Chapter 9
Product Theorems

In this section we investigate product Ramsey theorems. Recall that the pigeonhole principle implies that if we color $r(m-1)+1$ points with r many colors, then at least one color class contains m points. Ramsey's theorem generalizes this from points to k-subsets. Another generalization of the pigeonhole principle is from points to pairs of points:

Proposition 9.1. *Let m and r be positive integers. Then there exits an integer n such that for every r-coloring $\Delta : n \times n \to r$ there exist subsets $A \in [n]^m$ and $B \in [n]^m$ such that $\Delta \rceil A \times B$ is a constant coloring.*

Proof. Let $n_0 = r(m-1)+1$ and $n_1 = r \cdot \binom{n_0}{m} \cdot (m-1)+1$. Consider a coloring $\Delta : n_0 \times n_1 \to r$. By the pigeonhole principle, for every $i < n_1$ there exists a set $A_i \in [n_0]^m$ such that $\Delta \rceil A_i \times \{i\}$ is a constant coloring. Applying the pigeonhole principle once again on the coloring $\Delta' : n_1 \to \binom{n_0}{m} \cdot r$, given by $\Delta'(i) = \langle A_i, \Delta(A_i \times \{i\})\rangle$, there exists an m-element set $B \in [n_1]^m$ such that Δ' is constant on B. This in particular implies that for all $i, j \in B$ we have $A_i = A_j$ and all the restrictions $\Delta \rceil A_i \times \{i\}$ are constant in the same color. Choosing $n = n_1$ thus completes the proof. □

Erdős and Rado (1956) invented the so-called polarized partition arrow to abbreviate such product situations. The special case of Proposition 9.1, for example, is abbreviated by

$$\binom{n}{n} \to \binom{m}{m}_r^{1,1}.$$

In Sect. 9.1 we prove a finite product Ramsey theorem of the following form. Let m, r, t and $k_0, \ldots, k_{t-1}$ be positive integers. Then there exist positive integers $n_0, \ldots, n_{t-1}$ such that

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$$\begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_{t-1} \end{pmatrix} \to \begin{pmatrix} m \\ m \\ \vdots \\ m \end{pmatrix}_r^{k_0,k_1,\ldots,k_{t-1}},$$

meaning that for every r-coloring $\Delta : [n_0]^{k_0} \times [n_1]^{k_1} \times \ldots \times [n_{t-1}]^{k_{t-1}} \to r$ there exists sets $A_i \in [n_i]^m$, for $i < t$, such that $\Delta \rceil [A_0]^{k_0} \times [A_1]^{k_1} \times \ldots \times [A_{t-1}]^{k_{t-1}}$ is monochromatic.

In Sect. 9.2 we introduce the concept of diversification dealing with several unrestricted colorings acting on the same set. This concept turned out to be quite useful. As an application we deduce in Sect. 9.3 a product version of the finite Erdős-Rado canonization theorem originally due to Rado (1954).

9.1 A Product Ramsey Theorem

In the terminology of graph theory a rectangle $A \times B \in [n_0]^{k_0} \times [n_1]^{k_1}$ corresponds to a K_{k_0,k_1}-subgraph of the complete bipartite graph K_{n_0,n_1}. The product Ramsey theory in this special case $t = 2$ thus corresponds to the question: suppose we color K_{k_0,k_1}-subgraphs of the complete bipartite graph K_{n_0,n_1}, can we find a monochromatic $K_{m,m}$-subgraph. The following theorem shows that this is indeed true, whenever n is large enough.

Theorem 9.2 (Product Ramsey theorem). *Let t, $(k_i)_{i<t}$, m and r be positive integers. Then there exists a positive integer $n = n((k_i)_{i<t}, m, r)$ such that for every coloring $\Delta : \prod_{i<t}[n]^{k_i} \to r$ there exist m-subsets $(M_0, \ldots, M_{t-1}) \in \prod_{i<t}[n]^m$ such that*

$$\Delta(A_0, \ldots, A_{t-1}) = \Delta(B_0, \ldots, B_{t-1}),$$

for all $(A_0, \ldots , A_{t-1}), (B_0, \ldots , B_{t-1}) \in \prod_{i<t}[M_i]^{k_i}$.

Proof. We proceed by induction on t, the case $t = 1$ being Ramsey's theorem.

Let n be according to the inductive hypothesis with respect to $(k_i)_{i<t}, m$ and r, i.e.,

$$\begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} \to \begin{pmatrix} m \\ m \\ \vdots \\ m \end{pmatrix}_r^{k_0,k_1,\ldots,k_{t-1}},$$

and choose N according to Ramsey's theorem such that $N \to (m)^{k_t}_{r^p}$, where $p = \prod_{i<t}[n]^{k_i}$.

Now let $\Delta : \prod_{i<t}[n]^{k_i} \times [N]^{k_t} \to r$ be a coloring. We define $\Delta_t : [N]^{k_t} \to r^p$ by

$$\Delta_t(K_t) = \langle \Delta(K_0, \ldots, K_{t-1}, K_t) \mid (K_0, \ldots, K_{t-1}) \in \textstyle\prod_{i<t}[n]^{k_i} \rangle.$$

By choice of N there exists $M_t \in [N]^m$ such that $\Delta_t \rceil [M]^{k_t}$ is constant, which is to say that $\Delta \rceil (\prod_{i<t}[n]^{k_i} \times [M_t]^{k_t})$ is independent of the tth coordinate. Hence, by inductive hypothesis we get an $(M_0, \ldots, M_t) \in \prod_{i<t}[n]^m \times [N]^m$ monochromatic with respect to Δ. □

Notice that Theorem 9.2 remains valid if (at most) in one coordinate the m (and thus the n) is replaced by ω. However, even for $k = 1$ it becomes false if (at least) in two of the coordinates the m are replaced by ω, as the following example shows.

Let $\Delta : \omega \times \omega \to 2$ be given by

$$\Delta(x, y) = \begin{cases} 0, & \text{if } x \leq y \\ 1, & \text{otherwise.} \end{cases}$$

Then, obviously, no pair $(F_0, F_1) \in [\omega]^\omega \times [\omega]^\omega$ is colored monochromatically.

9.2 Diversification

Let $k \leq \ell$ and $\Delta_0 : [n]^k \to \omega$, $\Delta_1 : [n]^\ell \to \omega$ be colorings for some n sufficiently large. Then according to the Erdős-Rado canonization theorem (applied twice) there exists $M \in [n]^m$ such that $\Delta_0 \rceil [M]^k$ as well as $\Delta_1 \rceil [M]^\ell$ are canonical colorings. But in this way we do not get any information about dependencies between the colors used by $\Delta_0 \rceil [M]^k$ and $\Delta_1 \rceil [M]^\ell$. To obtain such information we introduce the concept of diversification:

Theorem 9.3. *Let $k \leq \ell$ and m be positive integers. Then there exists a positive integer n such that for each pair $\Delta_0 : [n]^k \to \omega$ and $\Delta_1 : [n]^\ell \to \omega$ of colorings there exists an $M \in [n]^m$ and there exists a pair $J_0 \subseteq k$ and $J_1 \subseteq \ell$ of sets such that*

(1) *$\Delta_0 \rceil [M]^k$ is canonical with respect to J_0,*
$\Delta_1 \rceil [M]^\ell$ is canonical with respect to J_1, and
(2) *Either $\Delta_0(A) \neq \Delta_1(B)$ for all $A \in [M]^k$ and $B \in [M]^\ell$ or $\Delta_0(A) = \Delta_1(B)$ if and only if $A : J_0 = B : J_1$ for all $A \in [M]^k$ and $B \in [M]^\ell$.*

Diversification, i.e., separating different colorings, was developed in Voigt (1985). In fact, more general results than Theorem 9.3 are true in this direction.

The key in proving the theorem is the following lemma for one-to-one colorings.

Lemma 9.4. *Let $i \leq j$ and m be positive integers. Then there exists a positive integer n such that for each pair $\Delta_0 : [n]^i \to \omega$ and $\Delta_1 : [n]^j \to \omega$ of one-to-one colorings there exists $M \in [n]^m$ such that one of the following possibilities holds:*

(1) $\Delta_0(A) \neq \Delta_1(B)$ *for all* $A \in [M]^i$, $B \in [M]^j$,
(2) $i = j$ *and* $\Delta_0(A) = \Delta_1(A)$ *for all* $A \in [M]^i$.

Proof. Let $m' = m + j - i$ and choose m^* such that $m^* \to (m')^j_3$. Finally, choose n such that $n \to (m^*)^j_2$.

Now assume Δ_0, Δ_1 are given as stated in the lemma. Recall that $i \leq j$ and that Δ_0 is defined on i-subsets of n. We extend Δ_0 to j-subsets of n as follows. Let $\Delta_0^1 : [n]^j \to \omega$ be defined by $\Delta_0^1(X) = \Delta_0(\{x_0, \ldots, x_{i-1}\})$, where $x_0, \ldots, x_{i-1}$ are the first i elements of X with respect to the natural order of n. Now define a coloring $\Delta^* : [n]^j \to 2$ by

$$\Delta^*(X) = \begin{cases} 1, & \text{if } \Delta_0^1(X) = \Delta_1(X) \\ 0, & \text{otherwise.} \end{cases}$$

By choice of n there exists $M^* \in [n]^{m^*}$ such that $\Delta^* \rceil [M^*]^j$ is a constant coloring. In case $\Delta^* \rceil [M^*]^j \equiv 1$ it follows from the fact that Δ_1 is one-to-one that necessarily $i = j$ and M^* thus satisfies (2).

So assume that $\Delta^* \rceil [M^*]^j \equiv 0$. Then we impose a directed graph on $[M^*]^j$ letting (X, Y) be an edge if $\Delta_0^1(X) = \Delta_1(\mathrm{Y})$. Clearly this graph has no loops and, since Δ_1 is one-to-one, the outdegree of every vertex is at most one. Therefore each connected component of this graph contains at most one cycle and hence, the underlying undirected graph is 3-colorable.

Given such a 3-coloring, by choice of m^* there exists a monochromatic m'-set $M' \in [M^*]^{m'}$. Choosing M as the first m elements of M' satisfies (1). □

Proof of Theorem 9.3. Let n' be such that the above lemma can be applied for every pair $i \leq k$ and $j \leq \ell$ and m. Further, let n be such that after applying the Erdős-Rado canonization theorem to colorings $\Delta_0 : [n]^k \to \omega$ and $\Delta_1 : [n]^\ell \to \omega$, we may assume that $\Delta_0 \rceil [n']^k$ and $\Delta_1 \rceil [n']^\ell$ are canonical colorings with respect to some $J_0 \subseteq k$ and $J_1 \subseteq \ell$, respectively.

Let $\Delta_0^* : [n']^{|J_0|} \to \omega$, resp. $\Delta_1^* : [n']^{|J_1|} \to \omega$, be such that

$$\Delta_0^*(A : J_0) = \Delta_0(A) \qquad \text{and} \qquad \Delta_1^*(B : J_1) = \Delta_1(B).$$

Observe that the assumption that $\Delta_0(A) = \Delta_0(B)$ if and only if $A : J_0 = B : J_0$ implies that Δ_0^* is well defined and one-to-one. (For sets $A^* \in [n']^{|J_0|}$ that cannot be written in the form $A : J_0$ we define $\Delta_0^*(A^*)$ arbitrarily, but so that the function remains one-to-one.) Similarly, we deduce that Δ_1^* is well-defined and one-to-one. Applying Lemma 9.4 we find some $M \in [n']^m$. If M satisfies property (1) of

Lemma 9.4, then we have for any $A \in [M]^k$ and $B \in [M]^\ell$ that $\Delta(A) = \Delta_0^*(A : J_0) \neq \Delta_1^*(B : J_1) = \Delta(B)$. Otherwise, from property (2) and the fact that Δ_0^* and Δ_1^* are one-to-one we have $\Delta(A) = \Delta_0^*(A : J_0) = \Delta_1^*(B : J_1) = \Delta(B)$ if and only if $A : J_0 = B : J_1$. Therefore M satisfies the theorem. □

Lemma 9.4 was independently obtained by Meyer auf der Heide and Wigderson (1987) in proving lower bounds for sorting networks. We have adopted some of their ideas here.

9.3 A Product Erdős-Rado Theorem

A t-dimensional version of the Erdős-Rado canonization theorem was established in Rado (1954). Loosely speaking it asserts that in each coordinate we have a canonical coloring.

Theorem 9.5. *Let t, $(k_i)_{i<t}$ and m be positive integers. Then there exists a positive integer $n = n((k_i)_{i<t}, m)$ such that for every coloring $\Delta : \prod_{i<t}[n]^{k_i} \to \omega$ there exist m-subsets $(M_0, \dots, M_{t-1}) \in \prod_{i<t}[n]^m$ and there exist (possibly empty) sets $J_i \subseteq k_i$ for $i < t$ such that*

$$\Delta(A_0, \dots, A_{t-1}) = \Delta(B_0, \dots, B_{t-1}) \quad \text{if and only if} \quad A_i : J_i = B_i : J_i \text{ for every } i < t,$$

for all $(A_0, \dots, A_{t-1})$ and $(B_0, \dots, B_{t-1}) \in \prod_{i<t}[M_i]^{k_i}$.

Proof. We proceed by induction on t, the case $t = 1$ being the Erdős-Rado canonization theorem. Let m^* be according to the inductive hypothesis with respect to $(k_i)_{i<t}$ and m. Furthermore, choose n according to the product Ramsey theorem such that

$$\begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} \to \begin{pmatrix} m^* \\ m^* \\ \vdots \\ m^* \end{pmatrix}^{k_0, k_1, \dots, k_{t-1}}_{2^{k_t}}.$$

Finally, choose N large enough so that Theorem 9.3 can be applied successively $\binom{\prod_{i<t}\binom{n}{k_i}}{2}$-times for colorings acting on k_t-sets, and yielding a set of size m after the last application of Theorem 9.3.

Let $\Delta : \prod_{i<t}[n]^{k_i} \times [N]^{k_t} \to \omega$ be a coloring. For every $\mathcal{K} = (K_0, \dots, K_{t-1})$, where $K_i \in [n]^{k_i}$ for $i < t$, let $\Delta_{\mathcal{K}} : [N]^k \to \omega$ be given by $\Delta_{\mathcal{K}}(K_t) = \Delta(K_0, \dots, K_{t-1}, K_t)$. By choice of N there exists $M_t \in [N]^m$ such that for every pair $\Delta_{\mathcal{K}}, \Delta_{\mathcal{K}'}$ the assertion of Theorem 9.3 is valid.

Observe that property (1) of Theorem 9.3 implies that for every $\mathcal{K}$ there exits a set $J_{\mathcal{K}} \subseteq k_t$ such that $\Delta_{\mathcal{K}}\rceil[M_t]^{k_t}$ is canonical with respect to $J_{\mathcal{K}}$. Define a coloring $\Delta^* : \prod_{i<t}[n]^{k_i} \to 2^{k_t}$ such that $\Delta^*(\mathcal{K}) = J_{\mathcal{K}}$ for every $\mathcal{K} = (K_0, \ldots, K_{t-1})$. By choice of n we can apply the product Ramsey theorem to find $(M_0^*, \ldots, M_{t-1}^*) \in \prod_{i<t}[n]^{m^*}$ such that there exists just one $J_t \subseteq k_t$ so that for every $\mathcal{K} \in \prod_{i<t}[M_i^*]^{k_i}$, it follows that $\Delta_{\mathcal{K}}(A) = \Delta_{\mathcal{K}}(B)$ if and only if $A : J_t = B : J_t$, whenever $A, B \in [M_t]^{k_t}$.

Finally, define a coloring $\Delta^{**} : \prod_{i<t}[M_i^*]^{k_i} \to \omega$ such that

$$\Delta^{**}(A_0, \ldots, A_{t-1}) = \Delta^{**}(B_0, \ldots, B_{t-1}) \quad \text{if and only if}$$

$$\Delta(A_0, \ldots, A_{t-1}, K_t) = \Delta(B_0, \ldots, B_{t-1}, K_t) \quad \text{for some } K_t \in [M_t]^{k_t}.$$

Observe that by property (2) of Theorem 9.3, Δ^{**} is well-defined. Then by induction hypothesis there exists $(M_0, \ldots, M_{t-1}) \in \prod_{i<t}[n]^m$ and there exist $J_i \subseteq k_i, i < t$, such that $\Delta^*(A_0, \ldots, A_{t-1}) = \Delta^*(B_0, \ldots, B_{t-1})$ if and only if $A_i : J_i = B_i : J_i$ for every $i < t$, for all $(A_0, \ldots, A_{t-1})$ and $(B_0, \ldots, B_{t-1})$ from $\prod_{i<t}[M_i]^{k_i}$.

An easy calculation shows that $(M_0, \ldots, M_{t-1}, M_t)$ and $J_0, \ldots, J_{t-1}, J_t$ satisfy Theorem 9.5. □

For more general product theorems compare, e.g., Graham and Spencer (1979) and Voigt (1985). Here we just write down the special case of Theorem 9.5 when all $k_i = 1$. This is the t-dimensional canonical pigeonhole principle.

Corollary 9.6. *Let t and m be positive integers. Then there exists a least positive integer $n = n(m, t)$ such that for every coloring $\Delta : [n]^t \to \omega$ there exist subsets $M_i \in [n]^m$, $i < t$, and there exists a (possibly empty) set $J \subseteq t$ such that*

$$\Delta(a_0, \ldots, a_{t-1}) = \Delta(b_0, \ldots, b_{t-1}) \quad \textit{if and only if} \quad a_j = b_j \;\; \textit{for all } j \in J$$

for all $(a_0, \ldots, a_{t-1}), (b_0, \ldots, b_{t-1}) \in \prod_{i<t} M_i$. □