Chapter 2
Intuitionistic Fuzzy Sets as a Generalization of Fuzzy Sets

Abstract. In the mid-1980s Atanassov introduced the concept of an intuitionistic fuzzy set. Basically, his idea was that unlike the conventional fuzzy sets in which imprecision is just modeled by the membership degree from [0,1], and for which the non-membership degree is just automatically the complementation to 1 of the membership degree, in an intuitionistic fuzzy set both the membership and non-membership degrees are numbers from [0,1], but their sum is not necessarily 1. Thus, one can express a well known psychological fact that a human being who expresses the degree of membership of an element in a fuzzy set, very often does not express, when asked, the degree of non-membership as the complementation to 1. This idea has led to an interesting theory whose point of departure is such a concept of intuitionistic fuzzy set. In this chapter we give brief introduction to intuitionistic fuzzy sets. After recalling main definitions, concepts, operations and relations over crisp sets, fuzzy sets, and intuitionistic fuzzy sets we discuss interrelationships among the three types of sets. Two geometrical representations of the intuitionistic fuzzy sets, useful in further considerations are discussed. Finally, two approaches of constructing the intuitionistic fuzzy sets from data are presented. First approach is via asking experts. Second one – the automatic, and mathematically justified method to construct the intuitionistic fuzzy sets from data seems to be especially important in the context of analyzing information in big data bases.

2.1 Main Definitions

As Atanassov's concept of an intuitionistic fuzzy set can be viewed as a generalization of a fuzzy set definition in the case when available information is not sufficient for the definition of an imprecise concept by means of a conventional fuzzy set, we start from the basic definitions of a set, and of a fuzzy set.

2.1.1 Crisp Sets (Classical Sets)

A set is one of the basic concepts in mathematics. Informally, a set is a collection of objects (elements) having similar properties (attributes). A classical set *A* (crisp set) has sharp boundaries, i.e., there are two possibilities only:

E. Szmidt, *Distances and Similarities in Intuitionistic Fuzzy Sets*,
Studies in Fuzziness and Soft Computing 307,
DOI: 10.1007/978-3-319-01640-5_2,

– an element x belongs to the set ($x \in A$), or
– an element does not belong to the set ($x \notin A$).

A classical set can be expressed by its characteristic function.

Definition 2.1. For a set X and a subset A of X ($A \subseteq X$) we call

$$\varphi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{2.1}$$

the characteristic function of the set A in X.

Using the notion of the characteristic function $\varphi_A(x)$, a crisp set A can be given as :

$$A = \{< x, \varphi_A(x) > /x \in X\} \tag{2.2}$$

For a conventional (crisp) set an element can not belong “to some extent” to a set.

2.1.2 Fuzzy Sets

A generalization of a crisp set is a fuzzy set. The notion of a fuzzy set was introduced by Zadeh [254]. A fuzzy set A' in a universe of discourse X is characterized by a membership function $\mu_{A'}$ which assigns to each element $x \in X$ a real number $\mu_{A'}(x) \in [0,1]$ expressing the membership grade of x in the fuzzy set A'.

Definition 2.2. (Zadeh [254])
A fuzzy set A' in $X = \{x\}$ is given by (Zadeh [254]):

$$A^{'} = \{< x, \mu_{A'}(x) > /x \in X\} \tag{2.3}$$

where $\mu_{A'} : X \to [0,1]$ is the membership function of the fuzzy set $A^{'}$; $\mu_{A'}$ for every element $x \in X$ describes its extent of membership to fuzzy set $A^{'}$.

As mentioned above, a fuzzy set $A^{'}$ is a generalization of a conventional (crisp) set A represented by its characteristic function $\varphi_A : X \to \{0,1\}$ (2.1). Full membership of x in $A^{'}$ occurs for $\mu_A^{'}(x) = 1$, full non-membership is for $\mu_A^{'}(x) = 0$ but opposite to a classical set other membership degrees are also allowed.

Every crisp set is a fuzzy set. The membership function of a crisp set $A \subseteq X$ can be expressed as its characteristic function

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{2.4}$$

2.1.3 Intuitionistic Fuzzy Sets

The notion of an intuitionistic fuzzy set was introduced by Atanassov (Atanassov [4]). An intuitionistic fuzzy set is a generalization of a fuzzy set.

Definition 2.3. (Atanassov [4], [6], [15] [22])
An intuitionistic fuzzy set A in X is given by:

$$A = \{< x, \mu_A(x), \nu_A(x) > /x \in X\} \tag{2.5}$$

where

$$\mu_A : X \to [0,1]$$

$$\nu_A : X \to [0,1]$$

with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \forall x \in X$$

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the degrees of membership and non-membership of the element $x \in X$ to the set A.

Obviously, every fuzzy set corresponds to the following intuitionistic fuzzy set:

$$FS : \{< x, \mu_A(x), 1 - \mu_A(x) > /x \in X\}. \tag{2.6}$$

Definition 2.4. (Atanassov [4], [6], [15] [22])
For an intuitionistic fuzzy set A we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{2.7}$$

the **intuitionistic fuzzy index** (hesitation margin) of the element x in the set A. The $\pi_A(x)$ expresses the lack of knowledge on whether x belongs to A or not.

It is obvious that

$$0 \leq \pi_A(x) \leq 1 \quad \text{for every } x.$$

For every fuzzy set A', where $x \in X$

$$\pi_A(x) = 1 - \mu_{A'}(x) - [1 - \mu_{A'}(x)] = 0.$$

The hesitation margin turns out to be important while considering the distances (Szmidt and Kacprzyk [165], [171], [188], entropy (Szmidt and Kacprzyk [175], [192]), similarity (Szmidt and Kacprzyk [193]) for the intuitionistic fuzzy sets, i.e., the measures that play crucial role in virtually all information processing tasks. The hesitation margin is shown to be indispensable also in the ranking of intuitionistic fuzzy alternatives as it indicates how reliable (sure) information presented for an alternative is (cf. Szmidt and Kacprzyk [198], [205]).

Making use of the intuitionistic fuzzy sets instead of fuzzy sets implies the introduction of additional degrees of freedom (non-memberships and hesitation margins) into the set description. Such a generalization of fuzzy sets gives us an additional possibility to represent imprecise knowledge which may lead to describing many real problems in a more adequate way.

It is woth stressing that from the point of view of the applications, taking into account the hesitation margins (besides non-membership values) is crucial in many areas exemplified by image processing (cf. Bustince et al. [45], [46]), classification of imbalanced and overlapping classes (cf. Szmidt and Kukier [229], [230], [231]), group decision making, negotiations, voting and other situations (cf. Szmidt and Kacprzyk [164], [167], [172], [174], [177], [178], [189]).

Intuitionistic fuzzy sets based models may be adequate mainly in the situations when we face human testimonies, opinions, etc. involving answers of three types:

– yes,
– no,
– abstaining i.e. which can not be classified (because of different reasons, eg. "I do not know", "I am not sure", "I do not want to answer", "I am not satisfied with any of the options" etc.).

Below we present an example given by Atanassov (Atanassov [15]). The example illustrates the essence of the intuitionistic fuzzy sets, and stresses the differences between them and the fuzzy sets.

Example 2.1. (Atanassov [15]) Let X be the set of all countries with elective governments. Assume that we know for every country $x \in X$ the percentage of the electorate who have voted for the corresponding government. Let it be denoted by $M(x)$ and let $\mu(x) = \frac{M(x)}{100}$. Let $\nu(x) = 1 - \mu(x)$. This number corresponds to that part of electorate who have not voted for the government. By means of the fuzzy set theory we cannot consider this value in more detail. However, if we define $\nu(x)$ as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all and the corresponding number will be $1 - \mu(x) - \nu(x)$. Thus, we can construct the set $\{\langle x, \mu(x), \nu(x)\rangle | x \in X\}$ and obviously, $0 \leq \mu(x) + \nu(x) \leq 1$. □

2.2 Brief Introduction to Fuzzy Sets

2.2.1 Basic Concepts

A fuzzy set $A^{'}$ in X is said to be empty, $A^{'} = \emptyset$, if and only if

$$\mu_A^{'}(x) = 0, \qquad \text{for each } x \in X \tag{2.8}$$

Two fuzzy sets $A^{'}$ and $B^{'}$ in the same universe of discourse X are said to be equal, i.e., $A^{'} = B^{'}$, if and only if

$$\mu_A^{'}(x) = \mu_B^{'}(x), \qquad \text{for each } x \in X \tag{2.9}$$

A fuzzy set $A^{'}$ defined in X is a subset of a fuzzy set $B^{'}$ in X, $A^{'} \subseteq B^{'}$, if and only if

$$\mu_A^{'}(x) \leq \mu_B^{'}(x), \qquad \text{for each } x \in X \tag{2.10}$$

A fuzzy set $A^{'}$ defined in X is said to be normal if and only if the membership function takes on the value of 1 for at least one value of its argument, i.e.

$$\max_{x \in X} \mu_A^{'}(x) = 1 \tag{2.11}$$

Otherwise, a fuzzy set is said to be subnormal.

The support supp$A^{'}$ of a fuzzy set $A^{'}$ in X is the following (nonfuzzy, i.e. crisp) set:

$$\text{supp}A^{'} = \{x \in X : \mu_A^{'}(x) > 0\} \tag{2.12}$$

where $\emptyset \subseteq \text{supp}A^{'} \subseteq X$.

The α-cut (or α-level set) A_α, of a fuzzy set $A^{'}$ in X is defined as the following (nonfuzzy) set:

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}, \qquad \text{for each } \alpha \in (0,1] \tag{2.13}$$

and if "$\geq$" in (2.13) is replaced by "$>$," then we have the strong α-cut (or strong α-level set), of a fuzzy set $A^{'}$ in X.

The α-cuts are quite important from the point of view of both theory and applications as they make it possible to uniquely replace a fuzzy set by a sequence of nonfuzzy sets. More details and properties of α-cuts can be found in any book on fuzzy set theory (cf. e.g., Dubois and Prade [61], Klir and Folger [107], or Klir and Yuan [109]).

Another important issue is to define how many elements are contained in a fuzzy set, i.e. to define its cardinality. The most often used definition is given below.

A nonfuzzy cardinality of a fuzzy set $A^{'} = \mu_A^{'}(x_1)/x_1 + \cdots + \mu_A^{'}(x_n)/x_n$, the so-called *sigma-count* $\sum\text{Count}(A^{'})$, is defined as (Zadeh [256], [257])

$$\sum\text{Count}(A^{'}) = \sum_{i=1}^{n} \mu_A^{'}(x_i) \tag{2.14}$$

Other definitions of cardinality making use of α-cuts were proposed by Zadeh [257] (see also Kacprzyk [97]). More discussion, criticism, and new definitions of cardinality are given by Ralescu [139], and Wygralak [247].

2.2.2 *Selected Operations on Fuzzy Sets*

Just like in the case of crisp sets, basic operations on fuzzy sets are complement, union, and intersection. They are defined in terms of the respective membership functions. The operations presented below correspond to the operations on intuitionistic fuzzy sets (cf. Section 2.3.2).

The complement $A^{'C}$ of a fuzzy set $A^{'}$ in X, corresponds to negation "not", and is defined as

$$\mu_{A'C}(x) = 1 - \mu_A^{'}(x), \qquad \text{for each } x \in X \tag{2.15}$$

The intersection of two fuzzy sets $A^{'}$ and $B^{'}$ in X, written $A^{'} \cap B^{'}$, corresponds to the connective "and", and is defined as

$$\mu_{A' \cap B'}(x) = \mu_A^{'}(x) \wedge \mu_B^{'}(x), \qquad \text{for each } x \in X \tag{2.16}$$

where "$\wedge$" is the minimum operation, i.e. $a \wedge b = \min(a,b)$.

The union of two fuzzy sets $A^{'}$ and $B^{'}$ in X, written $A^{'} + B^{'}$, corresponds to the connective "or", and is defined as

$$\mu_{A' + B'}(x) = \mu_A^{'}(x) \vee \mu_B^{'}(x), \qquad \text{for each } x \in X \tag{2.17}$$

where "$\vee$" is the maximum operation, i.e. $a \vee b = \max(a,b)$.

More general than the intersection and the union defined above, are so-called t-norms and s-norms (t-conorms).

A *t-norm* is defined as

$$t : [0,1] \times [0,1] \longrightarrow [0,1] \tag{2.18}$$

such that, for each $a,b,c \in [0,1]$ the following properties are fulfilled:

1. the value 1 is the unit element, i.e.

$$t(a,1) = a$$

2. monotonicity, i.e.

$$a \leq b \Longrightarrow t(a,c) \leq t(b,c)$$

3. commutativity, i.e.

$$t(a,b) = t(b,a)$$

4. associativity, i.e.

$$t[a,t(b,c)] = t[t(a,b),c]$$

A t-norm is monotone non-decreasing in both arguments, and $t(a,0) = 0$.

Among the most used t-norms there are:

- the minimum

$$t(a,b) = a \wedge b = \min(a,b) \tag{2.19}$$

  which is the most widely used,
- the algebraic product

$$t(a,b) = a \cdot b \tag{2.20}$$

- Łukasiewicz t-norm

$$t(a,b) = \max(0, a + b - 1) \tag{2.21}$$

An s-norm called also t-conorm is defined as

$$s : [0,1,] \times [0,1] \longrightarrow [0,1] \tag{2.22}$$

such that, for each $a,b,c \in [0,1]$ the following properties are fulfilled:

1. the value 0 is the unit element, i.e.
$$s(a,0) = a$$
2. monotonicity
$$a \leq b \Longrightarrow s(a,c) \leq s(b,c)$$
3. commutativity, i.e.
$$s(a,b) = s(b,a)$$
4. associativity, i.e.
$$s[a,s(b,c)] = s[s(a,b),c]$$

The most used s-norms include:

- the maximum
$$s(a,b) = a \vee b = \max(a,b) \tag{2.23}$$
which is the most widely used,
- the probabilistic product
$$s(a,b) = a + b - ab \tag{2.24}$$
- Łukasiewicz s-norm
$$s(a,b) = \min(a+b,1) \tag{2.25}$$

It is worth noticing that a t-norm is dual to an s-norm in that

$$s(a,b) = 1 - t(1-a,1-b) \tag{2.26}$$

Another concept, crucial from the point of view of theory and application, is that of a relation. It is discussed both for fuzzy sets and intuitionistic fuzzy sets in Section 2.3.3.

2.3 Brief Introduction to the Intuitionistic Fuzzy Sets

We will start this section by discussing two geometrical representations of intuitionistic fuzzy sets. The representation using two terms (membership values and non-membership values) describing the intuitionistic fuzzy sets leads to the so called 2D representation. Using three terms (membership values, non-membership values, and hesitation margins) in the intuitionistic fuzzy set description results in the so called 3D representation.

Other geometrical representations of the intuitionistic fuzzy sets are given by Atanassov [12], [13], [14], [15], [22]).

Later on, basic operators and relations over the intuitionistic fuzzy sets will be discussed.

2.3.1 Two Geometrical Representations of the Intuitionistic Fuzzy Sets

Having in mind that for each element x belonging to an intuitionistic fuzzy set A, the values of membership, non-membership and the intuitionistic fuzzy index sum up to one, i.e.

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1 \tag{2.27}$$

and that each of the membership, non-membership, and the intuitionistic fuzzy index belongs to $[0,1]$, we can imagine a unit cube (Figure 2.1) inside which there is an *MNH* triangle where the above equation is fulfilled (Szmidt and Kacprzyk [171]).

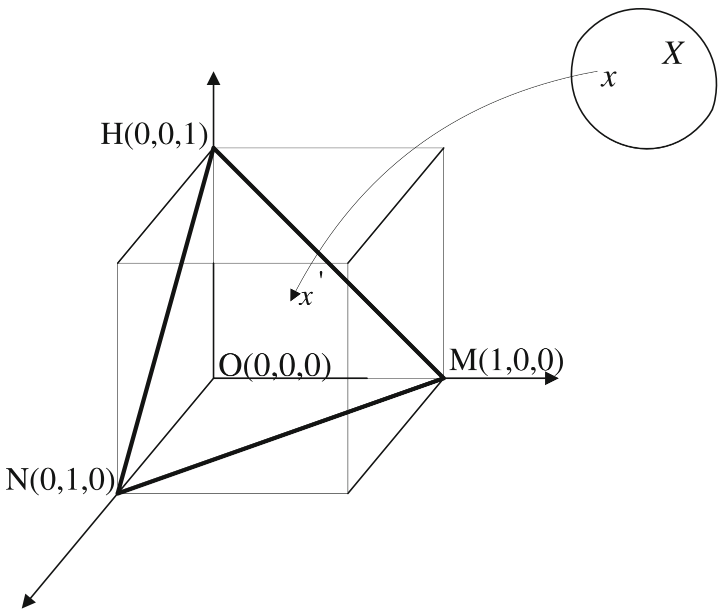

Fig. 2.1 Geometrical representation in 3D

In other words, the *MNH* triangle represents a surface where coordinates of any element belonging to an intuitionistic fuzzy set can be represented. Each point belonging to the *MNH* triangle is described via three coordinates: (μ, ν, π). Points M and N represent crisp elements. Point $M(1,0,0)$ represents elements fully belonging to an intuitionistic fuzzy set as $\mu = 1$. Point $N(0,1,0)$ represents elements fully not belonging to an intuitionistic fuzzy set as $\nu = 1$. Point $H(0,0,1)$ represents elements about which we are not able at all to say if they belong or not to an intuitionistic fuzzy set (intuitionistic fuzzy index $\pi = 1$). Such an interpretation is intuitively appealing and provides means for the representation of many aspects of imperfect information. The segment MN (where $\pi = 0$) represents elements belonging to classical fuzzy sets ($\mu + \nu = 1$).

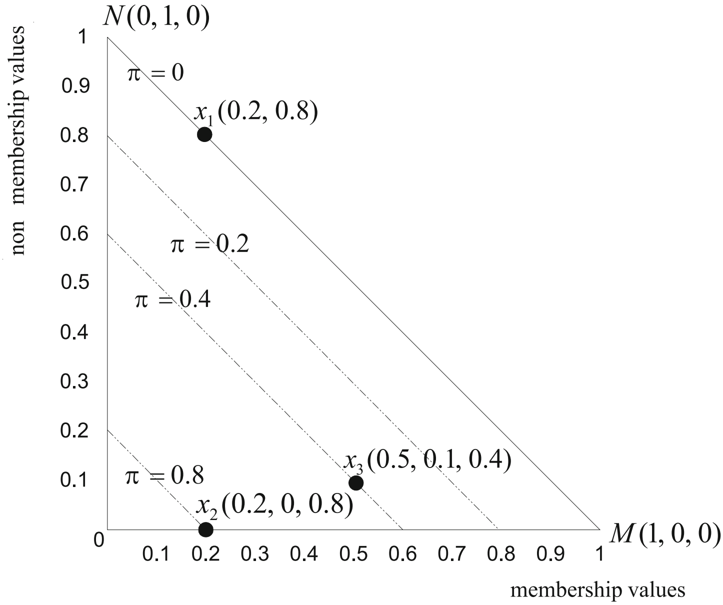

Fig. 2.2 Geometrical representation in 2D

Any other combination of the values characterizing an intuitionistic fuzzy set can be represented inside the triangle MNH. In other words, each element belonging to an intuitionistic fuzzy set can be represented as a point (μ, ν, π) belonging to the triangle MNH (Figure 2.1).

It is worth mentioning that the geometrical interpretation is directly related to the definition of an intuitionistic fuzzy set introduced by Atanassov [4], [15], and it does not need any additional assumptions.

Remark: We use the capital letters (e.g., M, N, H) for the geometrical representation of x_i's (Figure 2.2) on the plane. The same notation (capital letters) is used in this book for sets, but we always explain the current meaning of a symbol used.

Another possible geometrical representation of an intuitionistic fuzzy set can be in two dimensions (2D) – Figure 2.2 (cf. Atanassov [15]). It is worth noticing that although we use a 2D figure (which is more convenient to draw in many cases), we still adopt our approach (e.g., Szmidt and Kacprzyk [171], [188], [175], [192], [193], [218]) taking into account all three terms (membership, non-membership and hesitation margin values) describing the intuitionistic fuzzy sets. As previously, any element belonging to an intuitionistic fuzzy set may be represented inside an MNO triangle (O is projection of H in Figure 2.1). Each point belonging to the MNO triangle is still described by the three coordinates: (μ, ν, π), and points M and N represent, as previously, crisp elements. Point $M(1,0,0)$ represents elements fully belonging to an intuitionistic fuzzy set as $\mu = 1$, and point $N(0,1,0)$ represents elements fully not belonging to an intuitionistic fuzzy set as $\nu = 1$. Point $O(0,0,1)$

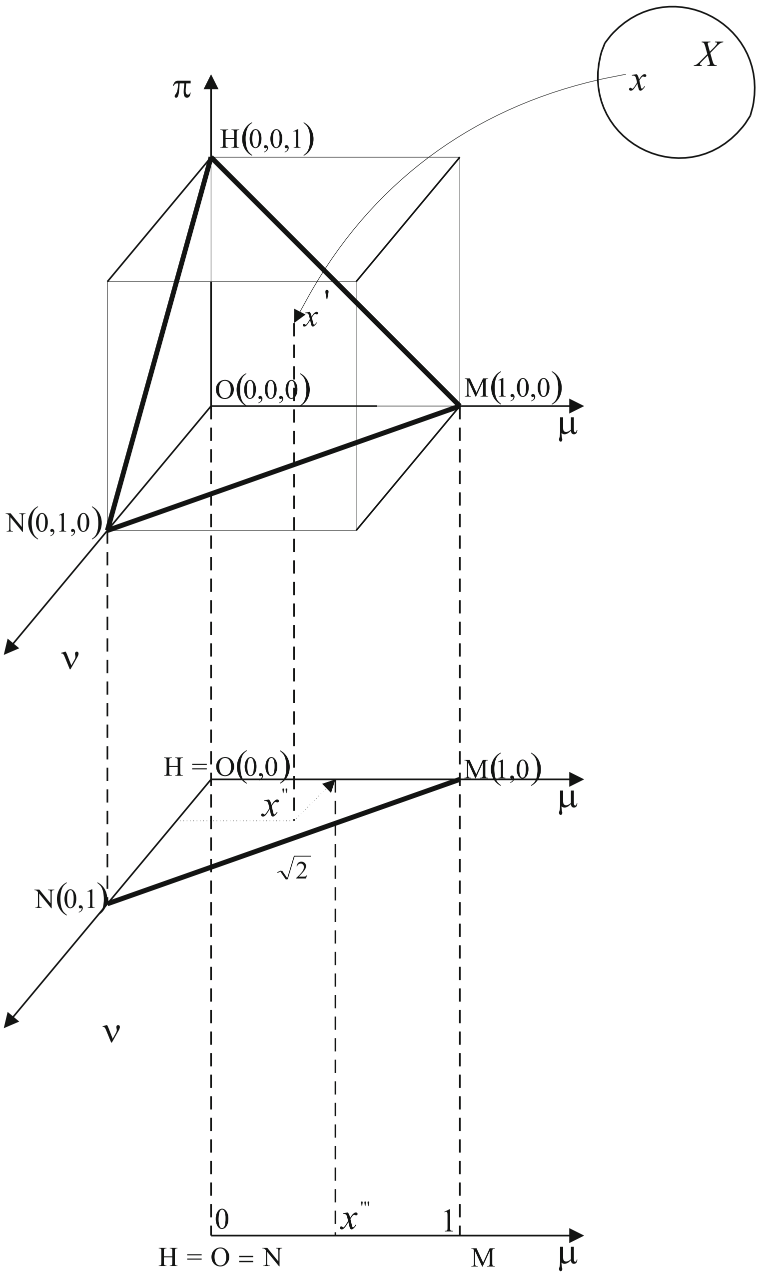

Fig. 2.3 Illustration of the interrelations between 3D and 2D representations of intuitionistic fuzzy sets

represents elements about which we cannot say if they belong or not to an intuitionistic fuzzy set (the intuitionistic fuzzy index $\pi = 1$). Segment MN (where $\pi = 0$) represents elements belonging to the classic fuzzy sets ($\mu + \nu = 1$). For example, point $x_1(0.2, 0.8, 0)$ (Figure 2.2), like any element of the segment MN, represents an element of a fuzzy set. A line parallel to MN describes the elements with the same values of the hesitation margin. In Figure 2.2 we can see point $x_3(0.5, 0.1, 0.4)$ representing an element with the hesitation margin equal 0.4, and point $x_2(0.2, 0, 0.8)$ representing an element with the hesitation margin equal 0.8. The closer a line that is parallel to MN is to O, the higher the hesitation margin.

In Figure 2.3 (Szmidt [158]) relations between the 2D and 3D representations are presented. It is worth stressing that 2D representation of the intuitionistic fuzzy sets (Figure 2.2), i.e., the triangle MNO is the orthogonal projection of the triangle MNH (3D representation – Figure 2.1) on the plane (Figure 2.3, the upper and the middle parts). The orthogonal projection transfers $x^{'} \in MNH$ into $x^{''} \in MNO$. Segment MN represents a fuzzy set described by μ and ν.

The orthogonal projection of the segment MN on the axis μ (the segment [0, 1] is only considered) gives the fuzzy set represented by μ only (Figure 2.3, its bottom part). This orthogonal projection transfers $x^{''} \in MNO$ into $x^{'''} \in OM$.

2.3.2 Operations Over the Intuitionistic Fuzzy Sets

This section contains results introduced by Atanassov [15] on operations and relations over the intuitionistic fuzzy sets. The point of departure is constituted by the respective definitions of relations and operations over fuzzy sets which are extended here. In the reverse perspective, relations and operations on fuzzy sets turn out to be particular cases of these new definitions.

Here is the definition of basic relations and operations on intuitionistic fuzzy sets.

Definition 2.5. (Atanassov [15])
For every two intuitionistic fuzzy sets A and B the following relations and operations can be defined ("iff" means "if and only if"):

$$A = B \text{ iff } (\forall x \in X)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)), \tag{2.28}$$

$$A^C = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}, \tag{2.29}$$

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in X\}, \tag{2.30}$$

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in X\}, \tag{2.31}$$

$$\begin{aligned} A + B = \{\langle x, \mu_A(x) \\ + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle, | \, x \in X\} \end{aligned} \tag{2.32}$$

$$A.B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle \mid x \in X\}, \tag{2.33}$$

$$A@B = \{\langle x, (\frac{\mu_A(x)+\mu_B(x))}{2}, \frac{(\nu_A(x)+\nu_B(x))}{2}\rangle | x \in X\}, \tag{2.34}$$

$$A\$B = \{\langle x, \sqrt{\mu_A(x).\mu_B(x)}, \sqrt{\nu_A(x).\nu_B(x)}\rangle | x \in X\}, \tag{2.35}$$

$$A * B = \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2.(\mu_A(x).\mu_B(x)+1)}, \frac{\nu_A(x)+\nu_B(x)}{2.(\nu_A(x).\nu_B(x)+1)}\rangle | x \in X\}, \tag{2.36}$$

$$A \bowtie B = \{\langle x, 2.\frac{\mu_A(x).\mu_B(x)}{\mu_A(x)+\mu_B(x)}, 2.\frac{\nu_A(x).\nu_B(x)}{(\nu_A(x)+\nu_B(x)}\rangle | x \in X\}, \tag{2.37}$$

for which we will accept that if

$\mu_A(x) = \mu_B(x) = 0,$ then $\dfrac{\mu_A(x).\mu_B(x)}{\mu_A(x)+\mu_B(x)} = 0$

and if $\nu_A(x) = \nu_B(x) = 0,$ then $\dfrac{\nu_A(x).\nu_B(x)}{\nu_A(x)+\nu_B(x)} = 0.$

Operations (2.28)–(2.37) have also their counterparts in the fuzzy set theory.

Example 2.2. (Atanassov [15])
Let $X = \{a,b,c,d,e\}$, let the intuitionistic fuzzy sets A and B have the form $\{< x_i, \mu(x_i), \nu(x_i) >\}$:

$$A = \{\langle a,0.5,0.3\rangle, \langle b,0.1,0.7\rangle, \langle c,1.0,0.0\rangle, \langle d,0.0,0.0\rangle, \langle e,0.0,1.0\rangle\},$$

$$B = \{\langle a,0.7,0.1\rangle, \langle b,0.3,0.2\rangle, \langle c,0.5,0.5\rangle, \langle d,0.2,0.2\rangle, \langle e,1.0,0.0\rangle\}.$$

Then

$$\overline{A} = \{\langle a,0.3,0.5\rangle, \langle b,0.7,0.1\rangle, \langle c,0.0,1.0\rangle, \langle d,0.0,0.0\rangle, \langle e,1.0,0.0\rangle\},$$

$$A \cap B = \{\langle a,0.5,0.3\rangle, \langle b,0.1,0.7\rangle, \langle c,0.5,0.5\rangle, \langle d,0.0,0.2\rangle, \langle e,0.0,1.0\rangle\},$$

$$A \cup B = \{\langle a,0.7,0.1\rangle, \langle b,0.3,0.2\rangle, \langle c,1.0,0.0\rangle, \langle d,0.2,0.0\rangle, \langle e,1.0,0.0\rangle\},$$

$$A + B = \{\langle a,0.85,0.03\rangle, \langle b,0.37,0.14\rangle, \langle c,1.0,0.0\rangle, \langle d,0.2,0.0\rangle, \langle e,1.0,0.0\rangle\},$$

$$\begin{aligned} A.B &= \{\langle a,0.35,0.37\rangle,\langle b,0.03,0.76\rangle,\langle c,0.5,0.5\rangle,\langle d,0.0,0.2\rangle,\\ &\quad \langle e,0.0,1.0\rangle\}, \\ A@B &= \{\langle a,0.6,0.2\rangle,\langle b,0.2,0.45\rangle,\langle c,0.75,0.25\rangle,\langle d,0.1,0.1\rangle,\\ &\quad \langle e,0.5,0.5\rangle\}, \\ A\$B &= \{\langle a,0.591...,0.173...\rangle,\langle b,0.173...,0.374...\rangle,\langle c,0.0707...,0.0\rangle,\\ &\quad \langle d,0.0,0.0\rangle,\langle e,0.0,0.0\rangle\}, \\ A*B &= \{\langle a,0.444...,0.194...\rangle,\langle b,0.194...,0.394...\rangle,\langle c,0.5,0.5\rangle,\\ &\quad \langle d,0.1,0.1\rangle,\langle e,0.5,0.5\rangle\}, \\ A\bowtie B &= \{\langle a,0.583...,0.15\rangle,\langle b,0.15,0.311...\rangle,\langle c,0.666...,0.0\rangle,\\ &\quad \langle d,0.0,0.0\rangle,\langle e,0.0,0.0\rangle\}. \end{aligned}$$

□

Proposition 2.1. (Atanassov [15])
For every three intuitionistic fuzzy sets A, B and C, following properties hold: :

$$A \cap B = B \cap A, \tag{2.38}$$

$$A \cup B = B \cup A, \tag{2.39}$$

$$A + B = B + A, \tag{2.40}$$

$$A.B = B.A, \tag{2.41}$$

$$A@B = B@A, \tag{2.42}$$

$$A\$B = B\$A, \tag{2.43}$$

$$A \bowtie B = B \bowtie A, \tag{2.44}$$

$$A * B = B * A, \tag{2.45}$$

$$A \cap B) \cap C = A \cap (B \cap C), \tag{2.46}$$

$$(A \cup B) \cup C = A \cup (B \cup C), \tag{2.47}$$

$$(A + B) + C = A + (B + C), \tag{2.48}$$

$$(A.B).C = A.(B.C), \tag{2.49}$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \tag{2.50}$$

$$(A \cap B) + C = (A + C) \cap (B + C), \tag{2.51}$$

$$(A \cap B).C = (A.C) \cap (B.C), \tag{2.52}$$

$$(A \cap B)@C = (A@C) \cap (B@C), \tag{2.53}$$

$$(A \cap B) \bowtie C = (A \bowtie C) \cap (B \bowtie C), \tag{2.54}$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \tag{2.55}$$

$$(A \cup B) + C = (A + C) \cup (B + C), \tag{2.56}$$

$$(A \cup B).C = (A.C) \cup (B.C), \tag{2.57}$$

$$(A \cup B)@C = (A@C) \cup (B@C), \tag{2.58}$$

$$(A \cup B) \bowtie C = (A \bowtie C) \cup (B \bowtie C), \tag{2.59}$$

$$(A + B).C \subset (A.C) + (B.C), \tag{2.60}$$

$$(A + B)@C \subset (A@C) + (B@C), \tag{2.61}$$

$$(A.B) + C \supset (A + C).(B + C), \tag{2.62}$$

$$(A.B)@C \supset (A@C).(B@C), \tag{2.63}$$

$$(A@B) + C = (A + C)@(B + C), \tag{2.64}$$

$$(A@B).C = (A.C)@(B.C), \tag{2.65}$$

$$A \cap A = A, \tag{2.66}$$

$$A \cup A = A, \tag{2.67}$$

$$A @ A = A, \tag{2.68}$$

$$A \$ A = A, \tag{2.69}$$

$$A \bowtie A = A, \tag{2.70}$$

$$\overline{\overline{A} \cap \overline{B}} = A \cup B, \tag{2.71}$$

$$\overline{\overline{A} \cup \overline{B}} = A \cap B, \tag{2.72}$$

$$\overline{\overline{A} + \overline{B}} = A.B, \tag{2.73}$$

$$\overline{\overline{A}.\overline{B}} = A + B, \tag{2.74}$$

$$\overline{\overline{A} @ \overline{B}} = A @ B, \tag{2.75}$$

$$\overline{\overline{A} \$ \overline{B}} = A \$ B, \tag{2.76}$$

$$\overline{\overline{A} \bowtie \overline{B}} = A \bowtie B, \tag{2.77}$$

$$\overline{\overline{A} * \overline{B}} = A * B. \tag{2.78}$$

2.3.2.1 The "necessity" and "possibility" Operators

The operators over intuitionistic fuzzy sets, presented above, correspond to the respective operators over fuzzy sets. Here we present two operators introduced by Atanassov in 1983, which are "meaningless" (Atanassov [4], [15]) in the case of fuzzy sets.

Definition 2.6. (Atanassov [4], [15]) Let us define, for every intuitionistic fuzzy set A, the following operators:

- the necessity operator

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}, \tag{2.79}$$

- the possibility operator

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in X\}. \tag{2.80}$$

If A is a classical fuzzy set, then

$$\Box A = A = \Diamond A. \tag{2.81}$$

From (2.81) it follows that both "$\Box$" (2.79) and "$\Diamond$" (2.80) are meaningless for a fuzzy set. Atanassov [15] considers in length the properties, modifications, and extensions of "$\Box$" (2.79) and "$\Diamond$" (2.80). Here we only recall their two extensions.

Let $\alpha \in [0,1]$ be a fixed number.

Definition 2.7. (Atanassov [15]) Given an intuitionistic fuzzy set A, an operator D_α is defined as follows:

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in X\} \tag{2.82}$$

where $\alpha \in [0,1]$.

From definition 2.7 we see that $D_\alpha(A)$ is a fuzzy set, namely:

$$\mu_A(x) + \alpha.\pi_A(x) + \nu_A(x) + (1-\alpha).\pi_A(x) = \mu_A(x) + \nu_A(x) + \pi_A(x) = 1.$$

Several interesting properties of $D_\alpha(A)$ (2.82) are given by proposition 2.2:

Proposition 2.2. *(Atanassov [15]) For every intuitionistic fuzzy set A and for every* $\alpha, \beta \in [0,1]$*:*

$$\textit{if } \alpha \leq \beta, \textit{ then } D_\alpha(A) \subset D_\beta(A), \tag{2.83}$$

$$D_0(A) = \Box A, \tag{2.84}$$

$$D_1(A) = \Diamond A. \tag{2.85}$$

The operator D_α is a generalization of the operators "necessity" and "possibility". The operator D_α has been be extended even further. Namely, Atanassov [15] introduced operator $F_{\alpha,\beta}$ (2.86).

Let $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$.

Definition 2.8. (Atanassov [7]) The operator $F_{\alpha,\beta}$, for an intuitionistic fuzzy set A, is defined as:

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in X\}. \tag{2.86}$$

The above operators are not only important from the theoretical point of view (indicating that the intuitionistic fuzzy sets are a generalization of the fuzzy sets) but they are also important from the point of view of applications. The operator $D_\alpha(A)$ was applied in constructing a classifier recognizing imbalanced classes (Szmidt and Kukier [229], [230], [231]). The operator $F_{\alpha,\beta}$ has been applied for image recognition (Bustince et al. [131], [46], [48]).

2.3.3 Intuitionistic Fuzzy Relations

As it was shown (cf. definitions in Section 2.1), one term, i.e., membership function, fully describes a fuzzy set, whereas two terms are necessary when we discuss the intuitionistic fuzzy sets. Similar differences hold when we define a fuzzy relation and an intuitionistic fuzzy relation.

Definition 2.9. A fuzzy relation between two non-fuzzy sets $X = \{x\}$ and $Y = \{y\}$ is defined on a Cartesian product $X \times Y$, i.e. $R \subset X \times Y = \{(x,y) : x \in X, y \in Y\}$, and given by

$$R = \{< (x,y), \mu_R(x,y) > /x \in X, y \in Y\} \tag{2.87}$$

where $\mu_R : X \times Y \to [0,1]$ is a membership function of a fuzzy relation R assigning to every pair (x,y), $x \in X$, $y \in Y$, its degree of membership $\mu_R(x,y) \in [0,1]$ describing the measure of intensity of a fuzzy relation R between x and y.

Example 2.3. If $X = \{Al, Bob, Clark\}$ and $Y = \{Paul, Jim\}$, the fuzzy relation R labelled "resemblance" may be exemplified by

$$\begin{aligned} R = & (Al, Paul)/0.5 + (Al, Jim)/0.3 + (Bob, Paul)/0.6 + \\ & + (Bob, Jim)/0.4 + (Clark, Paul)/0.9 + (Clark, Jim)/0.1 \end{aligned}$$

to be read as: there is resemblance between Paul and Clark (with respect to "our own" subjective aspects) to degree 0.9, i.e. to a very high extent, and rather low resemblance between Jim and Clark - to degree 0.1 only, etc.

Any fuzzy relation (in a finite $X \times Y$) may be represented in a matrix form. The following matrix corresponds to the above relation "resemblance"

$R = [r_{ij}] =$

	Paul	*Jim*
Al	0.5	0.3
Bob	0.6	0.4
Clark	0.9	0.1

□

Taking into account the definition of intuitionistic fuzzy set, the definition of an intuitionistic fuzzy relation R can be introduced as a counterpart of fuzzy relation.

Definition 2.10. (Atanassov [5], [23], [15])
An intuitionistc fuzzy relation between two non-fuzzy sets $X = \{x\}$ and $Y = \{y\}$ is defined on a Cartesian product $X \times Y$, i.e. $R \subset X \times Y = \{(x,y) : x \in X, y \in Y\}$, and given by

$$R = \{< (x,y), \mu_R(x,y), \nu_R(x,y) > /x \in X, y \in Y\} \tag{2.88}$$

with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \forall x \in X, y \in Y$$

where μ_R - as before, $\nu_R : X \times Y \to [0,1]$ is a non-membership function of a fuzzy relation R assigning to every pair (x,y), $x \in X$, $y \in Y$, its degree of non-membership $\nu_R(x,y) \in [0,1]$, being the measure of falsity of a fuzzy relation R between x and y.

Therefore, an intuitionistic fuzzy relation is described by any two terms from the triplet: membership function, non-membership function, intuitionistic fuzzy index function.

Example 2.4. If $X = \{Bob, Peter\}$ and $Y = \{Liz, Jim, John\}$, the intuitionistic fuzzy relation R labelled "cooperation" while preparing a new project, may be given as $\{(x,y), \mu_R(x,y), \pi_R(x,y)\}$, i.e. intuitionistic fuzzy relation can be also described by giving $\pi_R(x,y)$ instead of $\nu_R(x,y)$

$$R = (Bob, Liz)/1, 0 + (Bob, Jim)/0.7, 0.2 + (Bob, John)/0.5, 0.3 + \\ + (Peter, Liz)/0.7, 0.2 + (Peter, Jim)/0.9, 0 + (Peter, John)/0.4, 0.5$$

or, in a matrix form

$$\mu_R = \begin{array}{l|cc} & Bob & Peter \\ \hline Liz & 1 & 0.7 \\ Jim & 0.7 & 0.9 \\ John & 0.5 & 0.2 \end{array} \qquad \pi_R = \begin{array}{l|cc} & Bob & Peter \\ \hline Liz & 0 & 0.2 \\ Jim & 0.2 & 0 \\ John & 0.3 & 0.8 \end{array}$$

to be read as: excellent cooperation between Bob and Liz is foreseen (to the highest degree: 1), and any difficulties are not expected ($\pi = 0$) about their cooperation. Peter and Jim usually cooperate to a very high extent (to degree 0.9) but in a very rare situations it is known that they have quite different opinions ($\pi = 0$ which means $\nu = 0.1$). Peter and John usually at the beginning do not agree ($\mu = 0.2$ only), but they are open for arguments ($\pi = 0.8$) so it is possible in almost all cases to convince them to go for the same goal, etc. □

2.4 Interrelationships: Crisp Sets, Fuzzy Sets, Intuitionistic Fuzzy Sets

On the basis of the definitions and properties presented in this chapter, the following conclusions can be drawn:

- A membership function fully describes a fuzzy set (by specifying a membership function we automatically know the non-membership function).
- If we want to describe fully an intuitionistic fuzzy set, we must use any two terms from the triplet: {*membership function, non-membership function, intuitionistic fuzzy index function*}.

In other words, applying intuitionistic fuzzy sets instead of fuzzy sets means introducing another degree of freedom into the set description (apart from a function μ_A there appears a function ν_A or π_A).

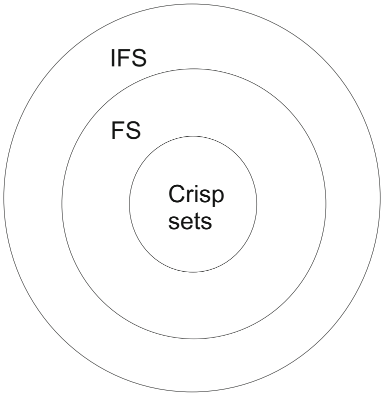

Fig. 2.4 An illustration of interrelations among conventional sets, fuzzy sets (FS) and intuitionistic fuzzy sets (IFS)

Interrelations among conventional sets, fuzzy sets (FS) and intuitionistic fuzzy sets (IFS) are given in Figure 2.4.

The meaning of Figure 2.4 is explained in Figure 2.5. For a crisp set, two points only in the coordinate system given in Figure 2.5 can represent the elements belonging to such a set. As an element corresponding to a crisp set fully belongs or fully does not belong to a crisp set, only the point with coordinates: $\mu = 1$ and $\nu = 0$ (fully belonging), or another point: $\mu = 0$ and $\nu = 1$ (fully not-belonging) can represent the elements from crisp sets. The case is illustrated in the upper part of Figure 2.5.

In the case of a fuzzy set, because of the fact that $\mu + \nu = 1$, besides previously described points (fully belonging, and fully not-belonging), also the entire segment connecting these points can be an image of elements belonging to a fuzzy set. The middle part of Figure 2.5 illustrates this fact.

Finally, for an intuitionistic fuzzy set, because of the condition: $0 \leq \mu + \nu \leq 1$, not only the points described above (the segment with its ends), but also the interior of the shaded triangle at the bottom part of Figure 2.5 can represent the elements belonging to an intuitionistic fuzzy set. Every one of the parallel lines (inside the triangle) is an image of the elements with the same value of intuitionistic fuzzy index.

In the above sense, the intuitionistic fuzzy sets contain fuzzy sets which, in turn, contain conventional (crisp) sets (cf. Figure 2.4). In terms of information it means that in the case of

- crisp sets – information is complete (elements fully belong or fully do not belong to a crisp set),
- fuzzy sets – information is also complete but the elements can belong to a fuzzy set to some degree; knowing the degree to which the elements belong to a fuzzy set (μ), we immediately know the degree to which they do not belong to the set $(1 - \mu)$,

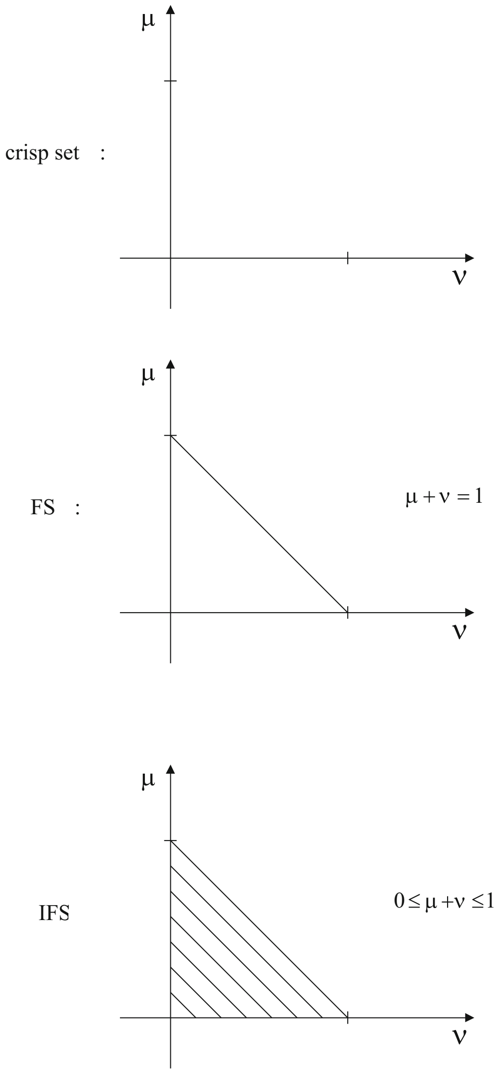

Fig. 2.5 Comparison of the possible images (in the same coordinate system) of elements belonging to crisp sets, fuzzy sets (FS) and intuitionistic fuzzy sets (IFS)

- intuitionistic fuzzy sets – information may be not complete because of the intuitionistic fuzzy indices; on the other hand, using the intuitionistic fuzzy sets we may express the same information as via crisp sets or fuzzy sets (cf. Figures 2.4 and 2.5).

In the next section, starting from relative frequency distributions, we describe the automatic algorithm of deriving the intuitionistic fuzzy sets from data (Szmidt and Baldwin [162]).

2.5 Deriving the Intuitionistic Fuzzy Sets from Data

In order to apply the intuitionistic fuzzy sets, which seem to be a very good tool for representation and processing of imperfect information, one should be able to construct their membership and non-membership functions. In this section we propose two ways of deriving the membership and non-membership functions for the intuitionistic fuzzy sets:

a) by asking experts;
b) from relative frequency distributions (histograms).

The second (automatic) method is justified by some similarities/parallels between intuitionistic fuzzy set theory and mass assignment theory – a well known tool for dealing with both probabilistic and fuzzy uncertainties. We also recall a semantic for membership functions – the interpretation having its roots in the possibility theory. Both mass assignment theory and a semantic interpretation of the membership functions made it possible to construct the automatic algorithm assigning the functions describing the intuitionistic fuzzy sets.

Uncertainty was identified and expressed for several centuries (starting from the mid-seventeenth century) in terms of probability theory only. Uncertainty was a synonym of randomness. This situation changed in the 1960s when other theories, distinct from probability theory, characterizing different aspects of uncertain situations, were introduced. Uncertainty started to be perceived as a multidimensional concept for which randomness became one of its dimension only. Other components turned out equally important from the point of view of representing and processing information.

The theory of fuzzy sets (Zadeh [254]), theory of evidence (Dempster-Shafer theory [59, 150]), possibility theory (Zadeh [256]), theory of fuzzy measures (Sugeno [155]) became the most visible theories dealing with different aspects of uncertainty. Here we explore two of the theories dealing with widely understood uncertainty, namely, intuitionistic fuzzy set theory (Atanassov [4]) which is a generalization of fuzzy set theory, and mass assignment theory (Baldwin [30, 29]) related to the theory of evidence (but the constraint that $m(0) = 0$ is not imposed). The theory of mass assignment we apply also differs from the Dempster-Shafer theory of evidence since the method of combining mass assignments is different. The theory discussed here is consistent with probability theory.

We start from showing some similarities/parallels between intuitionistic fuzzy set theory [Atanassov [6, 15]] and mass assignment theory (Baldwin [30], Baldwin et al. [38, 35]). The similarities we stress do not mean that one of the theories could replace the other or is better. To the contrary, the similarities we show seem to be important from the point of view of further development of both theories. Here, the noticed similarities made it possible to construct an algorithm of the automatic derivation of the membership and non-membership functions for intuitionistic fuzzy sets.

Making use of positive and (independently given) negative information, which is the core of the intuitionistic fuzzy set approach, is natural in real life, and as an obvious consequence, is well-known in psychology [e.g., Sutherland [157], Kahneman [98]]. The idea also attracted attention of the scientists in soft computing. It would be difficult to deal with machine learning (making use of examples and counter-examples), modeling of preferences or voting without taking into account positive and (independent) negative data.

Atanassov and Gargov [24]) noticed in 1989 that from the mathematical point of view intuitionistic fuzzy sets are equipollent to interval-valued fuzzy sets. However, from the point of view of solving problems (starting from the stage of collecting data), both kind of sets are different. The intuitionistic fuzzy sets make a user consider independently positive and negative information whereas when employing interval-valued fuzzy sets, user's attention is focused on positive information (in an interval) only. This fact, strongly connected with a psychological phenomenon called by the Nobel Prize winner Kahneman (cf. Kahneman [98]) "bounded rationality" (see also Sutherland [157]), caused among others by the fact that people tend to notice and take into account only most obvious aspects (e.g. advantages only), places the intuitionistic fuzzy sets among the up-to-date means of knowledge representation and processing.

To apply the intuitionistic fuzzy sets one should be able to assign respective membership values and non-membership values. Here we discuss two ways of assigning the membership and non-membership values for the intuitionistic fuzzy sets: by asking experts, and from the relative frequency distributions (histograms).

The models applying the intuitionistic fuzzy sets may be especially useful in the situations when we face human testimonies, opinions, etc. involving answers of three types:

– yes,
– no,
– abstaining, i.e. such that can not be classified in the former two (because of different reasons, eg. "I do not know", "I am not sure", "I do not want to answer", "I am not satisfied with any of the options", etc.).

Example 2.5. (Szmidt and Baldwin [162]) Let us assume that each individual x_i from a set X of n individuals who vote for/against building of nuclear power plant (electors voting for/against a given candidate or his opponent, judges voting for/against acquittal, consumers expressing/not expressing interest in buying a product) belongs to

- a set of individuals (judges, electors) voting for – to the extent $\mu(x_i)$,
- a set of individuals voting against – to the extent $\nu(x_i)$.

It is worth emphasizing that by means of the fuzzy set theory it is not possible to consider the situation in more details. By means of intuitionistic fuzzy set theory we can also point out

- a set of individuals who did not answer neither "yes" nor "no"– to the extent $\pi(x_i)$,
  whereas: $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$; $\pi(x_i)$ is an intuitionistic fuzzy index.

From the point of view of e.g. market analysts (election committees) it seems rather interesting to be able to assess the above data in terms of the possible final results of voting, by giving intervals containing

- probability of voting for

$$Pr_{for} \in [\mu, \mu + \pi]$$

where:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} \mu(x_i)$$

$$\pi = \frac{1}{n} \sum_{i=1}^{n} \pi(x_i)$$

- probability of voting against

$$Pr_{against} \in [\nu, \nu + \pi]$$

where:

$$\nu = \frac{1}{n} \sum_{i=1}^{n} \nu(x_i)$$

with the condition $Pr_{for} + Pr_{against} = 1$.

Interpreting the above results in terms of mass assignment (see Section 2.5.2) we could say that the necessary support for is equal to μ, the necessary support against is equal to ν, whereas the possible support for (the best possible result) Pos^{+} is equal to $\mu + \pi$, and the possible support against (the worst possible result) Pos^{-} is equal to $\nu + \pi$.

It is necessary to stress that we have made a simplifying assumption in the above example by assigning a sign of equality to probabilities and memberships/non-memberships. This assumption is valid under the condition that each value of membership/non-membership occurs with the same probability for each x_i. Here, for the sake of simpler notation, we follow this assumption. However, in general, probabilities for the intuitionistic fuzzy sets are calculated as discussed in (Szmidt and Kacprzyk [169, 170]) and recalled by Definitions 2.11 and 2.12.

Definition 2.11. (Szmidt and Kacprzyk [169, 170]) By an intuitionistic fuzzy event A we will mean an intuitionistic fuzzy subset belonging to the elementary event space X , i.e. $A \subset X$ whose membership function $\mu_A(x)$, non-membership function $\nu_A(x)$, and intuitionistic fuzzy index $\pi_A(x)$ are Borel measurable.

Definition 2.12. (Szmidt and Kacprzyk [169, 170]) Let us assign to every element of an intuitionistic fuzzy event $A \subset X = \{x_1, ..., x_n\}$ (where X is the elementary event space) its probability of occurrence, i.e. $p(x_1), ..., p(x_n)$.
Minimal probability $p_{\min}(A)$ of an intuitionistic fuzzy event A is equal to

$$p_{\min}(A) = \sum_{i=1}^{n} p(x_i)\mu(x_i)$$

Maximal probability of an intuitionistic fuzzy event A is equal to

$$p_{\max}(A) = p_{min}(A) + \sum_{i=1}^{n} p(x_i)\pi(x_i)$$

so probability of an event A is a number from the interval $[p_{min}(A), p_{max}(A)]$, or

$$p(A) \in [\sum_{i=1}^{n} p_A(x_i)\mu_A(x_i), \sum_{i=1}^{n} p_A(x_i)\mu_A(x_i) + + \sum_{i=1}^{n} p_A(x_i)\pi_A(x_i)], \tag{2.89}$$

probability of a complement event A^C is a number from the interval $[p_{min}(A^C)$, $p_{max}(A^C)]$, or

$$p(A^C) \in [\sum_{i=1}^{n} p_A(x_i)\nu_A(x_i), \sum_{i=1}^{n} p_A(x_i)\nu_A(x_i) + + \sum_{i=1}^{n} p_A(x_i)\pi_A(x_i)] \tag{2.90}$$

Applications of the intuitionistic fuzzy sets to group decision making, negotiations and other real situations are presented, e.g., in (Szmidt and Kacprzyk [163, 164, 167, 173, 177, 178, 179, 182]).

The question arises how to derive the membership and non-membership functions.

2.5.1 Derivation of the Intuitionistic Fuzzy Sets by Experts

We will discuss the problem of deriving membership and non-membership functions for the intuitionistic fuzzy models in the simplest case – when one person considers one decision only (this simple case can be easily extended to more complicated situations - with more persons and more decisions).

Assume that somebody considers a problem of changing his/her job. To decide if a new job is interesting enough to give up a previous one it seems reasonable to prepare a whole list of questions. The list would depend on the personal preferences but in general the following questions presented in Table 2.1 seem to be important (Szmidt and Baldwin [162].

Assuming that all the questions are equally important in Table 2.1, we can immediately conclude how to evaluate the considered case – just by summing up:

- all the positive answers (7/12) - this is the value of the membership for the considered option,
- all the negative answers (3/12) - this is the value of the non-membership for the considered option,

Table 2.1 The questions considered when changing a job

No	Questions	+/?/−
1	Is the job interesting	+
2	Salaries	−
3	Possibilities of promotion	?
4	Expected pension	−
5	Number of hours spent in work	?
6	Holidays – how long	+
7	Is the work safe	+
8	Responsibility	+
9	Time of the travel: home–work	−
10	Social reputation	+
11	Necessary creativity	+
12	Connected stress	+

- all the answers for which it was impossible to say "yes" or "not" (2/12) - this is the value of the intuitionistic fuzzy index for the considered option.

We can notice that employing the intuitionistic fuzzy sets just forces an individual to consider both advantages (membership values) and disadvantages (non-membership values) of a considered solution. Next, the imprecise area is taken into account as well. The importance of such an approach lies in the fact that most people concentrate usually on one or two "most visible" aspects of a problem. They do not try to find out the contrary arguments or to consider uncertain (in wide sense, i.e. not restricted to randomness) aspects of a situation (cf. Sutherland [157]).

The structure of the intuitionistic fuzzy sets make us consider a situation/problem taking into account more aspects. We refer again an interested reader to (Szmidt and Kacprzyk [163, 164, 167, 173, 177, 178, 179, 182]) where we exploit this fact - using the intuitionistic fuzzy sets to group decision making. In short, the problem boils down to selecting an option or a set of options which are best accepted by most of the individuals. The options are considered in pairs. Employing the intuitionistic fuzzy sets forces each individual to look at each pair (i,j) of the options considering: advantages of the first option over the second one (membership value), disadvantages of the first option over the second one (non-membership value), and taking into account lack of knowledge (intuitionistic fuzzy index) as far as the two options are concerned. In other words, the intuitionistic fuzzy sets force a user to explore a problem from different points of view – including all important aspects which should be taken into account but, unfortunately, are often omitted by people making decisions. This fact, strongly connected with a phenomenon called by the Nobel Prize winner Kahneman (cf. Kahneman [98]) "bounded rationality", caused, in particular, by the framing effect (explained in terms of salience and anchoring, playing a central role in treatment of judgements and choice), makes the intuitionistic fuzzy sets a highly effective means of knowledge representation and processing.

2.5.2 Automatic Method of Deriving Intuitionistic Fuzzy Sets from Relative Frequency Distributions (Histograms)

Baldwin (Baldwin [30], Baldwin et al. [38, 35]) developed the theory of mass assignment to provide a formal framework for manipulating both probabilistic and fuzzy uncertainty.

A fuzzy set can be converted into a mass assignment (Baldwin [28], Dubois and Prade [62]). This mass assignment represents a family of probability distributions.

Definition 2.13. (Mass Assignment) (Baldwin et al. [29], [35], [32], [37]) Let $A^{'}$ be a fuzzy subset of a finite universe X such that the range of the membership function of $A^{'}$, is $\{\mu_1, ..., \mu_n\}$ where $\mu_i > \mu_{i+1}$. Then the mass assignment of $A^{'}$ denoted $m_{A'}$, is a probability distribution on 2^X satisfying

$$m_{A'}(F_i) = \mu_i - \mu_{i+1} \text{ where } F_i = \{x \in X | \mu(x) \geq \mu_i\} \quad \text{for } i = 1, ..., n \tag{2.91}$$

We call the sets $F_1, ..., F_n$ the focal elements of $m_{A'}$. The details of mass assignment theory are presented by Baldwin et al. [38].

Example 2.6. (Baldwin [31])
For $X = \{x_1, x_2, x_3, x_4\}$,
if $A^{'} = x_1/1 + x_2/0.7 + x_3/0.4 + x_4/0.3$
then the associated mass assignment is
$m_{A'} = x_1 : 0.3, \quad \{x_1, x_2\} = 0.3, \quad \{x_1, x_2, x_3\} = 0.1, \quad \{x_1, x_2, x_3, x_4\} = 0.3$

The basic representation of uncertainty in the language FRIL [Baldwin et al. [38, 32]) are the so called Support Pairs which are associated with mass assignments and represent intervals containing unknown probabilities. Support Pairs are used to characterize uncertainty in facts and conditional probabilities in rules. A Support Pair (n, p) comprises a necessary and possible support and can be identified with an interval in which the unknown probability lies. Baldwin and Pilsworth [33] gave a voting interpretation of a support pair – the lower (necessary) support n represents the proportions of a sample population voting in favor of a proposition, whereas $(1-p)$ represents the proportion voting against; $(p-n)$ represents the proportion abstaining.

On the other hand, considering a voting model in terms of the intuitionistic fuzzy sets, [cf. Example 2.1 (Atanassov [15])] we have

- the membership values μ are equal to the proportion of a sample population voting in favour of a proposition,
- the non-membership values ν are equal to the proportion of a sample population voting against,
- the values of the hesitation margin π represents the proportion abstaining.

The interpretation of the parameters from Baldwin's voting model, and from intuitionistic fuzzy set (abbreviated IFS) voting model is presented in Table 2.2.

Table 2.2 Equality of the parameters for Baldwin's voting model and IFS voting model

	Baldwin's voting model	IFS voting model
voting in favour	n	μ
voting against	$1-p$	ν
abstaining	$p-n$	π

In other words, we can represent a Support Pair (n,p) using notation of the intuitionistic fuzzy sets by the following simple expression (Szmidt and Baldwin [162], [159], [160]):

$$(n,p) = (n, n+p-n) = (\mu, \mu+\pi) \tag{2.92}$$

i.e.: using notation of the intuitionistic fuzzy sets a Support Pair from the Baldwin's voting model can be expressed.

Moreover, one can note that the necessary support for the statement not being true is equal to one minus the possibility of the support for the statement being true, i.e. $1-p$. Similarly, the possible support for the statement being not true is one minus the necessary support for the statement being true i.e. $1-n$. Having in mind the correspondence of these parameters, we can express this fact making use of the intuitionistic fuzzy set notation, namely

$$(1-p, 1-n) = (\nu, \nu+\pi)$$

The following three Support Pairs (n,p) are especially interesting (Baldwin and Pilsworth [33]):

- (1, 1) expresses total support for the associated statement,
- (0, 0) characterizes total support against and
- (0, 1) represents complete uncertainty in the support.

Certainly, the meaning of the above Support Pairs is just the same in the models expressed in terms of the intuitionistic fuzzy sets (assuming that we consider probabilities for intuitionistic fuzzy membership/non-membership values as it was explained in the context of Definition 2.12):

- (1, 1) means that $\mu = 1$ and $\pi = 0$, i.e. total support,
- (0, 0) means $\mu = 0$ and $\pi = 0$ which involves $\nu = 1$, i.e. total support against,
- (0, 1) means $\mu = 0$ and $\pi = 1$ i.e.: complete lack of knowledge concerning support.

So, to sum up, both the Support Pairs and the intuitionistic fuzzy set models give the same intervals containing the probability of the fact being true. The difference between the upper and lower bounds of the intervals is a measure of the uncertainty associated with the fact [160], [159].

As Baldwin [34] has observed, the mass assignment structure is best used to represent knowledge that is statistically based in the sense that the values can be measured, even if the measurements themselves are approximate or uncertain.

Next notion, useful in our considerations, is so called least prejudiced distribution [62], [38].
For A', a fuzzy subset of a finite universe X, the least prejudiced distribution of A', denoted $lp_{A'}$, is a probability distribution on X given by

$$lp_{A'}(x) = \sum_{F_i : x \in F_i} \frac{m_{A'}(F_i)}{|F_i|} \tag{2.93}$$

A mass assignment corresponding to a normalized fuzzy subset of X naturally generates a family of probability distributions on X where each distribution corresponds to some redistribution of the masses associated with sets to elements of those sets. The most intuitive seems to redistribute a priori the mass associated with a set in the uniform manner among the elements of that set. In effect we obtain the distribution which coincides with Smet's pignistic probability [153], and with Dubois and Prade's [62] possibility-probability transformation based on a generalized Laplacean indifference principle.

It is worth emphasizing that the least prejudiced distribution (2.93) provides a mechanism by which we can, in a sense, convert a fuzzy set into a probability distribution. That is, in the absence of any prior knowledge, we might on being told A' naturally infer the distribution $lp_{A'}$ relative to a uniform prior. Certainly, if fuzzy sets are to serve as descriptions of probability distributions, the converse must also hold. The least prejudiced distribution provides the bijective possibility-probability transformation. In other words, for a probability distribution P on a finite universe X there is a unique fuzzy set A' conditioning on which yields this distribution (Baldwin et al. [37], Dubois and Prade [63]).

The mass assignment theory has been applied in some fields, such as induction of decision trees [36], computing with words among others, giving good results for real data.

Theorem 2.1. *(Baldwin et al. [37]) Let P be a probability distribution on a finite universe X taking the range of values* $\{p_1, \ldots, p_n\}$ *where* $0 \leq p_{i+1} < p_i \leq 1$ *and* $\sum_{i=1}^{n} p_i = 1$*. Then P is the least prejudiced distribution of a fuzzy set* A' *if and only if* A' *has a mass assignment given by*

$$\begin{aligned} m_{A'}(F_i) &= \mu_i - \mu_{i+1} \quad for\ i = 1, \ldots, n-1 \\ m_{A'}(F_n) &= \mu_n \end{aligned}$$

where

$$F_i = \{x \in X | P(x) \geq p_i\}$$
$$\mu_i = |F_i| p_i + \sum_{j=i+1}^{n} (|F_j| - |F_{j+1}|) p_j \tag{2.94}$$

The proof is given in (Baldwin at al. [37]).

Dubois and Prade [63] proposed a bijection method identical with the above algorithm, but it is worth mentioning that the motivation in [37] is quite different. A similar approach to mapping between probability and possibility was considered by Yager [248]. Yamada [251] has given a further justification for the transformation.

To sum up, Theorem 2.1 provides a general procedure converting a relative frequency distribution into a fuzzy set, i.e. gives us means for generating fuzzy sets from data. As the membership values of a fuzzy set univocally assign the non-membership values, Theorem 2.1 fully describes a fuzzy set.

Moreover, Theorem 2.1 gives an idea how to convert the relative frequency distributions into an intuitionistic fuzzy set. However, when discussing intuitionistic fuzzy sets we consider membership values and independent non-membership values [cf. (2.5)–(2.6)]. In result, Theorem 2.1 gives only a partial description we look for. To obtain a complete description of an intuitionistic fuzzy set (with independent membership and non-membership values), the procedure as in Theorem 2.1 should be carried out twice. Consequently, we obtain two fuzzy sets. To interpret the two fuzzy sets properly in terms of the intuitionistic fuzzy sets we recall first a semantic for membership functions.

Depending on the particular applications, Dubois and Prade [64] have explored three main semantics for membership functions. Here we make use of the interpretation proposed by Zadeh [256] when he introduced the possibility theory. Membership $\mu(x)$ is there the degree of possibility that a parameter x has the value μ (Zadeh [256]).

In effect of repeating the procedure as in Theorem 2.1 two times (first – for data representing membership values, second – for data representing non-membership values), and taking into account the interpretation that the obtained values are the degrees of possibility, we obtain the following results (Szmidt and Baldwin [162]).

- First time the algorithm from Theorem 2.1 is performed for the relative frequencies connected with membership values. In effect we obtain (fuzzy) possibilities $Pos^{+}(x) = \mu(x) + \pi(x)$ that x has the value Pos^{+}.
  $Pos^{+}(x)$ (left hand side of the above equation) means the values of a membership function for a fuzzy set (possibilities). In terms of intuitionistic fuzzy sets (right hand side of the above equation) these possibilities are equal to possible (maximal) memberships of an intuitionistic fuzzy set, i.e.
  $\mu(x) + \pi(x)$, where $\mu(x)$ – the values of the membership function for an intuitionistic fuzzy set, and $\mu(x) \in [\mu(x), \mu(x) + \pi(x)]$.
- Second time the algorithm from Theorem 2.1 is performed for the (independent) relative frequencies connected with non-membership values. In effect we obtain (fuzzy) possibilities $Pos^{-}(x) = \nu(x) + \pi(x)$ that x has not the value Pos^{-}.
  $Pos^{-}(x)$ (left hand side of the above equation) means the values of a membership function for another (than in the previous step) fuzzy set (possibilities). In terms of the intuitionistic fuzzy sets (right hand side of the above equation) these possibilities are equal to the possible (maximal) non-membership values, i.e.
  $\nu(x) + \pi(x)$, where $\nu(x)$ – the values of the non-membership function for an intuitionistic fuzzy set, and $\nu(x) \in [\nu(x), \nu(x) + \pi(x)]$.

The Algorithm of Constructing the Membership and Non-Membership Functions of Intuitionistic Fuzzy Sets (Szmidt and Baldwin [162])

1. Due to the explanations above, from Theorem 2.1 we calculate the values of the left hand sides of the equations:

$$Pos^{+}(x) = \mu(x) + \pi(x) \tag{2.95}$$

and

$$Pos^{-}(x) = \nu(x) + \pi(x). \tag{2.96}$$

2. Taking into account that $\mu(x) + \nu(x) + \pi(x) = 1$, from (2.95)–(2.96) we obtain the values $\pi(x)$

$$\begin{aligned} Pos^{+}(x) + Pos^{-}(x) = {} & \mu(x) + \pi(x) + \nu(x) + \\ & + \pi(x) = 1 + \pi(x) \end{aligned} \tag{2.97}$$

$$\pi(x) = Pos^{+}(x) + Pos^{-}(x) - 1 \tag{2.98}$$

3. Knowing the values $\pi(x)$, from (2.95) and (2.96) we obtain for each x: the values $\mu(x)$, and $\nu(x)$.

To illustrate the above procedure we will consider a simple example showing that starting from relative frequency distributions, and using Theorem 2.1, we obtain full description of an intuitionistic fuzzy set.

Example 2.7. (Szmidt and Baldwin [162]) The task is to classify products (taking into account presence of 10 different levels of an element) as legal and illegal. Relative frequencies obtained from data for legal and illegal products are respectively

- relative frequencies $p^{+}(i)$ for legal products (for each i-th level of the presence of the considered element), $i = 1, \ldots, 10$

$$\begin{aligned} & p^{+}(1) = 0., \quad p^{+}(2) = 0., \quad p^{+}(3) = 0.034, \\ & p^{+}(4) = 0.165, p^{+}(5) = 0.301, p^{+}(6) = 0.301, \\ & p^{+}(7) = 0.165, p^{+}(8) = 0.034, p^{+}(9) = 0., \\ & p^{+}(10) = 0. \end{aligned} \tag{2.99}$$

- relative frequencies $p^{-}(i)$ for illegal products (for each i-th level of the presence of the considered element), $i = 1, \ldots, 10$

$$\begin{aligned} & p^{-}(1) = 0.125, p^{-}(2) = 0.128, p^{-}(3) = 0.117, \\ & p^{-}(4) = 0.08, \ p^{-}(5) = 0.05, \ p^{-}(6) = 0.05, \\ & p^{-}(7) = 0.08, \ p^{-}(8) = 0.117, p^{-}(9) = 0.128, \\ & p^{-}(10) = 0.125 \end{aligned} \tag{2.100}$$

From the data (2.99), and Theorem 2.1 we obtain possibilities $Pos^{+}(i)$ for legal products

$$\begin{aligned}
&Pos^{+}(1) = 0., \quad Pos^{+}(2) = 0., \quad Pos^{+}(3) = 0.205, \\
&Pos^{+}(4) = 0.727, Pos^{+}(5) = 1., \quad Pos^{+}(6) = 1., \\
&Pos^{+}(7) = 0.727, Pos^{+}(8) = 0.205, Pos^{+}(9) = 0., \\
&Pos^{+}(10) = 0.
\end{aligned} \tag{2.101}$$

From the data (2.100) and Theorem 2.1 we obtain possibilities $Pos^{-}(i)$ for illegal products

$$\begin{aligned}
&Pos^{-}(1) = 1., \quad Pos^{-}(2) = 1., \quad Pos^{-}(3) = 0.961, \\
&Pos^{-}(4) = 0.737, Pos^{-}(5) = 0.503, Pos^{-}(6) = 0.503, \\
&Pos^{-}(7) = 0.737, Pos^{-}(8) = 0.961, Pos^{-}(9) = 1., \\
&Pos^{-}(10) = 1.
\end{aligned} \tag{2.102}$$

From (2.101), (2.102), and (2.98), we obtain the following values of $\pi(i)$

$$\begin{aligned}
&\pi(1) = 0., \quad \pi(2) = 0., \quad \pi(3) = 0.166, \\
&\pi(4) = 0.464, \pi(5) = 0.503, \pi(6) = 0.503, \\
&\pi(7) = 0.464, \pi(8) = 0.166, \pi(9) = 0., \\
&\pi(10) = 0.
\end{aligned} \tag{2.103}$$

Thus, (2.101) and (2.103) give $\mu(i)$

$$\begin{aligned}
&\mu(1) = 0., \quad \mu(2) = 0., \quad \mu(3) = 0.039, \\
&\mu(4) = 0.263, \mu(5) = 0.497, \mu(6) = 0.497, \\
&\mu(7) = 0.263, \mu(8) = 0.039, \mu(9) = 0., \\
&\mu(10) = 0.
\end{aligned} \tag{2.104}$$

next, from(2.102) and (2.103) we obtain $\nu(i)$

$$\begin{aligned}
&\nu(1) = 1., \quad \nu(2) = 1., \quad \nu(3) = 0.795, \\
&\nu(4) = 0.273, \nu(5) = 0., \quad \nu(6) = 0., \\
&\nu(7) = 0.273, \nu(8) = 0.795, \nu(9) = 1., \\
&\nu(10) = 1.
\end{aligned} \tag{2.105}$$

In result, making use of relative frequencies we have obtained the values μ (2.104), ν (2.105), and π (2.103) characterizing the corresponding intuitionistic fuzzy set.

Finally, we would like to emphasize the decisive difference as far as the approach discussed above is concerned, and the incorrect method of expressing an intuitionistic fuzzy set via two fuzzy sets constructed in a such way that membership values of the first fuzzy set are treated as the membership values of the intuitionistic fuzzy set, whereas membership values of the second fuzzy set are treated as the non-membership values of the same intuitionistic fuzzy set.

It is worth mentioning that the approach presented here was successfully applied for benchmark data from

UCI Machine Learning Repository (www.ics.uci.edu/ mlearn/).

The resulting intuitionistic fuzzy models were applied in:

- constructing a classifier for imbalanced and overlapping classes (cf. Szmidt and Kukier [229], [230], [231]),
- constructing intuitionistic fuzzy trees (Bujnowski [42]),

and were used for testing new approaches of calculating:

- Pearson correlation coefficient (Szmidt and Kacprzyk [211], [224]), (Szmidt et al. [220], [221], [226]),
- Spearman correlation coefficient (Szmidt and Kacprzyk [212]),
- Kendall correlation coefficient (Szmidt and Kacprzyk [222], [223]),
- Principal Component Analysis (Szmidt and Kacprzyk [225], Szmidt et al. [227]),
- ranking procedures (Szmidt and Kacprzyk [197], [198], [205], [206], [208], [214]).

2.6 Concluding Remarks

We have recalled basic definitions, and gave short introduction concerning the crisp sets, the fuzzy sets, and the intuitionistic fuzzy sets.

Two geometrical representations of the intuitionistic fuzzy sets were presented.

The interrelations among the crisp sets, the fuzzy sets, and the intuitionistic fuzzy sets were discussed.

Finally, two approaches to derivation of the intuitionistic fuzzy sets from data were presented. The first approach is by asking the experts. The second approach is automatic – starting from relative frequency distributions.

Both approaches seem to be useful. But the second one – the automatic, and mathematically justified method of deriving the intuitionistic fuzzy sets from data seems to be especially important in the context of analyzing information in big data bases. The approach has been proved to be useful in the context of widely used benchmark data.