Chapter 8
Rapidly Growing Ramsey Functions

Gödel's paper on formally undecidable propositions in first order Peano arithmetic (Gödel 1931) showed that any recursive axiomatic system containing Peano arithmetic still admits propositions which are not decidable. Gödel's original example of such a proposition was not that illuminating. It was merely a kind of formalization of the well known antinomy of the liar. This raised the problem to look for intuitively meaningful propositions which are independent of Peano arithmetic. Paris and Harrington (1977) showed that a straightforward variant of the finite Ramsey theorem is independent of Peano arithmetic, thus witnessing Gödel's first incompleteness theorem.

The original short and elegant proof of Paris and Harrington uses model theoretic tools. A different, purely combinatorial explanation of the unprovability by means of fast growing functions was given by Ketonen and Solovay (1981). In this section we present a simplification of the Ketonen-Solovay argument due to Loebl and Nešetřil (1991). We start with some background on fast growing hierarchies.

8.1 The Hardy Hierarchy

Let $\gamma_1 = \omega$ and $\gamma_{n+1} = \gamma_n^\omega$ for every $n < \omega$, i.e.,

$$\gamma_n = \omega^{\omega^{\cdot^{\cdot^{\omega}}}} \Big\}\text{n-times}.$$

Moreover set

$$\epsilon_0 = \omega^{\omega^{\cdot^{\cdot^{\cdot}}}} = \lim_{n\to\infty} \gamma_n.$$

Then ϵ_0 is the least ordinal solution to the equation $\omega^\lambda = \lambda$. Throughout this section we are only concerned with ordinals below ϵ_0.

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First note that every ordinal below ϵ_0 admits a unique representation known as the *Cantor normal form* of α:
Let $\alpha < \epsilon_0$ be a positive ordinal and k be a positive integer. Then α can be represented uniquely as

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \ldots + \omega^{\alpha_k} \cdot n_k,$$

where $\alpha > \alpha_1 > \alpha_2 > \ldots > \alpha_k \geq 0$ are ordinals and $n_1, \ldots, n_k$ are positive integers.
Such a coding of ordinals $\alpha < \epsilon_0$ by positive integers can be defined straightforwardly, compare for example Schütte (1977).

Next we define *fundamental sequences* which we will subsequently use in order to define the *Hardy hierarchy*. We need these fundamental sequences in order to handle limit ordinals properly. To every limit ordinal $\alpha < \epsilon_0$ we associate a strictly monotone sequence $\alpha[n]$, $n < \omega$, which approaches α from below. If $\alpha < \epsilon_0$ is given in Cantor normal form $\alpha = \alpha' + \omega^{\alpha_k} \cdot n_k$, where α_k is the minimal exponent, let

$$\alpha[n] = \begin{cases} \alpha' + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\alpha_k[n]}, & \text{if } \alpha_k \text{ is a limit ordinal,} \\ \alpha' + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\alpha_k - 1} \cdot (n + 1), & \text{if } \alpha_k \text{ is a successor ordinal.} \end{cases}$$

For example, $\omega[n] = n + 1$, $\omega^{\omega}[n] = \omega^{n+1}$, $\omega^{k+1}[n] = \omega^k \cdot (n + 1)$, and $\omega^k \cdot (k + 1)[n] = \omega^k \cdot k + \omega^{k-1} \cdot (n + 1)$.

With the help of these fundamental sequences we define functions $H_\alpha(\cdot)$ for all $\alpha < \epsilon_0$:

$$\begin{aligned} H_0(n) &= n, \\ H_{\alpha+1}(n) &= H_\alpha(n + 1), \\ H_\alpha(n) &= H_{\alpha[n]}(n) \qquad \text{for limit ordinals.} \end{aligned}$$

Finally, define H_{ϵ_0} by

$$H_{\epsilon_0}(n) = H_{\gamma_n}(n).$$

This is the *Hardy hierarchy,* introduced by Wainer (1972). This hierarchy is based on a sequence of functions first defined by Hardy (1904) to construct sets of real numbers of cardinality $\aleph_1$. It is not difficult to see that each H_α is strictly increasing and $H_\alpha(n) < H_{\alpha+1}(n)$ for every nonnegative integer n.

The significance of the Hardy hierarchy in connection with unprovability results stems from the following theorem, cf. Wainer (1970, 1972) and Buchholz and Wainer (1987).

Theorem 8.1. *Let $f : \omega \to \omega$ be a provably total and recursive function (provably total with respect to Peano arithmetic). Then f is eventually dominated by some H_α for an $\alpha < \epsilon_0$. Moreover, H_{ϵ_0} eventually dominates every provably total recursive function but it itself is not provably total.*

8.2 Paris-Harrington's Unprovability Result

A set $L \subseteq \omega$ is called large, if $L \neq \emptyset$ and $\min L \leq |L|$. So $\{4, 5, 6, 7\}$ is a large set but not $\{4, 10, 15\}$. Let k, n and r be positive integers. With this terminology at hand we can state the following variation of the classical Ramsey theorem that follows from the infinite Ramsey theorem using a compactness argument.

Theorem 8.2. *Let k and r be positive integers. Then there exists a least positive integer $n = PH(k, r)$ such that for every r-coloring $\Delta : [n]^k \to r$ there exists a large subset $L \subseteq n$ with $|L| > k$ such that $\Delta \rceil [L]^k$ is a constant coloring.* □

While for the classical Ramsey theorem it is difficult to obtain *tight* bounds it will turn out that for this seemingly small variation of the classical Ramsey theorem it is already difficult to obtain *any* kind of bound.

Theorem 8.3 (Paris and Harrington). *The statement*

(PH) for every pair k, r of positive integers there exists a least positive integer $n = PH(k, r)$ such that for every r-coloring $\Delta : [n]^k \to r$ there exists a large subset $L \subseteq n$ with $|L| > k$ such that $\Delta \rceil [L]^k$ is a constant coloring

is not provable in Peano arithmetic.

For the reader who is not used to work in Peano arithmetic we mention that for statements about natural numbers Peano arithmetic is equivalent to the result of replacing the axiom of infinity by its negation in the usual axioms of Zermelo-Fraenkel set theory (see, e.g., Jech (1978) for these axioms). Obviously, the principle (PH) can be formulated in this theory. In this way Theorem 8.3 should be understood as: the formula of Peano arithmetic corresponding to the principle (PH) is not provable in Peano arithmetic.

Intuitively, a reason for the unprovability of (PH) in Peano arithmetic is that the function $PH(k, r)$ grows too rapidly. Recall that a recursive function $f : \omega \to \omega$ is provably recursive if one can show in Peano arithmetic that f is total, i.e., defined for all natural numbers. Now it turns out that the function $PH(k, k)$ grows faster than any provably recursive function f, i.e., $f(k) < PH(k, k)$ for all but finitely many k. However, by Theorem 8.2 the function $PH(k, k)$ is total, hence, (PH) is not provable in Peano arithmetic.

The aim of this section is to prove the Paris-Harrington result by purely combinatorial means following an approach of Ketonen and Solovay (1981). Here we follow a simplified approach by Loebl and Nešetřil (1991).

Let $\alpha < \epsilon_0$ be an ordinal and let $\alpha = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \ldots + \omega^{\alpha_k} \cdot n_k$ be the Cantor normal form of α. Let $S_i(\alpha) = \omega^{\alpha_i} \cdot n_i$ be the ith summand in the Cantor normal form of α, let $C_i(\alpha) = n_i$ be the coefficient of the ith summand and let $E_i(\alpha) = \alpha_i$ be the corresponding exponent. If $\gamma_{h-1} \leq \alpha < \gamma_h$ then α is said to be of height h which is abbreviated by $h(\alpha) = h$. The weight $w(\alpha)$ of α is defined recursively as follows:

$$w(\alpha) = \begin{cases} \alpha, & \text{if } \alpha \text{ is an integer,} \\ \max\{n_1, \ldots, n_k, w(\alpha_1), \ldots, w(\alpha_k), k\}, & \text{otherwise.} \end{cases}$$

Let n be an integer. Then (α, n) is called a *good pair* if $n > w(\alpha) + h(\alpha)$. Let (α, n) be a good pair. We define a predecessor function $R(\alpha; n)$ as follows.

$$R(\alpha; n) = \begin{cases} (\alpha - 1; n + 1), & \text{if } \alpha \text{ is a successor ordinal,} \\ (\alpha[n - h(\alpha)]; n + 1), & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Since $w(\alpha[n - h(\alpha)]) \leq \max\{w(\alpha) + 1,\ n - h(\alpha)\}$, it follows that $R(\alpha; n)$ again is a good pair. As an example, consider γ_h, the stack of h many ω's. Observe that $h(\gamma_h) = h + 1$. Hence, $(\gamma_h;\ h + 3)$ is a good pair and so is $R(\gamma_h;\ h + 3) = (\gamma_h[2];\ h + 4)$.

Let $R^0(\alpha; n) = (\alpha; n)$ and $R^{k+1}(\alpha; n) = R(R^k(\alpha; n))$. By $\mathcal{R}(\alpha; n)$ we denote the family of all pairs which can be generated by successively applying this predecessor operation, i.e., $\mathcal{R}(\alpha; n) = \{R^i(\alpha; n) \mid i < \omega\}$. Finally, let $r(\alpha, n) = |\mathcal{R}(\alpha; n)|$. This function can be related to the Hardy hierarchy.

Lemma 8.4. *Let $\alpha < \epsilon_0$ be an ordinal and let n be a non-negative integer. Then*

$$r(\alpha,\ n + h(\alpha)) \ \geq\ H_\alpha(n) - n.$$

Proof. We apply transfinite induction on α. Obviously, for every natural number k, $r(k, n)$ is the length of the sequence $(k, n), (k - 1, n + 1), \ldots, (0, n + k)$. Thus $r(k, n + 1) = k + 1 > H_k(n) - n = k$.

In the induction step we have either $\alpha + 1$ being a successor ordinal. i.e.,

$$\begin{aligned} r(\alpha + 1, n + h(\alpha)) &= 1 + r(\alpha, n + h(\alpha) + 1) \\ &\geq 1 + H_\alpha(n + 1) - n - 1 \\ &= H_{\alpha+1}(n) - n, \end{aligned}$$

or α being a limit ordinal and therefore

$$r(\alpha,\ n + h(\alpha)) = r(\alpha[n],\ n + h(\alpha) + 1) \ \geq\ H_{\alpha[n]}(n) - n \ =\ H_\alpha(n) - n,$$

as claimed. □

A family of good pairs is called a *good family*. If there is a member of such a family of height h and, moreover, the height of each member is at most h then this family is said to be a *good family of height h*. A good family $(\beta_0; n_0), \ldots, (\beta_{t-1}; n_{t-1})$ is *monotone* if $\beta_i > \beta_j$ and $n_i < n_j$ for every pair $0 \leq i < j < t$. For instance, $\mathcal{R}(\gamma_h; h + 3)$ is a monotone family of height $h + 1$ for every $h < \omega$.

The following coloring lemma plays the key rôle in the proof of the Paris-Harrington result.

Lemma 8.5. *Let $h \geq 2$ be an integer and let $S = \{(\beta_0; n_0), \ldots, (\beta_{t-1}; n_{t-1})\}$ be a good family of height h such that $n_i > h + 1$ for every $i < t$. Then there exists a coloring of the $(h + 1)$-subsets of S with less than 3^h colors such that no monotone subfamily $S' = \{(\alpha_0; m_0), \ldots, (\alpha_{s-1}; m_{s-1})\}$ of S of size $|S'| > m_0$ is monochromatic.*

Proof. Let $\epsilon_0 > \alpha_0 > \alpha_1 > \alpha_2$ be ordinals in Cantor normal form. Then let $\Delta(\alpha_0, \alpha_1) = \min\{i \mid S_i(\alpha_0) \neq S_i(\alpha_1)\}$ be the index of the largest summand where α_0 and α_1 differ. Recall that $C_i(\alpha_1), C_i(\alpha_2)$ denotes the coefficient of the ith summand of α_1, α_2, respectively. We define $\delta(\alpha_0, \alpha_1, \alpha_2) < 3$ as follows.

$$\delta(\alpha_0, \alpha_1, \alpha_2) = \begin{cases} 0, & \text{if } \Delta(\alpha_0, \alpha_1) > \Delta(\alpha_1, \alpha_2), \\ 1, & \text{if } x = \Delta(\alpha_0, \alpha_1) \leq \Delta(\alpha_1, \alpha_2) = y \text{ and } C_y(\alpha_1) < C_x(\alpha_0), \\ 2, & \text{otherwise.} \end{cases}$$

Iterating this scheme we associate to every strictly monotone decreasing sequence $\alpha = (\alpha_0, \ldots, \alpha_{t-1})$ of ordinals a vector $\delta(\alpha) = (\delta_0, \ldots, \delta_{t-3}) \in 3^{t-2}$ where $\delta_i = \delta(\alpha_i, \alpha_{i+1}, \alpha_{i+2})$.

Let $S = \{(\beta_0; n_0), \ldots, (\beta_{t-1}; n_{t-1})\}$ be a good family of height h such that $\beta_0 > \ldots > \beta_{t-1}$. We define a coloring of the $(h + 1)$-subsets of S by induction on h.

First assume that S is of height 2. Then color every monotone 3-element subset $\{(\alpha_0; m_0), (\alpha_1; m_1), (\alpha_2; m_2)\}$ of S with color $\delta(\alpha_0, \alpha_1, \alpha_2)$. This is clearly a 3-coloring. Assume that $S' = \{(\alpha_0; m_0), \ldots, (\alpha_{s-1}; m_{s-1})\}$ is a monotone subfamily of S which is monochromatic. Recalling the definition of $w(\alpha_0)$ and the fact that $m_0 > w(\alpha_0) + h(\alpha_0)$ we show in the following that $|S'| \leq m_0$. The assumption that S' is monochromatic with color 0 implies that $|S'|$ is bounded by the number of summands in the Cantor normal form of α_0 plus one. The assumption that S' is monochromatic with color 1 implies that $|S'|$ is at most one more than the size of the coefficient of the largest summand in the Cantor normal form of α_0. Finally, the assumption that S' is monochromatic with color 2 implies that $|S'|$ is bounded by the size of the exponent of the first summand in the Cantor normal form of α_0 plus one. This is because $h(\alpha_0) = 2$, i.e., the exponent is an integer.

Next assume the validity of the lemma for all good families of height h for some $h \geq 2$ and assume that S is of height $h + 1$ and therefore $n_i > h + 2$ for every $i < t$.

We associate a family $H(S)$ of height h to S as follows. To any 2-subset of S, say $\{(\alpha_0; m_0), (\alpha_1; m_1)\}$, we associate a pair $(\eta_0; p_0)$ choosing $p_0 = m_0 - 1$ and $\eta_0 = E_x(\alpha_0)$ where $x = \Delta(\alpha_0, \alpha_1)$. Observe that each such pair is a good pair of height at most h. Let $H(S)$ be the set of all pairs which can be obtained this way. Then $H(S)$ is a good family and $p > h + 1$ for every pair $(\eta, p) \in H(S)$ is valid. Without loss of generality we can assume that $H(S)$ is of height h. Hence by inductive assumption there exists a coloring of the $(h+1)$-subsets of $H(S)$ with less than 3^h colors such that no monotone subfamily $H' = \{(\eta_0; p_0), \ldots, (\eta_{r-1}; p_{r-1})\}$ of $H(S)$ of size $|H'| > p_0$ is monochromatic.

Now color the monotone $(h+2)$-subfamilies of S as follows. Let $T = \{(\alpha_0; m_0), \ldots, (\alpha_{h+1}; m_{h+1})\}$ be such a family. Color T with $\delta(\alpha_0, \ldots, \alpha_{h+1}) \in 3^h$ if $\delta(\alpha_0, \ldots, \alpha_{h+1}) \neq (2, \ldots, 2)$. Otherwise consider the $(h+1)$-subfamily $H(T) = \{(\eta_0; p_0), \ldots, (\eta_h; p_h)\}$ of $H(S)$ and color T with the color assigned to $H(T)$ by the inductive assumption.

Obviously, this defines a coloring of all $(h+2)$-subfamilies of S with less than $2 \cdot 3^h < 3^{h+1}$ many colors. Assume that $S' = \{(\alpha_0; m_0), \ldots, (\alpha_{s-1}; m_{s-1})\}$ is a monotone subfamily of S which is monochromatic. Assume that S' is monochromatic in some color $\delta \in 3^h$ which is not a constant vector. Then $|S'| \leq h+2$, but $m_0 > h+2$. If S' is monochromatic with color $(0, \ldots, 0) \in 3^h$ or with color $(1, \ldots, 1) \in 3^h$ similar arguments as in the case $h = 2$ show that $|S'| \leq w(\alpha_0) + 1$ but $m_0 > w(\alpha_0) + h(\alpha_0)$. It remains to consider the case that S' is monochromatic with color $(2, \ldots, 2) \in 3^h$. But then the family $H(S') = \{(\eta_0; p_0), \ldots, (\eta_{s-2}; p_{s-2})\}$, where $p_i = m_i - 1$ and $\eta_i = E_x(\alpha_i)$ with $x = \Delta(\alpha_0, \alpha_1)$ for every $i \leq s-2$, is a monotone subfamily of $H(S)$ and, by definition of the coloring of S, monochromatic. Hence, by inductive assumption, $|H(S')| \leq p_0 = m_0 - 1$ and so $|S'| \leq m_0$. □

Lemma 8.6. *Let $h \geq 2$. Then*

$$PH(h+2,\ 3^{h+1} + 2h)\ >\ H_{\gamma_h}(h) + h.$$

Proof. Consider the monotone family $\mathcal{R}(\gamma_h; 2h+1) = \{(\alpha_0; m_0), \ldots, (\alpha_{t-1}; m_{t-1})\}$. Obviously, $\alpha_0 = \gamma_h$, $m_0 = 2h+1$ and $m_{i+1} = m_i + 1$ for every $i < t-1$. By Lemma 8.4 we have that $t \geq H_{\gamma_h}(h) - h$. Since $\mathcal{R}(\gamma_h; 2h+1)$ is a monotone family of height $h+1$ and $m_i > h+2$ for every $i < t$, by Lemma 8.5 there exists a coloring of the $(h+2)$-subsets of $\mathcal{R}(\gamma_h; 2h+1)$ with less than 3^{h+1} colors such that no monotone subfamily $S' \subseteq S$ of size $|S'| > 2h+1$ is monochromatic. This induces obviously a coloring of the $(h+2)$-subsets of $M = \{0, \ldots, 2h, 2h+1, \ldots, m_{t-1}\}$ with $3^{h+1} + 2h$ many colors having the property that there is no large subset of L which is monochromatic. Hence

$$\begin{aligned} PH(h+2,\ 3^{h+1} + 2h) &\geq m_{t-1} + 1\ \geq\ 2h + t + 1 \\ &\geq H_{\gamma_h}(h) + h + 1. \end{aligned}$$

□

Proof of Theorem 8.3.

$$PH(h+2,\ 3^{h+1} + 2h)\ >\ H_{\gamma_h}(h) + h + 1\ \geq\ H_{\epsilon_0}(h),$$

and so, by Theorem 8.1, $PH(h, h)$ is not a provably total and recursive function.

□

It should be mentioned that other variants of Ramsey-type theorems give rise to functions which grow even much faster than the Paris-Harrington function. For example, in Prömel et al. (1991) fast growing functions based on Ramsey's theorem are investigated which grow faster than any recursive function which can proved to be total in the formal system ATR_0.