Chapter 3
Distances

Abstract. In many theoretical and practical issues we face the following problem. Having two sets in the same universe, we want to calculate a difference between them exemplified by a distance. In this Chapter we consider distances between the intuitionistic fuzzy sets in two ways: while using the two term intuitionistic fuzzy set representation (membership values and non-membership values only are taken into account), and the three term intuitionistic fuzzy set representation (membership values, non-membership values, and hesitation margins are taken into account). We discuss norms and metrics for both types of representations. Both types are correct from the mathematical point of view but, in the practical perspective, the three term approach seems to be more justified. We discuss the problem in detail, considering its analytical, and geometrical aspects. We also show some problems with the Hausdorff distance, while the Hamming metric is applied when using the two term intuitionistic fuzzy set representation. We also show that the method of calculating the Hausdorff distances, which is correct for the interval-valued fuzzy sets, does not work for the intuitionistic fuzzy sets. Finally, we show the usefulness of the three term distances in a measure for ranking the intuitionistic fuzzy alternatives.

3.1 Basic Definitions

Definition 3.1. A distance on a set X is a positive function d (also called metric) from pairs of elements of X to the set R^+ of non-negative real numbers with the following properties, valid for all $x_1, x_2, x_3 \in X$:

1. $d(x_1, x_1) = 0$ (reflexivity);
2. $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$ (separability);
3. $d(x_1, x_2) = d(x_2, x_1)$ (symmetry);
4. $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$ (triangle inequality).

The pair (X, d) is called metric space.

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If a measure fulfills requirements 1, 3 and 4, it is called a pseudometric (separability does not hold).

A semimetric is defined with requirements 1, 2 and 3 (triangle inequality does not need to be satisfied).

A semi-pseudometric satisfies 1 and 3 only.

If a set of elements is identified with a vector space, the most known distances correspond to norms.

A norm of a vector corresponds in a sense to the absolute value (magnitude) of numbers.

Definition 3.2. (Bronshtein [41])

We assign a real positive number $\| x \|$ (*Norm* **x**) to the vector **x**. A number $\| x \|$, in order to be a norm, must satisfy the norm axioms which for any vector $\mathbf{x} \in \mathbf{R^n}$ are the following:

1. $\| \mathbf{x} \| \geq 0$ for every $\mathbf{x}$;
2. $\| \mathbf{x} \| = 0$ if and only if $\mathbf{x} = 0$;
3. $\| \alpha\mathbf{x} \| = |\alpha| \, \| \mathbf{x} \|$ for every $\mathbf{x}$ and every real number α;
4. $\| \mathbf{x}+\mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|$ for every $\mathbf{x}$ and $\mathbf{y}$.

Concrete norms are defined in many different ways.

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is a real vector of n dimensions, i.e., $\mathbf{x} \in \mathbf{R}^n$ then the most often used vector norms are:

Euclidean norm

$$\| \mathbf{x} \| = \| \mathbf{x} \|_2 = \sqrt{\sum_i^n x_i^2}. \tag{3.1}$$

Supremum or **Uniform Norm**

$$\| \mathbf{x} \| = \| \mathbf{x} \|_\infty = \max_{1 \leq i \leq n} |x_i|. \tag{3.2}$$

Sum Norm

$$\| \mathbf{x} \| = \| \mathbf{x} \|_1 = \sum_i^n |x_i|. \tag{3.3}$$

In applications, the so called l_r-norms and l^r-norms are often used, defined as follows.

Definition 3.3. For a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$, its l_r-norm, where r is a real number ≥ 1, is:

$$l_r(\mathbf{x}) = \parallel \mathbf{x} \parallel_r = (\sum_i^n |x_i|^r)^{\frac{1}{r}}. \tag{3.4}$$

Slight modification of the axioms in Definition 3.2 makes it possible to define i-th power of l_r-norm, i.e., the l^r-norm:

$$l^r(\mathbf{x}) = \parallel \mathbf{x} \parallel^r = \sum_i^n |x_i|^r. \tag{3.5}$$

Euclidean norm (3.1) is a special case of the l_r-norm (3.4).

Sum norm (3.3) is a special case of the l^r-norm (3.5).

Norms on vector spaces correspond to certain metrics, i.e., every norm determines a metric, and some metrics determine a norm.

Given a normed vector space $(X \parallel \cdot \parallel)$ we can define a metric on X by $d(x,y) = \parallel x - y \parallel$. The metric d is said to be induced by the norm $\parallel \cdot \parallel$.

We give below the most often used metrics $d_{i,j} = d(y_i, y_j)$ of vectors y_i and y_j having one extreme at the origin of the coordinate axes.

- Manhattan distance
$$d_{i,j} = \sum_{k=1}^n |y_{ik} - y_{jk}| \tag{3.6}$$
- Euclidean distance
$$d_{i,j} = \sqrt{\sum_{k=1}^n (y_{ik} - y_{jk})^2} \tag{3.7}$$
- Minkowski distance
$$d_{i,j} = (\sum_{k=1}^n |y_{ik} - y_{jk}|^p)^{\frac{1}{p}} \tag{3.8}$$
Minkowski distance is induced by the norm l_r, namely, for $r = 1$ (3.8) it becomes Manhattan distance (city block distance); for $r = 2$ it is equivalent to the Euclidean distance.
- Chebyshev distance
$$d_{i,j} = \max_k |y_{ik} - y_{jk}| \tag{3.9}$$
Chebyshev distance is also induced by the norm l_r when $r \to \infty$.
- Canberra distance
$$d_{i,j} = \sum_{k=1}^n \frac{|y_{ik} - y_{jk}|}{|y_{ik}| + |y_{jk}|} \tag{3.10}$$
Canberra distance (Lance and Williams [103]) is similar to the Manhattan distance. Each component of the sum (3.10) belongs to the interval $[0,1]$. If y_{ik} or y_{jk} is equal to 0, the respective component of the sum (3.10) is equal to 1 regardless of the value of the other component. The distance is rather sensitive to small changes when both components tend to zero (Apolloni et al. [2]). For practical purposes we assume value of 0 for both coordinates equal 0 (Emran and Ye [69]).

- Sorensen distance (also known as Bray Curtis)

$$d_{i,j} = \frac{\sum_{k=1}^{n} |y_{ik} - y_{jk}|}{\sum_{k=1}^{n} (y_{ik} + y_{jk})} \tag{3.11}$$

Sorensen distance (Bray and Curtis [40]) is a modified Manhattan distance. Sorensen distance value is between zero and one if all coordinates are positive. If denominator in (3.11) is zero, Sorensen distance is undefined.

- Mahalanobis distance

$$d_{i,j} = \sqrt{\sum_{k=1}^{n} (y_{ik} - y_{jk}) S^{-1} (y_{ik} - y_{jk})} \tag{3.12}$$

where S is a covariance matrix. Mahalanobis distance ([121]) can also be defined as a dissimilarity measure between two random vectors of the same distribution with the covariance matrix S. If S is the identity matrix, the Mahalanobis distance (3.12) ([121]) is equal to the Euclidean distance (3.7). For a diagonal covariance matrix S, the Mahalanobis distance (3.12) reduces to the normalized Euclidean distance. Mahalanobis distance is used in classification methods and cluster analysis (McLachlan [122]).

The above distances d_{ij} play often a role of measures corresponding to similarity measures s_{ij}, i.e., $s_{ij} = 1 - d_{ij}$, and are widely used for solving real problems (cf. e.g., Bray and Curtis [40], Apolloni et al. [2], McLachlan [122], Emran and Ye [69], Lance and Williams [103], Krebs [102], Hublek [84], Wolda [246], Clarke et al. [54], Field et al. [71]).

In vector spaces also other similarity measures are used, for example:

- Angular Separation

$$s_{i,j} = \frac{\sum_{k=1}^{n} y_{ik} y_{jk}}{\sqrt{\sum_{k=1}^{n} (y_{ik})^2 \sum_{k=1}^{n} (y_{jk})^2}} \tag{3.13}$$

Angular separation represents cosine between two vectors. The values of (3.13) belong to the interval [-1, 1]. The higher the values of (3.13), the more similar the vectors considered. If denominator is equal to zero, we assume 0 for angular separation.

- Correlation Coefficient

$$s_{i,j} = \frac{\sum_{k=1}^{n} (y_{ik} - \bar{y}_i)(y_{jk} - \bar{y}_j)}{\sqrt{\sum_{k=1}^{n} (y_{ik} - \bar{y}_i)^2 \sum_{k=1}^{n} (y_{jk} - \bar{y}_j)^2}} \tag{3.14}$$

Correlation coefficient is a standarized angular separation resulting from centering the coordinates with respect to the mean values.

3.2 Norms and Metrics Over the Intuitionistic Fuzzy Sets or their Elements – The Two Term Approach

It is worth stressing that this section will not be devoted to the usual set-theoretic properties of the intuitionistic fuzzy sets (i.e. the properties which are a direct result of the fact that the intuitionistic fuzzy sets are sets in the sense of set theory). For example, in a metric space X, one can study the metric properties of the intuitionistic fuzzy sets over X. This can be done directly by topological methods (cf. e.g., Schwartz [149]) without paying attention to the essential properties of the intuitionistic fuzzy sets. On the other hand, all intuitionistic fuzzy sets (and hence, all fuzzy sets) over a fixed universe X generate a space (in the sense of Schwartz [149]), but with a special metric (cf., e.g., Kaufmann [99]) which is not related to the elements of X but to the values of the functions μ_A and ν_A, defined for these elements.

We should have in mind that a "norm" of a given intuitionistic fuzzy element is actually not a norm in the sense of Schwartz [149], but rather a "pseudo-norm", assigning a number to every element $x \in X$. This number depends on the values of the functions μ_A and ν_A (which are calculated for this element).

In other words, the essential conditions for a norm, i.e.:

$$\|x\| = 0 \quad \text{iff} \quad x = 0, \tag{3.15}$$

and

$$\|x\| = \|y\| \quad \text{iff} \quad x = y, \tag{3.16}$$

are not fulfilled here.

Instead of (3.15)–(3.16), the following conditions hold:

$$\|x\| = \|y\| \text{ iff } \mu_A(x) = \mu_A(y) \tag{3.17}$$

and

$$\nu_A(x) = \nu_A(y). \tag{3.18}$$

For any element $x \in X$ in every fuzzy set over X, the value of $\mu_A(x)$ plays the role of a norm (more precisely, a pseudo-norm).

In the case of the intuitionistic fuzzy sets, the presence of the second functional component, namely, the function ν_A gives rise to different possibilities for the definition of a norm (in the sense of a pseudo-norm) over the subsets and the elements of a given universe X (Atanassov [8], [11], [15], [22]).

Definition 3.4. The first norm given by Atanassov (Atanassov [15]) for every $x \in X$ with respect to a fixed set $A \subset X$ is

$$\sigma_{1,A}(x) = \mu_A(x) + \nu_A(x). \tag{3.19}$$

Norm (3.19) represents the degree of *definiteness* (Atanassov [15]) of the element x. From

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

we can express (3.19) as

$$\sigma_{1,A}(x) = 1 - \pi_A(x).$$

For every two intuitionistic fuzzy sets A and B, and for every $x \in X$ the following properties of (3.19) hold (Atanassov [15]):

$$\sigma_{1,\overline{A}}(x) = \sigma_{1,A}(x), \tag{3.20}$$

$$\sigma_{1,A\cap B}(x) \geq \min(\sigma_{1,A}(x), \sigma_{1,B}(x)), \tag{3.21}$$

$$\sigma_{1,A\cup B}(x) \leq \max(\sigma_{1,A}(x), \sigma_{1,B}(x)), \tag{3.22}$$

$$\sigma_{1,A+B}(x) \leq 1, \tag{3.23}$$

$$\sigma_{1,A.B}(x) \leq 1, \tag{3.24}$$

$$\sigma_{1,A@B}(x) = \frac{(\sigma_{1,A}(x) + \sigma_{1,B}(x))}{2}, \tag{3.25}$$

$$\sigma_{1,A\$B}(x) \leq \frac{(\sigma_{1,A}(x) + \sigma_{1,B}(x))}{2}, \tag{3.26}$$

$$\sigma_{1,A\bowtie B}(x) \leq \max(\sigma_{1,A}(x), \sigma_{1,B}(x)), \tag{3.27}$$

$$\sigma_{1,A*B}(x) \geq \frac{\max(\sigma_{1,A}(x), \sigma_{1,B}(x))}{2}, \tag{3.28}$$

$$\sigma_{1,\Box A}(x) = 1, \tag{3.29}$$

$$\sigma_{1,\Diamond A}(x) = 1, \tag{3.30}$$

$$\sigma_{1,C(A)}(x) \geq \max_{x\in X} \sigma_{1,A}(x), \tag{3.31}$$

$$\sigma_{1,I(A)}(x) \leq \min_{x\in X} \sigma_{1,A}(x), \tag{3.32}$$

$$\sigma_{1,D_\alpha}(x) = 1 \tag{3.33}$$

for every $\alpha \in [0,1]$,

$$\sigma_{1,F_{\alpha,\beta}}(x) = \alpha + \beta + (1 - \alpha - \beta).\sigma_{1,A}(x) \tag{3.34}$$

for every $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$,

$$\sigma_{1,G_{\alpha,\beta}}(x) \leq \sigma_{1,A}(x), \tag{3.35}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{1,H_{\alpha,\beta}}(x) \leq \beta + (\alpha + \beta).\sigma_{1,A}(x), \tag{3.36}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{1,H^*_{\alpha,\beta}(A)}(x) \leq \beta + (1 - \beta).\sigma_{1,A}(x), \tag{3.37}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{1,J_{\alpha,\beta}}(x) \leq \alpha + (\alpha + \beta).\sigma_{1,A}(x), \tag{3.38}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{1,J^*_{\alpha,\beta}(A)}(x) \leq \alpha + (1 - \alpha).\sigma_{1,A}(x), \tag{3.39}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{1,!A}(x) \geq 0, \tag{3.40}$$

$$\sigma_{1,?A}(x) \geq 0, \tag{3.41}$$

$$\sigma_{1,K_{\alpha}}(x) \geq 0, \tag{3.42}$$

for every $\alpha \in [0,1]$,

$$\sigma_{1,L_{\alpha}}(x) \geq 0, \tag{3.43}$$

for every $\alpha \in [0,1]$,

$$\sigma_{1,P_{\alpha,\beta}}(x) \geq 0, \tag{3.44}$$

for every $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$,

$$\sigma_{1,Q_{\alpha,\beta}}(x) \geq 0, \tag{3.45}$$

for every $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$.

Definition 3.5. Another norm for every $x \in X$, with respect to a fixed $A \subset X$, is defined as follows (Atanassov [15]):

$$\sigma_{2,A}(x) = \sqrt{(\mu_A(x)^2 + \nu_A(x)^2)}. \tag{3.46}$$

The norms σ_1 (3.19) and σ_2 (3.46) are analogous to the basic classical types of norms.

For the norm σ_2 (3.46), the following properties are fulfilled for every two intuitionistic fuzzy sets A and B, and for every $x \in X$ (Atanassov [15]):

$$\sigma_{2,\overline{A}}(x) = \sigma_{2,A}(x), \tag{3.47}$$

$$\sigma_{2,A\cap B}(x) \geq \min(\sigma_{2,A}(x), \sigma_{2,B}(x)), \tag{3.48}$$

$$\sigma_{2,A\cup B}(x) \leq \max(\sigma_{2,A}(x), \sigma_{2,B}(x)), \tag{3.49}$$

$$\sigma_{2,A+B}(x) \leq 1, \tag{3.50}$$

$$\sigma_{2,A.B}(x) \leq 1, \tag{3.51}$$

$$\sigma_{2,A@B}(x) \leq \frac{1}{\sqrt{2}}.(\sigma_{2,A}(x) + \sigma_{2,B}(x)), \tag{3.52}$$

$$\sigma_{2,A\$B}(x) \leq \sqrt{\sigma_{2,A}(x).\sigma_{2,B}(x))}, \tag{3.53}$$

$$\sigma_{2,A\bowtie B}(x) \geq \min(\sigma_{2,A}(x), \sigma_{2,B}(x)), \tag{3.54}$$

$$\sigma_{2,A*B}(x) \geq \max(\sigma_{2,A}(x), \sigma_{2,B}(x))/2, \tag{3.55}$$

$$\sigma_{2,\Box A}(x) \leq 1, \tag{3.56}$$

$$\sigma_{2,\Diamond A}(x) \leq 1, \tag{3.57}$$

$$\sigma_{2,CA}(x) \leq \max_{x\in X} \sigma_{2,A}(x), \tag{3.58}$$

$$\sigma_{2,IA}(x) \geq \min_{x\in X} \sigma_{2,A}(x), \tag{3.59}$$

$$\sigma_{2,D_\alpha}(x) \geq \sigma_{2,A}(x), \tag{3.60}$$

for every $\alpha \in [0,1]$,

$$\sigma_{2,F_{\alpha,\beta}}(x) \geq \sigma_{2,A}(x), \tag{3.61}$$

for every $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$,

$$\sigma_{2,G_{\alpha,\beta}}(x) \leq \sigma_{2,A}(x), \tag{3.62}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{2,H_{\alpha,\beta}}(x) \geq \alpha.\sigma_{2,A}(x), \tag{3.63}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{2,H^*_{\alpha,\beta}(A)}(x) \geq \alpha.\sigma_{2,A}(x), \tag{3.64}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{2,J_{\alpha,\beta}}(x) \geq \beta.\sigma_{2,A}(x), \tag{3.65}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{2,J^*_{\alpha,\beta}(A)}(x) \geq \beta.\sigma_{2,A}(x), \tag{3.66}$$

for every $\alpha, \beta \in [0,1]$,

$$\sigma_{2,!A}(x) \geq \frac{1}{2}, \tag{3.67}$$

$$\sigma_{2,?A}(x) \geq \frac{1}{2}, \tag{3.68}$$

$$\sigma_{2,K_\alpha(A)}(x) \geq \alpha, \tag{3.69}$$

for every $\alpha \in [0,1]$,

$$\sigma_{2,L_\alpha(A)}(x) \geq \alpha, \tag{3.70}$$

for every $\alpha \in [0,1]$,

$$\sigma_{2,P_{\alpha,\beta}(A)}(x) \geq \alpha, \tag{3.71}$$

for every $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$,

$$\sigma_{2,Q_{\alpha,\beta}(A)}(x) \geq \beta, \tag{3.72}$$

for every $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$.

Definition 3.6. Tanev [235]) defined the third norm over the elements of a given intuitionistic fuzzy set A as:

$$\sigma_{3,A}(x) = \frac{\mu_A(x) + 1 - \nu_A(x)}{2}. \tag{3.73}$$

The properties of (3.73) are similar to the properties of the first norm (3.19), and the second one, (3.46).

Some other discrete norms introduced by Atanassov [15] are presented in Definition 3.7.

Definition 3.7. For a given finite universe X and for a given intuitionistic fuzzy set A, we have the following discrete norms (Atanassov [15]):

$$n_\mu(A) = \sum_{x \in X} \mu_A(x), \tag{3.74}$$

$$n_\nu(A) = \sum_{x \in X} \nu_A(x), \tag{3.75}$$

$$n_\pi(A) = \sum_{x \in X} \pi_A(x). \tag{3.76}$$

The above norms (3.74)–(3.76) can be extended to continuous norms by replacing the sum in (3.74)–(3.76) by an integral over X.

After normalizing the norms (3.74)–(3.76) on the interval $[0,1]$, we obtain for a given finite universe X and for a given intuitionistic fuzzy set A, the following normalized discrete norms (Atanassov [15]):

- corresponding to the norm "$n_\mu(A)$" (3.74)

$$n^*_\mu(A) = \frac{1}{card(X)} \sum_{x \in X} \mu_A(x), \tag{3.77}$$

- corresponding to the norm "$n_\nu(A)$" (3.76)

$$n^*_\nu(A) = \frac{1}{card(X)} \sum_{x \in X} \nu_A(x), \tag{3.78}$$

- corresponding to the norm "$n_\pi(A)$" (3.76)

$$n^*_\pi(A) = \frac{1}{card(X)} \sum_{x \in X} \pi_A(x), \tag{3.79}$$

where $card(X)$ is the cardinality of the set X.

The above norms have similar properties.

In the theory of fuzzy sets (see e.g. Kaufmann [99]) two different types of distances are defined, generated from the following metric

$$m_A(x,y) = | \mu_A(x) - \mu_A(y)|$$

and the Hamming and Euclidean metrics coincide (Atanassov [15]).

In the case of the intuitionistic fuzzy sets these metrics are different (Atanassov [15]):

Definition 3.8. For an intuitionistic fuzzy set A the Hamming metric is defined as (Atanassov [15]):

$$h_A(x,y) = \frac{1}{2}(\mid \mu_A(x) - \mu_A(y) + \nu_A(x) - \nu_A(y) \mid). \tag{3.80}$$

Definition 3.9. For an intuitionistic fuzzy set A the the Euclidean metric is defined as (Atanassov [15]):

$$e_A(x,y) = \sqrt{\frac{1}{2}((\mu_A(x) - \mu_A(y))^2 + (\nu_A(x) - \nu_A(y))^2)}. \tag{3.81}$$

Under the assumption that

$$\nu_A(x) = 1 - \mu_A(x)$$

both metrics, (3.80) and (3.81), reduce to the metric m_A (x, y) (Atanassov [15]). To show that h_A and e_A are pseudo-metrics over X (in the sense of [100, 149]), it is necessary to prove that for every three elements $x,y,z \in X$ we have (Atanassov [15]):

$$h_A(x,y) + h_A(y,z) \geq h_A(x,z), \tag{3.82}$$

$$h_A(x,y) = h_A(y,x), \tag{3.83}$$

$$e_A(x,y) + e_A(y,z) \geq e_A(x,z), \tag{3.84}$$

$$e_A(x,y) = e_A(y,x). \tag{3.85}$$

As conditions (3.82) and (3.84) do not hold (Atanassov [15]) for the metrics, h_A and e_A are pseudo-metrics. The proofs of the above equalities and inequalities are trivial.

The well known types of distances for the fuzzy sets A and B are:

- the Hamming distance

$$d(A,B) = \sum_{x \in X} \mid \mu_A(x) - \mu_B(x) \mid, \tag{3.86}$$

- the Euclidean distance

$$e(A,B) = \sqrt{\sum_{x \in X} (\mu_A(x) - \mu_B(x))^2}. \tag{3.87}$$

The distances (3.86) and (3.87) transformed into the intuitionistic fuzzy sets, have the following respective forms (Atanassov [15]):

Definition 3.10. For two intuitionistic fuzzy sets A and B over a universe X, the Hamming distance between A and B is defined as (Atanassov [15])

$$d_{IFS(2)}(A,B) = \frac{1}{2}\sum_{x\in X} \mid \mu_A(x) - \mu_B(x) \mid + \mid \nu_A(x) - \nu_B(x) \mid, \tag{3.88}$$

and the corresponding normalized Hamming distance is

$$l_{IFS(2)}(A,B) = \frac{1}{2n}\sum_{x\in X} \mid \mu_A(x) - \mu_B(x) \mid + \mid \nu_A(x) - \nu_B(x) \mid . \tag{3.89}$$

Definition 3.11. For two intuitionistic fuzzy sets A and B over a universe X, the Euclidean distance between A and B is defined as (Atanassov [15])

$$e_{IFS(2)}(A,B) = \sqrt{\frac{1}{2}(\sum_{x\in X} (\mu_A(x) - \mu_B(x))^2 + (\nu_A(x) - \nu_B(x))^2)}, \tag{3.90}$$

and the corresponding normalized Euclidean distance is

$$q_{IFS(2)}(A,B) = \sqrt{\frac{1}{2n}(\sum_{x\in X} (\mu_A(x) - \mu_B(x))^2 + (\nu_A(x) - \nu_B(x))^2)}. \tag{3.91}$$

Distances (3.88)–(3.91) correspond to the two term intuitionistic fuzzy set description – i.e. the membership values and the non-membership values are taken into account only. In Section 3.3 we will discuss another form of the Hamming and Euclidean distances, using the three term description of the intuitionistic fuzzy sets (besides the membership values and the non-membership values also the hesitation margin is taken into account):

$$l^1_{IFS}(A,B) = \frac{1}{2n}\sum_{x\in E} \mid \mu_A(x) - \mu_B(x) \mid + \mid \nu_A(x) - \nu_B(x) \mid + \mid \pi_A(x) - \pi_B(x) \mid \tag{3.92}$$

and

$$q^1_{IFS}(A,B) = \sqrt{\frac{1}{2n}(\sum_{x\in E} (\mu_A(x) - \mu_B(x))^2 + (\nu_A(x) - \nu_B(x))^2) + (\pi_A(x) - \pi_B(x))^2)}. \tag{3.93}$$

Distances (3.92)–(3.93) correspond to the three term intuitionistic fuzzy set description (membership values, non-membership values and hesitation margins are taken into account) and are useful from the point of view of practical applications.

In the next chapter we will discuss in details distances (3.88)–(3.90) and (3.92)–(3.93).

In (Atanassov [15]), other distances are also given (cf. [149]), which can be defined over the intutionistic fuzzy sets:

Definition 3.12. For two intuitionistic fuzzy sets A and B over a universe X, the following distances between A and B are defined (Atanassov [15]):

$$J_1(A,B) = \max_{x \in X} \mid \mu_A(x) - \mu_B(x) \mid, \tag{3.94}$$

$$J_2(A,B) = \max_{x \in X} \mid \nu_A(x) - \nu_B(x) \mid, \tag{3.95}$$

$$J(A,B) = \frac{1}{2}.(J_1(A,B) + J_2(A,B)), \tag{3.96}$$

$$J^*(A,B) = \frac{1}{2}.\max_{x \in X}(\mid \mu_A(x) - \mu_B(x) \mid + \mid \nu_A(x) - \nu_B(x) \mid). \tag{3.97}$$

It is easily seen that for every two intuitionistic fuzzy sets A and B we have (Atanassov [15]):

$$J^*(A,B) \leq J_1(A,B) + J_2(A,B).$$

In the distance $J_1(.,.)$ (3.94) only the membership values are taken into account, and so the distance is reduced directly to the distance for fuzzy sets. On the other hand, the distances $J_2(.,.)$ (3.95), $J(.,.)$ (3.96), and $J^*(.,.)$ (3.97) make use of both membership and non-membership values, and thus they do not reduce to the distances for fuzzy sets.

Atanassov [21], [22] introduced also norms following one of the most important ideas of Georg Cantor in set theory, calling the norms *"Cantor's intuitionistic fuzzy norms"*. Cantor's intuitionistic fuzzy norms are substantially different from the Euclidean and Hamming norms, existing in fuzzy set theory.

Let $x \in X$ be fixed universe and let

$$\mu_A(x) = 0.a_1a_2...$$

$$\nu_A(x) = 0.b_1b_2...$$

Next, Atanassov [22] bijectively constructed the numbers:

$$||x||_{\mu,\nu} = 0, a_1b_1a_2b_2...$$

and

$$||x||_{\nu,\mu} = 0, b_1a_1b_2a_2...$$

and noticed that the following properties hold for these numbers:

1. $||x||_{\mu,\nu}, ||x||_{\nu,\mu} \in [0,1]$
2. having both numbers it is possible to reconstruct directly the numbers $\mu_A(x)$ and $\nu_A(x)$.

The numbers $||x||_{\mu,\nu}$ and $||x||_{\nu,\mu}$ were called by Atanassov [22] Cantor norms of element $x \in X$.

Atanassov [22] denotes these norms by $||x||_{2,\mu,\nu}$ and $||x||_{2,\nu,\mu}$ in order to stress that they correspond to the two term intuitionistic fuzzy set description.

On the other hand, for the three term intuitionistic fuzzy set description introduced by Szmidt and Kacprzyk, for point x we have (Atanassov [22]):

$$\mu_A(x) = 0.a_1a_2...$$

$$\nu_A(x) = 0.b_1b_2...$$

$$\pi_A(x) = 0.c_1c_2...$$

with the condition: $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$. (Atanassov [22]) introduced six different Cantor norms:

$$||x||_{3,\mu,\nu,\pi} = 0.a_1b_1c_1a_2b_2c_2...,$$

$$||x||_{3,\mu,\pi,\nu} = 0.a_1c_1b_1a_2c_2b_2...,$$

$$||x||_{3,\nu,\mu,\pi} = 0.b_1a_1c_1b_2a_2c_2...,$$

$$||x||_{3,\nu,\pi,\mu} = 0.b_1c_1a_1b_2c_2a_2...,$$

$$||x||_{3,\pi,\mu,\nu} = 0.c_1a_1b_1c_2a_2b_2...,$$

$$||x||_{3,\pi,\nu,\mu} = 0.c_1b_1a_1c_2b_2a_2....$$

For the above three term Cantor norms it is possible, as previously, to reconstruct bijectively the three degrees of element $x \in X$.

3.3 Distances between the Intuitionistic Fuzzy Sets – The Three Term Approach

In this section we recall some new definitions of distances between intuitionistic fuzzy sets (Szmidt and Kacprzyk [171]). By taking into account the three term characterization of the intuitionistic fuzzy sets, and following the basic line of reasoning on which the definition of distances between the fuzzy sets is based, we define four basic distances between the intuitionistic fuzzy sets: Hamming distance, normalized Hamming distance, Euclidean distance, and normalized Euclidean distance. While deriving these distances a convenient geometric interpretation of the intuitionistic fuzzy sets is employed. It is shown that the definitions proposed are consistent with their counterparts traditionally used for the fuzzy sets, and that the consistency is ensured only under the condition that all three parameters characterizing the intuitionistic fuzzy sets are taken into account.

We will first reconsider some better known distances for the fuzzy sets in an intuitionistic setting, and then extend those distances to the intuitionistic fuzzy sets.

3.3.1 Distances between the Fuzzy Sets

The most widely used distances for fuzzy sets $A^{'}$, $B^{'}$ in $X = \{x_1, x_2, \ldots, x_n\}$ are (Kacprzyk, 1997):

- the Hamming distance $d(A^{'}, B^{'})$

$$d(A^{'}, B^{'}) = \sum_{i=1}^{n} \left| \mu_{A'}(x_i) - \mu_{B'}(x_i) \right| \tag{3.98}$$

- the normalized Hamming distance $l(A^{'}, B^{'})$:

$$l(A^{'}, B^{'}) = \frac{1}{n} \sum_{i=1}^{n} \left| \mu_{A'}(x_i) - \mu_{B'}(x_i) \right| \tag{3.99}$$

- the Euclidean distance $e(A^{'}, B^{'})$:

$$e(A^{'}, B^{'}) = \sqrt{\sum_{i=1}^{n} \left(\mu_{A'}(x_i) - \mu_{B'}(x_i) \right)^2} \tag{3.100}$$

- the normalized Euclidean distance $q(A^{'}, B^{'})$:

$$q(A^{'}, B^{'}) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(\mu_{A'}(x_i) - \mu_{B'}(x_i) \right)^2} \tag{3.101}$$

It is worth mentioning that in the above formulas, (3.98)-(3.101), only the membership functions are present. It is due to the fact that for a fuzzy set, $\mu(x_i) + \nu(x_i) = 1$.

In Chapter 2, we have introduced for a fuzzy set $A^{'}$ in X an equivalent intuitionistic-type representation given as

$$A^{'} = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > / x \in X\}.$$

The above representation will be employed while rewriting the distances (3.98)-(3.101).

So, first, taking into account an intuitionistic-type representation of a fuzzy set, we can express the very essence of the Hamming distance by putting

$$d'(A',B') = \sum_{i=1}^{n} \left(\left|\mu_{A'}(x_i) - \mu_{B'}(x_i)\right| + \left|\nu_{A'}(x_i) - \nu_{B'}(x_i)\right|\right) =$$
$$= \sum_{i=1}^{n} \left(\left|\mu_{A'}(x_i) - \mu_{B'}(x_i)\right| + \left|1 - \mu_{A'}(x_i) - 1 + \mu_{B'}(x_i)\right|\right) =$$
$$= 2\sum_{i=1}^{n} \left|\mu_{A'}(x_i) - \mu_{B'}(x_i)\right| = 2d(A',B') \tag{3.102}$$

i.e. the Hamming distance in an intuitionistic-type representation of the fuzzy sets is twice the Hamming distance between fuzzy sets calculated in a standard way, (3.98).

Similarly, the normalized Hamming distance $l'(A',B')$, when we take into account an intuitionistic-type representation of a fuzzy set, is in turn equal to

$$l'(A',B') = \frac{1}{n} \cdot d'(A',B') = \frac{2}{n}\sum_{i=1}^{n} \left|\mu_{A'}(x_i) - \mu_{B'}(x_i)\right| \tag{3.103}$$

i.e. the result of (3.103) is equal the well known normalized Hamming distance (3.99) between fuzzy sets, multiplied by two.

Then, by the same line of reasoning, the Euclidean distance, taking into account an intuitionistic-type representation of a fuzzy set, is equal to

$$e'(A',B') = \sqrt{\sum_{i=1}^{n} \left(\mu_{A'}(x_i) - \mu_{B'}(x_i)\right)^2 + \left(\nu_{A'}(x_i) - \nu_{B'}(x_i)\right)^2} =$$
$$= \sqrt{\sum_{i=1}^{n} \left(\mu_{A'}(x_i) - \mu_{B'}(x_i)\right)^2 + \left(1 - \mu_{A'}(x_i) - 1 + \mu_{B'}(x_i)\right)^2} =$$
$$= \sqrt{2\sum_{i=1}^{n} \left(\mu_{A'}(x_i) - \mu_{B'}(x_i)\right)^2} \tag{3.104}$$

i.e. it is just multiplied by $\sqrt{2}$ Euclidean distance for the usual representation of fuzzy sets given by (3.100).

The normalized Euclidean distance $q'(A',B')$ considering the intuitionistic-type representation of a fuzzy set is equal to

$$q'(A',B') = \sqrt{\frac{1}{n}} \cdot e'(A',B') = \sqrt{\frac{2}{n}\sum_{i=1}^{n} \left(\mu_{A'}(x_i) - \mu_{B'}(x_i)\right)^2} \tag{3.105}$$

so again the result of (3.105) is the expression from (3.101) multiplied by $\sqrt{2}$.

Example 3.1. (Szmidt and Kacprzyk [171]) For simplicity we consider "degenerate" fuzzy sets M, N, L, K, P in $X = \{1\}$. Complete description of each of them is given by $A = (\mu_A, \nu_A)/1$, namely:

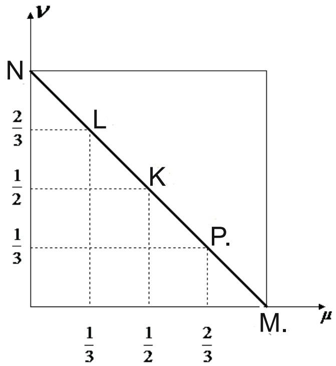


Fig. 3.1 A geometrical interpretation of one-element fuzzy sets considered in Example 3.1

$$M=(1,0)/1, \quad N=(0,1)/1, \quad L=(\frac{1}{3},\frac{2}{3})/1, \quad P=(\frac{2}{3},\frac{1}{3})/1, \quad K=(\frac{1}{2},\frac{1}{2})/1$$

Figure 3.1 gives a geometrical interpretation of these one-element fuzzy sets.

First, let us calculate the Euclidean distances between the fuzzy sets using their "normal" representation (i.e., taking into account the membership values only) as in (3.100)

$$e(L,P)=\sqrt{(\frac{1}{3}-\frac{2}{3})^2}=\frac{1}{3} \tag{3.106}$$

$$e(L,K)=\sqrt{(\frac{1}{3}-\frac{1}{2})^2}=\frac{1}{6} \tag{3.107}$$

$$e(P,K)=\sqrt{(\frac{2}{3}-\frac{1}{2})^2}=\frac{1}{6} \tag{3.108}$$

$$e(L,M)=\sqrt{(\frac{1}{3}-1)^2}=\frac{2}{3} \tag{3.109}$$

$$e(K,M)=\sqrt{(1-\frac{1}{2})^2}=\frac{1}{2} \tag{3.110}$$

$$e(N,K)=\sqrt{(0-\frac{1}{2})^2}=\frac{1}{2} \tag{3.111}$$

$$e(N,M)=\sqrt{1^2}=1 \tag{3.112}$$

The same Euclidean distances are calculated now using the intuitionistic-type representation of fuzzy sets (3.104)

$$e'(L,P) = \sqrt{(\frac{1}{3} - \frac{2}{3})^2 + (\frac{2}{3} - \frac{1}{3})^2} = \frac{\sqrt{2}}{3} \tag{3.113}$$

$$e'(L,K) = \sqrt{(\frac{1}{3} - \frac{1}{2})^2 + (\frac{2}{3} - \frac{1}{2})^2} = \frac{\sqrt{2}}{6} \tag{3.114}$$

$$e'(P,K) = \sqrt{(\frac{2}{3} - \frac{1}{2})^2 + (\frac{1}{3} - \frac{1}{2})^2} = \frac{\sqrt{2}}{6} \tag{3.115}$$

$$e'(L,M) = \sqrt{(\frac{1}{3} - 1)^2 + (\frac{2}{3} - 0)^2} = \frac{2\sqrt{2}}{3} \tag{3.116}$$

$$e'(K,M) = \sqrt{(\frac{1}{2} - 1)^2 + (\frac{1}{2})^2} = \frac{\sqrt{2}}{2} \tag{3.117}$$

$$e'(N,K) = \sqrt{(0 - \frac{1}{2})^2 + (1 - \frac{1}{2})^2} = \frac{\sqrt{2}}{2} \tag{3.118}$$

$$e'(N,M) = \sqrt{1^2 + 1^2} = \sqrt{2} \tag{3.119}$$

Thus, as has been already noticed, the above results are just those of (3.106)–(3.112) multiplied by the constant value equal to $\sqrt{2}$. Therefore, though the distances (3.106)–(3.112) and (3.113)–(3.119) are clearly different, their essence is the same.

Example 3.2. (Szmidt and Kacprzyk [171]) Let us consider two fuzzy sets A', B' in $X = \{1,2,3,4,5,6,7\}$. Their intuitionistic - type representation is $A' = (\mu_{A'}, \nu_{A'})/1$, given here as

$$A' = (0.7,\ 0.3)/1 + (0.2,\ 0.8)/2 + (0.6,\ 0.4)/4 + (0.5,\ 0.5)/5 + (1,\ 0)/6 \tag{3.120}$$

$$B' = (0.2,\ 0.8)/1 + (0.6,\ 0.4)/4 + (0.8,\ 0.2)/5 + (1,\ 0)/7 \tag{3.121}$$

The Hamming distance $d(A', B')$, accounting only for the membership functions (3.98), is

$$d(A',B') = |0.7 - 0.2| + |0.2 - 0| + |0.6 - 0.6| + |0.5 - 0.8| + |1 - 0| + |0 - 1| = 3 \tag{3.122}$$

while the normalized distance (3.99) $l(A', B')$ is equal to

$$l(A',B') = \frac{1}{7} \cdot d(A',B') = \frac{3}{7} = 0.43 \tag{3.123}$$

On the other hand, when both the membership and non-membership values are taken into account [cf.(3.102)], we obtain

$$d'(A',B') = |0.7-0.2|+|0.3-0.8|+|0.2-0|+|0.8-1|+|0.6-0.6|+ \\ + |0.4-0.4|+|0.5-0.8|+|0.5-0.2|+|1-0|+|0-1|+ \\ + |0-1|+|1-0| = 6 \quad (3.124)$$

i.e. we get the value from (3.122) multiplied by two. The normalized Hamming distance (3.103) is equal to

$$l'(A',B') = \frac{1}{n}d'(A',B') = \frac{6}{7} = 0.86 \quad (3.125)$$

Let us compare the Euclidean distances obtained from (3.100) and (3.104). From (3.100) we have

$$e(A',B') = ((0.7-0.2)^2+(0.2-0)^2+(0.6-0.6)^2+(0.5-0.8)^2+ \\ + (1-0.2)^2+(0-1)^2)^{\frac{1}{2}} = \sqrt{2.38} = 1.54 \quad (3.126)$$

while the counterpart normalized Euclidean distance (3.101) is

$$q(A',B') = \sqrt{\frac{1}{7}} \cdot e(A,B) = \sqrt{\frac{2.38}{7}} = 0.58 \quad (3.127)$$

From (3.104) we have the Euclidean distance, taking into account the intuitionistic-type representation of fuzzy sets, equal to

$$e'(A',B') = ((0.7-0.2)^2+(0.3-0.8)^2+(0.2-0)^2+(0.8-1)^2+(0.6-0.6)^2+ \\ + (0.4-0.4)^2+(0.5-0.8)^2+(0.5-0.2)^2+(1-0)^2+ \\ + (0-0)^2+(0-1)^2+(1-0)^2)^{\frac{1}{2}} = \sqrt{4.76} = 2.18 \quad (3.128)$$

whereas the counterpart, the normalized Euclidean distance (3.105), accounting for the intuitionistic-type representation of fuzzy sets is equal to

$$q'(A',B') = \sqrt{\frac{4.76}{7}} = 0.83 \quad (3.129)$$

Suppose we modify a little bit the fuzzy set B' (making it closer to A'), i.e., these two fuzzy sets are now

$$A' = (0.7,\ 0.3)/1+(0.2,\ 0.8)/2+(0.6,\ 0.4)/4+(0.5,\ 0.5)/5+(1,\ 0)/6 \quad (3.130)$$

$$B' = (0.2,\ 0.8)/1+(0.6,\ 0.4)/4+(0.8,\ 0.2)/5+(0.4,\ 0.6)/6+(1,\ 0)/7 \quad (3.131)$$

The Hamming distance calculated with (3.98) is

$$d(A',B') = |0.7-0.2|+|0.2-0|+|0.6-0.6|+|0.5-0.8|+|1-0.4|+|0-1| = \\ = 2.6 \quad (3.132)$$

whereas the normalized Hamming distance (3.99) is

$$l(A',B') = \frac{1}{7}\cdot d(A',B') = \frac{2.6}{7} = 0.37 \quad (3.133)$$

From (3.102), taking into account the intuitionistic-type representation of fuzzy sets, we obtain the Hamming distance equal to

$$d^{'}(A^{'},B^{'}) = |0.7-0.2|+|0.3-0.8|+|0.2-0|+|0.8-1|+|0.6-0.6|+ \\ + |0.4-0.4|+|0.5-0.8|+|0.5-0.2|+|1-0.4|+|0-0.6|+ \\ + |0-1|+|1-0| = 5.2 \quad (3.134)$$

while the normalized Hamming distance (3.103) taking into account the intuitionistic-type representation of fuzzy sets, is equal to

$$l^{'}(A^{'},B^{'}) = \frac{1}{7}\cdot 5.2 = 0.74 \quad (3.135)$$

Let us calculate the Euclidean distances now. From (3.100) we obtain

$$e(A^{'},B^{'}) = ((0.7-0.2)^2+(0.2-0)^2+(0.6-0.6)^2+(0.5-0.8)^2+ \\ + (1-0.4)^2+(0-1)^2)^{\frac{1}{2}} = \sqrt{1.74} = 1.32 \quad (3.136)$$

while from (3.101) we get the normalized Euclidean distance

$$q(A^{'},B^{'}) = \sqrt{\frac{1.74}{7}} = 0.5 \quad (3.137)$$

Taking into account the intuitionistic-type representation of fuzzy sets, from (3.104) we obtain the Euclidean distance

$$e^{'}(A^{'},B^{'}) = ((0.7-0.2)^2+(0.3-0.8)^2+(0.2-0)^2+(0.8-1)^2+ \\ + (0.6-0.6)^2+(0.4-0.4)^2+(0.5-0.8)^2+(0.5-0.2)^2+ \\ + (1-0.4)^2+(0-0.6)^2+(0-1)^2+(1-0)^2)^{\frac{1}{2}} = \\ = \sqrt{3.48} = 1.87 \quad (3.138)$$

while the normalized Euclidean distance (3.105), taking into account the intuitionistic-type representation of fuzzy sets, is equal to

$$q^{'}(A^{'},B^{'}) = \sqrt{\frac{1}{7}\cdot 3.48} = 0.705 \quad (3.139)$$

As we analyze the results of Examples 3.1 and 3.2 we may notice that:

- for any fuzzy sets A' and B', when we calculate the distances between them in a standard way (3.98)–(3.101), i.e., when we take into account the membership values only, we have

$$0 \leq d(A',B') \leq n \tag{3.140}$$

$$0 \leq l(A',B') \leq 1 \tag{3.141}$$

$$0 \leq e(A',B') \leq \sqrt{n} \tag{3.142}$$

$$0 \leq q(A',B') \leq 1 \tag{3.143}$$

- for any fuzzy sets A' and B', when we calculate distances between them taking into account the intuitionistic-type representation of fuzzy sets (3.102)-(3.105), we have

$$0 \leq d'(A',B') \leq 2n \tag{3.144}$$

$$0 \leq l'(A',B') \leq 2 \tag{3.145}$$

$$0 \leq e'(A',B') \leq \sqrt{2n} \tag{3.146}$$

$$0 \leq q'(A',B') \leq \sqrt{2} \tag{3.147}$$

We would like to emphasize that it is not our purpose to introduce a new way of calculating distances for fuzzy sets. To the contrary, we have shown that the intuitionistic-type representation of fuzzy sets results in multiplying the distances by constant values only. But similar reasoning for the case of the intuitionistic fuzzy sets (i.e. omitting one of the three terms) would lead to incorrect results, as this is discussed in detail in the next section.

3.3.2 Distances between the Intuitionistic Fuzzy Sets

Following the line of reasoning presented in Section 3.3.1, we will now extend the concepts of distances to the case of the intuitionistic fuzzy sets.

The Hamming distance between two intuitionistic fuzzy sets A and B in $X = \{x_1, x_2, ..., x_n\}$ is equal to (Szmidt and Kacprzyk [171])

$$d^1_{IFS}(A,B) = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|) \tag{3.148}$$

Having in mind that

$$\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i) \tag{3.149}$$

and

$$\pi_B(x_i) = 1 - \mu_B(x_i) - \nu_B(x_i) \tag{3.150}$$

we have

$$\begin{aligned}|\pi_A(x_i) - \pi_B(x_i)| &= |1 - \mu_A(x_i) - \nu_A(x_i) - 1 + \mu_B(x_i) + \nu_B(x_i)| \leq \\ &\leq |\mu_B(x_i) - \mu_A(x_i)| + |\nu_B(x_i) - \nu_A(x_i)|\end{aligned} \tag{3.151}$$

From inequality (3.151) it follows that the third term in (3.148) cannot be omitted as it was in the case of fuzzy sets, for which taking into account the second term would only result in the multiplication by a constant value.

For the Euclidean distance a similar situation occurs. Namely, for intuitionistic fuzzy sets A and B in $X = \{x_1, x_2, ..., x_n\}$, by following the line of reasoning as in Section 3.3.1, their Euclidean distance is equal to (Szmidt and Kacprzyk [171])

$$\begin{aligned}e^1_{IFS}(A,B) &= (\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + \\ &+ (\pi_A(x_i) - \pi_B(x_i))^2)^{\frac{1}{2}}\end{aligned} \tag{3.152}$$

Let us verify the effect of omitting the third term (π) in (3.152). Having in mind (3.149)–(3.150), we have (Szmidt and Kacprzyk [171]):

$$\begin{aligned}(\pi_A(x_i) - \pi_B(x_i))^2 &= (1 - \mu_A(x_i) - \nu_A(x_i) - 1 + \mu_B(x_i) + \nu_B(x_i))^2 = \\ &= (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + \\ &+ 2(\mu_A(x_i) - \mu_B(x_i))(\nu_A(x_i) - \nu_B(x_i))\end{aligned} \tag{3.153}$$

which means that taking into account the third term π when calculating the Euclidean distance for the intuitionistic fuzzy sets does have an influence on the final result. This is obvious, because a two-dimensional geometrical interpretation (Figure 2.2) is an orthogonal projection of a real situation presented in Figure 2.3.

Taking into account (3.149)-(3.153), in order to be more in agreement with the mathematical notion of normalization, the following distances for two intuitionistic fuzzy sets A and B in $X = \{x_1, x_2, ..., x_n\}$ are proposed (Szmidt and Kacprzyk [171])

- the Hamming distance:

$$\begin{aligned}d^1_{IFS}(A,B) &= \frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + \\ &+ |\pi_A(x_i) - \pi_B(x_i)|)\end{aligned} \tag{3.154}$$

- the Euclidean distance :

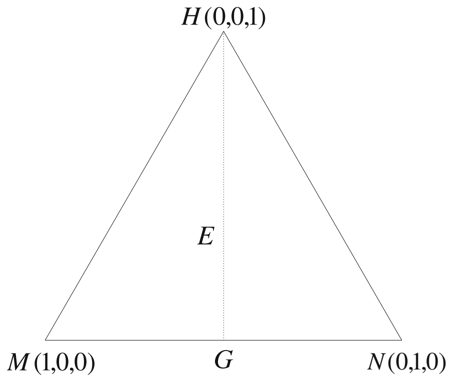


Fig. 3.2 Geometrical representation of the one-element intuitionistic fuzzy sets from Example 3.3

$$e^1_{IFS}(A,B) = (\frac{1}{2}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2 + (\nu_A(x_i)-\nu_B(x_i))^2 + \\ + (\pi_A(x_i)-\pi_B(x_i))^2)^{\frac{1}{2}} \tag{3.155}$$

- the normalized Hamming distance:

$$l^1_{IFS}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)| + |\nu_A(x_i)-\nu_B(x_i)| + \\ + |\pi_A(x_i)-\pi_B(x_i)|) \tag{3.156}$$

- the normalized Euclidean distance:

$$q^1_{IFS}(A,B) = (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2 + (\nu_A(x_i)-\nu_B(x_i))^2 + \\ + (\pi_A(x_i)-\pi_B(x_i))^2)^{\frac{1}{2}} \tag{3.157}$$

The above distances satisfy the conditions of the metric (cf. Kaufmann [99]).

Example 3.3. (Szmidt and Kacprzyk [171]) Let us consider for simplicity the "degenerate" intuitionistic fuzzy sets M, N, H, G, E in $X = \{1\}$. The full description of each intuitionistic fuzzy set, i.e. $A = (\mu_A, \nu_A, \pi_A)/1$, may be exemplified by

$$M = (1,0,0)/1, \quad N = (0,1,0)/1, \quad H = (0,0,1)/1, \\ G = (\frac{1}{2},\frac{1}{2},0)/1, \quad E = (\frac{1}{4},\frac{1}{4},\frac{1}{2})/1 \tag{3.158}$$

The geometrical interpretation of the above sets is presented in Figure 3.2.

Let us calculate the Euclidean distances between the above intuitionistic fuzzy sets omitting the third term, i.e., using the following formula:

$$e_{IFS(2)}(A,B) = \sqrt{\frac{1}{2}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2} \tag{3.159}$$

We obtain:

$$e_{IFS(2)}(M,H) = \sqrt{\frac{1}{2}((1-0)^2 + 0^2)} = \frac{1}{2} \tag{3.160}$$

$$e_{IFS(2)}(N,H) = \sqrt{\frac{1}{2}(0^2 + (0-1)^2)} = \frac{1}{2} \tag{3.161}$$

$$e_{IFS(2)}(M,N) = \sqrt{\frac{1}{2}((1-0)^2 + (0-1)^2)} = 1 \tag{3.162}$$

$$e_{IFS(2)}(M,G) = \sqrt{\frac{1}{2}((1-\frac{1}{2})^2 + (0-\frac{1}{2})^2)} = \frac{1}{2} \tag{3.163}$$

$$e_{IFS(2)}(N,G) = \sqrt{\frac{1}{2}((0-\frac{1}{2})^2 + (1-\frac{1}{2})^2)} = \frac{1}{2} \tag{3.164}$$

$$e_{IFS(2)}(E,G) = \sqrt{\frac{1}{2}((\frac{1}{4}-\frac{1}{2})^2 + (\frac{1}{4}-\frac{1}{2})^2)} = \frac{1}{4} \tag{3.165}$$

$$e_{IFS(2)}(H,G) = \sqrt{\frac{1}{2}((0-\frac{1}{2})^2 + (0-\frac{1}{2})^2)} = \frac{1}{4} \tag{3.166}$$

However, one can hardly agree with the above results. As it was shown (cf. Figure 2.3), the triangle MNH (Figure 3.2) has all edges equal to $\sqrt{2}$ (as they are diagonals of squares with sides equal to 1). So we should obtain $e_{IFS(2)}(M,H) = e_{IFS(2)}(N,H) = e_{IFS(2)}(M,N)$. But our results show only that $e_{IFS(2)}(M,H) = e_{IFS(2)}(N,H)$ [cf. (3.160)–(3.161)], but unfortunately $e_{IFS(2)}(M,H) \neq e_{IFS(2)}(M,N)$, and $e_{IFS(2)}(N,H) \neq e_{IFS(2)}(M,N)$. Also $e_{IFS(2)}(E,G)$, which is half of the height of triangle MNH multiplied in (3.159) by $\sqrt{1/2}$, is not the value we expect (it is too short, and the same concerns the height of $e_{IFS(2)}(H,G)$).

On the other hand, upon calculating the same Euclidean distances using (3.155), i.e., taking into account all three terms (membership values, non-membership values, and hesitation margins), we obtain:

$$e^1_{IFS}(M,H) = \sqrt{\frac{1}{2}((1-0)^2 + 0^2 + (0-1)^2)} = 1 \tag{3.167}$$

$$e^1_{IFS}(N,H) = \sqrt{\frac{1}{2}(0^2+(1-0)^2+(0-1)^2)} = 1 \tag{3.168}$$

$$e^1_{IFS}(M,N) = \sqrt{\frac{1}{2}((1-0)^2+(0-1)^2+0^2)} = 1 \tag{3.169}$$

$$e^1_{IFS}(M,G) = \sqrt{\frac{1}{2}((1-\frac{1}{2})^2+(0-\frac{1}{2})^2+0^2)} = \frac{1}{2} \tag{3.170}$$

$$e^1_{IFS}(N,G) = \sqrt{\frac{1}{2}((0-\frac{1}{2})^2+(1-\frac{1}{2})^2+0^2)} = \frac{1}{2} \tag{3.171}$$

$$e^1_{IFS}(E,G) = \sqrt{\frac{1}{2}((\frac{1}{4}-\frac{1}{2})^2+(\frac{1}{4}-\frac{1}{2})^2+(\frac{1}{2}-0)^2)} = \frac{\sqrt{3}}{4} \tag{3.172}$$

$$e^1_{IFS}(H,G) = \sqrt{\frac{1}{2}((0-\frac{1}{2})^2+(0-\frac{1}{2})^2+(1-0)^2)} = \frac{\sqrt{3}}{2} \tag{3.173}$$

From (3.155) we get the expected results, i.e.

$$e^1_{IFS}(M,H) = e^1_{IFS}(N,H) = e^1_{IFS}(M,N) = 2e^1_{IFS}(M,G) = 2e^1_{IFS}(N,G)$$

and $e^1_{IFS}(E,G)$ is equal to half the height of a triangle with all edges equal $\sqrt{2}$ multiplied by $1/\sqrt{2}$, i.e. $\frac{\sqrt{3}}{4}$.

Example 3.4. (Szmidt and Kacprzyk [171]) Let A and B in $X = \{1,2,3,4,5,6,7\}$ be the following intuitionistic fuzzy sets

$$\begin{aligned} A = {} & (0.5, 0.3, 0.2)/1 + (0.2, 0.6, 0.2)/2 + (0.3, 0.2, 0.5)/4 + \\ & + (0.2, 0.2, 0.6)/5 + (1, 0, 0)/6 \end{aligned} \tag{3.174}$$

$$B = (0.2, 0.6, 0.2)/1 + (0.3, 0.2, 0.5)/4 + (0.5, 0.2, 0.3)/5 + (0.9, 0, 0.1)/7 \tag{3.175}$$

Then, upon taking into account all three terms, we get the Hamming distance (3.154) equal to

$$\begin{aligned} d^1_{IFS}(A,B) = {} & \frac{1}{2}(|0.5-0.2|+|0.3-0.6|+|0.2-0.2|+|0.2-0|+|0.6-1|+ \\ & +|0.2-0|+|0.3-0.3|+|0.2-0.2|+|0.5-0.5|+|0.2-0.5|+ \\ & +|0.2-0.2|+|0.6-0.3|+|1-0|+|0-1|+|0-0|+ \\ & +|0-0.9|+|1-0|+|0-0.1|) = 3 \end{aligned} \tag{3.176}$$

Thus, taking into account all three terms, we get the normalized Hamming distance (3.156) as equal to

$$l_{IFS}^{1}(A,B) = \frac{3}{7} = 0.43 \tag{3.177}$$

The Hamming distance, when we account for two terms only, is equal to

$$\begin{aligned} d_{IFS(2)}(A,B) =& \frac{1}{2}(|0.5-0.2|+|0.3-0.6|+|0.2-0|+|0.6-1|+|0.3-0.3|+ \\ &+ |0.2-0.2|+|0.2-0.5|+|0.2-0.2|+|1-0|+|0-1|+ \\ &+ |0-0.9|+|1-0|) = 2.7 \end{aligned} \tag{3.178}$$

and the normalized Hamming distance accounting for two terms only is

$$l_{IFS(2)}(A,B) = \frac{1}{7} \cdot d(A,B) = \frac{2.7}{7} = 0.39 \tag{3.179}$$

The Euclidean distance (3.155) based on all three terms is equal to

$$\begin{aligned} e_{IFS}^{1}(A,B) =& 0.5^{0.5}((0.5-0.2)^2+(0.3-0.6)^2+(0.2-0.2)^2+(0.2-0)^2+ \\ &+ (0.6-1)^2+(0.2-0)^2+(0.3-0.3)^2+(0.2-0.2)^2+ \\ &+ (0.5-0.5)^2+(0.2-0.5)^2+(0.2-0.2)^2+(0.6-0.3)^2+ \\ &+ (1-0)^2+(0-1)^2+0^2+(0-0.9)^2+(1-0)^2+(0-0.1)^2)^{0.5} = \\ &= \sqrt{2.21} = 1.49 \end{aligned} \tag{3.180}$$

thus, the normalized Euclidean distance based on all three terms is

$$q_{IFS}^{1}(A,B) = \frac{e_{IFS}^{1}(A,B)}{\sqrt{7}} = \sqrt{\frac{2.21}{7}} = 0.56 \tag{3.181}$$

The Euclidean distance (3.159), calculated with two terms only is equal to

$$\begin{aligned} e_{IFS(2)}(A,B) =& 0.5^{0.5}((0.5-0.2)^2+(0.3-0.6)^2+(0.2-0)^2+(0.6-1)^2+ \\ &+ (0.3-0.3)^2+(0.2-0.2)^2+(0.2-0.5)^2+(0.2-0.2)^2+(1-0)^2+ \\ &+ (0-1)^2+(0-0.9)^2-(1-0)^2)^{0.5} = \sqrt{2.14} = 1.46 \end{aligned} \tag{3.182}$$

hence, the normalized Euclidean distance, based on only two terms is

$$q_{IFS(2)}(A,B) = \sqrt{\frac{1}{7} \cdot e(A,B)} = \sqrt{\frac{2.14}{7}} = 0.55 \tag{3.183}$$

It is easy to notice, when analyzing the results obtained in Examples 3.3 and 3.4 that distances between the intuitionistic fuzzy sets should be calculated by taking into account all three terms (membership values, non-membership values, and hesitancy margin values). It is also easy to notice that for the formulas (3.154) –(3.157) the following holds

$$0 \leq d^1_{IFS}(A,B) \leq n \tag{3.184}$$

$$0 \leq l^1_{IFS}(A,B) \leq 1 \tag{3.185}$$

$$0 \leq e^1_{IFS}(A,B) \leq \sqrt{n} \tag{3.186}$$

$$0 \leq q^1_{IFS}(A,B) \leq 1 \tag{3.187}$$

Using two terms only gives values of distances which are orthogonal projections of the real distances (Figure 2.3), and this implies that they are lower.

So to sum up, after analyzing several definitions of distances between the intuitionistic fuzzy sets, it was shown that the distances should be calculated taking into account all three terms describing an intuitionistic fuzzy set.

Taking into account all three terms describing the intuitionistic fuzzy sets when calculating distances ensures that the distances for fuzzy sets and intuitionistic fuzzy sets can be easily compared [cf. formulas (3.140)-(3.143) and formulas (3.184)-(3.187)].

3.3.2.1 Hausdorff Distances

The Hausdorff distances (cf. Grünbaum [77]) are important from the point of view of practical applications, namely, in image matching, image analysis, visual navigation of robots, motion tracking, computer-assisted surgery and so on (cf. e.g., Huttenlocher et al. [89], Huttenlocher and Rucklidge [90], Olson [127], Peitgen et al. [136], Rucklidge [142]-[146]). Although the definition of the Hausdorff distances is simple, the calculations needed to solve the real problems are complex. In result the efficiency of the algorithms for computing the Hausdorff distances may be crucial and the use of some approximations may be relevant and useful (e.g, Aichholzer [1], Atallah [3], Huttenlocher et al. [89], Preparata and Shamos [137], Rucklidge [146], Veltkamp [239]).

First of all, the formulas proposed for calculating the distances should be formally correct. This is the motivation of this section. Namely, we consider the results of using the Hamming distances between the intuitionistic fuzzy sets calculated in two possible ways – taking into account the two term representation (the membership and non-membership values) of the intuitionistic fuzzy sets, and next – taking into account the three term representation (the membership, non-membership values, and hesitation margin values) of the intuitionistic fuzzy sets. We will verify if the resulting distances fulfill the properties of the Hausdorff distances.

The next problem we consider concerns calculating the Hausdorff distance based on the Hamming metric for the interval-valued fuzzy sets. We prove that the formulas that are effective and efficient for the interval-valued fuzzy sets do not work well in the case of the intuitionistic fuzzy sets.

The Hausdorff distance is *the maximum distance of a set to the nearest point in the other set* (Rote [141]). More formal description is given by the following

Definition 3.13. Given two finite sets $A = \{a_1, ..., a_p\}$ and $B = \{b_1, ..., b_q\}$, the Hausdorff distance $H(A,B)$ is defined as:

$$H(A,B) = \max\{h(A,B), h(B,A)\} \tag{3.188}$$

where

$$h(A,B) = \max_{a \in A} \min_{b \in B} d(a,b) \tag{3.189}$$

where:

– a and b are elements belonging to sets A and B respectively,
– $d(a,b)$ is any metric between elements a and b,
– the two distances $h(A,B)$ and $h(B,A)$ (3.189) are called the directed Hausdorff distances.

The directed Hausdorff distance from A to B, i.e., the function $h(A,B)$ ranks each element of A based on its distance to the nearest element of B, and then the highest ranked element specifies the value of the distance. Usually, $h(A,B)$ and $h(B,A)$ can be different values (the directed distances are not symmetric).

Following the way of calculating the Hausdorff distances (Definition 3.13) we may notice that if A and B contain one element each (a_1 and b_1, respectively), the Hausdorff distance is just equal to $d(a_1, b_1)$. In other words, if for separate elements a formula which is expected to express the Hausdorff distance gives a result which is not consistent with the used metric d (e.g., the Hamming distance, the Euclidean distance, etc.), the formula considered is not a proper definition of the Hausdorff distance.

3.3.2.2 The Hausdorff Distance Between the InterVal-valued Fuzzy Sets

The Hausdorff distance between two intervals: $U = [u_1, u_2]$ and $W = [w_1, w_2]$ is (Moore [125]):

$$h(U,W) = \max\{|u_1 - w_1|, \ |u_2 - w_2|\} \tag{3.190}$$

Assuming the two-term representation for the intuitionistic fuzzy sets: $A = \{x, \mu_A(x), \nu_A(x)\}$ and $B = \{x, \mu_B(x), \nu_B(x)\}$, we may consider the two intuitionistic fuzzy sets, A and B, as two intervals, namely:

$$[\mu_A(x), 1 - \nu_A(x)] \text{ and } [\mu_B(x), 1 - \nu_B(x)] \tag{3.191}$$

then

$$h(A,B) = \max\{|\mu_A(x) - \mu_B(x)|, \ |\nu_A(x) - \nu_B(x)|\} \tag{3.192}$$

Later on we will verify if (3.192) is a properly calculated Hausdorff distance between the intuitionistic fuzzy sets while using the Hamming metric.

3.3.2.3 Two Term Representation of the Intuitionistic Fuzzy Sets and the Hausdorff Distance (Hamming Metric)

Following the algorithm of calculating the directed Hausdorff distances, when applying the two term type Hamming distance (3.88) between the intuitionistic fuzzy sets, we obtain:

$$d_h(A,B) = \frac{1}{n}\sum_{i=1}^{n} max\{|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|\} \tag{3.193}$$

If the above distance (3.193) is a properly calculated Hausdorff distance, then in the case of degenerate, i.e., one-element sets $A = \{< x, \mu_A(x), \nu_A(x) >\}$ and $B = \{< x, \mu_B(x), \nu_B(x) >\}$, it should give the same results as the two term type Hamming distance (3.88). It means that in the case of the two term type Hamming distance, for the degenerate, one element intuitionistic fuzzy sets, the following equations should give just the same results (Szmidt and Kacprzyk [209]):

$$l_{IFS(2)}(A,B) = \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|) \tag{3.194}$$

$$d_h(A,B) = \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|\} \tag{3.195}$$

where (3.194) is the normalized two term type Hamming distance, and (3.195) should be its counterpart Hausdorff distance.

We will verify on a simple example if (3.194) and (3.195) give the same results as they should do following the essence of the Hausdorff measures.

Example 3.5. (Szmidt and Kacprzyk [210]) Consider the following one-element intuitionistic fuzzy sets: $A, B, D, G, E \in X = \{x\}$

$$\begin{aligned} &A = \{< x, 1, 0 >\}, \quad B = \{< x, 0, 1 >\}, \quad D = \{< x, 0, 0 >\}, \\ &G = \{< x, \frac{1}{2}, \frac{1}{2} >\}, \quad E = \{< x, \frac{1}{4}, \frac{1}{4} >\} \end{aligned} \tag{3.196}$$

The results from (3.195) are:

$$d_h(A,B) = \max\{|1-0|, |0-1|\} = 1$$

$$d_h(A,D) = \max\{|1-0|, |0-0|\} = 1$$

$$d_h(B,D) = \max\{|0-0|, |1-0|\} = 1$$

$$d_h(A,G) = \max\{|1-1/2|, |0-1/2|\} = 0.5$$

$$d_h(A,E) = \max\{|1-1/4|, |0-1/4|\} = 0.75$$

$$d_h(B,G) = \max\{|0-1/2|,|1-1/2|\} = 0.5$$

$$d_h(B,E) = \max\{|0-1/4|,|1-1/4|\} = 0.75$$

$$d_h(D,G) = \max\{|0-1/2|,|0-1/2|\} = 0.5$$

$$d_h(D,E) = \max\{|0-1/4|,|1-1/4|\} = 0.25$$

$$d_h(G,E) = \max\{|1/2-1/4|,|1/2-1/4|\} = 0.25$$

Their counterpart Hamming distances calculated from (3.194) are:

$$l_{IFS(2)}(A,B) = 0.5(|1-0|+|0-1|) = 1$$

$$l_{IFS(2)}(A,D) = 0.5(|1-0|+|0-0||) = 0.5$$

$$l_{IFS(2)}(B,D) = 0.5(|0-0|+|1-0||) = 0.5$$

$$l_{IFS(2)}(A,G) = 0.5(|0-1/2|+|0-1/2|) = 0.5$$

$$l_{IFS(2)}(A,E) = 0.5(|1-1/4|+|0-1/4||) = 0.5$$

$$l_{IFS(2)}(B,G) = 0.5(|1-1/4|+|0-1/4|) = 0.5$$

$$l_{IFS(2)}(B,E) = 0.5(|1-1/4|+|0-1/4|) = 0.5$$

$$l_{IFS(2)}(D,G) = 0.5(|0-1/2|+|0-1/2|) = 0.5$$

$$l_{IFS(2)}(D,E) = 0.5(|0-1/4|+|0-1/4|) = 0.25$$

$$l_{IFS(2)}(G,E) = 0.5(|1/2-1/4|+|1/2-1/4|) = 0.25$$

i.e. the values of the Hamming distances (3.194) used to define the Hausdorff measures (3.195), and the values of the resulting Hausdorff distances (3.195) calculated for the separate elements are not consistent (as they should be). The differences are:

$$d_h(A,D) \neq l_{IFS(2)}(A,D) \tag{3.197}$$

$$d_h(B,D) \neq l_{IFS(2)}(B,D) \tag{3.198}$$

$$d_h(A,E) \neq l_{IFS(2)}(A,E) \tag{3.199}$$

$$d_h(B,E) \neq l_{IFS(2)}(B,E) \tag{3.200}$$

It is easy to show that the inconsistencies as shown above occur for an infinite number of other cases.

Now we will verify the conditions under which the equations (3.194) and (3.195) give consistent results, i.e., when for the separate elements we have (Szmidt and Kacprzyk [218]):

$$\frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|) = \\ = \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|\} \tag{3.201}$$

Taking into account that

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1 \tag{3.202}$$

$$\mu_B(x) + \nu_B(x) + \pi_B(x) = 1 \tag{3.203}$$

from (3.202) and (3.203) we obtain

$$(\mu_A(x) - \mu_B(x)) + (\nu_A(x) - \nu_B(x)) + (\pi_A(x) - \pi_B(x)) = 0 \tag{3.204}$$

It is easy to notice that (3.204) is not fulfilled for all elements belonging to an intuitionistic fuzzy set but for some elements only. Namely, equation (3.201) is fulfilled for the following conditions (Szmidt and Kacprzyk [218])

- for $\pi_A(x) - \pi_B(x) = 0$, from (3.204) we have

$$|\mu_A(x) - \mu_B(x)| = |\nu_A(x) - \nu_B(x)| \tag{3.205}$$

and having in mind (3.205), we can express (3.201) in the following way:

$$0.5(|\mu_A(x) - \mu_B(x)| + |\mu_A(x) - \mu_B(x)|) = \\ = \max\{|\mu_A(x) - \mu_B(x)|, |\mu_A(x) - \mu_B(x)|\} \tag{3.206}$$

- if $\pi_A(x) - \pi_B(x) \neq 0$, but, at the same time

$$\mu_A(x) - \mu_B(x) = \nu_A(x) - \nu_B(x) = -\frac{1}{2}(\pi_A(x) - \pi_B(x)) \tag{3.207}$$

then (3.201) boils down again to (3.206).

In other words, (3.201) is fulfilled (which means that the Hausdorff measure given by (3.195) is a natural counterpart of (3.194)) only for such elements belonging to an intuitionistic fuzzy set, for which some additional conditions are given, like $\pi_A(x) - \pi_B(x) = 0$ or (3.207). However in general, for an infinite numbers of elements, (3.201) is not valid.

In the above context it seems to be a bad idea to try constructing the Hausdorff distance using the two term type Hamming distance between the intuitionistic fuzzy sets.

An immediate conclusion is that, relating to the results concerning interval-valued fuzzy sets (3.190)–(3.192) the Hausdorff distance for the intuitionistic fuzzy sets can not be constructed in the same way as for the interval-valued fuzzy sets.

3.3.2.4 Three Term Hamming Distance Between the Intuitionistic Fuzzy Sets and the Hausdorff Metric

Now we will show that by applying the three term type Hamming distance for the intuitionistic fuzzy sets, we obtain correct (in the sense of Definition 3.13) Hausdorff distance.

Namely, if we calculate the three term type Hamming distance between two degenerate, i.e. one-element intuitionistic fuzzy sets, A and B in the spirit of Szmidt and Kacprzyk [171], [188], Szmidt and Baldwin [159], [160], i.e., in the following way:

$$\begin{aligned} l^1_{IFS}(A,B) &= \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)| + \\ &+ |\pi_A(x) - \pi_B(x)|) \end{aligned} \quad (3.208)$$

we can give a counterpart of the above distance in terms of the max function (Szmidt and Kacprzyk [218]):

$$\begin{aligned} H_3(A,B) &= \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, \\ &, \ |\pi_A(x) - \pi_B(x)|\} \end{aligned} \quad (3.209)$$

If $H_3(A,B)$ (3.209) is a properly specified Hausdorff distance (in the sense that for two degenerate, one element intuitionistic fuzzy sets, the result is equal to the metric used), the following condition should be fulfilled (Szmidt and Kacprzyk [218]):

$$\begin{aligned} &\frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|) + |\pi_A(x) - \pi_B(x)|) = \\ &= \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} \end{aligned} \quad (3.210)$$

Let us verify if (3.210) is valid. Without loss of generality we can assume

$$\begin{aligned} &\max\ \{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} = \\ &= \ |\mu_A(x) - \mu_B(x)| \end{aligned} \quad (3.211)$$

For $|\mu_A(x) - \mu_B(x)|$ fulfilling (3.211), and because of (3.202) and (3.203), we conclude that both $\nu_A(x) - \nu_B(x)$, and $\pi_A(x) - \pi_B(x)$ are of the same sign (both values are either positive or negative). Therefore

$$|\mu_A(x) - \mu_B(x)| = |\nu_A(x) - \nu_B(x)| + |\pi_A(x) - \pi_B(x)| \tag{3.212}$$

Applying (3.212) we can verify that (3.210) always is valid as

$$\begin{aligned} &0.5\{|\mu_A(x) - \mu_B(x)| + |\mu_A(x) - \mu_B(x)|\} = \\ &= \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} = \\ &= |\mu_A(x) - \mu_B(x)| \end{aligned} \tag{3.213}$$

Now we will use the above formulas, (3.208) and (3.209), for the data used in Example 1. But now, as we also take into account the hesitation margins $\pi(x)$ (2.7), instead of (3.196) we use the three term, "full" description of the data $\{< x, \mu(x), \nu(x), \pi(x) >\}$, i.e. employing all three functions (the membership, non-membership and hesitation margin) describing the considered intuitionistic fuzzy sets (Szmidt and Kacprzyk [210]):

$$\begin{aligned} &A = \{< x, 1, 0, 0 >\}, \quad B = \{< x, 0, 1, 0 >\}, \quad D = \{< x, 0, 0, 1 >\}, \\ &G = \{< x, \frac{1}{2}, \frac{1}{2}, 0 >\}, \quad E = \{< x, \frac{1}{4}, \frac{1}{4}, \frac{1}{2} >\} \end{aligned} \tag{3.214}$$

From (3.209) we have:

$$H_3(A,B) = \max(|1-0|, |0-1|, |0-0|) = 1$$

$$H_3(A,D) = \max(|1-0|, |0-0|, |0-1|) = 1$$

$$H_3(B,D) = \max(|0-0|, |1-0|, |0-1|) = 1$$

$$H_3(A,G) = \max(|0-1/2|, |0-1/2|, |0-0|) = 0.5$$

$$H_3(A,E) = \max(|1-1/4|, |0-1/4|, |0-1/2|) = 0.75$$

$$H_3(B,G) = \max(|1-1/4|, |0-1/4|, |0-1/2|) = 0.75$$

$$H_3(B,E) = \max(|1-1/4|, |0-1/4|, |0-1/2|) = 0.75$$

$$H_3(D,G) = \max(|0-1/2|, |0-1/2|, |1-0|) = 1$$

$$H_3(D,E) = \max(|0-1/4|, |0-1/4|, |1-1/2|) = 0.5$$

$$H_3(G,E) = \max(|1/2-1/4|,|1/2-1/4|,|0-1/2|) = 0.5$$

The counterpart Hamming distances obtained from (3.208) (with all three functions) are

$$l^1_{IFS}(A,B) = 0.5(|1-0|+|0-1|+|0-0|) = 1$$

$$l^1_{IFS}(A,D) = 0.5(|1-0|+|0-0|+|0-1|) = 1$$

$$l^1_{IFS}(B,D) = 0.5(|0-0|+|1-0|+|0-1|) = 1$$

$$l^1_{IFS}(A,G) = 0.5(|0-1/2|+|0-1/2|+|0-0|) = 0.5$$

$$l^1_{IFS}(A,E) = 0.5(|1-1/4|+|0-1/4|+|0-1/2|) = 0.75$$

$$l^1_{IFS}(B,G) = 0.5(|1-1/4|+|0-1/4|+|0-1/2|) = 0.75$$

$$l^1_{IFS}(B,E) = 0.5(|1-1/4|+|0-1/4|+|0-1/2|) = 0.75$$

$$l^1_{IFS}(D,G) = 0.5(|0-1/2|+|0-1/2|+|1-0|) = 1$$

$$l^1_{IFS}(D,E) = 0.5(|0-1/4|+|0-1/4|+|1-1/2|) = 0.5$$

$$l^1_{IFS}(G,E) = 0.5(|1/2-1/4|+|1/2-1/4|+|0-1/2|) = 0.5$$

As we can see, the Hausdorff distance (3.209) (using the membership values, non-membership values and hesitation margins) and the tree term Hamming distance (3.208) give for one-element intuitionistic fuzzy sets fully consistent results. The same situation occurs in a general case too.

In other words, for the normalized Hamming distance expressed in the spirit of (Szmidt and Kacprzyk [171], [188]), given by (3.154), we can give the following equivalent representation in terms of the max function:

$$H_3(A,B) = \frac{1}{n}\sum_{i=1}^{n} \max\left\{|\mu_A(x_i)-\mu_B(x_i)|, |\nu_A(x_i)-\nu_B(x_i)|, |\pi_A(x_i)-\pi_B(x_i)|\right\} \quad (3.215)$$

Unfortunately, it can be easily verified that it is impossible to give the counterpart pairs of the formulas like (3.154) and (3.215) for $r > 1$ in the Minkowski r-metrics

($r = 1$ is the Hamming distance, $r = 2$ is the Euclidean distance, etc.). More details are given in [25] and [236].

Now we will show that the three term distances between the intuitionistic fuzzy sets are useful in the ranking of intuitionistic fuzzy alternatives.

3.4 Ranking of the Intuitionistic Fuzzy Alternatives

Given their ability to model imperfect information, the intuitionistic fuzzy sets have found applications in many areas, in particular, in decision making. Ranking of the intuitionistic fuzzy alternatives (options), obtained, for example, as a result of decision analysis, aggregation, etc. is one of important problems. The intuitionistic fuzzy alternatives may be understood in different ways. Here we mean them as elements of a universe of discourse with their associated membership degrees, non-membership degrees, and hesitation margins. In the context of decision making each option fulfills a set of criteria to some extent $\mu(.)$, it does not fulfill this set of criteria to some extent $\nu(.)$ and, on the other hand we are not sure to the extent $\pi(.)$ if an option fulfills or does not fulfill a set of criteria. This implies that the alternatives can be expressed via the intuitionistic fuzzy sets. Here we will call such alternatives "intuitionistic fuzzy alternatives".

The intuitionistic fuzzy alternatives may be ranked only under some additional assumptions as there is no linear order among elements of the intuitionistic fuzzy sets. The situation is different from that for fuzzy sets (Zadeh [254]), for which elements of the universe of discourse are naturally ordered because their membership degrees are real numbers from $[0,1]$.

There are not many approaches for ranking the intuitionistic fuzzy alternatives in the literature. For instance, Chen and Tan [53], Hong and Choi [80], Li et al. [112], [114], and Hua-Wen Liu and Guo-Jun Wang [119] proposed some approaches.

Chen and Tan [53] proposed a score function for vague sets [72], but, as Bustince and Burillo [44] demonstrated that vague sets are equivalent to intuitionistic fuzzy sets, we can consider the concept of a score function for an intuitionistic fuzzy alternative $a = (\mu, \nu)$ meant as

$$S(a) = \mu - \nu, \tag{3.216}$$

and, clearly, $S(a) \in [-1, 1]$.

It is easy to notice that the score function $S(a)$ (3.216) can not alone evaluate the intuitionistic fuzzy alternatives as it produces the same result for such different intuitionistic fuzzy alternatives $a = (\mu, \nu)$ as, e.g.: $(0.5, 0.4)$, $(0.4, 0.3)$, $(0.3, 0.2)$, $(0.1, 0)$ – for all of them $S(a) = 0.1$, which seems counterintuitive.

Next, Hong and Choi [80] introduced, in addition to the score function (3.216), a so called accuracy function H

$$H(a) = \mu + \nu, \tag{3.217}$$

where $H(a) \in [0, 1]$.

Xu [253] made use of both (3.216) and (3.217), and proposed an algorithm ranking the intuitionistic fuzzy alternatives. In the case of two alternatives a_i and a_j , the algorithm is as follows [253]:

- if $S(a_i) \leq S(a_j)$, then a_i is smaller than a_j;
- if $S(a_i) = S(a_j)$, then:
  - if $H(a_i) = H(a_j)$, then a_i and a_j represent the same information (are equal);
  - if $H(a_i) \leq H(a_j)$, then a_i is smaller than a_j.

Unfortunately, the above method of ranking does not produce reliable results in many cases. Let us consider two intuitionistic fuzzy alternatives (Szmidt and Kacprzyk [205]) $a_1 = (0.5, 0.45)$ and $a_2 = (0.25, 0.05)$ for which we obtain $S(a_1) = 0.5 - 0.45 = 0.05$, $S(a_2) = 0.25 - 0.05 = 0.2$, suggesting that a_1 is smaller than a_2. However, information provided by a_1 (i.e. $0.5 + 0.45 = 0.95$) is certainly bigger than that provided by a_2 (i.e. $0.25 + 0.05 = 0.3$). In this context it is difficult to agree that a_1 is smaller than a_2. Later on, we will return to ranking of two intuitionistic fuzzy alternatives by the method we propose.

We give below an example showing some more weak sides of the above procedure. Let us consider the following intuitionistic fuzzy alternatives:

$$\begin{aligned} a_1 &= (0.1, 0, 0.9), \\ a_2 &= (0.2, 0.11, 0.69), \\ a_3 &= (0.3, 0.22, 0.48), \\ a_4 &= (0.4, 0.33, 0.27), \\ a_5 &= (0.5, 0.44, 0.06), \end{aligned}$$

for which the scores are:

$$\begin{aligned} S(a_1) &= 0.1 - 0 = 0.1, \\ S(a_2) &= 0.2 - 0.11 = 0.09, \\ S(a_3) &= 0.3 - 0.22 = 0.08, \\ S(a_4) &= 0.4 - 0.33 = 0.07, \\ S(a_5) &= 0.5 - 0.44 = 0.06, \end{aligned}$$

which, in the light of the above algorithm means that:

$$a_1 > a_2 > a_3 > a_4 > a_5$$

In other words, due to the above ranking procedure, in this particular case, the less we know, the better (it is worth noticing that the lack of knowledge is the biggest for the “best” alternative a_1 (equal to 0.9), and it decreases for the consecutive “worse” (according to the considered procedure) alternatives. Next, the membership values increase from 0.1 (for a_1) to 0.5 (for a_5). Thus, for increasing membership values and decreasing lack of knowledge we obtain (from the ranking procedure considered) worse alternatives, which is obviously counterintuitive.

Moreover, the ranking procedure considered produces answers that are not continuous. If we change a little the non-membership values in the above example, i.e.:

$$\begin{aligned} a_1 &= (0.1, 0, 0.9), \\ a_2 &= (0.2, 0.1, 0.7), \\ a_3 &= (0.3, 0.2, 0.5), \\ a_4 &= (0.4, 0.3, 0.3), \\ a_5 &= (0.5, 0.4, 0.1), \end{aligned}$$

we obtain the same score for each $a_i, i = 1, \ldots, 5$, and from the second part of the ranking procedure we obtain the reverse order, i.e.:

$$a_5 > a_4 > a_3 > a_2 > a_1$$

Certainly, it makes no sense for a ranking procedure to be so sensitive to so small changes of the parameters. Conclusion: the above ranking procedure should not be used (especially in decision making tasks).

We have already mentioned the possibility of using the intuitionistic fuzzy sets in voting models. Now we will consider some ways of ranking the voting alternatives expressed via the intuitionistic fuzzy elements.

Let an element x belonging to an intuitionistic fuzzy set characterized via (μ, ν, π) express a voting situation: μ represents the proportion (from $[0,1]$) of voters who vote for x, ν represents the proportion of those who vote against x, and π represents the proportion of those who abstain. The simplest idea of comparing different voting situations (ranking the alternatives) would be to use a distance measure from the ideal voting situation $M = (x, 1, 0, 0)$ (100% voting for, 0% vote against and 0% abstain) to the alternatives considered. We will call M the ideal positive alternative. Let

$A = (x, 0.5, 0.5, 0)$ – 50% vote for, 50% against, and 0% abstain,
$B = (x, 0.4, 0.4, 0.2)$ – 40% vote for, 40% vote against and 20% abstain,
$C = (x, 0.3, 0.3, 0.4)$ – 30% vote for, 30% vote against and 40% abstain.

First we confirm that the method of calculating distances between two intuitionistic fuzzy sets A and B described by two terms, i.e., the membership and non-membership values only (3.218) does not work properly (cf. Szmidt and Kacprzyk [171], [188], Szmidt and Baldwin [159], [160]) in this case, too:

$$l_{IFS(2)}(A, B) = \frac{1}{2n} \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|) \tag{3.218}$$

The results obtained with (3.218), i.e., the distances for the above voting alternatives represented by points A, B, C (cf. Figure 3.3) from the ideal positive alternative represented by $M(1, 0, 0)$ are, respectively (Szmidt and Kacprzyk [197]):

$$l_{IFS(2)}(M, A) = 0.5(|1 - 0.5| + |0 - 0.5|) = 0.5 \tag{3.219}$$

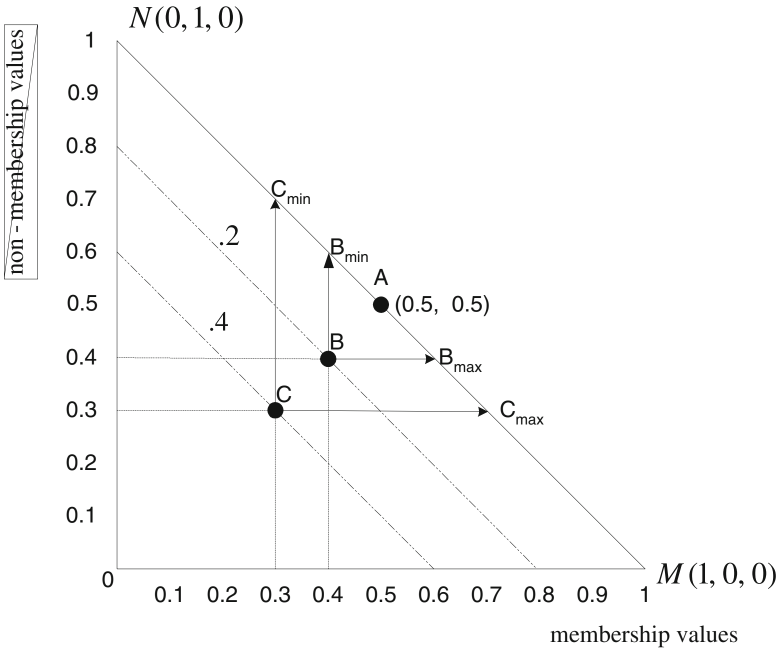


Fig. 3.3 Geometrical representation of the intuitionistic fuzzy alternatives

$$l_{IFS(2)}(M,B) = 0.5(|1-0.4| + |0-0.4|) = 0.5 \quad (3.220)$$
$$l_{IFS(2)}(M,C) = 0.5(|1-0.3| + |0-0.3|) = 0.5 \quad (3.221)$$

The results seem to be counterintuitive as (3.218) suggests that all the alternatives (represented by) A, B, C are "the same". On the other hand, the normalized Hamming distance (3.156), taking into account, besides the membership and non-membership, also the hesitation margin, gives:

$$l^1_{IFS}(M,A) = 0.5(|1-0.5| + |0-0.5| + |0-0|) = 0.5 \quad (3.222)$$
$$l^1_{IFS}(M,B) = 0.5(|1-0.4| + |0-0.4| + |0-0.2|) = 0.6 \quad (3.223)$$
$$l^1_{IFS}(M,C) = 0.5(|1-0.3| + |0-0.3| + |0-0.4|) = 0.7 \quad (3.224)$$

It is not difficult to accept the results (3.222)–(3.224), reflecting our intuition. Alternative A (cf. Figure 3.3) seems to be the best in the sense that the distance $l_{IFS}(M,A)$ is the smallest (we know for sure that 50% vote for, 50% vote against). The alternative represented by A is just a fuzzy alternative (A lies on MN where the values of the hesitation margin are equal 0). Alternatives B and C, on the other hand, are "less sure" (with the hesitation margins equal 0.2, and 0.4, respectively).

Unfortunately, a weak point in the ranking of alternatives by calculating the distances from the ideal positive alternative represented by M is that for a fixed membership value, from (3.156) we obtain just the same value (for example, if the membership value μ is equal 0.8, for any intuitionistic fuzzy element, i.e. such that

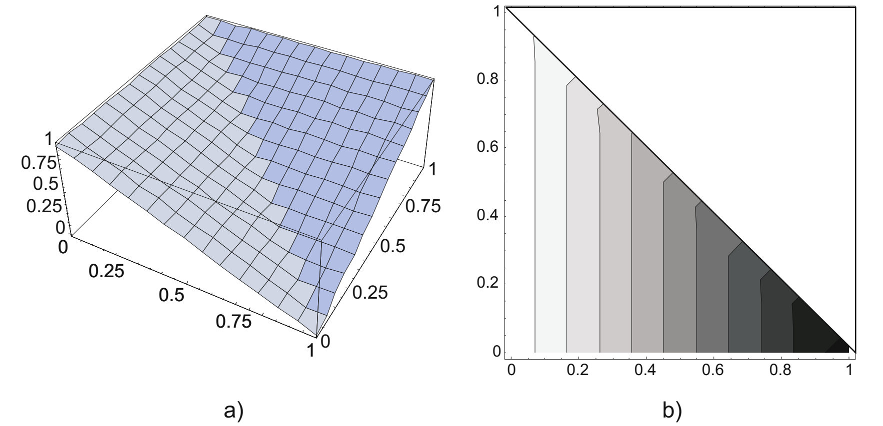


Fig. 3.4 a) Distances (3.156) of any intuitionistic fuzzy element from the ideal alternative M; b) contour plot

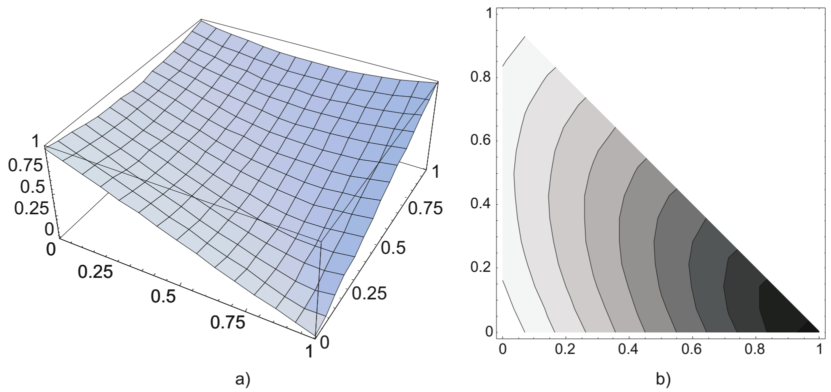


Fig. 3.5 a) Distances (3.157) of any intuitionistic fuzzy element from the ideal alternative M; b) contour plot

its non-membership degree ν and hesitation margin π fulfill $\nu + \pi = 0.2$, we obtain the value of 0.2). This fact is illustrated in Figure 3.4, a and b. To better see this, the distances (3.156) for any alternative from M (Figure 3.4a) are presented for μ and ν for the whole range $[0,1]$ (instead of showing them for $\mu + \nu \leq 1$ only). For the same reason (to better see the effect), in Figure 3.4b the contour plot of the distances (3.156) is given only for the range of μ and ν for which $\mu + \nu \leq 1$.

Now we will verify if the normalized Euclidean distance (3.157) from the ideal positive alternative represented by $M(1,0,0)$ gives better results from the point of view of ranking the alternatives.

Let $A = (x, 0.2, 0.8, 0)$ – 20% vote for, 80% against, and 0% abstain, $B = (x, 0.2, 0, 0.8)$ – 20% vote for, 0% vote against and 80% abstain, The normalized Euclidean distance (3.157) gives (Szmidt and Kacprzyk [214]) :

$$e^1_{IFS}(M,A) = (0.5((1-0.2)^2 + (0-0.8)^2 + (0-0)^2))^{0.5} = 0.8 \quad (3.225)$$
$$e^1_{IFS}(M,B) = (0.5((1-0.2)^2 + (0-0)^2 + (0-0.8)^2))^{0.5} = 0.8 \quad (3.226)$$

Making use of (3.157) for ranking the alternatives suggests [cf. (3.225)–(3.226)] that the alternatives (represented by) A, B seem to be "the same" which is counterintuitive. A general illustration of the above counterintuitive result is given in Fig. 3.5. We can see that the results of (3.157) are not univocally given for a given membership value μ; for clarity, the distances (3.157) for any x from M (Fig. 3.5a) are presented for μ and ν for $[0,1]$, and not for $\mu + \nu \leq 1$ only. For the same reason (to better see the effect), in Fig. 3.5b the contour plot of the distances (3.157) is given only for the range of μ and ν for which $\mu + \nu \leq 1$. So, the distances (3.157) (cf. also Szmidt and Kacprzyk [197]) from the ideal positive alternative alone do not make it possible to rank the alternatives in the intended way.

The analysis of the above examples shows that the distances from the ideal positive alternative alone do not make it possible to rank the alternatives in the intended way.

3.4.0.5 A New Method for Ranking Alternatives (Szmidt and Kacprzyk [205])

The sense of a voting alternative (expressed via an intuitionistic fuzzy element) can be analyzed by using the operators (cf. Atanassov [15]) of: *necessity* ($\Box$), *possibility* ($\Diamond$), $D_\alpha(A)$ and $F_{\alpha,\beta}(A)$ given as:

- The *necessity* operator ($\Box$)

$$\Box A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in X\} \quad (3.227)$$

- The *possibility* operator ($\Diamond$)

$$\Diamond A = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in X\} \quad (3.228)$$

- Operator $D_\alpha(A)$ (where $\alpha \in [0,1]$)

$$D_\alpha(A) = \{\langle x,\ \mu_A(x) + \alpha\pi_A(x),\ \nu_A(x)(1-\alpha)\pi_A(x)\rangle \,| x \in X\} \quad (3.229)$$

- Operator $F_{\alpha,\beta}(A)$ (where $\alpha, \beta \in [0,1]$; $\alpha + \beta \underline{<} 1$)

$$F_{\alpha,\beta}(A) = \{\langle x,\ \mu_A(x) + \alpha\pi_A(x), \nu_A(x)\beta\pi_A(x)\rangle \,| x \in X\} \quad (3.230)$$

Considering alternative $B(0.4, 0.4, 0.2)$, for example, and using the above operators we obtain $\Box B = B_{min}$, where $B_{min} = (0.4, 0.6)$, and $\Diamond B = B_{max}$, where $B_{max} = (0.6, 0.4)$ (Figure 3.3). Operator $F_{\alpha,\beta}(A)$ makes it possible for alternative B to become any alternative represented in triangle $BB_{max}B_{min}$. A similar reasoning leads to the conclusion that alternative C (Figure 3.3) might become any

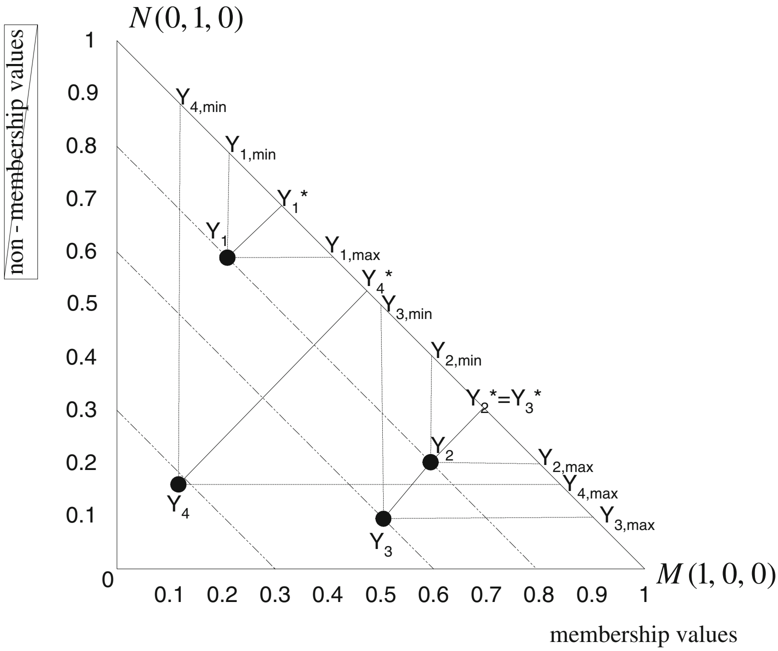


Fig. 3.6 Ranking of alternatives Y_i

alternative represented in triangle $CC_{max}C_{min}$, and alternative $O(0,0,1)$ (with the hesitation margin equal 1)

may become any alternative (the whole area of the triangle MNO).

In the context of the above considerations we could say that the smaller the area of the triangle $Y_iY_{i,min}Y_{i,max}$ (Figure 3.6) the better the alternative Y_i from a set Y of the alternatives considered. Alternatives having their representations on segment MN (i.e., fuzzy alternatives) are the best in the sense that:

- the alternatives are fully reliable in the sense of the information represented, as the hesitation margin is equal 0 here, and
- the alternatives are ordered – the closer an alternative to ideal positive alternative $M(1,0,0)$, the better it is (it is an obvious fact as fuzzy alternatives are univocally ordered).

The above reasoning suggests that a promising way of ranking the intuitionistic fuzzy alternatives Y_i with the same values of π_i is to convert them into the fuzzy alternatives (which may be easily ranked).

The simplest way of ranking the alternatives Y_i with different values of π_i seems to be to make use of the information carried by the triangles $Y_iY_{i,min}Y_{i,max}$.

The amount of information connected with Y_i is indicated by Y_i^*, i.e., by "the position" of triangle $Y_iY_{i,min}Y_{i,max}$ inside triangle MNO – expressed by the projection

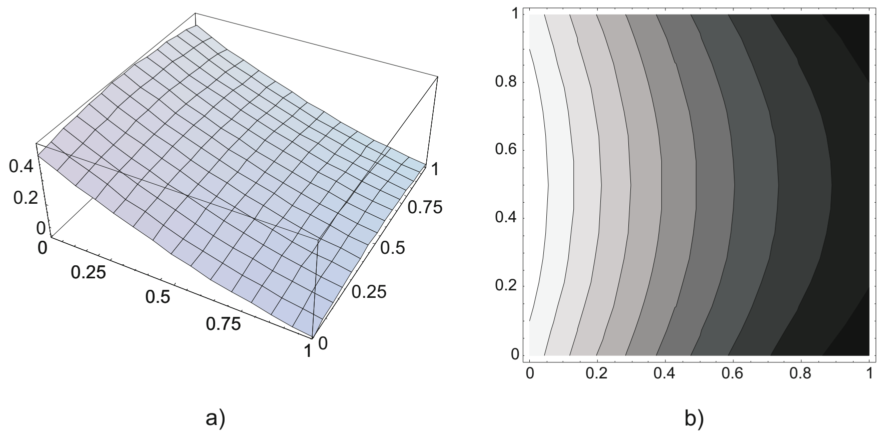


Fig. 3.7 a) $R(Y_i)$ as a function of distance $l_{IFS}(M,Y_i^*)$ (3.156) between Y_i^* and M, and the hesitation margin; b) contour plot

on the segment MN. The hesitation margin π_{Y_i} indicates how reliable (sure) is the information represented by Y_i^*.

Y_i^* are the orthogonal projections of Y_i on MN. Such an orthogonal projection of the intuitionistic fuzzy elements belonging to an intuitionistic fuzzy set A was considered by Szmidt and Kacprzyk [166]. This orthogonal projection may be obtained via operator $D_\alpha(A)$ (3.229) with parameter α equal 0.5.

We can see that all the elements from the segment OA (Figure 3.3) are transformed by $D_{0.5}(A)$ (3.229) into $A(0.5,0.5)$ which reflects the lack of differences between the membership and non-membership, irrespective of the value of the hesitation margin.

Having the above observations in mind, a reasonable measure R that can be used for ranking the alternatives (represented by) Y_i seems to be

$$R(Y_i) = 0.5(1+\pi_{Y_i})distance(M,Y_i^*) \tag{3.231}$$

where $distance(M,Y_i^*)$ is a distance from the ideal positive alternative $M(1,0,0)$, Y_i^* is the orthogonal projection of Y_i on MN. The constant 0.5 was introduced in (3.231) to ensure that $0 < R(Y_i) \leq 1$. The values of function R for any intuitionistic fuzzy element and the distance $l_{IFS}(M,Y_i^*)$ (3.156) are presented in Figure 3.7a, and the corresponding contour plot – in Figure 3.7b.

Unfortunately, the results obtained with (3.231) do not rank the alternatives in the intended way. (The maximum value of (3.231) is not obtained for the alternative $(0,0,1)$ but for $(0,1/2,1/2)$.)

Similarly, in the case of the normalized Euclidean distance (3.157) used in (3.231) instead of $l_{IFS}^1(M,Y_i^*)$ (3.156), the results of (3.231) do not meet our expectations in the sense of their relations to the areas of the triangles $Y_iY_{i,min}Y_{i,max}$ (cf. Figure 3.6). Let us consider the alternatives Y_i, $i = 1,\ldots,4$. of Figure 3.6. We might expect that the alternatives be ordered by (3.231) from Y_1 to Y_4 as just such an order renders the areas of the respective triangles. But the results from (3.231) obtained using the normalized Euclidean distance (3.157) for the different alternatives seem to be very much "the same". For example (Szmidt and Kacprzyk [214]), for Y_1=(0, 0.8, 0.2), $R_E(Y_1^*)$=0.54, for Y_2=(0, 0.6, 0.4), $R_E(Y_2^*)$=0.56, for Y_3=(0, 0.3, 0.7), $R_E(Y_3^*)$=0.55, for Y_4=(0, 0, 1), $R_E(Y_4^*)$=0.5.

So, again, the results obtained via (3.231) with the normalized Euclidean distance (3.157) do not rank the alternatives in the intended way.

It seems that a better measure than (3.231) for ranking the alternatives (represented by) Y_i might be the following measure R

$$R(Y_i) = 0.5(1+\pi_{Y_i})distance(M,Y_i) \tag{3.232}$$

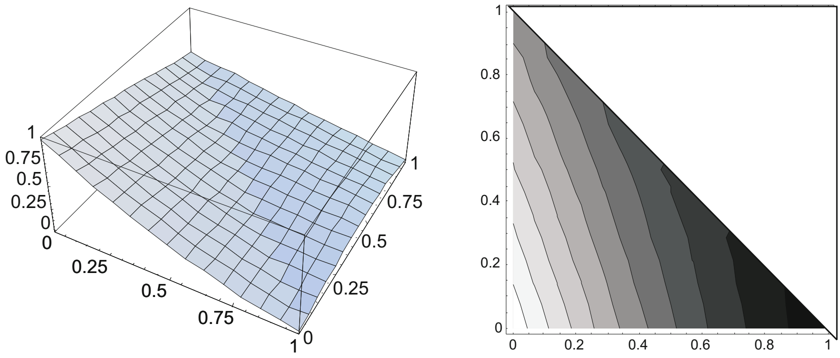
where *distance* means a distance (3.156) of Y_i from the ideal positive alternative $M(1,0,0)$.

Definition (3.232) tells us about the "quality" of an alternative – the lower the value of $R(Y_i)$, (3.232), the better the alternative in the sense of the amount of positive information included, and reliability of information.

For the distance $l^1_{IFS}(M,Y_i)$ (3.156), the best is alternative $M(1,0,0)$ for which $R(M)=0$. For the alternative $N(0,1,0)$ we obtain $R(N)=0.5$ (alternative N is fully reliable as the hesitation margin is equal 0, but the distance $l^1_{IFS}(M,N)=1$). The alternative A (Figures 3.3) gives $R(A)=0.25$. In general, on MN, the values of R decrease from 0.5 (for alternative N) to 0 (for the best alternative M). The maximum value of R, i.e. 1, is obtained for $O(0,0,1)$ for which both distances from M and the hesitation margin are equal 1 (alternative O "indicates" the whole triangle MNO). All other alternatives Y_i "indicate" smaller triangles $Y_iY_{i,min}Y_{i,max}$ (Figure 3.6), so their corresponding values of R are smaller (better in the sense of amount of reliable information).

The values of function R (3.232) for any intuitionistic fuzzy element are presented in Figure 3.8a, and the counterpart contour plot – in Figure 3.8b. Considering the numbers obtained via R (3.232), we may notice that the value 0.25 obtained for the alternative (0.5, 0.5, 0) constitutes the "border" of the "interesting" alternatives – in the sense of the amount of positive knowledge.

Let us consider again the ranking of two alternatives (which were ranked counter-intuitively by the algorithm presented in [253] as shown in the beginning of Section 3.4), namely $Y_1=(0.5,0.45,0.05)$ and $Y_2=(0.25,0.05,0.7)$ (we stress here that we take into account all three terms: the degrees of membership, non-membership and hesitation margin). From (3.232) we obtain: $R(Y_1)=0.26$, $R(Y_2)=0.64$ which means that Y_1 is better than Y_2 (previously, according to the algorithm from [253] Y_2 was better/bigger than Y_1). Obviously, Y_1 is not a "good" option as $R(Y_1)$ is bigger than 0.25 which follows from the fact that the non-membership value is quite big (equal 0.45). It might mean that we would not accept the option Y_1. But option Y_2 seems even less interesting – with the smaller membership value (equal 0.25 instead



a) b)

Fig. 3.8 a) $R(Y_i)$ as a function of distance $l_{IFS}(M,Y_i)$ (3.156) between Y_i and M, and the hesitation margin; b) contour plot

of 0.5 for Y_1), and with the bigger hesitation margin (equal 0.7 instead of 0.05 for Y_1).

Example 3.6. (Szmidt and Kacprzyk [205]) Let us evaluate (rank) six medical treatments. The treatments $C1 - C6$, affect a patient in the following way (Szmidt and Kacprzyk [205]):

- $C1 : (0.6, 0.2, 0.2)$ – influences in a positive way 60% of symptoms, in a negative way – 20% of symptoms, and its impact is unknown (was not confirmed) in the case of 20% of symptoms;
- $C2 : (0.7, 0.3, 0)$ – influences in a positive way 70% symptoms, in a negative way – 30% of symptoms, and its impact is unknown (was not confirmed) in the case of 0% of symptoms;
- $C3 : (0.7, 0.15, 0.15)$ – influences in a positive way 70% of symptoms, in a negative way – 15% of symptoms, and its impact is unknown (was not confirmed) in the case of 15% of symptoms;
- $C4 : (0.775, 0.225, 0)$ – influences in a positive way 77.5% of symptoms, in a negative way – 22.5% of symptoms, and its impact is unknown (was not confirmed) in the case of 0% of symptoms;
- $C5 : (0.8, 0.1, 0.1)$ – influences in a positive way 80% of symptoms, in a negative way – 10% of symptoms, and its impact is unknown (was not confirmed) in the case of 10% of symptoms;
- $C6 : (0.8, 0.2, 0)$ – influences in a positive way 80% of symptoms, in a negative way – 20% of symptoms, and its impact is unknown (was not confirmed) in the case of 0% of symptoms.

Table 3.1 shows the ranking of $C1, \ldots, C6$ with (3.232) – from the worst one, $C1$ to the best one, $C6$.

It is worth emphasizing that the ranking function R (3.232) is constructed taking strongly into account the lack of knowledge. Let us consider the pair: $C1$ and $C2$

Table 3.1 Ranking alternatives by R (3.232) – results for the data from Example 3.6

No.	$C_i : (\mu_i, \nu_i, \pi_i)$	$R_E(C_i)$
1	$C1 : (0.6, 0.2, 0.2)$	0.240
2	$C2 : (0.7, 0.3, 0)$	0.150
3	$C3 : (0.7, 0.15, 0.15)$	0.173
4	$C4 : (0.775, 0.225, 0)$	0.113
5	$C5 : (0.8, 0.1, 0.1)$	0.110
6	$C6 : (0.8, 0.2, 0)$	0.100

(Table 3.1). In the case of $C1$ the lack of knowledge is equal to 0.2, so that theoretically, we might expect "on the average" that the hesitation margin representing the lack of knowledge will be divided equally between the membership value and non-membership value giving as a result the case $C2$. Assuming that we wish to avoid the most disadvantageous cases, we will rank $C2$ higher than $C1$ so as to avoid the possibility which might by implied by $C1$, namely: $(0.6, 0.4, 0)$ (while the entire hesitation margin is added to the non-membership value). The best result which could happen (if the entire hesitation margin is added to the membership value of $C1$), namely $(0.8, 02, 0)$, (i.e. case $C6$ ranked as the best one – $R(C6) = 0.1$) does not influence the ranking of $C1$(3.232).

Analogous situation can be observed for the pairs: $C3$ and $C4$, and next for $C5$ and $C6$. It is easy to notice that the existence of the non-zero hesitation margin influences negatively the ranking.

The obtained results seem to meet our expectations pretty well.

Finally, we will verify the results produced by (3.232) with the normalized Euclidean distance (3.157).

At the beginning, we will rank the same alternatives using (3.232) as we have done previously using (3.231), i.e.: Y_1=(0, 0.8, 0.2), Y_2=(0, 0.6, 0.4), Y_3=(0, 0.3, 0.7), and Y_4=(0, 0, 1). We obtain $R_E(Y_1)$=0.55, $R_E(Y_2)$=0.61, $R_E(Y_3)$=0.85, $R_E(Y_4)$=1. The results seem to render our intuition now.

The results obtained via (3.232) for the most characteristic alternatives are still the same for the normalized Euclidean distance (3.157) as they were for the normalized Hamming distance (3.156). As previously (i.e., with the normalized Hamming distance (3.157)), the best is alternative $M(1, 0, 0)$ ($R_E(M) = 0$). For alternative $N(0, 1, 0)$, again, we obtain $R_E(N) = 0.5$ (N is fully reliable as the hesitation margin is equal 0 but the distance $e_{IFS}(M, N) = 1$). In general, on MN, the values of R_E decrease from 0.5 (for alternative N) to 0 (for the best alternative M). The maximal value of R_E, i.e. 1, is for $O(0, 0, 1)$ for which $e_{IFS}(M, O), \pi_O = 1$ (alternative O "indicates" the whole triangle MNO). All other alternatives Y_i "indicate" smaller triangles $Y_i Y_{i,min} Y_{i,max}$ (Figure 3.6), so that their R_E's are smaller (better as to the amount of reliable information).

It is worth emphasizing that the results obtained via (3.232), which reflect our intuition concerning ranking of the alternatives, are obtained using all three terms describing the intuitionistic fuzzy alternatives, i.e., membership values, non-membership values, and the hesitation margin values. Also the distances (3.157) in (3.232) are calculated taking into account all three terms. In other words, we use a 3D representation of the intuitionistic fuzzy sets.

Moreover, the proposed measure (3.232) strongly emphasizes the difference between knowledge (represented by the membership and non-membership values) and lack of knowledge (represented by the hesitation margins). Even if an alternative does not fulfill our criteria at all (alternative N), it is ranked higher ($R_E(N) = 0.5$) than an alternative about which we can say nothing (alternative O). Other examples are given in Table 3.2 (Szmidt and Kacprzyk [208]).

Table 3.2 Examples of results showing that (3.232) reflects differences between negative knowledge and lack of knowledge in the ranking of the alternatives

No.	Alternative (μ,ν,π)	$R_E(Y_i)$
1	$(0, 0.8, 0.2)$	0.550
2	$(0, 0.2, 0.8)$	0.825
3	$(0, 0.7, 0.3)$	0.578
4	$(0, 0.3, 0.7)$	0.755
5	$(0, 0.6, 0.4)$	0.610
6	$(0, 0.4, 0.6)$	0.697
7	$(0, 1, 0)$	0.5
8	$(0, 0, 1)$	1

The results provided in Table 3.2 make it possible to come to some conclusions concerning the situations for which we have a fixed membership value of the alternatives (membership value is equal to 0 in Table 3.2), namely:

- an alternative is ranked lower (which means bigger values from (3.232)) the smaller the non-membership function and the bigger the hesitation margin (cf. the sequence of cases: 1, 3, 5, 8);
- an alternative is ranked higher (i.e., the smaller the values from (3.232)) the higher the non-membership function and the lower the hesitation margin (cf. the sequence of cases: 2, 4, 6, 7);
- "negative knowledge" represented by the non-membership values, and lack of knowledge represented by the hesitation margins are different from the point of view of (3.232) (cf. the pairs: 1 and 2, 3 and 4, 5 and 6, 7 and 8).

Other examples, presented in Table 3.3 (Szmidt and Kacprzyk [208]) make it possible to notice that:

- an alternative is ranked higher (which means that the values from (3.232) are lower) for a fixed value of the non-membership function (cf. Table 3.3, the cases: 2, 4, 6, 8, for which the non-membership value is equal 0) the higher the values of the membership function (lower hesitation margins);

Table 3.3 Examples of results showing that (3.232) reflects differences between positive knowledge and lack of knowledge in the ranking of the alternatives

No.	Alternative (μ, ν, π)	$R_E(Y_i)$
1	$(0, 0.8, 0.2)$	0.550
2	$(0.8, 0, 0.2)$	0.12
3	$(0, 0.7, 0.3)$	0.578
4	$(0.7, 0, 0.3)$	0.195
5	$(0, 0.6, 0.4)$	0.610
6	$(0.6, 0, 0.4)$	0.280
7	$(0, 1, 0)$	0.5
8	$(1, 0, 0)$	0.

- the ranking function (3.232) does make a difference between the positive and negative knowledge (cf. Table 3.3, the pairs: 1 and 2, 3 and 4, 5 and 6, 7 and 8).

To sum up, the proposed ranking function (3.232) expresses differences both between knowledge and lack of knowledge, and between the positive and negative knowledge. In other words, the proposed function (3.232) seems to reflect the behavior of a human being in the process of ranking alternatives pretty well.

3.5 Concluding Remarks

We have considered distances between the intuitionistic fuzzy sets in two ways, employing:

– two term intuitionistic fuzzy set representation (membership values and non-membership values only were taken into account), and
– three term intuitionistic fuzzy set representation (membership values, non-membership values, and hesitation margins were taken into account).

We have discussed norms and metrics for both types of representations stressing their correctness from the mathematical point of view. However, the three term approach seems to be more justified and intuitively appealing from the practical point of view (which has its roots in some analytical and geometrical aspects).

Some problems have been shown concerning the Hausdorff distance while the Hamming metric was applied for the two term intuitionistic fuzzy set representation. It was shown, as well, that the method of calculating the Hausdorff distances in the same way which is correct for the interval-valued fuzzy sets does not work for the intuitionistic fuzzy sets.

Finally, the usefulness of the three term distances was emphasized in a measure of ranking of the intuitionistic fuzzy alternatives.