Algebraic Properties of Qualitative Spatio-temporal Calculi

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Abstract. Qualitative spatial and temporal reasoning is based on so-called qualitative calculi. Algebraic properties of these calculi have several implications on reasoning algorithms. But what exactly is a qualitative calculus? And to which extent do the qualitative calculi proposed meet these demands? The literature provides various answers to the first question but only few facts about the second. In this paper we identify the minimal requirements to binary spatio-temporal calculi and we discuss the relevance of the according axioms for representation and reasoning. We also analyze existing qualitative calculi and provide a classification involving different notions of relation algebra.

1 Introduction

Qualitative spatial and temporal reasoning is a sub-field of knowledge representation involved with representations of spatial and temporal domains that are based on finite sets of so-called qualitative relations. Qualitative relations serve to explicate knowledge relevant for a task at hand while at the same time they abstract from irrelevant knowledge. Often, these relations aim to relate to cognitive concepts. Qualitative spatial and temporal reasoning thus link cognitive approaches to knowledge representation with formal methods. Computationally, qualitative spatial and temporal reasoning is largely involved with constraint satisfaction problems over infinite domains where qualitative relations serve as constraints. Typical domains include, on the temporal side, points and intervals and, on the spatial side, regions or oriented points in the Euclidean plane. In the past decades, a vast number of qualitative representations have been developed that are commonly referred to as qualitative calculi (see [19] for a recent overview). Yet the literature provides us with several definitions of what a qualitative calculus exactly is. Nebel and Scivos [31] have introduced the rather general and weak notion of a *constraint algebra*, which is a set of relations closed under converse, finite intersection, and composition. Ligozat and Renz [20] focus on so-called non-associative algebras, which are relation algebras without associativity axioms, and which have a much richer structure. Both approaches assume

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that the converse operation is *strong*, which is not the case for calculi like the Cardinal Direction (Relations) Calculus (CDR) [39] or its recently introduced rectangular variant (RDR) [30].

The goal of this paper is to relate the existing definitions and to identify the essential representation-theoretic properties that characterize a qualitative calculus. It is achieved by the following contributions:

- We propose a definition of a qualitative calculus that includes existing spatio-temporal calculi by weakening the conditions usually imposed on the converse and composition relation (Section 2).
- We generalize the notions of constraint algebra and non-associative algebra to cover calculi with weak converse (Section 3.1).
- We discuss the role of algebraic properties of calculi for spatial reasoning, especially in connection with general-purpose reasoning tools like SparQ [44,43] and GQR [11] (Section 3.2).
- We experimentally evaluate the algebraic properties of calculi and derive a reasoning procedure that is sensitive to these properties (Section 4).
- We examine information preservation properties of calculi during reasoning, i.e., how general relations evolve after several compositions (Section 5).

2 Qualitative Representations

In this section, we formulate minimal requirements to a qualitative calculus, discuss their relevance to spatio-temporal representation and reasoning, and list existing calculi. We restrict ourselves to calculi with binary relations because we want to examine their algebraic properties using the notion of a relation algebra, which is best understood for binary relations.

2.1 Requirements to Qualitative Representations

We start with minimal requirements used in the literature. Let us first fix some notation. Let r, s, t range over binary relations over a non-empty universe $\mathcal{U}$, i.e., $r \subseteq \mathcal{U} \times \mathcal{U}$. We use $\cup$, $\cap$, $^{-}$, $^{\smile}$ and $\circ$ to denote the union, intersection, complement, converse, and composition of relations, as well as the identity and universal relations $\mathsf{id} = \{(u,u) \mid u \in \mathcal{U}\}$ and $\mathsf{u} = \mathcal{U} \times \mathcal{U}$. A relation $r \subseteq \mathcal{U} \times \mathcal{U}$ is called *serial* if, for every $u \in \mathcal{U}$, there is some $v \in \mathcal{U}$ such that $(u,v) \in r$.

Ligozat and Renz [20] note that most spatial and temporal calculi are based on a set of JEPD (jointly exhaustive and pairwise disjoint) relations. The following definition is standard in the QSR literature [20,4].

Definition 1. Let $\mathcal{U}$ be a non-empty universe and $\mathcal{R}$ a set of non-empty binary relations over $\mathcal{U}$. $\mathcal{R}$ is called a set of *JEPD relations* over $\mathcal{U}$ if the relations in $\mathcal{R}$ are pairwise disjoint and $\mathcal{U} \times \mathcal{U} = \bigcup_{r \in \mathcal{R}} r$.

An *abstract partition scheme* is a pair $(\mathcal{U}, \mathcal{R})$ where $\mathcal{R}$ is a set of *JEPD relations* over $\mathcal{U}$. $(\mathcal{U}, \mathcal{R})$ is called a *partition scheme* [20] if $\mathcal{R}$ contains the identity relation id and, for every $r \in \mathcal{R}$, there is some $s \in \mathcal{R}$ such that $r^{\smile} = s$.

The universe $\mathcal{U}$ represents the set of all spatial or temporal entities, and $\mathcal{R}$ being a set of JEPD relations ensures that each two entities are in exactly one relation from $\mathcal{R}$. Incomplete information about two entities is modeled by taking the union of base relations, with the universal relation (the union of all base relations) representing that no information is available. Disjointness of the base relations ensures that there is a unique way to represent an arbitrary relation, and exhaustiveness ensures that the empty relation can never occur in a consistent set of constraints (which are defined in Section 2.2).

Ligozat and Renz [20] base their definition of a qualitative calculus on the notion of a partition scheme. This excludes calculi like CDR and RDR which do not have strong converses. Hence, we take a more general approach based on the notion of an abstract partition scheme. This accommodates existing calculi with these weaker properties: some existing spatio-temporal representations do not require an identity relation, and some representations are deliberately kept coarse and thus do not guarantee that the converse of a base relation is again a (base) relation. Furthermore, the computation of the converse operation may be easier when weaker properties are postulated. The same rationale applies to the composition operation. Thus, the following definition of a spatial calculus, based on abstract partition schemes, contains minimal requirements.

Definition 2. A *qualitative calculus* with binary relations is a tuple $(\mathsf{Rel}, \mathsf{Int}, ^{\smile}, \diamond)$ with the following properties.

- Rel is a finite, non-empty set of *base relations*. The subsets of Rel are called *relations*. We use r, s, t to denote base relations and R, S, T to denote relations.
- $\mathsf{Int} = (\mathcal{U}, \varphi)$ is an *interpretation* with a non-empty universe $\mathcal{U}$ and a map $\varphi : \mathsf{Rel} \to 2^{\mathcal{U}\times\mathcal{U}}$ with $(\mathcal{U}, \{\varphi(r) \mid r \in \mathsf{Rel}\})$ being a weak partition scheme. The map φ is extended to arbitrary relations by setting $\varphi(R) = \bigcup_{r\in R} \varphi(r)$ for every $R \subseteq \mathsf{Rel}$.
- The *converse operation* $^{\smile}$ is a map $^{\smile} : \mathsf{Rel} \to 2^{\mathsf{Rel}}$ that satisfies

$$\varphi(r^{\smile}) \supseteq \varphi(r)^{\smile} \tag{1}$$

  for every $r \in \mathsf{Rel}$. The operation $^{\smile}$ is extended to arbitrary relations by setting $R^{\smile} = \bigcup_{r\in R} r^{\smile}$ for every $R \subseteq \mathsf{Rel}$.
- The *composition operation* $\diamond$ is a map $\diamond : \mathsf{Rel} \times \mathsf{Rel} \to 2^{\mathsf{Rel}}$ that satisfies

$$\varphi(r \diamond s) \supseteq \varphi(r) \circ \varphi(s) \tag{2}$$

  for all $r, s \in \mathsf{Rel}$. The operation $\diamond$ is extended to arbitrary relations by setting $R \diamond S = \bigcup_{r\in R} \bigcup_{s\in S} r \diamond s$ for every $R, S \subseteq \mathsf{Rel}$.

We call Properties (1) and (2) *abstract converse* and *abstract composition*, following Ligozat's naming [18]. Our notion of a qualitative calculus makes weaker requirements on the converse operation than Ligozat and Renz's notions of a weak representation [18,20]. We have already discussed a rationale behind choosing these "weaker than weak" variants and will name another one in Section 2.2.

On the other hand, our notion makes stronger requirements on the converse than Nebel and Scivos's notion of a constraint algebra [31]. The following definition gives the stronger variants of converse and composition existing in the literature.

Definition 3. Let $C = (\mathsf{Rel}, \mathsf{Int}, \breve{}, \diamond)$ be a qualitative calculus.
C has *weak converse* if, for all $r \in \mathsf{Rel}$:

$$r^{\smile} = \bigcap \{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(r)^{\smile}\} \tag{3}$$

C has *strong converse* if, for all $r \in \mathsf{Rel}$:

$$\varphi(r^{\smile}) = \varphi(r)^{\smile} \tag{4}$$

C has *weak composition* if, for all $r, s \in \mathsf{Rel}$:

$$r \diamond s = \bigcap \{T \subseteq \mathsf{Rel} \mid \varphi(T) \supseteq \varphi(r) \circ \varphi(s)\} \tag{5}$$

C has *strong composition* if, for all $r, s \in \mathsf{Rel}$:

$$\varphi(r \diamond s) = \varphi(r) \circ \varphi(s) \tag{6}$$

The following fact captures that Properties (1)–(6) immediately carry over to arbitrary relations; the straightforward proof is given in [8]. It has consequences for efficient spatio-temporal reasoning, which are explained in Section 2.2.

Fact 4. *Given a qualitative calculus* $(\mathsf{Rel}, \mathsf{Int}, \breve{}, \diamond)$ *and relations* $R, S \subseteq \mathsf{Rel}$*, the following hold:*

$$\varphi(R^{\smile}) \supseteq \varphi(R)^{\smile} \tag{7}$$
$$\varphi(R \diamond S) \supseteq \varphi(R) \diamond \varphi(S) \tag{8}$$

If C has weak converse, then, for all $R \subseteq \mathsf{Rel}$*:*

$$R^{\smile} = \bigcap \{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(R)^{\smile}\} \tag{9}$$

If C has strong converse, then, for all $R \subseteq \mathsf{Rel}$*:*

$$\varphi(R^{\smile}) = \varphi(R)^{\smile} \tag{10}$$

If C has weak composition, then, for all $R, S \subseteq \mathsf{Rel}$*:*

$$R \diamond S = \bigcap \{T \subseteq \mathsf{Rel} \mid \varphi(T) \supseteq \varphi(R) \circ \varphi(S)\} \tag{11}$$

If C has strong composition, then, for all $R, S \subseteq \mathsf{Rel}$*:*

$$\varphi(R \diamond S) = \varphi(R) \circ \varphi(S) \tag{12}$$

Since base relations are non-empty and JEPD, we have

Fact 5. *For any qualitative calculus, φ is injective.*

Comparing Definitions 1–3 with the basic notions of a qualitative calculus in [20], a *weak representation* is a calculus with identity relation, strong converse and abstract composition. Our basic notion of a qualitative calculus is more general: it does not require an identity relation, and it only requires abstract converse and composition. Conversely, [20] are slightly more general than we are, because the map φ need not be injective. However, this extra generality is not very meaningful: if base relations are JEPD, φ could only be non-injective in giving multiple names to the empty relation. Furthermore, in [20], a *representation* is a weak representation with strong composition and an injective map φ.

2.2 Spatio-temporal Reasoning

The most important flavor of spatio-temporal reasoning is constraint-based reasoning. Like with a classical constraint satisfaction problem (CSP), we are given a set of variables and constraints. The task of constraint satisfaction is to decide whether there exists a valuation of all variables that satisfies the constraints. In calculi for spatio-temporal reasoning, variables range over the specific spatial or temporal domain of a qualitative representation. The relations defined by the calculus serve as constraint relations. More formally, we have:

Definition 6 (QCSP). Let $(\mathsf{Rel}, \mathsf{Int}, \breve{\ }, \diamond)$ be a binary qualitative calculus with $\mathsf{Int} = (\mathcal{U}, \varphi)$ and let X be a set of variables ranging over $\mathcal{U}$. A *qualitative constraint* is a formula $x_i\, R_j\, x_k$ with variables $x_i, x_k \in X$ and relation $R_j \in \mathsf{Rel}$. We say that a valuation $\psi : X \to \mathcal{U}$ *satisfies* $x_i\, R_j\, x_k$ if $(\psi(x_i), \psi(x_k)) \in \phi(R_j)$ holds.

A *qualitative constraint satisfaction problem* (QCSP) is the task to decide whether there is a valuation ψ for a set of variables satisfying a set of constraints.

In the following we use X to refer to the set of variables and $r_{x,y}$ stands for the constraint relation between variables x and y. For simplicity and wlog. it is assumed that for every pair of variables exactly one constraint relation is given.

Several techniques originally developed for finite domain CSP can be adapted to spatio-temporal QCSPs. Since deciding CSP instances is already NP-complete for search problems with finite domains, heuristics are important. One particular valuable technique is constraint propagation which aims at making implicit constraints explicit in order to identify variable assignments that would violate some constraint. By pruning away these variable assignments, a consistent valuation can be searched more efficiently. A common approach is to enforce k-consistency.

Definition 7. A CSP is k-consistent if for all subsets of variables $X' \subset X$ with $|X'| = k - 1$ we can extend any valuation of X' that satisfies the constraints to a valuation of $X' \cup \{x\}$ also satisfying the constraints, where $x \in X \setminus X'$ is any additional variable.

QCSPs are naturally 1-consistent as the domains are infinite and there are no unary constraints. A QCSP is 2-consistent if $r_{x,y} = r_{y,x}{}^{\smile}$ and $r_{x,y} \neq \emptyset$ as relations are typically serial. A 3-consistent QCSP is also called *path-consistent* and Definition 7 can be rewritten using compositions as

$$r_{x,y} \subseteq \bigcap_{z \in X} r_{x,z} \circ r_{z,y} \tag{13}$$

and we can enforce the 3-consistency by iterating the refinement operation

$$r_{x,y} \leftarrow r_{x,y} \cap r_{x,z} \circ r_{z,y} \tag{14}$$

for all variables $x, y, z \in X$ until a fix point is reached. This procedure is known as the path-consistency algorithm [5]. For finite constraint networks the algorithm always terminates since the refinement operation is monotone and there are only finitely many relations.

If a qualitative calculus does not provide strong composition, iterating Equation (14) is not possible as it would lead to relations not contained in Rel. It is however straightforward to weaken Equation (14) using weak composition.

$$r_{x,y} \leftarrow r_{x,y} \cap r_{x,z} \diamond r_{z,y} \tag{15}$$

This procedure is called enforcing *algebraic closure* or *a-closure* for short. The reason why, in Definition 2, we require composition to be at least abstract is that the underlying inequality guarantees that reasoning via a-closure is sound.

Enforcing k-consistency or algebraic closure does not change the solutions of a CSP, as only impossible valuations are removed. If during application of Equation (15) an empty relation occurs, the QCSP is thus known to be inconsistent. By contrast, an algebraically closed QCSP may not be consistent though. However, for several qualitative calculi (or at least sub-algebras thereof) algebraic closure and consistency coincide.

Though we speak about composition in the following two paragraphs, the same statements hold for converse.

Fact 4 has the consequence that the composition operation of a calculus is uniquely determined if the composition of each pair of base relations is given. This information is usually stored in a table, the *composition table*. Then, computing the composition of two arbitrary relations is just a matter of table lookups which allows algebraic closure to be enforced efficiently. Speaking in terms of composition tables, abstract composition implies that each cell corresponding to $r \diamond s$ contains *at least* those base relations t whose interpretation intersects with $\varphi(r) \circ \varphi(s)$. In addition, weak composition implies that each cell contains *exactly* those t. If composition is strong, then Rel and φ even have to ensure that whenever $\varphi(t)$ intersects with $\varphi(r) \circ \varphi(s)$, it is contained in $\varphi(r) \circ \varphi(s)$ – i.e., the composition of the interpretation of any two base relations has to be the union of interpretations of certain base relations.

2.3 Existing Qualitative Spatio-temporal Representations

This paper is concerned with properties of binary spatio-temporal calculi that are described in the literature and implemented in the spatial representation and reasoning tool SparQ [44,43]. Table 1 lists these calculi.

Table 1. Overview of the binary calculi tested

Name	Ref.	Domain	#BR	RM
9-Intersection	[9]	simple 2D regions	8	I [12,16]
Allen's interval relations	[1]	intervals (order)	13	A [42]
Block Algebra	[2]	n-dimensional blocks	13^n	A [2]
Cardinal Dir. Calculus CDC	[10,17]	directions (point abstr.)	9	A [17]
Cardinal Dir. Relations CDR	[38]	regions	218	P
CycOrd, binary $\mathrm{CYC_b}$	[14]	oriented lines	4	U
Dependency Calculus	[33]	points (partial order)	5	A [33]
Dipole Calculus[a] $\mathrm{DRA_f}$	[25,24]	directions from line segm.	72	I [46]
$\mathrm{DRA_{fp}}$	[24]	directions from line segm.	80	I
DRA-connectivity	[45]	connectivity of line segm.	7	U
Geometric Orientation	[7]	relative orientation	4	U
INDU	[32]	intervals (order, rel. dur.n)	25	P
OPRA_m, $m = 1, \ldots, 8$ (Oriented Point Rel. Algebra)	[23,28]	oriented points	$4m \cdot (4m+1)$	I [46]
Point Calculus	[42]	points (total order)	3	A [42]
Qualitat. Traject. Calc. $\mathrm{QTC_{B11}}$	[40,41]	moving point obj.s in 1D	9	U
$\mathrm{QTC_{B12}}$	"	"	17	U
$\mathrm{QTC_{B21}}$	"	moving point obj.s in 2D	9	U
$\mathrm{QTC_{B22}}$	"	"	27	U
$\mathrm{QTC_{C12}}$	"	"	81	U
$\mathrm{QTC_{C22}}$	"	"	305	U
Region Connection Calc. RCC-5	[34]	regions	5	A [15]
RCC-8	[34]	regions	8	A [35]
Rectangular Cardinal Rel.s RDR	[30]	regions	36	A [30]
Star Algebra $\mathrm{STAR_4}$	[36]	directions from a point	9	P

[a]Variant $\mathrm{DRA_c}$ is not based on a weak partition scheme – JEPD is violated [24].

#BR: number of base relations
RM: reasoning method used to decide consistency of CSPs with base relns only:
A-closure; **P**olynomial: reducible to linear programming;
Intractable (assuming P $\neq$ NP); **U**nknown

3 Relation Algebras

3.1 Definition

If we focus our attention on spatio-temporal calculi with binary relations, it is reasonable to ask whether they are relation algebras (RAs). If a calculus is a RA, it is guaranteed to have properties that allow several optimizations in constraint reasoners. For example, associativity of the composition operation $\diamond$ ensures that, if the reasoner encounters a path $ArBsCtD$ of length 3, then the relation between A and D can be computed "from left to right". Without associativity, $(r \diamond s) \diamond t$ as well as $r \diamond (s \diamond t)$ would have to be computed. RAs have been considered in the literature for spatio-temporal calculi [20,6,26].

An (abstract) RA is defined in [22]; here we use the symbols $\cup$, $\diamond$, and id instead of $+$, ;, and $1'$. Let A be a set containing id and 1, and let $\cup$, $\diamond$ be binary

Table 2. Axioms for relation algebras and weaker variants [22]

R_1	$r \cup s = s \cup r$		$\cup$-commutativity
R_2	$r \cup (s \cup t) = (r \cup s) \cup t$		$\cup$-associativity
R_3	$\overline{\bar{r} \cup \bar{s}} \cup \overline{\bar{r} \cup s} = r$		Huntington's axiom
R_4	$r \diamond (s \diamond t) = (r \diamond s) \diamond t$		$\diamond$-associativity
R_5	$(r \cup s) \diamond t = (r \diamond t) \cup (s \diamond t)$		$\diamond$-distributivity
R_6	$r \diamond \mathsf{id} = r$		identity law
R_7	$(r^{\smile})^{\smile} = r$		$^{\smile}$-involution
R_8	$(r \cup s)^{\smile} = r^{\smile} \cup s^{\smile}$		$^{\smile}$-distributivity
R_9	$(r \diamond s)^{\smile} = s^{\smile} \diamond r^{\smile}$		$^{\smile}$-involutive distributivity
R_{10}	$r^{\smile} \diamond \overline{r \diamond s} \cup \bar{s} = \bar{s}$		Tarski/de Morgan axiom
WA	$((r \cap \mathsf{id}) \diamond 1) \diamond 1 = (r \cap \mathsf{id}) \diamond 1$		weak $\diamond$-associativity
SA	$(r \diamond 1) \diamond 1 = r \diamond 1$		$\diamond$ semi-associativity
R_{6l}	$\mathsf{id} \diamond r = r$		left-identity law
PL	$(r \diamond s) \cap t^{\smile} = \emptyset \Leftrightarrow (s \diamond t) \cap r^{\smile} = \emptyset$		Peircean law

and $^{-}$, $^{\smile}$ unary operations on A. The relevant axioms (R_1–R_{10}, WA, SA, and PL) are given in Table 2. All axioms except PL can be weakened to only one of two inclusions, which we denote by a superscript $\supseteq$ or $\subseteq$. For example, $\mathsf{R}_7^{\supseteq}$ denotes $(r^{\smile})^{\smile} \supseteq r$. Likewise, we use $\mathsf{PL}^{\Rightarrow}$ and $\mathsf{PL}^{\Leftarrow}$. Then, $\mathfrak{A} = (A, \cup, ^{-}, \diamond, ^{\smile}, \mathsf{id})$ is a

– *non-associative relation algebra (NA)* if it satisfies Axioms R_1–R_3, R_5–R_{10};
– *semi-associative relation algebra (SA)* if it is an NA and satisfies Axiom SA,
– *weakly associative relation algebra (WA)* if it is an NA and satisfies WA,
– *relation algebra (RA)* if it satisfies R_1–R_{10},

for all $r, s, t \in A$. Every RA is a WA; every WA is an SA; every SA is an NA.

In the literature, a different axiomatization is sometimes used, for example in [20]. The most prominent difference is that R_{10} is replaced by PL, "a more intuitive and useful form, known as the Peircean law or De Morgan's Theorem K" [13]. It is shown in [13, Section 3.3.2] that, given R_1–R_3, R_5, R_7–R_9, the axioms R_{10} and PL are equivalent. The implication $\mathsf{PL} \Rightarrow \mathsf{R}_{10}$ does not need R_5 and R_8.

Furthermore, Table 2 contains the redundant axiom R_{6l} because it may be satisfied when some of the other axioms are violated. It is straightforward to establish that R_6 and R_{6l} are equivalent given R_7 and R_9, see [8].

Due to our minimal requirements to a qualitative calculus given in Def. 2, certain axioms are always satisfied; see [8] for a proof of the following

Fact 8. *Every qualitative calculus satisfies* R_1–R_3, R_5, $\mathsf{R}_7^{\supseteq}$, R_8, $\mathsf{WA}^{\supseteq}$, $\mathsf{SA}^{\supseteq}$ *for all (base and complex) relations. This axiom set is maximal: each of the remaining axioms in Table 2 is not satisfied by some qualitative calculus.*

3.2 Discussion of the Axioms

We will now discuss the relevance of the above axioms for spatio-temporal representation and reasoning. Due to Fact 8, we only need to consider axioms R_4, R_6, R_7, R_9, R_{10} (or PL) and their weakenings R_{6l}, SA, WA.

R_4 (and SA, WA). Axiom R_4 is helpful for modeling. It allows for writing chains of compositions without parentheses, which have an unambiguous meaning. For example, consider the following statement in natural language about the relative length and location of two intervals A and D. *Interval A is before some equally long interval that is contained in some longer interval that meets the shorter D.* This statement is just a conjunction of relations between A, the unnamed intermediary intervals B, C, and D. When we evaluate it, it intuitively does not matter whether we give priority to the composition of the relations between A, B and B, C or to the composition of the relations between B, C and C, D.

However, INDU does not satisfy Axiom R_4 and, therefore, here the two ways of parenthesizing the above statement lead to different relations between A and D. This behavior is sometimes attributed to the absence of strong composition, which we will refute in Section 4. Conversely, strong composition implies R_4 since composition of binary relations over $\mathcal{U}$ is associative:

Fact 9. *Let $C = (\mathsf{Rel}, \mathsf{Int}, \breve{}, \diamond)$ be a qualitative calculus with strong composition. Then C satisfies R_4.*

Note that INDU still satisfies the weakenings SA and WA of R_4, and we already know from Fact 8 that the inequalities $\mathsf{SA}^{\supseteq}$ and $\mathsf{WA}^{\supseteq}$ are always satisfied.

Furthermore, Axiom R_4 is useful for optimizing reasoning algorithms: suppose a scenario that contains the constraints $\{W r X, X s Y, Y t Z, W r' Z\}$ with variables W, X, Y, Z needs to be checked for consistency. If *one* of the inclusions $\mathsf{R}_4^{\supseteq}$ and $\mathsf{R}_4^{\subseteq}$ is satisfied – say, $r \diamond (s \diamond t) \subseteq (r \diamond s) \diamond t$ – then it suffices to compute the "finer" composition result $r \diamond (s \diamond t)$ and check whether it contains r'. Otherwise, both results have to be computed and checked for containment of r'.

R_6 and R_{6l}. Axioms R_6 and R_{6l} do not seem to play a significant role in (optimizing) satisfiability checking, but the presence of an id relation is needed for the standard reduction from the correspondence problem to satisfiability: to test whether a constraint system admits the equality of two variables x, y, one can add an id-constraint between x, y and test the extended system for satisfiability.

Furthermore, the absence of an id relation may lead to an earlier loss of precision. For example, assume two variants of the 1D Point Calculus [42]: $\mathrm{PC}_=$ with the relations *less than* ($<$), *equal* ($=$), and *greater than* ($>$), interpreted as the natural relations $<, =, >$ over the domain of the reals, and its approximation $\mathrm{PC}_\approx$ with the relations *less than* ($<$), *approximately equal* ($\approx$), and *greater than* ($>$), where $\approx$ is interpreted as the set of pairs of points whose distance is below a certain threshold. Then, $=$ is the id-relation of $\mathrm{PC}_=$ and $= \diamond =$ results in $\{=\}$, whereas $\mathrm{PC}_\approx$ has no id-relation and $\approx \diamond \approx$ results in the universal relation.

R_7 and R_9. These axioms allow for certain optimizations in decision procedures for satisfiability based on algebraic operations like algebraic closure. If R_7 holds, the reasoning system does not need to store both constraints $A\,r\,B$ and $B\,r'\,A$, since r' can be reconstructed as $r^{\smile}$ if needed. Similarly, R_9 grants that, when enforcing algebraic closure by using Equation (15) to refine constraints between variable A and B, it is sufficient to compute composition once and, after applying converse, reuse it to refine the constraint between B and A too.

Current reasoning algorithms and their implementations use the described optimizations; they produce incorrect results for calculi violating R_7 or R_9.

R_{10} and PL. These axioms reflect that the relation symbols of a calculus indeed represent binary relations, i.e., pairs of elements of a universe. This can be explained from two different points of view.

1. If binary relations are considered as sets, R_{10} is equivalent to $r^{\smile} \diamond \overline{r \diamond s} \subseteq \bar{s}$. If we further assume the usual set-theoretic interpretation of the composition of two relations, the above inclusion reads as: *For any X, Y, if $Z\,r\,X$ for some Z and, $Z\,r\,U$ implies not $U\,s\,Y$ for any U, then not $X\,s\,Y$.* This is certainly true because X is one such U.
2. Under the same assumptions, each side of PL says (in a different order) that there can be no triangle $X\,r\,Y, Y\,s\,Z, Z\,t\,X$. The equality then means that the "reading direction" does not matter, see also [6]. This allows for reducing nondeterminism in the a-closure procedure, as well as for efficient refinement and enumeration of consistent scenarios.

3.3 Prerequisites for Being a Relation Algebra

The following correspondence between properties of a calculus and notions of a relation algebra is due to Ligozat and Renz [20].

Proposition 10. *Every calculus C based on a partition scheme is an NA. If, in addition, the interpretations of the base relations are serial, then C is an SA.*

Furthermore, R_7 is equivalent to the requirement that a calculus has strong converse. This is captured by the following lemma.

Lemma 11. *Let $C = (\mathsf{Rel}, \mathsf{Int}, {}^{\smile}, \diamond)$ be a qualitative calculus. Then the following properties are equivalent.*

1. *C has strong converse.*
2. *Axiom R_7 is satisfied for all base relations $r \in \mathsf{Rel}$.*
3. *Axiom R_7 is satisfied for all relations $R \subseteq \mathsf{Rel}$.*

Proof. Items (2) and (3) are equivalent due to distributivity of ${}^{\smile}$ over $\cup$, which is introduced with the cases for non-base relations in Definition 2.

For "(1) $\Rightarrow$ (2)", the following chain of equalities, for any $r \in \mathsf{Rel}$, is due to C having strong converse: $\varphi(r^{\smile\smile}) = \varphi(r^{\smile})^{\smile} = \varphi(r)^{\smile\smile} = \varphi(r)$. Since Rel is based on JEPD relations and φ is injective, this implies that $r^{\smile\smile} = r$.

For "(2) $\Rightarrow$ (1)", we show the contrapositive. Assume that C does not have strong converse. Then $\varphi(r^{\smile}) \supsetneq \varphi(r)^{\smile}$, for some $r \in \mathsf{Rel}$; hence $\varphi(r^{\smile})^{\smile} \supsetneq \varphi(r)^{\smile\smile}$. We can now modify the above chain of equalities replacing the first two equalities with inequalities, the first of which is due to Requirement (1) in the definition of the converse (Def. 2): $\varphi(r^{\smile\smile}) \supseteq \varphi(r^{\smile})^{\smile} \supsetneq \varphi(r)^{\smile\smile} = \varphi(r)$. Since $\varphi(r^{\smile\smile}) \neq \varphi(r)$, we have that $r^{\smile\smile} \neq r$. $\square$

4 Algebraic Properties of Existing Calculi

In this section, we report on tests for algebraic properties we have performed on spatio-temporal calculi. We want to answer the following questions. *(1) Which existing calculi correspond to relation algebras? (2) Which weaker notions of relation algebras correspond to calculi that do not fall under (1)?*

We examined the corpus of the 31 calculi[1] listed in Table 1. This selection is restricted to calculi with (a) binary relations – because the notion of a relation algebra is best understood for binary relations – and (b) an existing implementation in SparQ.

To answer Questions (1) and (2), we use the axioms for relation algebras listed in Table 2 using both the heterogeneous tool set HETS [27] and SparQ. Due to Fact 8, it suffices to test Axioms R_4, R_6, R_7, R_9, R_{10} (or PL) and, if necessary, the weakenings SA, WA, and R_{6l}. The weakenings are relevant to capture weaker notions such as semi-associative or weakly associative algebras, or algebras that violate either R_6 or some of the axioms that imply the equivalence of R_6 and R_{6l}. Because all axioms except R_{10} contain only operations that distribute over the union $\cup$, it suffices to test them for base relations only. Therefore, we have written a CASL specification of R_4, R_6, R_7, R_9, PL, SA, WA, and R_{6l}, and used a HETS parser that reads the definitions of the above listed calculi in SparQ to test them against our CASL specification. In addition, we have tested all definitions against R_4, R_6, R_7, R_9, PL, and R_{6l} using SparQ's built-in function `analyze-calculus`.

A part of the calculi have already been tested by Florian Mossakowski [26], using a different CASL specification based on an equivalent axiomatization from [20]. He comprehensively reports on the outcome of these tests, and on repairs made to the composition table where possible.

The results of our and Mossakowski's tests are summarized in Table 3; details are listed in [8]. With the exceptions of QTC, Cardinal Direction Relations (CDR) and Rectangular Direction Relations (RDR), all tested calculi are at least semi-associative relation algebras; most of them are even relation algebras. Hence, these calculi enjoy the advantages for representation and reasoning optimizations discussed in Section 3.2. In particular, current reasoning procedures, which already implement the optimizations described for R_7 and R_9, yield correct results for these calculi, and they could be optimized further by implementing the optimizations described for R_4, R_{10}, and PL.

The three groups of calculi that are SAs but not RAs are the Dipole Calculus variants $\mathrm{DRA_f}$ (variants $\mathrm{DRA_{fp}}$ and DRA-connectivity are even RAs!), as well as INDU and OPRA_m for $m = 1, \ldots, 8$. These calculi do not even satisfy one of the inclusions $\mathsf{R}_4^{\supseteq}$ and $\mathsf{R}_4^{\subseteq}$, which implies that the reasoning optimizations described in Section 3.2 for Axiom R_4 cannot be applied, but this is the only disadvantage of these calculi over the others. Our observations suggest that the meaning of the letter combination "RA" in the abbreviations "DRA" and "OPRA" should stand for "Reasoning Algebra", not for "Relation Algebra".

[1] For the parametrized calculi DRA, OPRA, QTC, we count every variant separately.

Table 3. Overview of calculi tested and their properties. The symbol "✓" means that the axiom is satisfied; otherwise the percentage of counterexamples (relations, pairs or triples violating the axiom) is given

Calculus	Tests[a]	R_4	SA	WA	R_6	R_{6l}	R_7	R_9	PL	R_{10}
Allen	MHS	✓	✓	✓	✓	✓	✓	✓	✓	✓
Block Algebra	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
Cardinal Direction *Calculus*	MHS	✓	✓	✓	✓	✓	✓	✓	✓	✓
CYC_b, Geometric Orientation	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
DRA_{fp}, DRA-conn.	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
Point Calculus	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
RCC-5, Dependency Calc.	MHS	✓	✓	✓	✓	✓	✓	✓	✓	✓
RCC-8, 9-Intersection	MHS	✓	✓	✓	✓	✓	✓	✓	✓	✓
$STAR_4$	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
DRA_f	MHS	19	✓	✓	✓	✓	✓	✓	✓	✓
INDU	MHS	12	✓	✓	✓	✓	✓	✓	✓	✓
$OPRA_n$, $n \leqslant 8$	MHS	21–91[b]	✓	✓	✓	✓	✓	✓	✓	✓
QTC_{Bxx}	MHS	✓	✓	✓	89–100		✓	✓	✓	✓
QTC_{C21}	HS	55	✓	✓	99	99	✓	2	<1	1
QTC_{C22}	HS	79	✓	✓	99	99	✓	3	<1	1
Rectang. Direction Relations	HS	✓	✓	✓	97	92	89	66	7	52
Cardinal Direction *Relations*	HS	28	17	✓	99	99	98	12	<1	88

[a]calculus was tested by: M = [26], H = HETS, S = SparQ

[b]21%, 69%, 78%, 83%, 86%, 88%, 90%, 91% for $OPRA_n$, $n = 1, \ldots, 8$

In principle, it cannot be completely ruled out that associativity is reported to be violated due to errors in either the implementation of the respective calculus or the experimental setup. This even applies to non-violations, although it is much more likely that errors cause sporadic violations than systematic non-violations. In the case of DRA_f, INDU and $OPRA_m$, $m = 1, \ldots, 8$, the relatively high percentage of violations make implementation errors seem unlikely to be the cause. However, to obtain certainty that these calculi indeed violate R_4, one has to find concrete counterexamples and verify them using the original definition of the respective calculus. For DRA_f and INDU, this has been done in the literature [24,3]. Interestingly, the violation of associativity has been attributed to the absence of strong converse and strong composition, respectively. We remark, however, that the latter cannot be responsible because, for example, DRA_{fp} has an associative, but only weak, composition operation. While DRA_{fp} has been proven to be associative due to strong composition in [24], for $OPRA_m$, it can be shown that *none* of the variants for any m are associative (see [29]).

The B-variants of QTC violate only the identity law R_6 and R_{6l}. As observed in [26], it is possible to equip them with a new id relation, modify the interpretation of the other relations such that they become JEPD, and adapt the converse and composition table accordingly. The thus modified calculi are then relation algebras.

The C-variants of QTC additionally violate R_4, R_9, R_{10}, and PL. We call the corresponding notion of algebra semi-associative Boolean algebra with converse-involution. As a consequence, most of the reasoning optimizations described in

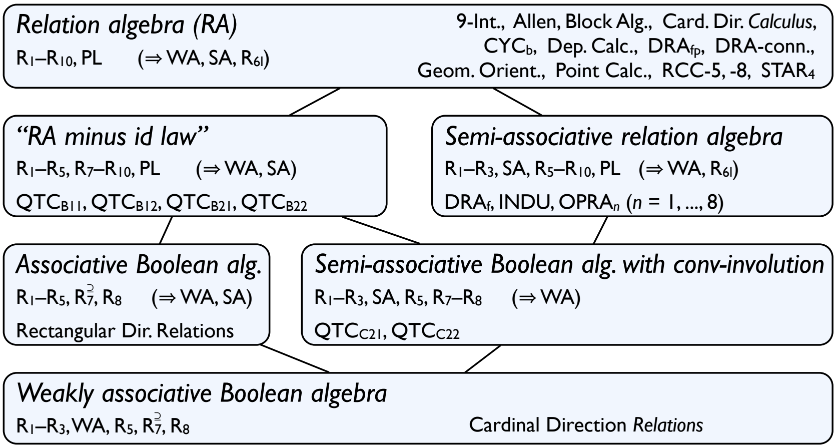

Fig. 1. Overview of algebra notions and calculi tested

Section 3.2 cannot be applied to the C-variants of QTC; hence, reasoning with these calculi is expected to be less efficient than with the calculi described so far. It is possible that the noticeably few violations of R_9, R_{10}, and PL are due to errors in the composition table; the non-trivial verification is part of future work.

Cardinal Direction Relations and Rectangular Direction Relations are the only calculi with weak converse that we have tested. The former satisfies only WA in addition to the axioms that are always satisfied by a Boolean algebra with distributivity. We call the corresponding notion of algebra weakly associative Boolean algebra. Hence, this calculus enjoys none of the advantages for representation and reasoning discussed in Section 3.2. Similarly to the C-variants of QTC, the relatively small number of violations of PL may be due to errors in the implementation. Rectangular Direction Relations additionally satisfies R_4 and therefore corresponds to what we call an associative Boolean algebra. Since both calculi satisfy neither R_7 nor R_9, current reasoning algorithms and their implementations yield incorrect results for them, as seen in Section 3.2.

An overview of the algebra notions identified is given in Figure 1.

When making use of algebraic closure as inference mechanism it is essential to acknowledge that some axiom violations require special procedures in order to compute algebraic closure. Our analysis reveals that there indeed exist calculi that do not meet axioms that have been taken for granted. For example, the current version of GQR can fail to compute algebraic closure correctly for calculi that violate R_9. In Algorithm 1 we present a universal algorithm to compute algebraic closure. For clarity and brevity of the presentation we stick to the well-known but simple control structure of PC-1. A real implementation would use an advanced control structure to avoid unnecessary invocations of the REVISE function, i.e., to use at least PC-2 [21]. Conformance with R_7 allows CSP storage to be restricted (flag s in the algorithm), while violation of R_9 requires two computations for the refinement operation Eq. 15, namely $C_{i,j} \diamond C_{j,k}$ and

Algorithm 1. Universal algebraic closure algorithm

```
function LOOKUP((C, i, j, s))
    if s ∨ (i < j) then
        return C_{i,j}
    else
        return C_{j,i}˘
end function

function REVISE((C, i, j, k, s))
    u ← false                    ▷ update flag to signal whether relation was updated
    r ← C_{i,j} ∩ LOOKUP(C, i, k, s) ◇ LOOKUP(C, k, j, s)
    if s ∨ R_9 does not hold then
        r′ ← LOOKUP(C, j, i, s) ∩ (LOOKUP(C, j, k, s) ◇ LOOKUP(C, k, i, s))
        r ← r ∩ r′˘
        r′ ← r′ ∩ r˘
        if r′ ≠ C_{j,i} then
            assert r′ ≠ ∅                    ▷ stop if inconsistency is detected
            u ← true
            C_{j,i} ← r′
    if r ≠ C_{i,j} then
        assert r ≠ ∅                         ▷ stop if inconsistency is detected
        u ← true
        C_{i,j} ← r
    return (C, u)
end function

function A-CLOSURE((n, {x_1 r_1 y_1, ..., x_m r_m y_m}))
    if R_7 does not hold then
        s ← true                  ▷ without R_7 we must store converse relations
        C_{i,j} ← 𝒰, i = 1, ..., n, j = 1, ..., n
    else
        s ← false ▷ for small calculi/CSPs storing converses may be more efficient
        C_{i,j} ← 𝒰, i = 1, ..., n, j = i + 1, ..., n   ▷ use triangular matrix storage
    C_{i,i} ← id, i = 1, ..., n
    for i = 1, ..., m do
        x ← x_i, r ← r_i, y ← y_i              ▷ process constraint x_i r_i y_i
        if ¬s ∧ (x > y) then
            (x, y) ← (y, x), r ← r˘        ▷ only write into upper half of matrix
        C_{x,y} ← C_{x,y} ∩ r
        assert (x = y) → (id ∈ C_{x,y})
    end for
    update ← true
    while update do
        update ← false
        for i = 1, ..., n, j = i+1, ..., n, k = 1, ... n, k ≠ i, k ≠ j do
            (u, C) ← REVISE(C, i, j, k, s)
            update ← update ∨ u
        end for
    return C                                   ▷ fix point reached
end function
```

$(C_{j,k}{}^{\smile} \diamond C_{i,j}{}^{\smile})^{\smile}$ (lines 10–17). R_4 and R_{10} are not used by the algorithm, since this would complicate the algorithm unduly.

5 A Quantitative Account of Qualitative Calculi

In this section, we report on computational properties of specific calculi which are beyond the computational complexity of constraint-based reasoning. For example, one might be interested to know how many relations are typically sufficient to describe a scene of n objects unequivocally or with a specific residual uncertainty. To this end, we developed two empirical measures that characterize certain aspects of qualitative calculi that are arguably relevant to applications. We want to answer two questions: (1) How well do calculi with many relations make use of the usually higher information content? (2) Does information content differ significantly between the six classes of calculi established in Section 4?

The first measure we consider is *information content* of the composition operation. Our motivation is to estimate how much additional information can be gained by applying a composition operation. This allows us to estimate whether, for example, having observed relations $r(A,B)$ and $r'(B,C)$ in a scene, it is worthwhile to observe $r''(A,C)$ too, as it may be improbable to derive r'' by composition ($r \circ r'$). To obtain more general results we consider sequences of compositions $r \circ r' \circ r'' \circ \ldots$ for several lengths. We define the information content I of a relation $R \subseteq \mathsf{Rel}$ to be

$$I(R) = 1 - \frac{|R|}{|\mathsf{Rel}|} \tag{16}$$

where $|\mathsf{Rel}|$ denotes the number of base relations of the calculus, and $|R|$ the number of base relations R consists of. In case of the universal relation this results in $I(U) = 0$ as nothing is known, $I(r) = 1 - \frac{1}{|\mathsf{Rel}|}$ for all base relations $r \in \mathsf{Rel}$, and $I(\emptyset) = 1$. Obviously, the more base relations a calculus involves, the higher the information content *can be* for base relations. Therefore, we define

$$I_C^k = \frac{\sum_{R \in r^k} I(R)}{|\mathsf{Rel}|^{k+1}} \tag{17}$$

for a calculus C with $r^k = \{r^0 \circ \ldots \circ r^k | r^0, \ldots, r^k \in \mathsf{Rel}\}$ to be the average information content after k composition operations, i.e., how restrictive relations are on average after information propagation with composition. In particular, I_C^k is 1 minus the average proportion of base relations in any cell in the composition table. For example, for QTC_{C22} ($|\mathsf{Rel}| = 209$) or the Cardinal Direction Relations (CDR) ($|\mathsf{Rel}| = 218$) $I^0 \approx 1$, whereas for the Point Calculus with three base relations $I_{PC}^0 \approx 0.67$. We apply an iterative method to derive the values of I_C^k that constructs r^k for $k = 0, 1, \ldots$ rather than looping across combinations of base relations. Despite the potentially exponential size of r^k, the calculation remains feasible in many cases. Only for OPRA_m with $m \geq 3$ and some QTC variants we were not able to derive values for higher k in reasonable time.

For the other calculi, computation was terminated after 14 compositions or if I_C^k drops below 0.5.

As a second measure we determine the average degree of overlap that occurs after k steps of composition for selected calculi. The degree of overlapping $O(R_i, R_j)$ is determined by counting the number of atomic relations shared by two relations, normalized by the total number of base relations:

$$O(R_i, R_j) = \frac{|R_i \cap R_j|}{|\mathsf{Rel}|} \tag{18}$$

For example, if two relations in a calculus with eight base relations share four base relations, the overlap is 0.5. This value indicates how the information content differs between dealing with base relations only versus dealing with arbitrary relations (and thus how the results on information content generalize to arbitrary relations). Similar to $I(R)$ and I_C^k, we define O_C^k to be the average overlap over all composition chains of length k.

$$O_C^k = \frac{\sum_{R_i, R_j \in r^k} O(R_i, R_j)}{|\mathsf{Rel}|^{k+1}} \tag{19}$$

The results of the two measures are summarized in Figure 2 and Table 4, showing information content versus length k of composition chains.

Figure 2 shows that the average information content for the Point Calculus after 1 step is ≈ 0.52 and additionally, the overlap of ≈ 0.33 is already quite high after a single composition. Therefore, in order to obtain detailed information it is reasonable to also observe r_{AC} between objects A and C even if r_{AB} and r_{BC} are already known. By contrast, the INDU calculus has a very high information

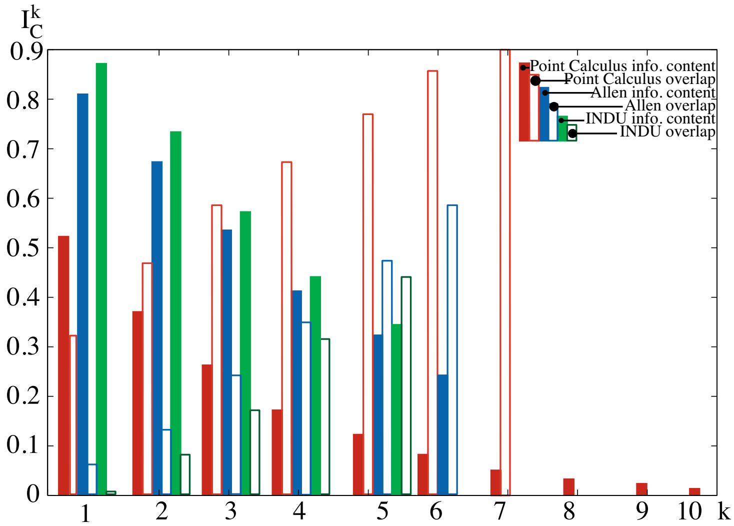

Fig. 2. Information content and overlap after k compositions for selected calculi

Table 4. Information content I_C^k for calculi in %

Calculus	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Allen	92.3	81.4	66.8	52.8	41.1	31.8	24.5	18.9	14.5	11.2	8.6	6.6	5.1	3.9	3.0
Block Algebra	99.4	96.5	89.0	77.7	65.3	53.4	43.0	34.1	27.0	21.1	16.4	12.8	9.9	7.7	5.9
CDC	88.9	76.8	60.4	44.5	31.6	21.9	14.9	10.1	6.8	4.6	3.1	2.0	1.4	0.9	0.6
CYC_b	75.0	62.5	46.9	32.8	21.9	14.1	8.8	5.4	3.2	1.9	1.1	0.6	0.4		
DRA_{fp}	98.8	89.9	69.0	45.0	25.8	13.4	6.5	3.0	1.3	0.6	0.2				
DRA-con	85.7	74.6	59.0	43.4	30.4	20.5	13.5	8.7	5.6	3.5	2.2	1.3	0.8	0.5	0.3
Point Calculus	66.7	51.9	37.0	25.5	17.3	11.6	7.8	5.2	3.5	2.3	1.5	1.0	0.7	0.5	
RCC-5	80.0	56.8	34.9	19.7	10.6	5.5	2.7	1.3	0.6	0.3					
RCC-8	87.5	62.3	38.0	21.1	11.0	5.5	2.6	1.2	0.6	0.3					
$STAR_4$	88.9	66.9	45.0	28.5	17.4	10.3	6.0	3.5	2.0	1.1	0.6	0.4			
DRA_f	98.6	90.6	70.4	46.3	26.7	13.9	6.7	3.0	1.3	0.6	0.2				
INDU	96.0	86.9	72.5	57.5	44.1	33.2	24.7	18.2	13.4	9.9	7.2	5.3	4.0	2.9	2.1
$OPRA_1$	95.0	82.0	55.8	30.8	14.5	6.2	2.4	0.9	0.3						
$OPRA_2$	98.6	90.3	64.1	32.9	13.0	4.3	1.3	0.3							
$OPRA_3$	99.4	93.1	71.4	40.2	16.7	5.6									
$OPRA_4$	99.6	94.6	76.7	48.0											
QTC_{B11}	88.9	90.0	93.2	95.8	97.5	98.6	99.1	99.5	99.7						
QTC_{B12}	94.1	91.2	90.5	91.3	92.8	94.2	95.6								
QTC_{B21}	88.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
QTC_{B22}	96.3	51.9	37.0	25.5	17.3	11.6	7.8	5.2	3.5	2.3	1.5	1.0	0.7	0.5	0.3
QTC_{C21}	98.8	92.5	76.6	68.6	69.5	73.0	76.5	79.4	81.8	83.7	85.2	86.4	87.4		
QTC_{C22}	99.5	95.1	78.0	69.3	51.2										
RDR	97.2	82.6	63.2	45.7	32.0	22.0	15.0	10.1	6.8	4.6	3.1	2.0	1.4	0.9	0.6
CDR	99.5	78.8	60.9	48.9	39.6	32.1	26.1	21.2	17.2	14.0	11.4	9.3	7.6	6.2	5.1

content (≈ 0.87) and a much smaller overlap. Therefore, it is not so informative to observe r_{AC} as a lot of information is preserved after a composition. It is clear that the O_C^k grows for increasing k as composition results become coarser step by step. Nevertheless, information loss for PC is much higher than compared to Allen and INDU calculus: I_{INDU}^5 and I_{Allen}^5 are close to I_{PC}^2 (O_C^k respectively).

Our results show that there is no evidence for a relation between the information content of a calculus and its classification as per Figure 1. The only exceptions are some of the QTC calculi as I_C^k starts to increase after some k with increasing k.

Although the calculi start with quite different values for I^0, most calculi have an information content less than 0.1 after six steps. The most notable exception is the Block Algebra where $I_{BA}^6 \approx 0.43$ and even after ten compositions it remains above 16%. Only Allen, INDU and CDR are somehow comparable. Concerning the classes we derived in Section 4 no uniform behavior can be observed. Thus, from a perspective of expressive power of calculi, there is no argument against working with calculi that are not relation algebras. We have to note that the comparison of the values for calculi where it is known that a-closure decides consistency and those where it does not (or is unknown) is problematic. The latter ones may contain relations which are not physically realizable and thus reduce the value of information content.

There are some interesting observations wrt. the various QTC variants. The QTC_{B1x}, QTC_{B21} and QTC_{C21} calculi behave differently from other calculi, whereas QTC_{B22} behaves 'normally', i.e., I^k increases, although it is very closely related to the other QTC variants. Interestingly, QTC_{B1x} and QTC_{C21} are the only calculi where I^k increases with growing k. From our perspective, the reason lies in the multimodal structure of the calculus. As it combines points with line segments, the composition table (CT) contains empty relations, since an object cannot be interpreted as a point and a line segment at the same time. Additionally, the CT contains only fairly small relations, i.e., with small $|R|$. For example, the CT of QTC_{B12} contains 29% empty relations, 29% atomic relations, and 42% other relations which have a maximal size of $|R| \leq 3$. The results for QTC_{B21} are not surprising as the composition table only contains the universal relation and thus, for all k, $I^k = 0.0$ and $O^k = 100.0$. For QTC_{C21} we observe that I^k decreases to 69.5% at step 4, but starts to increase for $k \geq 5$. We assume that this is also the case for QTC_{C22}, as it is a refinement of QTC_{C21}, but we were not able to calculate necessary values due to the high complexity. So far, we have no explanation for this decrease.

An additional observation is that PC and QTC_{B22} are similar with respect to information content, i.e., $I^k_{PC} \approx I^k_{QTC_{B22}}$ for $k \leq 14$. This congruence is interesting as the overlap values vary, the underlying partition scheme is different and the difference in base relations is significant (three for PC vs. 27 for QTC_{B22}). We leave the question of connections between these two calculi for future research.

6 Conclusion

We have looked at spatio-temporal representation and reasoning from an algebraic perspective, examining the implications of algebraic properties on modeling and reasoning algorithms, and testing these properties for a representative corpus of existing calculi. The resulting classification shows that calculi which have been described early in the literature tend to reside in the upper part of Figure 1; that is, they tend to have a rich algebraic structure. Few more recently developed calculi are based on generalizations and have a weaker structure. We have been able to conclude that common reasoning procedures are incorrect for the latter class of calculi, and have proposed a corrected universal a-closure algorithm that makes use of reasoning optimizations where they are allowed. Furthermore, we found that algebraic properties do not necessarily relate to how much information is preserved in successive reasoning steps.

An interesting and significant line of future work is to extend this study to ternary calculi, which requires an extension of binary relation algebras to ternary relations, see also [37].

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