Chapter 3
Stochastic Processes

For me problem-solving is the most interesting thing in life. To be handed something that's a complete mess and straighten it out. To organize where there is no organization. To give form to a medium that has no form.

—Sylvester Weaver

This chapter is an extension of the previous chapter. In the previous chapter, we focused essentially on random variables. In this chapter, we introduce the concept of *random* (or *stochastic*) *process* as a generalization of a random variable to include another dimension—time. While a random variable depends on the outcome of a random experiment, a random process depends on both the outcome of a random experiment and time. In other words, if a random variable X is time-dependent, X(t) is known as a *random process*. Thus, a random process may be regarded as any process that changes with time and controlled by some probabilistic law. For example, the number of customers N in a queueing system varies with time; hence N(t) is a random process

Figure 3.1 portrays typical *realizations* or *sample functions* of a random process. From this figure, we notice that a random process is a mapping from the sample space into an ensemble (family, set, collection) of time functions known as sample functions. Here $X(t,s_k)$ denotes the sample function or a realization of the random process for the s_k experimental outcome. It is customary to drop the s variable and use X(t) to denote a random process. For a fixed time t_1, $X(t_1) = X_1$ is a random variable. Thus,

A **random (or stochastic) process** is a family of random variables X(t), indexed by the parameter t and defined on a common probability space.

It should be note that the parameter t does not have to always represent time; it can represent any other variable such as space.

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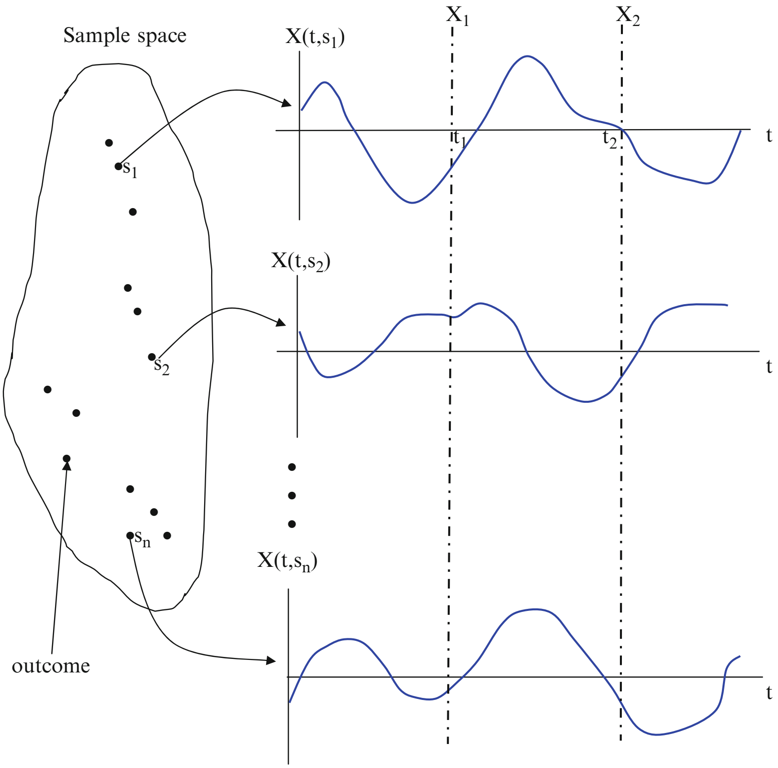


Fig. 3.1 Realizations of a random process

In this chapter, we discuss random processes, their properties, and the basic tools used for their mathematical analysis. Specifically, we will discuss random walks, Markov processes, birth-death processes, Poisson processes, and renewal processes. We will also consider how the concepts developed can be demonstrated using MATLAB.

3.1 Classification of Random Processes

It is expedient to begin our discussion of random processes by developing the terminology for describing random processes [1–3]. An appropriate way of achieving this is to consider the various types of random processes. Random processes may be classified as:

- Continuous or discrete
- Deterministic or nondeterministic
- Stationary or nonstationary
- Ergodic or nonergodic

3.1.1 Continuous Versus Discrete Random Process

A *continuous-time random process* is one that has both a continuous random variable and continuous time. Noise in transistors and wind velocity are examples of continuous random processes. So are Wiener process and Brownian motion. A *discrete-time random process* is one in which the random variables are discrete, i.e. it is a sequence of random variables. For example, a voltage that assumes a value of either 0 or 12 V because of switching operation is a sample function from a discrete random process. The binomial counting and random walk processes are discrete processes. It is also possible to have a mixed or hybrid random process which is partly continuous and partly discrete.

3.1.2 Deterministic Versus Nondeterministic Random Process

A *deterministic random process* is one for which the future value of any sample function can be predicted from a knowledge of the past values. For example, consider a random process described by

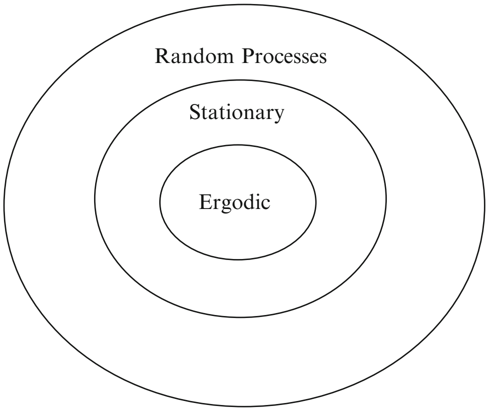
$$X(t) = A\cos(\omega t + \Phi) \tag{3.1}$$

where A and ω are constants and Φ is a random variable with a known probability distribution. Although X(t) is a random process, one can predict its future values and hence X(t) is deterministic. For a *nondeterministic random process*, each sample function is a random function of time and its future values cannot be predicted from the past values.

3.1.3 Stationary Versus Nonstationary Random Process

A *stationary random process* is one in which the probability density function of the random variable does not change with time. In other words, a random process is stationary when its statistical characteristics are time-invariant, i.e. not affected by a shift in time origin. Thus, the random process is stationary if all marginal and joint density functions of the process are not affected by the choice of time origin. A *nonstationary random process* is one in which the probability density function of the random variable is a function of time.

Fig. 3.2 Relationship between stationary and ergodic random processes



3.1.4 Ergodic Versus Nonergodic Random Process

An ergodic random process is one in which every member of the ensemble possesses the same statistical behavior as the entire ensemble. Thus, for ergodic processes, it is possible to determine the statistical characteristic by examining only one typical sample function, i.e. the average value and moments can be determined by time averages as well as by ensemble averages. For example, the nth moment is given by

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)dx = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^n(t)dt \tag{3.2}$$

This condition will only be satisfied if the process is stationary. This implies that ergodic processes are stationary as well. A nonergodic process does not satisfy the condition in Eq. (3.2). All non-stationary processes are nonergodic but a stationary process could also be nonergodic. Figure 3.2 shows the relationship between stationary and ergodic processes. These terms will become clearer as we move along in the chapter.

Example 3.1 Consider the random process

$$X(t) = \cos(2\pi t + \Theta)$$

where Θ is a random variable uniformly distributed on the interval $[0,2\pi]$.

Solution

We are given an analytic expression for the random process and it is evident that it is a continuous-time and deterministic random process. Figure 3.3 displays some sample functions or realizations of the process.

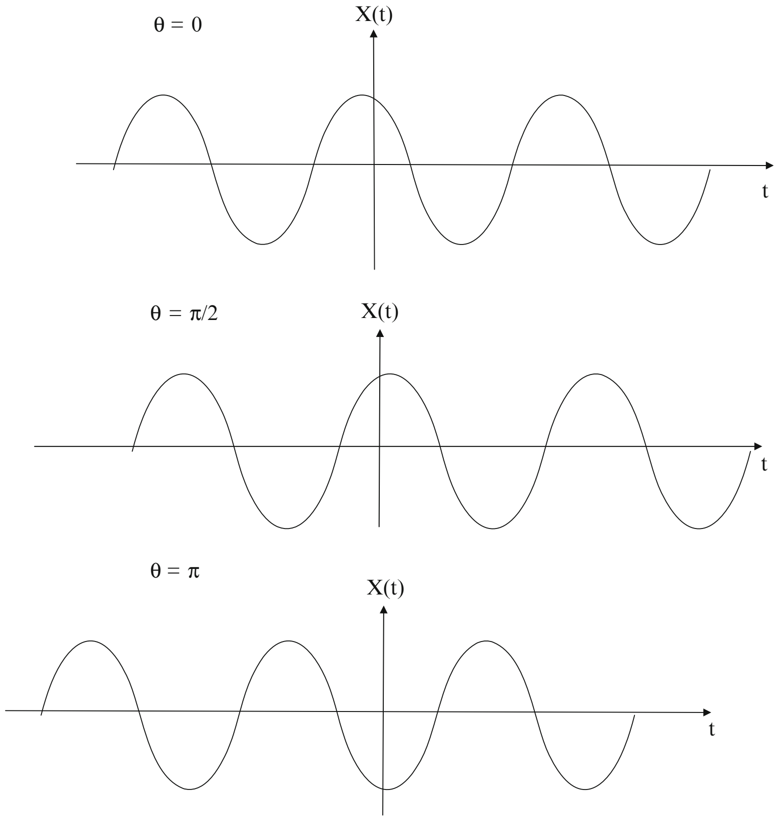


Fig. 3.3 For Example 3.1; sample functions of the random process

3.2 Statistics of Random Processes and Stationarity

Since a random process specifies a random variable at any given time, we can find the statistical averages for the process through the statistical averages of the corresponding random variables. For example, the first-order probability density function (PDF) for a random process X(t) is $f_X(x;t)$, while the corresponding first-order cumulative distribution function (CDF) of X(t) is

$$F_X(x;t) = P[X(t) \leq x] = \int_{-\infty}^{x} f_X(\lambda;t)d\lambda \tag{3.3}$$

or

$$f_X(x;t) = \frac{\partial F_X(x;t)}{\partial x} \tag{3.4}$$

Similarly, if $X(t_1) = X_1$ and $X(t_2) = X_2$ represent two random variables of a random process X(t), then their joint distributions are known as second-order PDF and CDF, which are related as

$$\begin{aligned} F_X(x_1, x_2; t_1, t_2) &= P[X(t_1) \le x_1, X(t_2) \le x_2] \\ &= \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_X(\lambda_1, \lambda_2; t_1, t_2) d\lambda_1 d\lambda_2 \end{aligned} \tag{3.5}$$

or

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2} \tag{3.6}$$

In general, the joint distributions of n random variables $X(t_1) = X_1, X(t_2) = X_2, \ldots, X(t_n) = X_n$ provide the nth-order PDF and CDF of a random process X(t) and are related as

$$\begin{aligned} F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) &= P\left[X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_n) \le x_n\right] \\ &= \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_X(\lambda_1, \lambda_2, \ldots, \lambda_n; t_1, t_2, \ldots, t_n) d\lambda_1 d\lambda_2 \ldots d\lambda_n \end{aligned} \tag{3.7}$$

or

$$f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{\partial F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)}{\partial x_1 \partial x_2 \ldots \partial x_n} \tag{3.8}$$

A random process X(t) is said to be *strictly stationary of order n* if its nth-order PDF and CDF are time-invariant, i.e.

$$\begin{aligned} &F_X\left(x_1, x_2, \ldots, x; t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau\right) \\ &= F_X\left(x_1, x_2, \ldots, x; t_1, t_2, \ldots, t_n\right) \end{aligned} \tag{3.9}$$

i.e. the CDF depends only on the relative location of $t_1, t_2, \ldots, t_n$ and not on their direct values.

We say that $\{X_k\},\ k = 0, 1, 2, \cdots, n$ is an independent process if and only if

$$F_X(x_0, x_1, \cdots, x_n; t_0, t_1, \cdots, t_n) = F_{X_0}(x_0; t_0) F_{X_1}(x_1; t_1) \cdots F_{X_n}(x_n; t_n)$$

In addition, if all random variables are drawn from the same distribution, the process is characterized by a single CDF, $F_{X_k}(x_k; t_k),\quad k = 0, 1, 2, \cdots, n$. In this case, we call $\{X_k\}$ a sequence of independent and identically distributed (IID) random variables.

Having defined the CDF and PDF for a random process X(t), we are now prepared to define the statistical (or ensemble) averages—the mean, variance, autocorrelation, and autocovariance of X(t). As in the case of random variables, these statistics play an important role in practical applications.

The *mean* or *expected value* of the random process X(t) is

$$m_X(t) = \overline{X(t)} = E[X(t)] = \int_{-\infty}^{-\infty} x f_X(x;t)dx \tag{3.10}$$

where E[•] denotes ensemble average, $f_X(x;t)$ is the PDF of X(t) and X(t) is regarded as a random variable for a fixed value of t. In general, the mean $m_X(t)$ is a function of time.

The *variance* of a random process X(t) is given by

$$\mathrm{Var}(X) = \sigma_X^2 = E\left[(X(t) - m_X(t))^2\right] = E\left[X^2\right] - m_X^2 \tag{3.11}$$

The *autocorrelation* of a random process X(t) is the joint moment of $X(t_1)$ and $X(t_2)$, i.e.

$$\boxed{R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)dx_1 dx_2} \tag{3.12}$$

where $f_X(x_1,x_2;t_1,t_2)$ is the second-order PDF of X(t). In general, $R_X(t_1,t_2)$ is a deterministic function of two variables t_1 and t_2. The autocorrection function is important because it describes the power-spectral density of a random process.

The *covariance* or *autocovariance* of a random process X(t) is the covariance of $X(t_1)$ and $X(t_2)$, i.e.

$$\mathrm{Cov}[X(t_1), X(t_2)] = C_X(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}] \tag{3.13a}$$

Or

$$\boxed{\mathrm{Cov}[X(t_1), X(t_2)] = R_X(t_1, t_2) - m_X(t_1)m_X(t_2)} \tag{3.13b}$$

indicating that the autocovariance can be expressed in terms of the autocorrelation and the means. Note that the variance of X(t) can be expressed in terms of its autocovariance, i.e.

$$\mathrm{Var}(X(t)) = C_X(t, t) \tag{3.14}$$

The *correlation coefficient* of a random process X(t) is the correlation coefficient of $X(t_1)$ and $X(t_2)$, i.e.

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1) C_X(t_2, t_2)}} \tag{3.15}$$

where $\left|\rho_X(t_1, t_2)\right| \leq 1$.

Finally, we define the *nth joint moment* of X(t) as

$$E[X(t_1)X(t_2)\dots X(t_n)] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 \dots x_n f_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) dx_1 dx_2 \dots dx_n \tag{3.16}$$

We should keep in mind that the mean, variance, autocorrelation, autocovariance, and nth joint moment are good indicators of the behavior of a random process but only partial characterizations of the process.

In terms of these statistics, a random process may be classified as follows.

1. A random process is *wide-sense stationary* (WSS) or weakly stationary if its mean is constant, i.e.

$$E[X(t)] = E[X(t_1)] = E[X(t_2)] = m_x = \text{constant} \tag{3.17}$$

and its autocorrelation depends only on the absolute time difference $\tau = |t_1 - t_2|$, i.e.

$$E[X(t)X(t + \tau)] = R_X(\tau) \tag{3.18}$$

Note that the autocovariance of a WSS process depends only on the time difference τ

$$C_X(\tau) = R_X(\tau) - m_x^2 \tag{3.19}$$

and that by setting $\tau = 0$ in Eq. (3.18), we get

$$E\left[X^2(t)\right] = R_X(0) \tag{3.20}$$

indicating that the mean power of a WSS process X(t) does not depend on t. The autocorrelation function has its maximum value when $\tau = 0$ so that we can write

$$-R_X(0) \leq R_X(\tau) \leq R_X(0) \tag{3.21}$$

2. A random process is said to be *strict-sense stationary* (SSS) if its statistics are invariant to shift in the time axis. Hence,

$$F_X(x_1, x_2, \dots, x_n; t_1 + \tau, t_2 + \tau, \dots, t_n + \tau) = F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) \tag{3.22}$$

An SSS random process is also WSS but the converse is not generally true.

In general terms, a random process is **stationary** if all its statistical properties do not vary with time.

Example 3.2 A random process is given by

$$X(t) = A\cos(\omega t + \Theta)$$

where A and ω are constants and Θ is uniformly distributed over (0,2π). (a) Find E[X(t)] , E[X^2(t)] and E[X(t)X(t+τ)]. (b) Is X(t) WSS?

Solution

(a) Since Θ has a uniform distribution, its PDF is given by

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$E[X(t)] = \int_{-\infty}^{\infty} x f_\Theta(\theta) d\theta = A\int_0^{2\pi} \cos(\omega t + \theta)\frac{1}{2\pi} d\theta = 0$$

$$E\left[X^2(t)\right] = \int_{-\infty}^{\infty} x^2 f_\Theta(\theta) d\theta = A^2\int_0^{2\pi} \cos^2(\omega t + \theta)\frac{1}{2\pi} d\theta$$

$$= A^2\int_0^{2\pi} \frac{1}{2}[1 + \cos 2(\omega t + \theta)]\frac{1}{2\pi} d\theta = \frac{A^2}{2}$$

where the trigonometric identity $\cos^2\alpha = \frac{1}{2}[1 + \cos 2\alpha]$ and the fact that $\omega = 2\pi/T$ have been applied.

$$E[X(t)X(t+\tau)] = \int_0^{2\pi} A\cos(\omega t + \theta)A\cos[\omega(t+\tau) + \theta]\frac{1}{2\pi} d\theta$$

$$= \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}[\cos(\omega\tau + 2\omega t + 2\theta) + \cos\omega\tau]d\theta = \frac{A^2}{2}\cos\omega\tau$$

where we have used the trigonometric identity $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$.

(b) Since the mean of X(t) is constant and its autocorrelation is a function of τ only, X(t) is a WSS random process.

Example 3.3 Let X(t) = A sin (πt/2), where A is a Gaussian or normal random variable with mean μ and variance σ^2, i.e. A = N(μ,σ). (a) Determine the mean, autocorrelation, and autocovariance of X(t). (b) Find the density functions for X(1) and X(3). (c) Is X(t) stationary in any sense?

Solution

Given that E[A] = μ and Var(A) = σ^2, we can obtain

$$E\left[A^2\right] = \text{Vax}(A) + E^2[A] = \sigma^2 + \mu^2$$

(a) The mean of X(t) is

$$m_X(t) = E[A \sin \pi t/2] = E[A] \sin \pi t/2 = \mu \sin \pi t/2$$

The autocorrelation of $X(t_1)$ and $X(t_2)$ is

$$\begin{aligned} R_X(t_1, t_2) &= E\left[A \sin\left(\pi t_1/2\right) A \sin\left(\pi t_2/2\right)\right] = E\left[A^2\right] \sin\left(\pi t_1/2\right) \sin\left(\pi t_2/2\right) \\ &= (\sigma^2 + \mu^2) \sin\left(\pi t_1/2\right) \sin\left(\pi t_2/2\right) \end{aligned}$$

The autocovariance is

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1) m_X(t_2) = \sigma^2 \sin(\pi t_1/2) \sin(\pi t_2/2)$$

(b) X(1) = A sin π(1)/2 = A

$$F_X(x_1, t_1) = P[X(1) \le x_1] = P[A \le x_1] = F_A(a)$$

where a = x_1

$$f_X(x_1) = \frac{\partial F(x_1; t_1)}{\partial x_1} = \frac{dF_A(a)}{da} \frac{da}{dx_1} = f_A(a)$$

Since A = N(μ,σ),

$$f_A(a) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(a-\mu)^2/\left(2\sigma^2\right)}$$

$$f_X(x_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_1-\mu)^2/\left(2\sigma^2\right)}$$

Similarly, X(3) = A sin π3/2 = −A

$$\begin{aligned} F_X(x_3, t_1) &= P[X(3) \le x_3] = P[-A \le x_3] = P[A \ge -x_3] = 1 - P[A \le -x_3] \\ &= 1 - F_A(a) \end{aligned}$$

where a = $-x_3$.

$$f_X(x_3) = \frac{\partial F(x_3; t_1)}{\partial x_3} = \frac{dF_A(a)}{da}\frac{da}{dx_3} = f_A(a)$$

Hence

$$f_X(x_3) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_3+\mu)^2/(2\sigma^2)}$$

(c) Since the mean of X(t) is a function of time, the process is not stationary in any sense.

3.3 Time Averages of Random Processes and Ergodicity

For a random process X(t), we can define two types of averages: ensemble and time averages. The ensemble averages (or statistical averages) of a random process X (t) may be regarded as "averages across the process" because they involve all sample functions of the process observed at a particular instant of time. The time averages of a random process X(t) may be regarded as "averages along the process" because they involve long-term sample averaging of the process.

To define the time averages, consider the sample function x(t) of random process X(t), which is observed within the time interval $-T \leq t \leq T$. The *time average* (or *time mean*) of the sample function is

$$\overline{x} = < x(t) > = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt \tag{3.23}$$

where <•> denotes time-averaging operation. Similarly, the *time autocorrelation* of the sample function x(t) is given by

$$\overline{R}_X(\tau) = < x(t)x(t+\tau) > = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt \tag{3.24}$$

Note that both $\overline{x}$ and $\overline{R}_X(\tau)$ are random variables since their values depend on the observation interval and on the sample function x(t) used.

If all time averages are equal to their corresponding ensemble averages, then the stationary process is *ergodic*, i.e.

$$\overline{x} = < x(t) > = E[X(t)] = m_X \tag{3.25}$$

$$\overline{R}_X(\tau) = < x(t)x(t+\tau) > = E[X(t)X(t+\tau)] = R_X(\tau) \tag{3.26}$$

An **ergodic process** is one for which time and ensemble averages are interchangeable.

The concept of ergodicity is a very powerful tool and it is always assumed in many engineering applications. This is due to the fact that it is impractical to have a large number of sample functions to work with. Ergodicity suggests that if a random process is ergodic, only one sample function is necessary to determine the ensemble averages. This seems reasonable because over infinite time each sample function of a random process would take on, at one time or another, all the possible values of the process. We will assume throughout this text that the random processes we will encounter are ergodic and WSS.

Basic quantities such as dc value, rms value, and average power can be defined in terms of time averages of an ergodic random process as follows:

1. $\overline{x} = m_X$ is the dc value of x(t).
2. $[\overline{x}]^2 = m_X^2$ is the normalized dc power.
3. $\overline{R}_X(0) = \overline{x^2}$ is the total average normalized power
4. $\overline{\sigma}_X^2 = \overline{x^2} - [\overline{x}]^2$ is the average normalized power in the ac or time-varying component of the signal.
5. $X_{\text{rms}} = \sqrt{\overline{x^2}} = \sqrt{\overline{\sigma}_X^2 + [\overline{x}]^2}$ is the rms value of x(t).

Example 3.4 Consider the random process in Example 3.2. Show that the process is stationary and ergodic.

Solution

We already showed that the process is stationary because the statistical or ensemble averages do not depend on time. To show that the process is ergodic, we compute the first and second moments. Since $\omega = 2\pi/T$,

$$\overline{x} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} A \cos(\omega t + \theta) dt = 0$$

$$\overline{x^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} A^2 \cos^2(\omega t + \theta) dt = \lim_{T \to \infty} \frac{A^2}{2T} \int_{-T}^{T} \frac{1}{2}[1 + \cos 2(\omega t + \theta)] dt = \frac{A^2}{2}$$

indicating that the time averages are equal to the ensemble averages we obtained in Example 3.2. Hence the process is ergodic.

3.4 Multiple Random Processes

The joint behavior of two or more random processes is dictated by their joint distributions. For example, two random processes X(t) and Y(t) are said to be *independent* if for all t_1 and t_2, the random variables $X(t_1)$ and $Y(t_2)$ are independent. That means that their nth order joint PDF factors, i.e.

$$\begin{aligned} &F_{XY}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n; t_1, t_2, \ldots, t_n) \\ &\quad = F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) F_Y(y_1, y_2, \ldots, y_n; t_1, t_2, \ldots, t_n) \end{aligned} \tag{3.27}$$

The *crosscorreleration* between two random processes X(t) and Y(t) is defined as

$$\boxed{R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]} \tag{3.28}$$

Note that

$$R_{XY}(t_1, t_2) = R_{YX}(t_2, t_1) \tag{3.29}$$

The processes X(t) and Y(t) are said to be *orthogonal* if

$$R_{XY}(t_1, t_2) = 0 \qquad \text{for all } t_1 \text{ and } t_2 \tag{3.30}$$

If X(t) and Y(t) are jointly stationary, then their crosscorrelation function becomes

$$R_{XY}(t_1, t_2) = R_{XY}(\tau)$$

where $\tau = t_2 - t_1$. Other properties of the crosscorrelation of jointly stationary processes are:

1. $R_{XY}(-\tau) = R_{XY}(\tau)$, i.e. it is symmetric.
2. $|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$, i.e. it is bounded.
3. $|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$, i.e. it is bounded.

The *crosscovariance* of X(t) and Y(t) is given by

$$\boxed{C_{XY}(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}] = R_{XY}(t_1, t_2) - m_X(t_1)m_Y(t_2)} \tag{3.31}$$

Just like with random variables, two random processes X(t) and Y(t) are *uncorrelated* if

$$C_{XY}(t_1, t_2) = 0 \qquad \text{for all } t_1 \text{ and } t_2 \tag{3.32}$$

which implies that

$$R_{XY}(t_1, t_2) = m_X(t_1)m_Y(t_2) \quad \text{for all } t_1 \text{ and } t_2 \tag{3.33}$$

Finally, for jointly ergodic random processes X(t) and Y(t),

$$\overline{R}_{XY}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt = R_{XY}(\tau) \tag{3.34}$$

Thus, two random processes X(t) and Y(t) are:

(a) **Independent** if their joint PDF factors.
(b) **Orthogonal** if $R_{XY}(t_1,t_2) = 0$ for all t_1 and t_2
(c) **Uncorrelated** if $R_{XY}(t_1,t_2) = m_X(t_1)m_Y(t_2)$ for all t_1 and t_2.

Example 3.5 Two random processes are given by

$$X(t) = \sin(\omega t + \Theta), \qquad Y(t) = \sin(\omega t + \Theta + \pi/4)$$

where Θ is random variable that is uniformly distributed over (0,2π). Find the cross correlation function $R_{XY}(t, t + \tau)$

Solution

$$\begin{aligned} R_{XY}(t, t+\tau) = R_{XY}(\tau) = E\left[X(t)Y(t+\tau)\right] &= \int_0^{2\pi} x(t)y(t+\tau)f_\Theta(\theta)d\theta \\ &= \int_0^{2\pi} \sin(\omega t + \theta)\sin[\omega(t+\tau) + \theta + \pi/4]\frac{1}{2\pi}d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left[\cos(\omega\tau + \pi/4) - \cos\left(2\omega t + \omega\tau + 2\theta + \pi/4\right]\frac{1}{2\pi}d\theta \\ &= \frac{1}{2}\cos(\omega\tau + \pi/4) \end{aligned}$$

where we have applied the trigonometric identity $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.

Example 3.6 A received signal X(t) consists of two components: desired signal S(t) and noise N(t), i.e.

$$X(t) = S(t) + N(t)$$

If the autocorrelation of the random signal is

$$R_S(\tau) = e^{-2|\tau|}$$

while that of the random noise is

$$R_N(\tau) = 3e^{-|\tau|}$$

Assume that they are independent and they both have zero mean.

(a) Find the autocorrelation of X(t). (b) Determine the cross correlation between X(t) and S(t).

Solution

(a) $$\begin{aligned} R_X(t_1, t_2) &= E\left[X(t_1)X(t_2)\right] = E[\{S(t_1) + N(t_1)\}\{S(t_2) + N(t_2)\}] \\ &= E\left[S(t_1)S(t_2)\right] + E\left[N(t_1)S(t_2)\right] + E\left[S(t_1)N(t_2)\right] + E\left[N(t_1)N(t_2)\right] \end{aligned}$$

Since S(t) and N(t) are independent and have zero mean,

$$E[N(t_1)S(t_2)] = E[N(t_1)]E[S(t_2)] = 0$$

$$E[S(t_1)N(t_2)] = E[S(t_1)]E[N(t_2)] = 0$$

Hence,

$$R_X(\tau) = R_S(\tau) + R_N(\tau) = e^{-2|\tau|} + 3e^{-|\tau|}$$

where $\tau = t_1 - t_2$.

(b) Similarly,

$$\begin{aligned} R_{XS}(t_1, t_2) &= E\big[X(t_1)S(t_2)\big] = E[\{S(t_1) + N(t_1)\}\{S(t_2)\}] \\ &= E\big[S(t_1)S(t_2)\big] + E\big[N(t_1)S(t_2)\big] \\ &= R_S(t_1, t_2) + 0 \end{aligned}$$

Thus,

$$R_{XS}(\tau) = R_S(\tau) = e^{-2|\tau|}$$

3.5 Sample Random Processes

We have been discussing random processes in general. Specific random processes include Poisson counting process, Wiener process or Brownian motion, random walking process, Bernoulli process, birth-and-death process, and Markov process [4, 5]. In this section, we consider some of these specific random processes.

3.5.1 Random Walks

A random walk (or drunkard's walk) is a stochastic process in which the states are integers X_n representing the position of a particle at time n. Each state change according to

$$X_n = X_{n-1} + Z_n \tag{3.35}$$

where Z_n is a random variable which takes values of 1 or −1. If $X_0 = 0$, then

$$X_n = \sum_{i=1}^{n} Z_i \tag{3.36}$$

A *random walk* on X corresponds to a sequence of states, one for each step of the walk. At each step, the walk switches from its current state to a new state or remains at the current state. Thus,

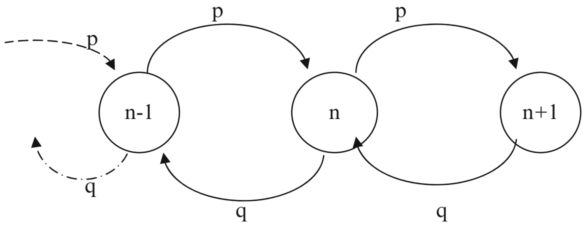


Fig. 3.4 A typical random walk

Random walks constitute a random process consisting of a sequence of discrete steps of fixed length.

Random walks are usually *Markovian*, which means that the transition at each step is independent of the previous steps and depends only on the current state. Although random walks are not limited to one-dimensional problems, the one-dimensional random walk is one of the simplest stochastic processes and can be used to model many gambling games. Random walks also find applications in potential theory. A typical one-dimensional random walk is illustrated in Fig. 3.4.

Example 3.7 Consider the following standard Markovian random walk on the integers over the range $\{0, \ldots, N\}$ that models a simple gambling game, where a player bets the same amount on each hand (i.e., step). We assume that if the player ever reaches 0, he has lost all his money and stops, but if he reaches N, he has won a certain amount of money and stops. Otherwise, at each step, one moves from state i (where $i \neq 0, N$) to $i + 1$ with probability p (the probability of winning the game), to $i - 1$ with probability q (the probability of losing the game), and stays at the same state with probability $1 - p - q$ (the probability of a draw).

3.5.2 Markov Processes

If the future state of a process depends only on the present (and independent of the past), the process is called a *Markov process.* A Markov process is made possible only if the state time has a memoryless (exponential) distribution. This requirement often limits the applicability of Markov processes.

Formally, a stochastic process X(t) is a Markov process if

$$\begin{aligned} \text{Prob}\left[X(t) = x \middle| X(t_n) = x_n, X(t_{n-1}) = x_{n-1} \cdots, X(t_o) = x_o\right] \\ = \text{Prob}\left[X(t) = x \middle| X(t_n) = x_n\right] \quad \text{for } t_o < t_1 < \cdots < t_n < t \end{aligned} \tag{3.37}$$

A discrete-state Markov process is called a *Markov chain* [4]. We use the state transition diagram to represent the evolution of a Markov chain. An example of three-state Markov chain is shown in Fig. 3.5.

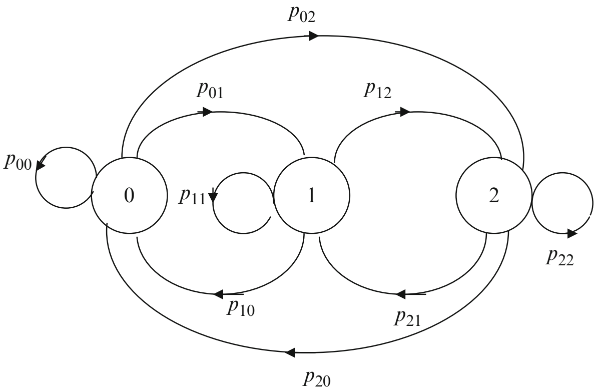


Fig. 3.5 State transition diagram for a three-state Markov chain

The conditional probability

$$\text{Prob}\left[X_{n+1} = i \middle| X_n = j\right] = p_n(i,j)$$

is called the *transition probability* from state i to state j. Since a Markov chain must go somewhere with a probability of 1, the sum of $p_n(i,j)$'s over all j's is equal to 1. If $p_n(i,j)$ is independent of n, the Markov chain is said to be time-homogeneous and in this case, the transition probability becomes p(i,j). When we arrange p(i,j) into an square array, the resulting matrix is called the *transition matrix*.

For a simple example, consider four possible states as 0, 1, 2, and 3. The transition matrix is

$$P = \begin{bmatrix} p(0,0) & p(0,1) & p(0,2) & p(0,3) \\ p(1,0) & p(1,1) & p(1,2) & p(1,3) \\ p(2,0) & p(2,1) & p(2,2) & p(2,3) \\ p(3,0) & p(3,1) & p(3,2) & p(3,3) \end{bmatrix} \tag{3.38}$$

3.5.3 Birth-and-Death Processes

Birth-death processes describe the stochastic evolution in time of a random variable whose value increases or decreases by one in a single event. These are discrete-space Markov processes in which the transitions are restricted to neighboring states only. A typical example is shown in Fig. 3.6.

For example, the number of jobs in a queue with a single server and the individual arrivals can be represented as a birth-death process. An arrival to the queue (a birth) causes the state to change by +1, while a departure (a death) causes the state to change by −1. Although the birth-death processes are used in modeling population, they are useful in the analysis of communication networks. They are also used in physics, biology, sociology, and economics.

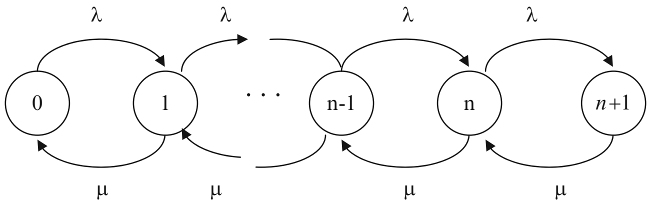


Fig. 3.6 The state transition diagram for a birth-and-death process

3.5.4 Poisson Processes

From application point of view, Poisson processes are very useful. They can be used to model a large class of stochastic phenomena. Poisson process is one in which the number of events which occur in any time interval t is distributed according to a Poisson random variable, with mean λt. In this process, the interarrival time is distributed exponentially.

A process is called a **Poisson process** when the time intervals between successive events are exponentially distributed.

Given a sequence of discrete events occurring at times $t_0, t_1, t_2, t_3, \ldots$, the intervals between successive events are $\Delta t_1 = (t_1 - t_0), \Delta t_2 = (t_2 - t_1), \Delta t_3 = (t_3 - t_2), \ldots,$ and so on. For a Poisson process, these intervals are treated as independent random variables drawn from an exponentially distributed population, i.e., a population with the density function $f(x) = \lambda e^{-\lambda x}$ for some fixed constant λ. The interoccurrence times between successive events of a Poisson process with parameter λ are independent identical distributed (IID) exponential random variable with mean 1/λ.

The Poisson process is a counting process for the number of randomly occurring point-events observed in a given time interval. For example, suppose the arrival process has a Poisson type distribution. If N(t) denotes the number of arrivals in time interval (0,t], the probability mass function for N(t) is

$$p_n(t) = P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \tag{3.39}$$

Thus, the number of events N(t) in the interval (0,t] has a Poisson distribution with parameter λt and the parameter λ is called the arrival rate of the Poisson process.

Two properties of the Poisson process are the superposition property and decomposition property [6, 7].

The superposition (additive) property states that the superposition of Poisson processes is also a Poisson process, as illustrated in Fig. 3.7.

Thus, the sum of n independent Poisson processes with parameters $\lambda_k, k = 1, 2, \cdots, n$ is a Poisson process with parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

The decomposition (splitting) property is just the reverse of the superposition property. If a Poisson stream is split into k substreams, each substream is also Poisson, as illustrated in Fig. 3.8.

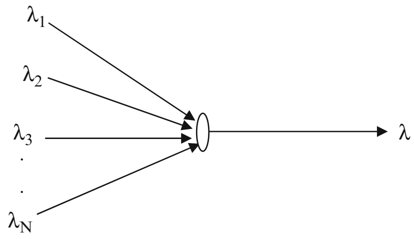


Fig. 3.7 Superposition of Poisson streams

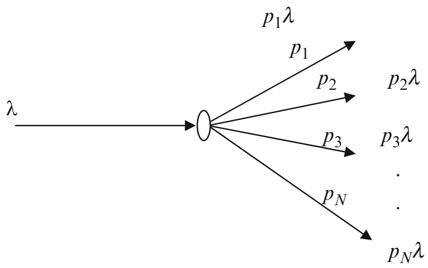


Fig. 3.8 Decomposition of a Poisson stream

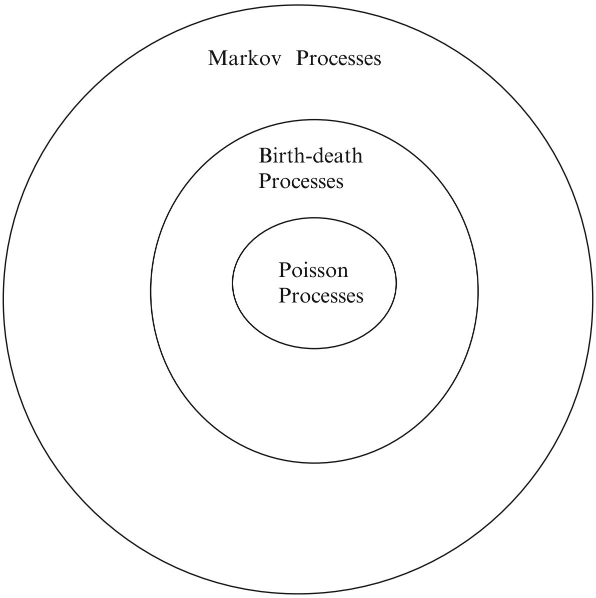


Fig. 3.9 Relationship between various types of stochastic processes

The Poisson process is related to the exponential distribution. If the interarrival times are exponentially distributed, the number of arrival-points in a time interval is given by the Poisson distribution and the process is a Poisson arrival process. The converse is also true—if the number of arrival-points in any interval is a Poisson process, the interarrival times are exponentially distributed.

The relationship among various types of stochastic processes is shown in Fig. 3.9.

3.6 Renewal Processes

A renewal process generalizes the notion of a Markov process. In a Markov process, the times between state transitions are exponentially distributed. Let X_1, X_2, X_3, $\cdots$ be times of successive occurrences of some phenomenon and let $Z_i = X_i - X_{i-1}$ be the times between ($i - 1$th) and ith occurrences, then if $\{Z_i\}$ are independent and identically distributed (IID), the process $\{X_i\}$ is called a *renewal process*. The study of renewal processes is called *renewal theory*.

One common example of a renewal process is the arrival process to a queueing system. The times between successive arrivals are IID. In a special case that the interarrival times are exponential, the nenewal process is a Poisson process. Poisson process, binomial process, and random walk process are special cases of renewal processes.

3.7 Computation Using MATLAB

The MATLAB software can be used to reinforce the concepts learned in this chapter. It can be used to generate a random process X(t) and calculate its statistics. It can also be used to plot X(t) and its autocorrelation function.

MATLAB provides command **rand** for generating uniformly distributed random numbers between 0 and 1. The uniform random number generator can then be used to generate a random process or the PDF of an arbitrary random variable. For example, to generate a random variable X uniformly distributed over (a,b), we use

$$\mathrm{X} = \mathrm{a} + (\mathrm{a} - \mathrm{b})\mathrm{U} \tag{3.40}$$

where U is generated by **rand**. A similar command **randn** generates a Gaussian (or normal) distribution with mean zero and variance one.

Suppose we are interested in generating a random process

$$X(t) = 10\cos(2\pi t + \Theta) \tag{3.41}$$

where Θ is a random variable uniformly distributed over (0,2π). We generate and plot X(t) using the following MATLAB commands.

```
» t=0:0.01:2; % select 201 time points between 0 and 2.
» n=length(t);
» theta=2*pi*rand(1,n);  % generates n=201 uniformly
  distributed theta
» x=10*cos(2*pi*t +theta);
» plot(t,x)
```

The plot of the random process is shown in Fig. 3.10.

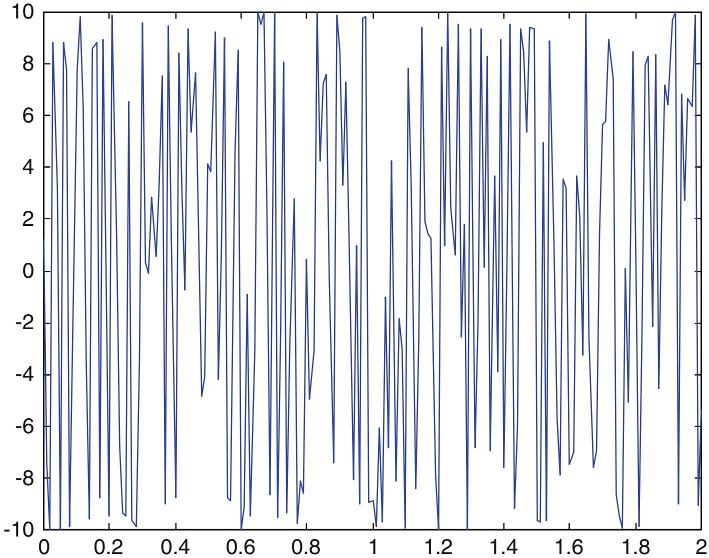


Fig. 3.10 MATLAB generation of the random process $X(t) = 10 \cos(2\pi t + \Theta)$

We may find the mean and standard deviation using MATLAB commands **mean** and **std** respectively. For example, the standard deviation is found using

```
» std(x)
ans =
  7.1174
```

where the result is a bit off from the exact value of 7.0711 obtained from Example 3.2. The reason for this discrepancy is that we selected only 201 points. If more points, say 10,000, are selected the two results should be very close.

We will now use MATLAB to generate a Bernoulli random process, which is used in data communication. The process consists of random variables which assume only two states or values: +1 and −1 (or +1 and 0). In this particular case, the process may also be regarded as *random binary process*. The probability of X(t) being +1 is p and −1 is q = 1 − p. Therefore, to generate a Bernoulli random variable X, we first use MATLAB **rand** to generate a random variable U that uniformly distributed over (0,1). Then, we obtain

$$X = \begin{cases} 1, & \text{if} \quad U \leq p \\ -1, & \text{if} \quad U > p \end{cases} \tag{3.42}$$

i.e. we have partitioned the interval (0,1) into two segments of length p and 1 − p. The following MATLAB program is used to generate a sample function for the random process. The sample function is shown in Fig. 3.11.

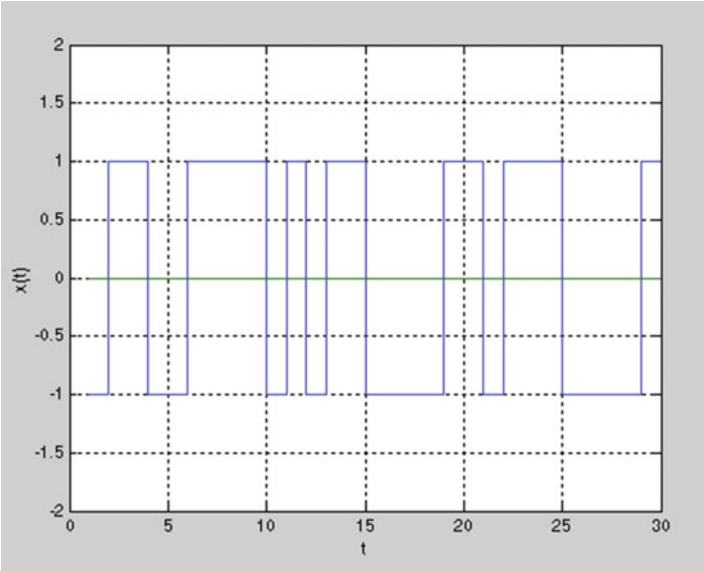


Fig. 3.11 A typical sample function of a Bernoulli random process

```
% Generation of a Bernoulli process
% Ref: D. G. Childers, "Probability of Random Processes,"
Irwin, 1997, p.164
p=0.6;     % probability of having +1
q=1-p;      % probability of having -1
n=30;        % length of the discrete sequence
t=rand(1,n);  %  generate  random  numbers  uniformly
distributed over (0,1)
x=zeros(length(t)); % set initial value of x equal to zero
for k=1:n
  if( t(k) <= p )
    x(k)=1;
  else
    x(k)= -1;
  end
end
stairs(x);
xlabel('t')
ylabel('x(t)')
a=axis;
axis([a(1) a(2) -2 2]);
grid on
```

3.8 Summary

1. A random process (also known as stochastic process) is a mapping from the sample space into an ensemble of time functions known as sample functions. At any instant of time, the value of a random process is a random variable.
2. A continuous-time random process X(t) is a family of sample functions of continuous random variables that are a function of time t, where t is a continuum of values. A random process is deterministic if future values of any sample function can be predicted from past values.
3. A random process is stationary if all its statistical properties does not change with time, i.e $m_X(t)$ is constant and $R_X(t_1,t_2)$ depends only on $\tau = |t_2 - t_1|$.
4. For an ergodic process, the statistical and time averages are the same and only one sample function is needed to compute ensemble averages.
5. For two stationary random processes X(t) and Y(t), the cross-correlation function is defined as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

Widely used random processes in communication include random walk, birth-and-death process, Poisson process, Markov process, and renewal process.
6. Some of the concepts covered in the chapter are demonstrated using MATLAB.

For more information on the material covered in this chapter, one should see [8–10].

Problems

3.1 If X(t) = A sin4t, where A is random variable uniformly distributed between 0 and 2, find E[X(t)] and $E[X^2(t)]$.

3.2 Given a random process X(t) = At + 2, where A is a random variable uniformly distributed over the range (0,1),

(a) sketch three sample functions of X(t),
(b) find $\overline{X(t)}$ and $\overline{X^2(t)}$,
(c) determine $R_X(t_1,t_2)$,
(d) Is X(t) WSS?

3.3 If a random process is given by

$$X(t) = A\cos\omega t - B\sin\omega t,$$

where ω is a constant and A and B are independent Gaussian random variables with zero mean and variance σ^2, determine: (a) E[X], $E[X^2]$ and Var(X), (b) the autocorrelation function $R_X(t_1,t_2)$.

3.4 Let Y(t) = X(t − 1) + cos3t, where X(t) is a stationary random process. Determine the autocorrelation function of Y(t) in terms of $R_X(\tau)$.

3.5 If Y(t) = X(t) − X(t − α), where α is a constant and X(t) is a random process. Show that

$$R_Y(t_1,t_2) = R_x(t_1,t_2) - R_x(t_1,t_2-\alpha) - R_x(t_1-\alpha,t_2) + R_x(t_1-\alpha,t_2-\alpha)$$

3.6 A random stationary process X(t) has mean 4 and autocorrelation functon

$$R_X(\tau) = 5e^{-2|\tau|}$$

(a) If Y(t) = X(t − 1), find the mean and autocorrelation function of Y(t).
(b) Repeat part (a) if Y(t) = tX(t).

3.7 Let Z(t) = X(t) + Y(t), where X(t) and Y(t) are two independent stationary random processes. Find $R_Z(\tau)$ in terms of $R_X(\tau)$ and $R_Y(\tau)$.

3.8 Repeat the previous problem if Z(t) = 3X(t) + 4Y(t).

3.9 If X(t) = Acosωt, where ω is a constant and A random variables with mean μ and variance σ^2, (a) find $<$ x(t) $>$ and $m_X(t)$. (b) Is X(t) ergodic?

3.10 A random process is defined by

$$X(t) = A\cos\omega t - B\sin\omega t,$$

where ω is a constant and A and B are independent random variable with zero mean. Show that X(t) is stationary and also ergodic.

3.11 N(t) is a stationary noise process with zero mean and autocorrelation function

$$R_N(\tau) = \frac{N_o}{2}\delta(\tau)$$

where N_o is a constant. Is N(t) ergodic?

3.12 X(t) is a stationary Gaussian process with zero mean and autocorrelation function

$$R_X(\tau) = \sigma^2 e^{-\alpha|\tau|}\cos\omega\tau$$

where σ, ω, and α are constants. Show that X(t) is ergodic.

3.13 If X(t) and Y(t) are two random processes that are jointly stationary so that $R_{XY}(t_1,t_2) = R_{XY}(\tau)$, prove that

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

where $\tau = |t_2 - t_1|$.

3.14 For two stationary processes X(t) and Y(t), show that

(a) $$|R_{XY}(\tau)| \le \frac{1}{2}[R_X(0) + R_Y(0)]$$

(b) $$|R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)}$$

3.15 Let X(t) and Y(t) be two random processes given by

$X(t) = \cos(\omega t + \Theta)$
$Y(t) = \sin(\omega t + \Theta)$

where ω is a constant and Θ is a random variable uniformly distributed over (0,2π). Find

$$R_{XY}(t, t+\tau) \text{ and } R_{YX}(t, t+\tau).$$

3.16 X(t) and Y(t) are two random processes described as

$X(t) = A \cos \omega t + B \sin \omega t$
$Y(t) = B \cos \omega t - A \sin \omega t$

where ω is a constant and $A = N(0,\sigma^2)$ and $B = N(0,\sigma^2)$. Find $R_{XY}(\tau)$.

3.17 Let X(t) be a stationary random process and $Y(t) = X(t) - X(t - a)$, where a is a constant. Find $R_{XY}(\tau)$.

3.18 Let $\{N(t), t \ge 0\}$ be a Poisson process with rate λ. Find $E[N(t).N(t+s)]$.

3.19 For a Poisson process, show that if $s < t$,

$$\text{Prob}\left[N(s) = k \middle| N(t) = n\right] = \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}, k = 0, 1, \cdots, n$$

3.20 Let N(t) be a renewal process where renewal epochs are Erlang with parameters (m,λ). Show that

$$\text{Prob}[N(t) = n] = \sum_{k=nm}^{nm+m-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

3.21 Use MATLAB to generate a random process $X(t) = A \cos(2\pi t)$, where A is a Gaussian random variable with mean zero and variance one. Take $0 < t < 4$ s.

3.22 Repeat the previous problem if A is random variable uniformly distributed over (−2, 2).

3.23 Given that the autocorrelation function $R_X(\tau) = 2 + 3e^{-\tau^2}$, use MATLAB to plot the function for $-2 < \tau < 2$.

3.24 Use MATLAB to generate a random process

$$X(t) = 2\cos\left(2\pi t + B[n]\frac{\pi}{4}\right)$$

where B[n] is a Bernoulli random sequence taking the values of +1 and −1. Take $0 < t < 3$ s.

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