Decentralized Robustification of Interconnected Time-Delay Systems Based on Integral Input-to-State Stability

Hiroshi Ito, Pierdomenico Pepe, and Zhong-Ping Jiang

Abstract. This article deals with interconnected systems described by retarded nonlinear equations with discontinuous right-hand side. The problem of feedback control redesign to achieve ISS (input-to-state stability) and iISS (integral input-to-state stability) with respect to additive disturbances acting on each subsystem is solved. It is shown that it is possible to design a decentralized controller accomplishing the robustification whenever a small-gain condition is satisfied.

1 Introduction

Lyapunov redesign is an important strategy in nonlinear control, which allows us to enhance system properties by additional feedback compensation exploiting the knowledge of a Lyapunov function. The ISS feedback control redesign was introduced by [14], for finite dimensional nonlinear systems. This methodology allows to attenuate the actuator disturbance in terms of ISS. It has been extended to different classes of systems with time-delays in [10], [11] and [12]. In particular, in [12], systems described by nonlinear functional differential equations with discontinuous right-hand side are considered, and the saturation problem of the input magnitude is

Hiroshi Ito
Department of Systems Design and Informatics, Kyushu Institute of Technology, Iizuka, Fukuoka 820-8502, Japan
e-mail: hiroshi@ces.kyutech.ac.jp

Pierdomenico Pepe
Information Engineering, Computer Science and Mathematics, University of L'Aquila, 67040 Poggio di Roio, L'Aquila, Italy
e-mail: pierdomenico.pepe@univaq.it

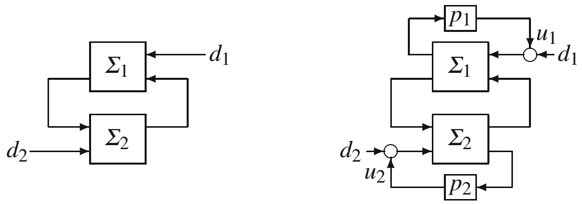
Zhong-Ping Jiang
Department of Electrical and Computer Engineering,
Polytechnic Institute of New York University, Brooklyn, NY 11201, USA
e-mail: zjiang@poly.edu

T. Vyhlídal, J.-F. Lafay, and R. Sipahi (eds.), *Delay Systems*,
Advances in Delays and Dynamics 1,
DOI: 10.1007/978-3-319-01695-5_15,

addressed, in both an iISS and an ISS fashion. The problem of iISS (ISS) feedback control redesign is based on the knowledge of a Lyapunov-Krasovskii functional which needs to be constructed a priori for the unforced (disturbance equal to zero) system. In [5], the construction of Lyapunov-Krasovskii functionals is addressed for ISS and iISS of interconnected systems under a small-gain condition. The small-gain condition allows to split the problem of finding an overall Lyapunov-Krasovskii functional for the whole system, into the problems of finding Lyapunov-Krasovskii functionals for each subsystem. This renders the original problem much easier. The reader can refer to [7] for an application of a similar, but different small-gain characterization to stabilization of a chemostat.

This article shows, for a class of nonlinear retarded interconnected systems, that it is possible, under a small-gain condition, to achieve the iISS (ISS) feedback control redesign by means of decentralized controllers. That is, the redesign allows us to attenuate the effect of disturbances acting on each subsystem by means of a feedback from the state of each subsystem itself. For this aim, we exploit the Lyapunov-Krasovskii functional which is proved to exist when the global asymptotic stability of the disturbance-free overall closed-loop is secured via a small-gain condition in [5]. We cover multiple discrete as well as distributed time-delays, and the maps describing the dynamics are allowed to be discontinuous. A preliminary version of this article has been presented in [6].

Notations. The symbol $\mathbb{R}$ denotes the set of real numbers $(-\infty,+\infty)$. $\overline{\mathbb{R}}$ denotes the extended real line $[-\infty,+\infty]$. We also use $\mathbb{R}_+ := [0,+\infty)$ and $\overline{\mathbb{R}}_+ := [0,+\infty]$. For a positive integer n, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with norm $|\cdot|$. A function $v : \mathbb{R}_+ \to \mathbb{R}^m$, with positive integer m, is said to be essentially bounded if $\operatorname{ess\,sup}_{t\geq 0} |v(t)| < \infty$. For given times $0 \leq T_1 < T_2$, we indicate with $v_{[T_1,T_2)} : \mathbb{R}_+ \to \mathbb{R}^m$ the function given by $v_{[T_1,T_2)}(t) = v(t)$ for all $t \in [T_1,T_2)$ and $= 0$ elsewhere. The function v is said to be locally essentially bounded if, for any $T > 0$, $v_{[0,T)}$ is essentially bounded. The essential supremum norm is indicated with the symbol $\|\cdot\|_\infty$. For a positive integer n and a positive real Δ: $\mathscr{C}_n$ denotes the space of continuous functions mapping $[-\Delta,0]$ into $\mathbb{R}^n$; $\mathscr{Q}_n$ denotes the space of bounded, continuous, except at a finite number of points, and right-continuous functions mapping $[-\Delta,0)$ into $\mathbb{R}^n$. For $\phi \in \mathscr{C}_n$, $\phi_{[-\Delta,0)}$ is the function in $\mathscr{Q}_n$ defined as $\phi_{[-\Delta,0)}(\tau) = \phi(\tau)$, $\tau \in [-\Delta,0)$. For a function $x : [-\Delta,c) \to \mathbb{R}^n$, with $0 < c \leq +\infty$, for any real $t \in [0,c)$, x_t is the function in $\mathscr{C}_n$ defined as $x_t(\tau) = x(t+\tau)$, $\tau \in [-\Delta,0]$. For given positive integers n,m, a map $f : \mathscr{C}_n \to \mathbb{R}^{n\times m}$ is said to be Lipschitz on bounded sets if, for any positive real q there exists a positive real L_q such that, for any $\phi_1,\phi_2 \in \mathscr{C}_n$ satisfying $\|\phi_i\|_\infty \leq q$, $i = 1,2$, the inequality holds $|f(\phi_1) - f(\phi_2)| \leq L_q\|\phi_1 - \phi_2\|_\infty$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be: of class $\mathscr{P}$ if it is continuous, zero at zero, and positive at any positive real; of class $\mathscr{K}$ if it is of class $\mathscr{P}$ and strictly increasing; of class $\mathscr{K}_\infty$ if it is of class $\mathscr{K}$ and it is unbounded; of class $\mathscr{L}$ if it is continuous and it monotonically decreases to zero as its argument tends to $+\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathscr{KL}$ if $\beta(\cdot,t)$ is of class $\mathscr{K}$ for each $t \geq 0$ and $\beta(s,\cdot)$ is of class $\mathscr{L}$ for each $s \geq 0$. The symbols $\vee$ and $\wedge$ denote logical sum and logical



(a) Interconnected subsystem (b) Decentralized compensation

Fig. 1 Decentralized robustification with respect to disturbances

product, respectively. For $x \in \mathbb{R}$, $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$. For $x \in \mathbb{R} \setminus \{0\}$, $\mathrm{sgn}(x) = x/|x|$ and $\mathrm{sgn}(0) = 0$. Proofs are omitted due to the space limitation.

2 Idea and Issues to Be Solved

Decentralized Robustification. Consider a finite-dimensional dynamical system Σ consisting of two subsystems Σ_1 and Σ_2, and suppose that the trivial solution $x = 0$ of the overall system Σ is globally asymptotically stable (GAS)[1]. If the GAS property is characterized by a Lyapunov function in a desirable form, we can secure robustness of the system Σ against the additional disturbances d_1 and d_2 shown in Fig. 1(a) by introducing decentralized compensators. Such decentralized robustification is to insert local feedback inputs in the places where the disturbances come in as shown in Fig. 1(b). To illustrate this idea, let subsystems Σ_1 and Σ_2 be

$$\dot{x}_i(t) = f_i(x_i(t), x_{3-i}(t)), \quad i = 1, 2 \tag{1}$$

and define $x = [x_1^T, x_2^T]^T$ and $f = [f_1^T, f_2^T]^T$. Let $V(x)$ be a Lyapunov function describing the GAS of Σ, i.e., $V(x)$ is a C^1 function satisfying

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|), \quad \dot{V}(t) \leq -\alpha(|x(t)|) \tag{2}$$

along the trajectories of (1) for some $\alpha \in \mathscr{P}$ and $\underline{\alpha}, \overline{\alpha} \in \mathscr{K}_\infty$. To assess robustness of the interconnected system Σ, we consider the disturbances d_1 and d_2 as

$$\dot{x}_i(t) = f_i(x_i(t), x_{3-i}(t)) + g_i(x_i(t))d_i(t), \quad i = 1, 2. \tag{3}$$

depicted in Fig. 1(a). Then along the trajectories of (3) with $d = [d_1^T, d_2^T]^T$, we have

$$\dot{V} = L_f V(x) + LgV(x)d \leq -\alpha(|x|) + LgV(x)d, \tag{4}$$

where $g = [g_1^T, g_2^T]^T$. A bounded α can yield a fair stability margin for GAS of the original system (1) without disturbance. However, we cannot derive either the ISS

[1] For brevity, a system without input is said to be GAS if an equilibrium of the system is GAS.

or iISS property of the system (3) with respect to the disturbance d for the bounded α if $LgV(x)$ is an unbounded function of x. To secure the robustness with respect to d, we can introduce a control input u_i at the place of d_i, i.e.,

$$\dot{x}_i(t) = f_i(x_i(t), x_{3-i}(t)) + g_i(x_i(t))(d_i(t) + u_i(t)) \ , i = 1, 2. \tag{5}$$

Indeed, applying the "L_gV-type" full state feedback

$$u(t) = [u_1^T(t), u_2^T(t)]^T = -a(LgV(x(t)))^T \tag{6}$$

with a real number $a > 0$ to (5) we obtain

$$\dot{V} = L_f V(x) - a(LgV(x))(LgV(x))^T + LgV(x)d \leq -\alpha(|x|) + \frac{1}{4a}|d|^2 \tag{7}$$

along the trajectories of (5) with the help of Young's inequality. The disadvantage of using (6) is its centralized structure. Since $L_gV(x)$ in (6) usually contain both x_1 and x_2, the control input $u_i(t)$ of subsystem i is based not only on the local state x_i, but also on the state x_{3-i} of the other subsystem $3-i$. To make the robustifying compensation decentralized, instead of $V(x)$, we consider C^1 functions $V_i(x_i)$ which only contain local information for $i = 1, 2$. Applying the local version

$$u_i(t) = -a_i(Lg_iV_i(x_i(t)))^T, \quad i = 1, 2 \tag{8}$$

with a real number $a_i > 0$ to (5) we obtain

$$\begin{aligned} \dot{V}_i &= L_{f_i}V_i(x_i, x_{3-i}) - a_i(Lg_iV_i(x_i))(Lg_iV_i(x_i))^T + Lg_iV_i(x_i)d_i \\ &\leq L_{f_i}V_i(x_i, x_{3-i}) + \frac{1}{4a_i}|d_i|^2, \quad i = 1, 2. \end{aligned} \tag{9}$$

At this point, the property (9) does not give us information about robustness of the overall system (5) with the decentralized state feedback (8). Indeed, it is true in general that $a(LgV(x)) = [a_1(Lg_1V_1(x_1)), a_2(Lg_2V_2(x_2))]$ does not hold for any choice of positive constants a, a_1 and a_2. If the function $V(x)$ fulfilling (2) happens to be in the form of $V(x) = V_1(x_1) + V_2(x_2)$, then the property (9) implies (7) for the choice $a = a_1 = a_2$. The larger the feedback gain a_i is, the stronger the robustness with respect to d is. However, general nonlinear systems consisting of (1) often disallow any V in the form of $V(x) = V_1(x_1) + V_2(x_2)$ to accomplish (2) even if the equilibrium $x = 0$ is GAS. In this way, the feedback input u_i which uses only local state x_i as in Fig. 1(b) achieves the desired robustness of the overall system Σ with respect to the disturbance d only if the construction of the Lyapunov function $V(x)$ and the selection of the local feedback control laws $u_i(x_i)$ are judiciously coordinated. Therefore, it is significantly useful to derive a condition under which such a desirable pair V and u can be constructed, and to provide the formulas of V and u.

iISS. If $\alpha \in \mathscr{K}_\infty$ holds in (2), the property (7) implies ISS of the system Σ with respect to the disturbance d. In the case of $\alpha \in \mathscr{P} \setminus \mathscr{K}_\infty$, the system Σ is iISS with respect to d. It is, however, not guaranteed to be ISS. The existence of $V(x)$ satisfying (2) ensures the existence of another C^1 function $V(x)$ satisfying (2) with a class $\mathscr{K}_\infty$ function α. Indeed, replacing $V(x)$ by $F(V(x))$ with an appropriate C^1 function $F : \mathbb{R}_+ \to \mathbb{R}_+$ always allows us to achieve (2) with $\alpha \in \mathscr{K}_\infty$. Here, it is important to notice that this transformation into $\alpha \in \mathscr{K}_\infty$ via redefinition of $V(x)$ does not preserve the decentralized structure of robustifying controllers. In fact, the redefinition of $V(x)$ yields the "L_gV-type" feedback (6) as

$$u(t) = -a(LgF(V(x(t))))^T = [u_1^T(x(t)), u_2^T(x(t))]^T \tag{10}$$

in which $u_i = u_i^T(x_i)$ does not hold true in general. The transformation by F results in the centralized feedback $u_i = u_i^T(x_1, x_2)$, $i = 1, 2$, even if the original V is in the form of $V(x) = V_1(x_1) + V_2(x_2)$. In addition, there are a lot of GAS systems for which no matter how we choose C^1 functions $F_1, F_2 : \mathbb{R}_+ \to \mathbb{R}_+$, the composite function $V(x) = F_1(V_1(x_1)) + F_2(V_2(x_2))$ never achieves (2) with $\alpha \in \mathscr{K}_\infty$. Such examples are found in the iISS framework (see e.g. [2]). Hence, it is unreasonable to expect that the interconnected system Σ achieves ISS with respect to the disturbance d. In this way, allowing $\alpha \notin \mathscr{K}_\infty$ is imperative to avoid unreasonably stringent constraints on systems Σ_i, and it is quite useful to develop a method of achieving iISS and including ISS as a special case.

Limitation of Input Magnitude. The control laws (6) and (8) are unbounded unless a strong constraint is imposed on the system Σ. In practical situations where the magnitude of control input is limited, the laws (6) and (8) need to be implemented with saturation functions. Then the property (7) does not hold true. From (5) it is obvious that, if $|d_i(t)|$ becomes larger than the upper limit of $|u_i(t)|$ an actuator can generate, such a control input cannot enhance the robustness against d. However, the upper bound of $|d_i(t)|$ is known and it is smaller than the actuator limitation, the robustness of Σ should be enhanced by applying appropriately saturated control input $u_i(t)$. Therefore, it is practically important to clarify how robust the system Σ can be by judiciously designing robustifying controllers meeting the input constraints.

***LgV* in the Presence of Delays and Discontinuities.** In addition to the inevitability of time delays in dynamical systems, discontinuity in the right-hand side often arises in practical models of control and sliding mode control laws. Such delays and discontinuities need to be incorporated into the right-hand side of (1), (3) or (5). Moreover, the map V needs to be extended to a functional in order to characterize the behavior of systems with delays whose solutions are defined as the evolution of segments defined on the delay interval along the time axis. In (6) and (8), the symbol LgV indicated the Lie derivative of the C^1 function V along g, i.e., $LgV = \frac{\partial V}{\partial x} g(x)$. When V is a functional, this definition is inapplicable. Furthermore, the relation between LgV and the estimation of the solutions $x(t)$ to the system Σ is not immediate at all for time-delay discontinuous right-hand side systems. It is necessary to redefine LgV in accordance with a feasible estimation of the behavior of Σ subject to time-delays and discontinuities.

An Approach. To address the above issues, we take an new approach based on

- Invariantly differential functionals to characterize the robustification in the form of LgV;
- A sum-type construction of a Lyapunov-Krasovskii functional to obtain V leading to the decentralized robustification;
- An iISS small-gain condition to formulate the robustification in the iISS framework in the presence of actuator limitations.

These tools have been investigated and developed very recently in [4, 5, 12]. This article demonstrates how successfully the problem of decentralized robustification for time-delay discontinuous right-hand side systems can be solved.

3 Invariantly Differentiable Functionals

This article borrows the definition of invariant differentiable functionals from [8] (see Definitions 2.2.1, 2.5.2 in Chapter 2). In the subsequent sections, we will assume that Lyapunov-Krasovskii functionals are invariantly differentiable. The formalism used in [8] is slightly modified here for the purpose of formalism uniformity throughout this article. For any given $x \in \mathbb{R}^n$, $\phi \in \mathcal{Q}_n$ and any continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^n$ with $\mathcal{Y}(0) = x$, let $\psi_h^{(x,\phi,\mathcal{Y})} \in \mathcal{Q}_n$, $h \in [0, \Delta)$, be defined as

$$\psi_0^{(x,\phi,\mathcal{Y})} = \phi$$
$$\psi_h^{(x,\phi,\mathcal{Y})}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h) \\ \mathcal{Y}(s+h), & s \in [-h, 0) \end{cases} \quad \text{for } h > 0. \tag{11}$$

Definition 1. (see [8]) A functional $V : \mathbb{R}^n \times \mathcal{Q}_n \to \mathbb{R}_+$ is said to be invariantly differentiable if, at any point $(x, \phi) \in \mathbb{R}^n \times \mathcal{Q}_n$:
i) for any continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^n$ with $\mathcal{Y}(0) = x$, there exists the finite right-hand derivative $\partial V\left(x, \psi_h^{(x,\phi,\mathcal{Y})}\right)/\partial h\Big|_{h=0}$ and such derivative is invariant with respect to the function $\mathcal{Y}$;
ii) there exists the finite derivative $\partial V(x, \phi)/\partial x$; iii) for any $z \in \mathbb{R}^n$, for any continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^n$ with $\mathcal{Y}(0) = x$, for any $h \in [0, \Delta)$,

$$V\left(x+z, \psi_h^{(x,\phi,\mathcal{Y})}\right) - V(x,\phi) = \frac{\partial V(x,\phi)}{\partial x} z + \left.\frac{\partial V\left(x, \psi_\ell^{(x,\phi,\mathcal{Y})}\right)}{\partial \ell}\right|_{\ell=0} h + o\left(\sqrt{|z|^2 + h^2}\right) \tag{12}$$

with $\lim_{s \to 0^+} o(\sqrt{s})/\sqrt{s} = 0$.

The first two terms in (12) serve as a differential of $V(x, \phi)$, and they are independent of $\mathcal{Y}$ defining the increment of the second argument ϕ of the functional $V(x, \phi)$. As explained in [12, 16], due to the invariant differentiability, we can define an appropriate derivative by which we can estimate the behavior of the trajectories of time-delay discontinuous right-hand side systems with a locally Lipschitz functional $V : \mathbb{R}^n \times \mathscr{Q}_n \to \mathbb{R}_+$ as in the classical Lyapunov theory for ordinary differential equations. Lemma 6 in [12] provides a tool to rescale invariantly differentiable functionals, which helps us evaluate robustness of interconnected systems effectively by means of invariantly differentiable functionals.

4 Interconnected Time-Delay Systems with Discontinuous Right-Hand Side

Consider an interconnected system Σ described by the following functional differential equations with discontinuous right-hand side

$$\Sigma \begin{cases} \Sigma_1 : \dot{x}_1(t) = f_1(x_{1,t}, x_{2,t}) + g_1(x_{1,t})(u_1(t) + d_1(t)) \\ \Sigma_2 : \dot{x}_2(t) = f_2(x_{2,t}, x_{1,t}) + g_2(x_{2,t})(u_2(t) + d_2(t)) \end{cases} \tag{13}$$
$$x_{1,0} = \xi_{1,0}, \quad x_{2,0} = \xi_{2,0},$$

where, for $i = 1,2$, $x_i(t) \in \mathbb{R}^{n_i}$; $d_i(t) \in \mathbb{R}^{m_i}$ is a disturbance adding to the control input (measurable, locally essentially bounded); n_i and m_i are positive integers. For $t \in \mathbb{R}_+$, $x_{i,t} : [-\Delta, 0] \to \mathbb{R}^{n_i}$ denotes the function $x_{i,t}(\tau) = x_i(t+\tau)$, where $\Delta > 0$ is the maximum involved delay. Suppose that $\xi_{i,0} \in \mathscr{C}_{n_i}$. The locally bounded maps $f_i : \mathscr{C}_{n_i} \times \mathscr{C}_{n_{3-i}} \to \mathbb{R}^{n_i}$ are continuous with respect to the second argument, and are allowed to be discontinuous with respect to the first argument, the maps $g_i : \mathscr{C}_{n_i} \to \mathbb{R}^{n_i \times m_i}$ are assumed to be Lipschitz on bounded sets. We combine vectors as $x(t) = [x_1(t)^T, x_2(t)^T]^T \in \mathbb{R}^n$, $n = n_1 + n_2$, $u(t) = [u_1(t)^T, u_2(t)^T]^T \in \mathbb{R}^m$, $d(t) = [d_1(t)^T, d_2(t)^T]^T \in \mathbb{R}^m$, $m = m_1 + m_2$, $\xi_0 = [\xi_{1,0}^T, \xi_{2,0}^T]^T \in \mathscr{C}_n$, $f() = [f_1()^T, f_2()^T]^T$, $\phi = [\phi_1^T, \phi_2^T]^T \in \mathscr{C}_n$ and $g() = [g_1()^T, g_2()^T]^T$. We define x_t as done for its i-th component $x_{i,t}$. It is assumed that $f_i(0,0) = 0$, $i = 1,2$. We use semi-norms $\|\cdot\|_{a,i} : \mathscr{C}_{n_i} \to \mathbb{R}_+$ and $\|\cdot\|_a : \mathscr{C}_n \to \mathbb{R}_+$, $i = 1,2$, respectively, for which there exist class $\mathscr{K}^\infty$ functions $\underline{\gamma}_{a,i}$, $\overline{\gamma}_{a,i}$, $\underline{\gamma}_a$ and $\overline{\gamma}_a$ such that

$$\underline{\gamma}_{a,i}(|\phi_i(0)|) \le \|\phi_i\|_{a,i} \le \overline{\gamma}_{a,i}(\|\phi_i\|_\infty), \quad \forall \phi_i \in \mathscr{C}_{n_i} \tag{14}$$

$$\underline{\gamma}_a(|\phi(0)|) \le \|\phi\|_a \le \overline{\gamma}_a(\|\phi\|_\infty), \qquad \forall \phi \in \mathscr{C}_n\ . \tag{15}$$

The retarded inclusions corresponding to Σ represented by (13) are given by

$$\begin{aligned} &\dot{x}_1(t) \in \Psi_1(x_{1,t}, x_{2,t}, u_1(t) + d_1(t)), \qquad t \ge 0,\ a.e., \\ &\dot{x}_2(t) \in \Psi_2(x_{2,t}, x_{1,t}, u_2(t) + d_2(t)), \qquad t \ge 0,\ a.e., \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \qquad \xi_0 \in \mathscr{C}_n, \end{aligned} \tag{16}$$

where, for $(\phi_i, \phi_{3-i}, v) \in \mathscr{C}_{n_i} \times \mathscr{C}_{n_{3-i}} \times \mathbb{R}^{m_i}$, $\Psi_i(\phi_i, \phi_{3-i}, v)$ is the set given by

$$\Psi_i(\phi_i, \phi_{3-i}, v) = \{\xi_i + g_i(\phi_i)v,\ \ \xi_i \in F_i[f_i](\phi_i, \phi_{3-i})\}, \tag{17}$$

and $F_i[f_i](\phi_i, \phi_{3-i})$ is the convex closure of all limit values of the map f_i at the point (ϕ_i, ϕ_{3-i}). We introduce here the following standard assumption on the maps f_i of subsystems in (13): For each $(\phi_i, \phi_{3-i}) \in \mathscr{C}_{n_i} \times \mathscr{C}_{n_{3-i}}$, the set $F_i[f_i](\phi_i, \phi_{3-i})$ is assumed to be compact in $\mathbb{R}^{n_i}$; for each bounded set $W \in \mathscr{C}_{n_i} \times \mathscr{C}_{n_{3-i}}$, the set $\cup_{(\phi_i, \phi_{3-i}) \in W} F_i[f_i](\phi_i, \phi_{3-i})$ is assumed to be bounded; the multimap $(\phi_i, \phi_{3-i}) \to F_i[f_i](\phi_i, \phi_{3-i})$ is assumed to satisfy the Carathéodory conditions (see Sections 4.2, 4.3, pp. 121-126, in [9]).

For the system (13), as in [12], we consider situations where essential bounds of the disturbance $d(t)$ are known in the following sense:

$$\underline{d}_{i,j} \leq \operatorname*{ess\,inf}_{t \in \mathbb{R}_+} d_{i,j}(t), \quad \overline{d}_{i,j} \geq \operatorname*{ess\,sup}_{t \in \mathbb{R}_+} d_{i,j}(t). \tag{18}$$

Here, $\underline{d}_{i,j}, \overline{d}_{i,j} \in \overline{\mathbb{R}}$, $i = 1,2$, $j = 1,2,...,m_i$ satisfying $\underline{d}_{i,j} \leq 0 \leq \overline{d}_{i,j}$ are given *a priori*. Note that, when do not have any *a priori* knowledge of the disturbance magnitude at Σ_i, we let $-\underline{d}_{i,j} = \overline{d}_{i,j} = \infty$, $i = 1,2$ for $j = 1,2,...,m_i$. The notions of ISS and iISS with the essential bounds are defined as follows:

Definition 2. The system (13) with $u(t) \equiv 0$ is said to be input-to-state stable (ISS) with respect to d with the essential bounds (18) if there exist a $\mathscr{KL}$ function β and a $\mathscr{K}$ function γ such that, for any initial state ξ_0 and any measurable, locally essentially bounded input d satisfying (18), any corresponding solution in the sense of (16) exists for all $t \geq 0$ and furthermore it satisfies

$$|x(t)| \leq \beta\left(\|\xi_0\|_\infty, t\right) + \gamma\left(\|d_{[0,t)}\|_\infty\right). \tag{19}$$

Definition 3. The system (13) with $u(t) \equiv 0$ is said to be integral input-to-state stable (iISS) with respect to d with the essential bounds (18) if there exist a $\mathscr{K}_\infty$ function χ, a $\mathscr{KL}$ function β and a $\mathscr{K}$ function γ such that, for any initial state ξ_0 and any measurable, locally essentially bounded input d satisfying (18), any corresponding solution in the sense of (16) exists for all $t \geq 0$ and furthermore it satisfies

$$\chi(|x(t)|) \leq \beta\left(\|\xi_0\|_\infty, t\right) + \int_0^t \gamma(|d(\tau)|)\, d\tau. \tag{20}$$

It is stressed that, in the situation where we take $-\underline{d}_{i,j} = \overline{d}_{i,j} = \infty$ for $i = 1,2$ and $j = 1,2,...,m_i$, the above definition reduces to the standard definitions of ISS and iISS without any bounds of the disturbance d (see [1, 13–15]). For example, the system (13) with $u(t) \equiv 0$ is ISS with respect to d with the essential bounds (18) for $-\underline{d}_{i,j} = \overline{d}_{i,j} = \infty$, $i = 1,2$, $j = 1,2,...,m_i$, if and only if the system (13) with $u(t) \equiv 0$ is ISS with respect to d. This equivalence also holds in the iISS case.

The following assumption is imposed on each unforced ($u_i(t) = d_i(t) \equiv 0$) subsystem in (13): For each subsystem Σ_i ($i = 1,2$) defined in (13) with $u_i(t) = d_i(t) \equiv 0$, we assume the existence of a Locally Lipschitz invariantly differentiable functional $V_i : \mathbb{R}^{n_i} \times \mathscr{Q}_{n_i} \to \mathbb{R}_+$ such that

$$\underline{\alpha}_i(\|\phi_i\|_{a,i}) \le V_i(\phi_i(0),(\phi_i)_{[-\Delta,0)}) \le \overline{\alpha}_i(\|\phi_i\|_{a,i}), \tag{21}$$

$$D^+V_i(\phi_i,\phi_{3-i}) \le \rho_i(\phi_i,\phi_{3-i}), \quad \forall\, \phi_j \in \mathscr{C}_j, j=1,2 \tag{22}$$

hold, where $\underline{\alpha}_i, \overline{\alpha}_i$ are $\mathscr{K}_\infty$ functions and $\rho_i : \mathscr{C}_{n_i} \times \mathscr{C}_{n_{3-i}} \to \mathbb{R}$ is a continuous functional given by

$$\rho_i(\phi_i,\phi_{3-i}) = -\alpha_i(\|\phi_i\|_{a,i}) + \sigma_{i,0}(\|\phi_{3-i}\|_{a,i}) + \sum_{j=1}^{h} \sigma_{i,j}\left(\underline{\gamma}_{a,3-i}|\phi_{3-i}(-\Delta_j)|\right)$$
$$+ \sum_{j=h+1}^{h+h_d} \int_{-\Delta_j}^{0} \sigma_{i,j}\left(\underline{\gamma}_{a,3-i}|\phi_{3-i}(\tau)|\right) d\tau. \tag{23}$$

Here, h and h_d are non-negative integers, α_i and $\sigma_{i,j}$ are class $\mathscr{K}$ functions, and $\Delta_j \in (0,\Delta]$ for $j = 0,1,\ldots,h+h_d$. The left hand side of (22) is defined with

$$D^+V_i(\phi_i,\phi_{3-i}) = \sup_{\xi_i \in F_i[f_i](\phi_i,\phi_{3-i})} \left.\frac{\partial V_i(x_i,\phi_i)}{\partial x_i}\right|_{x_i=\phi_i(0)} \xi_i + \left.\frac{\partial V_i(\phi_i(0),\phi_{i,h})}{\partial h}\right|_{h=0} \tag{24}$$

$$\phi_{i,h}(s) = \begin{cases} \phi_i(s+h), & s \in [-\Delta,-h) \\ \phi_i(0), & s \in [-h,0] \end{cases} \quad \text{for } h \in [0,\Delta). \tag{25}$$

5 Decentralized iISS and ISS Feedback Redesign

We introduce a few notations and definitions. Define an operator $\alpha_i^{\ominus} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ as

$$\alpha_i^{\ominus}(s) = \sup\{v \in \mathbb{R}_+ : s \ge \alpha_i(v)\}. \tag{26}$$

Thus, we have $\alpha_i^{\ominus}(s) = \infty$ for $s \ge \lim_{\tau\to\infty} \alpha_i(\tau)$, and $\alpha_i^{\ominus}(s) = \alpha_i^{-1}(s)$ elsewhere. For a class $\mathscr{K}$ function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, this article uses the extension ω: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ defined as

$$\omega(s) := \sup_{v \in \{w \in \mathbb{R}_+ : w \le s\}} \omega(v).$$

The reader may refer to [3] for the benefit of these extended operators. We define the following set $\mathscr{D}(\underline{w},\overline{w})$ of continuous functions:

Definition 4. Given $-\infty \le \underline{w} < 0 < \overline{w} \le +\infty$, a function $\omega : \mathbb{R} \to \mathbb{R}$ is said to belong to $\mathscr{D}(\underline{w},\overline{w})$ if it is a strictly increasing and locally Lipschitz function such that $\omega(0) = 0 \wedge \{\lim_{s\to-\infty} \omega(s) < \underline{w} \vee \lim_{s\to-\infty} \omega(s) = -\infty\} \wedge \{\overline{w} < \lim_{s\to+\infty} \omega(s) \vee \lim_{s\to+\infty} \omega(s) = +\infty\}$.

For a mapping $\omega \in \mathscr{D}(\underline{w},\overline{w})$ from $\mathbb{R}$ onto $(a,b) \subseteq \mathbb{R}$, the inverse of ω is a strictly increasing continuous function denoted by $\omega^{-1} : (a,b) \to \mathbb{R}$. For $\omega \in \mathscr{D}(\underline{w},\overline{w})$, the function $\omega^{-1}(s)s : (\underline{w},\overline{w}) \to \mathbb{R}$ is locally Lipschitz. For each $i = 1,2$, let N_i be

$$N_i = \sum_{j=0}^{h+h_d} \mathrm{sgn}(\sigma_{i,j}(1)) \ , \tag{27}$$

which describes the number of non-zero functions among $\sigma_{i,0}$, ..., $\sigma_{i,h+h_d}$, in (23). The following achieves decentralized robustification under a small-gain condition.

Theorem 1. *Define $\sigma_i \in \mathscr{K}$, $i = 1,2$, by*

$$\sigma_i(s) = N_i \max\left\{ \max_{j=0,1,\ldots,h} \sigma_{i,j}(s), \max_{j=h+1,\ldots,h+h_d} \Delta_j \sigma_{i,j}(s) \right\}. \tag{28}$$

Suppose that there exist $c_i > 1$, $i = 1,2$, such that

$$c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{\ominus} \circ c_2\sigma_2(s) \le \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \quad \forall s \in \mathbb{R}_+ \tag{29}$$

holds. Pick $\tau_i, \mu_i > 0$ and $\varphi \ge 0$ such that

$$1 < \tau_i < \frac{c_i}{1+\mu_i}, \quad \left(\frac{\tau_i(1+\mu_i)}{c_i}\right)^{\varphi} \le \tau_i - 1, \quad i = 1,2 \tag{30}$$

are satisfied. Define class $\mathscr{K}$ functions λ_i, $i = 1,2$, by

$$\lambda_i(s) = \left[\frac{1}{\tau_i}\alpha_i(\overline{\alpha}_i^{-1}(s))\right]^{\varphi} \left[(1+\mu_i)\sigma_{3-i}(\underline{\alpha}_i^{-1}(s))\right]^{\varphi+1}. \tag{31}$$

Assume that the mapping

$$\begin{aligned} h_i(\phi_i) &= [h_{i,1}(\phi_i), h_{i,2}(\phi_i), \ldots, h_{i,m_i}(\phi_i)] \\ &= \lambda_i(V_i(\phi_i(0), (\phi_i)_{[-\Delta,0)})) \cdot \left.\frac{\partial V_i(x_i, (\phi_i)_{[-\Delta,0)})}{\partial x_i}\right|_{x_i=\phi_i(0)} g_i(\phi_i) \end{aligned} \tag{32}$$

from $\mathscr{C}_{n_i}$ into $\mathbb{R}^{m_i}$ is Lipschitz on bounded sets for $i = 1,2$. Define

$$p_i(\phi_i) = -[Y_{i,1}(h_{i,1}(\phi_i)), Y_{i,2}(h_{i,2}(\phi_i)), \cdots, Y_{i,m_i}(h_{i,m_i}(\phi_i))]^T \tag{33}$$

for $Y_{i,j} \in \mathscr{D}(\underline{d}_{i,j}, \overline{d}_{i,j})$, $i = 1,2$, $j = 1,2,\ldots,m_i$. Then the decentralized feedback control laws ($i = 1,2$)

$$u_i(t) = p_i(x_{i,t}) \tag{34}$$

render the closed-loop system consisting of (13) *and* (34) *iISS with respect to the disturbance d with the essential bounds* (18)*. Moreover, if* $\lim_{s\to\infty} \alpha_i(s) = \infty$ *holds true for $i = 1,2$, then the closed-loop system is ISS with respect to the disturbance d with the essential bounds* (18)*.*

For any $c_i > 1$, there always exist $\tau_i, \mu_i > 0$ and $\varphi \ge 0$ fulfilling (30). Note that Theorem 1 establishes ISS and iISS even if $-\underline{d}_{i,j} = \overline{d}_{i,j} = \infty$ for $i = 1,2$, $j = 1,2,\ldots,m_i$. In such a case, $Y_{i,j}$'s are required to be unbounded and the magnitude of the robustifying inputs $u_i(t)$ become large arbitrarily for arbitrarily large disturbances $d_{i,j}(t)$. If time delays reside only in communication channels, the mappings V_i are functions which do not involve any terms for time delays. In such cases, equations (32), (33) and (34) yield the compensations $u_i(t)$ which are delay free. Theorem 1 is established by making use of the functional $V : \mathbb{R}^n \times \mathscr{Q}_n \to \mathbb{R}_+$:

$$\begin{aligned} V(\phi(0),(\phi)_{[-\Delta,0)}) = \sum_{i=1}^{2} \int_0^{V_i(\phi_i(0),(\phi_i)_{[-\Delta,0)})} \lambda_i(s) ds \\ + \sum_{j=1}^{h} \int_{-\Delta_j}^{0} F_{i,j}(\tau) \tilde{\sigma}_{i,j} \left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|) \right) d\tau \\ + \sum_{j=h+1}^{h+h_d} \int_{-\Delta_j}^{0} F_{i,j}(\tau) \int_{\tau}^{0} \tilde{\sigma}_{i,j} \left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\theta)|) \right) d\theta d\tau. \end{aligned} \tag{35}$$

where, for $i = 1,2$ and $j = 1,2,...,h+h_d$, the continuous functions $F_{i,j} : [-\Delta_j, 0] \to \mathbb{R}$ and the functions $\tilde{\sigma}_{i,j} \in \mathscr{K}$ are given by

$$F_{i,j}(\tau) = \frac{-\tau}{\Delta_j} + (1+\mu_i)\frac{\tau+\Delta_j}{\Delta_j}, \quad \tilde{\sigma}_{i,j}(s) = \lambda_i(\theta_{i,j}(s))\sigma_{i,j}(s)$$

$$\theta_{i,j}(s) = \begin{cases} \overline{\alpha}_i \circ \alpha_i^{\ominus} \circ N_i \tau_i \sigma_{i,j}(s) & , j = 0,1,...,h \\ \overline{\alpha}_i \circ \alpha_i^{\ominus} \circ N_i \tau_i \Delta_j \sigma_{i,j}(s) & , j = h+1, h+2, ..., h+h_d. \end{cases}$$

Let $\underline{\alpha}$, $\overline{\alpha} \in \mathscr{K}_\infty$ be such that $\underline{\alpha}(\|\phi\|_a) \le V(\phi(0),(\phi)_{[-\Delta,0)}) \le \overline{\alpha}(\|\phi\|_a)$. The functional V in (35) plays the role of a Lyapunov-Krasovskii functional to estimate the influence of the disturbance d on the resulting system as follows:

Corollary 1. *Suppose that all the assumptions in Theorem 1 are fulfilled. Then the closed-loop system consisting of* (13) *and* (34) *satisfies*

$$D^+V(\phi,d) \le -\alpha(\|\phi\|_a) + \sigma(|d|), \tag{36}$$

where $\alpha \in \mathscr{K}$ is given by (37)*, and σ is any class $\mathscr{K}$ function satisfying* (38)*:*

$$\begin{aligned} \alpha(s) = \min_{\left\{\phi=[\phi_1^T,\phi_2^T]^T \in \mathscr{C}_n : s=\|\phi\|_a\right\}} \Bigg\{ \sum_{i=1}^{n} \sum_{j=1}^{h} \frac{\mu_i}{\Delta_j} \int_{-\Delta_j}^{0} \tilde{\sigma}_{i,j} \left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|\right) d\tau + \\ \sum_{j=h+1}^{h+h_d} \frac{\mu_i}{\Delta_j} \int_{-\Delta_j}^{0} \int_{\tau}^{0} \tilde{\sigma}_{i,j} \left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\theta)|\right) d\theta d\tau + \\ \frac{\left(1-\frac{\tau_i}{c_i}\right)(\tau_i - 1)}{\tau_i} \lambda_i(V_i(\phi_i(0),(\phi_i)_{[-\Delta,0)}))[\alpha_i \circ \overline{\alpha}_i^{-1}(V_i(\phi_i(0),(\phi_i)_{[-\Delta,0)}))] \Bigg\} \end{aligned} \tag{37}$$

$$\sigma(s) \ge \sup_{\{d \in \mathbb{R}^m : s \ge |d|,\ d_{i,j} \in (\underline{d}_{i,j}, \overline{d}_{i,j})\}} \sum_{i=1}^{2} \sum_{j=1}^{m_i} Y_{i,j}^{-1}(d_{i,j}) d_{i,j}. \tag{38}$$

Furthermore, a pair of χ *and* γ *in* (20) *is given by* $\chi(s) = \underline{\alpha} \circ \underline{\gamma}_a(s)$ *and* $\gamma(s) = 2\sigma(s)$. *Moreover, if* $\lim_{s\to\infty} \alpha_i(s) = \infty$ *holds for* $i = 1, 2$, *a function* γ *satisfying* (19) *is* $\gamma(s) = \underline{\gamma}_a^{-1} \circ \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha^{-1}(2\sigma(s))$.

Equation (31) is a special case of the more general formula of λ_i presented in [4]. The free parameters in [4] allow us to replace (31) by the one presented in [5].

6 An Example

Consider the interconnection of two scalar subsystems:

$$\dot{x}_1(t) = -\frac{\mathrm{sgn}(x_1(t))}{1+|x_1(t)|} + \frac{\gamma_1}{1+|x_1(t)|} x_2(t-\Delta) + \cos(x_1(t))(u_1(t)+d_1(t)) \tag{39}$$
$$\dot{x}_2(t) = -x_2(t)(2+\mathrm{sgn}(x_2(t)-1)) + \gamma_2 \frac{x_1(t-\Delta)}{1+|x_1(t-\Delta)|} + x_2(t)(u_2(t)+d_2(t)),$$

where $\Delta > 0$ is a channel delay, $\gamma_i \in \mathbb{R}$, $i = 1, 2$, are interaction parameters. Choose $V_i(\phi_i(0), (\phi_i)_{[-\Delta,0)}) = \phi_i(0)^2$, $i = 1, 2$. For $u_i(t) \equiv 0$, $d_i(t) \equiv 0$, we obtain

$$D^+V_1 \le -\frac{2|\phi_1(0)|}{1+|\phi_1(0)|} + 2|\gamma_1||\phi_2(-\Delta)|,\ D^+V_2 \le -|\phi_2(0)|^2 + \gamma_2^2 \left(\frac{|\phi_1(-\Delta)|}{1+|\phi_1(-\Delta|)}\right)^2.$$

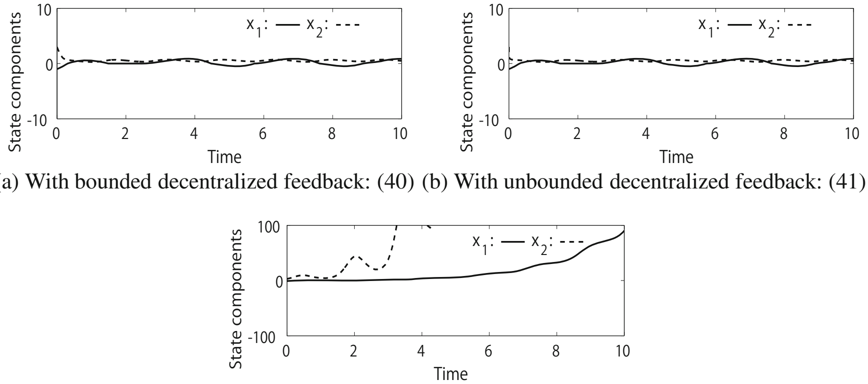
If $|\gamma_1\gamma_2| < 1$, (29) is satisfied. For example, in the case of $|\gamma_1| = |\gamma_2| = 1/2$, the formula (31) gives $\lambda_1(s) = \frac{1}{4}\left(\sqrt{s}/(1+\sqrt{s})\right)^2$ and $\lambda_2(s) = \sqrt{s}$ for $\varphi = 0$ and $\tau_i(1+\mu_i) = 17/8$, $i = 1, 2$. Assume that $|d_1(t)| \le 2$ and $|d_2(t)| \le 7$ hold for all $t \ge 0$. Setting $\underline{d}_{1,1} = \overline{d}_{1,1} = 2$ and $\underline{d}_{2,1} = \overline{d}_{2,1} = 7$, we can choose $Y_{1,1}(s) = 3\tanh(s)$ and $Y_{2,1}(s) = 10\tanh(s)$. Thus, equations (32)-(34) yield

$$\begin{array}{l} u_1(t) = -3\tanh(\lambda_1(x_1^2(t))2x_1(t)\cos(x_1(t))) \\ u_2(t) = -10\tanh(\lambda_2(x_2^2(t))2x_2^2(t)) \end{array} \tag{40}$$

which achieve iISS of the overall system (39) with respect to $d_i(t)$, $i = 1, 2$. Figure 2(a) illustrates the effectiveness of (40) for (39) with $\xi_0(\tau) = [-1, 3]^T$, $\tau \in [-\Delta, 0]$, $\Delta = 2$ and $\gamma_1 = \gamma_2 = 0.5$ in the presence of $d = [2\cos(2t), 4+3\cos(4t)]^T$ which satisfies $|d_1(t)| \le 2$ and $|d_2(t)| \le 7$ for all $t \ge 0$. Compared with Fig.2(c), the local feedback laws (40) with the input magnitude limitations significantly improve robustness with respect to the disturbance d. If no limitations of input magnitude are necessary, Theorem 1 yields the unbounded local feedback laws

$$\begin{array}{l} u_1(t) = -3\lambda_1(x_1^2(t))2x_1(t)\cos(x_1(t)), \\ u_2(t) = -10\lambda_2(x_2^2(t))2x_2^2(t) \end{array} \tag{41}$$

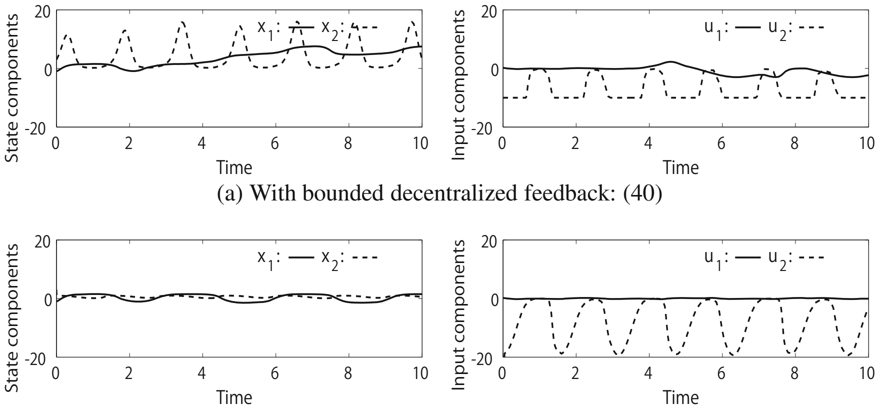
which produce state trajectories shown in Fig.2(b). The robustness achieved by the bounded control (40) is almost identical to the robustness achieved by the unbounded control (41). For $d = [7\cos(2t), 9+11\cos(4t)]^T$ exceeding the upper



(a) With bounded decentralized feedback: (40) (b) With unbounded decentralized feedback: (41)

(c) Without feedback: $u_i(t) \equiv 0$, $i = 1,2$

Fig. 2 State transition $x(t) = [x_1(t), x_2(t)]^T$ of (39) with $d = [2\cos(2t), 4+3\cos(4t)]^T$



(a) With bounded decentralized feedback: (40)

(b) With unbounded decentralized feedback: (41)

Fig. 3 State transition $x(t) = [x_1(t), x_2(t)]^T$ of (39) with $d = [7\cos(2t), 9+11\cos(4t)]^T$

bounds, the trajectories with the bounded laws (40) and the unbounded laws (41) are plotted in Figs.3(a) and (b), respectively. The parameters Δ, γ_1, γ_2 and ξ_0 are the same as those used in Figs. 2(a), (b) and (c). The control inputs (40) fulfill the magnitude constraints $|u_1(t)| \leq 3$ and $|u_2(t)| \leq 10$ for all $t \geq 0$. However, they cannot ensure the robustness the larger control inputs (41) can attain. In the case of no control inputs, the simulation exhibited a vertical increase of x_2 at $t = 1.80$.

7 Conclusions

For interconnected systems described by retarded nonlinear equations with discontinuous right-hand side, this article has proposed a methodology for decentralized redesign. In the iISS framework that does not require subsystems to be ISS, input magnitude limits and saturated decay rates of subsystems have been addressed. It has been shown that, if dissipation inequalities of subsystems satisfy the small-gain condition (29), the interconnected system can be rendered robust with respect to disturbances by adding local state feedback inputs. The notion of invariantly differential functionals allows us to carry out the robust compensation in the form of L_gV for retarded nonlinear equations with discontinuities. The sum-type construction of Lyapunov-Krasovskii functionals as in (35) enables us to obtain the robust compensation as decentralized controllers. The proposed controllers become delay free if time delays exist only in communication channels between subsystems.

Acknowledgements. The work of P. Pepe is supported by MIUR PRIN 2009 and DEWS. The work of Z.P. Jiang has been supported by NSF grant ECCS-1230040.

References

1. Heemels, W.P.M.H., Weiland, S.: Input-to-state stability and interconnections of discontinuous dynamical systems. Automatica 44, 3079–3086 (2008)
2. Ito, H., Jiang, Z.P.: Necessary and sufficient small gain conditions for integral input-to-state stable systems: a Lyapunov perspective. IEEE Trans. Autom. Control. 54, 2389–2404 (2009)
3. Ito, H., Jiang, Z.P., Dashkovskiy, S., Rüffer, B.: Robust stability of networks of iISS systems: construction of sum-type Lyapunov functions. IEEE Trans. Autom. Control 58 (2013), doi:10.1109/TAC.2012.2231552
4. Ito, H., Jiang, Z.P., Pepe, P.: A small-gain methodology for networks of iISS retarded systems based on Lyapunov-Krasovskii functionals. In: Proc. 18th IFAC World Congress, pp. 5100–5105 (2011)
5. Ito, H., Pepe, P., Jiang, Z.P.: A small-gain condition for iISS of interconnected retarded systems based on Lyapunov-Krasovskii functionals. Automatica 46, 1646–1656 (2010)
6. Ito, H., Pepe, P., Jiang, Z.P.: Decentralized iISS robustification of interconnected time-delay systems: a small-gain approach. In: Proc. 10th IFAC Workshop on Time Delay Systems, pp. 219–224 (2012)
7. Karafyllis, I., Jiang, Z.P.: A new small-gain theorem with an application to the stabilization of the chemostat. Int. J. Robust and Nonlinear Contr. 22, 1602–1630 (2012)
8. Kim, A.V.: Functional differential equations, application of i-smooth calculus. Kluwer, Dordrecht (1999)
9. Kolmanovskii, V., Myshkis, A.: Introduction to the theory and applications of functional differential equations. Kluwer, Dordrecht (1999)
10. Pepe, P.: Input-to-state stabilization of stabilizable, time-delay, control affine, nonlinear systems. IEEE Trans. Automat. Contr. 54, 1688–1693 (2009)
11. Pepe, P.: On the actuator disturbance attenuation for systems described by neutral equations. IMA J. Mathematical Control and Information 28, 163–181 (2011)

12. Pepe, P., Ito, H.: On saturation, discontinuities and delays in iISS and ISS feedback control redesign. IEEE Trans. Automat. Contr. 57, 1125–1140 (2012)
13. Pepe, P., Jiang, Z.P.: A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. Systems & Contr. Letters 55, 1006–1014 (2006)
14. Sontag, E.D.: Smooth stabilization implies coprime factorization. IEEE Trans. Automat. Contr. 34, 435–443 (1989)
15. Sontag, E.D.: Comments on integral variants of ISS. Systems & Contr. Letters 34, 93–100 (1998)
16. Surkov, A.V.: On the stability of functional-differential inclusions using invariantly differentiable Lyapunov functionals. Differential Equations 43, 1079–1087 (2007)