The KOSL Scaling, Invariant Measure and PDF of Turbulence

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In 1941 Kolmogorov and Obukhov [9, 12] proposed that there exists a statistical theory of turbulence that should allow the computation of all the statistical quantities that can be computed and measured in turbulent systems. These are quantities such as the moments, the structure functions and the probability density functions (PDFs) of the turbulent velocity field. The Kolmogorov-Obukhov '41 theory predicted that the structure functions of turbulence, that are the moments of the velocity differences at distances separated by a lag variable l, should scale with the lag variable to a power $p/3$ for the pth structure function, multiplied by a universal constant. This was found to be inconsistent with observations and in 1962 Kolomogorov and Obukhov [10, 13] presented a refined scaling hypothesis, where the multiplicative constants are not universal and the scaling exponents are modified to $\zeta_p = p/3 + \tau_p$, by the intermittency correction τ_p that are due to intermittency in the turbulent velocity. It was still not clear what the values of τ_p should be, because the log-normal exponents suggested by Kolmorogov turned out again to be inconsistent with observations. Then in 1994 She and Leveque [16] found the correct (log-Poissonian) formulas for τ_p that are consistent with modern simulations and experiments.

In this paper we will outline how the statistical theory of Kolmogorov and Obukhov is derived from the Navier-Stokes equation without getting into any of the technical details. We start with the classical Reynolds decomposition of the velocity into the mean (large scale) flow and the fluctuations or small scale flow. Then we develop a stochastic Navier-Stokes equation [6], for the small scale flow. If we assume that dissipation take place on all scales in the inertial range (defined below) then it turns out that the noise in this stochastic Navier-Stokes equation is determined by well-known theorems in probability. The additive noise in the stochastic Navier-Stokes equation is generic noise given by the central limit theorem and the large deviation principle. The multiplicative noise consists of jumps multiplying the

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velocity, modeling jumps in the velocity gradient. We will explain how this form of the noise follows from very general hypothesis.

Once the form of the noise in the stochastic Navier-Stokes equation for the small scales is determined, we can estimate the structure functions of turbulence and establish the Kolmogorov-Obukhov '62 scaling hypothesis with the She-Leveque intermittency corrections [5]. Then one can compute the invariant measure of turbulence writing the stochastic Navier-Stokes equation as an infinite-dimensional Ito process and solving the linear Kolmogorov-Hopf [8] functional differential equation for the invariant measure. Finally the invariant measure can be projected onto the PDF. The PDFs turn out to be the normalized inverse Gaussian (NIG) distributions of Barndorff-Nilsen [1, 2], and compare well with PDFs from simulations and experiments. The details of the proofs can be found in [5] and the background material can be found in [6].

A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$u_t + u\cdot\nabla u = \nu\Delta u - \nabla p, \qquad u(x,0) = u_0(x)$$

with the incompressibility condition $\nabla\cdot u = 0$. Eliminating the pressure using the incompressibility condition gives

$$u_t + u\cdot\nabla u = \nu\Delta u + \nabla\Delta^{-1}\text{trace}(\nabla u)^2, \qquad u(x,0) = u_0(x).$$

The turbulence is quantified by the dimensionless Taylor-Reynolds number $Re_\lambda = \frac{U\lambda}{\nu}$ [14].

Following the classical Reynolds decomposition [15], we decompose the velocity into mean flow U and the fluctuations u. Then the velocity is written as $U+u$, where U describes the large scale flow and u describes the small scale turbulence. We must also decompose the pressure into mean pressure P and the fluctuations p, then the equation for the large scale flow can be written as

$$U_t + U\cdot\nabla U = \nu\Delta U - \nabla P - \nabla\cdot(\overline{u\otimes u}), \tag{1}$$

where in coordinates $\nabla\cdot(\overline{u\otimes u}) = \frac{\partial\overline{u_i u_j}}{\partial x_j}$, that is ∇ is dotted with the rows of $\overline{u_i u_j}$ and $R_{ij} = \overline{u\otimes u}$ is the Reynolds stress, see [3]. The Reynolds stress has the interpretation of a turbulent momentum flux and the last term in (1) is also know as the eddy viscosity. It describes how the small scales influence the large scales. In addition we get divergence free conditions for U, and u

$$\nabla\cdot U = 0, \qquad \nabla\cdot u = 0.$$

Together, (1) and the divergence free condition on U give Reynolds Averaged Navier-Stokes (RANS) that forms the basis for most contemporary simulations of turbulent flow.

Finding a constitutive law for the Reynolds stress $\overline{u\otimes u}$ is the famous closure problem in turbulence and we will solve that by writing down a stochastic equation for the small scale velocity u. The hypothesis is that the large scale influence the small scales directly, through the fluid instabilities and the noise in fully developed turbulence. An example of this mechanics, how the instabilities magnify the tiny ambient noise to produce large noise, in given in [4], see also Chapter 1 in [6].

Now consider the inertial range in turbulence. In Fourier space this is the range of wave numbers k: $\frac{1}{L} \le |k| \le \frac{1}{\eta}$, where $\eta = (\nu^3/\varepsilon)^{1/4}$ is the Kolmogorov length scale, ε is the energy dissipation and L the size of the largest eddies, see [6]. If we assume that dissipation takes place on all length scale in the inertial range then the form of the dissipation processes are determined by the fundamental theorems of probability. Namely, if we impose periodic boundary conditions (different boundary conditions correspond to different basis vectores), then the central limit theorem and the large deviation principle stipulate that the additive noise in the Navier-Stokes equation for the small scale must be of the form:

$$\sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0} d_k |k|^{1/3} dt\ e_k(x),$$

where $e_k(x) = e^{2\pi i k\cdot x}$ are the Fourier coefficient and $c_k^{\frac{1}{2}}$ and d_k are coefficients that ensure the series converge in 3 dimensions. The first term describes the mean of weakly coupled dissipation processes given by the central limit theorem and the second term describes the large deviations of that mean, given by the large deviation principle, see [6]. Thus together the two terms give a complete description of the mean of the dissipation process similar to the mean of many processes in probability. The factor $|k|^{1/3}$ implies that the mean dissipation has only one scaling. The Fourier coefficients of the first series contain independent Brownian motions b_t^k and thus the noise is white in time in the infinitely many directions in function space. The noise cannot be white in space, hence the decaying coefficients $c_k^{1/2}$ and d_k, because if it was the small scale velocity u would be discontinuous in 3 dimension, see [5]. This is contrary to what is observed in nature.

The other part of the noise, in fully developed turbulence, is multiplicative and models the excursion (jumps) in the velocity gradient or vorticity concentrations. If we let N_t^k denote the integer number of velocity excursion, associated with kth wavenumber, that have occurred at time t, so that the differential $dN^k(t) = N^k(t+dt) - N^k(t)$ denotes the number of excursions in the time interval $(t, t+dt]$, then the process $df_t^3 = \sum_{k\neq 0}^M \int_{\mathbb{R}} h_k(t,z)\bar{N}^k(dt,dz)$, gives the multiplicative noise term. One can show that any noise corresponding to jumps in the velocity gradients must have this multiplicative noise to leading order, see [5]. A detailed derivation of both the noise terms can be found in [5] and [6].

Adding the additive noise and the multiplicative noise we get the stochastic Navier-Stokes equations describing the small scales in fully developed turbulence

$$du = (\nu\Delta u - u\cdot\nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt + \sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0} d_k|k|^{1/3} dt\ e_k(x)$$
$$+\ u\Big(\sum_{k\neq 0}^M \int_{\mathbb{R}} h_k \bar{N}^k(dt,dz)\Big) - U\cdot\nabla u - u\cdot\nabla U,\ \ u(x,0) = u_0(x), \quad (2)$$

where we have used the divergence free condition $\nabla \cdot u = 0$ to eliminate the small scale pressure p. Each Fourier component e_k comes with its own Brownian motion b_t^k and a deterministic bound $|k|^{1/3}dt$.

The next step is to figure out how the generic noise interacts with the Navier-Stokes evolution. This is determined by the integral form of the equation (2),

$$u = e^{Kt} e^{\int_0^t dq} M_t u^0 + \sum_{k \neq 0} \int_0^t e^{K(t-s)} e^{\int_s^t dq} M_{t-s} (c_k^{1/2} d\beta_s^k + d_k \mu_k ds) e_k(x), \qquad (3)$$

where K is the operator $K = \nu\Delta + \nabla\Delta^{-1} tr(\nabla u \nabla)$, and we have omitted the terms $-U \cdot \nabla u - u \cdot \nabla U$ in (2), to simplify the exposition. We solve (2) using the Feynmann-Kac formula, and the Cameron-Martin formula (or Girsanov's Theorem) from probability theory, see [6], to get (3). The Cameron-Martin formula gives the Martingale $M_t = exp\{-\int_0^t u(B_s, s) \cdot dB_s - \frac{1}{2}\int_0^t |u(B_s, s)|^2 ds\}$. The Feynmann-Kac formula gives the exponential of a sum of terms of the form $\int_s^t dq^k = \int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt, dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt, dz)$, see [5] or [6] Chapter 2 for details. The form of the processes

$$e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)} = e^{N_t^k \ln\beta + \gamma \ln|k|} = |k|^\gamma \beta^{N_t^k} \qquad (4)$$

was found by She and Leveque [16], for $h_k = \beta - 1$. It was pointed out by She and Waymire [17] and by Dubrulle [7] that they are log-Poisson processes. The upshot of this computation is that we see the Navier-Stokes evolution acting on the additive noise to give the Kolmogorov-Obukhov '41 scaling, and the Navier-Stokes evolution acting on the multiplicative noise to produce the intermittency corrections through the Feynmann-Kac formula. Together these two scaling combine to give the scaling of the structure functions in turbulence,

Lemma 1 (The Kolmogorov-Obukhov-She-Leveque scaling). *The scaling of the structure functions is*

$$S_p \sim C_p |x-y|^{\zeta_p}, \qquad \zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1 - (2/3)^{p/3}).$$

$\frac{p}{3}$ *being the Kolmogorov scaling and* τ_p *the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's* $4/5$ *law,* $S_3 = -\frac{4}{5}\varepsilon|x-y|$, *to leading order, were* $\varepsilon = -\frac{d\mathcal{E}}{dt}$ *is the energy dissipation.*

The first structure functions is estimated by

$$S_1(x,y,\infty) \leq \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|d_k|(1 - e^{-\lambda_k t})}{|k|^{\zeta_1}} |\sin(\pi k \cdot (x-y))|.$$

We get a stationary state as $t \to \infty$, and for $|x-y|$ small, $S_1(x,y,\infty) \sim \frac{2\pi^{\zeta_1}}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |d_k||x-y|^{\zeta_1}$, where $\zeta_1 = 1/3 + \tau_1 \approx 0.37$. Similarly, $S_2(x,y,\infty) \sim \frac{4\pi^{\zeta_2}}{C^2} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} [d_k{}^2 + (\frac{C}{2})c_k]|x-y|^{\zeta_2}$, when $|x-y|$ is small, where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$, and $S_3(x,y,\infty) \sim \frac{2^3\pi}{C^3} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} [|d_k|^3 + 3(C/2)c_k|d_k|]|x-y|$. For the pth structure functions, we get that S_p is estimated by

$$S_p \leq \frac{2^p}{C^p} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sigma^p \cdot (-i\sqrt{2}\mathrm{sgn}\, M)^p \; U\left(-\frac{1}{2}p, \frac{1}{2}, -\frac{1}{2}(M/\sigma)^2\right)}{|k|^{\zeta_p}} |\sin^p(\pi k \cdot (x-y))|.$$

where U is the confluent hypergeometric function, $M = |d_k|(1-e^{-\lambda_k t})$ and $\sigma = \sqrt{(C/2)c_k(1-e^{-2\lambda_k t})}$. The details of these estimates are given in [5].

The integral equation can be considered to be an infinite-dimensional Ito process, see [6]. This means that we can find the associated Kolmogorov backward equation for the Ito diffusion associated with the equation (3) and this equations that determines the invariant measure of turbulence, see [5], is linear. This was first attempted by Hopf [8] wrote down a functional differential equation for the characteristic function of the invariant measure of the deterministic Navier-Stokes equation. The Kolmogorov-Hopf (backward) equation for (2) is

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{tr}[P_t C P_t^* \Delta\phi] + \mathrm{tr}[P_t \bar{D} \nabla \phi] + < K(z)P_t, \nabla\phi >, \tag{5}$$

see [5] and [6] Chapter 3, where $\bar{D} = (|k|^{1/3} D_k)$, $\phi(z)$ is a bounded function of z, $P_t = e^{-\int_0^t \nabla u \, dr} M_t \prod_k^m |k|^{2/3}(2/3)^{N_t^k}$. The variance and drift are defined to be

$$Q_t = \int_0^t e^{K(s)} P_s C P_s^* e^{K^*(s)} ds, \quad E_t = \int_0^t e^{K(s)} P_s \bar{D} ds. \tag{6}$$

In distinction to the nonlinear Navier-Stokes equation (2) that cannot be solved explicitly, the linear equation (5) can be solved. The solution of the Kolmogorov-Hopf equation (5) is

$$R_t \phi(z) = \int_H \phi(e^{Kt} P_t z + EI + y) \mathcal{N}_{(0,Q_t)} * \mathbb{P}_{N_t}(dy),$$

$\mathbb{P}_{N_t}$ being the law of the log-Poisson process (4). The invariant measure of turbulence that appears in the last equation can now be expressed explicitly,

Theorem 1. *The invariant measure of the Navier-Stokes equation on* $H_c = H^{3/2^+}(\mathbb{T}^3)$ *is,*

$$\mu(dx) = e^{<Q^{-1/2}EI,\; Q^{-1/2}x> - \frac{1}{2}|Q^{-1/2}EI|^2} \mathcal{N}_{(0,Q)}(dx) \sum_k \delta_{k,l} \sum_{j=0}^{\infty} p_{m_l}^j \delta_{(N_l - j)}$$

where $Q = Q_\infty$, $E = E_\infty$, $m_k = \ln |k|^{2/3}$ *is the mean of the log-Poisson processes (4) and* $p_{m_k}^j = \frac{(m_k)^j e^{-m_k}}{j!}$ *is the the probability of* $N_\infty^k = N_k$ *having exactly j jumps,* $\delta_{k,l}$ *is the Kroncker delta function.*

This shows that the invariant measure of turbulence is simply a product of two measure, one an infinite-dimensional Gaussian that gives the Kolmogorov-Obukhov scaling and the other a discrete Poisson measure that gives the She-Leveque intermittency corrections. Together they produce the scaling of the structure functions in Lemma 1.

The quantity that can be compared directly to experiments is the probability density function (PDF). We take the trace of the Kolmogorov-Hopf equation (5), see [6] Chapter 3, to compute the differential equation satisfied by the PDE. The stationary equation satisfied by the PDF is

$$\frac{1}{2}\phi_{rr} + \frac{1+|c|}{r}\phi_r = \frac{1}{2}\phi. \tag{7}$$

Lemma 2. *The PDF is a Normalized Inverse Gaussian distribution NIG of Barndorff-Nilsen [1]:*

$$f(x) = \frac{(\delta/\gamma)}{\sqrt{2\pi}K_1(\delta\gamma)} \frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)e^{\beta(x-\mu)}}{\left(\sqrt{\delta^2+(x-\mu)^2}/\alpha\right)} \tag{8}$$

where K_1 is modified Bessel's function of the second kind, $\gamma = \sqrt{\alpha^2-\beta^2}$.

(8) is the solution of (7) and the PDF that can be compared a large class of experimental data.

We finally explain how we get around the famous non-uniqueness problem of the Navier-Stokes equation. It is well known that the fluid velocity u solving the (stochastic) Navier-Stokes equation may not be unique in 3 dimensions. However, the invariant measure in Theorem 1 exists by Leray's '34 [11] theory, see Theorem 2 below. If the velocity is not unique different velocities give equivalent statistics. Thus the statistical theory is unique although the velocity u may not be.

Theorem 2. *The solution of the stochastic Navier-Stokes equation (2) satisfies the estimates*

$$E(|u|_2^2)(t) \le |u|_2^2(0)e^{-at} + \frac{1}{a}\left(\frac{2|T|}{\varepsilon}\sum_{k\neq 0} d_k|k|^{\frac{1}{3}} + \sum_{k\neq 0} c_k\right) + \frac{|T|}{a}\ln\left(\prod_{k=1}^{m}|k|^2\right)^{\frac{1}{9}},$$

and

$$(1-\varepsilon D)\sup_{[0,t]} E(|u|_2^2)(t) + 2\nu\int_0^t E(|\nabla u|)(s)ds \le |u|_2^2(0) + \left(\frac{|T|}{\varepsilon}\sum_{k\neq 0} d_k|k|^{\frac{1}{3}} + \sum_{k\neq 0} c_k\right)t$$
$$+|T|\ln\left(\prod_{k=1}^{m}|k|^2\right)^{\frac{1}{9}},$$

where $D = \sum_{k\neq 0} d_k|k|^{1/3}$, E denotes the expectation, $a = 2\nu\lambda_1 - D$, λ_1 is the first eigenvalue of $-\Delta$, with vanishing boundary conditions, ε is a small number and $|T|$ is the volume of the torus (box with periodic boundary conditions).

The proof of the theorem is similar to the proof of the Leray theory in Chapter 4, in [6].

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