ &\geq 2(x_k - x_{k-1}) \left\| \frac{\tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}} \right\|_1^2. \end{aligned}$$

As a first consequence, we deduce that

$$\begin{aligned} F_\varepsilon^0(\tilde{\gamma}_\varepsilon) &\geq \sum_{k=1}^N (x_k - x_{k-1}) \left\| \frac{\tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}} \right\|_1^2 \\ &\quad + \sum_{k \text{ even}} (x_k - x_{k-1}) \left\| \frac{\tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}} \right\|_1^2 \\ &\geq \left\| \sum_{k=1}^N \tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1}) \right\|_1^2 \\ &\quad + \sum_{k \text{ even}} \frac{1}{x_k - x_{k-1}} \|\tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1})\|_1^2 \end{aligned}$$

$$\begin{aligned} &= \|\tilde{\gamma}_\varepsilon(1) - \tilde{\gamma}_\varepsilon(0)\|_1^2 + \sum_{k \text{ even}} \frac{1}{x_k - x_{k-1}} \|\tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1})\|_1^2 \\ &= F_\varepsilon^0(u_0) + \sum_{k \text{ even}} \frac{1}{x_k - x_{k-1}} \|\tilde{\gamma}_\varepsilon(x_k) - \tilde{\gamma}_\varepsilon(x_{k-1})\|_1^2. \end{aligned}$$

From the energy bound we then deduce that for each k even

$$(x_k - x_{k-1}) \left(1 + \left(\frac{\tilde{v}_\varepsilon(x_k) - \tilde{v}_\varepsilon(x_{k-1})}{x_k - x_{k-1}} \right) \right) \leq C\varepsilon^2.$$

so that both $(x_k - x_{k-1}) \leq C\varepsilon^2$ and $|\tilde{v}_\varepsilon(x_k) - \tilde{v}_\varepsilon(x_{k-1})| \leq C\varepsilon$. This implies that $\tilde{\gamma}_\varepsilon$ can be deformed with a perturbation with $o(\varepsilon^2)$ L^2 -norm to follow the 1-network between x_{k-1} and x_k . Hence, possible competitors essentially follow the 1-network (see Fig. 5.4).

If δ is small enough then in order that (5.7) hold we must have $v_\varepsilon \rightarrow 0$. This shows that the limit of \tilde{F}_ε is finite only at $v = 0$ on $B_\delta(0)$ as desired.

As a consequence of the computation above we deduce that for all $u \in X$ with $\|u'\|_\infty \leq 1$ we have a sequence $\{u_\varepsilon\}$ of local minimizers of F_ε converging to u .

Example 5.5 (Density of local minima for oscillating distances). We may consider a similar example to the one above for oscillating distances; i.e., length functionals defined on curves. Let the 1-periodic coefficient $a : \mathbb{R}^2 \rightarrow \{1, 2\}$ be defined as

$$a(v_1, v_2) = \begin{cases} 1 & \text{if either } v_1 \text{ or } v_2 \in \mathbb{Z} \\ 4 & \text{otherwise.} \end{cases} \tag{5.8}$$

This is the same type of coefficient as in the previous example up to a rotation and a scaling factor. Let

$$F_\varepsilon(u) = \int_0^1 a\left(\frac{u}{\varepsilon}\right) |u'| dx$$

be defined on

$$X = \{u \in W^{1,1}((0, 1); \mathbb{R}^2), u(0) = v_0, u(1) = v_1\}$$

equipped with the L^1 -convergence.

The Γ -limit of F_ε is

$$F(u) = \int_0^1 \|u'\|_1 dx,$$

where

$$\|z\|_1 = |z_1| + |z_2|.$$

This is easily checked after remarking that recovery sequences (u_ε) are such that $a(u_\varepsilon(t)/\varepsilon) = 1$ a.e. (except possibly close to 0 and 1 if $a(v_0/\varepsilon) \neq 1$ or $a(v_1/\varepsilon) \neq 1$) and then that $|u'_\varepsilon| = |(u_\varepsilon)'_1| + |(u_\varepsilon)'_2|$. For example, if both components of (u_ε) are monotone, then

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_0^1 a\left(\frac{u_\varepsilon}{\varepsilon}\right) |u'_\varepsilon| dx = \int_0^1 |u'_\varepsilon| dx + o(1) \\ &= \int_0^1 (|(u_\varepsilon)'_1| + |(u_\varepsilon)'_2|) dx + o(1) \\ &= \left| (v_1)_1 - (v_0)_1 \right| + \left| (v_1)_2 - (v_0)_2 \right| + o(1) \\ &= \int_0^1 (|u'_1| + |u'_2|) dx + o(1) = F(u) + o(1). \end{aligned}$$

For all these energies there are no strict local minimizers since energies are invariant with respect to reparameterization. Anyhow, if we consider equivalence classes with respect to reparameterization (e.g., by taking only functions in

$$X^1 = \{u \in X : \|u'\|_1 \text{ constant a.e.}\},$$

then an argument similar to the one in the previous example shows that local minimizers are L^1 dense, in the sense that for all $u \in X^1$ there exists a sequence of local minimizers of F_ε (restricted to X^1) converging to u .

As a technical remark, we note that in order to have coercivity the limit F should be extended to the space of curves with bounded variations. Anyhow, since functionals are invariant by reparameterization, it suffices to consider bounded sequences in $W^{1,\infty}$ after a change of variables.

Appendix

The use of Theorem 5.1 for proving the existence of local minimizers, together with Example 5.2 are due to Kohn and Sternberg [1].

Reference

1. Kohn, R.V., Sternberg, P.: Local minimizers and singular perturbations. Proc. Roy. Soc. Edinburgh A **111**, 69–84 (1989)

Chapter 6

Small-Scale Stability

In the previous chapter we have shown how it is possible to deduce the existence and convergence of local minimizers of F_ε from the existence of an isolated local minimizer of their Γ -limit F . On the contrary, the knowledge of the existence of local minimizers of F_ε is not sufficient to deduce the existence of local minimizers for F .

We now examine a notion of *stability* such that, loosely speaking, a point is *stable* if it is not possible to reach a lower-energy state from that point without crossing an energy barrier of a specified height. This notion is a quantification of the notion of local minimizer, which instead is ‘scale-independent’; i.e., it does not depend on the energy barrier that characterizes a local minimizer; e.g., it does not discriminate between energies

$$F_\varepsilon(x) = x^2 + \sin^2\left(\frac{x}{\varepsilon}\right) \quad \text{or} \quad F_\varepsilon(x) = x^2 + \sqrt{\varepsilon} \sin^2\left(\frac{x}{\varepsilon}\right).$$

In this case the local minimizers in the first of the two sequence of energies are stable as $\varepsilon \rightarrow 0$, while those in the second sequence are not.

This notion will allow us to state a convergence theorem for sequences of stable points.

6.1 Larsen’s Stable Points

We first introduce a notion of stability that often can be related to notions of local minimality. Note that in the literature the parameter ε is used in the place of δ (so that we have ε -slides, ε -stable points, etc.) while here the parameter ε is already used to label energies.

Definition 6.1 (Slide). Let $F : X \rightarrow [0, +\infty]$ and $\delta > 0$. A continuous function $\phi : [0, 1] \rightarrow X$ is a δ -slide for F at $u_0 \in X$ if

- $\phi(0) = u_0$ and $F(\phi(1)) < F(\phi(0)) = F(u_0)$.
- There exists $\delta' < \delta$ such that $F(\phi(t)) \leq F(\phi(s)) + \delta'$ if $0 \leq s \leq t \leq 1$.

Definition 6.2 (Stability). Let $F : X \rightarrow [0, +\infty]$ and $\delta > 0$. A point $u_0 \in X$ is δ -stable for F if no δ -slide exists for F at u_0 .

A point $u_0 \in X$ is *stable* for F if it is δ -stable for some $\delta > 0$ (and hence for all δ small enough).

Let $F_\varepsilon : X \rightarrow [0, +\infty]$. A sequence of points (u_ε) in X is *uniformly stable* for (F_ε) if there exists $\delta > 0$ such that all u_ε are δ -stable for ε small.

Example 6.1. (1) $F(x) = \begin{cases} 0 & x = 0 \\ \sin\left(\frac{1}{x}\right) & \text{otherwise.} \end{cases}$ The point 0 is not a local minimizer but it is δ -stable for $\delta < 1$.

(2) Similarly for $F(x) = \begin{cases} 0 & x = 0 \\ -x^2 + \sin^2\left(\frac{1}{x}\right) & \text{otherwise.} \end{cases}$

(3) Let $X = \mathbb{C}$ and $F(z) = F(\rho e^{i\theta}) = \begin{cases} \theta\rho & |z| \leq 1 \\ -1 & \text{otherwise,} \end{cases}$

where we have chosen the representation $z = \rho e^{i\theta}$ with $0 < \theta \leq 2\pi$. Then 0 is an isolated local minimum, but it is not stable; e.g., taking $\phi(t) = 2te^{i\delta/2}$. Note in fact that $\phi(0) = 0$, $F(\phi(1)) = -1 < 0$, and $\sup F(\phi(t)) = F(\phi(1/2)) = \delta/2$.

(4) We can generalize point (3) above to an infinite-dimensional example. Take $X = L^2(-\pi, \pi)$ and

$$F(u) = \begin{cases} \sum_k \frac{1}{k^2} |c_k|^2 & \text{if } u = \sum_k c_k e^{ikx} \text{ and } \|u\|_{L^2} < 1 \\ -1 & \text{otherwise.} \end{cases}$$

The constant 0 is an isolated minimum point. F is lower semicontinuous, and continuous in $\{\|u\|_{L^2} < 1\}$. Note that $F(e^{ikx}) = \frac{1}{k^2}$ so that $\phi_k(t) = 2te^{ikx}$ is a δ -slide for $k^2 > 1/\delta$.

- (5) $F_\varepsilon(x) = x^2 + \sin^2\left(\frac{x}{\varepsilon}\right)$. Each bounded sequence of local minimizers is uniformly stable.
- (6) $F_\varepsilon(x) = x^2 + \varepsilon^\alpha \sin^2\left(\frac{x}{\varepsilon}\right)$ with $0 < \alpha < 1$. No bounded sequence of local minimizers is uniformly stable (except the constant sequence of global minimizers $x_\varepsilon = 0$).

Remark 6.1 (Local minimality and stability).

- If $F : X \rightarrow \mathbb{R}$ is a continuous function and u is a stable point for F , then u is a local minimizer of F .
- Let F be a lower-semicontinuous and coercive function. Then every isolated local minimizer of F is stable.
- If u is just a local minimizer for F then u may not be stable for F .

To check (i) suppose that u is not a local minimum for F . Then let ρ be such that $|F(u) - F(w)| < \delta/2$ if $w \in B_\rho(u)$, and let $u_\rho \in B_\rho(u)$ be such that $F(u_\rho) < F(u)$. Then it suffices to take $\phi(t) = u + t(u_\rho - u)$.

To check (ii), let $\eta > 0$ be such that u_0 is an isolated minimum point for F in $B_\eta(u_0)$. If u_0 is not stable then there exist $1/k$ slides ϕ_k with final point outside $B_\eta(u_0)$. This implies that there exist $u_k = \phi_k(t_k)$ for some t_k with $u_k \in \partial B_\eta(u_0)$, so that $F(u_k) \leq F(u_0) + 1/k$. By coerciveness, upon extraction of a subsequence $u_\varepsilon \rightarrow \bar{u} \in \partial B_\eta(u_0)$, and by lower semicontinuity $F(\bar{u}) \leq \liminf_k F(u_k) \leq F(u_0)$, which is a contradiction.

For (iii) take for example $u = 0$ for $F(u) = (1 - |u|) \wedge 0$ on \mathbb{R} .

6.2 Stable Sequences of Functionals

We now give a notion of stability of parameterized functionals.

Definition 6.3 (Relative (sub)stability). We say that a sequence (F_ε) is *stable* (or *sub-stable*) *relative to* F if the following holds:

- If u_0 has a δ -slide for F and $u_\varepsilon \rightarrow u_0$, then each u_ε has a δ -slide for F_ε (for ε small enough).

Remark 6.2 (Relative (super)stability). The condition of sub-stability above can be considered as an analogue of the lower bound for Γ -convergence. With this analogy in mind we can introduce a notion of *(super)stability relative to* F by requiring that:

- If u_0 is an isolated local minimum for F then there exists $u_\varepsilon \rightarrow u_0$ such that (u_ε) is uniformly stable for F_ε .

This notion gives an analogue of the upper bound for Γ -convergence. However, we will not explore this notion further, since we already have a satisfactory sort of upper bound for local minimizers from Theorem 5.1.

Remark 6.3. (i) Note that if F is a constant then all (F_ε) are stable relative to F . (ii) In general if $F_\varepsilon = F$ for all ε then (F_ε) may not be stable relative to F . Take for example

$$F_\varepsilon(x) = F(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ x & \text{if } x \leq 0; \end{cases}$$

then $x_0 = 0$ has δ -slides for all $\delta > 0$, while taking $x_\varepsilon = (2\pi \lfloor \frac{1}{\varepsilon} \rfloor - \frac{\pi}{2})^{-1}$ we have $x_\varepsilon \rightarrow x_0$ and x_ε has no δ -slide for $\delta < 2$.

The following proposition is in a sense the converse of Theorem 5.1, with Γ -convergence substituted with stability.

Proposition 6.1. *Let (F_ε) be (sub)stable relative to F and u_ε be a sequence of uniformly stable points for F_ε with $u_\varepsilon \rightarrow u$. Then u is stable for F .*

Proof. If $u_\varepsilon \rightarrow u$ and u_ε is uniformly stable, then it is stable for some $\delta > 0$. By the (sub)stability of (F_ε) then u is δ' stable for all $0 < \delta' < \delta$; i.e., it is stable. \square

Remark 6.4. The main drawback of the notion of stability of energies is that it is not in general compatible with the addition of (continuous) perturbations. Take for example $F_\varepsilon(x) = \sin^2\left(\frac{x}{\varepsilon}\right)$ and $F = 0$. Then F_ε is stable relative to F , but $G_\varepsilon(x) = F_\varepsilon(x) + x$ is not stable with respect to $G(x) = x$. In fact, each x has a δ -slide for all $\delta > 0$, but if $x_\varepsilon \rightarrow x$ is a sequence of local minimizers of G_ε then they are δ -stable for $\delta < 1$.

6.3 Stability and Γ -Convergence

In this section we will couple stability with Γ -convergence, and try to derive some criteria in order to guarantee the compatibility with respect to the addition of continuous perturbations. The main issue is to construct δ -slides for the approximating functionals starting from δ -slides for the Γ -limit.

Example 6.2. We consider the one-dimensional energies

$$F_\varepsilon(u) = \int_0^1 a\left(\frac{x}{\varepsilon}\right) |u'|^2 dx,$$

where a is a 1-periodic function with $0 < \inf a < \sup a < +\infty$, so that F_ε Γ -converge to the Dirichlet integral

$$F(u) = \underline{a} \int_0^1 |u'|^2 dx.$$

We will also consider a perturbation of F_ε with

$$G(u) = \int_0^1 g(x, u) dx,$$

where g is a Carathéodory function with $|g(x, u)| \leq C(1 + |u|^2)$ (this guarantees that G is L^2 -continuous).

We want to check that $F_\varepsilon + G$ is stable relative to $F + G$. To this end consider a point u_0 such that a δ -slide ϕ for $F + G$ exists at u_0 , and points $u_\varepsilon \rightarrow u_0$. We wish to construct a δ -slide for $F_\varepsilon + G$ at u_ε .

With fixed $K \in \mathbb{N}$ we consider points $x_i^K = i/K$ for $i = 0, \dots, K$ and denote for every t with $\phi^K(t)$ the piecewise affine interpolation of $\phi(t)$ on the points x_i^K . Note that we have:

- For all K we have $F(\phi^K(t)) \leq F(\phi(t))$ by Jensen's inequality.
- $F(\phi^K(t)) \rightarrow F(\phi(t))$ as $K \rightarrow +\infty$.
- For fixed K the map $t \mapsto \phi^K(t)$ is continuous with respect to the *strong* H^1 -convergence. Indeed its gradient is piecewise constant and is weakly continuous in t , hence it is strongly continuous.

We fix $\delta' < \delta$ such that

$$F(\phi(t)) + G(\phi(t)) \leq F(\phi(s)) + G(\phi(s)) + \delta' \text{ if } 0 \leq s \leq t \leq 1,$$

choose $\delta'' > 0$ such that $\delta' + 2\delta'' < \delta$ and

$$F(\phi(1)) + G(\phi(1)) < F(u_0) + G(u_0) - 2\delta'.$$

Let K be large enough so that (if $u_0^K = \phi^K(0)$ denotes the interpolation of u_0)

$$F(u_0^K) + G(u_0^K) \geq F(u_0) + G(u_0) - \delta''$$

and

$$|G(\phi^K(t)) - G(\phi(t))| < \delta''$$

for all t . We then have

$$F(\phi^K(t)) + G(\phi^K(t)) \leq F(\phi(t)) + G(\phi(t)) + \delta''.$$

We then claim that, up to a reparameterization, ϕ^K is a δ -slide for $F + G$ from u_0^K .

Indeed, let $M = \inf\{t : F(\phi^K(t)) + G(\phi^K(t)) < F(u_0^K) + G(u_0^K)\}$. This set is not empty since it contains the point 1. If $0 \leq s \leq t \leq M$ then we have

$$\begin{aligned} & F(\phi^K(t)) + G(\phi^K(t)) - F(\phi^K(s)) + G(\phi^K(s)) \\ & \leq \sup\{F(\phi^K(r)) + G(\phi^K(r)) : 0 \leq r \leq M\} - F(u_0^K) + G(u_0^K) \\ & \leq \sup\{F(\phi(r)) + G(\phi(r)) : 0 \leq r \leq M\} + 2\delta'' - F(u_0) + G(u_0) \\ & \leq \delta' + 2\delta'' < \delta \end{aligned}$$

By the continuity of $t \mapsto F(\phi^K(t)) + G(\phi^K(t))$ we can then find $\bar{t} > M$ such that $F(\phi^K(\bar{t})) + G(\phi^K(\bar{t})) < F(u_0^K) + G(u_0^K)$ and $s \mapsto \bar{\phi}^K(s\bar{t})$ is a δ -slide. For the following, we suppose that $\bar{t} = 1$, so that we do not need any reparameterization.

Next, we construct a δ -slide for $F_\varepsilon + G$. To this end, for the sake of simplicity, we assume that $N = \frac{1}{\varepsilon K} \in \mathbb{N}$. Let v be a function in $H_0^1(0, 1)$ such that

$$\int_0^1 a(y)|v' + 1|^2 dy = \min\left\{\int_0^1 a(y)|w' + 1|^2 dy : w \in H_0^1(0, 1)\right\} = \underline{a}.$$

Note that we also have

$$\int_0^N a(y)|v' + 1|^2 dy = \min \left\{ \int_0^N a(y)|w' + 1|^2 dy : w \in H_0^1(0, 1) \right\} = N\underline{a}.$$

We then define the function $\phi_\varepsilon^K(t)$ by setting on $[x_i^K, x_{i+1}^K]$

$$\phi_\varepsilon^K(t)(x_i^K + s) = \phi(t)(x_i^K) + K(\phi(t)(x_{i+1}^K) - \phi(t)(x_i^K)) \left(s + \varepsilon v \left(\frac{s}{\varepsilon} \right) \right)$$

for $0 \leq s \leq \frac{1}{K}$, so that

$$F_\varepsilon(\phi_\varepsilon^K(t)) = F(\phi^K(t)).$$

Note again that we may suppose ε small enough so that $|G(\phi_\varepsilon^K(t)) - G(\phi^K(t))| = o(1)$ uniformly in t so that ϕ_ε^K is a δ -slide for $F_\varepsilon + G$ at $\phi_\varepsilon^K(0)$.

It now remains to construct a L^2 -continuous function $\psi_\varepsilon : [0, 1] \rightarrow H^1(0, 1)$ with $\psi_\varepsilon(0) = u_\varepsilon$ and $\psi_\varepsilon(1) = \phi_\varepsilon^K(0)$ such that concatenating ψ_ε with ϕ_ε^K we have a δ -slide. This is achieved by taking the affine interpolation (in t) of u_ε and u_ε^K defined by setting on $[x_i^K, x_{i+1}^K]$

$$u_\varepsilon^K(x_i^K + s) = u_\varepsilon(x_i^K) + K(u_\varepsilon(x_{i+1}^K) - u_\varepsilon(x_i^K)) \left(s + \varepsilon v \left(\frac{s}{\varepsilon} \right) \right), \quad 0 \leq s \leq \frac{1}{K},$$

on $(0, 1/2)$ and of u_ε^K and $\phi_\varepsilon^K(0)$ on $(1/2, 1)$.

Example 6.3. We consider the oscillating perimeter functionals F_ε and F of Example 4.3. We now show that if A has a δ -slide for F and $A_\varepsilon \rightarrow A$, then each A_ε has a $(\delta + o(1))$ -slide for F_ε (and so a δ -slide for ε sufficiently small). It is easily checked that the same argument can be used if we add to F_ε a continuous perturbation

$$G(A) = \int_A f(x) dx,$$

where f is a (smooth) bounded function, so that the stability can be used also for $F_\varepsilon + G$.

We first observe that an arbitrary sequence A_ε of Lipschitz sets converging to a set A can be substituted by a sequence in \mathcal{A}_ε with the same limit. To check this, consider a connected component of ∂A_ε . Note that for ε small enough every portion of ∂A_ε parameterized by a curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ such that $a(\gamma(0)/\varepsilon) = a(\gamma(1)/\varepsilon) = 1$ and $a(\gamma(t)/\varepsilon) = 2$ for $0 < t < 1$ can be deformed continuously to a curve lying on $\varepsilon a^{-1}(1)$ and with the same endpoints. If otherwise a portion of ∂A_ε lies completely inside a cube Q_i^ε it can be shrunk to a point or expanded to the whole cube Q_i^ε . In both cases this process can be obtained by a $O(\varepsilon)$ -slide, since either the lengths

of the curves are bounded by 2ε , or the deformation can be performed so that the lengths are decreasing.

We can therefore assume that $A_\varepsilon \in \mathcal{A}_\varepsilon$ and that there exist a δ -slide for E at A obtained by a continuous family $A(t)$ with $0 \leq t \leq 1$.

We fix $N \in \mathbb{N}$ and set $t_j^N = j/N$. For all $j \in \{1, \dots, N\}$ let $A_\varepsilon^{N,j}$ be a recovery sequence in \mathcal{A}_ε for $A(t_j^N)$. Furthermore we set $A_n^{N,0} = A_\varepsilon$. Note that, since $A_\varepsilon^{N,j} \rightarrow A(t_j^N)$ and $A(t)$ is continuous, we have $|A_\varepsilon^{N,j} \Delta A_\varepsilon^{N,j+1}| = o(1)$ as $N \rightarrow +\infty$. We may suppose that the set $A_\varepsilon^{N,j+1}$ is the union of $A_\varepsilon^{N,j}$ and a family of cubes $Q_i^{N,j}$. We may order the indices i and construct a continuous family of sets $A^{N,j,i}(t)$ such that $A^{N,j,i}(0) = A_\varepsilon^{N,j} \cup \bigcup_{k < i} Q_k^{N,j}$, $A^{N,j,i}(1) = A_\varepsilon^{N,j} \cup \bigcup_{k \leq i} Q_k^{N,j}$, and

$$\begin{aligned} \left(\mathcal{H}^1(A_\varepsilon^{N,j}) \wedge \mathcal{H}^1(A_\varepsilon^{N,j+1}) \right) - C\varepsilon &\leq \mathcal{H}^1(A^{N,j,i}(t)) \\ &\leq \left(\mathcal{H}^1(A_\varepsilon^{N,j}) \vee \mathcal{H}^1(A_\varepsilon^{N,j+1}) \right) + C\varepsilon. \end{aligned}$$

Since also $|A^{N,j,i}(t)|$ differs from $|A_\varepsilon^{N,j}|$ and $|A_\varepsilon^{N,j+1}|$ by at most $o(1)$ as $N \rightarrow +\infty$, by concatenating all these families, upon reparametrization we obtain a family $A_n^N(t)$ such that $A_n^N(0) = A_\varepsilon$, $A_n^N(1) = A_\varepsilon(1)$, and, if $s < t$ then we have, for some $j < k$

$$\begin{aligned} F_\varepsilon(A_n^N(s)) &\geq F(A(t_j^N)) - C\varepsilon - o(1), \\ F_\varepsilon(A_n^N(t)) &\leq F(A(t_k^N)) + C\varepsilon + o(1). \end{aligned}$$

Since $A(t)$ is a δ -slide for E , we have

$$F(A(t_k^N)) \leq F(A(t_j^N)) + \varepsilon,$$

so that

$$F_\varepsilon(A_n^N(t)) \leq F_\varepsilon(A_n^N(s)) + \delta + C\varepsilon + o(1).$$

By choosing N large enough and ε small enough we obtain the desired $(\delta + o(1))$ -slide.

The previous example suggests a criterion for ‘strong’ stability (i.e., compatible with continuous perturbations), which is sometimes satisfied by Γ -converging sequences. We have constructed δ -slides for the approximating functionals in two steps: one in which we have transformed a limit δ -slide $\phi(t)$ considering recovery sequences (essentially, setting $\phi_\varepsilon(t) = u_\varepsilon^t$, where (u_ε^t) is a recovery sequence for $\phi(t)$), another where we have constructed an ‘almost-decreasing’ path from u_ε to $\phi_\varepsilon(0)$. Note that this step, conversely, is possible thanks to the liminf inequality.

Theorem 6.1 (A criterion of strong stability). *Suppose that F_ε and F satisfy the following requirements:*

if ϕ is a path from u (i.e., $\phi : [0, 1] \rightarrow X$, $\phi(0) = u$, and ϕ is continuous) and $u_\varepsilon \rightarrow u$, then there exist paths ψ_ε from u_ε and ϕ_ε from $\psi_\varepsilon(1)$ such that

(i) $\tau \mapsto F_\varepsilon(\psi_\varepsilon(\tau))$ is decreasing up to $o(1)$ as $n \rightarrow +\infty$; i.e.,

$$\sup_{0 \leq \tau_1 < \tau_2 \leq 1} \left(F_\varepsilon(\psi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1)) \right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

(ii) $\sup_{\tau \in [0,1]} \text{dist}(\phi_\varepsilon(\tau), \phi(\tau)) = o(1)$

(iii) *there exist $0 = \tau_1^\varepsilon < \tau_2^\varepsilon < \dots < \tau_n^\varepsilon = 1$ with $\max_i [\tau_i^\varepsilon - \tau_{i-1}^\varepsilon] = o(1)$ such that $\max_i |F_\varepsilon(\phi_\varepsilon(\tau_i^\varepsilon)) - F(\phi(\tau_i^\varepsilon))| = o(1)$ and $F_\varepsilon(\phi_\varepsilon(\tau))$ is between $F_\varepsilon(\phi_\varepsilon(\tau_i^\varepsilon))$ and $F_\varepsilon(\phi_\varepsilon(\tau_{i+1}^\varepsilon))$ for $\tau \in (\tau_i^\varepsilon, \tau_{i+1}^\varepsilon)$, up to $o(1)$; i.e., there exist infinitesimal $\beta_n > 0$ such that*

$$\begin{aligned} \min \left\{ F_\varepsilon(\phi_\varepsilon(\tau_i^\varepsilon)), F_\varepsilon(\phi_\varepsilon(\tau_{i+1}^\varepsilon)) \right\} - \beta_n &\leq F_\varepsilon(\phi_\varepsilon(\tau)) \\ &\leq \max \left\{ F_\varepsilon(\phi_\varepsilon(\tau_i^\varepsilon)), F_\varepsilon(\phi_\varepsilon(\tau_{i+1}^\varepsilon)) \right\} + \beta_n \end{aligned}$$

Then $(F_\varepsilon + G)$ is stable relative to $(E + G)$ for every continuous G such that $(F_\varepsilon + G)$ is coercive.

Proof. Suppose that u has a δ -slide ϕ for $F + G$ (and therefore a $(\delta - \delta')$ -slide for some $\delta' > 0$) and $u_\varepsilon \rightarrow u$. Then we choose $\psi_\varepsilon, \phi_\varepsilon$ as in (i)–(iii) above and set $\phi'_\varepsilon(\tau) := \psi_\varepsilon(\tau)$ for $\tau \in [0, 1]$, and $\phi'_\varepsilon(\tau) := \phi_\varepsilon(\tau - 1)$ for $\tau > 1$. We then consider $\tau_1 < \tau_2 \in [0, T]$. If $\tau_1, \tau_2 \in [0, 1]$, then

$$F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) = F_\varepsilon(\psi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1)) \leq o(1).$$

If $\tau_1, \tau_2 > 1$, then

$$\begin{aligned} F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) &= F_\varepsilon(\phi_\varepsilon(\tau_2)) - F_\varepsilon(\phi_\varepsilon(\tau_1)) \\ &\leq E(\phi(\tau_2^\varepsilon)) - E(\phi(\tau_1^\varepsilon)) + o(1) \end{aligned}$$

for some $\tau_i^\varepsilon \leq \tau_j^\varepsilon$. If $\tau_1 < 1 < \tau_2$, then

$$\begin{aligned} F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) &= F_\varepsilon(\phi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1)) \\ &\leq E(\phi(\tau_2^\varepsilon)) - E(\phi(0)) + o(1) \end{aligned}$$

for some τ_i^ε , so that in any case

$$\begin{aligned} &(F_\varepsilon(\phi'_\varepsilon(\tau_2)) + G(\phi'_\varepsilon(\tau_2))) - (F_\varepsilon(\phi'_\varepsilon(\tau_1)) + G(\phi'_\varepsilon(\tau_1))) \\ &\leq (E(\phi(\tau_j)) + G(\phi(\tau_j))) - (E(\phi(\tau_i)) + G(\phi(\tau_i))) + o(1) \\ &< \delta - \delta' + o(1) \end{aligned}$$

for some $\tau_i \leq \tau_j$, where we used the continuity of G together with (ii) and (iii), as well as the fact that ϕ is a δ -slide for u . The same argument gives

$$(F_\varepsilon + G)(\phi'_\varepsilon(1)) - (F_\varepsilon + G)(\phi'_\varepsilon(0)) \leq (E + G)(\phi(1)) - (E + G)(\phi(0)) + o(1),$$

so that ϕ'_ε is a δ -slide for $F_\varepsilon + G$, for ε sufficiently small. \square

6.4 Delta-Stable Evolution

The notion of δ -slide (or some of its modification) can be used to define evolutions in a similar way as for quasistatic motion, in cases when the presence of energy barriers may be relevant in the model under consideration, and can be similarly characterized. In order to construct such an evolution one can proceed by discrete approximation as in Remark 3.2.1:

- With fixed $\mathcal{F}(t, U)$ and $\mathcal{D}(U)$ energy and dissipation as in Sect. 3.2, time step $\tau > 0$ and *maximal barrier height* $\delta > 0$, define U_k^τ recursively by setting $U_0^\tau = U_0$, and choosing U_k^τ as a solution of the minimum problem

$$\min_U \left\{ \mathcal{F}(\tau k, U) + \mathcal{D}(U - U_{k-1}^\tau) \right\}$$

on the class of U such that there exists a path ϕ from U_{k-1}^τ to U such that

$$\mathcal{F}(\tau k, \phi(t)) + \mathcal{D}(\phi(s) - U_{k-1}^\tau) \leq \mathcal{F}(\tau k, \phi(s)) + \mathcal{D}(\phi(s) - U_{k-1}^\tau) + \delta \quad (6.1)$$

if $0 \leq s < t \leq 1$.

- Define the continuous trajectory $U^{\delta, \tau}(t) = U_{\lfloor t/\tau \rfloor}^\tau$.
- Define the *δ -stable evolutions* as the limits U^δ of (subsequences of) $U^{\delta, \tau}$ (which exist, under suitable assumptions).

In order to ensure the existence of the minimizer U_k^τ some additional properties of the functionals

$$F_k(U) = \mathcal{F}(\tau k, U) + \mathcal{D}(U - U_{k-1}^\tau)$$

must be assumed; namely, that if U_j is a sequence converging to \bar{U} and $F_k(U_j) < F_k(U_{k-1}^\tau) - C$ for some positive constant C such that there exists paths ϕ_j from U_{k-1}^τ to U_j satisfying (6.1), then there exists a path ϕ satisfying (6.1) from U_{k-1}^τ to \bar{U} .

Remark 6.5. It must be noted that stable evolution gives a different notion from the global minimization approach even when $\mathcal{D} = 0$, in which case the quasistatic approach just gives a parameterized choice of minimizers of $F(t, \cdot)$.

As a simple example, take the one-dimensional energy

$$F(t, x) = \min\{x^2, (x - 1)^2\} - 2tx,$$

and $x_0 = 0$. Then the trajectory of parameterized minimizers of $F(t, x)$ from x_0 is

$$u(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 + t & \text{if } t > 0. \end{cases}$$

On the contrary, the limit \bar{u} as $\delta \rightarrow 0$ of the corresponding δ -stable evolutions $u_\delta(t)$ is

$$\bar{u}(t) = \begin{cases} t & \text{if } t < \frac{1}{2} \\ 1 + t & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Example 6.4 (The long-bar paradox in Fracture Mechanics). As shown in Remark 3.8, for one-dimensional fracture problems with applied boundary displacement; i.e., for the energies and dissipations

$$\mathcal{F}(t, u) = \int_0^1 |u'|^2 dt, \quad \mathcal{D}(S(u)) = \#(S(u))$$

defined on piecewise- H^1 functions u with $u(0^-) = 0$ and $u(1^+) = t$ (and $S(u)$ denotes the set of discontinuity points of u), fracture is brutal and appears at a critical value of the displacement t . If instead of a bar of unit length we take a bar of length L and we consider the normalized boundary conditions $u(0^-) = 0$ and $u(L^+) = tL$, then the critical value for fracture is $t = \frac{1}{\sqrt{L}}$ for which the energy of the (unfractured) linear solution $u(t) = tx$ equals the energy of a piecewise-constant solution with one discontinuity. In other words, a long bar fractures at lower values of the strain (the gradient of the linear solution). In order to overcome this drawback of the theory, one may consider δ -stable evolutions, or, rather, a small variation from it necessary due to the fact that the domains of the functionals $\mathcal{F}(t, \cdot)$ are disjoint for different t . In the iterated minimization scheme above we consider minimization among functions u such that there exists a L^2 -continuous path ϕ from the elastic solution $u_{k\tau}(x) = k\tau x$ (we again consider only the case $L = 1$) to u such that $\phi(t)(0) = 0$, $\phi(t)(1) = k\tau$ for all t and

$$\mathcal{F}(k\tau, \phi(t)) + \#(S(\phi(t))) \leq \mathcal{F}(k\tau, \phi(s)) + \#(S(\phi(s))) + \delta \quad (6.2)$$

if $0 \leq s < t \leq 1$. This set of u is contained in H^1 . Indeed, otherwise there would be a $\bar{t} \in (0, 1]$ such that $\#(S(\phi(t_j))) \geq 1$ for a non-increasing sequence of t_j converging to \bar{t} , and $\phi(t) \in H^1$ for $t \leq \bar{t}$. By the lower semicontinuity of \mathcal{F} and the minimality of $u_{k\tau}$ we have

$$\mathcal{F}(k\tau, u_{k\tau}) \leq \mathcal{F}(k\tau, \phi(\bar{t})) \leq \mathcal{F}(k\tau, \phi(t_j)) + o(1),$$

which gives

$$\mathcal{F}(k\tau, \phi(0)) + 1 \leq \mathcal{F}(k\tau, \phi(t_j)) + \#(S(\phi(t_j))) + o(1)$$

contradicting (6.2) for $s = 0$ and $t = t_j$ if $\delta < 1$.

We conclude that for all k the minimizer is exactly $u_{k\tau}$, and we may pass to the limit obtaining the elastic solution $u(t, x) = tx$. As a conclusion we have that no fracture appears, and this conclusion is clearly independent of the length of the bar.

Appendix

The notion of stable points has been introduced by Larsen in [3], where also stable fracture evolution has been studied; in particular there it is shown that the scheme in Sect. 6.4 can be applied to Griffith fracture energies.

The notions of stability for sequences of functionals have been analyzed by Braides and Larsen in [1], and are further investigated by Focardi in [2].

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Chapter 7

Minimizing Movements

In this chapter we give a brief account of the variational motion defined by the limit of Euler schemes at vanishing time step. This notion is linked to the study of local minimizers, which provide stationary solutions for such motions. For the sake of simplicity of exposition, we limit our analysis to a Hilbert setting, even though many results can be proven in general metric spaces.

7.1 An Energy-Driven Implicit-Time Discretization

We now introduce a notion of energy-based motion which generalizes an implicit-time scheme for the approximation of solutions of gradient flows to general (also non differentiable) energies. We will use the terminology of *minimizing movements*, introduced by De Giorgi, even though we will not follow precisely the notation used in the literature.

Definition 7.1 (Minimizing movements). Let X be a separable Hilbert space and let $F : X \rightarrow [0, +\infty]$ be coercive and lower semicontinuous. Given $x_0 \in X$ and $\tau > 0$ we define x_k recursively as a minimizer for the problem

$$\min \left\{ F(x) + \frac{1}{2\tau} \|x - x_{k-1}\|^2 \right\}, \tag{7.1}$$

and consider the piecewise-constant trajectory $u^\tau : [0, +\infty) \rightarrow X$ given by

$$u^\tau(t) = x_{\lfloor t/\tau \rfloor}. \tag{7.2}$$

The construction of this trajectory is pictured in Figs. 7.1 and 7.2.

A *minimizing movement* for F from x_0 is any limit of a subsequence u^{τ_j} uniform on compact sets of $[0, +\infty)$.

Fig. 7.1 Computation of the first step of a minimizing movement

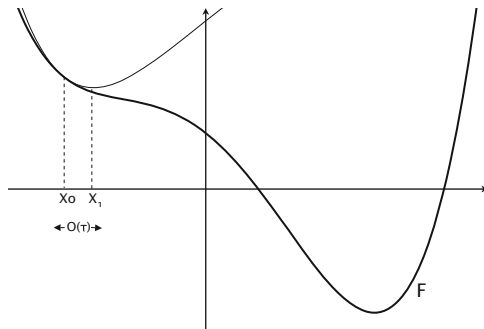
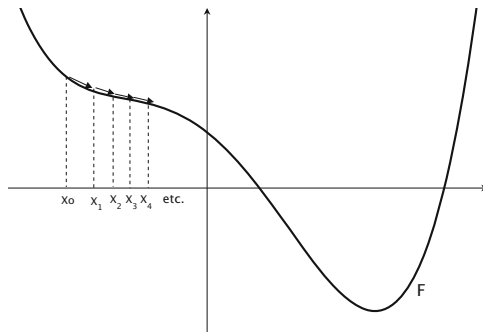


Fig. 7.2 Pictorial description of the trajectory at time step τ



As in the rest of these notes, we are not focusing on the general topological assumptions on function spaces and convergences. In this definition we have taken $F \geq 0$ and X Hilbert for the sake of simplicity. More in general, we can take X a metric space and the (power of the) distance in place of the squared norm. Note, in addition, that the topology on X with respect to which F is lower semicontinuous can be taken weaker than the one of the related distance.

Remark 7.1. A heuristic explanation of the definition above is given when F is smooth. In this case, with the due notation, a minimizer for (7.1) solves the equation

$$\frac{x_k - x_{k-1}}{\tau} = -\nabla F(x_k); \tag{7.3}$$

i.e., u^τ solves the equation

$$\frac{u^\tau(t) - u^\tau(t - \tau)}{\tau} = -\nabla F(u^\tau(t)). \tag{7.4}$$

If we can pass to the limit in this equation as $u^\tau \rightarrow u$ then we obtain

$$\frac{\partial u}{\partial t} = -\nabla F(u). \tag{7.5}$$

This is easily shown to hold if $X = \mathbb{R}^n$ and $F \in C^2(\mathbb{R}^n)$. In this case, by taking any $\varphi \in C_0^\infty((0, T); \mathbb{R}^n)$ we have

$$\begin{aligned} -\int_0^T \langle \nabla F(u^\tau), \varphi \rangle dt &= \int_0^T \left\langle \frac{u^\tau(t) - u^\tau(t - \tau)}{\tau}, \varphi \right\rangle dt \\ &= -\int_0^T \left\langle u^\tau(t), \frac{\varphi(t) - \varphi(t + \tau)}{\tau} \right\rangle dt, \end{aligned}$$

from which, passing to the limit

$$\int_0^T \langle \nabla F(u), \varphi \rangle dt = \int_0^T \langle u, \varphi' \rangle dt;$$

i.e., (7.5) is satisfied in the sense of distributions, and hence in the classical sense.

Remark 7.2 (Stationary solutions). Let x_0 be a local minimizer for F , then the only minimizing movement for F from x_0 is the constant function $u(t) = x_0$.

Indeed, if x_0 is a minimizer for F on the set of x such that $\|x - x_0\| \leq \delta$, then, by the positiveness of F , it is the only minimizer of $F(x) + \frac{1}{2\tau}\|x - x_0\|^2$ for $\tau \leq \delta^2/F(x_0)$ if $F(x_0) > 0$ (for any τ if $F(x_0) = 0$), so that for these τ $x_k = x_0$ for all k .

Proposition 7.1 (Existence of minimizing movements). *For all F and x_0 as above there exists a minimizing movement $u \in C^{1/2}([0, +\infty); X)$.*

Proof. By the coerciveness and lower semicontinuity of F we obtain that x_k are well defined for all k . Moreover, since

$$F(x_k) + \frac{1}{2\tau}\|x_k - x_{k-1}\|^2 \leq F(x_{k-1}),$$

we have $F(x_k) \leq F(x_{k-1})$ and

$$\|x_k - x_{k-1}\|^2 \leq 2\tau(F(x_{k-1}) - F(x_k)), \quad (7.6)$$

so that for $t > s$

$$\begin{aligned} \|u^\tau(t) - u^\tau(s)\| &\leq \sum_{k=\lfloor s/\tau \rfloor + 1}^{\lfloor t/\tau \rfloor} \|x_k - x_{k-1}\| \\ &\leq \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \sqrt{\sum_{k=\lfloor s/\tau \rfloor + 1}^{\lfloor t/\tau \rfloor} \|x_k - x_{k-1}\|^2} \end{aligned}$$

$$\begin{aligned}
&\leq \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \sqrt{2\tau \sum_{k=\lfloor s/\tau \rfloor+1}^{\lfloor t/\tau \rfloor} (F(x_{k-1}) - F(x_k))} \\
&= \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \sqrt{2\tau (F(x_{\lfloor s/\tau \rfloor}) - F(x_{\lfloor t/\tau \rfloor}))} \\
&\leq \sqrt{2F(x_0)} \sqrt{\tau(\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor)} \\
&\leq \sqrt{2F(x_0)} \sqrt{t - s + \tau}.
\end{aligned}$$

This shows that the functions u^τ are (almost) equicontinuous and equibounded in $C([0, +\infty); X)$. Hence, they converge uniformly. Moreover, passing to the limit we obtain

$$\|u(t) - u(s)\| \leq \sqrt{2F(x_0)} \sqrt{|t - s|},$$

so that $u \in C^{1/2}([0, +\infty); X)$. \square

Remark 7.3 (Growth conditions). The positiveness of F can be substituted by the requirement that for all \bar{x} the functionals

$$x \mapsto F(x) + \frac{1}{2\tau} \|x - \bar{x}\|^2$$

be bounded from below; i.e., that there exists $C > 0$ such that

$$x \mapsto F(x) + C \|x - \bar{x}\|^2$$

be bounded from below.

Example 7.1 (Non-uniqueness of minimizing movements). If F is not C^2 we may have more than one minimizing movement.

- (i) *Bifurcation at times with multiple minimizers.* A simple example is $F(x) = -\frac{1}{\alpha}|x|^\alpha$ with $0 < \alpha < 2$, which is not C^2 at $x = 0$. In this case, for $x_0 = 0$ we have a double choice for minimum problem (7.1); i.e.,

$$x_1 = \pm \tau^{1/(2-\alpha)}.$$

Once x_1 is chosen all other values are determined, and it can be seen that either $x_k > 0$ for all k or $x_k < 0$ for all k (for $\alpha = 1$, e.g., we have $x_k = \pm k\tau$), and that in the limit we have the two solutions of

$$\begin{cases} u' = |u|^{(\alpha-2)}u \\ u(0) = 0 \end{cases}$$

with $u(t) \neq 0$ for $t > 0$. Note in particular that we do not have the trivial solution $u(t) = 0$ for all t . In this example we do not have to pass to a subsequence of τ .

- (ii) *Different movements depending on subsequences of τ* . Discrete trajectories can be different depending on the time step τ . We give an explicit example, close in spirit to the previous one. In this example the function F is asymmetric, so that x_1 is unique but may take positive or negative values depending on τ .

We define F as the Lipschitz function taking value 0 at $x = 0$, for $x > 0$

$$F'(x) = \begin{cases} -1 & \text{if } 2^{-2k-1} < x < 2^{-2k}, k \in \mathbb{N} \\ -2 & \text{otherwise for } x > 0 \end{cases}$$

and $F'(x) = 3 + F'(-x)$ for $x < 0$. It is easily seen that for $x_0 = 0$ we may have a unique minimizer x_1 with $x_1 > 0$ or $x_1 < 0$ depending on τ . In particular we have $x_1 = -2^{-2k} < 0$ for $\tau = 2^{-2k-1}$ and $x_1 = 2^{-2k+1} > 0$ for $\tau = 2^{-2k}$. In the two cases we then have again the solutions to

$$\begin{cases} u' = -F'(u) \\ u(0) = 0 \end{cases}$$

with $u(t) < 0$ for all $t > 0$ or $u(t) > 0$ for all $t > 0$, respectively.

Example 7.2 (Heat equation). Taking $X = L^2(\Omega)$ and the Dirichlet integral $F(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx$, with fixed $u_0 \in H^1(\Omega)$ and $\tau > 0$ we can solve iteratively

$$\min \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2\tau} \int_{\Omega} |u - u_{k-1}|^2 dx \right\},$$

whose unique minimizer u_k solves the Euler–Lagrange equation

$$\frac{u_k - u_{k-1}}{\tau} = \Delta u_k, \quad \frac{\partial u_k}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad (7.7)$$

where ν is the inner normal to Ω . We then set $u^\tau(x, t) = u_{\lfloor t/\tau \rfloor}(x)$, which converges, up to subsequences, to $u(x, t)$. We can then pass to the limit in (7.7) in the sense of distributions to obtain the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad (7.8)$$

combined with the initial datum $u(x, 0) = u_0(x)$. Due to the uniqueness of the solution to the heat equation we also obtain that the whole sequence converges as $\tau \rightarrow 0$.

Example 7.3 (One-dimensional fracture energies). In dimension one, we still consider $X = L^2(0, 1)$ and the Griffith (or Mumford–Shah) energy

$$F(u) = \frac{1}{2} \int_0^1 |u'|^2 dx + \#(S(u))$$

with domain piecewise- H^1 functions. We fix u_0 piecewise- H^1 and $\tau > 0$. In this case we solve iteratively

$$\min \left\{ \frac{1}{2} \int_0^1 |u'|^2 dx + \#(S(u)) + \frac{1}{2\tau} \int_0^1 |u - u_{k-1}|^2 dx \right\}.$$

This problem is not convex, and may have multiple minimizers. Nevertheless in this simpler case we can prove iteratively that for τ small enough we have $S(u_k) = S(u_0)$ for all k , and hence reduce to the independent iterated minimization problems of the Dirichlet integral on each component of $(0, 1) \setminus S(u_0)$, giving the heat equation in $(0, 1) \setminus S(u_0)$ with Neumann boundary conditions on $0, 1$ and $S(u_0)$. This description holds until the first time T such that $u(x^-, T) = u(x^+, T)$ at some point $x \in S(u(\cdot, T))$.

We check this with some simplifying hypotheses:

- (1) That $\int_0^1 |u_0'|^2 dx < 2$. This implies that $\#(S(u_k)) \leq \#S(u_0)$ since by the monotonicity of the energy we have $\#(S(u_k)) \leq F(u_k) \leq F(u_0) < \#S(u_0) + 1$. This hypothesis can be removed with a localization argument.
- (2) That there exists $\eta > 0$ such that $|u_0(x) - u_0(x')| \geq \eta$ if $(x, x') \cap S(u_0) \neq \emptyset$. This will imply that $|u_k(x^+) - u_k(x^-)| \geq \eta$ at all $x \in S(u_k)$ so that $T = +\infty$ in the notation above.

Furthermore, we suppose that $S(u_0) = \{x_0\}$ (the jump set is a single point) and $u_0(x_0^+) > u_0(x_0^-)$, for simplicity of notation.

We reason by induction. We first examine the properties of u_1 ; checking that it has a jump point close to x_0 . Suppose otherwise that there exists $\delta > 0$ such that $u \in H^1(x_0 - \delta, x_0 + \delta)$. We take δ small enough so that

$$\begin{aligned} u_0(x) &\leq u(x_0^-) + \frac{1}{4}(u(x_0^+) - u(x_0^-)) \text{ for } x_0 - \delta < x < x_0 \\ u_0(x) &\geq u(x_0^+) - \frac{1}{4}(u(x_0^+) - u(x_0^-)) \text{ for } x_0 < x < x_0 + \delta. \end{aligned}$$

In this case

$$\frac{1}{2} \int_{x_0-\delta}^{x_0+\delta} |u_1'|^2 dx + \frac{1}{2\tau} \int_{x_0-\delta}^{x_0+\delta} |u_1 - u_0|^2 dx$$

$$\begin{aligned} &\geq \frac{1}{2} \min \left\{ \int_0^\delta |v'|^2 dx + \frac{1}{\tau} \int_0^\delta |v|^2 dx : v(\delta) = \frac{1}{4}(u(x_0^+) - u(x_0^-)) \right\} \\ &= \frac{(u(x_0^+) - u(x_0^-))^2}{8\sqrt{\tau}} \tanh\left(\frac{\delta}{\sqrt{\tau}}\right), \end{aligned}$$

the last equality being easily obtained by computing the solution of the Euler–Lagrange equation. This shows that for each such $\delta > 0$ fixed we have $(x_0 - \delta, x_0 + \delta) \cap S(u_1) \neq \emptyset$ for τ sufficiently small. Note that the smallness of τ depends only on the size of $u(x_0^+) - u(x_0^-)$ (which is larger than η). Since $\#S(u_1) \leq \#S(u_0)$ we then have $(x_0 - \delta, x_0 + \delta) \cap S(u_1) = \{x_1\}$; we may suppose that $x_1 \geq x_0$.

We now check that $x_1 = x_0$. Suppose otherwise; then note that by the Hölder continuity of u_1 we have that for δ small enough (depending only on the size of $u(x_0^+) - u(x_0^-)$) we have

$$u_1(x) \leq u_0(x_0^-) + \frac{1}{8}(u(x_0^+) - u(x_0^-)) \text{ for } x_0 < x < x_1$$

and

$$u_1(x_1^+) \geq u_0(x_1^+) - \frac{1}{8}(u(x_0^+) - u(x_0^-)).$$

We may then consider the function \tilde{u} coinciding with u_1 on $(0, x_0)$, with $\tilde{u}' - U_1'$ and $S(\tilde{u}) = \{x_0\}$. Then $F(\tilde{u}) = F(u_1)$ and

$$\int_0^1 |\tilde{u} - u_0|^2 dx < \int_0^1 |u_1 - u_0|^2 dx,$$

contradicting the minimality of u_1 . Hence, we have $S(u_1) = S(u_0)$.

Note that u_1 is obtained by separately minimizing the problems with the Dirichlet integral on $(0, x_0)$ and $(x_0, 1)$, and in particular that on each such interval $\sup u_1 \leq \sup u_0$ and $\inf u_1 \geq \inf u_0$, so that the condition that $|u_1(x) - u_1(x')| \geq \eta$ if $(x, x') \cap S(u_1) \neq \emptyset$ still hold. This shows that we can iterate the scheme obtaining u_k which satisfy $\inf u_0 \leq u_k \leq \sup u_0$ on each component of $(0, 1) \setminus S(u_0)$ and $u_k' = 0$ on $0, 1$ and $S(u_0)$. In particular $|u_k^+ - u_k^-| \geq \eta$ on $S(u_0)$, which shows that the limit satisfies the heat equation with Neumann conditions on $S(u_0)$ for all times.

Remark 7.4 (Higher-dimensional fracture energies: an open problem). In dimension higher than one, we may apply the minimizing-movement scheme to the Mumford–Shah functional (which can also be interpreted as a Griffith energy in an anti-plane setting). To this scheme we may also add the constraint of increasing crack in the same way as in Sect. 3.3. We then obtain a minimizing movement u in $L^2(\Omega)$ from an initial datum u_0 . The solution u solves a weak form of the heat equation almost everywhere in Ω , and in some cases it can be proved that Neumann boundary conditions still hold on $S(u)$, but it remains open the problem

of the characterization of the (possible) motion of the crack set. As compared with the Francfort and Marigo quasistatic theory, here an analog of the Francfort–Larsen transfer lemma is missing.

7.2 Time-Dependent Minimizing Movements

By considering time-parameterized energies $F(x, t)$ we can generalize the definition of minimizing movement to include forcing terms or varying boundary conditions. Then, given τ and an initial datum x_0 , we recursively choose x_k as a minimizer of

$$\min \left\{ F(x, k\tau) + \frac{1}{2\tau} \|x - x_{k-1}\|^2 \right\}, \quad (7.9)$$

and eventually set $u^\tau(t) = x_{\lfloor t/\tau \rfloor}$. If some technical hypothesis is added to F then we may define a limit u of u^τ as $\tau \rightarrow 0$, up to subsequences. One such hypothesis is that

$$F(u_k, k\tau) + \frac{1}{2\tau} \|x_k - x_{k-1}\|^2 \leq (1 + C\tau)F(u_{k-1}, (k-1)\tau) + C\tau, \quad (7.10)$$

for some C in the scheme above (at least if $k\tau$ remains bounded). With such a condition we can repeat the convergence argument as for the time-independent case and obtain a limit minimizing movement u .

Indeed, with such a condition we have

$$\|x_k - x_{k-1}\|^2 \leq 2\tau \left((1 + C\tau)F(u_{k-1}, (k-1)\tau) - F(u_k, k\tau) + C\tau \right), \quad (7.11)$$

and the inequality (for τ small enough)

$$\begin{aligned} F(u_k, k\tau) &\leq (1 + C\tau)F(u_{k-1}, (k-1)\tau) + C\tau \\ &\leq (1 + C\tau)(F(u_{k-1}, (k-1)\tau) + 1), \end{aligned} \quad (7.12)$$

that implies that $F(u_k, k\tau)$ is equibounded for $k\tau$ bounded. We fix $T > 0$; from (7.11) we obtain (for $0 \leq s \leq t \leq T$)

$$\begin{aligned} \|u^\tau(t) - u^\tau(s)\| &\leq \sum_{k=\lfloor s/\tau \rfloor + 1}^{\lfloor t/\tau \rfloor} \|x_k - x_{k-1}\| \\ &\leq \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \sqrt{\sum_{k=\lfloor s/\tau \rfloor + 1}^{\lfloor t/\tau \rfloor} \|x_k - x_{k-1}\|^2} \end{aligned}$$

$$\begin{aligned}
 &\leq \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \\
 &\quad \times \sqrt{2\tau \sum_{k=\lfloor s/\tau \rfloor+1}^{\lfloor t/\tau \rfloor} \left(F(x_{k-1}, (k-1)\tau) - F(x_k, k\tau) + C_T\tau \right)} \\
 &= \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \sqrt{2\tau F(x_0, 0) + C_T\tau(\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor)} \\
 &\leq \sqrt{2F(x_0, 0)(t-s+\tau) + C_T(t-s+\tau)^2},
 \end{aligned}$$

which gives an equicontinuity condition sufficient to pass to the limit as $\tau \rightarrow 0$.

Note that from (7.12) for $s < t$ we obtain the estimate

$$F(u^\tau(t), \lfloor t/\tau \rfloor \tau) \leq e^{C(t-s+\tau)}(F(u^\tau(s), \lfloor s/\tau \rfloor \tau) + C\tau). \tag{7.13}$$

Example 7.4 (Heat equation with varying boundary conditions). We can take

$$F(u, t) = \frac{1}{2} \int_0^1 |u'|^2 dx,$$

with domain all H^1 -functions satisfying $u(0) = 0$ and $u(1) = t$.

Then if we can test the problem defining u_k with the function $\tilde{u} = u_{k-1} + \tau x$. We then have

$$\begin{aligned}
 F(u_k, k\tau) + \frac{1}{2\tau} \|u_k - u_{k-1}\|^2 &\leq F(\tilde{u}, k\tau) + \frac{1}{2\tau} \|\tilde{u} - u_{k-1}\|^2 \\
 &= \frac{1}{2} \int_0^1 |u'_{k-1} + \tau|^2 dx + \frac{1}{6} \tau^2 \\
 &\leq (1 + \tau) \frac{1}{2} \int_0^1 |u'_{k-1}|^2 dx + \tau + \frac{1}{6} \tau^2 \\
 &\leq (1 + \tau) \frac{1}{2} F(u_{k-1}, (k-1)\tau) + \left(1 + \frac{1}{6} \tau\right) \tau.
 \end{aligned}$$

which gives (7.10).

We then have the convergence of u^τ to the solution u of the equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(0, t) = 0, u(1, t) = t \\ u(x, 0) = u_0(x). \end{cases} \tag{7.14}$$

Clearly in this example we may take any Lipschitz function $g(t)$ in place of t as boundary condition.

Example 7.5 (Minimizing movements vs quasistatic evolution for fracture). We can take

$$F(u, t) = \frac{1}{2} \int_0^1 |u'|^2 dx + \#S(u),$$

with domain all piecewise- H^1 -loc functions with $u(x) = 0$ if $x \leq 0$ and $u(x) = t$ for $x \geq 1$, so that $S(u) \subset [0, 1]$, and the fracture may also appear at the boundary points 0 and 1. As in the previous example, we can test the problem defining u_k with the function $\tilde{u} = u_{k-1} + \tau x$ since $\#S(\tilde{u}) = \#S(u_{k-1})$, and obtain (7.10).

We consider the initial datum $u_0 = 0$. Note that the minimum problems for $F(\cdot, t)$ correspond to the definition of quasistatic evolution in Remark 3.8. We now show that for problems (7.9) the solution does not develop fracture.

Indeed, consider

$$k_\tau = \min\{k : u_k \notin H_{\text{loc}}^1(\mathbb{R})\},$$

and suppose that $\tau k_\tau \rightarrow \bar{t} \in [0, +\infty)$. Then we have that u^τ converges on $[0, \bar{t}]$ to u described by (7.14) in the previous example. Moreover we may suppose that

$$\lim_{\tau \rightarrow 0} \int_0^1 |u'_{k_\tau-1}|^2 dx = \int_0^1 |u'(x, \bar{t})|^2 dx,$$

and since also $u_{k_\tau} \rightarrow u(\cdot, \bar{t})$ as $\tau \rightarrow 0$, we have

$$\int_0^1 |u'(x, \bar{t})|^2 dx \leq \liminf_{\tau \rightarrow 0} \int_0^1 |u'_{k_\tau}|^2 dx.$$

We deduce that

$$F(u_{k_\tau}, k_\tau \tau) \geq F(u_{k_\tau-1}, (k_\tau - 1)\tau) + 1 + o(1)$$

as $\tau \rightarrow 0$, which contradicts (7.12).

From the analysis above we can compare various ways to define the evolutive response of a brittle elastic material to applied increasing boundary displacements (at least in a one-dimensional setting):

- **(quasistatic motion)** the response is purely elastic until a threshold (depending on the size of the specimen) is reached, after which we have brutal fracture.
- **(stable evolution)** the response is purely elastic, and corresponds to minimizing the elastic energy at fixed boundary displacement. No fracture occurs.
- **(minimizing movement)** in this case the solution does not develop fracture, but follows the heat equation with given boundary conditions.

Appendix

The terminology ‘(generalized) minimizing movement’ has been introduced by De Giorgi in a series of papers devoted to mathematical conjectures (see [5]). We also refer to the original treatment by Ambrosio [1].

A theory of gradient flows in metric spaces using minimizing movements is described in the book by Ambrosio et al. [3].

Minimizing movements for the Mumford–Shah functional in more than one space dimension (and hence also for the Griffith fracture energy) with the condition of increasing fracture have been defined by Ambrosio and Braides [2], and partly analyzed in a two-dimensional setting by Chambolle and Doveri [4].

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Chapter 8

Minimizing Movements Along a Sequence of Functionals

Gradient flows, and hence minimizing movements, and the convergence of functionals trivially do not commute even when the convergence is uniform. As a simple example, take $X = \mathbb{R}$ and

$$F_\varepsilon(x) = x^2 - \rho \sin\left(\frac{x}{\varepsilon}\right),$$

with $\rho = \rho_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, uniformly converging to $F(x) = x^2$. If also

$$\varepsilon \ll \rho,$$

then for fixed x_0 the solutions u_ε to the equation

$$\begin{cases} u'_\varepsilon = -2u_\varepsilon + \frac{\rho}{\varepsilon} \cos\left(\frac{u_\varepsilon}{\varepsilon}\right) \\ u_\varepsilon(0) = x_0 \end{cases}$$

converge to the constant function $u_0(t) = x_0$ as $\varepsilon \rightarrow 0$. This is easily seen by studying the stationary solutions of

$$-2x + \frac{\rho}{\varepsilon} \cos\left(\frac{x}{\varepsilon}\right) = 0.$$

Conversely, the gradient flow of the limit is

$$\begin{cases} u' = -2u \\ u(0) = x_0, \end{cases}$$

for which the constant functions are not solutions if $x_0 \neq 0$.

With the remark above in mind, in order to give a meaningful limit for the energy-driven motion along a sequence of functionals it may be useful to vary the definition

of minimizing movement. This will be done in the following section. As in the previous chapter, we will limit our analysis to a Hilbert setting for simplicity.

8.1 Minimizing Movements Along a Sequence

In this section we will give a notion of minimizing movement along a sequence F_ε , which will depend in general on the interaction between the time scale τ and the parameter ε in the energies.

Definition 8.1 (minimizing movements along a sequence). Let X be a separable Hilbert space, let $F_\varepsilon : X \rightarrow [0, +\infty]$ be equicoercive and lower semicontinuous, $x_0^\varepsilon \rightarrow x_0$ with

$$F_\varepsilon(x_0^\varepsilon) \leq C < +\infty, \quad (8.1)$$

and let $\tau_\varepsilon > 0$ converge to 0 as $\varepsilon \rightarrow 0$. With fixed $\varepsilon > 0$ we define x_k^ε recursively as a minimizer for the problem

$$\min \left\{ F_\varepsilon(x) + \frac{1}{2\tau} \|x - x_{k-1}^\varepsilon\|^2 \right\}, \quad (8.2)$$

and the piecewise-constant trajectory $u^\varepsilon : [0, +\infty) \rightarrow X$ given by

$$u^\varepsilon(t) = x_{\lfloor t/\tau_\varepsilon \rfloor}. \quad (8.3)$$

A *minimizing movement* for F_ε from x_0^ε is any limit of a subsequence u^{ε_j} uniform on compact sets of $[0, +\infty)$.

After remarking that the Hölder continuity estimates in Proposition 7.1 only depend on the bound on $F_\varepsilon(x_0^\varepsilon)$, with the same proof we can show the following result.

Proposition 8.1. *For every F_ε and x_0^ε as above there exist minimizing movements for F_ε from x_0^ε in $C^{1/2}([0, +\infty); X)$.*

Remark 8.1 (Growth conditions). As for the case of a single functional, the positiveness of F_ε can be substituted by the requirement that for all \bar{x} the functionals

$$x \mapsto F_\varepsilon(x) + \frac{1}{2\tau} \|x - \bar{x}\|^2$$

be bounded from below; i.e., that there exists $C > 0$ such that

$$x \mapsto F_\varepsilon(x) + C \|x - \bar{x}\|^2$$

be bounded from below.

Example 8.1. We give a simple example that shows how the limit minimizing movement may depend on the choice of the mutual behavior of ε and τ . We consider the functions

$$F_\varepsilon(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 0 & \text{if } 0 \leq x \leq \varepsilon \\ \varepsilon - x & \text{if } x \geq \varepsilon, \end{cases}$$

which converge uniformly to $F(x) = -x$. Note that the energies are not bounded from below, but their analysis falls within the framework in the previous remark. For this example a direct computation is immediately carried on. We consider a fixed initial datum x_0 .

If $x_0 > 0$, then for $\varepsilon < x_0$ we have $x_k^\varepsilon = x_{k-1}^\varepsilon + \tau$ for all $k \geq 0$.

If $x_0 \leq 0$ then we have $x_k^\varepsilon = x_{k-1}^\varepsilon + \tau$ if $x_{k-1}^\varepsilon \leq -\tau$. If $0 \geq x_{k-1}^\varepsilon > -\tau$ then $x_k^\varepsilon - x_{k-1}^\varepsilon$ is obtained by minimizing the function

$$f(y) = \begin{cases} -y + \frac{1}{2\tau}y^2 & \text{if } 0 \leq y \leq -x_{k-1}^\varepsilon \\ x_{k-1}^\varepsilon + \frac{1}{2\tau}y^2 & \text{if } -x_{k-1}^\varepsilon \leq y \leq -x_{k-1}^\varepsilon + \varepsilon \\ \varepsilon - y + \frac{1}{2\tau}y^2 & \text{if } y \geq -x_{k-1}^\varepsilon + \varepsilon, \end{cases}$$

whose minimizer is always $y = \tau + x_{k-1}^\varepsilon$ if $\varepsilon - x_{k-1}^\varepsilon > \tau$. In this case $x_k^\varepsilon = 0$. If otherwise $\varepsilon - x_{k-1}^\varepsilon \leq \tau$ the other possible minimizer is $y = \tau$. We then have to compare the values

$$f(-x_{k-1}^\varepsilon) = x_{k-1}^\varepsilon + \frac{1}{2\tau}(x_{k-1}^\varepsilon)^2, \quad f(\tau) = \varepsilon - \frac{1}{2}\tau.$$

We have three cases:

- $\varepsilon - \frac{1}{2}\tau > 0$. In this case we have $x_k^\varepsilon = 0$ (and this holds for all subsequent steps).
- $\varepsilon - \frac{1}{2}\tau < 0$. In this case we either have $f(\tau) < f(-x_{k-1}^\varepsilon)$, in which case $x_k^\varepsilon = x_{k-1}^\varepsilon + \tau$ (and this then holds for all subsequent steps); otherwise $x_k^\varepsilon = 0$ and $x_{k+1}^\varepsilon = x_k^\varepsilon + \tau$ (and this holds for all subsequent steps).
- $\varepsilon - \frac{1}{2}\tau = 0$. If $x_{k-1}^\varepsilon < 0$ then $x_k^\varepsilon = 0$ (otherwise we already have $x_{k-1}^\varepsilon = 0$). Then, since we have the two solutions $y = 0$ and $y = \tau$, we have $x_j^\varepsilon = 0$ for $k \leq j \leq k_0$ for some $k_0 \in \mathbb{N} \cup +\infty$ and $x_j^\varepsilon = x_{j-1}^\varepsilon + \tau$ for $j > k_0$.

We can summarize the possible minimizing movements with initial datum $x_0 \leq 0$ as follows:

- (i) If $\tau < 2\varepsilon$ then the unique minimizing movement is $x(t) = \min\{x_0 + t, 0\}$.
- (ii) If $\tau > 2\varepsilon$ then the unique minimizing movement is $x(t) = x_0 + t$.
- (iii) If $\tau = 2\varepsilon$ then we have the family of minimizing movements (parameterized by $x_1 \leq x_0$) $x(t) = \max\{\min\{x_0 + t, 0\}, x_1 + t\}$.
For $x_0 > 0$ we always have the only minimizing movement $x(t) = x_0 + t$.

8.2 Commutability Along ‘Fast-Converging’ Sequences

We now show that, by suitably choosing the ε - τ regimes, the minimizing movement along the sequence F_ε from x_ε converges to a minimizing movement for the limit F from x_0 (‘fast-converging ε ’), while for other choices (‘fast-converging τ ’) the minimizing movement converges to a limit of minimizing movements for F_ε as $\varepsilon \rightarrow 0$. Heuristically, minimizing movements for all other regimes are ‘trapped’ between these two extrema.

Theorem 8.1. *Let F_ε be a equi-coercive sequence of (non-negative) lower-semicontinuous functionals on a Hilbert space X Γ -converging to F , let $x_\varepsilon \rightarrow x_0$. Then:*

- (i) *There exists a choice of $\varepsilon = \varepsilon(\tau)$ such that every minimizing movement along F_ε (and with time-step τ) with initial data x_ε is a minimizing movement for F from x_0 on $[0, T]$ for all T .*
- (ii) *There exists a choice of $\tau = \tau(\varepsilon)$ such that every minimizing movement along F_ε (and with time-step τ) with initial data x_ε is a limit of a sequence of minimizing movements for F_ε (for ε fixed) from x_ε on $[0, T]$ for all T .*

Proof. (i) Note that if $y_\varepsilon \rightarrow y_0$ then the solutions of

$$\min\left\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - y_\varepsilon\|^2\right\} \quad (8.4)$$

converge to solutions of

$$\min\left\{F(x) + \frac{1}{2\tau}\|x - y_0\|^2\right\} \quad (8.5)$$

since we have a continuously converging perturbation of a Γ -converging sequence.

Let now $x_\varepsilon \rightarrow x_0$. Let τ be fixed. We consider the sequence $\{x_k^{\tau, \varepsilon}\}$ defined by iterated minimization of F_ε with initial point x_ε . Since $x_\varepsilon \rightarrow x_0$, up to subsequences we have $x_1^{\tau, \varepsilon} \rightarrow x_1^{\tau, 0}$, which minimizes

$$\min\left\{F(x) + \frac{1}{2\tau}\|x - x_0\|^2\right\}. \quad (8.6)$$

The point $x_2^{\tau,\varepsilon}$ converge to $x_2^{\tau,0}$. Since they minimize

$$\min\left\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_1^{\tau,\varepsilon}\|^2\right\} \tag{8.7}$$

and $x_1^{\tau,\varepsilon} \rightarrow x_1^{\tau,0}$, their limit is a minimizer of

$$\min\left\{F(x) + \frac{1}{2\tau}\|x - x_1^{\tau,0}\|^2\right\}. \tag{8.8}$$

This operation can be repeated iteratively, obtaining (upon subsequences) $x_k^{\tau,\varepsilon} \rightarrow x_k^{\tau,0}$, and $\{x_k^{\tau,0}\}$ iteratively minimizes F with initial point x_0 . Since up to subsequences the trajectories $\{x_k^{\tau,0}\}$ converge to a minimizing movement for F with initial datum x_0 , the thesis follows by a diagonal argument.

- (ii) For fixed ε , the piecewise-constant functions $u^{\varepsilon,\tau}(t) = x_{\lfloor t/\tau \rfloor}^{\varepsilon,\tau}$ converge uniformly to a minimizing movement u^ε for F_ε with initial datum x_ε . By compactness, these u^ε converge uniformly to some function u as $\varepsilon \rightarrow 0$. Again, a diagonal argument gives the thesis. \square

Remark 8.2. Note that, given x_ε and F_ε , if F has more than one minimizing movement from x_0 then the approximation gives a choice criterion. As an example, take $F(x) = -|x|$, $F_\varepsilon(x) = -|x + \varepsilon|$ and $x_0 = x_\varepsilon = 0$.

Remark 8.3 (The convex case). If all F_ε are convex then it can be shown that, actually, the minimizing movement along the sequence F_ε always coincides with the minimizing movement for their Γ -limit. This (exceptional) case will be dealt with in detail separately in Chap. 11.

Example 8.2. In dimension one, we can take

$$F_\varepsilon(x) = \frac{1}{2}x^2 + \varepsilon W\left(\frac{x}{\varepsilon}\right),$$

where W is a one-periodic odd Lipschitz function with $\|W'\|_\infty = 1$. Up to addition of a constant is not restrictive to suppose that the average of W is 0. We check that the critical regime for the minimizing movements along F_ε is $\varepsilon \sim \tau$. Indeed, if $\varepsilon \ll \tau$ then from the estimate

$$\left|F_\varepsilon(x) - \frac{1}{2}x^2\right| \leq \frac{\varepsilon}{2}$$

we deduce that

$$\frac{x_k - x_{k-1}}{\tau} = -x_k + O\left(\frac{\varepsilon}{\tau}\right),$$

and hence that the limit minimizing movement satisfies $u' = -u$, so that it corresponds to the minimizing movement of the limit $F_0(x) = \frac{1}{2}x^2$.

Conversely, if $\tau \ll \varepsilon$ then it may be seen that for $|x_0| \leq 1$ the motion is *pinned*; i.e., the resulting minimizing movement is the trivial solution $u(t) = x_0$ for all t . If $W \in C^2$ this is easily checked, since in this case the stationary solutions, corresponding to x satisfying

$$x + W'\left(\frac{x}{\varepsilon}\right) = 0$$

tend to be dense in the interval $[-1, 1]$ as $\varepsilon \rightarrow 0$. Moreover, in this regime the minimizing movement corresponds to the limit as $\varepsilon \rightarrow 0$ of the minimizing movements of F_ε for ε fixed; i.e., solutions u_ε of the gradient flow

$$u'_\varepsilon = -u_\varepsilon - W'\left(\frac{u_\varepsilon}{\varepsilon}\right).$$

Integrating between t_1 and t_2 we have

$$\int_{u_\varepsilon(t_1)}^{u_\varepsilon(t_2)} \frac{1}{s + W'(s/\varepsilon)} ds = t_1 - t_2.$$

By the uniform convergence $u_\varepsilon \rightarrow u$ we can pass to the limit, recalling that the integrand weakly converges to the function $1/g$ defined by

$$\frac{1}{g(s)} = \int_0^1 \frac{1}{s + W'(\sigma)} d\sigma,$$

and obtain the equation

$$u' = -g(u).$$

This equation corresponds to the minimizing movement for the even energy \tilde{F}_0 given for $x \geq 0$

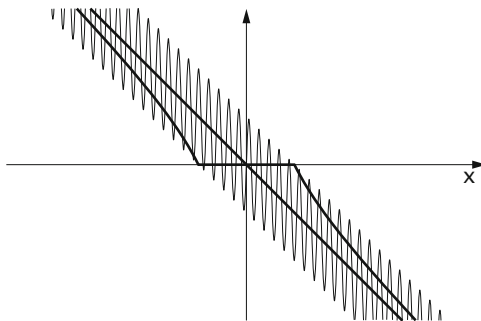
$$\tilde{F}_0(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \int_1^x g(w) dw & \text{if } x \geq 1. \end{cases}$$

The plot of the derivatives of F_ε , F_0 and \tilde{F}_0 is reproduced in Fig. 8.1

We can explicitly compute the minimizing movement for $\tau \ll \varepsilon$; e.g., in the case

$$W(x) = \frac{1}{2\pi} \sin(2\pi x),$$

Fig. 8.1 The derivatives of F_ε , F_0 and \tilde{F}_0



which gives the equation

$$u' = \sqrt{u^2 - 1},$$

for $|x_0| \geq 1$, and

$$\tilde{F}_0(x) = \frac{1}{2} \left(|x| \sqrt{x^2 - 1} - \log \left(|x| + \sqrt{x^2 - 1} \right) \right)$$

for $|x| > 1$, and in the case

$$W(x) = \left| x - \frac{1}{2} \right| - \frac{1}{4} \quad \text{for } 0 \leq x \leq 1. \tag{8.9}$$

In the latter, the solutions with initial datum $x_0 > 1$ satisfy the equation

$$u' = \frac{1}{u} - u.$$

Integrating this limit equation we conclude that the minimizing movement along F_ε corresponds to that of the effective energy

$$\tilde{F}_0(x) = \left(\frac{1}{2} x^2 - \log |x| - \frac{1}{2} \right)^+.$$

Example 8.3 (Pinning threshold). In the previous example we have computed the critical regime $\varepsilon \sim \tau$, but we have not computed the minimizing movement for a fixed ratio ε/τ . In this case, a simpler interesting problem is the computation of the pinning threshold; i.e., the maximal value T such that $|x_0| \leq T$ gives in the limit a stationary minimizing movement. We have seen that for $\varepsilon \ll \tau$ we have $T = 0$, while for $\tau \ll \varepsilon$ we have $T = 1$. After considering the linearization of the problem above, the pinning threshold can be characterized as the greatest value T such that we have only stationary minimizing movements for the energies

$$F_\varepsilon^T(x) = Tx + \varepsilon W\left(\frac{x}{\varepsilon}\right).$$

In order to have an explicit description of $T = T(\gamma)$ in terms of $\gamma := \varepsilon/\tau$, we only treat the case of

$$W(x) = |x| \quad \text{for } |x| \leq \frac{1}{2}, \quad (8.10)$$

which gives the same limit as the one in (8.9). By comparison with the case $\tau \ll \varepsilon$, we have $T(\gamma) \leq 1$ for all γ .

By a comparison argument, it is not restrictive to suppose that $x_0 \in \varepsilon\mathbb{Z}$, and then by translation that $x_0 = 0$. The problem is then translated in the existence of negative minimizers for the problem

$$\min\left\{Tx + \varepsilon W\left(\frac{x}{\varepsilon}\right) + \frac{1}{2\tau}x^2\right\}.$$

Since $T \leq 1$ and $W' = -1$ in $[-\varepsilon/2, 0]$, this holds only if we have a negative value in $[-\varepsilon, -\varepsilon/2]$, or equivalently if

$$\begin{aligned} 0 &> \min\left\{Tx + \varepsilon W\left(\frac{x}{\varepsilon}\right) + \frac{1}{2\tau}x^2 : -\varepsilon \leq x \leq -\varepsilon/2\right\} \\ &= \min\left\{(T+1)x + \varepsilon + \frac{1}{2\tau}x^2 : -\varepsilon \leq x \leq -\varepsilon/2\right\}. \end{aligned}$$

Taking again into account that $T \leq 1$, it is easily seen that this minimum must be taken for $x = -\varepsilon$, so that the condition is equivalent to

$$0 > -T\varepsilon + \frac{1}{2\tau}\varepsilon^2; \text{ i.e., } T > \frac{\varepsilon}{2\tau}.$$

This proves that we have pinning for $T \leq \gamma/2$. In conclusion, the pinning threshold is

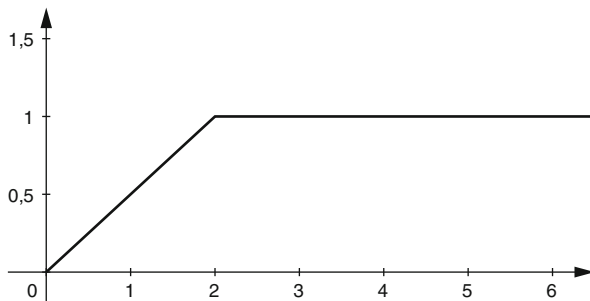
$$T(\gamma) = \min\left\{\frac{\gamma}{2}, 1\right\}$$

(see Fig. 8.2). As $\gamma \rightarrow 0$ and $\gamma \rightarrow +\infty$ we recover the thresholds in the limit cases.

8.2.1 Relaxed Evolution

In Theorem 8.1 we have considered, as usual for simplicity, the Γ -convergence with respect to the topology in X . In this way we characterize the convergence of solutions to problems (8.4) to solutions of problems (8.5) in terms of the Γ -limit. This is the only argument where we have used the definition of F in the proof of

Fig. 8.2 Pinning threshold in dependence of the ratio ε/τ



Theorem 8.1(i). We may consider the Γ -limits with respect to weaker topologies, for which we have coerciveness but the distance term is not a continuous perturbation. In analogy with what already observed for quasistatic motions in Chap. 3 (see, e.g., Sect. 3.1.5), the proof of Theorem 8.1(i) can be repeated, upon defining a *relaxed limit motion*, where the minimizing movement for F is replaced by the limit of u^τ defined by successive minimizing

$$\min_X \mathcal{F}_\tau^{x_{k-1}}(x),$$

where

$$\mathcal{F}_\tau^y(x) = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \left(F_\varepsilon(x) + \frac{1}{2\tau} \|x - y\|^2 \right). \tag{8.11}$$

The study of this more general minimizing movements is beyond the scope of these notes. We only give a simple example.

Example 8.4. Consider $X = L^2(0, 1)$ and

$$F_\varepsilon(u) = \int_0^1 a\left(\frac{x}{\varepsilon}\right) u^2 dx,$$

where a is 1-periodic and $0 < \alpha \leq a(y) \leq \beta < +\infty$ for some constants α and β . Then F_ε is equicoercive with respect to the *weak- L^2* topology, and its limit is $\underline{a} \int_0^1 u^2 dx$ (\underline{a} the harmonic mean of a). However, the perturbations with the L^2 -distance are not continuous, and the limits in (8.11) with respect to the weak topology are easily computed as

$$\begin{aligned} \mathcal{F}_\tau^v(u) &= \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \left(F_\varepsilon(u) + \frac{1}{2\tau} \|u - v\|^2 \right) \\ &= \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \int_0^1 \left(a\left(\frac{x}{\varepsilon}\right) + \frac{1}{2\tau} \right) u^2 + \frac{(v^2 - 2uv)}{2\tau} dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \left(\underline{a}_\tau u^2 + \frac{(v^2 - 2uv)}{2\tau} \right) dx \\
&= \int_0^1 \left(\underline{a}_\tau - \frac{1}{2\tau} \right) u^2 dx + \frac{1}{2\tau} \|u - v\|^2,
\end{aligned}$$

where

$$\underline{a}_\tau = \left(\int_0^1 \frac{1}{\left(a(y) + \frac{1}{2\tau} \right)} dy \right)^{-1}.$$

A series expansion argument easily yields that

$$\begin{aligned}
\underline{a}_\tau &= \frac{1}{2\tau} \left(\int_0^1 \frac{1}{2\tau a(y) + 1} dy \right)^{-1} \\
&= \frac{1}{2\tau} \left(\int_0^1 \left(1 - 2\tau a(y) + O(\tau^2) \right) dy \right)^{-1} \\
&= \frac{1}{2\tau} \left(1 + 2\tau \int_0^1 a(y) dy + O(\tau^2) \right) = \frac{1}{2\tau} + \bar{a} + O(\tau),
\end{aligned}$$

where \bar{a} is the arithmetic mean of a . We then obtain that the limit of u^τ coincides with the minimizing motion for \tilde{F} given by

$$\tilde{F}(u) = \bar{a} \int_0^1 u^2 dx.$$

The same argument leading to an effective motion can be applied to varying distances as in the following example.

Example 8.5. We consider $X_\varepsilon = X = L^2(0, 1)$ equipped with the distance d_ε given by

$$d_\varepsilon^2(u, v) = \int_0^1 a\left(\frac{x}{\varepsilon}\right) |u - v|^2 dx,$$

and $F_\varepsilon(u) = F(u) = \int_0^1 |u'|^2 dx$. For fixed v the square distances can be seen as functionals depending on v , weakly equicoercive in L^2 and Γ -converging to $\frac{1}{2\tau} \|u - v\|^2$ ($\|u\|$ the L^2 -norm). Nevertheless, in this case the functionals $F_\varepsilon(u) + \frac{1}{2\tau} d_\varepsilon^2(u, v)$ are coercive with respect to the strong L^2 -norm and Γ -converge to $F(u) + \frac{1}{2\tau} \bar{a} \|u - v\|^2$. As a conclusion, the minimizing movement coincide with the minimizing movement for F with respect to the norm $\sqrt{\bar{a}} \|u\|$ or, equivalently, with the minimizing movement for $\frac{1}{\bar{a}} F$ with respect to the L^2 -norm.

8.3 An Example: ‘Overdamped Dynamics’ of Lennard-Jones Interactions

We now give an example of a sequence of non-convex energies which commute with the minimizing movement procedure.

Let J be as in Sect. 4.4 and $\frac{1}{\varepsilon} = N \in \mathbb{N}$. We consider the energies

$$F_\varepsilon(u) = \sum_{i=1}^N J\left(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\right)$$

with the periodic boundary condition $u_N = u_0$. As proved in Sect. 4.4, after identification of u with a piecewise-constant function on $[0, 1]$, these energies Γ -converge to the energy

$$F(u) = \int_0^1 |u'|^2 dt + \#(S(u) \cap [0, 1)), \quad u^+ > u^-,$$

defined on piecewise- H^1 functions, in this case extended 1-periodically on the whole real line.

In this section we apply the minimizing movements scheme to F_ε as a sequence of functionals in $L^2(0, 1)$. In order to have initial data u_0^ε with equibounded energy, we may suppose that these are the discretization of a single piecewise- H^1 function u_0 (with a slight abuse of notation we will continue to denote all those discrete functions by u_0).

With fixed ε and τ , the time-discretization scheme consists in defining recursively u^k as a minimizer of

$$u \mapsto \sum_{i=1}^N J\left(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\right) + \frac{1}{2\tau} \sum_{i=1}^N \varepsilon |u_i - u_i^{k-1}|^2. \quad (8.12)$$

By Proposition 8.1, upon extraction of a subsequence, the functions $u^\tau(t) = u_{\lfloor t/\tau \rfloor}$ converge uniformly in L^2 to a function $u \in C^{1/2}([0, +\infty); L^2(0, 1))$. Moreover, since we have $F(u(t)) \leq F(u_0) < +\infty$, $u(t)$ is a piecewise- H^1 function for all t .

We now describe the motion of the limit u . For the sake of simplicity we suppose that u_0 is a piecewise-Lipschitz function and that $S(u_0) \cap \{\varepsilon i : i \in \{1, \dots, N\}\} = \emptyset$ (so that we do not have any ambiguity in the definition of the interpolations of u_0).

We first write down the Euler-Lagrange equations for u^k , which simply amount to a N -dimensional system of equations obtained by deriving (8.12) with respect to u_i

$$\frac{1}{\sqrt{\varepsilon}} \left(J' \left(\frac{u_i^k - u_{i-1}^k}{\sqrt{\varepsilon}} \right) - J' \left(\frac{u_{i+1}^k - u_i^k}{\sqrt{\varepsilon}} \right) \right) + \frac{\varepsilon}{\tau} (u_i^k - u_i^{k-1}) = 0. \quad (8.13)$$

- With fixed $i \in \{1, \dots, N\}$ let v_k be defined by

$$v_k = \frac{u_i^k - u_{i-1}^k}{\varepsilon}.$$

For simplicity of notation, we set

$$J_\varepsilon(w) = \frac{1}{\varepsilon} J(\sqrt{\varepsilon} w).$$

By (8.13) and the corresponding equation for $i - 1$, which can be rewritten as

$$J'_\varepsilon\left(\frac{u_{i-1}^k - u_{i-2}^k}{\varepsilon}\right) - J'_\varepsilon\left(\frac{u_i^k - u_{i-1}^k}{\varepsilon}\right) + \frac{\varepsilon}{\tau}(u_{i-1}^k - u_{i-1}^{k-1}) = 0,$$

we have

$$\begin{aligned} \frac{v_k - v_{k-1}}{\tau} &= \frac{1}{\tau} \left(\frac{u_i^k - u_{i-1}^k}{\varepsilon} - \frac{u_i^{k-1} - u_{i-1}^{k-1}}{\varepsilon} \right) \\ &= \frac{1}{\varepsilon} \left(\frac{u_i^k - u_i^{k-1}}{\tau} - \frac{u_{i-1}^k - u_{i-1}^{k-1}}{\tau} \right) \\ &= \frac{1}{\varepsilon^2} \left(\left(J'_\varepsilon\left(\frac{u_{i-1}^k - u_{i-2}^k}{\varepsilon}\right) - J'_\varepsilon\left(\frac{u_i^k - u_{i-1}^k}{\varepsilon}\right) \right) \right. \\ &\quad \left. - \left(J'_\varepsilon\left(\frac{u_i^k - u_{i-1}^k}{\varepsilon}\right) - J'_\varepsilon\left(\frac{u_{i+1}^k - u_i^k}{\varepsilon}\right) \right) \right), \end{aligned}$$

so that

$$\begin{aligned} \frac{v_k - v_{k-1}}{\tau} - \frac{2}{\varepsilon^2} J'_\varepsilon(v_k) &= -\frac{1}{\varepsilon^2} \left(J'_\varepsilon\left(\frac{u_{i-1}^k - u_{i-2}^k}{\varepsilon}\right) + J'_\varepsilon\left(\frac{u_{i+1}^k - u_i^k}{\varepsilon}\right) \right) \\ &\geq -\frac{2}{\varepsilon^2} J'_\varepsilon\left(\frac{w_0}{\sqrt{\varepsilon}}\right). \end{aligned} \tag{8.14}$$

We recall that we denote by w_0 the maximum point of J' .

We can read (8.14) as an inequality for the difference system

$$\frac{v_k - v_{k-1}}{\eta} - 2J'_\varepsilon(v_k) \geq -2J'_\varepsilon\left(\frac{w_0}{\sqrt{\varepsilon}}\right),$$

where $\eta = \tau/\varepsilon^2$ is interpreted as a discretization step. Note that $v_k = w_0/\sqrt{\varepsilon}$ for all k is a stationary solution of the equation

$$\frac{v_k - v_{k-1}}{\eta} - 2J'_\varepsilon(v_k) = -2J'_\varepsilon\left(\frac{w_0}{\sqrt{\varepsilon}}\right)$$

and that J'_ε are equi-Lipschitz functions on $[0, +\infty)$. If $\eta \ll 1$ this implies that if $v_{k_0} \leq w_0/\sqrt{\varepsilon}$ for some k_0 then

$$v_k \leq \frac{w_0}{\sqrt{\varepsilon}} \quad \text{for } k \geq k_0,$$

or, equivalently, that if $\tau \ll \varepsilon^2$ the set

$$S_\varepsilon^k = \left\{ i \in \{1, \dots, N\} : \frac{u_i^k - u_{i-1}^k}{\varepsilon} \geq \frac{w_0}{\sqrt{\varepsilon}} \right\}$$

is decreasing with k . By our assumption on u_0 , for ε small enough we then have

$$S_\varepsilon^0 = \left\{ i \in \{1, \dots, N\} : [\varepsilon(i-1), \varepsilon i] \cap S(u_0) \neq \emptyset \right\},$$

so that, passing to the limit

$$S(u(t)) \subseteq S(u_0) \quad \text{for all } t \geq 0. \quad (8.15)$$

- Taking into account that we may define

$$u^\tau(t, x) = u_{\lfloor x/\varepsilon \rfloor}^{\lfloor t/\tau \rfloor},$$

we may choose functions $\phi \in C_0^\infty(0, T)$ and $\psi \in C_0^\infty(x_1, x_2)$, with $(x_1, x_2) \cap S(u_0) = \emptyset$, and obtain from (8.13)

$$\begin{aligned} & \int_0^T \int_{x_1}^{x_2} u^\tau(t, x) \left(\frac{\phi(t) - \phi(t + \tau)}{\tau} \right) \psi(x) dx dt \\ &= - \int_0^T \int_{x_1}^{x_2} \left(\frac{1}{\sqrt{\varepsilon}} J' \left(\sqrt{\varepsilon} \frac{u^\tau(t, x) - u^\tau(t, x - \varepsilon)}{\varepsilon} \right) \right) \\ & \quad \times \phi(t) \left(\frac{\psi(x) - \psi(x + \varepsilon)}{\varepsilon} \right) dx dt. \end{aligned}$$

Taking into account that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\varepsilon}} J'(\sqrt{\varepsilon} w) = 2w,$$

we can pass to the limit and obtain that

$$- \int_0^T \int_{x_1}^{x_2} u(t, x) \phi'(t) \psi(x) dx dt = \int_0^T \int_{x_1}^{x_2} 2 \frac{\partial u}{\partial x} \phi(t) \psi'(x) dx dt;$$

i.e., that

$$\frac{\partial u}{\partial t} = -2 \frac{\partial^2 u}{\partial x^2} \quad (8.16)$$

in the sense of distributions (and hence also classically) in $(0, T) \times (x_1, x_2)$. By the arbitrariness of the interval (x_1, x_2) we have that equation (8.16) is satisfied for x in $(0, 1) \setminus S(u_0)$.

- We now derive boundary conditions on $S(u(t))$. Let $i_0 + 1$ belong to S_ε^0 , and suppose that $u^+(t, x) - u^-(t, x) \geq c > 0$. Then we have

$$\lim_{\tau \rightarrow 0} \frac{1}{\sqrt{\varepsilon}} J' \left(\frac{u_{i_0}^{\lfloor t/\tau \rfloor} - u_{i_0-1}^{\lfloor t/\tau \rfloor}}{\sqrt{\varepsilon}} \right) = 0.$$

If $i < i_0$, from (8.13) it follows, after summing up the indices from i to i_0 , that

$$\sum_{j=i}^{i_0} \frac{\varepsilon}{\tau} (u_j^k - u_{j-1}^k) = -\frac{1}{\sqrt{\varepsilon}} J' \left(\frac{u_i^k - u_{i-1}^k}{\sqrt{\varepsilon}} \right). \quad (8.17)$$

We may choose $i = i_\varepsilon$ such that $\varepsilon i_\varepsilon \rightarrow \bar{x}$ and we may deduce from (8.17) that

$$\int_{\bar{x}}^{x_0} \frac{\partial u}{\partial t} dx = -2 \frac{\partial u}{\partial x}(\bar{x}),$$

where $x_0 \in S(u(t))$ is the limit of εi_0 . Letting $\bar{x} \rightarrow x_0^-$ we obtain

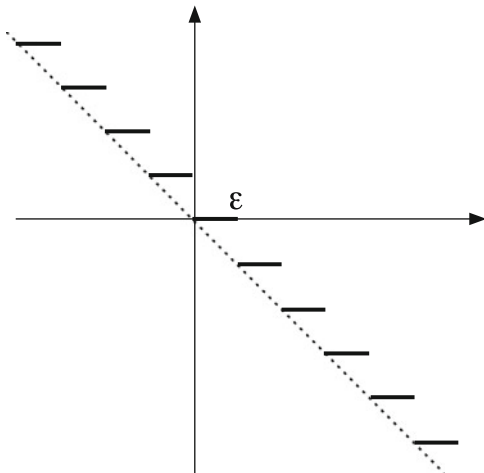
$$\frac{\partial u}{\partial x}(x_0^-) = 0.$$

Similarly we obtain the homogeneous Neumann condition at x_0^+ .

Summarizing, the minimizing movement along the scaled Lennard-Jones energies F_ε from a piecewise- H^1 function consists in a piecewise- H^1 motion, following the heat equation on $(0, 1) \setminus S(u_0)$, with homogeneous Neumann boundary conditions on $S(u_0)$ (as long as $u(t)$ has a discontinuity at the corresponding point of $S(u_0)$).

Note that for $\varepsilon \rightarrow 0$ sufficiently fast Theorem 8.1 directly ensures that the minimizing movement along F_ε coincides with the minimizing movement for the functional F . The computation above shows that this holds also for $\tau \ll \varepsilon^2$ (i.e., $\varepsilon \rightarrow 0$ ‘sufficiently slow’), which then must be regarded as a technical condition.

Fig. 8.3 The function F_ε



8.4 Homogenization of Minimizing Movements

We now examine minimizing movements along oscillating sequences (with many local minima), treating two model cases in the real line.

8.4.1 Minimizing Movements for Piecewise-Constant Energies

We apply the minimizing-movement scheme to the functions

$$F_\varepsilon(x) = -\left\lfloor \frac{x}{\varepsilon} \right\rfloor \varepsilon$$

converging to $F(x) = -x$ (see Fig. 8.3). This is a prototype of a function with many local minimizers (actually, in this case all points are local minimizers) converging to a function with few local minimizers (actually, none).

Note that, with fixed ε , for any initial datum x_0 the minimizing movement for F_ε is trivial: $u(t) = x_0$, since all points are local minimizers. Conversely, the corresponding minimizing movement for the limit is $u(t) = x_0 + t$.

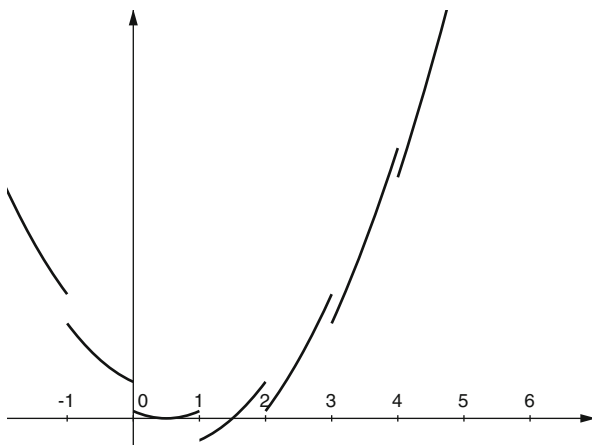
We now fix an initial datum x_0 , the space scale ε and the time scale τ , and examine the successive-minimization scheme from x_0 . Note that it is not restrictive to suppose that $0 \leq x_0 < 1$ up to a translation in $\varepsilon\mathbb{Z}$.

The first minimization, giving x_1 , is

$$\min\left\{F_\varepsilon(x) + \frac{1}{2\tau}(x - x_0)^2\right\}. \tag{8.18}$$

The function to minimize is pictured in Fig. 8.4 in normalized coordinates ($\varepsilon = 1$); note that it equals $-x + \frac{1}{2\tau}(x - x_0)^2$ if $x \in \varepsilon\mathbb{Z}$.

Fig. 8.4 The function in the minimization problem (8.18)



Apart from some exceptional cases that we deal separately below, we have two possibilities:

- (i) If $\frac{\tau}{\varepsilon} < \frac{1}{2}$ then the motion is trivial. The value $1/2$ is the **pinning threshold**.

Indeed, after setting set $x_0 = s\varepsilon$ with $0 \leq s < 1$, we have two sub-cases:

- (a) The minimizer x_1 belongs to $[0, \varepsilon)$. This occurs exactly if $F_\varepsilon(\varepsilon) + \frac{1}{2\tau}(\varepsilon - x_0)^2 > 0$; i.e.,

$$\tau < \frac{(s-1)^2\varepsilon}{2}. \quad (8.19)$$

In this case the only minimizer is the initial datum x_0 . This implies that we have $x_k = x_0$ for all k .

- (b) We have that $x_1 = \varepsilon$. This implies that, up to a translation we are in the case $x_0 = 0$ with $s = 0$, and (8.19) holds since $\tau < \frac{\varepsilon}{2}$. Hence, $x_k = x_1$ for all $k \geq 1$.

- (ii) If $\frac{\tau}{\varepsilon} > \frac{1}{2}$ then for ε small the minimum is taken on $\varepsilon\mathbb{Z}$. So that again we may suppose that $x_0 = 0$.

Note that we are leaving out for the time being the case when $x_0 = 0$ and $\frac{\tau}{\varepsilon} = \frac{1}{2}$. In that case we have a double choice for the minimizer; such situations will be examined separately.

If $x_0 = 0$ then x_1 is computed by solving

$$\min\left\{F_\varepsilon(x) + \frac{1}{2\tau}x^2 : x \in \varepsilon\mathbb{Z}\right\}, \quad (8.20)$$

and is characterized by

$$x_1 - \frac{1}{2}\varepsilon \leq \tau \leq x_1 + \frac{1}{2}\varepsilon.$$

We then have

$$x_1 = \left\lfloor \frac{\tau}{\varepsilon} + \frac{1}{2} \right\rfloor \varepsilon \quad \text{if } \frac{\tau}{\varepsilon} + \frac{1}{2} \notin \mathbb{Z}$$

(note again that we have two solutions for $\frac{\tau}{\varepsilon} + \frac{1}{2} \in \mathbb{Z}$, which also includes the case $\frac{\tau}{\varepsilon} = \frac{1}{2}$ already set aside, and we examine those cases separately). The same computation is repeated at each k giving

$$\frac{x_k - x_{k-1}}{\tau} = \left\lfloor \frac{\tau}{\varepsilon} + \frac{1}{2} \right\rfloor \frac{\varepsilon}{\tau}.$$

We can now choose τ and ε tending to 0 simultaneously and pass to the limit. The behaviour of the limit minimizing movements is governed by the quantity

$$w = \lim_{\varepsilon \rightarrow 0} \frac{\tau}{\varepsilon}, \tag{8.21}$$

which we may suppose exists up to subsequences. If $w + \frac{1}{2} \notin \mathbb{Z}$ then the minimizing movement along F_ε from x_0 is uniquely defined by

$$u(t) = x_0 + vt, \text{ with } v = \left\lfloor w + \frac{1}{2} \right\rfloor \frac{1}{w}, \tag{8.22}$$

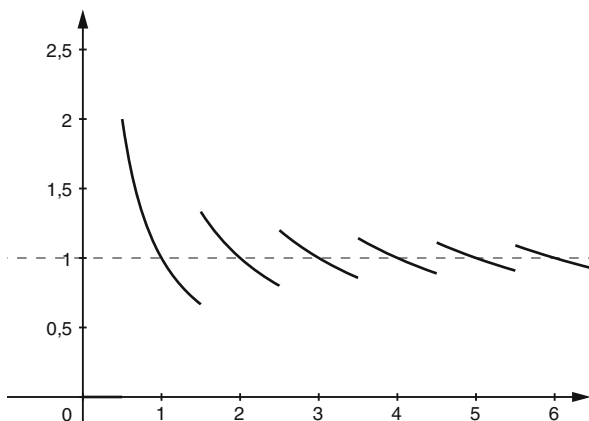
so that the whole sequence converges if the limit in (8.21) exists. Note that

- **(pinning)** we have $v = 0$ exactly when $\frac{\tau}{\varepsilon} < \frac{1}{2}$ for ε small. In particular this holds for $\tau \ll \varepsilon$ (i.e., for $w = 0$).
- **(limit motion for slow times)** if $\varepsilon \ll \tau$ then the motion coincides with the gradient flow of the limit, with velocity 1.
- **(discontinuous dependence of the velocity)** the velocity is a discontinuous function of w at points of $\frac{1}{2} + \mathbb{Z}$. Note moreover that it may be actually greater than the limit velocity 1. The graph of v is pictured in Fig. 8.5.
- **(non-uniqueness at $w \in \frac{1}{2} + \mathbb{Z}$)** in these exceptional cases we may have either of the two velocities $1 + \frac{1}{2w}$ or $1 - \frac{1}{2w}$ in the cases $\frac{\varepsilon}{\tau} + \frac{1}{2} > w$ or $\frac{\varepsilon}{\tau} + \frac{1}{2} < w$ for all ε small respectively, but we may also have any $u(t)$ with

$$1 - \frac{1}{2w} \leq u'(t) \leq 1 + \frac{1}{2w}$$

if we have precisely $\frac{\varepsilon}{\tau} + \frac{1}{2} = w$ for all ε small, since in this case at every time step we may choose any of the two minimizers giving the extremal velocities, and then obtain any such u' as a weak limit of piecewise constant functions taking only those two values. Note therefore that in this case the limit is not determined only by w , and in particular it may depend on the subsequence even if the limit (8.21) exists.

Fig. 8.5 The velocity v in terms of w



We remark that the functions F_ε above can be substituted by functions with isolated local minimizers; e.g. by taking ($\alpha > 0$)

$$F_\varepsilon(x) = -\left\lfloor \frac{x}{\varepsilon} \right\rfloor \varepsilon + \alpha \left(x - \left\lfloor \frac{x}{\varepsilon} \right\rfloor \varepsilon \right),$$

with isolated local minimizers at $\varepsilon\mathbb{Z}$ (for which the computations run exactly as above), or

$$F_\varepsilon(x) = -x + (1 + \alpha)\varepsilon \sin\left(\frac{x}{\varepsilon}\right).$$

Note that the presence of an energy barrier between local minimizers does not influence the velocity of the final minimizing movement, that can always be larger than 1 (the velocity as $\varepsilon \ll \tau$).

We also remark that the same result can be obtained by a ‘discretization’ of F ; i.e., taking

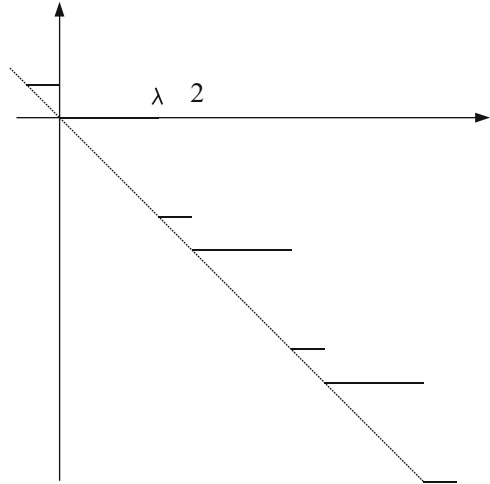
$$F_\varepsilon(x) = \begin{cases} -x & \text{if } x \in \varepsilon\mathbb{Z} \\ +\infty & \text{otherwise.} \end{cases} \quad (8.23)$$

8.4.2 A Heterogeneous Case

We briefly examine a variation of the previous example obtained by introducing a heterogeneity parameter $1 \leq \lambda \leq 2$ and defining

$$F^\lambda(x) = \begin{cases} -2\left\lfloor \frac{x}{2} \right\rfloor & \text{if } 2\left\lfloor \frac{x}{2} \right\rfloor \leq x < 2\left\lfloor \frac{x}{2} \right\rfloor + \lambda \\ -2\left\lfloor \frac{x}{2} \right\rfloor - \lambda & \text{if } 2\left\lfloor \frac{x}{2} \right\rfloor + \lambda \leq x < 2\left\lfloor \frac{x}{2} \right\rfloor + 1. \end{cases} \quad (8.24)$$

Fig. 8.6 The function F^λ



If $\lambda = 1$ we are in the previous situation; for general λ the function F^λ is pictured in Fig. 8.6.

We apply the minimizing-movement scheme to the functions

$$F_\varepsilon(x) = F_\varepsilon^\lambda(x) = \varepsilon F^\lambda\left(\frac{x}{\varepsilon}\right).$$

Arguing as above, we can reduce to the two cases

$$(a) x_k \in 2\varepsilon\mathbb{Z}, \quad \text{or} \quad (b) x_k \in 2\varepsilon\mathbb{Z} + \varepsilon\lambda.$$

Taking into account that x_{k+1} is determined as the point in $2\varepsilon\mathbb{Z} \cup (2\varepsilon\mathbb{Z} + \varepsilon\lambda)$ closer to τ (as above, we only consider the cases when we have a unique solution to the minimum problems in the iterated procedure), we can characterize it as follows.

In case (a) we have the two sub cases:

(a₁) If we have

$$2n < \frac{\tau}{\varepsilon} - \frac{\lambda}{2} < 2n + 1$$

for some $n \in \mathbb{N}$ then

$$x_{k+1} = x_k + (2n + \lambda)\varepsilon.$$

In particular $x_{k+1} \in 2\varepsilon\mathbb{Z} + \varepsilon\lambda$.

(a₂) If we have

$$2n - 1 < \frac{\tau}{\varepsilon} - \frac{\lambda}{2} < 2n$$

for some $n \in \mathbb{N}$ then

$$x_{k+1} = x_k + 2n\varepsilon.$$

In particular $x_{k+1} \in 2\varepsilon\mathbb{Z}$. Note that $x_{k+1} = x_k$ (pinning) if $\frac{\tau}{\varepsilon} < \frac{\lambda}{2}$.

In case (b) we have the two sub cases:

(b₁) If we have

$$2n < \frac{\tau}{\varepsilon} + \frac{\lambda}{2} < 2n + 1$$

for some $n \in \mathbb{N}$ then

$$x_{k+1} = x_k + 2n\varepsilon.$$

In particular $x_{k+1} \in 2\varepsilon\mathbb{Z} + \varepsilon\lambda$. Note that $x_{k+1} = x_k$ (pinning) if $\frac{\tau}{\varepsilon} < 1 - \frac{\lambda}{2}$, which is implied by the pinning condition in (a₂).

(b₂) If we have

$$2n - 1 < \frac{\tau}{\varepsilon} + \frac{\lambda}{2} < 2n$$

for some $n \in \mathbb{N}$ then

$$x_{k+1} = x_k + 2n\varepsilon - \varepsilon\lambda.$$

In particular $x_{k+1} \in 2\varepsilon\mathbb{Z}$.

Eventually, we have the two cases:

(1) When

$$\left| \frac{\tau}{\varepsilon} - 2n \right| < \frac{\lambda}{2}$$

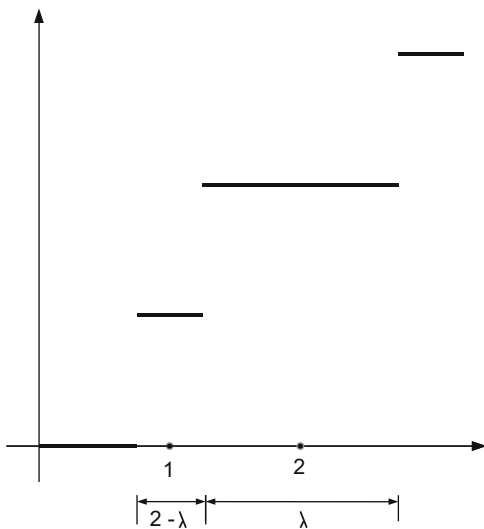
for some $n \in \mathbb{N}$ then, after possibly one iteration, we are either in the case (a₂) or (b₁). Hence, either $x_k \in 2\varepsilon\mathbb{Z}$ or $x_k \in 2\varepsilon\mathbb{Z} + \varepsilon\lambda$ for all k . The velocity in this case is then

$$\frac{x_{k+1} - x_k}{\tau} = 2n \frac{\varepsilon}{\tau}.$$

(2) When

$$\left| \frac{\tau}{\varepsilon} - (2n + 1) \right| < 1 - \frac{\lambda}{2}$$

Fig. 8.7 The function f describing the effective velocity



for some $n \in \mathbb{N}$ then we are alternately in case (a₁) or (b₂). In this case we have an

- **averaged velocity:** the speed of the orbit $\{x_k\}$ oscillates between two values with an average speed given by

$$\frac{x_{k+2} - x_k}{2\tau} = \frac{2n\varepsilon + \lambda\varepsilon}{2\tau} + \frac{2(n + 1)\varepsilon - \lambda\varepsilon}{2\tau} = (2n + 1)\frac{\varepsilon}{\tau}.$$

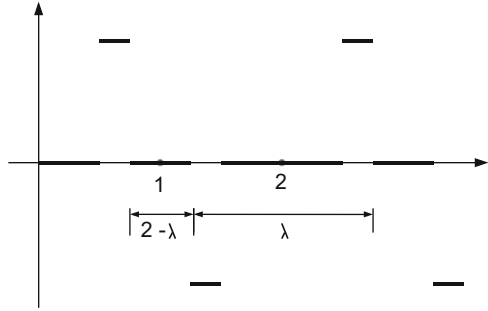
This is an additional feature with respect to the previous example.

Summarizing, if we define w as in (8.21) then (taking into account only the cases with a unique limit) the minimizing movement along the sequence F_ε with initial datum x_0 is given by $x(t) = x_0 + vt$ with $v = \frac{1}{w} f(w)$, and f is given by

$$f(w) = \begin{cases} 2n & \text{if } |w - 2n| \leq \frac{\lambda}{2}, n \in \mathbb{N} \\ 2n + 1 & \text{if } |w - (2n + 1)| < 1 - \frac{\lambda}{2}, n \in \mathbb{N} \end{cases}$$

(see Fig. 8.7). Note that the pinning threshold is now $\lambda/2$. We can compare this minimizing movement with the one given in (8.22) by examining the graph of $w \mapsto \lfloor w + 1/2 \rfloor - f(w)$ in Fig. 8.8. For $2n + 1/2 < w < 2n + \lambda/2$ the new minimizing movement is slower, while for $2n + 2 - \lambda/2 < w < 2n + 2 - 1/2$ it is faster.

Fig. 8.8 Comparison with the homogeneous case



8.4.3 A Proposal for Some Random Models

From the heterogeneous example above we may derive two possible random models, of which we may then study the corresponding minimizing movement. We only give a heuristic proposal, which can then be correctly formalized by introducing suitable random variables.

1. Random environment. Let $\lambda \in (1/2, 1)$ and $p \in [0, 1]$. We consider a random array of points $\{x_i^\omega\}$ in \mathbb{R} such that, e.g.,

$$x_i^\omega - x_{i-1}^\omega = \begin{cases} \lambda & \text{with probability } p \\ 2 - \lambda & \text{with probability } 1 - p. \end{cases} \tag{8.25}$$

With fixed ω we may consider the minimizing movement related to

$$F_\varepsilon^\omega(x) = \begin{cases} -x & \text{if } x \in \{\varepsilon x_i^\omega : i \in \mathbb{Z}\} \\ +\infty & \text{otherwise,} \end{cases}$$

or equivalently (as in the definition (8.23))

$$F_\varepsilon^\omega(x) = -\varepsilon x_i^\omega \text{ if } x \in [\varepsilon x_i^\omega, \varepsilon x_{i+1}^\omega), i \in \mathbb{Z}.$$

In the case $p = 0$ or $p = 1$ we almost surely have a homogeneous environment as in Sect. 8.4.1. For $p = 1/2$ we have a random version of the heterogeneous model of Sect. 8.4.2. Note that in this case for all $p \in (0, 1)$ the pinning threshold for the ratio τ/ε is almost surely $\lambda/2$, since below that value, the motion will be pinned at the first index i with $x_i^\omega - x_{i-1}^\omega = \lambda$; i.e., almost surely after a finite number of steps. For $\tau/\varepsilon = \lambda/2$ and $\lambda < 2/3$ (with this condition we always move of one index) then the (maximal) velocity after pinning is $v = \lambda p + (1-p)$ (for $\lambda > 2/3$ the computation of the velocity involves the probability of m -consecutive points x_i^ω at distance $2 - \lambda$).

2. Random movements. Let $\lambda \in (1/2, 1)$ and $p \in [0, 1]$. Contrary to the model above, we suppose that at every time step k we may make a random choice of points $\{x_i^{\omega_k}\}$ satisfying (8.25) such that $x_k^\varepsilon \in \{x_i^{\omega_k}\}$; i.e., this choice now represents the random possibility of motion of the point itself (and not a characteristic of the medium). Note that in this case for $p \in (0, 1)$ the pinning threshold for the ratio τ/ε is almost surely the lower value $1 - \frac{\lambda}{2}$, and the (maximal) velocity after pinning is $v = (2 - \lambda)(1 - p)$.

8.5 Time-Dependent Minimizing Movements

Following the arguments of Sect. 7.2 we can define a minimizing movement along a time-dependent sequence of energies $F_\varepsilon(x, t)$, upon some technical assumptions as in (7.10). In this case we fix a sequence of initial data x_0^ε and $\tau = \tau_\varepsilon \rightarrow 0$, and define recursively x_k^ε as minimizing

$$\min \left\{ F_\varepsilon(x, k\tau) + \frac{1}{2\tau} \|x - x_{k-1}^\varepsilon\|^2 \right\}. \tag{8.26}$$

A minimizing movement is then any limit u of u^ε defined by $u^\varepsilon(t) = x_{\lfloor t/\tau \rfloor}^\varepsilon$.

We only give a simple one-dimensional example with a time-dependent forcing term.

Example 8.6. We consider

$$F_\varepsilon(x, t) = \varepsilon W\left(\frac{x}{\varepsilon}\right) - tx$$

with W as in Example 8.2. Similarly to that example we can check that $\varepsilon \sim \tau$ is the critical case, and we can explicitly describe the minimizing movement in the extreme cases:

- ($\varepsilon \ll \tau$) the minimizing movement is that corresponding to $F_0(x, t) = -tu$; i.e., to the equation $u' = t$.
- ($\tau \ll \varepsilon$) the minimizing movement is that corresponding to the function

$$\tilde{F}_0(t, x) = \begin{cases} 0 & \text{if } t \leq 1 \\ g(t)u & \text{if } t \geq 1, \end{cases}$$

where g is now defined by

$$\frac{1}{g(t)} = \int_0^1 \frac{1}{W'(\sigma) - t} d\sigma.$$

8.6 Varying Dissipations: BV-Solutions of Evolution Equations

In the previous sections of this chapter, we have limited ourselves to a Hilbert setting. This often rules out interesting applications, in particular a viscosity approach to quasistatic motion as a limit of gradient flows, which is obtained by perturbing a positively one-homogeneous dissipation \mathcal{D} by a sequence $\mathcal{D} + \frac{1}{\tau}\mathcal{D}_\varepsilon$ for which a gradient flow-type motion can be defined using the minimizing-movement approach. In general, the limit of these gradient flows gives a motion, called *BV-solution*, which is different from the energetic solution as defined in Sect. 3.2, and can be characterized in a variational way different from the energy balance. A treatment of this subject is beyond the scope of these notes, since it would need a too refined introduction to the theory of gradient flows in metric spaces, even though it would fit the spirit of the book since it may be stated in terms of Γ -limits. Many of the arguments followed above for varying energies also hold for varying dissipations.

We only deal with a simple example, in order to highlight the differences with energetic solutions.

Example 8.7 (Nonconvex mechanical play). We can consider the double-well potential in Example 3.3 and the perturbed dissipations

$$\mathcal{F}(t, x) = \frac{1}{2} \min\{(x-1)^2, (x+1)^2\} - tx, \quad \mathcal{D}_{\varepsilon, \tau}(x) = |x| + \frac{\varepsilon}{2\tau} x^2,$$

with $x_0 \in [-2, -1]$. Then the sequence x_k^τ is increasing and minimizes

$$\min \left\{ \frac{1}{2} \min\{(x-1)^2, (x+1)^2\} - (k\tau - 1)x - x_{k-1}^\tau + \frac{\varepsilon}{2\tau} (x - x_{k-1}^\tau)^2 : x \geq x_{k-1}^\tau \right\}.$$

We fix the ratio

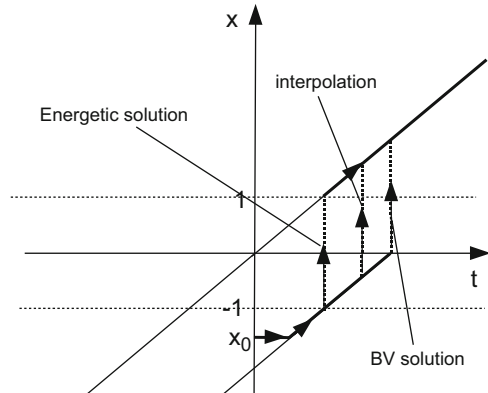
$$\gamma = \frac{\varepsilon}{\tau}. \tag{8.27}$$

With a computation similar to the one in Example 3.3, we obtain as limit the solution

$$x(t) = \begin{cases} x_0 & \text{if } t \leq x_0 + 2 \\ t - 2 & \text{if } x_0 \leq t \leq 2 - \frac{1}{\gamma+1} \\ t & \text{if } t > 2 - \frac{1}{\gamma+1} \end{cases}$$

or the one equal to this except for $t = 2 - \frac{1}{\gamma+1}$ where $x = t$.

Fig. 8.9 Interpolation between energetic and BV solutions



Remark 8.4 (Interpolations of energetic and BV solutions). In the previous example, the case $\varepsilon \ll \tau$ (formally, $\gamma = 0$) gives the energetic solution obtained in Example 3.3. The case $\tau \ll \varepsilon$ (formally, $\gamma = +\infty$) corresponds to the BV-solution hinted at above. The case in which (8.27) holds can be interpreted as an interpolation between these two extreme case, and is pictured in Fig. 8.9.

Appendix

The definition of minimizing movement along a sequence of functionals formalizes a natural extension to the notion of minimizing movement, and follows the definition given in the paper by Braides et al. [2].

The energies in Examples 8.2 and 8.6 have been taken as a prototype to model plastic phenomena by Puglisi and Truskinovsky [7]. More recently, that example has been recast in the framework of quasistatic motion in the papers by Mielke and Truskinovsky [4, 6].

The example of the minimizing movement for Lennard-Jones interactions is part of results of Braides et al. [1]. It is close in spirit to a semi-discrete approach (i.e., the study of the limit of the gradient flows for the discrete energies) by Gobbino [3].

For the notion of BV-solution we refer to Mielke et al. [5].

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Chapter 9

Geometric Minimizing Movements

We now examine some minimizing movements describing the motion of sets. Such a motion can be framed in the setting of the previous chapter after identification of a set A with its characteristic function $u = \chi_A$. The energies we are going to consider are of perimeter type; i.e., with

$$F(A) = \mathcal{H}^{n-1}(\partial A) \tag{9.1}$$

as a prototype in the notation of the previous chapter.

9.1 Motion by Mean Curvature

The prototype of a geometric motion is *motion by mean curvature*; i.e., a family of sets $A(t)$ whose boundary moves in the normal direction with velocity proportional to its curvature (inwards in convex regions and outwards in concave regions). In the simplest case when the initial datum is a ball in \mathbb{R}^2 , $A(0) = A_0 = B_{R_0}(0)$, the motion is given by concentric balls with radii satisfying

$$\begin{cases} R' = -\frac{c}{R} \\ R(0) = R_0; \end{cases} \tag{9.2}$$

i.e., $R(t) = \sqrt{R_0^2 - 2ct}$, valid until the *extinction time* $t = R_0^2/2c$, when the radius vanishes.

A heuristic argument suggests that the variation of the perimeter be linked to the notion of curvature; hence, we expect to be able to obtain motion by mean curvature as a minimizing movement for the perimeter functional. We will see that, in order to obtain geometric motions as minimizing movements, we will have to modify the procedure described in the previous chapter.

Example 9.1 (Pinning for the perimeter motion). Let $n = 2$. We apply the minimizing-movement procedure to the perimeter functional (9.1) and the initial datum $A_0 = B_{R_0}(0)$ in \mathbb{R}^2 .

With fixed τ , since

$$\int_{\mathbb{R}^2} |\chi_A - \chi_B|^2 dx = |A \Delta B|,$$

the minimization to determine A_1 is

$$\min \left\{ \mathcal{H}^1(\partial A) + \frac{1}{2\tau} |A \Delta A_0| \right\}. \quad (9.3)$$

We note that we can restrict our attention to sets A contained in A_0 , since otherwise taking $A \cap A_0$ as test sets in their place would decrease both terms in the minimization. Once this is observed, we also note that, given $A \subset A_0$, if $B_R(x) \subset A_0$ has the same measure as A then it decreases the perimeter part of the energy (strictly, if A itself is not a ball) while keeping the second term fixed. Hence, we can limit our analysis to balls $B_R(x) \subset A_0$, for which the energy depends only on R . The incremental problem is then given by

$$\min \left\{ 2\pi R + \frac{\pi}{2\tau} (R_0^2 - R^2) : 0 \leq R \leq R_0 \right\}, \quad (9.4)$$

whose minimizer is either $R = 0$ (with value $\frac{\pi}{2\tau} R_0^2$) or $R = R_0$ (with value $2\pi R_0$), since in (9.4) we are minimizing a concave function of R . For τ small the minimizer is then R_0 . This means that the motion is trivial: $A_k = A_0$ for all k , and hence also the resulting minimizing movement is trivial.

9.2 A First (Unsuccessful) Generalization

We may generalize the scheme of the minimizing movements by taking a more general distance term in the minimization; e.g., considering x_k as a minimizer of

$$\min \left\{ F(x) + \frac{1}{\tau} \Phi(\|x - x_{k-1}\|) \right\}, \quad (9.5)$$

where Φ is a continuous increasing function with $\Phi(0) = 0$. As an example, we can consider

$$\Phi(z) = \frac{1}{p} |z|^p.$$

Note that in this case we obtain the estimate

$$\|x_k - x_{k-1}\|^p \leq p \tau (F(x_{k-1}) - F(x_k))$$

for the minimizer x_k . Using Hölder's inequality as in the case $p = 2$, we end up with (for $j > h$)

$$\begin{aligned} \|x_j - x_h\| &\leq (j - h)^{(p-1)/p} \left(\sum_{k=h+1}^j \|x_k - x_{k-1}\|^p \right)^{1/p} \\ &\leq (p F(x_0))^{1/p} (\tau^{1/(p-1)} (j - h))^{(p-1)/p}. \end{aligned}$$

In order to obtain the $(1 - \frac{1}{p})$ -Hölder continuity for the interpolated function u^τ , we have to define it as

$$u^\tau(t) = u_{\lfloor t/\tau^{1/(p-1)} \rfloor}.$$

Note that we may use the previous definition $u^\tau(t) = u_{\lfloor t/\tau \rfloor}$ with time step τ for the interpolated function if we change the parameter τ in (9.5) and, to define x_k , consider the problem

$$\min \left\{ F(x) + \frac{1}{\tau^{p-1}} \Phi(\|x - x_{k-1}\|) \right\} \quad (9.6)$$

instead.

Example 9.2 ((non-)geometric minimizing movements). We use the scheme above, with a slight variation in the exponents, since we will be interested in the description of the motion in terms of the radius of a ball in \mathbb{R}^2 (which is the square root of the L^2 -norm and not the norm itself). As in the previous example, we take the initial datum $A_0 = B_{R_0} = B_{R_0}(0)$, and consider A_k defined recursively as a minimizer of

$$\min \left\{ \mathcal{H}^1(\partial A) + \frac{1}{p\tau^{p-1}} |A \Delta A_0|^p \right\}, \quad (9.7)$$

with $p > 1$. As above, at each step the minimizer is given by balls

$$B_{R_k}(x_k) \subset B_{R_{k-1}}(x_{k-1}). \quad (9.8)$$

The value of R_k is determined by solving

$$\min \left\{ 2\pi R + \frac{\pi^p}{p\tau^{p-1}} (R_{k-1}^2 - R^2)^p : 0 \leq R \leq R_{k-1} \right\}, \quad (9.9)$$

which gives

$$\frac{R_k - R_{k-1}}{\tau} = -\frac{1}{\pi R_k^{1/(p-1)}(R_k + R_{k-1})}. \quad (9.10)$$

Note that, in this case, the minimum value is not taken at $R_k = R_{k-1}$ (this can be verified, e.g., by checking that the derivative of the function to be minimized in (9.9) is positive at R_{k-1}). By passing to the limit in (9.10) we deduce the equation

$$R' = -\frac{1}{2\pi R^{p/(p-1)}} \quad (9.11)$$

(valid until the extinction time).

Despite having obtained an equation for R , we notice that this approach is not satisfactory, since we have:

- **(non-geometric motion)** in (9.8) we have infinitely many solutions; namely, all balls centered in x_k with

$$|x_{k-1} - x_k| \leq R_{k-1} - R_k.$$

This implies that we may have moving centres $x(t)$ provided that $|x'| \leq R'$ and $x(0) = 0$; in particular, we may choose $x(t) = (R_0 - R(t))z$ for any $z \in B_1(0)$ which converges to R_0z ; i.e., the point where the sets concentrate at the vanishing time may be any point in \bar{B}_{R_0} at the extinction time. This implies that the motion is not a geometric one: sets do not move according to geometric quantities.

- **(failure to obtain mean-curvature motion)** even if we obtain an equation for R we never obtain the mean curvature flow since $p/(p-1) > 1$.

9.3 A Variational Approach to Curvature-Driven Motion

In order to obtain motion by curvature, Almgren, Taylor and Wang have introduced a variation of the implicit-time scheme described above, where the term $|A \Delta A_k|$ is substituted by an integral term which favours variations which are ‘uniformly distant’ to the boundary of A_k . The problem defining A_k is then

$$\min \left\{ \mathcal{H}^1(\partial A) + \frac{1}{\tau} \int_{A \Delta A_{k-1}} \text{dist}(x, \partial A_{k-1}) dx \right\}. \quad (9.12)$$

Note that the integral term can be indeed interpreted as an L^2 distance between the boundaries of the sets.

We will not prove a general convergence result for an arbitrary initial datum A_0 , but we will check the convergence to mean-curvature motion for $A = B_{R_0}$ in \mathbb{R}^2 .

In this case we note that if A_{k-1} is a ball centered in 0 then we have:

- A_k is contained in A_{k-1} . To check this, note that, given a test set A , considering $A \cap A_{k-1}$ as a test set in its place decreases the energy in (9.12), strictly if $A \setminus A_{k-1} \neq \emptyset$.
- A_k is convex and with baricenter in 0. To check this, first, note that each connected component of A_k is convex. Otherwise, considering the convex envelopes decreases the energy (strictly, if one of the connected components is not convex). Then note that if 0 is not the baricenter of a connected component of A_k then a small translation towards 0 strictly decreases the energy (this follows by computing the derivative of the volume term along the translation). In particular, we only have one (convex) connected component.

From these properties we can conclude that A_k is indeed a ball centered in 0. Were it not so, there would be a line through 0 such that the boundary of A_k does not intersect perpendicularly this line. By a reflection argument, we then obtain a non-convex set \tilde{A}_k with total energy not greater than the one of A_k (note that the line considered subdivides A_k into two subsets with equal total energy). Its convexification would then strictly decrease the energy. This shows that each A_k is of the form

$$A_k = B_{R_k} = B_{R_k}(0).$$

We can now compute the equation satisfied by R_k , by minimizing (after passing to polar coordinates)

$$\min \left\{ 2\pi R + \frac{2\pi}{\tau} \int_R^{R_{k-1}} (R_{k-1} - \rho) \rho \, d\rho \right\}, \tag{9.13}$$

which gives

$$\frac{R_k - R_{k-1}}{\tau} = -\frac{1}{R_k}. \tag{9.14}$$

Passing to the limit gives the desired mean curvature equation (9.2).

9.4 Homogenization of Flat Flows

We now consider geometric functionals with many local minimizers (introduced in Example 4.3) which give a more refined example of homogenization of minimizing movements. The functionals we consider are defined on (sufficiently regular) subsets of \mathbb{R}^2 by

$$F_\varepsilon(A) = \int_{\partial A} a\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^1, \tag{9.15}$$

where

$$a(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z} \text{ or } x_2 \in \mathbb{Z} \\ 2 & \text{otherwise.} \end{cases}$$

The Γ -limit of the energies F_ε is the *crystalline perimeter energy*

$$F(A) = \int_{\partial A} \|v\|_1 d\mathcal{H}^1, \quad (9.16)$$

with $\|(v_1, v_2)\|_1 = |v_1| + |v_2|$. A minimizing movement for F is called a *flat flow*. We will first briefly describe it, and then compare it with the minimizing movements for F_ε .

9.4.1 Motion by Crystalline Curvature

The incremental problems for the minimizing-movement scheme for F in (9.16) are of the form

$$\min \left\{ F(A) + \frac{1}{\tau} \int_{A \Delta A_{k-1}} \text{dist}_\infty(x, \partial A_{k-1}) dx \right\}, \quad (9.17)$$

where for technical reasons we consider the ∞ -distance

$$\text{dist}_\infty(x, B) = \inf \{ \|x - y\|_\infty : y \in B \}.$$

However, in the simplified situation below this will not be relevant in our computations.

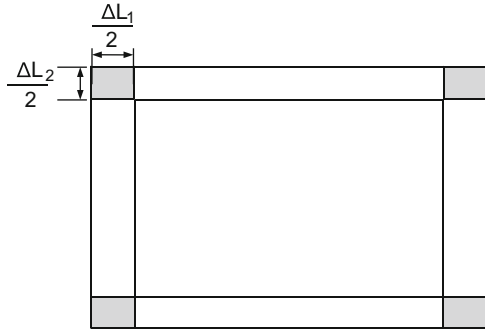
We only consider the case when the initial datum A_0 is a rectangle, which plays the role played by a ball for motion by mean curvature. Note that, as in Sect. 9.3, we can prove that if A_{k-1} is a rectangle, then we can limit the computation in (9.17) to

- A contained in A_{k-1} (otherwise $A \cap A_{k-1}$ strictly decreases the energy).
- A with each connected component a rectangle (otherwise taking the least rectangle containing a given component would decrease the energy, strictly if A is not a rectangle).
- A connected and with the same center as A_0 (since translating the center towards 0 decreases the energy).

Hence, we may suppose that

$$A_k = \left[-\frac{L_{k,1}}{2}, \frac{L_{k,1}}{2} \right] \times \left[-\frac{L_{k,2}}{2}, \frac{L_{k,2}}{2} \right]$$

Fig. 9.1 Incremental crystalline minimization



for all k . In order to iteratively determine L_k , we have to minimize the energy

$$\min \left\{ 2(L_{k,1} + \Delta L_1) + 2(L_{k,2} + \Delta L_2) + \frac{1}{\tau} \int_{A \Delta A_{k-1}} \text{dist}_\infty(x, \partial A_{k-1}) dx \right\}. \quad (9.18)$$

In this computation it is easily seen that for τ small the integral term can be substituted by

$$\frac{L_{k,1}}{4} (\Delta L_2)^2 + \frac{L_{k,2}}{4} (\Delta L_1)^2.$$

This argument amounts to noticing that the contribution of the small rectangles at the corners highlighted in Fig. 9.1 is negligible as $\tau \rightarrow 0$. The optimal increments (more precisely, decrements) ΔL_j are then determined by the conditions

$$\begin{cases} 1 + \frac{L_{k,2}}{4\tau} \Delta L_1 = 0 \\ 1 + \frac{L_{k,1}}{4\tau} \Delta L_2 = 0. \end{cases} \quad (9.19)$$

Hence, we have the difference equations

$$\frac{\Delta L_1}{\tau} = -\frac{4}{L_{k,2}}, \quad \frac{\Delta L_2}{\tau} = -\frac{4}{L_{k,1}}, \quad (9.20)$$

which finally gives the system of ODEs for the limit rectangles, with edges of length $L_1(t)$ and $L_2(t)$ respectively,

$$\begin{cases} L_1' = -\frac{4}{L_2} \\ L_2' = -\frac{4}{L_1}. \end{cases} \quad (9.21)$$

Geometrically, each edge of the rectangle moves inwards with velocity inversely proportional to its length; more precisely, equal to twice the inverse of its length (so that the other edge contracts with twice this velocity). Hence, in this context the inverse of the length of an edge plays the role of the curvature (crystalline curvature).

It is worth noticing that by (9.21) all rectangles are homothetic, since $\frac{d}{dt} \frac{L_1}{L_2} = 0$, and with area satisfying

$$\frac{d}{dt} L_1 L_2 = -8,$$

so that $L_1(t)L_2(t) = L_{0,1}L_{0,2} - 8t$, which gives the extinction time $t = L_{0,1}L_{0,2}/8$. In the case of an initial datum a square of side length L_0 , the sets are squares whose side length at time t is given by $L(t) = \sqrt{L_0^2 - 8t}$, in analogy with the evolution of balls by mean curvature flow.

9.5 Homogenization of Oscillating Perimeters

We consider the sequence F_ε in (9.15). Note that, for any (sufficiently regular) initial datum A_0 , we have that $A'_\varepsilon \subset A_0 \subset A''_\varepsilon$, where A'_ε and A''_ε are such that $F_\varepsilon(A'_\varepsilon) = \mathcal{H}^1(\partial A'_\varepsilon)$ and $F_\varepsilon(A''_\varepsilon) = \mathcal{H}^1(\partial A''_\varepsilon)$ and $|A''_\varepsilon \setminus A'_\varepsilon| = O(\varepsilon)$. Such sets are local minimizers for F_ε and hence the minimizing movement of F_ε from either of them is trivial. As a consequence, if $A_\varepsilon(t)$ is a minimizing movement for F_ε from A_0 we have

$$A'_\varepsilon \subset A_\varepsilon(t) \subset A''_\varepsilon.$$

This shows that for any set A_0 the only limit $\lim_{\varepsilon \rightarrow 0} A_\varepsilon(t)$ of minimizing movements for F_ε from A_0 is the trivial motion $A(t) = A_0$.

We now compute the minimizing movements along the sequence F_ε with initial datum a rectangle, and compare them with the flat flow described in the previous section.

For simplicity of computation we deal with a constrained case, when:

- For every ε the initial datum $A_0 = A_0^\varepsilon$ is a rectangle centered in 0 such that $F_\varepsilon(A) = \mathcal{H}^1(\partial A)$ (i.e., its edge lengths $L_{0,j}$ belong to $2\varepsilon\mathbb{Z}$). In analogy with x_0 in the example in Sect. 8.4, if this does not hold then either it does after one iteration or we have a pinned state $A_k = A_0$ for all k .
- All competing A are rectangles with $F_\varepsilon(A) = \mathcal{H}^1(\partial A)$ centered in 0. The fact that all competing sets are rectangles follows as for the flat flow in the previous section. The fact that $F_\varepsilon(A_k) \leq F_\varepsilon(A_{k-1})$ then implies that the minimal rectangles satisfy $F_\varepsilon(A_k) = \mathcal{H}^1(\partial A_k)$. The only real assumption at this point is that they are centered in 0. This hypothesis can be removed, upon a slightly more complex computation, which would only make the arguments less clear.

After this simplifications, the incremental problem is exactly as in (9.17) since for competing sets we have $F_\varepsilon(A) = F(A)$, the only difference being that now $L_{k,1}, L_{k,2} \in 2\varepsilon\mathbb{Z}$. The problem in terms of ΔL_j , using the same simplification for (9.18) as in the previous section, is then

$$\min\left\{2(L_{k,1} + \Delta L_1) + 2(L_{k,2} + \Delta L_2) + \frac{L_{k,1}}{4\tau}(\Delta L_2)^2 + \frac{L_{k,2}}{4\tau}(\Delta L_1)^2 : \Delta L_j \in 2\varepsilon\mathbb{Z}\right\}. \tag{9.22}$$

This is a minimization problem for a parabola as the ones in Sect. 8.4 that gives

$$\Delta L_1 = -\left\lfloor \frac{4\tau}{\varepsilon L_{k,2}} + \frac{1}{2} \right\rfloor \varepsilon \quad \text{if } \frac{4\tau}{\varepsilon L_{k,2}} + \frac{1}{2} \notin \mathbb{Z} \tag{9.23}$$

(the other cases giving two solutions), and an analogous equation for ΔL_2 . Passing to the limit, we have the system of ODEs governed by the parameter

$$w = \lim_{\varepsilon \rightarrow 0} \frac{\tau}{\varepsilon}$$

(which we may suppose exists, up to subsequences), which reads as

$$\begin{cases} L'_1 = -\frac{1}{w} \left\lfloor \frac{4w}{L_2} + \frac{1}{2} \right\rfloor \\ L'_2 = -\frac{1}{w} \left\lfloor \frac{4w}{L_1} + \frac{1}{2} \right\rfloor. \end{cases} \tag{9.24}$$

Note that the right-hand side is a discontinuous function of L_j , so some care must be taken at times t when $\frac{4w}{L_j(t)} + \frac{1}{2} \in \mathbb{Z}$. However, apart from some exceptional cases, this condition holds only for a countable number of t , and is therefore negligible.

We can compare the resulting minimizing movements with the crystalline curvature flow, related to F .

- **(total pinning)** if $\tau \ll \varepsilon$ ($w = 0$) then we have $A(t) = A_0$.
- **(crystalline curvature flow)** if $\varepsilon \ll \tau$ then we have the minimizing movements described in the previous section.
- **(partial pinning/asymmetric curvature flow)** if $0 < w < +\infty$ then we have
 - (i) *(total pinning)* if both $L_{0,j} > 8w$ then the motion is trivial $A(t) = A_0$.
 - (ii) *(partial pinning)* if $L_{0,1} > 8w$, $L_{0,2} < 8w$ and $\frac{4w}{L_{0,2}} + \frac{1}{2} \notin \mathbb{Z}$ then the horizontal edges do not move, but they contract with constant velocity until $L_1(t) = 8w$.
 - (iii) *(asymmetric curvature flow)* if $L_{0,1} \leq 8w$ and $L_{0,2} < 8w$ then we have a unique motion with $A(t) \subset\subset A(s)$ if $t > s$, up to a finite extinction time. Note, however, that the sets $A(s)$ are not homothetic, except for the trivial case when A_0 is a square.

Some cases are not considered above, namely those when we do not have uniqueness of minimizers in the incremental problem. This may lead to a multiplicity of minimizing movements, as remarked in Sect. 8.4.

It is worthwhile to highlight that we may rewrite the equations for L'_j as a variation of the crystalline curvature flow; e.g., for L'_1 we can write it as

$$L'_1 = -f\left(\frac{L_2}{w}\right) \frac{4}{L_2}, \quad \text{with } f(z) = \frac{z}{4} \left[\frac{4}{z} + \frac{1}{2} \right].$$

This suggests that the ‘relevant’ homogenized problem is the one obtained for $\frac{z}{\varepsilon} = 1$, as all the others can be obtained from this one by a scaling argument.

We note that the scheme can be applied to the evolution of more general sets, but the analysis of the rectangular case already highlights the new features deriving from the microscopic geometry.

9.6 Flat Flow with Oscillating Forcing Term

We now consider another minimizing-movement scheme linked to the functional F in (9.16). In this case, the oscillations are given by a lower-order forcing term. We consider, in \mathbb{R}^2 ,

$$G_\varepsilon(A) = \int_{\partial A} \|v\|_1 d\mathcal{H}^1 + \int_A g\left(\frac{x_1}{\varepsilon}\right) dx, \quad (9.25)$$

where g is 1-periodic and even, given by

$$g(s) = \begin{cases} \alpha & \text{if } \text{dist}(s, \mathbb{Z}) < \frac{1}{4} \\ \beta & \text{otherwise,} \end{cases}$$

with $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. Note that the additional term may be negative, so that this functional is not positive; however, the minimizing-movement scheme can be applied unchanged.

Since the additional term converges continuously in L^1 as $\varepsilon \rightarrow 0$, the Γ -limit is simply

$$G(A) = \int_{\partial A} \|v\|_1 d\mathcal{H}^1 + \frac{\alpha + \beta}{2} |A|. \quad (9.26)$$

9.6.1 Flat Flow with Forcing Term

We now consider minimizing movements for G . As in Sect. 9.4.1, we only deal with a constrained problem, when both the initial datum and the competing sets are

rectangles centered in 0. With the notation of Sect. 9.4.1, we are led to the minimum problem

$$\min\left\{2(L_{k,1} + \Delta L_1 + L_{k,2} + \Delta L_2) + \frac{L_{k,1}}{4\tau}(\Delta L_2)^2 + \frac{L_{k,2}}{4\tau}(\Delta L_1)^2 + \frac{\alpha + \beta}{2}(L_{k,1} + \Delta L_1)(L_{k,2} + \Delta L_2)\right\}.$$

The minimizing pair $(\Delta L_1, \Delta L_2)$ satisfies

$$\frac{\Delta L_1}{\tau} = -\left(\frac{4}{L_{k,2}} + (\alpha + \beta)\left(1 + \frac{\Delta L_2}{L_{k,2}}\right)\right) \tag{9.27}$$

and the analogous equation for $\frac{\Delta L_2}{\tau}$. Passing to the limit we have

$$\begin{cases} L'_1 = -\left(\frac{4}{L_2} + \alpha + \beta\right) \\ L'_2 = -\left(\frac{4}{L_1} + \alpha + \beta\right), \end{cases} \tag{9.28}$$

so that each edge moves with velocity $\frac{2}{L_2} + \frac{\alpha + \beta}{2}$, with the convention that it moves inwards if this number is positive, outwards if it is negative.

Note that if $\alpha + \beta \geq 0$ then L_1 and L_2 are always decreasing and we have finite-time extinction, while if $\alpha + \beta < 0$ then there is an equilibrium for $L_j = 4/|\alpha + \beta|$, and we have expanding rectangles, with an asymptotic velocity of each side of $|\alpha + \beta|/2$ as the side length diverges.

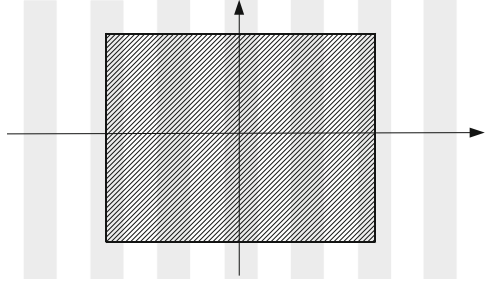
9.6.2 Homogenization of Forcing Terms

We treat the case $\tau \ll \varepsilon$ only, in which we may highlight new homogenization phenomena. Again, we consider the constrained case when both the initial datum and the competing sets are rectangles centered in 0 and adopt the notation of Sect. 9.4.1. The geometry of the problem is pictured in Fig. 9.2, where the two colors in the background represent the two values of the forcing term.

Taking into account that $\tau \ll \varepsilon$, the incremental minimum problem can be approximated by

$$\min\left\{2(L_{k,1} + \Delta L_1 + L_{k,2} + \Delta L_2) + \frac{L_{k,1}}{4\tau}(\Delta L_2)^2 + \frac{L_{k,2}}{4\tau}(\Delta L_1)^2 + \frac{\alpha + \beta}{2}L_{k,1}L_{k,2} + \frac{\alpha + \beta}{2}L_{k,1}\Delta L_2 + g\left(\frac{L_{k,1}}{2\varepsilon}\right)L_{k,2}\Delta L_1\right\}. \tag{9.29}$$

Fig. 9.2 Rectangle in a layered environment



In considering the term $g(L_{k,1}/2\varepsilon)$ we implicitly assume that τ is so small that both $L_{k,1}/2\varepsilon$ and $(L_{k,1} + \Delta L_1)/2\varepsilon$ belong to the same interval where g is constant. This can be assumed up to a number of k that is negligible as $\tau \rightarrow 0$.

For the minimizing pair of (9.29) we have

$$\begin{cases} 2 + \frac{L_{k,2}}{2\tau} \Delta L_1 + g\left(\frac{L_{k,1}}{2\varepsilon}\right) L_{k,2} = 0 \\ 2 + \frac{L_{k,1}}{2\tau} \Delta L_2 + \frac{\alpha + \beta}{2} L_{k,1} = 0; \end{cases} \quad (9.30)$$

that is,

$$\begin{cases} \frac{\Delta L_1}{\tau} = -\left(\frac{4}{L_{k,2}} + 2g\left(\frac{L_{k,1}}{2\varepsilon}\right)\right) \\ \frac{\Delta L_2}{\tau} = -\left(\frac{4}{L_{k,1}} + (\alpha + \beta)\right). \end{cases} \quad (9.31)$$

This systems shows that the horizontal edges move with velocity $\frac{2}{L_{k,1}} + \frac{\alpha + \beta}{2}$, while the velocity of the vertical edges depends on the location of the edge and is

$$\frac{2}{L_{k,2}} + g\left(\frac{L_{k,1}}{2\varepsilon}\right).$$

We then deduce that the limit velocity for the horizontal edges of length L_1 is

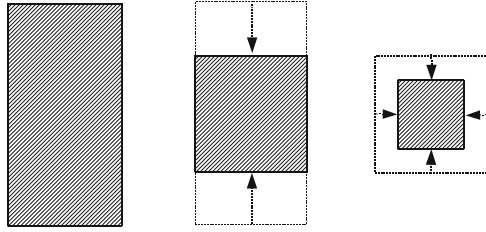
$$\frac{2}{L_1} + \frac{\alpha + \beta}{2}. \quad (9.32)$$

As for the vertical edges, we have:

- **(mesoscopic pinning)** if L_2 is such that

$$\left(\frac{2}{L_2} + \alpha\right)\left(\frac{2}{L_2} + \beta\right) < 0$$

Fig. 9.3 Stages in the motion according to system (9.34)



then the vertical edge is eventually pinned in the minimizing-movement scheme. This pinning is not due to the equality $L_{k+1,1} = L_{k,1}$ in the incremental problem, but to the fact that the vertical edge moves in contrasting directions depending on the value of g .

- **(homogenized velocity)** if, on the contrary, the vertical edge length satisfies

$$\left(\frac{2}{L_2} + \alpha\right)\left(\frac{2}{L_2} + \beta\right) > 0$$

then we have a limit *effective velocity* of the vertical edge given by the harmonic mean of the two velocities $\frac{2}{L_2} + \alpha$ and $\frac{2}{L_2} + \beta$; namely,

$$\frac{(2 + \alpha L_2)(2 + \beta L_2)}{L_2\left(2 + \frac{\alpha + \beta}{2} L_2\right)}. \tag{9.33}$$

We examine some cases explicitly.

- (i) Let $\alpha = -\beta$. Then we have

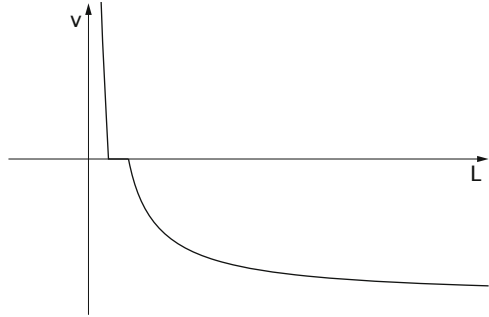
$$\begin{cases} L'_2 = -\frac{4}{L_1} \\ L'_1 = -2\frac{(2 - \beta L_2) \vee 0}{L_2}; \end{cases} \tag{9.34}$$

i.e., the vertical edges are pinned if their length is larger than $2/\beta$. In this case, the horizontal edges move inwards with constant velocity $\frac{2}{L_{0,1}}$. In this way the vertical edges shrink with rate $\frac{4}{L_{0,1}}$ until their length is $2/\beta$. After this, the whole rectangle shrinks in all directions. The stages of this evolution are pictured in Fig. 9.3.

- (ii) Let $\alpha < \beta < 0$. Then for the vertical edges we have an interval of ‘mesoscopic pinning’ corresponding to

$$\frac{2}{|\beta|} \leq L_2 \leq \frac{2}{|\alpha|} \tag{9.35}$$

Fig. 9.4 Velocity of the vertical hedges with an interval of mesoscopic pinning



The velocity of the vertical edges in dependence of their length is then given by

$$v = \begin{cases} 0 & \text{if (9.35) holds} \\ \frac{(2 + \alpha L_2)(2 + \beta L_2)}{L_2(2 + \frac{\alpha + \beta}{2} L_2)} & \text{otherwise} \end{cases}$$

and is pictured in Fig. 9.4. Instead, the velocity of the horizontal edges is given by (9.32), so that they move inwards if

$$L_1 < \frac{4}{|\alpha + \beta|},$$

and outwards if $L_1 > \frac{4}{|\alpha + \beta|}$.

In this case we can consider as initial datum a square of side length L_0 . We have the following cases:

- If $L_0 \leq \frac{2}{|\beta|}$ then all edges move inwards until a finite extinction time.
- If $\frac{2}{|\beta|} < L_0 < \frac{4}{|\alpha + \beta|}$ then first only the horizontal edges move inwards until the vertical edge reaches the length $\frac{2}{|\beta|}$, after which all edges move inwards.
- If $\frac{4}{|\alpha + \beta|} < L_0 < \frac{2}{|\alpha|}$ then first only the horizontal edges move outwards until the vertical edge reaches the length $\frac{2}{|\alpha|}$, after which all edges move outwards.
- If $L_0 \geq \frac{2}{|\alpha|}$ then all edges move outwards, and the motion is defined for all times. The asymptotic velocity of the vertical edges as the length of the edges diverges is

$$\left| \frac{2\alpha\beta}{\alpha + \beta} \right|,$$

lower than $\left| \frac{\alpha + \beta}{2} \right|$ (the asymptotic velocity for the horizontal edges).

The critical case can be shown to be $\varepsilon \sim \tau$, so that for $\varepsilon \ll \tau$ we have the flat flow with averaged forcing term described in Sect. 9.6.1. The actual description in the case $\varepsilon \sim \tau$ would involve a complicated homogenization argument for the computation of the averaged velocity of vertical sides.

Appendix

The variational approach for motion by mean curvature is due to Almgren et al. [2]. The variational approach for crystalline curvature flow is contained in a paper by Almgren and Taylor [1].

The homogenization of the flat flow essentially follows the discrete analog contained in the paper by Braides et al. [3]. In that paper more effects of the microscopic geometry are described for more general initial sets. The homogenization with forcing term is part of ongoing work with A. Malusa and M. Novaga.

Geometric motions with a non-trivial homogenized velocity are described in the paper by Braides and Scilla [4], where example are shown of geometries which do not influence the crystalline perimeter obtained as Γ -limit, but do influence various features of the evolution.

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Chapter 10

Different Time Scales

In this chapter we treat some variations on the minimizing-movement scheme motivated by some time-scaling argument. In particular, we highlight that a scaling of the energy entails a time scaling, from which we may define long-time and backwards motion.

10.1 Long-Time Behaviour

We will consider a new parameter $\lambda > 0$ and follow the iterative minimizing scheme from an initial datum x_0 by considering x_k defined recursively as a minimizer of

$$\min \left\{ \frac{1}{\lambda} F_\varepsilon(x) + \frac{1}{2\tau} \|x - x_{k-1}\|^2 \right\}, \tag{10.1}$$

and setting $u^\tau(t) = u^{\tau,\lambda}(t) = x_{\lfloor t/\tau \rfloor}$. Equivalently, we may view this as applying the minimizing-movement scheme to

$$\min \left\{ F_\varepsilon(x) + \frac{\lambda}{2\tau} \|x - x_{k-1}\|^2 \right\}. \tag{10.2}$$

Note that we may compare this scheme with the one for unscaled energies, where x_k are defined as minimizers of the minimizing-movement scheme with time scale $\eta = \tau/\lambda$. This other scheme gives a function u^η which is a discrete function on a lattice of time step η . Then we have

$$u^\tau(t) = x_{\lfloor t/\tau \rfloor} = x_{\lfloor t/\lambda\eta \rfloor} = u^\eta\left(\frac{t}{\lambda}\right).$$

Hence, the introduction of λ corresponds to a scaling of time.

Remark 10.1. Note that the process described above may be meaningful also if $F_\varepsilon = F$ is independent of ε . In this case, as $\tau \rightarrow 0$ we obtain the minimizing movement along $F_\lambda = \frac{1}{\lambda}F$ with λ in place of ε in the notation used hitherto (of course, being a matter of notation, up to a change of parameters; i.e., considering $1/\lambda$ in place of λ , we can always suppose that $\lambda \rightarrow 0$).

Conversely, if F_ε Γ -converges to F and $\lambda = \lambda_\varepsilon$ is a scale along which there exists a non-trivial Γ -development; i.e., the limit

$$F^{(1)}(x) = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(x) - \min F}{\lambda_\varepsilon}$$

exists and is not trivial, then from (10.1) we argue that such λ_ε is a meaningful scale (which may be related to minimizing movements for $F^{(1)}$). More in general, such a scale is the one suggested by the existence of a non-trivial

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(x) - m_\varepsilon}{\lambda_\varepsilon}$$

for some choice of m_ε .

Finally, if $F_\varepsilon = F$, note that taking $\lambda = 0$ in (10.2) we obtain a global minimum problem for F , while the trajectories $u^\tau(t/\lambda)$ may remain uniformly distant from a global minimizer for τ small (e.g., if the initial datum x_0 is a local minimizer for F). Similarly, for time-dependent minimizing movements, time scaling in general does not lead to a quasistatic motion (as in Example 7.5).

We now first give some simple examples which motivate the study of time-scaled problems, also when the unscaled problems already give a non trivial minimizing movement.

Example 10.1. Consider in \mathbb{R}^2 the energy

$$F_\varepsilon(x, y) = \frac{1}{2}(x^2 + \varepsilon y^2).$$

The corresponding gradient flow is then

$$\begin{cases} x' = -x \\ y' = -\varepsilon y, \end{cases}$$

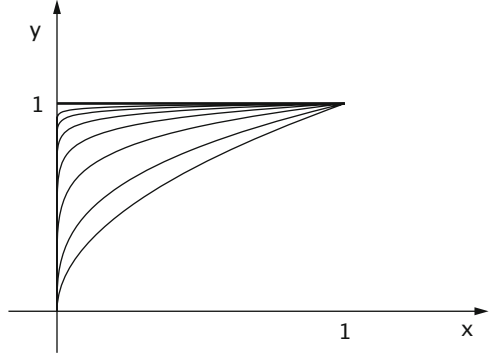
with solutions of the form

$$(x_\varepsilon(t), y_\varepsilon(t)) = (x_0 e^{-t}, y_0 e^{-\varepsilon t}).$$

These solutions converge to $(x(t), y(t)) = (x_0 e^{-t}, y_0)$, solving

$$\begin{cases} x' = -x \\ y' = 0, \end{cases}$$

Fig. 10.1 Trajectories of the solutions, and their pointwise limit



which is the gradient flow of the limit $F(x, y) = \frac{1}{2}x^2$. Note that

$$\lim_{t \rightarrow +\infty} (x_\varepsilon(t), y_\varepsilon(t)) = (0, 0) \neq (0, y_0) = \lim_{t \rightarrow +\infty} (x(t), y(t)).$$

The trajectories of the solutions $(x_\varepsilon, y_\varepsilon)$ lie on the curves

$$\frac{y}{y_0} = \left(\frac{x}{x_0}\right)^\varepsilon$$

and are pictured in Fig. 10.1.

The solutions can be seen as a superposition of $(x(t), y(t))$ and $\varepsilon(x_\infty(t), y_\infty(t))$, where

$$(x_\infty(t), y_\infty(t)) := (0, e^{-t})$$

is the solution of

$$\begin{cases} x' = 0 \\ y' = -y \\ (x(0), y(0)) = (0, y_0). \end{cases}$$

The solution (x_∞, y_∞) can be obtained by scaling $(x_\varepsilon, y_\varepsilon)$; namely,

$$(x_\infty(t), y_\infty(t)) = \lim_{\varepsilon \rightarrow 0} (x_\varepsilon(t/\varepsilon), y_\varepsilon(t/\varepsilon)).$$

In this case the scaling of time is $\lambda = \varepsilon$. Note that the limit of the scaled solutions does not satisfy the original initial condition, which instead is the ‘projection’ of the initial condition on the set of (local) minimizers of the limit energy F (or, in other words, the domain of the limit of the energies $\frac{1}{\varepsilon}F_\varepsilon$).

Example 10.2. A similar example can be constructed in one-dimension, taking, e.g.,

$$F_\varepsilon(x) = \frac{\varepsilon}{2}x^2 + \frac{1}{2}((|x| - 1) \vee 0)^2.$$

If $x_0 < -1$ then the corresponding solutions x_ε satisfy:

- The limit $x(t) = \lim_{\varepsilon \rightarrow 0} x_\varepsilon(t)$ solves

$$\begin{cases} x' = -x + 1 \\ x(0) = x_0, \end{cases}$$

which corresponds to the gradient flow of the energy

$$F(x) = \frac{1}{2}((|x| - 1) \vee 0)^2.$$

- The scaled limit $x_\infty(t) = \lim_{\varepsilon \rightarrow 0} x_\varepsilon(t/\varepsilon)$ solves

$$\begin{cases} x' = -x \\ x(0) = -1, \end{cases}$$

which corresponds to the gradient flow of the energy

$$F_\infty(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} F_\varepsilon(x).$$

In this case, the initial datum is the projection of x_0 on the domain of F_∞ .

Remark 10.2. In the previous examples we faced the problem of defining a minimizing movement for a sequence of functionals F_ε (Γ -)converging to a limit F when the initial data x_0^ε converge to a point $x_0 \notin \text{dom} F$. Note that in this case the approximating trajectories u^ε are always defined if one can define x_1^ε ; i.e., a solution of

$$\min \left\{ F_\varepsilon(x) + \frac{1}{2\tau} \|x - x_0\|^2 \right\},$$

or equivalently of

$$\min \{ 2\tau F_\varepsilon(x) + \|x - x_0\|^2 : x \in \text{dom} F \},$$

after which $x_1^\tau \in \text{dom} F$ and we apply the theory already studied. Note that if $\text{dom} F$ is a closed set in X then x_1^τ converge to the projection x_1 of x_0 on $\text{dom} F$, so it may be meaningful to directly study the minimizing movements from that point.

Note however that, as always, the choice of initial data $x_1^f \rightarrow x_1$ may provide a choice among the minimizing movements from x_1 .

We now give more examples with families of energies F_ε Γ -converging to a limit F . Since we are mainly interested in highlighting the existence of a time scale $\lambda = \lambda_\varepsilon$ at which the scaled motion is not trivial, we will make some simplifying assumptions, one of which is that the initial datum be a local minimizer for F , so that the (unscaled) minimizing movement for the limit from that point is trivial.

Example 10.3. We take as F the 1D Mumford–Shah functional on $(0, 1)$ defined by

$$F(u) = \int_0^1 |u'|^2 dt + \#(S(u)),$$

with domain the set of piecewise- H^1 functions. We take

$$F_\varepsilon(u) = \int_0^1 |u'|^2 dt + \sum_{S(u)} g\left(\frac{|u^+ - u^-|}{\varepsilon}\right),$$

where g is a positive concave function with

$$\lim_{z \rightarrow +\infty} g(z) = 1.$$

We also consider the boundary conditions

$$u(0-) = 0, \quad u(1+) = 1.$$

We suppose that:

- u_0 is a local minimizer for F ; i.e., it is piecewise constant.
- $\#(S(u_0)) = \{x_0, x_1\}$ (the simplest non-trivial local minimizer) with $0 \leq x_0 < x_1 \leq 1$.
- Competing functions are also piecewise constant.

With these conditions, all minimizers u_k obtained by iterative minimization satisfy:

- $S(u_k) \subset \{x_0, x_1\}$.

We may use the constant value z_k of u_k on (x_0, x_1) as a one-dimensional parameter. The minimum problem defining z_k is then (supposing that $z_0 > 0$ so that all $z_k > 0$)

$$\min \left\{ \frac{1}{\lambda} \left(g\left(\frac{z}{\varepsilon}\right) + g\left(\frac{1-z}{\varepsilon}\right) \right) + \frac{1}{2\tau} (x_1 - x_0)(z - z_{k-1})^2 \right\},$$

which gives

$$(x_1 - x_0) \frac{z_k - z_{k-1}}{\tau} = -\frac{1}{\varepsilon \lambda} \left(g' \left(\frac{z_k}{\varepsilon} \right) - g' \left(\frac{1 - z_k}{\varepsilon} \right) \right).$$

As an example, we may take

$$g(z) = \frac{z}{1+z},$$

so that the equation for z_k becomes

$$(x_1 - x_0) \frac{z_k - z_{k-1}}{\tau} = -\frac{\varepsilon}{\lambda} \left(\frac{1}{\varepsilon^2 + z_k^2} - \frac{1}{\varepsilon^2 + (z_k - 1)^2} \right).$$

This suggests the scale

$$\lambda = \varepsilon,$$

and with this choice gives the limit equation for $z(t)$

$$z' = -\frac{1-2z}{(x_1 - x_0)z^2(z-1)^2}.$$

In this time scale, unless we are in the equilibrium $z = \frac{1}{2}$, the middle value moves towards the closest value between 0 and 1.

As a side remark, note that a simple qualitative study of this equation shows that if the initial datum is not the constant $1/2$ then $z = 0$ or $z = 1$ after a finite time, after which the motion is trivial. Note that the limit state is a local minimum with only one jump.

Remark 10.3 (Long-time behaviour as a choice criterion). Note that we may apply the time-scale approach above also to the scaled Lennard-Jones energies, which we proved to converge to the Griffith fracture energy in Example 2.6. The resulting long-time behaviour can be compared to the one described in the previous example for Barenblatt fracture energies. Requiring that the two long-time behaviours be the same gives further conditions on the function g in addition to those obtained in Sect. 4.4.

Example 10.4. We consider the same functionals F and F_ε as in Example 10.3, with an initial datum with three jumps satisfying the same Dirichlet boundary conditions $u(0^-) = 0$, $u(1^+) = 1$ and the same assumptions as before.

With the notation used above, the minimum problem is

$$\min \left\{ \frac{1}{\lambda} \left(g \left(\frac{z_0 - 0}{\varepsilon} \right) + g \left(\frac{z_1 - z_0}{\varepsilon} \right) + g \left(\frac{1 - z_1}{\varepsilon} \right) \right) \right\}$$

$$+ \frac{1}{2\tau} \left((x_1 - x_0) |z_0 - z_0^{k-1}|^2 + (x_2 - x_1) |z_1 - z_1^{k-1}|^2 \right). \quad (10.3)$$

Differently from the previous case, now we have to compute a gradient as a function of z_0 and z_1 , the constant values of u respectively on (x_0, x_1) and (x_1, x_2) . Hence, the Euler equations for (10.3) give the following system for z_0^k and z_1^k :

$$(x_1 - x_0) \frac{z_0^k - z_0^{k-1}}{\tau} = -\frac{1}{\lambda \varepsilon} \left(g' \left(\frac{z_0^k}{\varepsilon} \right) - g' \left(\frac{z_1^k - z_0^k}{\varepsilon} \right) \right), \quad (10.4)$$

$$(x_2 - x_1) \frac{z_1^k - z_1^{k-1}}{\tau} = -\frac{1}{\lambda \varepsilon} \left(g' \left(\frac{z_1^k - z_0^k}{\varepsilon} \right) - g' \left(\frac{1 - z_1^k}{\varepsilon} \right) \right). \quad (10.5)$$

For the sake of illustration, we may take the same g as in the previous example, so that (10.4) and (10.5) become

$$(x_1 - x_0) \frac{z_0^k - z_0^{k-1}}{\tau} = -\frac{\varepsilon}{\lambda} \left(\frac{1}{(\varepsilon + z_0^k)^2} - \frac{1}{(\varepsilon + z_1^k - z_0^k)^2} \right), \quad (10.6)$$

$$(x_2 - x_1) \frac{z_1^k - z_1^{k-1}}{\tau} = -\frac{\varepsilon}{\lambda} \left(\frac{1}{(\varepsilon + z_1^k - z_0^k)^2} - \frac{1}{(\varepsilon + 1 - z_1^k)^2} \right). \quad (10.7)$$

This suggests the scale

$$\lambda = \varepsilon, \quad (10.8)$$

and with this choice the limit equations for $z_0(t)$ and $z_1(t)$ are

$$z_0' = -\frac{z_1(z_1 - 2z_0)}{(x_1 - x_0)z_0^2(z_1 - z_0)^2}, \quad (10.9)$$

$$z_1' = -\frac{1 - z_0^2 - 2z_1(1 - z_0)}{(x_2 - x_1)(z_1 - z_0)^2(1 - z_1)^2}. \quad (10.10)$$

In this time scale, it is easy to see that the gradient is zero when $(z_0, z_1) = (\frac{z_1}{2}, \frac{1+z_0}{2})$, so we can have the following different behaviors:

- Equilibrium point. For the initial datum $(\bar{z}_0, \bar{z}_1) = (\frac{1}{3}, \frac{2}{3})$ the motion is trivial.
- If z_0 is larger than the equilibrium point, then $z_0' > 0$ and the constant value z_0 will increase towards z_1 , otherwise it will decrease towards zero. The same holds for z_1 between z_0 and 1.

It must be noted that if the initial datum is not an equilibrium point then after a finite time one of the jump sizes vanishes, after which we are back to the previous example. In Figs. 10.2–10.5 we picture four stages of the evolution computed numerically.

Fig. 10.2 Initial conditions

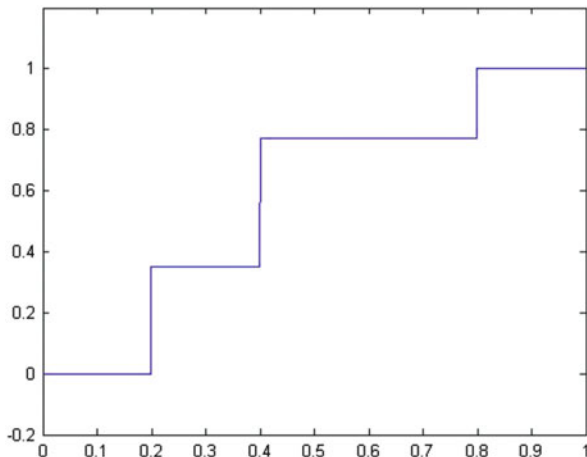
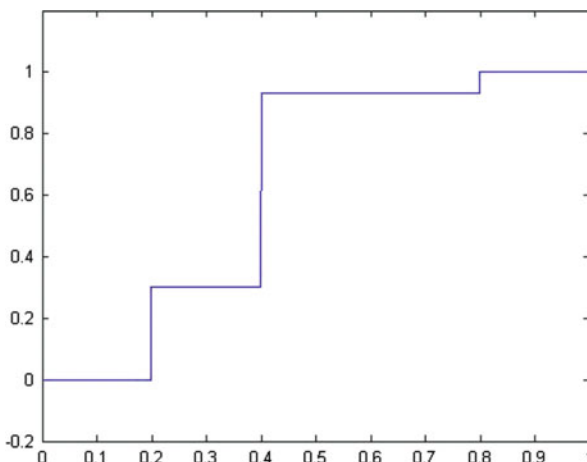


Fig. 10.3 Iteration n. 30



A further simplified example is obtained by taking symmetric initial data $x_2 - x_1 = x_1 - x_0 =: L$ and $z_0(0) = \frac{1}{2} - w_0$ and $z_1(0) = \frac{1}{2} + w_0$ with $0 < w_0 < 1/2$, for which the motion is described by a single parameter $w(t)$ satisfying

$$w' = \frac{3(\frac{1}{2} + w)(w - \frac{1}{6})}{4L(\frac{1}{2} - w)w^2},$$

in which case the equilibrium point corresponds to $w_0 = 1/6$, and otherwise after a finite either we have $w = 0$ (which gives $z_0 = z_1 = 1/2$; i.e., the equilibrium point with two jumps) or $w = \frac{1}{2}$ (which gives $z_0 = 0$ and $z_1 = 1$; i.e., a final state with only one jump point).

Fig. 10.4 Iteration n. 60

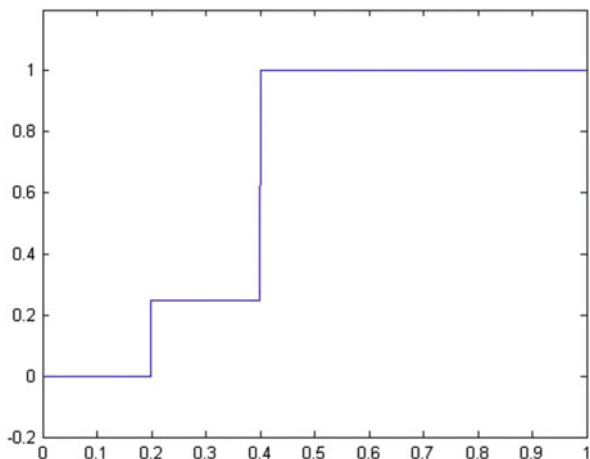
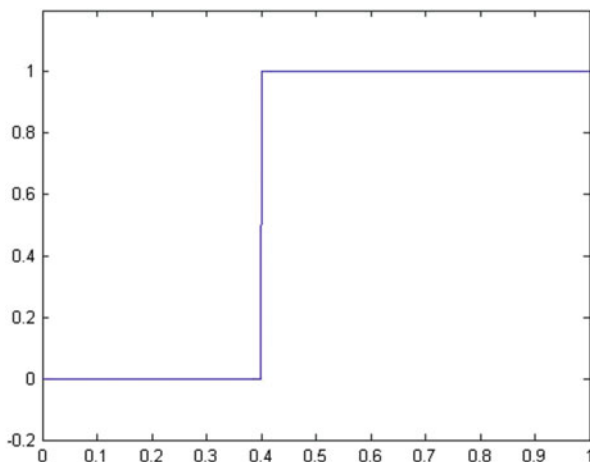


Fig. 10.5 Iteration n. 100



Example 10.5. We consider another approximation of the Mumford–Shah functional: the (scaled) Perona–Malik functional. In the notation for discrete functionals (see Sect. 4.4), we may define

$$F_\varepsilon(u) = \sum_{i=1}^N \frac{1}{|\log \varepsilon|} \log \left(1 + \varepsilon |\log \varepsilon| \left| \frac{u_i - u_{i-1}}{\varepsilon} \right|^2 \right).$$

Note that also the pointwise limit on piecewise- H^1 functions gives the Mumford–Shah functional since

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon |\log \varepsilon|} \log \left(1 + \varepsilon |\log \varepsilon| z^2 \right) = z^2$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \log \left(1 + |\log \varepsilon| \frac{w^2}{\varepsilon} \right) = 1$$

for all $w \neq 0$.

As in Example 10.3, we consider the case when competing functions are non-negative piecewise constants with $S(u) \subset S(u_0) = \{x_0, x_1\}$ and satisfying the boundary conditions $u(0-) = 0, u(1+) = 1$. The computation is then reduced to a one-dimensional problem with unknown the constant value z_k defined by the minimization

$$\min \left\{ \frac{1}{\lambda |\log \varepsilon|} \left(\log \left(1 + |\log \varepsilon| \frac{z^2}{\varepsilon} \right) + \log \left(1 + |\log \varepsilon| \frac{(z-1)^2}{\varepsilon} \right) \right) + \frac{1}{2\tau} (x_1 - x_0)(z - z_{k-1})^2 \right\},$$

which gives the equation

$$(x_1 - x_0) \frac{z_k - z_{k-1}}{\tau} = -\frac{2}{\lambda} \left(\frac{z}{\varepsilon + |\log \varepsilon| z^2} + \frac{z-1}{\varepsilon + |\log \varepsilon| (z-1)^2} \right).$$

This suggests the time scale

$$\lambda = \frac{1}{|\log \varepsilon|},$$

and gives the equation for $z(t)$

$$z' = -\frac{2}{(x_1 - x_0)} \cdot \frac{1-2z}{z(1-z)},$$

which provides a qualitative behaviour of z similar to the previous example.

Example 10.6. We now consider the sharp-interface model with

$$F(u) = \#(S(u) \cap [0, 1))$$

defined on all piecewise-constant 1-periodic functions with values in ± 1 . For F all functions are local minimizers.

We take

$$F_\varepsilon(u) = \#(S(u) \cap [0, 1)) - \sum_{x_i \in [0, 1) \cap S(u)} e^{-\frac{x_i + 1 - x_i}{\varepsilon}},$$

where $\{x_i\} = S(u)$ is a numbering of $S(u)$ with $x_i < x_{i+1}$.

We take as initial datum u_0 with $\#(S(u_0)) = 2$; hence, $S(u_0) = \{x_0, y_0\}$, and, after identifying u_0 with $A_0 = [x_0, y_0]$, apply the Almgren–Taylor–Wang variant of the iterative minimization process, where the distance term $\frac{1}{2\tau}\|u - u_{k-1}\|^2$ is substituted by

$$\frac{1}{\tau} \int_{A \Delta A_{k-1}} \text{dist}(x, \partial A_{k-1}) dx.$$

The computation of $A_1 = [x_1, y_1]$ is obtained by the minimization problem

$$\min \left\{ -\frac{1}{\lambda} \left(e^{-\frac{(y-x)}{\varepsilon}} + e^{-\frac{(1+x-y)}{\varepsilon}} \right) + \frac{1}{2\tau} ((x-x_0)^2 + (y-y_0)^2) \right\},$$

which gives

$$\begin{aligned} \frac{x_1 - x_0}{\tau} &= \frac{1}{\varepsilon \lambda} \left(e^{-\frac{(y_1-x_1)}{\varepsilon}} - e^{-\frac{(1+x_1-y_1)}{\varepsilon}} \right) \\ \frac{y_1 - y_0}{\tau} &= -\frac{1}{\varepsilon \lambda} \left(e^{-\frac{(y_1-x_1)}{\varepsilon}} - e^{-\frac{(1+x_1-y_1)}{\varepsilon}} \right). \end{aligned}$$

Let $y_0 - x_0 < 1/2$; we argue that the scaled time scale is

$$\lambda = \frac{1}{\varepsilon} e^{-\frac{y_0-x_0}{\varepsilon}},$$

for which we have

$$\begin{aligned} \frac{x_1 - x_0}{\tau} &= \left(e^{-\frac{(y_1-y_0-x_1+x_0)}{\varepsilon}} - e^{-\frac{(1+x_1-x_0-y_1+y_0)}{\varepsilon}} \right) \\ \frac{y_1 - y_0}{\tau} &= -\left(e^{-\frac{(y_1-y_0-x_1+x_0)}{\varepsilon}} - e^{-\frac{(1+x_1-x_0-y_1+y_0)}{\varepsilon}} \right). \end{aligned}$$

In terms of $L_k = y_k - x_k$ this can be written as

$$\frac{L_1 - L_0}{\tau} = -2 \left(e^{-\frac{(L_1-L_0)}{\varepsilon}} - e^{-\frac{(1+L_0-L_1)}{\varepsilon}} \right).$$

Under the assumption $\tau \ll \varepsilon$ we have in the limit

$$L' = -2 \left(e^{o(1)} - e^{-\frac{1}{\varepsilon} + o(1)} \right) = -2,$$

which shows that the two closer interfaces move towards each other shortening linearly their distance.

10.2 Reversed Time

In a finite-dimensional setting, a condition that ensures the possibility of defining a minimizing movement for F is that

$$u \mapsto F(u) + \frac{1}{2\tau}|u - \bar{u}|^2 \quad (10.11)$$

be lower semicontinuous and coercive for all \bar{u} and for τ sufficiently small. This is not in contrast with requiring that also

$$u \mapsto -F(u) + \frac{1}{2\tau}|u - \bar{u}|^2 \quad (10.12)$$

satisfy the same conditions; for example if F is continuous and of quadratic growth. Note that this can be seen as a further extension of the time-scaling argument in the previous sections with $\lambda = -1$. If the iterative scheme gives a solution for the gradient flow, a minimizing movement u for the second scheme produces a solution $v(t) = u(-t)$ to the backward problem

$$\begin{cases} v'(t) = -\nabla F(v(t)) & \text{for } t \leq 0 \\ v(0) = u_0 \end{cases}$$

In an infinite-dimensional setting, the two requirements of being able to define both the minimizing movement (10.11) and (10.12) greatly limits the choice of F , and rules out all interesting cases. A possible approach to the definition of a backward minimizing movement is then to introduce a (finite-dimensional) approximation F_ε to F , for which we can define a minimizing motion along $-F_\varepsilon$.

We now give an example in the context of crystalline motion, where we consider a negative scaling of time.

Example 10.7 (Nucleation). We consider in \mathbb{R}^2

$$F(A) = \int_{\partial A} \|v\|_1 d\mathcal{H}^1,$$

and F_ε the restriction of F to the sets of the form

$$\bigcup_{i \in B} \left(\varepsilon i + \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right)^2 \right),$$

where B is a subset of \mathbb{Z}^2 . Hence, we may identify these union of ε -squares with the corresponding B . Even though this is not a finite-dimensional space, we will be able to apply the Almgren–Taylor–Wang scheme.

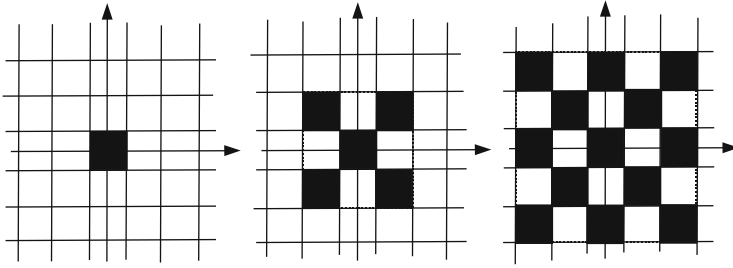


Fig. 10.6 Nucleating sets

We choose (with the identifications with subsets of \mathbb{Z}^2) as initial datum

$$A_0^\varepsilon = \{(0, 0)\} = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)^2,$$

and solve iteratively

$$\min \left\{ -\frac{1}{\lambda} F_\varepsilon(A) + \frac{1}{\tau} \int_{A \setminus A_{k-1}^\varepsilon} \text{dist}_\infty(x, \partial A_{k-1}^\varepsilon) dx \right\},$$

with $\lambda = \lambda_\varepsilon > 0$ to be determined. In the interpretation as a reversed-time scheme, this means that we are solving a problem imposing the extinction at time 0.

Note that taking F in place of F_ε would immediately give the value $-\infty$ in the minimum problem above; e.g., by considering sets of the form (in polar coordinates)

$$A_j = \{(\rho, \theta) : \rho \leq 3\varepsilon + \varepsilon \sin(j\theta)\},$$

which contain A_0^ε , are contained in $B_{4\varepsilon}(0)$ and have a perimeter larger than $4j\varepsilon$.

Under the assumption that $\varepsilon \ll \tau$, all minimizing sets are the checkerboard structure corresponding to indices $i \in \mathbb{Z}^2$ with $i_1 + i_2$ even contained in a square Q_k centered in 0 (see Fig. 10.6). We may take the sides L_k of those squares as unknown. The incremental problems can be rewritten as

$$\min \left\{ -\frac{2}{\varepsilon\lambda} ((L_{k-1} + \Delta L)^2 - L_{k-1}^2) + \frac{1}{\tau} (L_{k-1}(\Delta L)^2 + r_k(\Delta L)^2) \right\},$$

with r_k negligible as $\tau \rightarrow 0$. For the interfacial part, we have taken into account that for ε small the number of ε -squares contained in a rectangle is equal to its area divided by $2\varepsilon^2$ and each of the squares gives an energy contribution of 4ε ; for the distance part, we note that the integral can be equivalently taken on half of $Q_k \setminus Q_{k-1}$. Minimization in ΔL gives

$$\frac{\Delta L}{\tau} = \frac{2}{\varepsilon\lambda} \left(1 + \frac{\Delta L}{L_{k-1}} \right).$$

Choosing $\lambda = \frac{1}{\varepsilon}$, we obtain a linear growth

$$L(s) = 2s.$$

The checkerboard structures converge (meaning, e.g., that their convex envelopes converge as sets) to squares, whose side length L increases linearly, $L(s) = 2s$.

What we have obtained is the description of the structure of ε -squares (the checkerboard one) along which the increase of the perimeter is maximal (and, in a sense, the decrease of the perimeter is maximal for the reverse-time problem).

Appendix

The literature on long-time behaviour and backward equations, even though not by the approach by minimizing movements, is huge. The long-time motion of interfaces in one space dimension by energy methods has been studied in a paper by Bronsard and Kohn [1].

Example 10.4 has been part of the course exam of C. Sargentone and S. Tozza at Sapienza University in Rome, who kindly provided the pictures for the numerical simulations.

Example 10.7 is contained in a paper by Braides and Scilla [2]. It is a pleasure to acknowledge the suggestion of J.W. Cahn to use finite-dimensional approximations to define backward motion of sets.

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Chapter 11

Stability Theorems

We now face the problem of determining conditions under which the minimizing-movement scheme commutes with Γ -convergence. Let F_ε Γ -converge to F , with initial data x_ε converging to x_0 . We have seen in Sect. 8.2 that by suitably choosing $\varepsilon = \varepsilon(\tau)$ the minimizing movement along the sequence F_ε from x_ε converges to a minimizing movement for the limit F from x_0 . A further issue is whether, by assuming some further properties on F_ε , we may deduce that the same thing happens for any choice of ε . In order to give an answer, we will use results from the theory of gradient flows recently elaborated by Ambrosio, Gigli and Savaré, and by Sandier and Serfaty.

11.1 Stability for Convex Energies

In this section we use general results in the theory of gradient flows to deduce a stability property for functionals satisfying some convexity assumptions. For the sake of simplicity, we will assume that X is a Hilbert space and all F_ε are convex.

11.1.1 Convergence Estimates

We first recall some results on minimizing movements for a single convex functional F .

Proposition 11.1. *Let F be convex, $z \in X$ and let w be a minimizer of*

$$\min\left\{F(x) + \frac{1}{2\eta}\|x - z\|^2\right\}. \tag{11.1}$$

Then

$$\|x - w\|^2 - \|x - z\|^2 \leq 2\eta(F(x) - F(w)) \quad (11.2)$$

for all $x \in X$.

Proof. We recall that the inequality

$$\|sx + (1-s)w - z\|^2 \leq s\|x - z\|^2 + (1-s)\|w - z\|^2 - s(1-s)\|x - w\|^2 \quad (11.3)$$

holds for all $x, w, z \in X$ and $s \in [0, 1]$. Using this property and the convexity of F , thanks to the minimality of w we have

$$\begin{aligned} F(w) + \frac{1}{2\eta}\|w - z\|^2 &\leq F(sx + (1-s)w) + \frac{1}{2\eta}\|sx + (1-s)w - z\|^2 \\ &\leq sF(x) + (1-s)F(w) \\ &\quad + \frac{1}{2\eta}(s\|x - z\|^2 + (1-s)\|w - z\|^2 - s(1-s)\|x - w\|^2). \end{aligned}$$

After regrouping and dividing by s , from this we have

$$\frac{1}{2\eta}(\|w - z\|^2 + (1-s)\|x - w\|^2 - \|x - z\|^2) \leq F(x) - F(w),$$

and then the desired inequality (11.2) after letting $s \rightarrow 0$ and dropping the positive term $\|w - z\|^2$. \square

Remark 11.1. Let $\{z_k\} = \{z_k^\eta\}$ be a minimizing scheme for F from z_0 with time-step η . Then (11.2) gives

$$\|x - z_{k+1}\|^2 - \|x - z_k\|^2 \leq 2\eta(F(x) - F(z_{k+1})) \quad (11.4)$$

for all $x \in X$.

We now fix $\tau > 0$ and two initial data x_0 and y_0 . We will compare the resulting $\{x_k\} = \{x_k^\tau\}$ obtained by iterated minimization with time-step τ and initial datum x_0 and $\{y_k\} = \{y_k^{\tau/2}\}$ with time-step $\tau/2$ and initial datum y_0 . Note that the corresponding continuous-time interpolations are

$$u^\tau(t) := x_{\lfloor t/\tau \rfloor}, \quad v^{\tau/2}(t) = y_{\lfloor 2t/\tau \rfloor}, \quad (11.5)$$

so that the comparison must be performed between x_k and y_{2k} .

Proposition 11.2. *For all $j \in \mathbb{N}$ we have*

$$\|x_j - y_{2j}\|^2 - \|x_0 - y_0\|^2 \leq 2\tau F(x_0).$$

Proof. We first give an estimate between x_1 and y_2 . We apply (11.4) with $\eta = \tau$, $z_k = x_0$, $z_{k+1} = y_1$ and $x = y_2$ which gives

$$\|y_2 - x_1\|^2 - \|y_2 - x_0\|^2 \leq 2\tau(F(y_2) - F(x_1)). \quad (11.6)$$

If instead we apply (11.4) with $\eta = \tau/2$, $z_k = y_0$, $z_{k+1} = y_1$ and $x = x_0$, or $z_k = y_1$, $z_{k+1} = y_2$ and $x = x_0$ we get, respectively,

$$\begin{aligned} \|x_0 - y_1\|^2 - \|x_0 - y_0\|^2 &\leq \tau(F(x_0) - F(y_1)) \\ \|x_0 - y_2\|^2 - \|x_0 - y_1\|^2 &\leq \tau(F(x_0) - F(y_2)), \end{aligned}$$

so that, summing up,

$$\|x_0 - y_2\|^2 - \|x_0 - y_0\|^2 \leq 2\tau F(x_0) - \tau F(y_1) - F(y_2) \leq 2\tau(F(x_0) - F(y_2)), \quad (11.7)$$

where we have used that $F(y_2) \leq F(y_1)$ in the last inequality. Summing up (11.6) and (11.7) we obtain

$$\|x_1 - y_2\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_1)). \quad (11.8)$$

We can repeat the same argument with x_0 and y_0 substituted by x_1 and y_2 , so that we get

$$\|x_2 - y_4\|^2 - \|x_1 - y_2\|^2 \leq 2\tau(F(x_1) - F(x_2)), \quad (11.9)$$

and, summing (11.8),

$$\|x_2 - y_4\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_2)). \quad (11.10)$$

By iterating this process we get

$$\|x_j - y_{2j}\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_j)) \leq 2\tau F(x_0) \quad (11.11)$$

as desired. \square

Theorem 11.1. *Let F be convex and let $F(x_0) < +\infty$. Then there exists a unique minimizing movement u for F from x_0 such that, if u^τ is defined by (11.5), then*

$$\|u^\tau(t) - u(t)\| \leq 6\sqrt{F(x_0)}\sqrt{\tau}$$

for all $t \geq 0$.

Proof. With fixed τ we first prove the convergence of $u^{2^{-j}\tau}$ as $j \rightarrow +\infty$. By Proposition 11.2 applied with $y_0 = x_0$ and $2^{-j}\tau$ in place of τ we have

$$\|u^{2^{-j}\tau}(t) - u^{2^{-j-1}\tau}(t)\| \leq 2^{-j/2} \sqrt{2\tau} \sqrt{F(x_0)} \quad (11.12)$$

for all t . This shows the convergence to a limit $u_\tau(t)$, which in particular satisfies

$$\|u^\tau(t) - u_\tau(t)\| \leq \sqrt{2} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{\tau} \sqrt{F(x_0)} \leq 6\sqrt{F(x_0)} \sqrt{\tau}. \quad (11.13)$$

The limit u_τ can be characterized as follows: with fixed x , inequality (11.4) applied to $z_k = u^{2^{-j}\tau}((k-1)2^{-j}\tau)$ ($k \geq 1$) can be seen as describing in the sense of distribution the derivative

$$\frac{d}{dt} \frac{1}{2} \|x - u^{2^{-j}\tau}(t)\|^2 \leq \sum_{k=1}^{\infty} \left(F(x) - F\left(u^{2^{-j}\tau}((k-1)2^{-j}\tau)\right) \right) 2^{-j}\tau \delta_{k2^{-j}\tau}. \quad (11.14)$$

Note in fact that $x \mapsto \frac{1}{2} \|x - u^{2^{-j}\tau}\|^2$ is a piecewise-constant function with discontinuities in $2^{-j}\tau\mathbb{Z}$, whose size is controlled by (11.4). Since the measures

$$\mu_j = \sum_{k=1}^{\infty} 2^{-j}\tau \delta_{k2^{-j}\tau}$$

converge to the Lebesgue measure, and $u^{2^{-j}\tau}(t) \rightarrow u_\tau(t)$ for all t , so that by the lower semicontinuity of F

$$F(u_\tau(t)) \leq \liminf_{j \rightarrow +\infty} F\left(u^{2^{-j}\tau}(t)\right),$$

we deduce that

$$\frac{d}{dt} \frac{1}{2} \|x - u_\tau(t)\|^2 \leq F(x) - F(u_\tau(t)) \quad (11.15)$$

for all x . Equation (11.15) is sufficient to characterize u_τ . We only sketch the argument: suppose otherwise that (11.15) is satisfied by some other $v(t)$. Then we have

$$\langle x - u_\tau, \nabla u_\tau \rangle \leq F(x) - F(u_\tau) \quad \text{and} \quad \langle x - v, \nabla v \rangle \leq F(x) - F(v)$$

for all x . Inserting $x = v(t)$ and $x = u_\tau(t)$ respectively, and summing the two inequalities we have

$$\frac{d}{dt} \frac{1}{2} \|v(t) - u_\tau(t)\|^2 = \langle v - u_\tau, \nabla v - \nabla u_\tau \rangle \leq 0.$$

Since $v(0) = u_\tau(0)$ we then have $v = u_\tau$.

This argument shows that $u = u_\tau$ does not depend on τ . We then have the convergence of the whole sequence. Inequality (11.13) provides the desired estimate of $\|u^\tau - u\|$. \square

11.1.2 Stability Along Sequences of Convex Energies

From the estimates in the previous section, and the convergence argument in Sect. 8.2, we can deduce the following stability results.

Theorem 11.2. *Let F_ε be a equi-coercive sequence of lower-semicontinuous coercive positive convex energies Γ -converging to F , and let $x_0^\varepsilon \rightarrow x_0$ with $\sup_\varepsilon F_\varepsilon(x_0^\varepsilon) < +\infty$. Then*

- (i) *For every choice of τ and ε converging to 0, the family u^ε introduced in Definition 8.1 converges to the unique u given by Theorem 11.1.*
- (ii) *The sequence of minimizing movements u_ε for F_ε from x_0^ε (given by Theorem 11.1 with F_ε in place of F) also converge to the same minimizing movement u .*

Proof. We first show (ii). Indeed, by the estimate in Theorem 11.1 we have that, after defining u_ε^τ following the notation of that theorem,

$$\|u^\tau - u\|_\infty \leq M \sqrt{\tau}, \quad \|u_\varepsilon^\tau - u_\varepsilon\|_\infty \leq M \sqrt{\tau},$$

where

$$M = 6 \sup_\varepsilon F_\varepsilon(x_0^\varepsilon).$$

In order to show that $u_\varepsilon \rightarrow u$ it suffices to show that $u_\varepsilon^\tau \rightarrow u^\tau$ for fixed τ . That has already been noticed to hold in Sect. 8.2 by the Fundamental Theorem of Γ -convergence.

In order to prove (i) it suffices to use the triangular inequality

$$\|u_\varepsilon^\tau - u\| \leq \|u_\varepsilon^\tau - u_\varepsilon\| + \|u_\varepsilon - u\| \leq M \sqrt{\tau} + o(1)$$

by Theorem 11.1 and (ii). \square

Remark 11.2 (Compatible topologies). We may weaken the requirement that F_ε be equi-coercive with respect to the X -convergence. It suffices to require that the Γ -limit be performed with respect to a topology with respect to which the energies are equi-coercive and be *compatible* with the X -norm; i.e., such that there exist X -converging recovery sequences. In this way, the Γ -convergence of F_ε to F ensures that $F_\varepsilon(x) + C \|x - x_0\|^2$ Γ -converges to $F(x) + C \|x - x_0\|^2$ for fixed C and x_0 , and we still have $u_\varepsilon^r \rightarrow u^r$ in the proof above.

Example 11.1 (Parabolic homogenization). We can consider $X = L^2(0, T)$,

$$F_\varepsilon(u) = \int_0^T a\left(\frac{x}{\varepsilon}\right) |u'|^2 dx, \quad F(u) = \underline{a} \int_0^T |u'|^2 dx$$

with the notation of Sect. 2.4. We take as initial datum u_0 independent of ε . Since all functionals are convex, lower semicontinuous and coercive, from Theorem 11.2 we deduce the convergence of the corresponding minimizing movements. From this we deduce the convergence of the solutions of the parabolic problem with oscillating coefficients

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \frac{\partial}{\partial x} \left(a\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x} \right) \\ u_\varepsilon(x, 0) = u_0(x) \end{cases}$$

to the solution of the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \underline{a} \frac{\partial^2 u}{\partial x^2} \\ u_\varepsilon(x, 0) = u_0(x). \end{cases}$$

Example 11.2 (High-contrast media). We consider a discrete system parameterized on $\{0, \dots, N\}$ with N even. We set $\varepsilon = 1/N$ and consider the energies

$$F_\varepsilon(u) = \frac{1}{2} \sum_{l=1}^{N/2} \varepsilon \left| \frac{u_{2l} - u_{2l-2}}{\varepsilon} \right|^2 + \frac{c_\varepsilon}{2} \sum_{j=1}^N \varepsilon \left| \frac{u_j - u_{j-1}}{\varepsilon} \right|^2$$

with the periodic boundary condition $u_N = u_0$.

This is a simple model where two elliptic energies interact, possibly on different scales. The critical regime is $c_\varepsilon \sim \varepsilon^2$. We will assume that

$$c_\varepsilon = \varepsilon^2$$

in the rest of the example. The first sum is a strong next-to-nearest-neighbor interaction between even points, and the second one is a weak nearest-neighbor interaction between all points.

Upon identifying u_i with the piecewise-constant function $u \in L^2(0, 1)$ with $u(x) = u_{\lfloor x/\varepsilon \rfloor}$, we may regard F_ε as defined on $X = L^2(0, 1)$ and consider the minimizing movement of F_ε with respect to the L^2 -norm, which we can write

$$\|u\|^2 = \sum_{j=1}^N \varepsilon |u_j|^2$$

on the domain of F_ε , so that the iterated minimum problem giving u^k reads

$$\min \left\{ \frac{1}{2} \sum_{l=1}^{N/2} \varepsilon \left| \frac{u_{2l} - u_{2l-2}}{\varepsilon} \right|^2 + \frac{1}{2} \sum_{j=1}^N \varepsilon^3 \left| \frac{u_j - u_{j-1}}{\varepsilon} \right|^2 + \frac{1}{2\tau} \sum_{j=1}^N \varepsilon (u_j - u_j^{k-1})^2 \right\}.$$

We consider as initial datum (the sampling on $\varepsilon\mathbb{Z} \cap [0, 1]$ of) a smooth 1-periodic datum u^0 (for simplicity independent of ε).

Since all F_ε are convex, we may describe their minimizing movement through the gradient flow of their Γ -limit. Since F_ε is not equi-coercive with respect to the L^2 norm, we have to choose a different topology for the Γ -limit.

Among the different choices we may consider the following two:

- (1) We choose the strong L^2 -convergence of the even piecewise-constant interpolations only; i.e.,

$$\|u - v\|_{even}^2 = \sum_{j=1}^N \varepsilon |u_{2j} - v_{2j}|^2.$$

Note that F_ε are equi-coercive and their Γ -limit is simply

$$F^s(u) = \int_0^1 |u'|^2 dx.$$

To check this it suffices to remark that, if we consider the even piecewise-affine interpolation \tilde{u} of u_i , then we have

$$\sum_{l=1}^{N/2} \varepsilon \left| \frac{u_{2l} - u_{2l-2}}{\varepsilon} \right|^2 = 2 \sum_{l=1}^{N/2} 2\varepsilon \left| \frac{u_{2l} - u_{2l-2}}{2\varepsilon} \right|^2 = 2 \int_0^1 |\tilde{u}'|^2 dx,$$

so that F^s is a lower bound, while a recovery sequence is simply obtained by taking u_ε the interpolation of u , for which

$$F_\varepsilon(u_\varepsilon) = \int_0^1 |u'|^2 dx + \frac{\varepsilon^2}{2} \int_0^1 |u'|^2 dx + o(1).$$

- (2) We choose the strong L^2 -convergence of the even piecewise-constant interpolations and the weak L^2 -convergence of the odd piecewise-constant interpolations. A function u is then identified with a pair (u_e, u_o) (even and odd piecewise-constant interpolations), so that

$$F_\varepsilon(u) = F^w(u_e, u_o) := \int_0^1 |u'_e|^2 dx + \frac{1}{2} \int_0^1 |u_e - u_o|^2 dx.$$

The functional F^w thus defined is the Γ -limit in this topology, which is compatible with the L^2 -distance (interpreted as the sum of the L^2 -distances of the even/odd piecewise-constant interpolations).

We can apply Theorem 11.2, together with Remark 11.2, and deduce that the minimizing movement for F_ε is given by the solution $(u_e, u_o) = (u_e(x, t), u_o(x, t))$ of the gradient flow for F^w , which is

$$\begin{cases} \frac{\partial u_e}{\partial t} = 2 \frac{\partial^2 u_e}{\partial x^2} - u_e + u_o \\ \frac{\partial u_o}{\partial t} = u_o - u_e \\ u_o(x, 0) = u_e(x, 0) = u^0(x), \end{cases}$$

with periodic boundary conditions for u_e .

Note that F^s is not compatible with the L^2 -norm since it does not contain the odd interpolations, and its gradient flow is simply a heat equation. Note however that we may use u_e as a single parameter with respect to which to describe the minimizing movement of F_ε , as suggested by the choice of F^s as Γ -limit. Indeed, we may integrate the second equation of the system above expressing u_o in terms of u_e . Plugging its expression in the first equation we obtain the integro-differential problem satisfied by u_e

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = 2 \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) + u^0(x)e^{-t} + \int_0^t e^{s-t} u(x, s) ds \\ u(x, 0) = u^0(x) \end{cases}$$

with periodic boundary conditions.

11.2 Sandier–Serfaty Theory

We have already remarked that for some non-convex problems minimizing movements commute with Γ -convergence, as for approximations of the Mumford–Shah functional. We conclude this section by giving a brief (and simplified) account of

another very fruitful approach to gradient flows that allows to prove the stability of certain solutions with respect to Γ -convergence related to non-convex energies.

11.2.1 Convergence of Gradient Flows

We consider a family of Hilbert spaces X_ε and functionals $F_\varepsilon : X_\varepsilon \rightarrow (-\infty, +\infty]$, which are C^1 on their domain. We denote by $\nabla_{X_\varepsilon} F_\varepsilon$ the gradient of F_ε in X_ε .

Definition 11.1. Let $T > 0$; we say that $u_\varepsilon \in H^1([0, T]; X_\varepsilon)$ is a *a.e. solution for the gradient flow of F_ε* if

$$\frac{\partial u_\varepsilon}{\partial t} = -\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon)$$

almost everywhere on $(0, T)$. Such solution for the a gradient flow is *conservative* if

$$F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(s)) = \int_0^s \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{X_\varepsilon}^2 dt$$

for all $\tau \in (0, T)$.

We suppose that there exist a Hilbert space X and a notion of metrizable convergence $x_\varepsilon \rightarrow x$ of families of elements of X_ε to an element of X . With respect to that convergence, we suppose that F_ε Γ -converge to a functional F , which is also C^1 on its domain.

Theorem 11.3 (Sandier–Serfaty Theorem). *Let F_ε and F be as above with F_ε Γ -converging to F , let u_ε be a family of conservative solutions for the gradient flow of F_ε with initial data $u_\varepsilon(0) = u^\varepsilon$ converging to u^0 . Suppose furthermore that:*

- (well-preparedness of initial data) u^ε is a recovery sequence for $F(u^0)$.
- (lower bound) upon subsequences u_ε converges to some $u \in H^1((0, T); X)$ and

$$\liminf_{\varepsilon \rightarrow 0} \int_0^s \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{X_\varepsilon}^2 dt \geq \int_0^s \left\| \frac{\partial u}{\partial t} \right\|_X^2 dt \tag{11.16}$$

$$\liminf_{\varepsilon \rightarrow 0} \left\| \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(s)) \right\|_{X_\varepsilon}^2 \geq \left\| \nabla_X F(u(s)) \right\|_X^2 \tag{11.17}$$

for all $s \in (0, T)$.

Then u is a solution for the gradient flow of F with initial datum u^0 , $u_\varepsilon(t)$ is a recovery sequence for $F(u(t))$ for all t and the inequalities in (11.16) and (11.17) are equalities.

Proof. Using the fact that u_ε is conservative and that for all t

$$\left\| \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon) + \frac{\partial u_\varepsilon}{\partial t} \right\|_{X_\varepsilon}^2 = 0,$$

and hence

$$-\left\langle \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(t)), \frac{\partial u_\varepsilon}{\partial t} \right\rangle = \frac{1}{2} \left(\left\| \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(t)) \right\|_{X_\varepsilon}^2 + \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{X_\varepsilon}^2 \right),$$

we then get

$$\begin{aligned} F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t)) &= \int_0^t \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{X_\varepsilon}^2 ds \\ &= - \int_0^t \left\langle \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon), \frac{\partial u_\varepsilon}{\partial t} \right\rangle_{X_\varepsilon} ds \\ &= \frac{1}{2} \int_0^t \left(\left\| \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon) \right\|_{X_\varepsilon}^2 + \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{X_\varepsilon}^2 \right) ds \end{aligned}$$

By the lower-bound assumption then we have

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} (F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t))) &\geq \frac{1}{2} \int_0^t \left(\left\| \nabla_X F(u) \right\|_X^2 + \left\| \frac{\partial u}{\partial t} \right\|_X^2 \right) ds \\ &\geq - \int_0^t \left\langle \nabla_X F(u), \frac{\partial u}{\partial t} \right\rangle_X ds. \end{aligned} \quad (11.18)$$

The last term equals

$$- \int_0^t \frac{d}{dt} F(u) ds = F(u(0)) - F(u(t)),$$

so that we have

$$\liminf_{\varepsilon \rightarrow 0} (F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t))) \geq F(u(0)) - F(u(t)).$$

Since $u_\varepsilon(0)$ is a recovery sequence for $F(u(0))$ we then have

$$F(u(t)) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon(t)), \quad (11.19)$$

so that $u_\varepsilon(t)$ is a recovery sequence for $u(t)$ and indeed we have equality in (11.19) and hence both inequalities in (11.18) are equalities. The second one of those shows that

$$\left\| \nabla_X F(u) + \frac{\partial u}{\partial t} \right\|_X^2 = 0,$$

for all t , and hence the thesis. \square

Example 11.3 (Ginzburg–Landau vortices). The theory outlined above has been successfully applied by Sandier and Serfaty to obtain the motion of vortices as the limit of the gradient flows of Ginzburg–Landau energies. We give a short account of their setting without entering into details.

Let Ω be a bounded regular open subset of \mathbb{R}^2 and $N \in \mathbb{N}$; the Hilbert spaces X_ε and X are chosen as

$$X_\varepsilon = L^2(\Omega; \mathbb{R}^2), \quad X = \mathbb{R}^{2N}$$

with scalar products

$$\langle u, v \rangle_{X_\varepsilon} = \frac{1}{|\log \varepsilon|} \int_\Omega \langle u(x), v(x) \rangle_{\mathbb{R}^2} dx, \quad \langle x, y \rangle_X = \frac{1}{\pi} \langle x, y \rangle_{\mathbb{R}^{2N}},$$

respectively.

The energies $F_\varepsilon : H^1(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}$ are defined as

$$F_\varepsilon(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + \frac{1}{\varepsilon^2} (1 - |u|^2)^2 \right) dx.$$

The convergence of u_ε is defined as follows: if we write in polar coordinates

$$u_\varepsilon(x) = \rho_\varepsilon(x) e^{i\varphi_\varepsilon(x)}$$

then $u_\varepsilon \rightarrow (x^1, \dots, x^N)$ if we have

$$\lim_{\varepsilon \rightarrow 0} \operatorname{curl}(\rho_\varepsilon^2 \nabla \varphi_\varepsilon) = 2\pi \sum_{j=1}^N d_j \delta_{x^j}$$

weak* in the sense of measures for some integers d_j , where

$$\operatorname{curl}(A_1, A_2) = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1}.$$

This convergence describes the location of *vortices* at the points x^j with a *degree* d_j . For $u_\varepsilon(x) \rightarrow x/|x|$ we have $N = 1$, $x^1 = 0$ and $d_1 = 1$.

It can be proved that there exists a function $W = W_{(d_1, \dots, d_N)}$ such that

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \left(F_\varepsilon(u) - \pi N |\log \varepsilon| \right) = W(x^1, \dots, x^N).$$

The function W can be characterized in terms of the Green function of Ω . Its precise definition is not relevant to this example.

The well-preparedness condition for the initial data amounts to requiring that

$$u_\varepsilon^0 \rightarrow (x_0^1, \dots, x_0^N) \quad \text{and } d_j \in \{-1, 1\}.$$

Under these conditions we may apply Theorem 11.3 to the scaled energies $F_\varepsilon - \pi N |\log \varepsilon|$. This yields solutions $u_\varepsilon = u_\varepsilon(x, t)$ to the equation

$$\begin{cases} \frac{1}{|\log \varepsilon|} \frac{\partial u_\varepsilon}{\partial t} = \Delta u + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x) \end{cases}$$

converging to $x(t) = (x^1(t), \dots, x^N(t)) = (x_1(t), \dots, x_{2N}(t))$. The limit vortices move following the system of ODE

$$\frac{dx_i}{dt} = -\frac{1}{\pi} \frac{\partial W(x)}{\partial x_i}.$$

This description is valid until the first *collision time* T^* when $x_j(T^*) = x_k(T^*)$ for some j and k with $j \neq k$.

11.2.2 Convergence of Stable Critical Points

The Sandier–Serfaty approach can be extended to analyze the convergence of critical points, for which Theorem 11.3 is trivial.

Theorem 11.4. *Let u_ε be a family of critical points of F_ε with $u_\varepsilon \rightarrow u$, such that the following holds: for any $V \in X$, we can find $v_\varepsilon(t)$ defined in a neighborhood of $t = 0$, such that $\partial_t v_\varepsilon(0)$ depends on V in a linear and one-to-one manner, and*

$$v_\varepsilon(0) = u_\varepsilon(0) \tag{11.20}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{d}{dt} F_\varepsilon(v_\varepsilon(t))|_{t=0} = \frac{d}{dt} F(u + tV)|_{t=0} = \langle dF(u), V \rangle \tag{11.21}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{d^2}{dt^2} F_\varepsilon(v_\varepsilon(t))|_{t=0} = \frac{d^2}{dt^2} F(u + tV)|_{t=0} = Q(u)(V). \tag{11.22}$$

Then

- If (11.20)–(11.21) are satisfied, then u is a critical point of F .
- If (11.20)–(11.21)–(11.22) are satisfied, then if u_ε are critical points of E_ε with positive Hessian, u is a critical point of F with positive Hessian.

We do not enter in the details of this result, but only remark that in this case the Γ -convergence of F_ε to F is not required, in analogy to the conditions analyzed in Chap. 6.

Appendix

The results in Sect. 11.1.1 and part (ii) of Theorem 11.1 are a simplified version of the analogous results for geodesic-convex energies in metric spaces, that can be found in the notes by Ambrosio and Gigli [1]. Example 11.2 is a simplified version of a result by Braides et al. [2].

The result by Sandier and Serfaty (with weaker hypotheses than those reported here) is contained in the seminal paper [4]. An account of their approach is contained in the notes by Serfaty [6].

The convergence of stable points has been considered by Serfaty in [5] and further analyzed by Jerrard and Sternberg in [3].

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