

A QUADRATIC C^0 INTERIOR PENALTY METHOD FOR AN ELLIPTIC OPTIMAL CONTROL PROBLEM WITH STATE CONSTRAINTS

S.C. BRENNER, L.-Y. SUNG, AND Y. ZHANG*

Abstract. We consider an elliptic distributed optimal control problem on convex polygonal domains with pointwise state constraints and solve it as a fourth order variational inequality for the state by a quadratic C^0 interior penalty method. The error for the state in an H^2 -like energy norm is $O(h^\alpha)$ on quasi-uniform meshes (where $\alpha \in (0, 1]$ is determined by the interior angles of the domain) and $O(h)$ on graded meshes. The error for the control in the L_2 norm has the same behavior. Numerical results that illustrate the performance of the method are also presented.

Key words. Elliptic distributed optimal control problem, Pointwise state constraints, Simply supported plate, Fourth order variational inequality, Finite element, C^0 interior penalty method, Discontinuous Galerkin, Graded meshes

AMS(MOS) subject classifications. 65K15, 65N30.

1. Introduction. Let Ω be a bounded convex polygonal domain in \mathbb{R}^2 , $y_d \in L_2(\Omega)$, $\gamma \geq 0$ and $\beta > 0$ be constants. The following problem [33] is a model elliptic distributed optimal control problem with pointwise state constraints:

Find the minimizer of the functional

$$J(y, u) = \frac{\gamma}{2} \int_{\Omega} (y - y_d)^2 dx + \frac{\beta}{2} \int_{\Omega} u^2 dx, \quad (1.1)$$

where $(y, u) \in H_0^1(\Omega) \times L_2(\Omega)$ are subjected to the constraints

$$\int_{\Omega} \nabla y \cdot \nabla v dx = \int_{\Omega} uv dx \quad \forall v \in H_0^1(\Omega), \quad (1.2)$$

$$\psi_1 \leq y \leq \psi_2 \quad \text{a.e. in } \Omega. \quad (1.3)$$

Here the functions $\psi_1(x), \psi_2(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\psi_1 < \psi_2 \quad \text{in } \Omega, \quad (1.4a)$$

$$\psi_1 < 0 < \psi_2 \quad \text{on } \partial\Omega. \quad (1.4b)$$

Since Ω is convex, elliptic regularity [36, 45, 58] implies that (1.2) is equivalent to $y \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u = -\Delta y$. Note that [46, Theorem 2.2.1]

$$\int_{\Omega} (\Delta v)(\Delta w) dx = \int_{\Omega} (D^2 v : D^2 w) dx \quad \forall v, w \in H^2(\Omega) \cap H_0^1(\Omega), \quad (1.5)$$

*Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA, brenner@math.lsu.edu; sung@math.lsu.edu; yghan112@utk.edu

where

$$D^2v : D^2w = \sum_{1 \leq i, j \leq 2} \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right) \left(\frac{\partial^2 w}{\partial x_i \partial x_j} \right)$$

is the (Frobenius) inner product of the Hessian matrices of v and w . Therefore we can solve the optimal control problem (1.1)–(1.3) by looking for the minimizer of the reduced functional

$$\hat{J}(y) = \frac{\gamma}{2} \int_{\Omega} (y - y_d)^2 dx + \frac{\beta}{2} \int_{\Omega} (D^2y : D^2y) dx$$

in the set

$$K = \{v \in H^2(\Omega) \cap H_0^1(\Omega) : \psi_1 \leq v \leq \psi_2 \text{ in } \Omega\}. \quad (1.6)$$

A simple calculation shows that this is equivalent to the following problem:

$$\text{Find } \bar{y} = \operatorname{argmin}_{y \in K} \left[\frac{1}{2} \mathcal{A}(y, y) - (f, y) \right], \quad (1.7)$$

where $f = \gamma y_d$, (\cdot, \cdot) is the inner product of $L_2(\Omega)$, and

$$\mathcal{A}(v, w) = \int_{\Omega} [\beta(D^2v : D^2w) + \gamma vw] dx. \quad (1.8)$$

Since (1.4) implies that K is a nonempty closed convex subset of $H^2(\Omega) \cap H_0^1(\Omega)$ and the bilinear form $\mathcal{A}(\cdot, \cdot)$ is symmetric, bounded, and coercive on $H^2(\Omega) \cap H_0^1(\Omega)$, we can apply the standard theory [43, 52, 54, 59] to conclude that the problem (1.7) has a unique solution $\bar{y} \in K$ characterized by the variational inequality

$$\mathcal{A}(\bar{y}, y - \bar{y}) \geq (f, y - \bar{y}) \quad \forall y \in K. \quad (1.9)$$

The solution of the optimal control problem is then given by (\bar{y}, \bar{u}) , where $\bar{u} = -\Delta \bar{y}$. Note that (1.7) becomes the displacement obstacle problem for simply supported Kirchhoff plates if we take γ to be 0. For this reason we will also refer to (1.7) as an obstacle problem.

According to the regularity results in [32, 41, 42] for fourth order obstacle problems, the solution \bar{y} of (1.7) belongs to $H_{\text{loc}}^3(\Omega) \cap C^2(\Omega)$ under our assumptions on the functions y_d , ψ_1 , and ψ_2 . Note that (1.4b) implies that the constraints are inactive near $\partial\Omega$ and hence

$$\beta \Delta^2 \bar{y} + \gamma \bar{y} = f$$

near $\partial\Omega$. It then follows from the elliptic regularity theory for the biharmonic equation (cf. [8] and Appendix A) that there exists $\alpha \in (0, 1]$ (determined by the interior angles of Ω) such that $\bar{y} \in H^{2+\alpha}(\mathcal{N})$ in a

neighborhood \mathcal{N} of $\partial\Omega$ disjoint from the active set. Thus globally \bar{y} belongs to $H^{2+\alpha}(\Omega)$. We shall refer to α as the index of elliptic regularity for the obstacle problem (1.7).

A main difficulty in the analysis of finite element methods for fourth order obstacle problems is that the solutions in general do not belong to $H_{\text{loc}}^4(\Omega)$ even for smooth data, which means that the complementarity form of the variational inequality (1.9) in general only exists in a weak sense. In contrast, the solutions of second order obstacle problems belong to $H^2(\Omega)$ under appropriate assumptions on the data (cf. [29, 53]). Hence the complementarity forms of the variational inequalities arising from second order obstacle problems exist in the strong sense, which is a crucial ingredient for the derivations of optimal error estimates in [30, 31, 40].

A new approach to the obstacle problem for clamped Kirchhoff plates on convex polygonal domains was introduced in [25], where optimal error estimates were obtained for C^1 finite element methods, classical non-conforming finite element methods, and discontinuous Galerkin methods. The results were later extended to general domains and general Dirichlet boundary conditions in [15, 23, 24]. This new approach does not rely on the complementarity forms of the variational inequalities and hence can bypass the aforementioned difficulty. The goal of this paper is to extend the results in [23] to (1.7)/(1.9), which covers both obstacle problems for simply supported plates and optimal control problems with pointwise state constraints. We will show that the magnitude of the error in the energy norm is $O(h^\alpha)$ on quasi-uniform meshes and $O(h)$ on graded meshes.

Finite element methods for state constrained elliptic optimal control problems were investigated in [37, 56], where the finite element approximation (\bar{y}_h, \bar{u}_h) of (\bar{y}, \bar{u}) is obtained from discrete versions of the optimal control problems. In this approach the error analysis for the state and the error analysis for the control are coupled and hence the estimates for $\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}$ and $\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)}$ have the same magnitude, which in the case of a rectangle with quasi-uniform meshes is $O(h^{1-\epsilon})$. In our approach we obtain instead an error estimate for the approximation \bar{y}_h of \bar{y} in an H^2 -like energy norm, which then implies an error estimate in the L_2 norm for the approximation \bar{u}_h of \bar{u} (generated from \bar{y}_h by a postprocessing procedure) with the same magnitude. In the case of a rectangle with quasi-uniform meshes, the magnitudes of these errors are $O(h)$. On the other hand, the convergence of \bar{y}_h in the $H^1(\Omega)$ norm and the $L_\infty(\Omega)$ norm, which are weaker than the energy norm, can be expected to be of higher order. This is indeed observed in our numerical experiments, where the magnitudes of the errors of \bar{y}_h in the $H^1(\Omega)$ norm and the $L_\infty(\Omega)$ norm are $O(h^2)$ for a rectangle.

The optimal control problem defined by (1.1)–(1.3) is solved as a fourth order variational inequality in [55] by a Morley finite element method and in [44] by a mixed finite element method. However the analyses in [44, 55] rely on additional assumptions on the active set first introduced in [7]. Our new approach for fourth order obstacle problems may provide an error

analysis for the finite element methods in [44, 55] without the additional assumptions on the active set.

Other numerical methods for (1.1)–(1.3) are investigated, for example, in [5, 6, 34, 48–51, 57, 60].

The rest of the paper is organized as follows. We introduce a quadratic C^0 interior penalty method for (1.7) in Sect. 2 and an intermediate obstacle problem that connects the continuous and discrete obstacle problems in Sect. 3. Section 4 contains several preliminary estimates which are useful for the convergence analysis carried out in Sect. 5. Numerical results that illustrate the performance of our method are presented in Sect. 6, followed by some concluding remarks in Sect. 7. Elliptic regularity results for simply supported plates, which play an important role in the error analysis, are summarized in Appendix A. Some technical results concerning an enriching operator that connects the discrete and continuous spaces are given in Appendix B.

We will follow the notation for Sobolev spaces and norms in [20, 35]. Throughout the paper we will denote by C a generic positive constant independent of mesh sizes that can take different values at different occurrences. To avoid the proliferation of constants, we will also use $A \lesssim B$ (or $B \gtrsim A$) to denote the statement that $A \leq (\text{constant})B$, where the positive constant is independent of mesh sizes. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

2. A Quadratic C^0 Interior Penalty Method. C^0 interior penalty methods were introduced in [39] for fourth order elliptic boundary value problems. They were further studied in [13, 16, 18, 21] and fast solvers for C^0 interior methods were developed in [22, 26, 27]. Adaptive [17] and isoparametric [19] versions of C^0 interior penalty methods are also available. Below we will recall the notation for C^0 interior penalty methods and introduce the discrete obstacle problem for (1.7).

2.1. Triangulation. Let \mathcal{T}_h be a simplicial triangulation of Ω that is regular (i.e., \mathcal{T}_h satisfies a minimum angle condition). We will use the following notation throughout the paper.

- h_T is the diameter of the triangle T .
- h is a mesh parameter proportional to $\max_{T \in \mathcal{T}_h} h_T$.
- v_T is the restriction of the function v to the triangle T .
- \mathcal{E}_h is the set of the edges of the triangles in \mathcal{T}_h .
- \mathcal{E}_h^i is the subset of \mathcal{E}_h consisting of edges interior to Ω .
- \mathcal{E}_h^b is the subset of \mathcal{E}_h consisting of edges along $\partial\Omega$.
- $|e|$ is the length of an edge e .
- \mathcal{V}_h is the set of the vertices of the triangles in \mathcal{T}_h .
- \mathcal{V}_T is the set of the three vertices of T .
- $\mathcal{E}_{\mathcal{V}_T}^i$ is the set of the edges in \mathcal{E}_h^i emanating from the vertices of T .
- \mathcal{T}_T is the set of triangles sharing a vertex with T .
- \mathcal{S}_T is the interior of the closure of $\cup_{T' \in \mathcal{T}_T} T'$.
- \mathcal{T}_p is the set of the triangles in \mathcal{T}_h that share the common vertex p .

- \mathcal{T}_e is the set of the triangles in \mathcal{T}_h that share the common edge e .
- $|\mathcal{T}_p|$ (resp. $|\mathcal{T}_e|$) is the number of triangles in \mathcal{T}_p (resp. \mathcal{T}_e).
- Let $e \in \mathcal{E}_h^b$. Then T_e is the triangle in \mathcal{T}_h such that $\mathcal{T}_e = \{T_e\}$.

We will consider both quasi-uniform and graded triangulations. For a quasi-uniform triangulation \mathcal{T}_h , we have

$$h_T \approx h \quad \forall T \in \mathcal{T}_h. \tag{2.1}$$

Let p_1, \dots, p_L be the corners of Ω and ω_ℓ be the interior angle at p_ℓ for $1 \leq \ell \leq L$. For a graded triangulation \mathcal{T}_h , we have

$$h_T \approx h\Phi(c_T) \quad \forall T \in \mathcal{T}_h, \tag{2.2}$$

where c_T is the center of T ,

$$\Phi(x) = \prod_{\ell=1}^L |p_\ell - x|^{1-\alpha_\ell}, \tag{2.3}$$

and the grading parameters $\alpha_\ell > 0$ are determined as follows:

$$\begin{cases} \alpha_\ell = 1 & \text{if } \omega_\ell \leq \frac{\pi}{2}, \\ \alpha_\ell < \left(\frac{\pi}{\omega_\ell}\right) - 1 & \text{if } \frac{\pi}{2} < \omega_\ell < \pi. \end{cases} \tag{2.4}$$

Note that (2.2) and (2.3) imply

$$h_T^{\alpha_\ell} \approx h \tag{2.5}$$

if $T \in \mathcal{T}_h$ touches the corner p_ℓ .

REMARK 2.1. We can take $\alpha = \min_{1 \leq \ell \leq L} \alpha_\ell$ to be the index of elliptic regularity (cf. Appendix A).

REMARK 2.2. The construction of regular triangulations that satisfy (2.2) is discussed, for example, in [1, 10, 14].

2.2. Jumps and Averages. The jumps and averages of the normal derivatives for functions in the piecewise Sobolev spaces

$$H^s(\Omega, \mathcal{T}_h) = \{v \in L_2(\Omega) : v_T = v|_T \in H^s(T) \quad \forall T \in \mathcal{T}_h\}$$

are defined as follows.

Let $e \in \mathcal{E}_h^i$ be the common edge of $T_\pm \in \mathcal{T}_h$ and n_e be the unit normal of e pointing from T_- to T_+ . We define on e

$$\left\{ \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \right\} = \frac{1}{2} \left(\left. \frac{\partial^2 v_+}{\partial n_e^2} \right|_e + \left. \frac{\partial^2 v_-}{\partial n_e^2} \right|_e \right) \quad \forall v \in H^s(\Omega, \mathcal{T}_h), s > \frac{5}{2}, \tag{2.6a}$$

$$\left[\left[\frac{\partial v}{\partial n} \right] \right] = \left. \frac{\partial v_+}{\partial n_e} \right|_e - \left. \frac{\partial v_-}{\partial n_e} \right|_e \quad \forall v \in H^2(\Omega, \mathcal{T}_h), \tag{2.6b}$$

where $v_{\pm} = v|_{T_{\pm}}$. Similarly, we define on e

$$\left\{ \left\{ \frac{\partial v}{\partial n_e} \right\} \right\} = \frac{1}{2} \left(\frac{\partial v_+}{\partial n_e} \Big|_e + \frac{\partial v_-}{\partial n_e} \Big|_e \right) \quad \forall v \in H^2(\Omega, \mathcal{T}_h), \quad (2.7a)$$

$$\left[\left[\frac{\partial^2 v}{\partial n_e^2} \right] \right] = \frac{\partial^2 v_+}{\partial n_e^2} \Big|_e - \frac{\partial^2 v_-}{\partial n_e^2} \Big|_e \quad \forall v \in H^s(\Omega, \mathcal{T}_h), s > \frac{5}{2}. \quad (2.7b)$$

REMARK 2.3. Note that the definitions for the average $\left\{ \left\{ \partial^2 v / \partial n^2 \right\} \right\}$ and the jump $\left[\left[\partial v / \partial n \right] \right]$ in (2.6), which appear in C^0 interior penalty methods, are independent of the choice of T_{\pm} (or n_e). On the other hand, the definitions in (2.7) for $\left\{ \left\{ \partial v / \partial n_e \right\} \right\}$ and $\left[\left[\partial^2 v / \partial n_e^2 \right] \right]$, which appear only in the analysis, do depend on the choice of T_{\pm} (or n_e). However their product is also independent of the choice of T_{\pm} (or n_e).

Let $e \in \mathcal{E}_h^b$ be a boundary edge and n_e be the unit normal of e pointing towards the outside of Ω . We define on e

$$\left\{ \left\{ \frac{\partial v}{\partial n_e} \right\} \right\} = \frac{\partial v}{\partial n_e} \Big|_e \quad \forall v \in H^2(\Omega, \mathcal{T}_h), \quad (2.8a)$$

$$\left[\left[\frac{\partial^2 v}{\partial n_e^2} \right] \right] = - \frac{\partial^2 v}{\partial n_e^2} \Big|_e \quad \forall v \in H^s(\Omega, \mathcal{T}_h), s > \frac{5}{2}. \quad (2.8b)$$

2.3. The Discrete Obstacle Problem. Let $V_h \subset H_0^1(\Omega)$ be the \mathbb{P}_2 Lagrange finite element space associated with \mathcal{T}_h whose members vanish on $\partial\Omega$. We define the bilinear form $a_h(\cdot, \cdot)$ on $V_h \times V_h$ by

$$\begin{aligned} a_h(v, w) &= \sum_{T \in \mathcal{T}_h} \int_T (D^2 v : D^2 w) dx + \sum_{e \in \mathcal{E}_h^i} \int_e \left\{ \left\{ \partial^2 v / \partial n^2 \right\} \right\} \left[\left[\partial w / \partial n \right] \right] ds \\ &\quad + \sum_{e \in \mathcal{E}_h^i} \int_e \left\{ \left\{ \partial^2 w / \partial n^2 \right\} \right\} \left[\left[\partial v / \partial n \right] \right] ds \\ &\quad + \sigma \sum_{e \in \mathcal{E}_h^i} |e|^{-1} \int_e \left[\left[\partial v / \partial n \right] \right] \left[\left[\partial w / \partial n \right] \right] ds, \end{aligned} \quad (2.9)$$

where $\sigma > 0$ is a penalty parameter. Note that $a_h(\cdot, \cdot)$ is a consistent bilinear form for the biharmonic equation with the boundary conditions of simply supported plates.

It follows from (2.6a) and scaling that

$$\sum_{e \in \mathcal{E}_h^i} |e| \left\| \left\{ \left\{ \partial^2 v / \partial n^2 \right\} \right\} \right\|_{L_2(e)}^2 \lesssim \sum_{T \in \mathcal{T}_h} |v|_{H^2(T)}^2 \quad \forall v \in V_h. \quad (2.10)$$

Therefore, for sufficiently large σ , we have (cf. [21])

$$a_h(v, v) \gtrsim \left(\sum_{T \in \mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_h^i} |e|^{-1} \left\| \left[\left[\partial v / \partial n \right] \right] \right\|_{L_2(e)}^2 \right) \quad \forall v \in V_h. \quad (2.11)$$

The discrete bilinear form that approximates $\mathcal{A}(\cdot, \cdot)$ is then given by

$$\mathcal{A}_h(v, w) = \beta a_h(v, w) + \gamma(v, w), \tag{2.12}$$

and

$$\|v\|_h = \left[\beta \left(\sum_{T \in \mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_h^i} |e|^{-1} \|[\![\partial v / \partial n]\!]\|_{L_2(e)}^2 \right) + \gamma \|v\|_{L_2(\Omega)}^2 \right]^{\frac{1}{2}} \tag{2.13}$$

is the mesh-dependent energy norm. It follows from (2.10)–(2.13) that

$$|\mathcal{A}_h(v, w)| \lesssim \|v\|_h \|w\|_h \quad \forall v, w \in V_h, \tag{2.14}$$

$$\mathcal{A}_h(v, v) \gtrsim \|v\|_h^2 \quad \forall v \in V_h, \tag{2.15}$$

provided that σ is sufficiently large, which we assume to be the case.

Note that

$$\|v\|_{H^1(\Omega)}^2 \lesssim \sum_{T \in \mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_h^i} |e|^{-1} \|[\![\partial v / \partial n]\!]\|_{L_2(e)}^2 \tag{2.16}$$

for all $v \in H^2(\Omega, \mathcal{T}_h) \cap H_0^1(\Omega)$ by a Poincaré–Friedrichs inequality [28, Example 5.4], and hence

$$\|v\|_{H^1(\Omega)} \lesssim \|v\|_h \tag{2.17}$$

for all $v \in H^2(\Omega, \mathcal{T}_h) \cap H_0^1(\Omega) (\supset V_h + H^2(\Omega) \cap H_0^1(\Omega))$.

We can now define the discrete obstacle problem for (1.7):

$$\text{Find } \bar{y}_h = \operatorname{argmin}_{y_h \in K_h} \left[\frac{1}{2} \mathcal{A}_h(y_h, y_h) - (f, y_h) \right], \tag{2.18}$$

where

$$K_h = \{v \in V_h : \psi_1(p) \leq v(p) \leq \psi_2(p) \quad \forall p \in \mathcal{V}_h\}. \tag{2.19}$$

Let Π_h be the nodal interpolation operator for the \mathbb{P}_2 Lagrange finite element space. Then Π_h maps $H^2(\Omega) \cap H_0^1(\Omega)$ into V_h and K into K_h . Therefore K_h is a nonempty closed convex subset of V_h . Moreover the bilinear form $\mathcal{A}_h(\cdot, \cdot)$ is symmetric positive definite by (2.15). Hence the discrete problem (2.18) has a unique solution $\bar{y}_h \in K_h$ characterized by the discrete variational inequality:

$$\mathcal{A}_h(\bar{y}_h, y_h - \bar{y}_h) \geq (f, y_h - \bar{y}_h) \quad \forall y_h \in K_h. \tag{2.20}$$

Let Π_T be the nodal interpolation operator for the \mathbb{P}_2 Lagrange finite element on a triangle T . We have a standard local interpolation error estimate [20, 35]

$$\sum_{m=0}^2 h_T^{m-2} |\zeta - \Pi_T \zeta|_{H^m(T)} \lesssim h_T^s |\zeta|_{H^{2+s}(T)} \quad (2.21)$$

for all $\zeta \in H^{2+s}(T)$, $T \in \mathcal{T}_h$ and $s \in [0, 1]$.

The following lemma provides global interpolation error estimates for the solution \bar{y} of (1.7)/(1.9).

LEMMA 2.1. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \Pi_h \bar{y}\|_h \leq Ch^\tau,$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

Proof. Since the estimate for a quasi-uniform \mathcal{T}_h is standard (cf. [21]), we will focus on a graded \mathcal{T}_h . Let \mathcal{T}_h^I be the set of triangles in \mathcal{T}_h that do not touch any corner of Ω and $\mathcal{T}_h^C = \mathcal{T}_h \setminus \mathcal{T}_h^I = \cup_{1 \leq \ell \leq L} \mathcal{T}_{h,\ell}^C$, where $\mathcal{T}_{h,\ell}^C$ is the set of the triangles that touch the corner p_ℓ .

Since $\bar{y}_T \in H^3(T)$ for $T \in \mathcal{T}_h^I$ (cf. Appendix A), we have, by (2.21),

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h^I} \sum_{m=0}^2 h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2 \\ & \lesssim \sum_{T \in \mathcal{T}_h^I} (\Phi^{-2}(c_T) h_T^2) \Phi^2(c_T) |\bar{y}|_{H^3(T)}^2 \end{aligned} \quad (2.22)$$

where the function Φ is defined in (2.3).

It follows from (2.2), (2.3), (2.22), and (A.7) that

$$\sum_{T \in \mathcal{T}_h^I} \sum_{m=0}^2 h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2 \lesssim h^2. \quad (2.23)$$

Let $T \in \mathcal{T}_{h,\ell}^C$ be a triangle that touches a corner p_ℓ . Then $\bar{y} \in H^{2+\alpha_\ell}(T)$ (cf. Appendix A) and we have, by (2.21),

$$\sum_{m=0}^2 h_T^{m-2} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)} \lesssim h_T^{\alpha_\ell} |\bar{y}|_{H^{2+\alpha_\ell}(T)} \quad \forall T \in \mathcal{T}_{h,\ell}^C. \quad (2.24)$$

It follows from (2.5) and (2.24) that

$$\sum_{T \in \mathcal{T}_h^C} \sum_{m=0}^2 h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2 \quad (2.25)$$

$$= \sum_{\ell=1}^L \sum_{T \in \mathcal{T}_{h,\ell}^C} \sum_{m=0}^2 h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2 \lesssim h^2.$$

Combining (2.23) and (2.25), we find

$$\sum_{T \in \mathcal{T}_h} \sum_{m=0}^2 h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2 \lesssim h^2. \tag{2.26}$$

By the trace theorem with scaling, (2.6b) and (2.26), we also have

$$\begin{aligned} & \sum_{e \in \mathcal{E}_h^i} |e|^{-1} \|[\partial(\bar{y} - \Pi_h \bar{y})/\partial n]\|_{L_2(e)}^2 \\ & \lesssim \sum_{T \in \mathcal{T}_h} \left(h_T^{-2} |\bar{y} - \Pi_h \bar{y}|_{H^1(T)}^2 + |\bar{y} - \Pi_h \bar{y}|_{H^2(T)}^2 \right) \lesssim h^2. \end{aligned} \tag{2.27}$$

The lemma for a graded \mathcal{T}_h follows from (2.13), (2.16), (2.26), and (2.27). \square

3. An Intermediate Obstacle Problem. As mentioned in Sect. 1, the difficulties due to the lack of $H_{\text{loc}}^4(\Omega)$ regularity can be bypassed if the convergence analysis does not rely on the complementarity form of the variational inequality (1.9). We can accomplish this by introducing the following intermediate obstacle problem:

$$\text{Find } \bar{y}_h^* = \operatorname{argmin}_{y_h^* \in K_h^*} \left[\frac{1}{2} \mathcal{A}(y_h^*, y_h^*) - (f, y_h^*) \right], \tag{3.1}$$

where

$$K_h^* = \{v \in H^2(\Omega) \cap H_0^1(\Omega) : \psi_1(p) \leq v(p) \leq \psi_2(p) \quad \forall p \in \mathcal{V}_h\}. \tag{3.2}$$

By the standard theory (3.1) has a unique solution \bar{y}_h^* characterized by the variational inequality

$$\mathcal{A}(\bar{y}_h^*, y_h^* - \bar{y}_h^*) \geq (f, y_h^* - \bar{y}_h^*) \quad \forall y_h^* \in K_h^*. \tag{3.3}$$

Note that, on the one hand, $\bar{y}_h^* \in H^2(\Omega) \cap H_0^1(\Omega)$ minimizes the same functional as \bar{y} but on the larger set $K_h^* \supset K$, and, on the other hand, \bar{y}_h^* shares the same pointwise constraints as \bar{y}_h . Thus the intermediate obstacle problem connects the continuous obstacle problem (1.7) and the discrete obstacle problem (2.18). We will carry out the convergence analysis using (1.9), (2.20), and (3.3), but not their complementarity forms.

3.1. Relation Between \bar{y} and \bar{y}_h^* . Using the fact that $H^2(\Omega)$ is compactly embedded in $C(\bar{\Omega})$, it was shown in [25] that there exist two nonnegative functions $\phi_1, \phi_2 \in C_0^\infty(\Omega)$ and a positive number h_0 such that for any $h \leq h_0$ we can find two positive numbers $\delta_{h,1}$ and $\delta_{h,2}$ with the following properties:

$$\hat{y}_h := \bar{y}_h^* + \delta_{h,1}\phi_1 - \delta_{h,2}\phi_2 \in K \quad \text{and} \quad \delta_{h,i} \lesssim h^2. \quad (3.4)$$

Note that we can treat \bar{y} as an internal approximation of \bar{y}_h^* since $K \subset K_h^*$. It then follows from (3.4) and a standard result [3] that

$$\|\bar{y}_h^* - \bar{y}\|_{H^2(\Omega)} \lesssim \left[\inf_{y \in K} \|\bar{y}_h^* - y\|_{H^2(\Omega)} \right]^{\frac{1}{2}} \lesssim \|\bar{y}_h^* - \hat{y}_h\|_{H^2(\Omega)}^{\frac{1}{2}} \lesssim h. \quad (3.5)$$

REMARK 3.1. *Even though the results in [25] are obtained for clamped Kirchhoff plates on convex polygonal domains, these results are also valid for general boundary conditions and general polygonal domains because they are interior results that only require the following ingredients: (i) The set K_h^* is a closed convex subset of $H^2(\Omega)$. (ii) The constraints and the boundary conditions are separated. (iii) The obstacle functions ψ_1, ψ_2 and the solution \bar{y} belong to $C^2(\Omega)$.*

3.2. Connection Between K_h and K_h^* . We can connect K_h and K_h^* by an enriching operator E_h that maps V_h into $H^2(\Omega) \cap H_0^1(\Omega)$. By construction E_h is a linear operator that preserves the nodal values at the vertices of \mathcal{T}_h , i.e.,

$$(E_h v)(p) = v(p) \quad \forall p \in \mathcal{V}_h, \quad v \in V_h, \quad (3.6)$$

which, in view of (2.19) and (3.2), implies

$$E_h K_h \subset K_h^*. \quad (3.7)$$

Moreover we have (cf. the notation in Sect. 2.1),

$$\begin{aligned} & \sum_{m=0}^2 h_T^{2m} |v - E_h v|_{H^m(T)}^2 \\ & \lesssim h_T^4 \left(\sum_{T' \in \mathcal{T}_T} |v|_{H^2(T')}^2 + \sum_{e \in \mathcal{E}_T^i} |e|^{-1} \|[\![\partial v / \partial n]\!\!] \|_{L_2(e)}^2 \right) \end{aligned} \quad (3.8)$$

for any $v \in V_h$ and $T \in \mathcal{T}_h$, and

$$\sum_{m=0}^2 h_T^{m-2} |\zeta - E_h \Pi_h \zeta|_{H^m(T)} \lesssim h_T^s |\zeta|_{H^{2+s}(S_T)} \quad (3.9)$$

for all $\zeta \in H^{2+s}(\mathcal{S}_T)$, $T \in \mathcal{T}_h$ and $s \in [0, 1]$. The construction of E_h , which is similar to the constructions of the enriching operators in [16, 21] under different boundary conditions, is given in Appendix B, where we also derive the estimates (3.8) and (3.9).

The estimate (3.8) implies

$$\sum_{T \in \mathcal{T}_h} h_T^{2(m-2)} |v - E_h v|_{H^m(T)}^2 \lesssim \|v\|_h^2 \quad \forall v \in V_h, \quad (3.10)$$

and in particular,

$$|E_h v|_{H^2(\Omega)} \lesssim \|v\|_h \quad \forall v \in V_h. \quad (3.11)$$

Combining (2.7a), (2.8a), (3.10) and the trace theorem with scaling, we also have

$$\sum_{e \in \mathcal{E}_h} |e|^{-1} \|\{\{\partial(v - E_h v)/\partial n_e\}\}\|_{L_2(e)}^2 \lesssim \|v\|_h^2 \quad \forall v \in V_h. \quad (3.12)$$

Finally the quasi-local estimate (3.9) implies the following result for the solution \bar{y} of (1.7). We omit the proof due to its similarity with the proof of Lemma 2.1.

LEMMA 3.1. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - E_h \Pi_h \bar{y}\|_{L_2(\Omega)} + h \|\bar{y} - E_h \Pi_h \bar{y}\|_{H^1(\Omega)} + h^2 \|\bar{y} - E_h \Pi_h \bar{y}\|_{H^2(\Omega)} \leq Ch^{2+\tau},$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

4. Preliminary Estimates. In this section we derive some preliminary estimates that are useful for the convergence analysis in Sect. 5. We begin by stating the following integration by parts formula that holds for $v, w \in V_h$:

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2(w - E_h w) dx \\ &= \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left[\left(\frac{\partial^2 v}{\partial n^2} \right) \left(\frac{\partial(w - E_h w)}{\partial n} \right) \right. \\ & \quad \left. + \left(\frac{\partial^2 v}{\partial n \partial t} \right) \left(\frac{\partial(w - E_h w)}{\partial t} \right) \right] ds \\ &= - \sum_{e \in \mathcal{E}_h} \int_e \left[\left[\frac{\partial^2 v}{\partial n_e^2} \right] \left\{ \left\{ \frac{\partial(w - E_h w)}{\partial n_e} \right\} \right\} \right] ds \\ & \quad - \sum_{e \in \mathcal{E}_h^i} \int_e \left\{ \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \right\} \left[\left[\frac{\partial(w - E_h w)}{\partial n} \right] \right] ds. \end{aligned} \quad (4.1)$$

Note that $v \in \mathbb{P}_2(T)$ and hence on any edge e of $T \in \mathcal{T}_h$ we have

$$\int_e \left(\frac{\partial^2 v_T}{\partial n \partial t} \right) \left(\frac{\partial(w_T - E_h w)}{\partial t} \right) ds = \left(\frac{\partial^2 v_T}{\partial n \partial t} \right) \int_e \frac{\partial(w_T - E_h w)}{\partial t} ds = 0$$

because of (3.6).

Next we derive a basic estimate for $\bar{y} - \bar{y}_h$, where \bar{y} (resp. \bar{y}_h) is the solution of (1.7)/(1.9) [resp. (2.18)/(2.20)].

LEMMA 4.1. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_h^2 \leq 2\|\bar{y} - \Pi_h \bar{y}\|_h^2 + C[\mathcal{A}_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (f, \Pi_h \bar{y} - \bar{y}_h)]. \quad (4.2)$$

Proof. Since $\Pi_h \bar{y} \in K_h$, we deduce from (2.15) and (2.20) that

$$\begin{aligned} \|\bar{y} - \bar{y}_h\|_h^2 &\leq 2\|\bar{y} - \Pi_h \bar{y}\|_h^2 + 2\|\Pi_h \bar{y} - \bar{y}_h\|_h^2 \\ &\leq 2\|\bar{y} - \Pi_h \bar{y}\|_h^2 + C\mathcal{A}_h(\Pi_h \bar{y} - \bar{y}_h, \Pi_h \bar{y} - \bar{y}_h) \\ &\leq 2\|\bar{y} - \Pi_h \bar{y}\|_h^2 + C[\mathcal{A}_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (f, \Pi_h \bar{y} - \bar{y}_h)]. \end{aligned}$$

□

In view of Lemmas 2.1 and 4.1, we can complete the error analysis by bounding the second term on the right-hand side of (4.2). This will be carried out in Sect. 5 after we have developed several technical lemmas in the remaining part of this section.

LEMMA 4.2. *There exists a positive constant C independent of h such that*

$$\sum_{e \in \mathcal{E}_h} |e| \|\llbracket \partial^2(\Pi_h \bar{y}) / \partial n_e^2 \rrbracket\|_{L_2(e)}^2 \leq Ch^{2\tau},$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

Proof. We will split the estimate into two cases. Let $\mathcal{E}_h^R = \{e \in \mathcal{E}_h : e \text{ is not an edge of any triangle that touches a corner of } \Omega \text{ where the angle is strictly greater than } \pi/2\}$ and $\mathcal{E}_h^S = \mathcal{E}_h \setminus \mathcal{E}_h^R$. Note that the number of edges in \mathcal{E}_h^S is bounded by a constant determined by the minimum angle of \mathcal{T}_h .

Since away from the corners of Ω where the angles are strictly greater than $\pi/2$ the function \bar{y} belongs to H^3 and $\partial^2 \bar{y} / \partial n^2 = \Delta \bar{y}$ vanishes on $\partial \Omega$ (cf. Appendix A), we have, by (2.7b), (2.21) and the trace theorem with scaling,

$$\begin{aligned} \sum_{e \in \mathcal{E}_h^R} |e| \|\llbracket \partial^2(\Pi_h \bar{y}) / \partial n_e^2 \rrbracket\|_{L_2(e)}^2 &= \sum_{e \in \mathcal{E}_h^R} |e| \|\llbracket \partial^2(\Pi_h \bar{y} - \bar{y}) / \partial n_e^2 \rrbracket\|_{L_2(e)}^2 \\ &\lesssim \sum_{e \in \mathcal{E}_h^R} \sum_{T \in \mathcal{T}_e} (h_T^2 \Phi^{-2}(c_T)) \Phi^2(c_T) |\bar{y}|_{H^3(T)}^2, \end{aligned}$$

where the function Φ is defined in (2.3). It then follows from (2.1)–(2.3) and (A.7) that

$$\sum_{e \in \mathcal{E}_h^R} |e| \left\| \left[\partial^2(\Pi_h \bar{y}) / \partial n_e^2 \right] \right\|_{L_2(e)}^2 \lesssim h^{2\tau}, \tag{4.3}$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h satisfies (2.2)–(2.4).

Let $e \in \mathcal{E}_h^S$ be an edge of a triangle that touches a corner p_ℓ of Ω where the angle $\omega_\ell \in (\pi/2, \pi)$. It follows from scaling that

$$\begin{aligned} |e| \left\| \left[\partial^2(\Pi_h \bar{y}) / \partial n_e^2 \right] \right\|_{L_2(e)}^2 &\lesssim \sum_{T \in \mathcal{T}_e} |\Pi_h \bar{y}|_{H^2(T)}^2 \\ &\lesssim \sum_{T \in \mathcal{T}_e} \left(|\Pi_h \bar{y} - \bar{y}|_{H^2(T)}^2 + |\bar{y}|_{H^2(T)}^2 \right). \end{aligned}$$

Let $T \in \mathcal{T}_e$. Since $\bar{y} \in H^{2+\alpha_\ell}(T)$ (cf. Appendix A), we have

$$|\Pi_h \bar{y} - \bar{y}|_{H^2(T)} \lesssim |\bar{y}|_{H^{2+\alpha_\ell}(T)} h_T^{\alpha_\ell}.$$

Moreover we have

$$\begin{aligned} |\bar{y}|_{H^2(T)} &\approx \sum_{|\mu|=2} \|\partial^\mu \bar{y}\|_{L_2(T)} \\ &= \sum_{|\mu|=2} \|\Psi^{-1}(\Psi(\partial^\mu \bar{y}))\|_{L_2(T)} \lesssim h_T^{\alpha_\ell} \sum_{|\mu|=2} \|\Psi(\partial^\mu \bar{y})\|_{L_2(T)}, \end{aligned}$$

where Ψ is define in (A.9). Therefore it follows from (2.1), (2.5), Remark 2.1 and (A.8) that

$$\sum_{e \in \mathcal{E}_h^S} |e| \left\| \left[\partial^2(\Pi_h \bar{y}) / \partial n_e^2 \right] \right\|_{L_2(e)}^2 \lesssim h^{2\tau}, \tag{4.4}$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h satisfies (2.2)–(2.4).

The lemma follows from (4.3) and (4.4). \square

LEMMA 4.3. *There exists a positive constant C independent of h such that*

$$\begin{aligned} &\left| a_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - \int_\Omega D^2 \bar{y} : D^2 E_h(\Pi_h \bar{y} - \bar{y}_h) dx \right| \\ &\leq Ch^\tau \|\Pi_h \bar{y} - \bar{y}_h\|_h, \end{aligned} \tag{4.5}$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

Proof. Since both $[\partial E_h(\Pi_h \bar{y} - \bar{y}_h) / \partial n]$ and $[\partial \bar{y} / \partial n]$ equal 0, we have, from (2.9),

$$\begin{aligned}
& a_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) \\
&= \sum_{T \in \mathcal{T}_h} \int_T D^2 \bar{y} : D^2 E_h(\Pi_h \bar{y} - \bar{y}_h) dx \\
&+ \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h \bar{y} - \bar{y}) : D^2 E_h(\Pi_h \bar{y} - \bar{y}_h) dx \\
&+ \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h \bar{y}) : D^2 [(\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)] dx \\
&+ \sum_{e \in \mathcal{E}_h^i} \int_e \left\{ \frac{\partial^2(\Pi_h \bar{y})}{\partial n^2} \right\} \left[\frac{\partial [(\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)]}{\partial n} \right] ds \quad (4.6) \\
&+ \sum_{e \in \mathcal{E}_h^i} \int_e \left\{ \frac{\partial^2(\Pi_h \bar{y} - \bar{y}_h)}{\partial n^2} \right\} \left[\frac{\partial(\Pi_h \bar{y} - \bar{y})}{\partial n} \right] ds \\
&+ \sum_{e \in \mathcal{E}_h^i} \frac{\sigma}{|e|} \int_e \left[\frac{\partial(\Pi_h \bar{y} - \bar{y})}{\partial n} \right] \left[\frac{\partial(\Pi_h \bar{y} - \bar{y}_h)}{\partial n} \right] ds,
\end{aligned}$$

and we can use (2.6), (2.13), Lemma 2.1, (3.11) and scaling to estimate the second, fifth, and sixth terms on the right-hand side of (4.6) as follows:

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h \bar{y} - \bar{y}) : D^2 E_h(\Pi_h \bar{y} - \bar{y}_h) dx \right| \\
& \leq \left(\sum_{T \in \mathcal{T}_h} |\Pi_h \bar{y} - \bar{y}|_{H^2(T)}^2 \right)^{\frac{1}{2}} |E_h(\Pi_h \bar{y} - \bar{y}_h)|_{H^2(\Omega)} \quad (4.7) \\
& \lesssim h^\tau \|\Pi_h \bar{y} - \bar{y}_h\|_h,
\end{aligned}$$

$$\begin{aligned}
& \left| \sum_{e \in \mathcal{E}_h^i} \int_e \left\{ \frac{\partial^2(\Pi_h \bar{y} - \bar{y}_h)}{\partial n^2} \right\} \left[\frac{\partial(\Pi_h \bar{y} - \bar{y})}{\partial n} \right] ds \right| \\
& \leq \left(\sum_{e \in \mathcal{E}_h^i} |e| \left\| \left\{ \frac{\partial^2(\Pi_h \bar{y} - \bar{y}_h)}{\partial n^2} \right\} \right\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
& \quad \times \left(\sum_{e \in \mathcal{E}_h^i} \frac{1}{|e|} \left\| \left[\frac{\partial(\Pi_h \bar{y} - \bar{y})}{\partial n} \right] \right\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \quad (4.8) \\
& \lesssim \left(\sum_{T \in \mathcal{T}_h} |\Pi_h \bar{y} - \bar{y}_h|_{H^2(T)}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_h^i} \frac{1}{|e|} \left\| \left[\frac{\partial(\Pi_h \bar{y} - \bar{y})}{\partial n} \right] \right\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
& \lesssim h^\tau \|\Pi_h \bar{y} - \bar{y}_h\|_h,
\end{aligned}$$

$$\begin{aligned}
 & \left| \sum_{e \in \mathcal{E}_h^i} \frac{\sigma}{|e|} \int_e \left[\frac{\partial(\Pi_h \bar{y} - \bar{y})}{\partial n} \right] \left[\frac{\partial(\Pi_h \bar{y} - \bar{y}_h)}{\partial n} \right] ds \right| \\
 & \lesssim \left(\sum_{e \in \mathcal{E}_h^i} \frac{1}{|e|} \| [\partial(\Pi_h \bar{y} - \bar{y}) / \partial n] \|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
 & \quad \times \left(\sum_{e \in \mathcal{E}_h^i} \frac{1}{|e|} \| [\partial(\Pi_h \bar{y} - \bar{y}_h) / \partial n] \|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
 & \lesssim h^\tau \| \Pi_h \bar{y} - \bar{y}_h \|_h.
 \end{aligned} \tag{4.9}$$

Now we use (3.12), the integration by parts formula (4.1) together with Lemma 4.2 to estimate the sum of the third and fourth terms on the right-hand side of (4.6) by

$$\begin{aligned}
 & \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h \bar{y}) : D^2[(\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)] dx \\
 & \quad + \sum_{e \in \mathcal{E}_h^i} \int_e \left\{ \frac{\partial^2(\Pi_h \bar{y})}{\partial n^2} \right\} \left[\frac{\partial[(\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)]}{\partial n} \right] ds \\
 & = - \sum_{e \in \mathcal{E}_h} \int_e \left[\frac{\partial^2(\Pi_h \bar{y})}{\partial n_e^2} \right] \left\{ \frac{\partial[(\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)]}{\partial n_e} \right\} ds \tag{4.10} \\
 & \leq \left(\sum_{e \in \mathcal{E}_h} |e| \| [\partial^2(\Pi_h \bar{y}) / \partial n_e^2] \|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
 & \quad \times \left(\sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \| \{ \partial[(\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)] / \partial n_e \} \|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
 & \lesssim h^\tau \| \Pi_h \bar{y} - \bar{y}_h \|_h.
 \end{aligned}$$

The lemma follows from (4.6)–(4.10). \square

LEMMA 4.4. *There exists a positive constant C independent of h such that*

$$\mathcal{A}(\bar{y}, E_h \Pi_h \bar{y} - \bar{y}) \leq Ch^{1+\tau},$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

Proof. Since $\Delta \bar{y} \in H_0^1(\Omega)$ (cf. Appendix A), we have, by (1.5) and Lemma 3.1,

$$\begin{aligned}
\int_{\Omega} D^2 \bar{y} : D^2 (E_h \Pi_h \bar{y} - \bar{y}) \, dx &= \int_{\Omega} (\Delta \bar{y}) (\Delta (E_h \Pi_h \bar{y} - \bar{y})) \, dx \\
&= - \int_{\Omega} \nabla (\Delta \bar{y}) \cdot \nabla (E_h \Pi_h \bar{y} - \bar{y}) \, dx \quad (4.11) \\
&\lesssim |E_h \Pi_h \bar{y} - \bar{y}|_{H^1(\Omega)} \lesssim h^{1+\tau}.
\end{aligned}$$

Moreover Lemma 3.1 also implies

$$(\bar{y}, E_h \Pi_h \bar{y} - \bar{y}) \lesssim h^{2+\tau}. \quad (4.12)$$

The lemma follows from (1.8), (4.11), and (4.12). \square

5. Convergence Analysis. In this section we complete the error analysis by finding a bound for the second term on the right-hand side of (4.2). We will show that

$$\mathcal{A}_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (f, \Pi_h \bar{y} - \bar{y}_h) \lesssim h^{2\tau} + h^\tau \|\Pi_h \bar{y} - \bar{y}_h\|_h, \quad (5.1)$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h satisfies (2.2)–(2.4). But first we use (5.1) to establish the main result of this paper.

THEOREM 5.1. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_h \leq Ch^\tau, \quad (5.2)$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

Proof. It follows from Lemma 2.1, (4.2), (5.1), and the inequality of arithmetic and geometric means that

$$\begin{aligned}
\|\bar{y} - \bar{y}_h\|_h^2 &\leq C(h^{2\tau} + h^\tau \|\Pi_h \bar{y} - \bar{y}_h\|_h) \\
&\leq C(h^{2\tau} + h^\tau \|\bar{y} - \bar{y}_h\|_h) \leq Ch^{2\tau} + \frac{1}{2} \|\bar{y} - \bar{y}_h\|_h^2,
\end{aligned}$$

which implies (5.2). \square

The following lemma reduces the derivation of (5.1) to an estimate at the continuous level.

LEMMA 5.1. *There exists a positive constant C independent of h such that*

$$\begin{aligned}
&\mathcal{A}_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (f, \Pi_h \bar{y} - \bar{y}_h) \\
&\leq Ch^\tau \|\Pi_h \bar{y} - \bar{y}_h\|_h + \mathcal{A}(\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) - (f, E_h(\Pi_h \bar{y} - \bar{y}_h)),
\end{aligned}$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

Proof. From (1.8) and (2.12) we have

$$\begin{aligned}
 & \mathcal{A}_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (f, \Pi_h \bar{y} - \bar{y}_h) \\
 &= \beta \left[a_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - \int_{\Omega} (D^2 \bar{y} : D^2 E_h(\Pi_h \bar{y} - \bar{y}_h)) dx \right] \\
 & \quad + \gamma [(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h))] \\
 & \quad - (f, (\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)) \\
 & \quad + \mathcal{A}(\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) - (f, E_h(\Pi_h \bar{y} - \bar{y}_h)),
 \end{aligned} \tag{5.3}$$

and we can bound the second and third terms on the right-hand side of (5.3) as follows:

$$\begin{aligned}
 & (\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) \\
 &= (\Pi_h \bar{y} - \bar{y}, \Pi_h \bar{y} - \bar{y}_h) + (\bar{y}, (\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)) \\
 &\lesssim h^2 \|\Pi_h \bar{y} - \bar{y}_h\|_h
 \end{aligned} \tag{5.4}$$

by (2.21) and (3.10); and

$$|(f, (\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h))| \lesssim h^2 \|\Pi_h \bar{y} - \bar{y}_h\|_h \tag{5.5}$$

by (3.10).

The lemma follows from Lemma 4.3 and (5.3)–(5.5). \square

In view of Lemma 5.1, it only remains to show that

$$\mathcal{A}(\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) - (f, E_h(\Pi_h \bar{y} - \bar{y}_h)) \lesssim h^{2\tau} + h^\tau \|\Pi_h \bar{y} - \bar{y}_h\|_h. \tag{5.6}$$

We will use the relation $A \leq B$ to streamline the derivation of (5.6), where

$$A \leq B \quad \text{means that} \quad A - B \lesssim h^{2\tau} + h^\tau \|\Pi_h \bar{y} - \bar{y}_h\|_h.$$

The estimate (5.6) can then be written as

$$\mathcal{A}(\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) \leq (f, E_h(\Pi_h \bar{y} - \bar{y}_h)). \tag{5.7}$$

It follows from (1.9), (3.3), (3.4), (3.7), and Lemma 4.4 that

$$\begin{aligned}
 & \mathcal{A}(\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) = \mathcal{A}(\bar{y}, E_h \Pi_h \bar{y} - \bar{y}) + \mathcal{A}(\bar{y}, \bar{y} - E_h \bar{y}_h) \\
 & \leq \mathcal{A}(\bar{y}, \bar{y} - E_h \bar{y}_h) \\
 & = \mathcal{A}(\bar{y}, \bar{y} - \hat{y}_h) + \mathcal{A}(\bar{y}, \hat{y}_h - E_h \bar{y}_h) \\
 & \leq (f, \bar{y} - \hat{y}_h) + \mathcal{A}(\bar{y}, \bar{y}_h^* - E_h \bar{y}_h) + \mathcal{A}(\bar{y}, \delta_{h,1} \phi_1 - \delta_{h,2} \phi_2) \\
 & \leq (f, \bar{y} - \hat{y}_h) + \mathcal{A}(\bar{y}, \bar{y}_h^* - E_h \bar{y}_h) \\
 & = (f, \bar{y} - \hat{y}_h) + \mathcal{A}(\bar{y}_h^*, \bar{y}_h^* - E_h \bar{y}_h) + \mathcal{A}(\bar{y} - \bar{y}_h^*, \bar{y}_h^* - E_h \bar{y}_h) \\
 & \leq (f, \bar{y} - \hat{y}_h) + (f, \bar{y}_h^* - E_h \bar{y}_h) + \mathcal{A}(\bar{y} - \bar{y}_h^*, \bar{y}_h^* - E_h \bar{y}_h) \\
 & \leq (f, \bar{y} - E_h \bar{y}_h) + \mathcal{A}(\bar{y} - \bar{y}_h^*, \bar{y}_h^* - E_h \bar{y}_h).
 \end{aligned} \tag{5.8}$$

Moreover we have

$$\begin{aligned} (f, \bar{y} - E_h \bar{y}_h) &= (f, E_h(\Pi_h \bar{y} - \bar{y}_h)) + (f, \bar{y} - E_h \Pi_h \bar{y}) \\ &\leq (f, E_h(\Pi_h \bar{y} - \bar{y}_h)) \end{aligned} \quad (5.9)$$

by Lemma 3.1, and

$$\begin{aligned} &\mathcal{A}(\bar{y} - \bar{y}_h^*, \bar{y}_h^* - E_h \bar{y}_h) \\ &= \mathcal{A}(\bar{y} - \bar{y}_h^*, \bar{y}_h^* - \bar{y}) + \mathcal{A}(\bar{y} - \bar{y}_h^*, \bar{y} - E_h \bar{y}_h) \\ &\leq \mathcal{A}(\bar{y} - \bar{y}_h^*, \bar{y} - E_h \Pi_h \bar{y}) + \mathcal{A}(\bar{y} - \bar{y}_h^*, E_h(\Pi_h \bar{y} - \bar{y}_h)) \\ &\leq 0 \end{aligned} \quad (5.10)$$

by (3.5), (3.11), and Lemma 3.1. The relation (5.7) then follows from (5.8)–(5.10). Therefore we have established (5.6) and hence (5.1).

The following corollary is an immediate consequence of (2.17) and Theorem 5.1.

COROLLARY 5.1. *There exists a positive constant C independent of h such that*

$$|\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq Ch^\tau,$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

Since the energy norm $\|\cdot\|_h$ is an H^2 -like norm, we can also deduce an L_∞ norm error estimate from Theorem 5.1. The proof of the following theorem, which is based on Lemmas 2.1, 3.1, Theorem 5.1, standard inverse estimates and the Sobolev inequality, is identical to the proof of Theorem 4.1 in [23] and thus omitted.

THEOREM 5.2. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq Ch^\tau,$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

REMARK 5.1. *Since the norms $\|\cdot\|_{L_\infty(\Omega)}$ and $|\cdot|_{H^1(\Omega)}$ are weaker than the energy norm $\|\cdot\|_h$, the order of convergence in these norms should be higher than the order of convergence in $\|\cdot\|_h$. This is confirmed by the numerical results in Sect. 6. Therefore the estimates for $\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)}$ and $|\bar{y} - \bar{y}_h|_{H^1(\Omega)}$ in Corollary 5.1 and Theorem 5.2 are not sharp.*

For the optimal control problem defined by (1.1)–(1.3), we can take the approximation for the optimal control \bar{u} to be the function $\bar{u}_h \in V_h$ defined by

$$\int_{\Omega} \nabla \bar{y}_h \cdot \nabla v \, dx = \int_{\Omega} \bar{u}_h v \, dx \quad \forall v \in V_h. \quad (5.11)$$

THEOREM 5.3. *There exists a positive constant C independent of h such that*

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} \leq Ch^\tau, \tag{5.12}$$

where $\tau = \alpha$ if \mathcal{T}_h is quasi-uniform and $\tau = 1$ if \mathcal{T}_h is graded according to (2.2)–(2.4).

Proof. Let $Q_h : L_2(\Omega) \rightarrow V_h$ be the orthogonal projection. From (1.2) we have

$$\int_{\Omega} \nabla \bar{y} \cdot \nabla v \, dx = \int_{\Omega} \bar{u} v \, dx = \int_{\Omega} (Q_h \bar{u}) v \, dx \quad \forall v \in V_h. \tag{5.13}$$

Let $v \in V_h$ be arbitrary. Using integration by parts, the Cauchy–Schwarz inequality, scaling, (2.7b), (2.13), and Theorem 5.1, we find

$$\begin{aligned} & \int_{\Omega} \nabla(\bar{y} - \bar{y}_h) \cdot \nabla v \, dx \\ &= - \sum_{e \in \mathcal{E}_h^i} \int_e \llbracket \partial(\bar{y} - \bar{y}_h) / \partial n \rrbracket v \, ds - \sum_{T \in \mathcal{T}_h} \int_T [\Delta(\bar{y} - \bar{y}_h)] v \, dx \\ &\leq \left(\sum_{e \in \mathcal{E}_h^i} |e|^{-1} \|\llbracket \partial(\bar{y} - \bar{y}_h) / \partial n \rrbracket\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_h^i} |e| \|v\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\ &\quad + \left(\sum_{T \in \mathcal{T}_h} |\bar{y} - \bar{y}_h|_{H^2(T)}^2 \right)^{\frac{1}{2}} \|v\|_{L_2(\Omega)} \\ &\lesssim \|\bar{y} - \bar{y}_h\|_h \|v\|_{L_2(\Omega)} \lesssim h^\tau \|v\|_{L_2(\Omega)}. \end{aligned}$$

It then follows from (5.11), (5.13) and duality that

$$\begin{aligned} & \|Q_h \bar{u} - \bar{u}_h\|_{L_2(\Omega)} \\ &= \sup_{v \in V_h \setminus \{0\}} \left(\int_{\Omega} (Q_h \bar{u} - \bar{u}_h) v \, dx \right) / \|v\|_{L_2(\Omega)} \\ &= \sup_{v \in V_h \setminus \{0\}} \left(\int_{\Omega} \nabla(\bar{y} - \bar{y}_h) \cdot \nabla v \, dx \right) / \|v\|_{L_2(\Omega)} \lesssim h^\tau. \end{aligned} \tag{5.14}$$

Furthermore, we have, by a standard interpolation error estimate [61],

$$\|Q_h \bar{u} - \bar{u}\|_{L_2(\Omega)} \lesssim |\bar{u}|_{H^1(\Omega)} h. \tag{5.15}$$

The estimate (5.12) follows from (5.14) and (5.15). \square

REMARK 5.2. One can also take the piecewise constant function $\bar{u}_h = -\Delta_h \bar{y}_h$ to be an approximation of the optimal control \bar{u} , where Δ_h is the piecewise Laplacian with respect to \mathcal{T}_h . The estimate (5.12) then immediately follows from (2.13) and Theorem 5.1. But numerical results indicate that the approximation of \bar{u} defined by (5.11) is a better choice.

REMARK 5.3. By tracing the constants in all the estimates (including (3.4)) one can show (using (A.7), (A.8), and (A.10)) that the constant C in Theorem 5.1, Corollary 5.1, Theorem 5.2, and Theorem 5.3 is of the form

$$\mathfrak{C} \left(\left(\|f\|_{L_2(\Omega)} + \sum_{i=1}^2 \|\psi_i\|_{W_\infty^2(K)} + |\Delta \bar{y}|_{H^1(\Omega)} + \sum_{|\mu|=3} \|\Phi(\partial^\mu \bar{y})\|_{L_2(\Omega)} + \sum_{|\mu|=2} \|\Psi(\partial^\mu \bar{y})\|_{L_2(\Omega)} + \|\bar{y}\|_{W_\infty^2(K)} \right) \right),$$

where Φ (resp. Ψ) is defined in (2.3) (resp. (A.9)), $K \subset\subset \Omega$ is a compact neighborhood of the contact set where $(\bar{y} - \psi_1)(\bar{y} - \psi_2) = 0$, and the positive constant \mathfrak{C} depends only on Ω and the shape regularity of \mathcal{T}_h .

6. Numerical Results. In this section we present several numerical examples for the obstacle problem (1.7) with $\psi_1(x) = -\infty$. The computational domain for the first four examples is the square $(-0.5, 0.5) \times (-0.5, 0.5)$. The discrete problems are defined on uniform triangulations \mathcal{T}_j with mesh parameter $h_j = 2^{-j}$ (= the length of the horizontal and vertical edges) for $1 \leq j \leq 8$, and the penalty parameter σ is chosen to be 5 which ensures the coercivity of the discrete bilinear form on uniform meshes. The solutions of the discrete problems are denoted by \bar{y}_j ($1 \leq j \leq 8$), which are obtained by a primal–dual active set algorithm [4, 47].

Example 1. In this example we validate our numerical scheme by solving (1.7)/(1.9) with a known solution. We begin with the obstacle problem on the disc $\{x : |x| < 2\}$ with $\gamma = 0$, $\beta = 1$, $f = 0$ and $\psi_2(x) = \frac{1}{2}|x|^2 - 1$. This problem can be solved analytically because of rotational symmetry and the exact solution is given by

$$y_\dagger(x) = \begin{cases} C_1|x|^2 \ln|x| + C_2|x|^2 + C_3 \ln|x| + C_4 & |x| > r_0 \\ \frac{1}{2}|x|^2 - 1 & |x| \leq r_0 \end{cases}, \tag{6.1}$$

where $r_0 = 0.31078820\dots$, $C_1 = -0.26855864\dots$, $C_2 = 0.45470930\dots$, $C_3 = -0.02593989\dots$, and $C_4 = -1.05625438\dots$.

Let \bar{y} be the restriction of y_\dagger to $\Omega = (-0.5, 0.5)^2$. Then we have

$$\bar{y} = \operatorname{argmin}_{y \in \tilde{K}} \left[\frac{1}{2} \int_\Omega (D^2 y : D^2 y) dx - \int_{\partial\Omega} \left(\frac{\partial^2 y_\dagger}{\partial n^2} \right) \left(\frac{\partial y}{\partial n} \right) ds \right], \tag{6.2}$$

where n is the unit outer normal on $\partial\Omega$ and

$$\tilde{K} = \{v \in H^2(\Omega) : v - y_\dagger \in H_0^1(\Omega) \text{ and } v \leq \psi_2 \text{ in } \Omega\},$$

i.e., \bar{y} is the solution of an obstacle problem for a simply supported plate with nonhomogeneous boundary conditions.

As in the case of clamped plates [23], our results for simply supported plates with homogeneous boundary conditions (Theorems 5.1 and 5.2) can be extended to the nonhomogeneous case. Let \tilde{V}_h be the \mathbb{P}_2 Lagrange finite element space associated with the triangulation \mathcal{T}_h . The discrete problem for (6.2) is to find

$$\bar{y}_h = \operatorname{argmin}_{y_h \in \tilde{K}_h} \left[\frac{1}{2} a_h(y_h, y_h) - \sum_{e \in \mathcal{E}_h^b} \int_e \left(\frac{\partial^2 y_h}{\partial n^2} \right) \left(\frac{\partial y_h}{\partial n} \right) ds \right], \quad (6.3)$$

where

$$\tilde{K}_h = \{v \in \tilde{V}_h : v - \Pi_h y_h \in H_0^1(\Omega) \quad \text{and} \quad v(p) \leq \psi_2(p) \quad \forall p \in \mathcal{V}_h\}.$$

Let \bar{y}_j be the solution of (6.3) for the j th level triangulation and Π_j be the Lagrange nodal interpolation operator for the j th level finite element space V_j . We evaluate the error $e_j = \Pi_j \bar{y} - \bar{y}_j$ in the energy norm $\|\cdot\|_{h_j}$ and in the ℓ_∞ norm $\|\cdot\|_\infty$ defined by

$$\|e_j\|_\infty = \max_{p \in \mathcal{N}_j} |e_j(p)|,$$

where \mathcal{N}_j is the set of the vertices and midpoints of \mathcal{T}_j . We also compute the order of convergence in these norms by the formulas

$$\ln(\|e_{j-1}\|_{h_{j-1}}/\|e_j\|_{h_j})/\ln 2 \quad \text{and} \quad \ln(\|e_{j-1}\|_\infty/\|e_j\|_\infty)/\ln 2.$$

The numerical results are presented in Table 1. The order of convergence in the energy norm is observed to be 1.5, which is better than the order of 1 predicted by Theorem 5.1. This is likely due to the fact that \bar{y} is actually a C^∞ function on Ω away from the circle with radius r_0 and therefore superconvergence occurs since we use uniform triangulations. We also observe that the order of convergence in the ℓ_∞ norm is close to 2, better than the order of 1 predicted by Theorem 5.2.

We plot the discrete coincidence sets I_7 and I_8 in Fig. 1, where

$$I_j = \{p \in \mathcal{N}_j : \bar{y}_j(p) \geq \psi_2(p) - \|e_j\|_\infty\}.$$

The black circle represents the exact free boundary $F = \{x \in \Omega : |x| = r_0\}$ (cf. (6.1)). It is evident that the discrete coincidence sets (resp. free boundaries) are converging to the exact coincidence set (resp. free boundary).

The second set of examples are optimal control problems with state constraints that come from [6, 55]. The value of γ is taken to be 1. Since the exact solutions are not known, we take $\tilde{e}_{\bar{y},j} = \bar{y}_{j-1} - \bar{y}_j$ and evaluate $\|\tilde{e}_{\bar{y},j}\|_{h_j}$ (the error of the state in the energy norm), $|\tilde{e}_{\bar{y},j}|_{H^1}$ (the error of

TABLE 1
Energy and l_∞ errors for Example 1

j	$\ e_j\ _{h_j}/\ \bar{y}_8\ _{h_8}$	Order	$\ e_j\ _\infty$	Order
1	2.1840×10^{-1}		7.0940×10^{-3}	
2	6.8348×10^{-2}	1.68	5.9691×10^{-4}	3.57
3	3.1394×10^{-2}	1.12	5.7224×10^{-4}	0.06
4	9.8571×10^{-3}	1.67	1.1579×10^{-4}	2.31
5	3.8462×10^{-3}	1.36	3.5461×10^{-5}	1.71
6	1.4533×10^{-3}	1.40	1.0669×10^{-5}	1.73
7	5.4157×10^{-4}	1.42	3.3085×10^{-6}	1.69
8	1.9884×10^{-4}	1.45	8.9654×10^{-7}	1.88

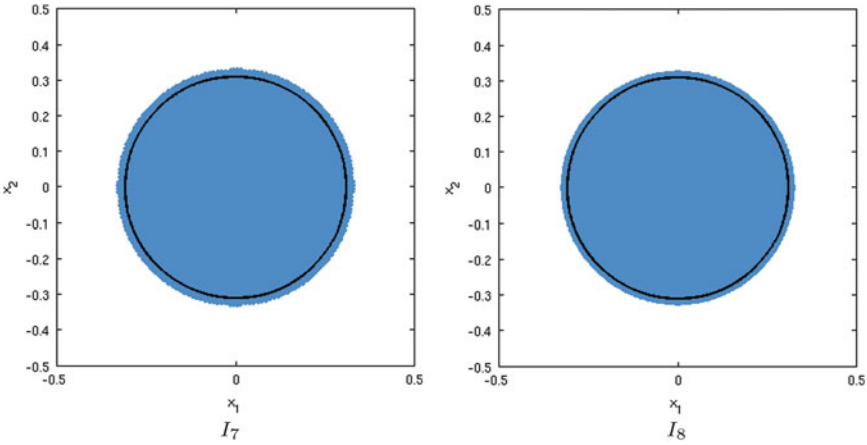


FIG. 1. Discrete coincidence sets I_7 (left) and I_8 (right) for Example 1

the state in the $H^1(\Omega)$ seminorm), and $\|\tilde{e}_{\bar{y},j}\|_\infty$ (the error of the state in the l_∞ norm) defined by

$$\|\tilde{e}_{\bar{y},j}\|_\infty = \max_{p \in \mathcal{N}_j} |\tilde{e}_{\bar{y},j}(p)|.$$

The approximations of the optimal control in these examples are given by the piecewise quadratic functions $\bar{u}_j \in V_j$ defined by (5.11). We take $\tilde{e}_{\bar{u},j} = \bar{u}_{j-1} - \bar{u}_j$ and evaluate $\|\tilde{e}_{\bar{u},j}\|_{L_2}$ (the error of the control in the $L_2(\Omega)$ norm). The orders of convergence in these examples are generated by the formulas

$$\ln(\|\tilde{e}_{\bar{y},j-1}\|/\|\tilde{e}_{\bar{y},j}\|)/\ln(2) \quad \text{and} \quad \ln(\|\tilde{e}_{\bar{u},j-1}\|/\|\tilde{e}_{\bar{u},j}\|)/\ln(2).$$

Example 2. In this example we take $y_d(x) = 10(\sin(2\pi(x_1 + 0.5)) + (x_2 + 0.5))$, $\psi_2(x) = 0.01$ and $\beta = 0.1$. The errors for the approximations of the state and the control are reported in Tables 2 and 3. The discrete state \bar{y}_8 and control \bar{u}_8 are depicted in Fig. 2.

TABLE 2
Energy and ℓ_∞ state errors for Example 2

j	$\ \tilde{e}_{\bar{y},j}\ _{h_j}/\ \bar{y}_8\ _{h_8}$	Order	$\ \tilde{e}_{\bar{y},j}\ _\infty$	Order
1	3.3661×10^0		1.1842×10^{-1}	
2	1.9062×10^0	0.82	3.9252×10^{-2}	1.59
3	7.4142×10^{-1}	1.36	6.5358×10^{-3}	2.59
4	4.4582×10^{-1}	0.73	2.0856×10^{-3}	1.66
5	2.2066×10^{-1}	1.01	6.2389×10^{-4}	1.74
6	1.0916×10^{-1}	1.02	1.8209×10^{-4}	1.78
7	5.4174×10^{-2}	1.01	4.5582×10^{-5}	2.00
8	2.7011×10^{-2}	1.00	1.1677×10^{-5}	1.96

TABLE 3
 H^1 state errors and L_2 control errors for Example 2

j	$ \tilde{e}_{\bar{y},j} _{H^1}/ \bar{y}_8 _{H^1}$	Order	$\ \tilde{e}_{\bar{u},j}\ _{L_2}/\ \bar{u}_8\ _{L_2}$	Order
1	4.9436×10^0		3.6418×10^0	
2	1.9541×10^0	1.34	2.1388×10^0	0.77
3	3.6305×10^{-1}	2.43	7.9054×10^{-1}	1.44
4	1.1593×10^{-1}	1.65	3.3568×10^{-1}	1.24
5	3.4745×10^{-2}	1.74	1.2506×10^{-1}	1.42
6	9.7768×10^{-3}	1.83	4.0060×10^{-2}	1.64
7	2.5550×10^{-3}	1.94	1.3141×10^{-2}	1.61
8	6.4538×10^{-4}	1.99	4.4595×10^{-3}	1.56

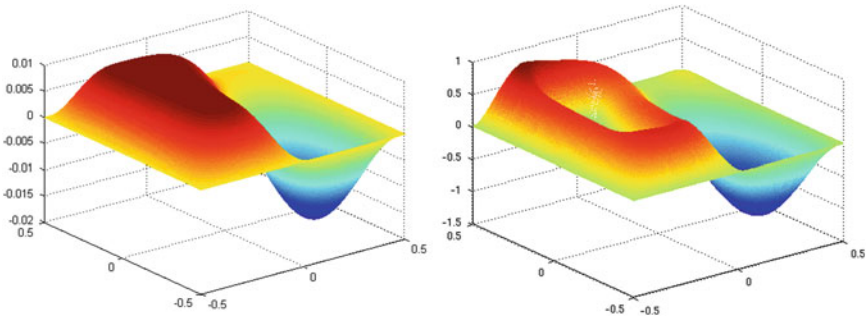


FIG. 2. Discrete state \bar{y}_8 (left) and control \bar{u}_8 (right) for Example 2

Example 3. In this example we take $y_d(x) = \sin(2\pi(x_1 + 0.5)(x_2 + 0.5))$, $\psi_2(x) = 0.1$ and $\beta = 10^{-3}$. The errors for the approximations of the state and the control are given in Tables 4 and 5. Figure 3 contains the plots for the discrete state \bar{y}_8 and the discrete control \bar{u}_8 .

TABLE 4
Energy and ℓ_∞ state errors for Example 3

j	$\ \tilde{e}_{\bar{y},j}\ _{h_j}/\ \bar{y}_8\ _{h_8}$	Order	$\ \tilde{e}_{\bar{y},j}\ _\infty$	Order
1	2.6968×10^0		6.3179×10^{-1}	
2	1.2439×10^0	1.12	1.2247×10^{-1}	2.37
3	6.7643×10^{-1}	0.88	3.7137×10^{-2}	1.72
4	3.4552×10^{-1}	0.97	7.2368×10^{-3}	2.36
5	1.7485×10^{-1}	0.98	2.5667×10^{-3}	1.50
6	8.6434×10^{-2}	1.01	7.3986×10^{-4}	1.79
7	4.2673×10^{-2}	1.02	1.9661×10^{-4}	1.91
8	2.1230×10^{-2}	1.01	4.9541×10^{-5}	1.99

TABLE 5
 H^1 state errors and L_2 control errors for Example 3

j	$ \tilde{e}_{\bar{y},j} _{H^1}/ \bar{y}_8 _{H^1}$	Order	$\ \tilde{e}_{\bar{u},j}\ _{L_2}/\ \bar{u}_8\ _{L_2}$	Order
1	3.2873×10^0		2.6748×10^0	
2	1.1542×10^0	1.51	1.5626×10^0	0.78
3	3.3936×10^{-1}	1.77	9.4478×10^{-1}	0.73
4	9.2061×10^{-2}	1.88	3.5294×10^{-1}	1.42
5	2.6639×10^{-2}	1.79	1.2171×10^{-1}	1.54
6	7.2818×10^{-3}	1.87	4.1983×10^{-2}	1.54
7	1.8560×10^{-3}	1.97	1.3128×10^{-2}	1.68
8	4.6711×10^{-4}	1.99	4.4419×10^{-3}	1.56

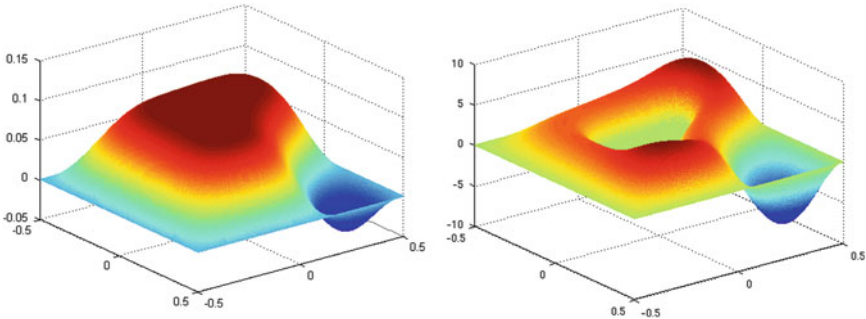


FIG. 3. Discrete state \bar{y}_8 (left) and control \bar{u}_8 (right) for Example 3

Example 4. In this example we take $y_d(x) = \sin(4\pi(x_1 + 0.5)(x_2 + 0.5)) + 1.5$, $\psi_2(x) = 1$ and $\beta = 10^{-4}$. The errors for the approximations in the state and the control are presented in Tables 6 and 7. The plots of the discrete state \bar{y}_8 and the discrete control \bar{u}_8 are given in Fig. 4.

TABLE 6
Energy and ℓ_∞ state errors for Example 4

j	$\ \tilde{e}_{\bar{y},j}\ _{h_j}/\ \bar{y}_8\ _{h_8}$	Order	$\ \tilde{e}_{\bar{y},j}\ _\infty$	Order
1	8.6145×10^{-1}		1.9915×10^0	
2	7.0373×10^{-1}	0.29	1.3112×10^0	0.60
3	3.6102×10^{-1}	0.96	3.7238×10^{-1}	1.82
4	2.3689×10^{-1}	0.61	8.4619×10^{-2}	2.14
5	1.1894×10^{-1}	0.99	1.6099×10^{-2}	2.39
6	5.9093×10^{-2}	1.01	5.6989×10^{-3}	1.50
7	2.9179×10^{-2}	1.02	1.5619×10^{-3}	1.87
8	1.4505×10^{-2}	1.01	3.4243×10^{-4}	2.19

TABLE 7
 H^1 state errors and L_2 control errors for Example 4

j	$ \tilde{e}_{\bar{y},j} _{H^1}/ \bar{y}_8 _{H^1}$	Order	$\ \tilde{e}_{\bar{u},j}\ _{L_2}/\ \bar{u}_8\ _{L_2}$	Order
1	1.3273×10^0		1.2796×10^0	
2	8.1485×10^{-1}	0.70	1.2466×10^0	0.04
3	2.9527×10^{-1}	1.46	8.4385×10^{-1}	0.56
4	1.0283×10^{-1}	1.52	4.0479×10^{-1}	1.06
5	3.3447×10^{-2}	1.62	1.6078×10^{-1}	1.33
6	8.4522×10^{-3}	1.98	5.3193×10^{-2}	1.60
7	2.2334×10^{-3}	1.92	1.7536×10^{-2}	1.60
8	5.5791×10^{-4}	2.00	5.7008×10^{-3}	1.62

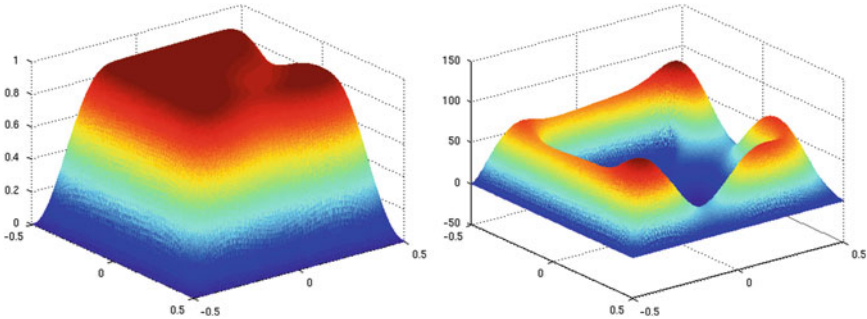


FIG. 4. Discrete state \bar{y}_8 (left) and control \bar{u}_8 (right) for Example 4

The numerical results in Tables 2–7 confirm the error estimate for $\|\bar{y} - \bar{y}_h\|_h$ in Theorem 5.1, since the index of elliptic regularity $\alpha = 1$ for a rectangular domain. On the other hand, the order of convergence for \bar{y}_h is 2 for both the ℓ_∞ norm and the $H^1(\Omega)$ seminorm, which is better than the first order convergence predicted by Theorem 5.2 and Corollary 5.1; and the order of convergence for \bar{u}_h is around 1.5, which is also better than the

first order convergence predicted by Theorem 5.3. The plots of the state and control in Figs. 2–4 also agree with the ones reported in [6, 55].

Example 5. In this example we take Ω to be the pentagonal domain obtained from the square $(-0.5, 0.5)^2$ by deleting the triangle with vertices $(0.5, 0)$, $(0.5, 0.5)$ and $(0, 0.5)$. We use the same data as Example 3, i.e., $\psi_2(x) = 0.1$, $y_d(x) = \sin(2\pi(x_1 + 0.5)(x_2 + 0.5))$, $\beta = 10^{-3}$ and $\gamma = 1$. The mesh parameter for the j th level uniform triangulation \mathcal{T}_j is $h_j = 2^{-(j+1)}$. The errors for the approximate state \bar{y}_j and approximate control \bar{u}_j are presented in Tables 8 and 9. Since the index of elliptic regularity α for the pentagonal domain can be taken to be any number less than $1/3$ (cf. Remark 2.1), the results in Tables 8 and 9 agree with Theorems 5.1 and 5.3. However, for this example the magnitude of the l_∞ error of the state seems to be $O(h^{2\alpha})$ and the magnitude of the $H^1(\Omega)$ error of the state seems to be $O(h)$.

We also plot the discrete state \bar{y}_8 and control \bar{u}_8 in Fig. 5. The singular nature of \bar{y} near the corners at $(0.5, 0)$ and $(0, 0.5)$ can be observed in the plot of \bar{u}_6 .

TABLE 8
Energy and ℓ_∞ state errors for Example 5

j	$\ \tilde{e}_{\bar{y},j}\ _{h_j}/\ \bar{y}_8\ _{h_8}$	Order	$\ \tilde{e}_{\bar{y},j}\ _\infty$	Order
1	1.2749×10^0		1.2541×10^{-1}	
2	7.3054×10^{-1}	0.80	3.7113×10^{-2}	1.76
3	3.6072×10^{-1}	1.02	4.6868×10^{-3}	2.99
4	1.9576×10^{-1}	0.88	1.3685×10^{-3}	1.78
5	1.1763×10^{-1}	0.73	3.4423×10^{-4}	1.99
6	7.8971×10^{-2}	0.57	1.4986×10^{-4}	1.20
7	5.7723×10^{-2}	0.45	7.7115×10^{-5}	0.96
8	4.4159×10^{-2}	0.39	4.6283×10^{-5}	0.74

TABLE 9
 H^1 state errors and L_2 control errors for Example 5

j	$ \tilde{e}_{\bar{y},j} _{H^1}/ \bar{y}_8 _{H^1}$	Order	$\ \tilde{e}_{\bar{u},j}\ _{L_2}/\ \bar{u}_8\ _{L_2}$	Order
1	1.1561×10^0		1.7291×10^0	
2	3.4840×10^{-1}	1.73	1.0841×10^0	0.67
3	8.2785×10^{-2}	2.07	3.9368×10^{-1}	1.46
4	2.2172×10^{-2}	1.90	1.3651×10^{-1}	1.53
5	6.5259×10^{-3}	1.76	5.3314×10^{-2}	1.36
6	2.1309×10^{-3}	1.61	2.5368×10^{-2}	1.07
7	8.7597×10^{-4}	1.28	1.7158×10^{-2}	0.56
8	4.5704×10^{-4}	0.94	1.3122×10^{-2}	0.39

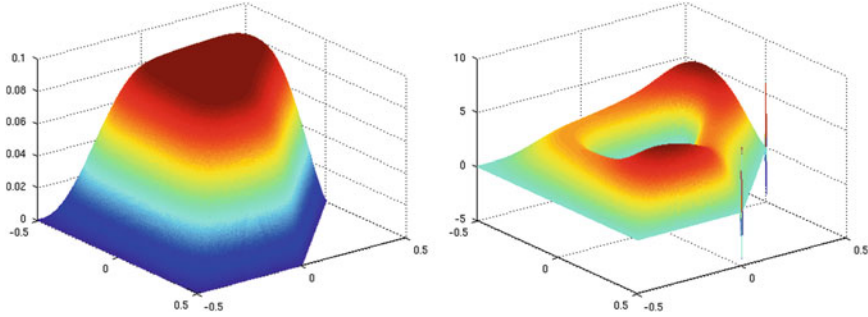


FIG. 5. Discrete state \bar{y}_8 (left) and control \bar{u}_8 (right) for Example 5

TABLE 10
Energy and ℓ_∞ state errors for Example 6

j	$\ \tilde{e}_{\bar{y},j}\ _{h_j}/\ \bar{y}_7\ _{h_7}$	Order	$\ \tilde{e}_{\bar{y},j}\ _\infty$	Order
1	5.1533×10^{-1}		2.1625×10^{-2}	
2	3.4325×10^{-1}	0.59	8.5234×10^{-3}	1.34
3	1.9843×10^{-1}	0.79	3.9322×10^{-3}	1.12
4	1.0957×10^{-1}	0.86	1.5313×10^{-3}	1.36
5	6.0125×10^{-2}	0.87	6.2314×10^{-4}	1.30
6	3.2836×10^{-2}	0.87	2.2180×10^{-4}	1.49
7	1.7795×10^{-2}	0.88	7.5681×10^{-5}	1.55

Example 6. In this example we solve the same problem in Example 5 on graded meshes obtained from a uniform triangulation \mathcal{T}_0 of the pentagonal domain by the refinement process in [10] (cf. Fig. 6), and we take the penalty parameter σ to be 20.

The errors of the approximate state \bar{y}_j and approximate control \bar{u}_j are reported in Tables 10 and 11. It is observed that the order of convergence for the state in the energy norm and for the control in the $L_2(\Omega)$ norm is about 1, which agrees with Theorems 5.1 and 5.3. On the other hand, the order of convergence for the state in the ℓ_∞ norm and the $H^1(\Omega)$ seminorm is about 1.5, which is better than the order of convergence predicted by Theorem 5.2 and Corollary 5.1.

The discrete state \bar{y}_7 and control \bar{u}_3 are depicted in Fig. 7. By comparing Figs. 5 and 7 we see that the graphs of the optimal states computed by a uniform mesh and a graded mesh are very similar. But the graph of the optimal control computed by graded meshes exhibited a more pronounced singular behavior near the corners $(0, 0.5)$ and $(0.5, 0)$ since the triangles at these corners are much smaller than the corresponding ones in a uniform mesh.

TABLE 11
 H^1 state errors and L_2 control errors for Example 6

j	$ \tilde{e}_{\bar{y},j} _{H^1}/ \bar{y}_8 _{H^1}$	Order	$\ \tilde{e}_{\bar{u},j}\ _{L_2}/\ \bar{u}_8\ _{L_2}$	Order
1	2.2461×10^{-1}		4.7166×10^{-1}	
2	1.0985×10^{-1}	1.03	2.9528×10^{-1}	0.68
3	4.5006×10^{-2}	1.29	1.5324×10^{-1}	0.95
4	1.5012×10^{-2}	1.58	7.7635×10^{-2}	0.98
5	5.7283×10^{-3}	1.39	4.0842×10^{-2}	0.93
6	1.9491×10^{-3}	1.56	2.1646×10^{-2}	0.92
7	6.6083×10^{-4}	1.56	1.1353×10^{-2}	0.93

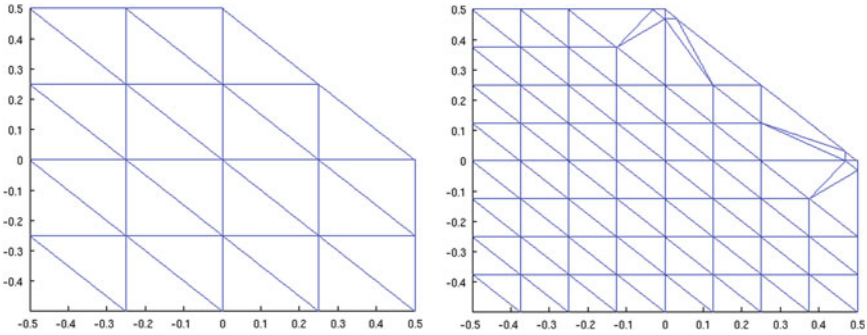


FIG. 6. Triangulation \mathcal{T}_0 (left) and \mathcal{T}_1 (right) for the pentagonal domain

7. Concluding Remarks. In this paper we have only considered the optimal control problem (1.1)–(1.3) on convex polygonal domains. It is possible to treat this problem on general polygonal domains, in which case the space $H^2(\Omega) \cap H_0^1(\Omega)$ will be replaced by the space $\{v \in H_0^1(\Omega) : \Delta v \in L_2(\Omega)\}$ that has been thoroughly analyzed in [45, 46] and the discretization will involve singular functions.

The three-dimensional version of (1.1)–(1.3) can also be solved as fourth order variational inequalities by finite element methods. For smooth domains, a straightforward extension of the approach in [15, 23–25] and this paper will lead to $O(h^{\frac{1}{2}})$ errors for the state in the energy norm and the control in the $L_2(\Omega)$ norm, similar to the error estimates in [37, 56]. Again we expect the convergence of the state in the $H^1(\Omega)$ norm and the $L_\infty(\Omega)$ norm to be of higher order.

These and other topics, such as the solution of optimal control problems with both state and control constraints as fourth order variational inequalities are subjects of ongoing investigations.

Acknowledgment. The authors would like to thank an anonymous referee for suggesting the piecewise quadratic approximation of the optimal control defined by (5.11). The work of S.C. Brenner, L.-Y. Sung, and Y.

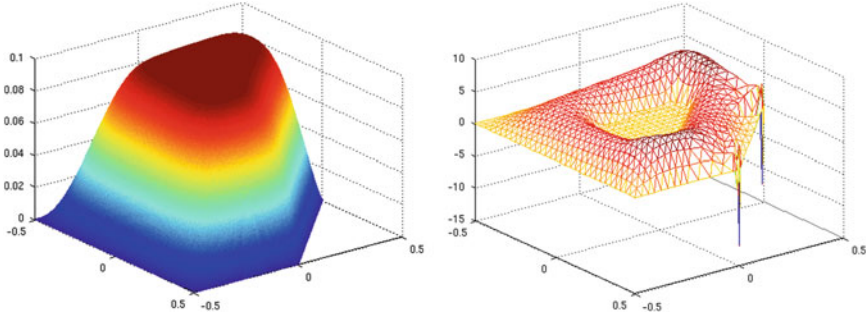


FIG. 7. Discrete state \bar{y}_7 (left) and control \bar{u}_3 (right) for Example 6

Zhang was supported in part by the National Science Foundation under Grant No. DMS-10-16332.

APPENDIX

A. Elliptic Regularity for Simply Supported Plates. In this appendix we summarize elliptic regularity results for the biharmonic equation on convex polygonal domains with the boundary conditions of simply supported plates and also discuss related results for the solution \bar{y} of the obstacle problem (1.7). We will focus on the H^3 regularity (or lack thereof) for the solution since $\bar{y} \in H^3_{\text{loc}}(\Omega)$.

Let Ω be a convex polygonal domain with corners p_1, \dots, p_L and ω_ℓ be the interior angle of Ω at p_ℓ . Let $g \in L_2(\Omega)$ and $z \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfy

$$\int_{\Omega} D^2 z : D^2 v \, dx = \int_{\Omega} g v \, dx \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega). \tag{A.1}$$

It follows from (A.1) that $w = -\Delta z \in L_2(\Omega)$ has the following properties: (i) w is an H^2 function away from the corners of Ω , (ii) w vanishes on $\partial\Omega \setminus \{p_1, \dots, p_L\}$. These two conditions then imply that

$$w = -\Delta z \text{ belongs to } H^2(\Omega) \cap H^1_0(\Omega) \tag{A.2}$$

and that z also satisfies

$$\int_{\Omega} \nabla z \cdot \nabla v \, dx = \int_{\Omega} w v \, dx \quad \forall v \in H^1_0(\Omega).$$

Thus we can deduce the elliptic regularity of z from the elliptic regularity theory for the Laplace operator [36, 45, 58].

First of all,

$$z \text{ is an } H^4 \text{ function away from the corners of } \Omega, \tag{A.3}$$

which also follows directly from (A.1). Secondly we have

$$z \in H^3(\mathcal{N}_\ell) \quad \text{if } \omega_\ell \leq \pi/2, \tag{A.4}$$

where $\mathcal{N}_\ell \subset \Omega$ is a neighborhood of p_ℓ . Finally, at a corner p_ℓ where $\omega_\ell > \pi/2$, we have

$$z - \kappa_\ell \varphi_\ell \in H^3(\mathcal{N}_\ell), \tag{A.5}$$

where $\mathcal{N}_\ell \subset \Omega$ is a neighborhood of p_ℓ , κ_ℓ is a constant (generalized stress intensity factor), and the singular function φ_ℓ is defined by

$$\varphi_\ell = r_\ell^{\pi/\omega_\ell} \sin((\pi/\omega_\ell)\theta_\ell). \tag{A.6}$$

Here (r_ℓ, θ_ℓ) are the polar coordinates at p_ℓ such that the two edges of Ω emanating from p_ℓ are given by $\theta_\ell = 0$ and $\theta_\ell = \omega_\ell$. Note that φ_ℓ is a harmonic function and $\varphi_\ell \in H^{1+(\pi/\omega_\ell)-\epsilon}(\mathcal{N}_\ell)$ for any $\epsilon > 0$.

Now we turn to the solution \bar{y} of (1.7)/(1.9). Since the constraints in (1.3) are not active near $\partial\Omega$ because of (1.4b), we have

$$\begin{aligned} \int_\Omega [\beta(D^2(\rho_1\bar{y}) : D^2w)] \, dx &= \int_\Omega \beta [D^2(\rho_1\bar{y}) : D^2((1 - \rho_2)w)] \, dx \\ &\quad + \int_\Omega \rho_2(f - \gamma\bar{y})w \, dx \end{aligned}$$

for all $w \in H^2(\Omega) \cap H_0^1(\Omega)$, where $\rho_1 = \rho_2 = 1$ near $\partial\Omega$, $\rho_1 = 1$ on the support of ρ_2 , and the support of ρ_1 is disjoint from the active set where $\bar{y}(x) = \psi_1(x)$ or $\psi_2(x)$. Note that standard interior elliptic regularity [62, Sect. 20] implies

$$\int_\Omega [D^2(\rho_1\bar{y}) : D^2((1 - \rho_2)w)] \, dx = \int_\Omega (1 - \rho_2)[\Delta^2(\rho_1\bar{y})]w \, dx,$$

where $(1 - \rho_2)\Delta^2(\rho_1\bar{y}) \in L_2(\Omega)$.

Therefore $z = \rho_1\bar{y}$ satisfies (A.1) with $g = \rho_2(f - \gamma\bar{y})/\beta + (1 - \rho_2)\Delta^2(\rho_1\bar{y}) \in L_2(\Omega)$. Combining (A.2)–(A.6) and the fact that $\bar{y} \in H_{\text{loc}}^3(\Omega)$, we can draw the following conclusions about \bar{y} .

- The function $\Delta\bar{y}$ belongs to $H_0^1(\Omega)$. Therefore $\bar{u} = -\Delta\bar{y}$ belongs to $H_0^1(\Omega)$ for the optimal control problem (1.1)–(1.3).
- Let α_ℓ be chosen according to (2.4). Then $\bar{y} \in H^{2+\alpha_\ell}(\mathcal{N}_\ell)$, where $\mathcal{N}_\ell (\subset \Omega)$ is a neighborhood of p_ℓ . Globally we have $\bar{y} \in H^{2+\alpha}(\Omega)$ where $\alpha = \min_{1 \leq \ell \leq L} \alpha_\ell$.
- We can write $\bar{y} = \bar{y}_s + \bar{y}_R$, where $\bar{y}_R \in H^3(\Omega) \cap H_0^1(\Omega)$, $\Delta\bar{y}_R \in H_0^1(\Omega)$ and \bar{y}_s have the following properties.
 - \bar{y}_s is an H^3 function away from the corners of Ω where the angles are $> \pi/2$.

- \bar{y}_s is a multiple of φ_ℓ in a neighborhood \mathcal{N}_ℓ of a corner p_ℓ where $\omega_\ell > \pi/2$.
- $\Delta \bar{y}_s$ belongs to $H_0^1(\Omega)$.
- Since $r_\ell^{1-\alpha_\ell}(\partial^\mu \varphi_\ell) \in L_2(\mathcal{N}_\ell)$ for $|\mu| = 3$, we have $\Phi(\partial^\mu \bar{y}_s) \in L_2(\Omega)$ for $|\mu| = 3$ and hence

$$\Phi(\partial^\mu \bar{y}) \in L_2(\Omega) \quad \text{for } |\mu| = 3, \tag{A.7}$$

where the function Φ is defined by (2.3).

- Since $r_\ell^{-\alpha_\ell}(\partial^\mu \varphi_\ell) \in L_2(\mathcal{N}_\ell)$ for $|\mu| = 2$, we have $\Psi(\partial^\mu \bar{y}_s) \in L_2(\Omega)$ for $|\mu| = 2$ and hence

$$\Psi(\partial^\mu \bar{y}) \in L_2(\Omega) \quad \text{for } |\mu| = 2, \tag{A.8}$$

where the function Ψ is defined by

$$\Psi(x) = \prod_{\ell=1}^L |p_\ell - x|^{-\alpha_\ell}. \tag{A.9}$$

Finally we note that (cf. [36, Theorem AA.3 and Theorem AA.7])

$$|\bar{y}|_{H^{2+\alpha}(\Omega)} \leq C_\Omega \sum_{|\mu|=3} \|\Phi(\partial^\mu \bar{y})\|_{L_2(\Omega)}. \tag{A.10}$$

B. An Enriching Operator. In this appendix we construct the enriching operator introduced in Sect. 3.2. Such operators have played an important role in the design and analysis of fast solvers for nonconforming finite element methods [11, 12, 22, 26].

Let $\tilde{V}_h \subset H^1(\Omega)$ be the \mathbb{P}_2 Lagrange finite element space associated with \mathcal{T}_h and $\tilde{W}_h \subset H^2(\Omega)$ be the \mathbb{P}_6 Argyris finite element space [2] associated with \mathcal{T}_h . The degrees of freedom of $w \in \tilde{W}_h$ (cf. Fig. 8) consist of the values of the derivatives of w up to second order at the vertices of \mathcal{T}_h , the values of w at the midpoints of the edges of \mathcal{T}_h and at the centers of the triangles of \mathcal{T}_h , and the values of the normal derivative of w at two nodes on each edge in \mathcal{E}_h .

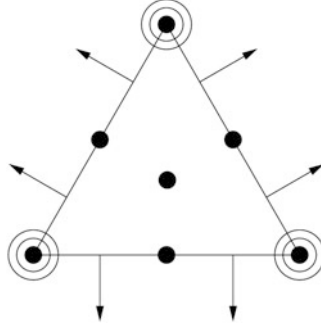
The enriching operator $E_h : \tilde{V}_h \rightarrow \tilde{W}_h$ is defined by averaging as follows (cf. Sect. 2.1 for the notation).

- (i) Let N be a degree of freedom associated with an interior node p . We define

$$N(E_h v) = \frac{1}{|\mathcal{T}_p|} \sum_{T \in \mathcal{T}_p} N(v_T).$$

- (ii) Let N be a degree of freedom involving the normal derivative associated with a boundary node interior to an edge $e \in \mathcal{E}_h^b$. We define

$$N(E_h v) = N(v_{T_e}).$$

FIG. 8. Degrees of freedom for the \mathbb{P}_6 Argyris finite element

- (iii) Let p be a boundary node which is not a corner of Ω such that p is the common endpoint of two edges $e_1, e_2 \in \mathcal{E}_h^b$. For any degree of freedom N associated with p , we define

$$N(E_h v) = \frac{1}{2} [N(v_{T_{e_1}}) + N(v_{T_{e_2}})].$$

- (iv) Let p be a corner of Ω . Then p is the common endpoint of $e_1, e_2 \in \mathcal{E}_h^b$. Let t_j (resp. n_j) be a unit tangent (resp. normal) of e_j . We define

$$\begin{aligned} (E_h v)(p) &= v(p), \\ (\partial(E_h v)/\partial t_j)(p) &= (\partial v_{T_{e_j}}/\partial t_j)(p) && \text{for } j = 1, 2, \\ (\partial^2(E_h v)/\partial t_j^2)(p) &= (\partial^2 v_{T_{e_j}}/\partial t_j^2)(p) && \text{for } j = 1, 2, \\ (\partial^2(E_h v)/\partial t_1 \partial n_1)(p) &= (\partial^2 v_{T_{e_1}}/\partial t_1 \partial n_1)(p). \end{aligned}$$

REMARK B.1. We can also replace the last equation in (iv) by

$$(\partial^2(E_h v)/\partial t_2 \partial n_2)(p) = (\partial^2 v_{T_{e_2}}/\partial t_2 \partial n_2)(p).$$

Since v is continuous at the vertices, the relation (3.6) follows immediately from (i), (iii), and (iv). It is also easy to check that

$$E_h v \in W_h = \tilde{W}_h \cap H_0^1(\Omega) \subset H^2(\Omega) \cap H_0^1(\Omega) \quad \text{if} \quad v \in V_h = \tilde{V}_h \cap H_0^1(\Omega).$$

We now turn to the derivations of (3.8) and (3.9). Let $T \in \mathcal{T}_h$ be arbitrary. Since $v = E_h v$ at the vertices and the center of T , we have, by scaling,

$$\begin{aligned} \|v - E_h v\|_{L^2(T)}^2 &\lesssim h_T^4 \left(\sum_{p \in \mathcal{V}_T} |\nabla(v - E_h v)(p)|^2 \right. \\ &\quad \left. + \sum_{p \in \mathcal{N}_T} \left| \frac{\partial(v - E_h v)}{\partial n}(p) \right|^2 + \sum_{p \in \mathcal{V}_T} h_T^2 |D^2(v - E_h v)(p)|^2 \right) \end{aligned} \quad (\text{B.1})$$

for all $v \in \tilde{V}_h$, where \mathcal{N}_T is the set of the six nodes on ∂T associated with the degrees of freedom of the \mathbb{P}_6 Argyris finite element that involve the normal derivative (cf. Fig. 8).

Let $p \in \mathcal{V}_T$ be interior to Ω . Since the tangential derivative of $v - E_h v$ is continuous across element boundaries, we have, by the definition of E_h and a standard inverse estimate,

$$\begin{aligned} |\nabla(v - E_h v)(p)|^2 &= \left| \frac{1}{|\mathcal{T}_p|} \sum_{T' \in \mathcal{T}_p} (\nabla v_{T'}(p) - \nabla v_{T'}(p)) \right|^2 \\ &\lesssim \sum_{e \in \mathcal{E}_p^i} |e|^{-1} \|[\partial v / \partial n]\|_{L^2(e)}^2 \end{aligned} \quad (\text{B.2})$$

where \mathcal{E}_p^i is the set of the edges in \mathcal{E}_h^i sharing p as a common endpoint. Similarly, we have

$$\begin{aligned} |D^2(v - E_h v)(p)|^2 &= \left| \frac{1}{|\mathcal{T}_p|} \sum_{T' \in \mathcal{T}_p} D^2(v_{T'} - v_{T'})(p) \right|^2 \\ &\lesssim \sum_{T' \in \mathcal{T}_p} h_{T'}^{-2} |v|_{H^2(T')}. \end{aligned} \quad (\text{B.3})$$

The estimates (B.2) and (B.3) are also valid for $p \in \partial\Omega$ by similar arguments.

Now we consider $p \in \mathcal{N}_T$. If p is a boundary node, then $|(\partial(v - E_h v) / \partial n)(p)| = 0$ by the definition of E_h . Otherwise we have, by a standard inverse estimate,

$$|\partial(v - E_h v) / \partial n(p)|^2 \lesssim |e|^{-1} \|[\partial v / \partial n]\|_{L^2(e)}^2 \quad (\text{B.4})$$

for some $e \in \mathcal{E}_h^i$.

Combining (B.1)–(B.4), we obtain the estimate (3.8) for $m = 0$, which then implies the estimates for $m = 1$ and 2 through standard inverse estimates.

For the operator $E_h \circ \Pi_h$, first we observe that it is a bounded linear operator from $H^{2+s}(\mathcal{S}_T)$ into $H^2(T)$ because of (2.21) and (3.8). Furthermore, by construction, $E_h \Pi_h \zeta = \zeta$ on T if $\zeta \in \mathbb{P}_2(\mathcal{S}_T)$. Hence the estimate (3.9) follows from the Bramble–Hilbert lemma (cf. [9, 38]).

REFERENCES

- [1] T. APEL, A.-M. SÄNDIG, AND J. WHITEMAN, *Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains*, Math. Methods Appl. Sci., 19 (1996), pp. 63–85.
- [2] J. ARGYRIS, I. FRIED, AND D. SCHARPF, *The TUBA family of plate elements for the matrix displacement method*, Aero. J. Roy. Aero. Soc., 72 (1968), pp. 701–709.
- [3] J. AUBIN, *Approximation of variational inequations*, in Functional Analysis and Optimization, Academic Press, New York, 1966, pp. 7–14.
- [4] M. BERGOUNIOUX, K. ITO, AND K. KUNISCH, *Primal-dual strategy for constrained optimal control problems*, SIAM J. Control Optim., 37 (1999), pp. 1176–1194 (electronic).
- [5] M. BERGOUNIOUX AND K. KUNISCH, *Augmented Lagrangian techniques for elliptic state constrained optimal control problems*, SIAM J. Control Optim., 35 (1997), pp. 1524–1543.
- [6] ———, *Primal-dual strategy for state-constrained optimal control problems*, Comput. Optim. Appl., 22 (2002), pp. 193–224.
- [7] ———, *On the structure of Lagrange multipliers for state-constrained optimal control problems*, Systems Control Lett., 48 (2003), pp. 169–176.
- [8] H. BLUM AND R. RANNACHER, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Methods Appl. Sci., 2 (1980), pp. 556–581.
- [9] J. BRAMBLE AND S. HILBERT, *Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation*, SIAM J. Numer. Anal., 7 (1970), pp. 113–124.
- [10] J. BRANNICK, H. LI, AND L. ZIKATANOV, *Uniform convergence of the multigrid V-cycle on graded meshes for corner singularities*, Numer. Linear Algebra Appl., 15 (2008), pp. 291–306.
- [11] S.C. BRENNER, *A two-level additive Schwarz preconditioner for nonconforming plate elements*, Numer. Math., 72 (1996), pp. 419–447.
- [12] ———, *Convergence of nonconforming multigrid methods without full elliptic regularity*, Math. Comp., 68 (1999), pp. 25–53.
- [13] ———, *C^0 Interior Penalty Methods*, in Frontiers in Numerical Analysis-Durham 2010, J. Blowey and M. Jensen, eds., vol. 85 of Lecture Notes in Computational Science and Engineers, Springer-Verlag, Berlin-Heidelberg, 2012, pp. 79–147.
- [14] S.C. BRENNER AND C. CARSTENSEN, *Finite Element Methods*, in Encyclopedia of Computational Mechanics, E. Stein, R. de Borst, and T. Hughes, eds., Wiley, Weinheim, 2004, pp. 73–118.
- [15] S.C. BRENNER, C. DAVIS, AND L.-Y. SUNG, *A generalized finite element method for the displacement obstacle problem of clamped Kirchhoff plates*, arXiv:1212.3026 [math.NA].
- [16] S.C. BRENNER, S. GU, T. GUDI, AND L.-Y. SUNG, *A quadratic C^0 interior penalty method for linear fourth order boundary value problems with boundary conditions of the Cahn-Hilliard type*, SIAM J. Numer. Anal., 50 (2012), pp. 2088–2110.
- [17] S.C. BRENNER, T. GUDI, AND L.-Y. SUNG, *An a posteriori error estimator for a quadratic C^0 interior penalty method for the biharmonic problem*, IMA J. Numer. Anal., 30 (2010), pp. 777–798.
- [18] S.C. BRENNER AND M. NEILAN, *A C^0 interior penalty method for a fourth order elliptic singular perturbation problem*, SIAM J. Numer. Anal., 49 (2011), pp. 869–892.
- [19] S.C. BRENNER, M. NEILAN, AND L.-Y. SUNG, *Isoparametric C^0 interior penalty methods for plate bending problems on smooth domains*, Calcolo, 49 (2012), pp. 35–67.

- [20] S.C. BRENNER AND L.R. SCOTT, *The Mathematical Theory of Finite Element Methods (Third Edition)*, Springer-Verlag, New York, 2008.
- [21] S.C. BRENNER AND L.-Y. SUNG, C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains, *J. Sci. Comput.*, 22/23 (2005), pp. 83–118.
- [22] ———, *Multigrid algorithms for C^0 interior penalty methods*, *SIAM J. Numer. Anal.*, 44 (2006), pp. 199–223.
- [23] S.C. BRENNER, L.-Y. SUNG, H. ZHANG, AND Y. ZHANG, *A quadratic C^0 interior penalty method for the displacement obstacle problem of clamped Kirchhoff plates*, *SIAM J. Numer. Anal.*, 50 (2012), pp. 3329–3350.
- [24] ———, *A Morley finite element method for the displacement obstacle problem of clamped Kirchhoff plates*, *J. Comp. Appl. Math.*, DOI:10.1016/j.cam.2013.02.028, published online May 29, 2013.
- [25] S.C. BRENNER, L.-Y. SUNG, AND Y. ZHANG, *Finite element methods for the displacement obstacle problem of clamped plates*, *Math. Comp.*, 81 (2012), pp. 1247–1262.
- [26] S.C. BRENNER AND K. WANG, *Two-level additive Schwarz preconditioners for C^0 interior penalty methods*, *Numer. Math.*, 102 (2005), pp. 231–255.
- [27] ———, *An iterative substructuring algorithm for a C^0 interior penalty method*, *Elect. Trans. Numer. Anal.*, 39 (2012), pp. 313–332.
- [28] S.C. BRENNER, K. WANG, AND J. ZHAO, *Poincaré-Friedrichs inequalities for piecewise H^2 functions*, *Numer. Funct. Anal. Optim.*, 25 (2004), pp. 463–478.
- [29] H. BRÉZIS AND G. STAMPACCHIA, *Sur la régularité de la solution d'inéquations elliptiques*, *Bull. Soc. Math. France*, 96 (1968), pp. 153–180.
- [30] F. BREZZI, W. HAGER, AND P.-A. RAVIART, *Error estimates for the finite element solution of variational inequalities*, *Numer. Math.*, 28 (1977), pp. 431–443.
- [31] ———, *Error estimates for the finite element solution of variational inequalities. II. Mixed methods*, *Numer. Math.*, 31 (1978/79), pp. 1–16.
- [32] L. CAFFARELLI AND A. FRIEDMAN, *The obstacle problem for the biharmonic operator*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 6 (1979), pp. 151–184.
- [33] E. CASAS, *Control of an elliptic problem with pointwise state constraints*, *SIAM J. Control Optim.*, 24 (1986), pp. 1309–1318.
- [34] S. CHEREDNICHENKO, K. KRUMBIEGEL, AND A. RÖSCH, *Error estimates for the Lavrentiev regularization of elliptic optimal control problems*, *Inverse Problems*, 24 (2008), p. 055003.
- [35] P. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
- [36] M. DAUGE, *Elliptic Boundary Value Problems on Corner Domains*, Lecture Notes in Mathematics 1341, Springer-Verlag, Berlin-Heidelberg, 1988.
- [37] K. DECKELNICK AND M. HINZE, *Convergence of a finite element approximation to a state-constrained elliptic control problem*, *SIAM J. Numer. Anal.*, 45 (2007), pp. 1937–1953 (electronic).
- [38] T. DUPONT AND R. SCOTT, *Polynomial approximation of functions in Sobolev spaces*, *Math. Comp.*, 34 (1980), pp. 441–463.
- [39] G. ENGEL, K. GARIKIPATI, T. HUGHES, M. LARSON, L. MAZZEI, AND R. TAYLOR, *Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity*, *Comput. Methods Appl. Mech. Engrg.*, 191 (2002), pp. 3669–3750.
- [40] R. FALK, *Error estimates for the approximation of a class of variational inequalities*, *Math. Comp.*, 28 (1974), pp. 963–971.
- [41] J. FREHSE, *Zum Differenzierbarkeitsproblem bei Variationsungleichungen höherer Ordnung*, *Abh. Math. Sem. Univ. Hamburg*, 36 (1971), pp. 140–149.
- [42] ———, *On the regularity of the solution of the biharmonic variational inequality*, *Manuscripta Math.*, 9 (1973), pp. 91–103.
- [43] A. FRIEDMAN, *Variational Principles and Free-Boundary Problems*, Robert E. Krieger Publishing Co. Inc., Malabar, FL, second ed., 1988.

- [44] W. GONG AND N. YAN, *A mixed finite element scheme for optimal control problems with pointwise state constraints*, J. Sci. Comput., 46 (2011), pp. 182–203.
- [45] P. GRISVARD, *Elliptic Problems in Non Smooth Domains*, Pitman, Boston, 1985.
- [46] ———, *Singularities in Boundary Value Problems*, Masson, Paris, 1992.
- [47] M. HINTERMÜLLER, K. ITO, AND K. KUNISCH, *The primal-dual active set strategy as a semismooth Newton method*, SIAM J. Optim., 13 (2002), pp. 865–888 (2003).
- [48] M. HINTERMÜLLER AND K. KUNISCH, *PDE-constrained optimization subject to pointwise constraints on the control, the state, and its derivative*, SIAM J. Optim., 20 (2009), pp. 1133–1156.
- [49] M. HINTERMÜLLER AND W. RING, *A level set approach for the solution of a state-constrained optimal control problem*, Numer. Math., 98 (2004), pp. 135–166.
- [50] M. HINZE AND C. MEYER, *Variational discretization of Lavrentiev-regularized state constrained elliptic optimal control problems*, Comput. Optim. Appl., 46 (2010), pp. 487–510.
- [51] M. HINZE AND A. SCHIELA, *Discretization of interior point methods for state constrained elliptic optimal control problems: optimal error estimates and parameter adjustment*, Comput. Optim. Appl., 48 (2011), pp. 581–600.
- [52] D. KINDERLEHRER AND G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, Society for Industrial and Applied Mathematics, Philadelphia, 2000.
- [53] H. LEWY AND G. STAMPACCHIA, *On the regularity of the solution of a variational inequality*, Comm. Pure Appl. Math., 22 (1969), pp. 153–188.
- [54] J.-L. LIONS AND G. STAMPACCHIA, *Variational inequalities*, Comm. Pure Appl. Math., 20 (1967), pp. 493–519.
- [55] W. LIU, W. GONG, AND N. YAN, *A new finite element approximation of a state-constrained optimal control problem*, J. Comput. Math., 27 (2009), pp. 97–114.
- [56] C. MEYER, *Error estimates for the finite-element approximation of an elliptic control problem with pointwise state and control constraints*, Control Cybernet., 37 (2008), pp. 51–83.
- [57] C. MEYER, A. RÖSCH, AND F. TRÖLTZSCH, *Optimal control of PDEs with regularized pointwise state constraints*, Comput. Optim. Appl., 33 (2006).
- [58] S. NAZAROV AND B. PLAMENEVSKY, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, de Gruyter, Berlin-New York, 1994.
- [59] J.-F. RODRIGUES, *Obstacle Problems in Mathematical Physics*, North-Holland Publishing Co., Amsterdam, 1987.
- [60] A. SCHIELA AND A. GÜNTHER, *An interior point algorithm with inexact step computation in function space for state constrained optimal control*, Numer. Math., 119 (2011), pp. 373–407.
- [61] L. SCOTT AND S. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.
- [62] J. WLOKA, *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.