Consensus in Networks of Discrete-Time Multi-agent Systems: Dynamical Topologies and Delays

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Abstract. A stability analysis of general consensus algorithms in discrete-time networks of multi-agents is presented. Here, the networks can have time-varying topologies and delays, as well as nonlinearities. The Hajnal diameter approach is developed for synchronization analysis and sufficient conditions for both consensus at uniform value and synchronization at periodic trajectories are derived, which show how the periods depend on the transmission delay patterns.

1 Introduction

Consensus problems have been recognized to be important in coordination of dynamic agent systems and are widely applied in distributed computing [1], management science [2], flocking/swarming theory [3], distributed control [4], and sensor networks [5]. In these applications, the multi-agent systems need to agree on a common value for a certain quantity of interest that depends on the states of the interests of all agents or is a preassigned value. In this chapter, we consider the following dynamical system of multi-agents:

$$x_i^{t+1} = \phi_i^t\left(x_1^{t-\tau_{i1}(t)},\dots,x_m^{t-\tau_{im}(t)}\right),\ i=1,\dots,m;\ t\in\mathbb{Z}_{\geq 0}, \tag{1}$$

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where $x_i^t \in \mathbb{R}$ denotes the state of agent i at time t, $\phi_i^t : \mathbb{R}^m \to \mathbb{R}$ is a differentiable map for each t $\tau_{ij}(t)$ is the time-varying delay from agent j to agent i and $\mathbb{Z}_{\geq 0}$ denotes the discrete time, the set nonnegative integers. We suppose that the delays are uniformly bounded, i.e., $\sup_{i,j,t} \tau_{ij}(t) = \tau_M$ for some finite $\tau_M > 0$.

Let $x^t = [x_1^t, \dots, x_m^t]^\top \in \mathbb{R}^m$ and $w(t) = [x^{t\top}, x^{t-1\top}, \dots, x^{t-\tau_M\top}]^\top \in \mathbb{R}^{m(\tau_M+1)}$. We first rewrite (1) in the more abstract form

$$w(t+1) = \Phi^t(w(t)) \tag{2}$$

with $\Phi^t(\cdot) = [\Phi_0^t(\cdot)^\top, \dots, \Phi_{\tau_M}^t(\cdot)^\top]^\top$, where

$$\begin{cases} \Phi_0^t(w) & = [\phi_1^t(w), \dots, \phi_m^t(w)]^\top \\ \Phi_\tau^t(w) & = x^{t-\tau+1} \;\; \tau \geq 1. \end{cases}$$

We assume that all $\phi_i^t(\cdot)$, $i = 1, \cdots, m$, are $C^{1+\alpha}$ continuous for some $\alpha > 0$ and

$$\phi_i^t(s, \dots, s) = s \tag{3}$$

for all $s \in D(\subset \mathbb{R})$, i, and t. Eq. (2) is an abstraction and simplification of *consensus algorithm/protocol*, an interaction rule specifying the information communication between each agent and its neighborhood. In the present work, we address the question of consensus when the right-hand side of (2) contains time variations in both couplings and delays.

The condition (3) guarantees that global consensus is a solution of (1). A concept related to consensus, namely *synchronization* [6–8] , indicates that the system's diagonal, i.e. the set

$$\mathscr{S} = \left\{ u \in \mathbb{R}^m : \; u_i = u_j \in \mathbb{R}, \text{ for all } i, j = 1, \dots, m \right\}$$

is invariant under the dynamics and asymptotically attracting. Due to fact that the transmission delays $\tau_{ij}(t)$ from agent j to agent i depend on the receiver agent i, the scenario is different from the systems without delays. To specify the argument, let

$$\mathbb{S} = \left\{ w = [w^{0\top}, \cdots, w^{\tau_M\top}]^\top \in \mathbb{R}^{m(\tau_M+1)} : \; w^\tau \in \mathscr{S}, \; \forall \; \tau = 0, 1, \dots, \tau_M \right\}.$$

Under hypothesis (3), $\mathscr{S}$ may contain subsets that are invariant with respect to (2). However, the more general condition used in [9], namely $\phi_i^t(s, \dots, s) = \phi(s)$ for some function ϕ independent of index i, does not guarantee that $\mathscr{S}$ contains invariant subsets with respect to Eq. (1).

Actually, the trajectory of system (2), constrained on $\mathbb{S}$, depends on the pattern of the delays. First, let

$$\mathscr{S}_1 = \left\{ w = [w^{0\top}, \dots, w^{\tau_M\top}]^\top \in \mathbb{S} : \; w_\tau = w_{\tau'}, \text{ for all } \tau, \tau' = 0, \dots, \tau_M \right\}.$$

Each $s^* = [s, \cdots, s]^\top \in \mathscr{S}$ is an equilibrium of system (2). Next, if

$$P = \gcd\{\tau_{ij}(t) + 1 : \ i, j = 1, \ldots, m; t \in \mathbb{Z}_{\geq 0}\} > 1, \tag{4}$$

where gcd stands for the greatest common divisor, then the set

$$\mathscr{S}_P = \Big\{ w = [w^{0\top}, \ldots, w^{\tau_M\top}]^\top \in \mathbb{S} : w^k = w^{k+P}, \ \forall\, k = 0, 1, \ldots, \tau_M - P \Big\}$$

consists of invariant periodic solutions of system (2) (with period P). It can be seen that $\mathscr{S}_1$ is a special case of $\mathscr{S}_P$ when $P = 1$. In addition, restricting $\mathbb{S}$ on a local region, for example, the region D where (3) holds, we define

$$\mathscr{S}(C) = \big\{ u \in \mathbb{R}^m : \ u_i = u_j \in C, \ \text{for all } i, j = 1, \cdots, m \big\}$$

for some $C \subset \mathbb{R}$. In the same fashion, we define $\mathbb{S}(C)$, $\mathscr{S}_1(C)$ and $\mathscr{S}_P(C)$ as well.

The relationship and difference between consensus and synchronization was presented in [10]. The question we consider is whether the invariant set $\mathscr{S}_0$ or $\mathscr{S}_P$ (according to the delays' gcd) is attracting for dynamical states $[x_m^t, \ldots, x_m^t]$ outside of it, at least locally. First, this question can be translated into synchronization problem as we did in [9]. Then, upon reaching synchronization, hypothesis (3) guarantees that the synchronized trajectory should be an equilibrium or a periodic trajectory (depending of the delay patterns), instead of a general attractor on $\mathbb{S}$.

The motivation for studying (1) (or its abstract form (2)) comes initially from the basic discrete-time consensus algorithm:

$$x_i^{t+1} = \sum_{j=1}^{m} G_{ij} x_j^t, \ i = 1, \cdots, m, \tag{5}$$

where $x_i^t \in \mathbb{R}$ denotes the state variable of the agent i and $G_{ij} \geq 0$ is the nonnegative coupling strength from agent j to agent i and satisfies: $\sum_{j=1}^m G_{ij} = 1$. Define $G = [G_{ij}]_{i,j=1}^m$, which is related to the underlying connecting graph of the system, in the sense that $G_{ij} > 0$ if there is a link from node (agent) j to i and $G_{ij} = 0$ otherwise. It can be seen that G is a stochastic matrix. Then, (5) can be rewritten as

$$x^{t+1} = G x^t, \tag{6}$$

where $x^t = [x_1^t, \ldots, x_m^t]^\top$. Eq. (6) is a general model of the consensus algorithm on a network with fixed topology, which can be a directed graph and may have weights. Additionally, in many real-world applications, the connection structure may change in time, for instance when the agents are moving in physical space. One must then consider time-varying topologies under link failure or creation. Furthermore, delays occur inevitably due to limited information transmission speed. To sum up, the linear model of consensus with transmission delays can be described as

$$x_i^{t+1} = \sum_{j=1}^{m} G_{ij}(t) x_j^{t - \tau_{ij}(t)}, \tag{7}$$

where $\tau_{ij}(t)$, $i,j=1,\cdots,m$, denotes the time-dependent delay from agent j to agent i. We say that a link from j to i is *instantaneous* if $\tau_{ij}(t)\equiv 0$, and *delayed* otherwise. We will associate $G(t)=[G_{ij}(t)]_{i,j=1}^{m}$ with a directed graph sequence (see Sec. 3).

Stability analysis of the consensus in multi-agent networks (the special forms of (7) for discrete-time model) has been intensively investigated in control theory [11–19]. In our recent work [20], we have investigated consensus in dynamic networks and delays under a general stochastic framework, which provides a theoretical method to analyze stability of Eq. (7), and applied the results to analyze consensus in a mobile agent network model [21]. In this paper, we shall address this problem in the context of the general form (1).

The time variation of the connections and delays can be either deterministic or stochastic, which may have a special form, or may be driven by some other dynamical system. Let $\mathscr{Y}=\{\Omega,\mathscr{F},P(\cdot),\theta^t\}$ denote a metric dynamical system, where Ω is the metric state space, $\mathscr{F}$ is the σ-algebra, $P(\cdot)$ is the probability measure, and θ^t is a measure-preserving shift satisfying: $\theta^{t+s}=\theta^t\circ\theta^s$ and $\theta^0=id$, where id denotes the identity map. Then Eq. (7) can be regarded as a random dynamical system (RDS) driven by $\mathscr{Y}$:

$$x_i^{t+1}=\sum_{j=1}^{m}G_{ij}(\theta^t\omega)x_j^{t-\tau_{ij}(\theta^t\omega)},\ i=1,\ldots,m,\ t\in\mathbb{Z}_{\geq 0};$$

or the abstract form (2) can be rewritten as:

$$w^{t+1}=\Phi(w^t,\theta^t\omega),t\in\mathbb{Z}_{\geq 0},\omega\in\Omega. \tag{8}$$

The consensus problem under this scenario is defined in forward and almost-sure sense, i.e., convergence is attained except for a subset of ω of zero probability. For details on random dynamical systems and attractors, we refer the reader to [22].

2 Stability Analysis

In this section we present a linear stability analysis of the invariant sets $\mathscr{S}_1$ and $\mathscr{S}_P$ according to delay patterns. We first consider $\mathscr{S}_1$ in the deterministic time-varying case. We start with a boundedness condition of system (1). The notation $\pi_A(\cdot)$ denotes the orthogonal projection operator from $\mathbb{R}^{m(\tau_M+1)}$ onto a subset A.

$\mathbf{B}_1$: *There exists a neighborhood U containing $\mathscr{S}_1(D)$ such that any trajectory $w(t)$ of (2) starting in U is bounded and $\pi_{\mathscr{S}_1}(w(t))\in\mathscr{S}_1(D)$ for all t.*

Due to hypothesis (3), each point $s^*=[s,\ldots,s]^\top\in S$ is an equilibrium of (1). Using the approach in [9] the variational equations of $z(t)=x(t)-s^*$ near an equilibrium point $s^*\in\mathscr{S}$ are

$$z_i^{t+1}=\sum_{j=1}^{m}\frac{\partial\phi_i^t}{\partial x_j}(s^*)z_j^{t-\tau_{ij}(t)},\ i=1,\ldots,m. \tag{9}$$

Hypothesis (3) implies that

$$\sum_{j=1}^{m} \frac{\partial \phi_i^t}{\partial x_j}(s^*) = 1,\ i = 1,\dots,m,$$

for all t and $s^* \in \mathscr{S}(D)$. However, the Jacobian matrix $J(t) = [\frac{\partial \phi_i^t}{\partial x_j}(s^*)]_{i,j=1}^m$ is not necessary a stochastic matrix since some elements may be negative.

With $\tau_M = \sup_{i,j,t} \tau_{ij}(t)$, assumed to be finite as above, partition $J(t)$ into $J_0(t), J_1(t), \dots, J_{\tau_M}(t)$, according to the delays, such that $J(t) = \sum_{\tau=0}^{\tau_M} J_\tau(t)$, and (9) can be rewritten in the general form

$$z^{t+1} = \sum_{\tau=0}^{\tau_M} J_\tau(t) z^\tau, \tag{10}$$

where $z(t) = [z_1(t), \dots, z_m^t]^\top$. Eq. (10) can further be rewritten as

$$y^{t+1} = B(t) y^t,$$

where $y^t = [z^{t\top}, z^{t-1\top}, \dots, z^{t-\tau_M\top}]^\top$ and

$$B(t) = \begin{bmatrix} J_0(t) & J_1(t) & J_2(t) & \cdots & J_{\tau_M}(t) \\ I_m & 0 & 0 & \cdots & 0 \\ 0 & I_m & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_m & 0 \end{bmatrix}$$

with all row sums equal to 1. To state the main results, we use the concept of the Hajnal diameter introduced in [23, 24]: For a matrix A with row vectors $a_1, \dots, a_m$ and a vector norm $\|\cdot\|$ in $\mathbb{R}^m$, the Hajnal diameter of A is defined by $\mathrm{diam}(A) = \max_{i,j} \|a_i - a_j\|$. The Hajnal diameter of an infinite product of a deterministically time-varying matrix sequence $\{B(t)\}$ is defined as [9]:

$$\mathrm{diam}(B(\cdot)) = \overline{\lim_{T\to\infty}} \sup_{t_0 \ge 0} \left[\mathrm{diam}\Big(\prod_{t=t_0}^{t_0+T} B(t)\Big) \right]^{1/T}.$$

From Theorem 3.1 in [9], the following result can be concluded.

Theorem 1. *Under the hypothesis* $\mathbf{B}_1$, *if* $\sup_{s^* \in \mathscr{S}_1(D)} \mathrm{diam}(B(\cdot)) < 1$, *then system (1) is (locally) stable with respect to* $\mathscr{S}_1(D)$, *that is, there exists a sufficiently small neighborhood* U *of* $\mathscr{S}_1(D)$ *such that for any initial condition in* U, *the trajectory converges to an equilibrium in* $\mathscr{S}_1(D)$.

In fact, Theorem 3.1 in [9] assumes that there exists an attractor for the system restricted to $\mathscr{S}_1$, which is needed to guarantee that the projection of the trajectory on $\mathscr{S}_1$ are kept in the bounded region defined by the attractor. Here, condition $\mathscr{B}_1$ guarantees that the projection of the trajectory of Eq. (2) with initial condition in U onto $\mathscr{S}_1$ is still in D. So, the proof of Theorem 3.1 in [9] is valid for this theorem, and in addition, this condition also guarantees that hypothesis (3) holds for the system restricted to $\mathscr{S}_1$. When (1) converges to $\mathscr{S}_1(D)$, according to the form of (3), the synchronized trajectory should be an equilibrium. In other words, system (2) reaches consensus, as all agents converge to a uniform value.

If the time-variation is driven by a stochastic process, then the system (1) or (2) becomes the random dynamical system (8). Let

$$V_\lambda^t = \left\{ \omega \in \Omega : \left[\mathrm{diam}(\prod_{k=0}^{t} B(\theta^k \omega)) \right]^{1/t} < \lambda \right\}.$$

From Theorem 4.3 in [25], we have:

Theorem 2. *Under hypothesis* $\mathbf{B}_1$*, if there exists some* $\lambda \in (0,1)$ *such that* $\sum_{t=0}^{\infty} P(V_\lambda^t) < +\infty$*, then (8) is (locally) stable with respect to* $\mathscr{S}_1(D)$ *in the almost sure sense, that is, for almost every* $\omega \in \Omega$*, there exists a sufficient small neighborhood* $U(\omega)$ *(possibly depending on* ω*) of* $\mathscr{S}_1(D)$ *such that for any initial condition in* U*, the trajectory of (8) converges to an equilibrium in* $\mathscr{S}_1(D)$*.*

We note that the equilibrium depends also on ω. In [25], a sufficient condition was stated in terms of the normal Lyapunov exponent, which was proved to be equivalent to the Hajnal diameter in [9].

We next consider synchronized periodic solutions under condition (4). Note that each $t \in \mathbb{Z}_{\geq 0}$ can be written as $t = kP + l$ for some $k \geq 0$ and $l = 0, \dots, P-1$. Eq. (1) implies that the state at $t+1$ depends on the states at $t - \tau_{ij}(t)$, $i, j = 1, \dots, m$. We can write $\tau_{ij}(t) = z_{ij}(t)P - 1$, owing to hypothesis (3). Therefore, $\mathrm{mod}\,(t - \tau_{ij}(t), P) = l + 1$, which is equal to $\mathrm{mod}\,(t+1, P)$ (if $l = P-1$, then $l+1$ equals to 0 in modulus), where $\mathrm{mod}(a,b)$ denotes the remainder of a divided by b. In other words, hypothesis (4) implies that the state of node at time $t+1$ depends only on those states at the time points that have the same remainder with respect to P. Therefore, after permutation of the $\tau_M + 1$ components in $w^t = [x^{t\top}, \dots, x^{t-\tau_M\,\top}]^\top$ such that the time with the same remainder with respect to P are brought together, i.e., $\tilde{w}^t = \left[(\tilde{w}_0^t)^\top, \dots, (\tilde{w}_{P-1}^t)^\top \right]^\top$ with $\tilde{w}_k^t = \left[(x^{t-k})^\top, (x^{t-P-k})^\top, \dots, (x^{t-(\tau_M+1)+P-k})^\top \right]^\top$ for all $k = 0, \dots, P-1$, system (2) has the following block form:

$$\tilde{w}_k^{t+1} = \tilde{\Phi}_k^t \left(\tilde{w}_k^t \right), \; k = 0, \dots, P-1, \; t \in \mathbb{Z}_{\geq 0},$$

with $\tilde{\Phi}_k^t = \left[\tilde{\Phi}_{k,0}^t, \dots, \tilde{\Phi}_{k,n}^t \right]^\top$ $(n = (\tau_M + 1)/P - 1)$, and

$$\tilde{\Phi}_{k,z}^t = \begin{cases} [\phi_1^{t-k}(\cdot)^\top, \dots, \phi_m^{t-k}(\cdot)^\top]^\top & z = 0, \\ x^{t-z\tau_0-k} & z > 0. \end{cases}$$

By linearization, with the same permutation of y^t with that of w^t, we can bring the variational equation near each periodic solution w^* into the form

$$\tilde{y}^{t+1} = \tilde{B}(t)\tilde{y}^t \tag{11}$$

with block-diagonal $\tilde{B}(t)$:

$$\tilde{B}(t) = \mathrm{diag}[\tilde{B}_r(t)]_{r=0}^{P-1}.$$

Thus, after a partition of $\tilde{y} = [\tilde{y}_0^\top, \cdots, \tilde{y}_{P-1}^\top]^\top$, (11) has the block form

$$\tilde{y}_r^{t+1} = \tilde{B}^r(t)\tilde{y}_r^t,\ r = 0, \ldots, P-1. \tag{12}$$

A similar hypothesis to $\mathbf{B}_1$ can be stated as

$\mathbf{B}_2$: *There exists neighborhood U containing $\mathscr{S}_P(D)$ such that any trajectory $w(t)$ of (2) starting in U is bounded and $\pi_{\mathscr{S}_P}(w(t)) \in \mathscr{S}_P(D)$ for all $t \in \mathbb{Z}_{\geq 0}$.*

Then, from Theorem 3.1 in [9], we have

Theorem 3. *Under the hypothesis $\mathbf{B}_2$, if $\sup_{w^* \in \mathscr{S}_P(D)} \max_{r=0,\ldots,P-1} \mathrm{diam}(\tilde{B}^r(\cdot)) < 1$, then system (2) is (locally) stable with respect to $\mathscr{S}_P(D)$, that is, there exists a sufficiently small neighborhood U of $\mathscr{S}_P(D)$ such that from any initial condition in U, the trajectory converges to a periodic trajectory in $\mathscr{S}_P(D)$.*

In a similar fashion as in Theorem 2, if the time-variation is driven by a metric dynamical system $(\Omega, \mathscr{F}, P, \theta^t)$, i. e., Eq. (10) becomes a RDS:

$$\tilde{y}^{t+1} = \tilde{B}(\theta^t \omega)\tilde{y}^t, \tag{13}$$

then letting

$$\tilde{V}_\lambda^t = \left\{ \omega \in \Omega : \max_{r=0,\ldots,P-1} \left[\mathrm{diam}\left(\prod_{k=0}^{t} \tilde{B}^r(\theta^k \omega)\right) \right]^{1/t} < \lambda \right\},$$

we can state the following result.

Theorem 4. *Under hypothesis $\mathbf{B}_2$, if there exists some $\lambda \in (0,1)$ such that $\sum_{t=0}^{\infty} P(\tilde{V}_\lambda^t) < +\infty$, then (8) is (locally) stable with respect to $\mathscr{S}_P(D)$ in the almost sure sense, that is, for almost every $\omega \in \Omega$, there exists a sufficient small neighborhood $U(\omega)$ (possibly depending on ω) of $\mathscr{S}_P(D)$ such that for any initial condition in U, the trajectory of (8) converges to a periodic trajectory in $\mathscr{S}_P(D)$.*

Theorems 1 and 2 can be regarded as special cases of Theorems 3 and 4, respectively, when $P = 1$.

Remark 1. Hypotheses $\mathbf{B}_{1,2}$ can be satisfied if system (2) is essentially bounded, i.e., there exists a bounded region $Q \subset \mathbb{R}^{m(\tau_M+1)}$ such that any trajectory enters Q for all $t \geq T$ for a sufficiently large T. Then the set D can be derived by projecting the convex closure of Q onto $\mathscr{S}_1$ or $\mathscr{S}_P$, respectively.

3 Linear Model

Eq. (7) can be regarded as a special case of (1) with $\Phi_i^t = \sum_{j=1}^m G_{ij}(t) x_j^{t-\tau_{ij}^t}$. However, in such a linear model the stability is always global, instead of local for nonlinear systems. In this section, we provide the main results in terms of matrix and graph theories for linear models (7) or (8). The link between stochastic matrices and graphs is an essential feature here.

A stochastic (or simply nonnegative) matrix $A = [a_{ij}]_{i,j=1}^{m} \in \mathbb{R}^{m,m}$ defines a graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, where $\mathscr{V} = \{1,\dots,m\}$ denotes the *node (agent) set* with m nodes and $\mathscr{E}$ denotes the *link set* where there exists a directed link from node j to i (i.e., $e(i,j)$ exists) if and only if $a_{ij} > 0$. We denote this graph corresponding to the stochastic matrix A by $\mathscr{G}(A)$. The node i is said to be self-linked if $e(i,i)$ exists, i.e., $a_{ii} > 0$. The node i can *access* the node j, or equivalently, the node j is *accessible* from the node i, if there exists a path from i to j. The graph $\mathscr{G}$ has a *spanning tree* if there exists a node i which can access all other nodes. The graph $\mathscr{G}$ is said to be *strongly connected* if each node is a root. We refer the interested reader to the book [26] for more details. Due to the relationship between nonnegative matrices and graphs, we can call upon and switch between their respective properties as needed. For example, the indecomposability of a nonnegative matrix A is equivalent to that $\mathscr{G}(A)$ has a spanning tree, and the aperiodicity of a graph is associated with the aperiodicity of its corresponding matrix [27]. For a sequence of nonnegative matrices $A(t)$, we can define a graph sequence associated with $A(t)$: $\mathscr{G}(t) = \mathscr{G}(A(t))$. The union of several graphs $\{\mathscr{G}_i,\ i = 1,\dots,p\}$ on the same node set is the union of their link sets.

For a nonnegative matrix A and a given $\delta > 0$, the δ-*matrix* of A, denoted by A^{δ}, is defined as

$$[A^{\delta}]_{ij} = \begin{cases} \delta, \text{ if } A_{ij} \geq \delta; \\ 0, \ \text{if } A_{ij} < \delta. \end{cases}$$

The δ-*graph* of A is the directed graph corresponding to the δ-matrix of A. We can then state the following result for the stability of $\mathscr{S}_1$ (noting that in the linear model, $D = \mathbb{R}$).

Theorem 5. *[12] Suppose there exist $\mu > 0$, $L \in \mathbb{Z}_{\geq 0}$, and $\delta > 0$ such that $G^0(\sigma) > \mu I_m$ for all $\sigma \in \Omega$ and the δ-graph of $\sum_{k=n+1}^{n+L} G(k)$ has a spanning tree for all $n \in \mathbb{Z}_{\geq 0}$. Then system (7) is (globally) stable with respect to $\mathscr{S}_1$, i. e., it reaches consensus.*

In fact, with $D = \mathbb{R}$, if the condition in this theorem is satisfied, there exist a sufficiently large integer T and $\lambda \in (0,1)$ such that $\operatorname{diam}\left(\prod_{k=n+1}^{n+T'} G(k)\right) < \lambda^{T'}$ for any $T' > T$. Hence, the conditions in Theorem 1 hold.

We rewrite system (7) in the general form

$$x_i^{t+1} = \sum_{\tau=0}^{\tau_M} \sum_{j=1}^{m} G_{ij}^{\tau}(t) x_j^{t-\tau}, \quad i = 1,\dots,m, \tag{14}$$

by partitioning the inter-links according to delays, as well as in the matrix form

$$x^{t+1} = \sum_{\tau=0}^{\tau_M} G^{\tau}(t) x^{t-\tau}, \tag{15}$$

where $G^{\tau}(t) = [G^{\tau}_{ij}(t)]^{m}_{i,j=1}$. In some cases delays occur at self-links, for example when it takes time for each agent to process its own information. Suppose that the self-linking delay for each node is identical, that is, $\tau_{ii} = P-1 > 0$. We classify each integer t in the discrete-time set $\mathbb{Z}_{\geq 0}$ (or the whole integer set $\mathbb{Z}$) via $\mathrm{mod}(t+1,P)$ as the quotient group of $(\mathbb{Z}+1)/P$. As a default set-up, we denote $\langle i\rangle_P$ by $\langle i\rangle$. Let $\hat{G}^{i}(\cdot) = \sum_{j\in\langle i\rangle} G^{j}(\cdot)$. We have the following result for the stability of $\mathscr{S}_P$.

Theorem 6. *Assume that*

(1) Hypothesis (4) holds for $P > 0$*;*
(2) $\tau_{ii}(t) = P-1$ *for all* $i = 1,\dots,m$*;*
(3) $G^{P-1}(t) > \mu I_m$ *for some* $\mu > 0$ *and all* $t \in \mathbb{Z}_{\geq 0}$*.*

Suppose further that there exist $L \in \mathbb{Z}_{\geq 0}$ *and* $\delta > 0$ *such that the* δ*-graph of* $\sum_{k=n+1}^{n+L} \hat{G}^{0}(k)$ *is strongly connected for all* $n \in \mathbb{Z}_{\geq 0}$*. Then system (14) is (globally) stable with respect to* $\mathscr{S}_P$*, i. e., it synchronizes to a P-periodic trajectory.*

This theorem can be proved as a consequence from Theorem 3 in a similar fashion as the proof of Theorem 3.4 in [21], but by removing the discussion of randomness, since here we consider deterministic time-variation.

The time-variation can be random, e. g., induced by a stochastic process σ^t. In [20, 21], we considered the case when $\{\sigma^t\}$ is an *adapted stochastic process*: Let $\{A_k\}$ be a stochastic process defined on the basic probability space $\{\Omega, \mathscr{F}, P\}$, with the state space Ω, the σ-algebra $\mathscr{F}$, and the probability function $\mathbb{P}$. Let $\{\mathscr{F}^k\}$ be a *filtration*, i. e., a sequence of nondecreasing sub-σ-algebras of $\mathscr{F}$. If A_k is measurable with respect to (w.r.t.) $\mathscr{F}^k$, then the sequence $\{A_k, \mathscr{F}^k\}$ is called an adapted process. Let $\mathbb{E}(\cdot|\mathscr{F}^t)$ denote the conditional expectation with respect to σ-algebra $\mathscr{F}^t$. Then, Eq. (15) becomes

$$x^{t+1} = \sum_{\tau=0}^{\tau_M} G^{\tau}(\sigma^t) x^{t-\tau}. \tag{16}$$

This adapted process can be regarded as a metric dynamical system with invariant probability, $\{\boldsymbol{\Omega}, \mathbb{F}, \mathbb{P}, \theta^t\}$, where $\boldsymbol{\Omega} = \Omega^{\mathbb{Z}_{\geq 0}}$, i. e., each element is the sequence $\{\sigma^t\}_{t\geq 0}$, $\mathbb{F} = \mathscr{F}^{\mathbb{Z}_{\geq 0}}$ is the infinite Cartesian product of $\mathscr{F}$, $\mathbb{P}$ coincides with the intrinsic probability P, and θ^t is the shift map: $\theta\omega = \{\sigma^t\}_{t\geq 1}$. The following results are the stochastic versions of Theorems 5 and 6.

Theorem 7. *[20, 21] Suppose there exist* $\mu > 0$*,* $L \in \mathbb{Z}_{\geq 0}$*, and* $\delta > 0$ *such that* $G^{0}(\sigma) > \mu I_m$ *for all* $\sigma \in \Omega$ *and the* δ*-graph of* $\mathbb{E}\{\sum_{k=n+1}^{n+L} G(\sigma^k)|\mathscr{F}^n\}$ *has a spanning tree for all* $n \in \mathbb{Z}_{\geq 0}$ *almost surely. Then (16) is stable with respect to* $\mathscr{S}_1$ *almost surely (i.e. with probability one).*

Theorem 8. *[20, 21] Assume that*

(1) Hypothesis (4) holds;
(2) $\tau_{ii}(t) = P-1$ *for all* $i = 1,\dots,m$*;*
(3) $G^{P-1}(t) > \mu I_m$ *for some* $\mu > 0$ *and all* $t \in \mathbb{Z}_{\geq 0}$*.*

Suppose further that there exist $L \in \mathbb{Z}_{\geq 0}$ *and* $\delta > 0$ *such that the* δ*-graph of* $\mathbb{E}\{\sum_{k=n+1}^{n+L} \hat{G}^{0}(\sigma^k)|\mathscr{F}^n\}$ *is strongly connected for all* $n \in \mathbb{Z}_{\geq 0}$ *almost surely. Then (16) is stable with respect to* $\mathscr{S}_P$ *almost surely (i.e. with probability one).*

4 Multi-agent Model with Nonlinear Coupling

In this section, we present a stability analysis of a class of nonlinear multi-agent models

$$x_i^{t+1} = \sum_{j=1}^{m} \psi_{ij}^t \left(x_j^{t-\tau_{ij}(t)} - x_i^{t-\tau_{ii}(t)} \right) x_j^{t-\tau_{ij}(t)}, \; i = 1, \ldots, m, \; t \in \mathbb{Z}_{\geq 0}, \tag{17}$$

where $\psi_{ij}^t(\cdot)$ is a (time-dependent) nonlinear function that denotes the coupling strength from agent j to agent i, acting on the difference between the states of the two nodes under the presence of delays. We assume that $\psi_{ij}^t(s)$ is $C^{1+\alpha}$ continuous for some $\alpha > 0$ and attains its maximum value, which is assumed to be nonzero, at $s = 0$. In other words, the coupling strength is maximum when the (delayed) states are equal. Thus, $d\psi_{ij}^t(s)/ds|_{s=0} = 0$ for all $i, j = 1, \ldots, m$ and $t \in \mathbb{Z}_{\geq 0}$. For example, $\psi_{ij}^t(\cdot)$ can be chosen from a class of Gaussian-type kernel functions. In addition, to guarantee that (3) holds, we also assume that $\sum_{j=1}^m \psi_{ij}^t(0) = 1$ for all i and t.

The variational equation near $\mathscr{S}_1$ or $\mathscr{S}_P$ under the assumption (4) is:

$$\delta x_i^{t+1} = \sum_{j=1}^{m} \psi_{ij}^t(0) \delta x_j^{t-\tau_{ij}(t)}, \; i = 1, \cdots, m \; , t \in \mathbb{Z}_{\geq 0}. \tag{18}$$

It has the similar form of (7). Let $\Psi^0(t) = [\psi_{ij}^t(0)]_{i,j=1}^m$ and $\tilde{\Psi}_r^0(t)$ be defined in the same fashion as done in Eqs. (11) and (12). Then we have the following result.

Theorem 9. *Assume all conditions mentioned above for* $\psi_{ij}^t(\cdot)$ *hold.*

(1) Under hypothesis $\mathbf{B}_1$ *for some* $D \subset \mathbb{R}$, *if* $\operatorname{diam}\left(\Psi^0(\cdot)\right) < 1$, *then system (17) is (locally) stable with respect to* $\mathscr{S}_1(D)$;

(2) Under the hypothesis $\mathbf{B}_2$ *for some* $D \subset \mathbb{R}$, *if* $\operatorname{diam}\left(\tilde{\Psi}_r^0(\cdot)\right) < 1$ *for all* $r = 0, \ldots, P-1$, *then system (17) is (locally) stable with respect to* $\mathscr{S}_P(D)$.

In addition, if $\psi_{ij}^t(0)$ are all nonnegative, then $\{\Psi^0(t)\}$ are stochastic matrices, and the conditions in Theorem 9 can be "translated" in terms of graphs associated with the stochastic matrix sequence $\{\Psi^0(t)\}$, namely, into the conditions in Theorems 6 and 7.

When the time-variation is induced by a stochastic process, or generally by a metric dynamical system, the results of Theorems 2 and 4 can be applied to derive sufficient conditions for consensus in the almost surely sense for system (17). Combined with the graph theory used in [20], if $\{\Psi^0(t)\}$ are stochastic matrices, we can derive sufficient conditions for consensus like Theorem 7 and 8. We omit the details due to space constraints.

5 Numerical Examples: Dynamical Networks for Random Waypoint Model

We perform numerical examples to illustrate the results by the "random waypoint" (RWP) model, which is a widely used model in performance evaluation of protocols of ad hoc networks, first introduced in [29]. We use the same set-up of the model as done in [21] to mimic time-varying graph topologies and realize the random waypoint model in a 1000×1000 (m^2) square area, where the agent i moves towards a randomly selected target in this area following the uniform distribution. The velocity of movement is also random, with a uniform distribution in $[10,20]$ (m/sec). After approaching the target, the agent waits for a random time period following the uniform distribution in $[1,5]$ (sec). Moreover, each agent's behavior is stochastically independent of the others. The links between agents are generated such that each agent is linked to the agents whose distance is not more than R. We take $R = 120$ (m). There are 50 independent mobile agents in the network, whose location and status of the agents can be modeled as a homogeneous Markov chain [21].

We set up two models of multi-agent systems on the RWP network. The first one is a linear model (stated in the form of (7)):

$$x_i^{t+1} = \frac{1}{\#\mathcal{N}_i^t} \sum_{j \in \mathcal{N}_i(t)} x_j^{t-\tau_{ij}^t}, \; i = 1, \ldots, m, \tag{19}$$

where $\mathcal{N}_i(t)$ denotes the neighborhood of agent i at time t and $\#F$ denotes the number of the elements in a finite set F. The second model is a special case of (17) with coupling functions:

$$\psi_{ij}^t(s) = \frac{1}{\#\mathcal{N}_i(t)} \exp(-\frac{s^2}{2}). \tag{20}$$

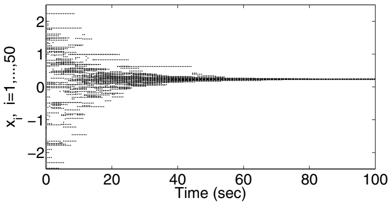
It can be verified that all conditions of $\psi_{ij}^t(\cdot)$ in Section 4 are satisfied. We assume that the self-links exist for all nodes. Thus $\mathcal{N}_i(t)$ is always nonempty.

We consider discrete time with a 0.01 (sec) time interval. Each agent operates according to the algorithm (19). Transmission delays exist due to finite information transmission speed and storage buffer. Since the speed of information transmission is typically much higher than the movement of agents, we omit the displacement of the information transmission caused by the movement of agents and define the delays (0.01 sec) as:

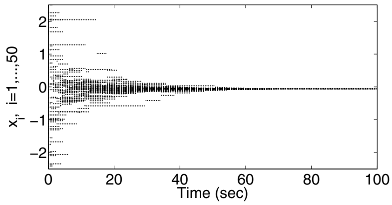
$$\tau_{ij}(\sigma^t) = \lambda \lfloor \frac{d_{ij}^t}{v_s} \rfloor + \tau_0, \tag{21}$$

where d_{ij}^t (m) denotes the distance between agents i and j in the two-dimensional space at time t, v_s denotes the transmission speed of information, $\lfloor \cdot \rfloor$ denotes the floor function, i.e., the largest integer less than or equal to its argument, λ is a scaling parameter representing the ratio of the time scale of movement of the agents and that of the information transmission and processing among agents, and τ_0 (0.01 sec) denotes the identical self-linking delay.

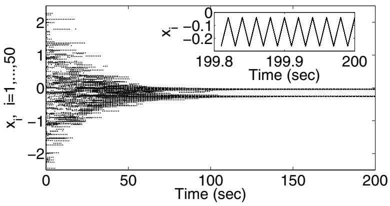
Following the arguments in [21], the network has a positive probability of being a complete network, with respect to the stationary probability distribution. This implies that the expectation, with respect to the stationary probability distribution, of the graph topology is a complete graph. Hence, for the case of existence of self-links, the conditions of Theorems 7 and 8 are satisfied. In the absence of self-links, for any initial network graph, there are a path of finite length and a positive probability such that all agents enter a disc with radius less than R. So, the conditional expectation of product of the matrices has a positive probability of being complete. This implies that the conditional expectation is complete. In a similar way, conditions for consensus can be verified for system (17) with (20) as well.



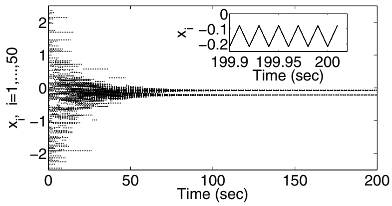
(a) Convergence of (19) with $\lambda = 1$ and $\tau_0 = 0$.



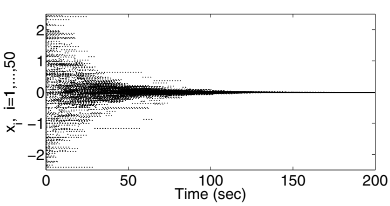
(b) Convergence of (17) with (20), $\lambda = 1$ and $\tau_0 = 0$.



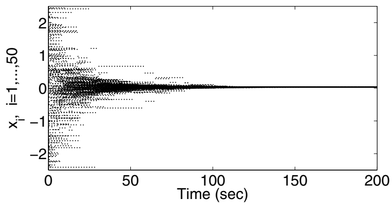
(c) Convergence of (19) with $\lambda = 2$ and $\tau_0 = 1$.



(d) Convergence of (17) with (20), $\lambda = 2$ and $\tau_0 = 1$.



(e) Convergence of (19) with $\lambda = 2$ and $\tau_0 = 2$.



(f) Convergence of (17) with (20), $\lambda = 2$ and $\tau_0 = 2$.

Fig. 1 Convergence dynamics of the multi-agent systems (19) (left column) and (17) with (20) (right column) in RWP networks. The insets show the terminal synchronous orbits.

We fix $v_s = 3000$ (m/sec) and pick different values of τ_0 and λ to illustrate the synchronous or consensus dynamics as mentioned in Theorems 7, 8, and 9.

First, we choose $\lambda = 1$ and $\tau_0 = 0$. Theorem 7 indicates that the multi-agent system (19) reaches consensus. Fig. 1(a) depicts the consensus dynamics of (19) with the delays (21) with respect to $\mathscr{S}_1$. We also observe that system (17) with coupling function (20) reaches consensus, as shown in Fig. 1(b).

We next take $\lambda = 2$ and $\tau_0 = 1$. Thus, the delays can be picked only in the set $\{1,3,5,7,9\}$ and each value in this set can be a possible delay in (21). One can see that $\gcd(\tau_{ij}+1: \ i,j = 1,\dots,m; t \in \mathbb{Z}_{\geq 0}) = 2$. Theorem 8 yields that (19) cannot reach consensus but must instead synchronize to a 2-periodic trajectory. The same conclusion holds also for system (17) with coupling function (20). Fig. 1(c) and 1(d) show the synchronous dynamics of systems (19) and (17) with (20) and the delays (21), $\lambda = 2$, and $\tau_0 = 1$.

Finally, we choose $\lambda = 2$ and $\tau_0 = 2$. Thus, the delays can be picked only from the set $\{2,4,6,8,10\}$. We have $\gcd(\tau_{ij}+1): \ i,j = 1,\dots,m; t \in \mathbb{Z}_{\geq 0}) = 1$. From Theorem 8, similar arguments indicate that (19) reaches consensus, i. e., synchronizes at a periodic trajectory with period $P = 1$. The same conclusion holds for the system (17) with (20). Fig. 1(e) and 1(f) indicate the consensus dynamics with the delays (21), $\lambda = 2$, and $\tau_0 = 2$.

6 Conclusion

We have presented an analysis of consensus problem in discrete-time networks of multi-agent systems, based on our previous results in [9,20,21,25]. Here the model is general, including the linear consensus model as a special example. When the time variation is driven by a metric dynamical system, multi-agent systems become random dynamic systems. Based on a Hajnal diameter approach that we developed for synchronization analysis, we have presented sufficient conditions for both consensus at a uniform value and synchronization at a periodic trajectory, and shown how the periods depend on the transmission delay patterns. As special examples, we have re-derived the stability results for the consensus of the linear model and derived sufficient conditions for the stability of a class of delayed multi-agent systems with nonlinear coupling. To illustrate the theoretical results, we have presented two consensus algorithms in a mobile-agent model under transmission delays.

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