Chapter 14
Hypergraphs on Parameter Sets

So far in this chapter, we have studied graphs which are defined on sets. Now we start studying (hyper)graphs which are defined on more complex structures. In particular, in this section we study hypergraphs on parameter sets.

In Sect. 14.1 we prove an induced version of Hales-Jewett's theorem and, as corollaries, we obtain results for sets of integers carrying an arithmetic structure like, e.g., arithmetic progressions or (m, p, c)-sets. In Sect. 14.2 we give an alternative proof of the Ramsey theorem for finite ordered graphs (Theorem 12.13). Though it doesn't exactly fit the theme of the section, it will serve us as a motivating example for a technique which we will then use in Sect. 14.3 to prove an induced version of the Graham-Rothschild's theorem on parameter sets. The induced Graham-Rothschild's theorem gives, in a sense, a complete analogue of the Ramsey theorem for finite ordered graphs.

Before we state these results, we first fix some notation. Given an alphabet A and integers k and n we build an (ordered) hypergraph $\mathcal{H}^k(n)$ as follows. The vertices are all words of length n over A, i.e. $V(\mathcal{H}^k(n)) = A^n$. The set of edges is given by all i-parameter words in $[A]\binom{n}{i}$, for all $0 \le i \le k$. More precisely, every $f \in [A]\binom{n}{i}$ corresponds to a hyperedge e_f given by

$$e_f = \{f \cdot g \mid g \in A^i\},$$

and

$$E(\mathcal{H}^k(n)) = \bigcup_{\substack{0 \le i \le k \\ f \in [A]\binom{n}{i}}} e_f.$$

Note that we do allow $i = 0$ in the above definition, i.e., all vertices of $\mathcal{H}^k(n)$ are also considered as edges. In this section we will mostly be concerned with finding an appropriate subgraph $\mathcal{F}$ of $\mathcal{H}^k(n)$ that has some nice Ramsey properties. It is important to note that here we do consider weak subgraphs. That is, a subgraph $\mathcal{F} \subseteq \mathcal{H}^k(n)$ can have the property that some vertices of $V(\mathcal{F})$ do not belong to $E(\mathcal{F})$.

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Similarly as in the graph case, for a hypergraph $\mathcal{F} \subseteq \mathcal{H}^k(n)$ and a subset $\mathcal{A} \subseteq A^n$, we denote by $\mathcal{F}[\mathcal{A}]$ the subgraph of $\mathcal{F}$ induced by $\mathcal{A}$, i.e., the vertex set $V(\mathcal{F}[\mathcal{A}])$ is given by $V(\mathcal{F}) \cap \mathcal{A}$ and for all $e_f \in E(\mathcal{F})$ we have

$$e_f \in \mathcal{F}[\mathcal{A}] \quad \text{if and only if} \quad e_f \subseteq \mathcal{A}.$$

We will mostly be interested in subgraphs induced by an m-subspace of A^n, i.e., by some $f \in [A]\binom{n}{m}$. To shorten notation we use $\mathcal{F}[f]$ to denote the subgraph induced by such an m-space:

$$\mathcal{F}[f] := \mathcal{F}[\{f \cdot g \mid g \in A^m\}].$$

14.1 An Induced Hales-Jewett Theorem

For this section let A be a finite set containing at least two elements. As Hales-Jewett's theorem itself, induced versions of Hales-Jewett's theorem consider colorings of A^n, i.e., of vertices. Without loss of generality, we restrict to colorings of vertices which exist as hyperedges.

More precisely, for hypergraphs $\mathcal{E} \subseteq \mathcal{H}^k(m)$ and $\mathcal{F} \subseteq \mathcal{H}^k(n)$, let the Ramsey arrow $\mathcal{F} \to (\mathcal{E})^0_r$ abbreviate the following statement:
For every coloring $\Delta : A^n \to r$ there exists $f \in [A]\binom{n}{m}$ such that $\mathcal{F}[f]$ is isomorphic to $\mathcal{E}$ and $\Delta(f \cdot y) = \Delta(f \cdot x)$ for all $x, y \in A^m$ with $e_x, e_y \in E(\mathcal{E})$.

Note that we require monochromaticity only for those vertices that form an edge in $\mathcal{E}$. Clearly, if all vertices form edges then we get monochromaticity in the usual sense. It is an easy observation that $\mathcal{F}[f]$ is isomorphic to $\mathcal{E}$ if and only if for every $g \in [A]\binom{m}{i}$, $i \leq k$, we have $e_{f \cdot g} \in E(\mathcal{F})$ iff $e_g \in E(\mathcal{E})$. Note that this condition needs to hold for all edges, also those which form vertices.

With this notation at hand, we can state the induced version of Hales-Jewett's theorem:

Theorem 14.1 (Induced Hales-Jewett theorem). *Let r, m and k be positive integers and let $\mathcal{E} \subseteq \mathcal{H}^k(m)$ be given. Then there exists a positive integer n and a subgraph $\mathcal{F} \subseteq \mathcal{H}^k(n)$ such that $\mathcal{F} \to (\mathcal{E})^0_r$.*

Recall that with respect to ordinary graphs the corresponding vertex partition theorem can be established using a simple product construction (cf. Sect. 12.1). Essentially the same idea applies here.

Convention. Recall that in Sect. 4.1 we introduced $\times$ to concatenate two parameter words. In order to get a subspace whose dimension is the sum of the two subspaces we there shifted the parameters in the second word. In this section we only need the *formal* concatenation of two parameter words. With abuse of notation we thus let $\times$ denote in this section the formal concatenation, i.e., for $g = (g_0, \dots, g_{m-1}) \in [A]\binom{m}{k}$ and $h = (h_0, \dots, h_{\tilde{m}-1}) \in [A]\binom{\tilde{m}}{i}$ with $i \leq k$ we let

$g \times h = (g_0, \ldots, g_{m-1}, h_0, \ldots, h_{\tilde{m}-1}) \in [A]\binom{m+\tilde{m}}{k}$. In this section we will mostly be concerned with a product space given by the concatenations of a set of parameter words. More precisely, for $B \subseteq \bigcup_{i \leq k}[A]\binom{m}{i}$ let $(B)_M = \{f_0 \times \ldots \times f_{M-1} \mid f_i \in B\}$ which, then, is a subset of $\bigcup_{i \leq k}[A]\binom{m \cdot M}{i}$.

Proof of Theorem 14.1. Consider the set $B = \{x \in A^m \mid e_x \in E(\mathcal{E})\}$. According to Hales-Jewett's theorem (Theorem 4.2) let the positive integer N be such that $N \geq HJ(|B|, 1, r)$. Let $n = N \cdot m$.

We define $\mathcal{F} \subseteq \mathcal{H}^k(n)$ as follows. The vertex set of $\mathcal{F}$ is $V(\mathcal{F}) = A^n$, i.e., it is identical to that of $\mathcal{H}^k(n)$. For $g \in [A]\binom{m}{i}$ such that $e_g \in E(\mathcal{E})$, add $e_{\tilde{g}}$ to $E(\mathcal{F})$ for all $\tilde{g} \in (B \cup \{g\})_N$ such that $\tilde{g}_j = g$ for some $j < N$. Note that in this case $\tilde{g} = (g_0, \ldots, g_{N-1})$ is an element of $[A]\binom{n}{i}$. It remains to verify that $\mathcal{F} \to (\mathcal{E})^0_r$.

Let $\Delta : A^n \to r$ be a coloring. As $(B)_N \subseteq A^n$, by abuse of language this can be viewed as a coloring $\Delta : B^N \to r$. By choice of N there exists a one-parameter word $\tilde{f} \in [B]\binom{N}{1}$ such that the set $\{\tilde{f} \cdot x \mid x \in B\}$ is monochromatic with respect to Δ. Consider an m-parameter word $f \in [A]\binom{n}{m}$ defined as $f = \tilde{f}_0^* \times \ldots \times \tilde{f}_{N-1}^*$ where

$$\tilde{f}_i^* = \begin{cases} \tilde{f}_i, & \text{if } \tilde{f}_i \in B, \\ (\lambda_0, \ldots, \lambda_{m-1}), & \text{if } \tilde{f}_i = \lambda_0. \end{cases}$$

It is clear from the construction of f that $\Delta(f \cdot x) = \Delta(f \cdot y)$ for every $x, y \in A^m$ such that $e_x, e_y \in E(\mathcal{E})$. Moreover, for every $g \in [A]\binom{m}{i}$ we have $f \cdot g \in (B \cup \{g\})_N$, thus $e_{f \cdot g} \in E(\mathcal{F})$ iff $e_g \in E(\mathcal{E})$ and so $\mathcal{F}[f]$ is isomorphic to $\mathcal{E}$. □

14.1.1 Applications

Apparently (Spencer 1975b) first considered induced partition theorems for other structures than graphs defined on sets, by proving an induced version of van der Waerden's theorem on arithmetic progressions. We have seen in Sect. 4.2.1 that van der Waerden's theorem on arithmetic progressions can be easily deduced from Hales-Jewett's theorem. Basically following the lines of this proof we show how an induced version of van der Waerden's theorem can be deduced from Theorem 14.1.

Theorem 14.2 (Induced van der Waerden). *Let r and m be positive integers and let $\mathcal{E} = (m, E)$ be a hypergraph on the vertex set m. Then there exists a positive integer n and a hypergraph $\mathcal{F} = (n, F)$ on the vertex set n, such that for every r-coloring $\Delta : n \to r$ there exists an arithmetic progression $A = \{a + j \cdot b \mid 0 \leq j < m\} \subseteq n$ such that*

(1) *The subgraph of $\mathcal{F}$ spanned by A is isomorphic to $\mathcal{E}$, and*
(2) *$\Delta\rceil\{a + j \cdot b \mid j < m \text{ and } j \in E\}$ is a constant coloring.*

Remark 14.3. Observe that Theorem 14.2 generalizes the particular case of vertex colorings from the Ramsey theorem for ordered graphs in a somewhat unexpected direction. Considering hypergraphs whose vertex sets are integers (i.e., carry an arithmetic structure) the additional requirement is that the vertex set of the monochromatic hypergraph forms an arithmetic progression.

Proof of Theorem 14.2. Let the positive integer r and the hypergraph $\mathcal{E} = (m, E)$ be given. Let $A = m$ and consider the hypergraph $\mathcal{E}_0 \subseteq \mathcal{H}^0(1)$ such that $i \in E(\mathcal{E}_0)$ if and only if $i \in E(\mathcal{E})$, for $i < m$. Now we apply Theorem 14.1 and find a positive integer n_0 and a hypergraph $\mathcal{F}_0 \subseteq \mathcal{H}^0(n_0)$ such that $\mathcal{F}_0 \to (\mathcal{E}_0)^0_r$. Let $n = m^{n_0}$ and recall that $\mathcal{P}(n)$ denotes the power set of n. We define the required hypergraph $\mathcal{F}$ with vertex set $n = \{0, \ldots, n-1\}$ and edges $E(\mathcal{F}) \subseteq \mathcal{P}(n)$ as follows.

Let $\varphi : A^{n_0} \to n$ be such that $\varphi(a_0, \ldots, a_{n_0-1}) = \sum_{i<n_0} a_i \cdot m^i$. Note that φ is a bijection. For every $(a_0, \ldots, a_{n_0-1}) \in A^{n_0}$ let $\varphi(a_0, \ldots, a_{n_0}) \in E(\mathcal{F})$ iff $(a_0, \ldots, a_{n_0-1}) \in E(\mathcal{F}_0)$. Furthermore, for every $f \in [A]\binom{n_0}{1}$ and $J \in \mathcal{P}(m)$, $|J| \geq 2$, let

$$\{\varphi(f \cdot j) \mid j \in J\} \in E(\mathcal{F}) \quad \text{if and only if} \quad J \in E(\mathcal{E}),$$

where $f \cdot j$ refers to composition of parameter words. Observe that $\mathcal{F}$ is well-defined since any two distinct one-parameter sets intersect in at most one point and the mapping φ is a bijection. It remains to verify that the hypergraph $\mathcal{F}$ has the desired properties.

Let $\Delta : n \to r$ be an r-coloring. This defines a coloring $\Delta^* : A^{n_0} \to r$ by $\Delta^*(a_0, \ldots, a_{n_0-1}) = \Delta(\sum_{i<n_0} a_i \cdot m^i)$. By choice of the parameter-graph $\mathcal{F}_0$ there exists $f \in [A]\binom{n_0}{1}$ such that $\mathcal{E}_0$ is isomorphic to $\mathcal{F}_0[f]$ and

$$\Delta^* \rceil \{f \cdot j \mid j < m \text{ and } j \in E(\mathcal{E}_0)\} = \Delta \rceil \{\varphi(f \cdot j) \mid j < m \text{ and } j \in E(\mathcal{E})\}$$

is a constant coloring. By construction, then, the arithmetic progression $A = \{\varphi(f \cdot j) \mid j < m\}$ has the desired properties. □

Note that in the above proof the induced version of Hales-Jewett is only applied to the subhypergraph of $\mathcal{E}$ that contains exactly all singleton edges. For the case that all vertices of $\mathcal{E}$ do form an edge one easily checks that the use of the induced Hales-Jewett theorem can be replaced by applying just the classical Hales-Jewett theorem.

Recall that a subset $M \subseteq \mathbb{Z}$ is an (m, p, c)-set if there exist integers $x_0, \ldots, x_m$ such that $M = M_{p,c}(x_0, \ldots, x_m) = \{cx_i + \sum_{j=i+1}^{m} \xi_j x_j \mid -p \leq \xi_j \leq p,\ \xi_j \in \mathbb{Z}$ for $j = 1, \ldots, m\}$. As seen in Chap. 2, (m, p, c)-sets are a basic tool in studying partition regular systems of equations. Thereby, arithmetic progressions can be viewed as special (m, p, c)-sets, in fact as $(1, p, 1)$-sets. Extending the method of proof used for the induced van der Waerden theorem, Deuber et al. (1982) proved an induced partition theorem for (m, p, c)-sets.

Theorem 14.4. *Let m, p, c and r be positive integers and let (M, E) be a hypergraph on the set $M = M_{p,c}(x_0, \dots, x_m)$. Then there exist positive integers n, q, d and there exists a hypergraph (N, F) on the set $N = M_{q,d}(x_0, \dots, x_n)$ such that for every r-coloring $\Delta : N \to r$ there exists an (m, p, c)-subset $M' \subseteq N$ such that the subgraph of (N, F) spanned by M' is isomorphic to (M, E) and such that $\Delta \rceil \{x \in M' \mid x \in F\}$ is a constant coloring.* □

The proof basically combines ideas from the proof of the (non induced) partition theorem for (m, p, c)-set (cf. Sect. 4.2.3) and the induced Hales-Jewett resp. van der Waerden theorem. We omit this proof.

14.2 Colorings of Subgraphs: An Alternative Proof

We now reprove Theorem 12.13, the Ramsey theorem for ordered graphs. Instead of using a powerful tool like the Graham-Rothschild theorem for parameter sets (as we did in Sect. 12.3), we now give an elementary proof that uses only Ramsey's theorem and a clever construction. This proof is due to Prömel and Voigt (1989). Recall that the Ramsey theorem for ordered graphs states that for any two ordered two finite graphs $(H, \leq)$ and $(F, \leq)$ and any positive integer r there exists a finite ordered graph $(G, \leq)$ such that

$$(G, \leq) \overset{\text{ind}}{\to} (F, \leq)_r^{(H,\leq)}.$$

Throughout the remainder of this section we assume that all graphs are supplied with an underlying vertex ordering, and that all embeddings and subgraphs respect this ordering, but for ease of notation we will not state these orderings explicitly. In this section the term 'subgraph' also always refers to an *induced* subgraph. In particular, we only color H-subgraphs that are induced H-copies.

Let us first give a high-level overview of our proof strategy. Instead of looking directly for an F-subgraph in G which is monochromatic with respect to H-subgraphs, we define another graph F_0. We want that, roughly speaking, F_0 has the following property: if there exists an F_0-subgraph such that the coloring of its H-subgraphs satisfy a certain condition which is, this is the crucial point, much weaker than being monochromatic, then we are guaranteed to find a monochromatic F-subgraph in F_0. Additionally, F_0 will have a strong structural property, namely it is *partite*, which, as we will see, conveniently allows us to find a desired F_0-subgraph iteratively.

14.2.1 Partite Graphs

As usual in graph theory, we say that a graph is *m-partite* if its vertex set can be split into m mutually disjoint and nonempty sets, each inducing an independent set. We impose another strong structural property, namely that it is *left-rectified*.

Definition 14.5. A left-rectified m-partite graph is a pair $((V_\nu)_{\nu<m}, E)$, where $V = \bigcup_{i<m} V_i$ is the set of vertices (we assume that the sets V_i are nonempty and mutually disjoint) and

(1) Each V_i induces an independent set, i.e., no edge has both endpoints in the same set V_i,
(2) If $a \in V_{i'}$ and $b \in V_i$ for $i' < i$, then $a \leq b$,
(3) If $\{a, b\} \in E$ for some $a \leq b$ and $a \in V_i$, then $\{a', b\} \in E$ for every $a' \in V$.

Henceforth, we will also call the sets V_i the *parts* of the partition $V = \bigcup_{i<m} V_i$.

Naturally, we want that embeddings of partite graphs preserve the ordering of vertices as well as respect partitions.

Definition 14.6. Let $G = ((V_\nu)_{\nu<m}, E)$ and $F = ((\tilde{V}_\nu)_{\nu<\tilde{m}})$ be partite graphs. We call a subgraph F of G a *partite F-subgraph of G* if it satisfies the following three conditions: (i) $G[V(F)]$ is isomorphic to F, i.e., F is an *induced* F-subgraph of G, (ii) every part of F is a subset of some part of G and (iii) no two parts of F are subsets of the same part of G. By $\binom{G}{F}_{part}$ we denote the set of all partite F-subgraphs of G.

We say that an m-partite graph is *crossing* if $|V_\nu| = 1$ for every $\nu < m$. Note that every graph on m vertices can be viewed as a crossing m-partite graph. Note also that a crossing m-partite graph can easily made left-rectified by ordering the parts in such a way that (2) is satisfied.

Lemma 14.7 (Partite lemma). *Let F and H be left-rectified m-partite graphs with H being crossing, and let r be a positive integer. Then there exists a left-rectified m-partite graph G such that $G \overset{part}{\rightarrow} (F)^H_r$, meaning that for every coloring $\Delta : \binom{G}{H}_{part} \to r$ there exists a $F \in \binom{G}{F}_{part}$ such that $\Delta \rceil \binom{F}{H}_{part}$ is a constant coloring.*

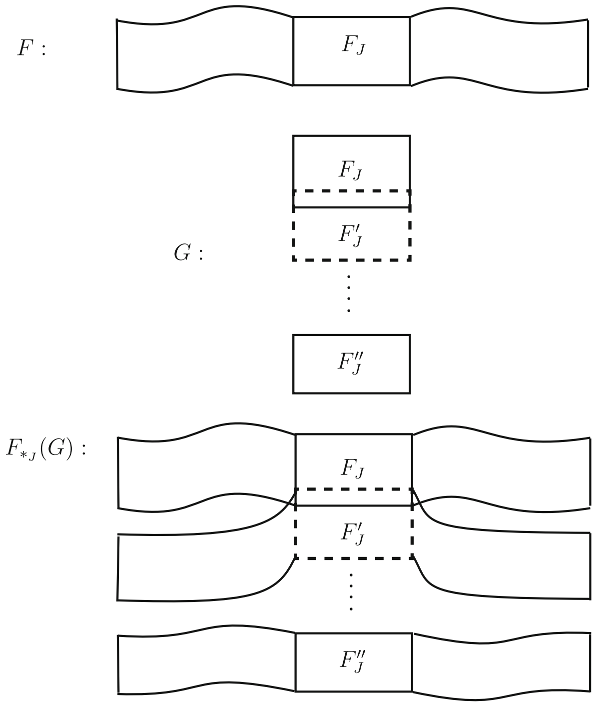
Proof. We proceed by induction on m. For $m = 1$ the statement reduces to the pigeonhole principle. We prove it for $m + 1$.

Let $F = ((V_\nu)_{\nu<m+1}, E_F)$ and $H = (m+1, E_H)$ be $(m+1)$-partite left-rectified graphs where H is crossing, and let r be a positive integer. As H is crossing we may assume that it has $m + 1$ as the set of vertices with parts $\{i\}$ for $i < m + 1$.

Since F is left-rectified, any two vertices $x, x' \in V_m$ which belong to an H-subgraph have the same "profile", i.e., for any $a \in \bigcup_{i<m} V_i$ we have $\{a, x\} \in E_F$ if and only $\{a, x'\} \in E_F$. Let $V_H \subseteq V_m$ be the set of all vertices in V_m which belong to an H-subgraph, and set $z = |V_H|$. Furthermore, let H' and F' be subgraphs of H, resp. F, spanned by the first m parts.

By the induction hypothesis, there exists an m-partite graph G' such that $G' \overset{\text{part}}{\rightarrow} (F')^{H'}_{r^{z^*}}$, where $z^* = r \cdot (z - 1) + 1$. Now we extend G' by a set X_m to an $(m + 1)$-partite graph G as follows. First we add vertices $y_0, \dots, y_{z^*-1}$ to X_m, such that they respect the property of being left-rectified and they all form an H-subgraph with the vertices from G'. Secondly, for each vertex $\hat{x} \in V_m \setminus V_H$ add a vertex to X_m and connect it to the parts in G' in exactly the same way as $\hat{x}$ is connected to the parts V_i, $i < m$ in F. Note that this guarantees that every z-element subset of

Fig. 14.1 The $*_J$-amalgamation



$y_0, \ldots, y_{z^*-1}$ can be extended to a copy of F in G. We claim that the so constructed graph G has the desired properties.

Let $\Delta : \binom{G}{H}_{part} \to r$ be an r-coloring. This induces an r^{z^*} coloring $\Delta^* : \binom{G'}{H'}_{part} \to r^{z^*}$ by $\Delta^*(\tilde{H}') = \langle \Delta(\tilde{H}' \cup \{y_i\}) \mid i < z^* \rangle$. Let $\tilde{G}' \in \binom{F'}{G'}_{part}$ be monochromatic with respect to Δ^*. This induces an r-coloring of the vertices $\{y_0, \ldots, y_{z^*-1}\}$ and by choice of z^* and the pigeonhole principle there exist z of them in the same color. Extending $\tilde{G}'$ with such z vertices and the corresponding $\hat{x}$ vertices yields a partite F-subgraph monochromatic with respect to Δ. □

14.2.2 Amalgamation of Partite Graphs

Having the partite lemma available, we explain our second tool, the $*_J$-*amalgamation*.

Let $F = ((X_\nu)_{\nu<m}, E_F)$ be a left-rectified m-partite graph and let $J \subset m$ be a nonempty subset. By F_J we denote the subgraph of F spanned by the parts X_j, $j \in J$. Additionally, let $G = ((Y_\nu)_{\nu \in J}, E_G)$ be a left-rectified $|J|$-partite graph that contains many partite F_J-subgraphs. The idea of the $*_J$-amalgamation is to extend every partite F_J-subgraph of G to an F-graph in a vertex disjoint way, cf. Fig. 14.1.

Finally, we add edges (as few as possible) to ensure that the newly constructed graph is again left-rectified.

Formally, we define the amalgamation $F_{*_J}(G)$ of F with G along F_J as follows:

Definition 14.8. The subgraph of the amalgamation which is spanned by the parts $j \in J$ is precisely G, i.e., $(F_{*_J}(G))_J = G$. Moreover, every $\tilde{F}_J \in \binom{G}{F_J}$ extends to an m-partite graph isomorphic to F such that every two such graphs are mutually disjoint up to the intersection in $(F_{*_J}(G))_J$. The graph $F_{*_J}(G)$ is m-partite and left-rectified.

A moment of thought reveals that such a graph can indeed be constructed. For our need the following property, which can easily be seen to follow from the definitions, is of importance.

Property 14.9. Let F be an m-partite left-rectified graph and $J \subset m$. Let H and G be $|J|$-partite left-rectified graphs, where in addition H is crossing, and assume $G \overset{\text{part}}{\rightarrow} (F_J)^H_r$. Then for every r-coloring $\Delta : \binom{F_{*_J}(G)}{H}_{part} \rightarrow r$ there exists an $\tilde{F} \in \binom{F_{*_J}(G)}{F}_{part}$ such that $\Delta \rceil \binom{\tilde{F}_J}{H}_{part}$ is a constant coloring.

With these tools at hand, we can now reprove the Ramsey theorem for ordered graphs.

Proof of Theorem 12.13. Let F and H be given graphs. As observed earlier, we can treat them as m-partite, resp. k-partite graphs, where m and k are the number of vertices of F, resp. H. According to Ramsey's theorem let n be such that $n \rightarrow (m)^k_r$.

Instead of looking directly for a monochromatic F-subgraph, we define a left-rectified n-partite graph F_0 such that for every $J \in [n]^m$ there exists a (partite) F-subgraph in the partite subgraph $(F_0)_J$ of F_0 spanned by the parts $j \in J$. Such an F_0 can be obtained straightforwardly by placing the required F-subgraphs vertex disjointly and, eventually, adding edges to make it left-rectified. We aim at finding an F_0-subgraph $\tilde{F}_0$ which satisfies the following coloring property,

($\star$) For all $J \in [n]^k$: all H-subgraphs in $(\tilde{F}_0)_J$ are colored monochromatically, i.e. for all $\tilde{H}, \tilde{H}' \in \binom{(\tilde{F}_0)_J}{H}_{part}$ we have $\Delta(\tilde{H}) = \Delta(\tilde{H}')$.

Note that the existence of such an F_0-subgraph implies, by choice of n, that there exists an F-subgraph $\tilde{F}$ such that $\Delta \rceil \binom{\tilde{F}}{H}_{part}$ is a constant coloring. As F is crossing it follows that $\binom{\tilde{F}}{H}_{part}$ coincides with $\binom{\tilde{F}}{H}$, thus we have found the desired monochromatic F-subgraph.

Next we construct an n-partite left-rectified graph G such that for every coloring $\Delta : \binom{G}{H}_{part} \rightarrow r$ there exists an F_0-subgraph which satisfies property ($\star$).

Let $(J_i)_{i<q}$ be an enumeration of $[n]^k$. By Lemma 14.7 (partite lemma) there exists a left-rectified n-partite graph F_0^* such that

$$F_0^* \overset{\text{part}}{\rightarrow} ((F_0)_{J_0})^H_r,$$

where $(F_0)_{J_0}$ denotes the subgraph of F_0 spanned by parts $j \in J_0$. Let $F_1 = (F_0)_{*J_0}(F_0^*)$, and observe that by Property 14.9, F_1 contains an F_0-subgraph $\tilde{F}_0$ which satisfies property $(\star)$ when restricted to J_0 instead of all $J \in [n]^k$.

We continue the construction in the same way. Assume that we have constructed a graph F_i such that for any coloring of $\binom{F_i}{H}_{part}$ there exists an F_0-subgraph $\tilde{F}_0$ which satisfies property $(\star)$ when restricted to sets $J_0, \ldots, J_{i-1}$. Then let F_i^* be such that $F_i^* \overset{\text{part}}{\rightarrow} ((F_i)_{J_i})_r^H$ and set $F_{i+1} = (F_i)_{*J_i}(F_i^*)$. Now we have that for any coloring of partite H-subgraphs of F_{i+1} there exists an F_i-subgraph $\tilde{F}_i$ such that $\Delta\rceil\binom{(\tilde{F}_i)_{J_i}}{H}_{part}$ is a constant coloring. However, such $\tilde{F}_i$ now contains an F_0 subgraph $\tilde{F}_0$ which satisfies property $(\star)$ when restricted to restricted to $J_0, \ldots, J_i$.

Repeating the same argument inductively, we have that for any coloring $\Delta : \binom{F_q}{H}_{part} \rightarrow r$ there exists an F_0-subgraph which satisfies property $(\star)$. By the earlier observation, this implies the existence of a monochromatic F-subgraph, thus setting $G = F_q$ proves the theorem. □

Remark 14.10. The approach presented in this section can be extended to also obtain a *restricted* version of the Ramsey theorem for ordered graphs, cf. Prömel and Voigt (1989). In Chap. 16 we consider restricted Ramsey theorems from a different view point.

14.3 An Induced Graham-Rothschild Theorem

In this section we prove an induced version of the Graham-Rothschild theorem. This generalizes the Graham-Rothschild partition theorem for parameter sets in the same way as the Ramsey's theorem for ordered graphs defined on sets generalizes Ramsey's theorem.

The induced Graham-Rothschild theorem has been proved originally in Prömel (1985). Somewhat simpler proofs, then, have been given in Frankl et al. (1987) and Prömel and Voigt (1988).

Definition 14.11. For hypergraphs $\mathcal{F} \subseteq \mathcal{H}^k(m)$ and $\mathcal{G} \subseteq \mathcal{H}^k(n)$, by $\binom{\mathcal{G}}{\mathcal{F}}$ we denote the set of all m-parameter words $f \in [A]\binom{n}{m}$ such that $\mathcal{G}[f]$ is isomorphic to $\mathcal{F}$.

Theorem 14.12 (Induced Graham-Rothschild theorem). *Let A be an alphabet of size $|A| \geq 2$, k and r be positive integers, and let $\mathcal{F} \subseteq \mathcal{H}^k(m)$ and $\mathcal{E} \subseteq \mathcal{H}^k(t)$ be given hypergraphs. Then there exists a positive integer n and a hypergraph $\mathcal{G} \subseteq \mathcal{H}^k(n)$ such that $\mathcal{G} \rightarrow (\mathcal{F})_r^{\mathcal{E}}$, i.e., for every $\Delta : \binom{\mathcal{G}}{\mathcal{E}} \rightarrow r$ there exists an $f \in \binom{\mathcal{G}}{\mathcal{F}}$ such that $\Delta\rceil\binom{\mathcal{G}[f]}{\mathcal{E}}$ is a constant coloring.*

The assumption $|A| \geq 2$ is just for convenience. For $|A| = 1$ the proof requires some additional twists, cf. Prömel and Voigt (1988).

Recall that with respect to hypergraph $\mathcal{E} = \mathcal{H}^0(0)$, i.e., the case of vertex colorings, the theorem reduces to the induced Hales-Jewett theorem which has been proved in Sect. 14.1.

As the proof of Theorem 14.12 is quite involved, let us first give a very high-level overview of our proof strategy. In fact, the general approach is very similar to the one that we just saw for the graph case in the previous section. In order to transfer these ideas to the hypergraph case we first need to generalize the notations of 'partiteness' and 'amalgamation' from the graph setting to hypergraphs define on parameters sets. In a second step we will use these notions to define an appropriate hypergraph $\mathcal{F}_0$ (that takes over the rôle of F_0 in the graph case). The structural properties of $\mathcal{F}_0$ will then allows us, again similar as in the graph case, to construct the desired hypergraph $\mathcal{G}$ iteratively.

14.3.1 Partite Hypergraphs

As a first step in the proof of the induce Graham-Rothschild theorem, we define an appropriate notion of 'partiteness'. While we will eventually have the property that the 'parts' are stable (contain no edges), we here use a different approach of defining the 'parts'. Consider $\mathcal{H}^k(m+n)$. Its vertices are words of length $m+n$ over the alphabet A. The idea is to use the first m letters to describe the 'part' and the remaining n letters to describe the vertices within a part. Note that in this way an m-partite graph will actually consist of $|A|^m$ parts. We also want that edges in an m-partite graph are 'crossing', meaning that they contain at most one vertex from each part. We now give a formal definition.

Let $f \in [A]\binom{m+n}{j}$ be a parameter word. We write $\dim f = j$ indicating that f is a j-parameter word. By $f\rceil m$ we denote the restriction of f to the first m entries (coordinates). Recall that, formally, f is a mapping $f : m+n \to A \cup \{\lambda_0, \ldots, \lambda_{j-1}\}$. So the restriction $f\rceil m$ again is a parameter word, this time of length m. Observe that $\dim f\rceil m \le j$.

Definition 14.13. A hypergraph $\mathcal{E} \subseteq \mathcal{H}^k(m+n)$ is m-partite if $e_g \notin \mathcal{E}$ whenever $\dim g\rceil m < \dim g$. A partite embedding of an m-partite hypergraph $\mathcal{E} \subseteq \mathcal{H}^k(m+n)$ into $\tilde{\mathcal{E}} \subseteq \mathcal{H}^k(m+\tilde{n})$ is given by an $f \in [A]\binom{m+\tilde{n}}{m+n}$ such that $\tilde{\mathcal{E}}[f]$ is isomorphic to $\mathcal{E}$ and $\dim f\rceil m = m$. By $\binom{\tilde{\mathcal{E}}}{\mathcal{E}}_{part}$ we denote the set of partite $\mathcal{E}$-subgraphs of $\tilde{\mathcal{E}}$, i.e., the set of all partite embeddings of $\mathcal{E}$ into $\tilde{\mathcal{E}}$.

With respect to sets A having at least two elements, an m-partite hypergraph $\mathcal{E} \subseteq \mathcal{H}^k(m+n)$ can be visualized as follows. The set of vertices $[A]\binom{m+n}{0}$ is split into sets $x \times [A]\binom{n}{0}, x \in [A]\binom{m}{0}$, which we call the *parts* of $\mathcal{E}$. Then the edges have to be *crossing*, i.e., intersect each partition at most once. In other words, ignoring the hyperedges containing only a single vertex, each partition then forms an independent set. Being crossing is reflected by the requirement that $e_g \in \mathcal{E}$ only if $\dim g\rceil m = \dim g$. In particular every hypergraph $\mathcal{E} \subseteq \mathcal{H}^k(m)$ can be viewed as a (crossing) m-partite hypergraph. Finally, the requirement on partite embeddings ensures that each part of $\mathcal{E}$ is inscribed into some (unique) part of $\tilde{\mathcal{E}}$.

Lemma 14.14 (Partite lemma). *Let $\mathcal{F} \subseteq \mathcal{H}^k(m+n)$ and $\mathcal{E} \subseteq \mathcal{H}^k(m)$ be m-partite hypergraphs and let r be a positive integer. Then there exists a positive integer $\tilde{n}$ and an m-partite hypergraph $\mathcal{G} \subseteq \mathcal{H}^k(m+\tilde{n})$ satisfying $\mathcal{G} \overset{part}{\rightarrow} (\mathcal{F})^{\mathcal{E}}_r$, meaning that for every $\Delta : \binom{\mathcal{G}}{\mathcal{E}}_{part} \rightarrow r$ there exists a partite embedding $f \in \binom{\mathcal{G}}{\mathcal{F}}_{part}$ such that $\Delta\rceil\binom{\mathcal{G}[f]}{\mathcal{E}}_{part}$ is a constant coloring.*

Proof. The proof of Lemma 14.14 just uses Hales-Jewett's theorem and is somewhat similar to the proof of the induced Hales-Jewett theorem (Theorem 14.1).

Recall that $\binom{\mathcal{F}}{\mathcal{E}}_{part} \subseteq \{f \in [A]\binom{m+n}{m} \mid f\rceil m = (\lambda_0, \dots, \lambda_{m-1})\}$. In particular we have that $f\rceil m = \tilde{f}\rceil m$ for any two $f, \tilde{f} \in \binom{\mathcal{F}}{\mathcal{E}}$. We cut off the first m entries of each such f and let

$$\mathcal{T} = \{g \in (A \cup \{\lambda_0, \dots, \lambda_{m-1}\})^n \mid (\lambda_0, \dots, \lambda_{m-1}) \times g \in \binom{\mathcal{F}}{\mathcal{E}}_{part}\}$$

be the set of tails. Let the positive integer s be such that $s \geq HJ(|\mathcal{T}|, 1, r)$, and consider the set

$$\mathcal{T}^* = \{g_0 \times \dots \times g_{s-1} \mid g_i \in \mathcal{T} \text{ or } g_i = (\lambda_m, \dots, \lambda_{m+n-1}) \text{ for all } i < s \text{ and } g_j = (\lambda_m, \dots, \lambda_{m+n-1}) \text{ for at least one } j < s\}.$$

Observe that $\mathcal{T}^*$ corresponds to the set of one-parameter words $[\mathcal{T}]\binom{s}{1}$, where, for convenience, the parameter is replaced by $(\lambda_m, \dots, \lambda_{m+n-1})$. Also observe that $(\lambda_0, \dots, \lambda_{m-1}) \times \mathcal{T}^* \subseteq [A]\binom{m+n\cdot s}{m+n}$.

We now define a hypergraph $\mathcal{G} \subseteq \mathcal{H}^k(m+n\cdot s)$. For a $h \in \mathcal{T}^*$ let $\overline{\lambda h} = (\lambda_0, \dots, \lambda_{m-1}) \times h$. Then for every $h \in \mathcal{T}^*$ and for every $g \in [A]\binom{m+n}{i}$ set

$$e_{\overline{\lambda h}\cdot g} \in \mathcal{G} \text{ if and only if } e_g \in \mathcal{F}.$$

The following claim shows, and this is where the property of being partite comes into play, that $\mathcal{G}$ is well-defined.

Claim. Let $g, g' \in [A]\binom{m+n}{i}$ and let $h, h' \in \mathcal{T}^*$. Assume that $g \neq g'$ and $\overline{\lambda h} \cdot g = \overline{\lambda h'} \cdot g'$. Then $e_g \in \mathcal{F}$ iff $e_{g'} \in \mathcal{F}$.

Proof of Claim. First observe that $\overline{\lambda h} \cdot g = \overline{\lambda h'} \cdot g'$ implies that $g\rceil m = g'\rceil m$, so g and g' differ only in their tail sequence. If $\dim g\rceil m < i$, then by the definition we have $e_g \notin \mathcal{G}$ and $e_{g'} \notin \mathcal{G}$, thus we are done.

Otherwise, let $h = h_0 \times \dots \times h_{s-1}$ and $h' = h'_0 \times \dots \times h'_{s-1}$. Since $h \in \mathcal{T}^*$ there exists an $j < s$ such that $h_j = (\lambda_m, \dots, \lambda_{m+n-1})$. Then $h'_j \in \mathcal{T}$, as otherwise we would have $((\lambda_0, \dots, \lambda_{m-1}) \times h_j) \cdot g \neq ((\lambda_0, \dots, \lambda_{m-1}) \times h'_j) \cdot g'$ and so $\overline{\lambda h} \cdot g \neq \overline{\lambda h'} \cdot g'$. Thus $((\lambda_0, \dots, \lambda_{m-1}) \times h'_j) \cdot (g'\rceil m) = g$. Moreover, as $h'_j \in \mathcal{T}$

we know that $(\lambda_0, \ldots, \lambda_{m-1}) \times h'_j \in \binom{\mathcal{F}}{\mathcal{E}}_{part}$, hence $e_{g'\rceil m} \in \mathcal{E}$ iff $e_g \in \mathcal{F}$. Using the same argument we deduce $e_{g\rceil m} \in \mathcal{E}$ iff $e_{g'} \in \mathcal{F}$, which together with the observation $g\rceil m = g'\rceil m$ proves the claim.

It remains to verify that indeed $\mathcal{G} \overset{\text{part}}{\rightarrow} (\mathcal{F})^{\mathcal{H}}_r$. Let $\Delta : \binom{\mathcal{G}}{\mathcal{E}}_{part} \rightarrow r$ be an r-coloring. This induces an r-coloring of $[\mathcal{T}]\binom{s}{0}$ and thus, by choice of s, there exists a monochromatic line which can be identified with some $h \in \mathcal{T}^*$. Now by the construction of $\mathcal{G}$ we have that $\mathcal{G}[\overline{\lambda h}]$ is isomorphic to $\mathcal{F}$, yielding the desired monochromatic $\mathcal{F}$-subgraph. □

14.3.2 Amalgamation of Partite Graphs

In this section we describe the concept of *amalgamation*. Again, we first fix some notation.

Let $\mathcal{F} \subseteq \mathcal{H}^k(m+n)$ be an m-partite hypergraph. Then for $h \in [A]\binom{m}{t}$, by $\mathcal{F}_h$ we denote the t-partite hypergraph isomorphic to $\mathcal{F}[h \times (\lambda_t, \ldots, \lambda_{t+n-1})]$, or, more precisely,

$$\mathcal{F}_h = \mathcal{F}[\{h \cdot x \mid x \in \tbinom{t}{0}\} \times [A]\tbinom{n}{0}].$$

Intuitively, $\mathcal{F}_h$ is a subgraph spanned by a subset of the partition of $\mathcal{F}$ specified by the parameter word h.

Additionally, let $\mathcal{G} \subseteq \mathcal{H}^k(t+\tilde{n})$ be a t-partite hypergraph. The idea of an $*h$-amalgamation is exactly as in the similar notion of a $*_J$-amalgamation in the graph case: we want an m-partite graph $\mathcal{F}_{*h}(\mathcal{G})$ that extends every $\mathcal{F}_h$-subgraph in $\binom{\mathcal{G}}{\mathcal{F}_h}_{part}$ to an $\mathcal{F}$-graph in a 'vertex-disjoint way'. For a formal definition let $g_0, \ldots, g_{z-1}$ be an enumeration of the partite $\mathcal{F}_h$-subgraphs in $\mathcal{G}$.

Definition 14.15. A hypergraph $\mathcal{F}_{*h}(\mathcal{G}) \subseteq \mathcal{H}^k(m+n')$ is an $*h$-amalgamation of $\mathcal{F}$ with $\mathcal{G}$ along h if the following holds: $\mathcal{F}_{*h}(\mathcal{G})$ is m-partite and there exist $f_0^*, \ldots, f_{z-1}^* \in \binom{\mathcal{F}_{*h}(\mathcal{G})}{\mathcal{F}}_{part}$ such that the intersection of $\mathcal{F}_{*h}[f_i^*]$ and $\mathcal{F}_{*h}[f_j^*]$ is isomorphic to the intersection of $\mathcal{G}[g_i]$ and $\mathcal{G}[g_j]$. In particular, we require that $(\mathcal{F}_{*h}(\mathcal{G}))_h$ is isomorphic to $\mathcal{G}$.

The next lemma shows that such a hypergraph $\mathcal{F}_{*h}(\mathcal{G})$ indeed exists.

Lemma 14.16. *Let $\mathcal{G} \subseteq \mathcal{H}^k(t+\tilde{n})$ and $\mathcal{F} \subseteq \mathcal{H}^k(m+n)$ be given t-partite, resp. m-partite hypergraphs and let $h \in [A]\binom{m}{t}$. Then there exists an $*h$-amalgamation $\mathcal{F}_{*h}(\mathcal{G}) \subseteq \mathcal{H}^k(m+\tilde{n}+(z+1)\cdot m)$ of $\mathcal{F}$ with $\mathcal{G}$ along h, where z denotes the cardinality of $\binom{\mathcal{G}}{\mathcal{F}_h}_{part}$.*

As in the graph case the importance of this amalgamation technique stems from its strong coloring properties. The following proposition (that follows immediately from the definition of the amalgamation) captures this feature. This proposition is all we need in the subsequent section for the proof of the induced Graham-Rothschild theorem.

Proposition 14.17 (Coloring property of $*h$-amalgamation). *Let $\mathcal{E} \subseteq \mathcal{H}^k(t)$ be a t-partite hypergraph and assume that $\mathcal{G} \overset{part}{\longrightarrow} (\mathcal{F}_h)^{\mathcal{E}}_r$. Then for every coloring $\Delta : \binom{\mathcal{F}_{*h}(\mathcal{G})}{\mathcal{E}}_{part} \overset{part}{\longrightarrow} r$ there exists an $f \in \binom{\mathcal{F}_{*h}(\mathcal{G})}{\mathcal{F}}_{part}$ such that for $\tilde{\mathcal{F}} = \mathcal{F}_{*h}(\mathcal{G})[f]$ we have that $\Delta\rceil\binom{\tilde{\mathcal{F}}_h}{\mathcal{E}}_{part}$ is a constant coloring.* □

The remainder of this section is devoted to the (somewhat technical) proof of Lemma 14.16. The first lemma shows that for every $h \in [A]\binom{m}{t}$ and every positive integer z there exist z distinct m-parameter sets in $[A]\binom{(z+1)m}{m}$ which mutually intersect in their h-subspace.

Lemma 14.18. *Let $h \in [A]\binom{m}{t}$ and let z be a positive integer. Then there exist parameter words $f_i \in [A]\binom{(1+z)\cdot m}{m}$ for $i < z$ with the following properties.*

(1) $f_i \cdot x = f_j \cdot x$ *for all $i < j < z$ and all $x \in h \cdot [A]\binom{t}{0}$,*

(2) $f_i \cdot x \neq f_j \cdot x'$ *for all $i < j < z$ and all $x \in [A]\binom{m}{0} \setminus h \cdot [A]\binom{t}{0}$ and all $x' \in [A]\binom{m}{0}$.*

To understand the proof of the lemma properly some familiarity with the formal calculus of parameter words may be helpful. As we slightly extend the composition of parameter words also to non-parameter words let us recall the basic definition.

Let $g = (g_0, \dots, g_{n-1}) \in (A \cup \{\lambda_0, \dots, \lambda_{m-1}\})^n$ and let $h = (h_0, \dots, h_{m-1}) \in (A \cup \{\lambda_0, \dots, \lambda_{t-1}\})^m$. Note that neither g nor h are required to be parameter words in the sense of Sect. 3.1. Still we define the composition $g \cdot h \in (A \cup \{\lambda_0, \dots, \lambda_{t-1}\})^n$ straightforwardly, viz., $g \cdot h = (f_0, \dots, f_{n-1})$ where

$$f_i = \begin{cases} g_i, & \text{if } g_i \in A, \\ h_j, & \text{if } g_i = \lambda_j. \end{cases}$$

Proof of Lemma 14.18. Let $h = (h_0, \dots, h_{m-1}) \in [A]\binom{m}{t}$. For every $j < t$ we define j' as the minimal index at which λ_j appears: $j' = \min\{i < m \mid h_i = \lambda_j\}$.

Consider $y = (y_0, \dots, y_{m-1}) \in (A \cup \{\lambda_{j'} \mid j < t\})^m$ which is defined by

$$y_i = \begin{cases} h_i, & \text{if } h_i \in A, \\ \lambda_{j'}, & \text{if } h_i = \lambda_j. \end{cases}$$

We now show that $y \cdot x = x$ if and only if $x \in h \cdot [A]\binom{t}{0}$. Since by construction we have $y \cdot h = h$, it easily follows that $x \in h \cdot [A]\binom{t}{0}$ implies $y \cdot x = x$. On the other hand, y and h have the same pattern: if $h_i = h_j = \lambda_k$ then $y_i = y_j = \lambda_{k'}$. Thus, $y \cdot x = x$ implies $(y \cdot x)_i = h_i = x_i$ if $h_i \in A$ and $(y \cdot x)_i = x_{k'} = x_i$ if $h_i = \lambda_k$. Hence, $x \in h \cdot [A]\binom{t}{0}$.

Now we define $f_i \in [A]\binom{(1+z)\cdot m}{m}$ by

$$f_i = (\lambda_0, \ldots, \lambda_{m-1}) \times \underbrace{y \times \ldots \times y}_{i \text{ times}} \times (\lambda_0, \ldots, \lambda_{m-1}) \times \underbrace{y \times \ldots \times y}_{z-1-i \text{ times}}.$$

As each f_i starts with $(\lambda_0, \ldots, \lambda_{m-1})$ assertion (2) is obviously satisfied for $x \neq x'$. The remaining cases follow from the fact that $y \cdot x = x$ if and only if $x \in h \cdot [A]\binom{t}{0}$. □

Also the next lemma sounds somewhat technical. Its significance will be clear in the construction of the amalgamation.

The problem is the following: consider the embedding $g_i \in \binom{\mathcal{F}_h}{\mathcal{G}}$, so $g_i \in [A]\binom{t+\tilde{n}}{t+n}$. We want to find a $g_i^* \in [A]\binom{m+\tilde{n}}{m+n}$ such that $g_i^* \cdot (h \times (\lambda_k, \ldots, \lambda_{k+n-1}))$ behaves like g_i. Recall that $h \times (\lambda_k, \ldots, \lambda_{k+n-1}) \in \binom{\mathcal{F}}{\mathcal{F}_h}$.

Lemma 14.19. *Let $g \in [A]\binom{t+\tilde{n}}{t+n}$ be such that $\dim g\rceil t = t$, thus g can be written as $g = (\lambda_0, \ldots, \lambda_{t-1}) \times g_{tail}$. Let $h \in [A]\binom{m}{t}$. Then there exists $g^* \in [A]\binom{m+\tilde{n}}{m+n}$ which can be written as $g^* = (\lambda_0, \ldots, \lambda_{m-1}) \times g^*_{tail}$ such that for all $f \in [A]\binom{t+n}{i}$ it follows that*

$$g^* \cdot ((h \times (\lambda_t, \ldots, \lambda_{t+n-1})) \cdot f) = (h \cdot f\rceil t) \times (g_{tail} \cdot f).$$

Proof. As before, let $j' = \min\{i < m \mid h_i = \lambda_j\}$, for all $j < t$. Let $g_{tail} = (\alpha_0, \ldots, \alpha_{\tilde{n}-1})$. Then setting $g^* = (\lambda_0, \ldots, \lambda_{m-1}) \times (\alpha_0^*, \ldots, \alpha_{\tilde{n}-1}^*)$, where

$$\alpha_i^* = \begin{cases} \alpha_i, & \text{if } \alpha_i \in A \\ \lambda_{m+j}, & \text{if } \alpha_i = \lambda_{t+j}, \\ \lambda_{j'}, & \text{if } \alpha_i = \lambda_j \text{ for } j < t \end{cases}$$

proves the lemma. □

Now we are in the position to prove Lemma 14.16.

Proof of Lemma 14.16. Let $h \in [A]\binom{m}{t}$ and $(g_i)_{i<z}$ be an enumeration of $\binom{\mathcal{F}_h}{\mathcal{G}}_{part}$. Let the parameter words $f_i \in [A]\binom{(z+1)\cdot m}{m}$ for $i < z$ be as in Lemma 14.18. Also let $g_i^* \in [A]\binom{m+\tilde{n}}{m+n}$ be as in Lemma 14.19 with respect to h and g_i. Now we define $\mathcal{F}_{*h}(\mathcal{G})$ as follows:

$$e_{(g_j^* \times f_j)\cdot g} \in \mathcal{F}_{*h}(\mathcal{G}) \;\text{ iff }\; e_g \in \mathcal{F}$$

for all $i \leq k$ and $g \in [A]\binom{m+n}{i}$ and all $j < z$. The following claim shows that this is a proper definition.

Claim. Let $i \leq k$ and $g, g' \in [A]\binom{m+n}{i}$ and let $j < j' < z$ be such that $(g_j^* \times f_j) \cdot g = (g_{j'}^* \times f_{j'}) \cdot g'$. Then $e_g \in \mathcal{F}$ if and only if $e_{g'} \in \mathcal{F}$.

Proof of the Claim. As $g_j^*\rceil m = g_{j'}^*\rceil m = (\lambda_0, \ldots, \lambda_{m-1})$ we see that $g\rceil m = g'\rceil m$. Without loss of generality we can assume that g and g' are crossing, i.e., $g\rceil m \in [A]\binom{m}{i}$. From Lemma 14.18 we conclude that $g\rceil m \in h \cdot [A]\binom{t}{i}$. In other words, there exist $f, f' \in [A]\binom{t+n}{i}$ such that $g = (h \times (\lambda_t, \ldots, \lambda_{m+n-1})) \cdot f$, resp., $g' = (h \times (\lambda_t, \ldots, \lambda_{m+n-1})) \cdot f'$. From Lemma 14.19 it follows that

$$g_j^* \cdot g = (h \cdot f\rceil t) \times (g_{j,tail} \cdot f), \quad \text{resp.,} \quad g_{j'}^* \cdot g' = (h \cdot f\rceil t) \times (g_{j',tail} \cdot f'),$$

where $g_j = (\lambda_0, \ldots, \lambda_{t-1}) \times g_{j,tail}$ and $g_{j'} = (\lambda_0, \ldots, \lambda_{t-1}) \times g_{j',tail}$.

It follows from $g_j^* \cdot g = g_j^*$ that $g_j \cdot f = g_{j'} \cdot f = g_{j'} \cdot f'$, hence $e_{g_j \cdot f} \in \mathcal{G}$ iff $e_{g_{j'} \cdot f'} \in \mathcal{G}$. On the other hand, as $h \times (\lambda_t, \ldots, \lambda_{t+n-1}) \in \binom{\mathcal{F}}{\mathcal{F}_h}$ and $g_j, g_{j'} \in \binom{\mathcal{F}_h}{\mathcal{G}}$, we see that

$$\begin{aligned} e_g \in \mathcal{F} &\Leftrightarrow e_{(h \times (\lambda_t, \ldots, \lambda_{t+n-1})) \cdot f} \in \mathcal{F} \\ &\Leftrightarrow e_f \in \mathcal{F}_h \Leftrightarrow e_{g_j \cdot g} \in \mathcal{G} \Leftrightarrow e_{g_{j'} \cdot f'} \in \mathcal{G} \Leftrightarrow e_{f'} \in \mathcal{F}_h \\ &\Leftrightarrow e_{(h \times (\lambda_t, \ldots, \lambda_{t+n-1})) \cdot f'} \in \mathcal{F} \Leftrightarrow e_{g'} \in \mathcal{F}, \end{aligned}$$

as desired. □

14.3.3 Proof of the Induced Graham-Rothschild Theorem

With these tools at hand, namely induced Graham-Rothschild theorem for partite graphs (Lemma 14.14) and the notion of an $*h$-amalgamation, we can now prove the induced Graham-Rothschild theorem. Actually, the proof is very similar to the one for the ordered Ramsey theorem from the previous section. First we define an appropriate hypergraph $\mathcal{F}_0$ that will allow us to always find the desired monochromatic $\mathcal{F}$-subgraph. In order to construct $\mathcal{F}_0$ we use now the Graham-Rothschild theorem (Theorem 5.1) instead of the classical Ramsey theorem. In the second part of the proof we then proceed almost word by word as before: we just use the new partite Lemma 14.14 and the new amalgamation technique instead of the ones from the graph case.

Proof of Theorem 14.12. Let $\mathcal{E} \subseteq \mathcal{H}^k(t)$ and $\mathcal{F} \subseteq \mathcal{H}^k(m)$. Choose a positive integer n such that $n \geq GR(|A|, k, m, r)$, where $GR(\cdot)$ is as defined by the Graham-Rothschild partition theorem for parameter sets (Theorem 5.1).

We first construct a suitable hypergraph $\mathcal{F}_0$ satisfying certain coloring properties. Let $(f_i)_{i<z}$ be an enumeration of $[A]\binom{n}{m}$. Furthermore, let a and b be any two distinct elements of A and let for $i < z$ the z-tuple $y_i \in [A]\binom{z}{0}$ be defined by

$$y_i = (\underbrace{a, \ldots, a}_{i \text{ times}}, b, \underbrace{a, \ldots, a}_{(z-1-i) \text{ times}}).$$

Consider the m-parameter word $f_i^* = f_i \times y_i \in [A]\binom{n+z}{m}$. Each f_i^* describes an m-subspace of $[A]\binom{n+z}{0}$. Moreover, $f_i^* \cdot [A]\binom{m}{0} \cap f_j^* \cdot [A]\binom{m}{0} = \emptyset$ for $i < j < z$, i.e., all these subspaces are mutually disjoint. Hence we can define a hypergraph $\mathcal{F}_0 \subseteq \mathcal{H}^k(n+z)$ such that each f_i^* is an embedding of $\mathcal{F}$, viz., let

$$e_{f_j^* \cdot g} \in \mathcal{F}_0 \quad \text{iff} \quad e_g \in \mathcal{F},$$

for all $j < z, i \leq k$ and $g \in [A]\binom{m}{i}$. Observe that each f_j^* induces a crossing subgraph of $\mathcal{F}_0$ isomorphic to $\mathcal{F}$, with one vertex in each partition, and $\mathcal{F}_0$ itself is n-partite. Of course, if we can find a monochromatic $\mathcal{F}_0$-subgraph, then it clearly implies the existence of a monochromatic $\mathcal{F}$-subgraph. However, the trick lies in the following much weaker coloring requirement:

($\star$) For any $h, h' \in [A]\binom{\mathcal{F}_0}{\mathcal{E}}_{part}$ such that $h \rceil n = h' \rceil n$, we have $\Delta(\mathcal{F}_0[h]) = \Delta(\mathcal{F}_0[h'])$.

In other words, instead of requiring that $\mathcal{F}_0$ is monochromatic, we require that any two partite $\mathcal{E}$-subgraphs of $\mathcal{F}_0$ spanned by the same parts have the same color.

To see that this suffices, consider a coloring $\Delta : \binom{\mathcal{G}}{\mathcal{E}} \to r$ and assume that an $\mathcal{F}_0$-subgraph $\tilde{\mathcal{F}}_0$ of $\mathcal{G}$ satisfying property ($\star$) is given. Then this induces a coloring $\Delta' : [A]\binom{n}{t} \to r$ given by

$$\Delta'(h') = \begin{cases} \Delta(\tilde{\mathcal{F}}_0[h]) & \text{if there exists } h \in \binom{\tilde{\mathcal{F}}_0}{\mathcal{E}} \text{ such that } h\rceil n = h' \\ 0 & \text{otherwise.} \end{cases}$$

Note that property ($\star$) implies that Δ' is well-defined. Then by the Graham-Rothschild theorem and choice of n, there exists $f \in [A]\binom{n}{m}$ such that $\Delta' \rceil \{f \cdot x \mid x \in [A]\binom{m}{t}\}$ is a constant coloring. As we enumerated $[A]\binom{n}{m}$ we know that $f = f_i$ for some $i < z$. But then $\Delta \rceil \binom{\tilde{\mathcal{F}}_0[f_i^*]}{\mathcal{E}}_{part}$ is also a monochromatic coloring, and by the construction $\tilde{\mathcal{F}}_0[f_i^*]$ is isomorphic to $\mathcal{F}$. As each vertex of $\tilde{\mathcal{F}}_0[f_i^*]$ belongs to a distinct partition, we have that $\binom{\tilde{\mathcal{F}}_0[f_i^*]}{\mathcal{E}}_{part}$ coincides with $\binom{\tilde{\mathcal{F}}_0[f_i^*]}{\mathcal{E}}$ and thus we have found a monochromatic $\mathcal{F}$-subgraph.

Next, we construct an n-partite hypergraph $\mathcal{G} \subseteq \mathcal{H}^k(n+n')$ such that for every coloring $\Delta : \binom{\mathcal{G}}{\mathcal{E}}_{part} \to r$ there exists an $\mathcal{F}_0$-subgraph with property ($\star$).

Let $(h_i)_{i<q}$ be an enumeration of $[A]\binom{n}{t}$. According to the partite lemma (Lemma 14.14), let $\mathcal{F}_0^*$ be a t-partite hypergraph satisfying

$$\mathcal{F}_0^* \overset{\text{part}}{\to} ((\mathcal{F}_0)_{h_0})_r^{\mathcal{E}},$$

where $\mathcal{E}$ is viewed as a t-partite graph. Now let $\mathcal{F}_1 = (\mathcal{F}_0)_{*h_0}(\mathcal{F}_0^*)$.

Observe that $\mathcal{F}_1$ has the following property. For any coloring $\Delta : \binom{\mathcal{F}_1}{\mathcal{E}}_{part} \to r$ there exists $f \in \binom{\mathcal{F}_1}{\mathcal{F}_0}_{part}$ which satisfies that for any $h, h' \in \binom{\mathcal{F}_1[f]}{\mathcal{E}}_{part}$ such that $h\rceil n = h'\rceil n = h_0$, we have $\Delta(\mathcal{F}_1[f \cdot h]) = \Delta(\mathcal{F}_1[f \cdot h'])$. Therefore, we have an $\mathcal{F}_0$-subgraph which satisfies property ($\star$) when restricted to the $\mathcal{E}$-subgraphs spanned by the partition given by h_0.

Let us assume that we have constructed a hypergraph $\mathcal{F}_i \subseteq \mathcal{H}^k(n + n_i)$ with the similar property as for $\mathcal{F}_1$: for any coloring $\Delta : \binom{\mathcal{F}_i}{\mathcal{E}}_{part} \to r$ there exists $f \in \binom{\mathcal{F}_i}{\mathcal{F}_0}_{part}$ which satisfies that for any $h, h' \in \binom{\mathcal{F}_i[f]}{\mathcal{E}}_{part}$ such that $h\rceil n = h'\rceil n = h_j$ for some $j < i$, we have $\Delta(\mathcal{F}_i[f \cdot h]) = \Delta(\mathcal{F}_i[f \cdot h'])$. Then, again by the partite lemma (Lemma 14.14), let $\mathcal{F}_i^*$ be a t-partite hypergraph satisfying

$$\mathcal{F}_i^* \overset{\text{part}}{\to} ((\mathcal{F}_i)_{h_i})_r^{\mathcal{E}},$$

and let $\mathcal{F}_{i+1} = (\mathcal{F}_i)_{*h_i}(\mathcal{F}_i^*)$. A moment of thought now reveals that $\mathcal{F}_{i+1}$ always contains an $\mathcal{F}_i$-subgraph which is monochromatic with respect to $\mathcal{E}$-copies spanned by partitions given by h_i. But now this $\mathcal{F}_i$ copy further contains an $\mathcal{F}_0$-subgraph for which property ($\star$) holds for all $\mathcal{E}$-copies spanned by partitions given by $h_0, \ldots, h_{i-1}$ and, by previous observation, also h_i.

Inductively repeating the same argument, we get that $\mathcal{F}_q$ always contains an $\mathcal{F}_0$-subgraph which satisfies property ($\star$). By the previous observations, this implies the existence of a monochromatic $\mathcal{F}$-subgraph, which finishes the proof. □