Chapter 13
Infinite Graphs

Considering infinite graphs, the picture is, even in the case of countable graphs, far from being complete. We discuss some of the pieces which are known. Section 13.1 deals with vertex colorings of Rado's graph R. We show that $R \to (R)_r^v$ for every positive integer r. In Sect. 13.2 we consider K_ℓ-free subgraphs of Rado's graph. Section 13.3 is concerned with edge colorings. Most of the results of this section are contained in the important paper (Erdős et al. 1975). We show that countable graphs do not have the edge partition property.

Graphs and embeddings of graphs are defined as in Chap. 12. The cardinality of a graph is the cardinality of its vertex set. The Ramsey arrow is used as introduced in Chap. 12.

13.1 Rado's Graph

Rado (1964) describes a construction of a universal countable graph, let us call it R, which has a lot of interesting properties. Being universal means that every countable graph can be embedded into R. The crucial property of R is that it is ω-good.

Definition 13.1. A graph $G = (V, E)$ is ω-good if for any two finite and disjoint sets X and Y of vertices there exists a vertex z not belonging to $X \cup Y$ such that z is joined by an edge to all $x \in X$ and not joined to any $y \in Y$.

Proposition 13.2. *Let $G = (V, E)$ be an ω-good graph. Then G is universal for countable graphs, i.e., every countable graph can be embedded into G.*

Proof. Let F be a countable graph. Without loss of generality we assume that $F = (\omega, E_F)$, i.e., the vertices of F are the nonnegative integers. We construct an embedding $f : \omega \to V$ inductively, one vertex at a time.

Let $f(0)$ be any vertex in V and suppose that $f(0), \ldots, f(n-1)$ have been defined. Consider the vertex n and let $A = \{k < n \mid \{k, n\} \in E_F\}$ be the set of

H.J. Prömel, *Ramsey Theory for Discrete Structures*,
DOI 10.1007/978-3-319-01315-2_13,

previous vertices which are joined to n, resp., let $B = \{k < n \mid \{k,n\} \notin E_F\}$ be its complement. Let $X = f(A)$ and $Y = f(B)$ be the corresponding sets in G. As G is ω-good there exists a vertex $z \in V \setminus (X \cup Y)$ which is joined to all $x \in X$ and not joined to any $y \in Y$. So, define $f(n) = z$ for any such z and continue as before. □

Actually, the above proof establishes slightly more, namely: any embedding of a finite subgraph of F into G can be extended to an embedding of F into G. This is to say that the automorphism group of any ω-good graph acts transitively on finite subgraphs, this property is sometimes called ultrahomogeneity.

Using the argument in the proof of Proposition 13.2 *back and forth* yields:

Proposition 13.3. *Any two countable ω-good graphs F and G are isomorphic.*

Proof. Proceed as in the proof of Proposition 13.2, however, 'back and forth'. At even-numbered steps try to embed F into G and at odd-numbered steps try to embed G into F. Eventually, any f constructed in such a way is an isomorphism. □

Knowing that, up to isomorphisms, there is just one countable ω-good graph we call this graph Rado's graph R. Still we are lacking some kind of explicit description, resp. a proof of the existence of countable ω-good graphs. Such an explicit construction has been given in Rado (1964).

Definition 13.4. Let the set $\mathcal{R} \subseteq [\omega]^2$ be defined as follows. Given $k < m$ put $\{k,m\} \in \mathcal{R}$ if and only if 2^k occurs in the binary expansion of m. Let $R = (\omega, \mathcal{R})$ be the graph which has as vertices nonnegative integers and $\mathcal{R}$ as the set of edges. One easily observes that this graph R is, in fact, ω-good.

Remark 13.5. About at the same time when Rado gave his construction, Erdős and Rényi (1963) showed that if one considers countably infinite random graphs by inserting edges independently with probability $1/2$ then almost surely any such random graph is ω-good. Thus, almost surely a countable random graph is isomorphic to Rado's graph R. For further interesting properties of Rado's graph compare, e.g., Cameron (1984).

Theorem 13.6. *For every positive integer r we have*

$$R \xrightarrow{ind} (R)^v_r.$$

Proof. Let $\Delta : \omega \to r$ be a coloring of the vertices of R. Let $V_i := \{n < \omega \mid \Delta(n) = i\}$ denote the set of vertices that are colored with color i. If the graph induced by V_i is ω-good then Proposition 13.2 implies that it contains an induced R-subgraph which is monochromatic in color i and we are done. Otherwise there exist finite and disjoint sets $X_i, Y_i \subseteq V_i$ such that V_i contains no vertex that is connected to all vertices in X_i and to no vertex in Y_i. Now assume that no set V_i induces an ω-good graph. Then consider $X := \bigcup_{i<r} X_i$ and $Y := \bigcup_{i<r} Y_i$. By construction, X and Y are finite and disjoint. As the Rado graph is ω-good, there exists a vertex z that is connected to all vertices in X and to no vertex in Y. As z has to be colored with some color this contradicts the definition of the sets X_i and Y_i. $\square$

With slightly more effort we also obtain a canonical version:

Theorem 13.7. *For every (unbounded) coloring $\Delta : \omega \to \omega$ of the vertices of the Rado graph $R = (\omega, \mathcal{R})$, there exists $X \subseteq \omega$ spanning a subgraph which is isomorphic to Rado's graph such that $\Delta \rceil X$ is constant or one-to-one.*

Proof. For finite and disjoint sets X and Y in ω, let $\Gamma(X, Y)$ be the set of vertices that are joined by an edge to all vertices in X and to no vertex in Y,

$$\begin{aligned}\Gamma(X, Y) = \{z < \omega \mid z \notin X \cup Y, \{x, z\} \in \mathcal{R} \ \text{ for all } \ x \in X,\\ \{y, z\} \notin \mathcal{R} \ \text{ for all } \ y \in Y\}.\end{aligned}$$

We first prove that for any finite and disjoint sets X and Y in ω, $\Gamma(X, Y)$ spans a ω-good graph. Assuming otherwise, there exists finite (and disjoint) subsets $C_1, C_2 \subseteq \Gamma(X, Y)$, for some finite (and disjoint) $X, Y \subseteq \omega$, such that $\Gamma(C_1, C_2) \cap \Gamma(X, Y) = \emptyset$. Since R is ω-good there exists some z not in $X \cup C_1 \cup Y \cup C_2$ such that z is joined to all vertices in $X \cup C_1$ and to no vertex in $Y \cup C_2$. But then $z \in \Gamma(X \cup C_1, Y \cup C_2) = \Gamma(C_1, C_2) \cap \Gamma(X, Y)$, yielding the desired contradiction.

Using this observation, we inductively find a set of vertices $\{x_0, \ldots, x_{n-1}\}$ such that

(1) $\Delta(x_i) \neq \Delta(x_j)$ for all $i < j < n$,
(2) $\{x_0, \ldots, x_{n-1}\}$ spans a graph which is isomorphic to the one spanned by $\{0, \ldots, n-1\}$; in other words, $\{x_0, \ldots, x_{n-1}\}$ yields a one-to-one colored initial segment of R,

or deduce that there exists a monochromatic subgraph isomorphic to Rado's graph. Note that for $n = 0$ these assertions hold vacuously, yielding the beginning of the induction. Having vertices $\{x_0, \ldots, x_{n-1}\}$ which satisfy (1) and (2), let $A_n \subseteq \{x_0, \ldots, x_{n-1}\}$ resp., $B_n = \{x_0, \ldots, x_{n-1}\} \setminus A_n$ be such that for every $x \in \Gamma(A_n, B_n)$ the set $\{x_0, \ldots, x_{n-1}, x\}$ is isomorphic to $\{0, \ldots, n\}$. If there exists a vertex $x \in \Gamma(A_n, B_n)$ such that $\Delta(x) \neq \Delta(x_i)$ for $0 \leq i < n$, then setting $x_n = x$ finishes the induction step. Otherwise, the subgraph $\Gamma(A_n, B_n)$ is colored with at most n different colors. By the above observation it is also ω-good. It thus follows from Theorem 13.6 that in this case there exist a monochromatic subgraph of $\Gamma(A_n, B_n)$ isomorphic to R, which finishes the proof. $\square$

13.2 Countable-Universal K_ℓ-Free Graphs

In this section we consider subgraphs of Rado's graph which do not contain complete graphs on ℓ vertices.

Definition 13.8. Let $\ell \geq 3$ be a positive integer. By U_ℓ we denote the subgraph of Rado's graph R which is spanned by the vertices $V_\ell = \{n < \omega \mid$ whenever $X \in [n]^{\ell-1}$ spans a $K_{\ell-1}$ in R then there exists $x \in X$ with $\{x, n\} \notin \mathcal{R}\}$.

Obviously, U_ℓ does not contain any complete graph on ℓ vertices, it is K_ℓ-free. Moreover, U_ℓ is universal with respect to the class of all countable K_ℓ-free graphs and its automorphism group acts transitively on finite subgraphs. This is summarized in the next proposition.

Proposition 13.9. *The graph U_ℓ satisfies the following properties:*

(1) *U_ℓ is K_ℓ-free,*
(2) *For any two finite and disjoint sets X and Y in U_ℓ such that X does not contain a complete graph on $(\ell - 1)$ vertices there exists a vertex $z \in V_\ell \setminus (X \cup Y)$ which is joined to all $x \in X$ and not joined to any $y \in Y$.*
(3) *Every countable K_ℓ-free F can be embedded into U_ℓ, moreover, every finite subgraph of U_ℓ which is isomorphic to a subgraph G of F can be extended to an F-subgraph.*
(4) *Any two countable graphs satisfying* (1) *and* (2) *are isomorphic.*

Proof. (1) is obvious from the construction, (3) follows from (2) using the same method as in the proof of Proposition 13.2, (4) follows, then, from a back and forth argument. So it remains to show (2). Consider $n_y = 2^{\max Y}$ and $n_x = \sum_{x \in X} 2^x + 2^{n_y}$. By definition of the Rado graph we have

$$\{w < n_x \mid \{w, n_x\} \in \mathcal{R}\} = X \cup \{n_y\}.$$

As n_y is only joined by an edge to $\max Y \notin X$, $X \cup \{n_y\}$ induces no $K_{\ell-1}$ thus we know that $n_x \in V_\ell$. $\square$

Henson (1971) showed that for every r-coloring of the vertices of U_ℓ, where $\ell \geq 3$ and r is a positive integer, one of the color-classes contains a copy of every finite K_ℓ-free graph. Alternatively, this can also be deduced from Folkman's result (Theorem 12.3). Henson, then, raised the question whether $U_\ell \to (U_\ell)^v_r$. This question was answered positively by El-Zahar and Sauer (1989). Here we only give a proof of the special case when $\ell = 3$ and $r = 2$, which is due to Komjáth and Rödl (1986).

Theorem 13.10.

$$U_3 \stackrel{ind}{\to} (U_3)^v_2.$$

Proof. For finite and disjoint subsets $X, Y \subset V_3$, let $\Gamma_3(X,Y) = \Gamma(X,Y) \cap V_3$ denote the set of those vertices in $V_3 \setminus (X \cup Y)$ that are completely connected to X and are not connected to any vertex in Y. From property (2) of Proposition 13.9 we deduce that for every finite subset $Y \subset V_3$ the graph induced by $\Gamma_3(\emptyset, Y)$ is ω-good. As in the proof of Theorem 13.7 it thus follows that the graph induced by $\Gamma_3(\emptyset, Y)$ is isomorphic to U_3.

For a given red-blue coloring of $V_3 = \{v_0, v_1, \ldots\}$, we may assume that both colors appear infinitely many times, as otherwise the previous observation implies that there exists a monochromatic subgraph isomorphic to U_3. Furthermore, assume that there is no red subgraph isomorphic to U_3. We show that this implies the existence of a blue subgraph isomorphic to U_3.

In order to see this we inductively define a sequence of vertices $z_0, z_1, \ldots \in V_3$ and sequences $Y_0, Y_1, \ldots \subset V_3$, $S_0, S_1, \ldots \subset V_3$ and $A_0, A_1, \ldots \subset V_3$ such that the following properties hold for all $n < \omega$:

(1) $S_n = \bigcup_{j<n}(Y_j \cup \{z_j\})$, $Y_n \cap S_n = \emptyset$, and all vertices in Y_i are colored red,
(2) The subgraph spanned by Y_n is isomorphic to the subgraph spanned by $\{v_0, \ldots, v_{k_n}\}$ for some $0 < k_n < \omega$,
(3) $A_n \subseteq Y_n$ such that $A_n = \{j \le k_n \mid \{v_j, v_{k_n+1}\} \in E(U_3)\}$,
(4) Y_n is a maximal subset of V_3 (maximal by set inclusion) with respect to properties (1)–(3),
(5) $z_n \notin S_n \cup Y_n$, and z_n is colored blue,
(6) Let $B_n := \{z_j < n \mid \{v_j, v_n\} \in E(U_3)\}$; if $B_n = \emptyset$ then z_n is not joined to any vertex in $S_n \cup Y_n$, otherwise z_n is joined to all vertices in $B_n \cup A_{n_0}$ and to no vertex in $(S_n \cup Y_n) \setminus (B_n \cup A_{n_0})$, where $n_0 = \min\{j \mid z_j \in B_n\}$.

Clearly, properties (5) and (6) imply that for all $n < \omega$ the subgraph spanned by $\{z_0, \ldots, z_n\}$ is monochromatic (in blue) and isomorphic to the subgraph induced by $\{v_0, \ldots, v_n\}$. Therefore, if we can show that such an infinite sequence exists this will finish the proof.

Assume that we have found a family of subsets $Y_0, \ldots, Y_{n-1}$ and a set of vertices $z_0, \ldots, z_{n-1}$ which satisfy (1)–(6). In order to construct Y_n start with $Y_n = \{v\}$, where v is any red vertex such that $v \notin S_n$ (which exists as we have infinitely many vertices that are colored red). Then greedily add vertices to Y_n so that (2) and (3) remain satisfied. As we assumed that there exits no red monochromatic subgraph isomorphic to U_3, this process will stop with a finite set Y_n satisfying (2)–(4).

If v_n is not joined by an edge to any $v_0, \ldots, v_{n-1}$, then the fact that $\Gamma_3(\emptyset, Y_n \cup S_n)$ is isomorphic to U_3 implies that it cannot contain only red vertices colored; thus taking z_n to be any blue vertex in $\Gamma_3(\emptyset, Y_n \cup S_n)$ suffices.

Otherwise, let B_n be as defined in (6) and let $n_0 := \min\{j \mid z_j \in B_n\}$. If $\Gamma_3(B_n \cup A_{n_0}, (S_n \cup Y_n) \setminus (B_n \cup A_{n_0}))$ is not empty, then by maximality of Y_{n_0} and the definition of A_{n_0} it contains only vertices colored with blue, and it can easily be seen any such vertex can be taken as z_n, satisfying properties (5) and (6). Therefore it only remains to argue that $\Gamma_3(B_n \cup A_{n_0}, (S_n \cup Y_n) \setminus (B_n \cup A_{n_0}))$ cannot be the empty set.

Observe that, by property (2) of Proposition 13.9, it suffices to show that there is no edge in the subgraph induced by $B_n \cup A_{n_0}$. By the definition of A_{n_0} and the fact that U_3 is K_3-free we know that there exists no edge in the subgraph induced by A_{n_0}. By the definition go B_n we know that any edge between two vertices $x, y \in B_n$ spans a triangle with V_n, which can't be. Finally, assume there is an edge between some vertex $x \in A_{n_0}$ and a vertex $z_i \in B_n$, for some $n_0 < i < n$. Observe that (6) implies that the only case that this can happen is when $i_0 = n_0$. But then we have an edge joining v_{i_0} and v_i, again closing a triangle with v_n. Therefore there is no edge in the subgraph induced by $B_n \cup A_{n_0}$ and we can thus find a vertex z_n which satisfies all properties. $\square$

13.3 Colorings of Edges

Considering colorings of edges it turns out that Rado's graph no longer has the property to arrow itself. In fact an even stronger negative result is known (Erdős et al. 1975).

Proposition 13.11. *Let $K_{\omega,\omega}$ be the complete bipartite graph with both parts being countably infinite. Then there exists a 2-coloring of the edges of Rado's graph R such that no induced $K_{\omega,\omega}$-subgraph is monochromatic. In other words,*

$$R \overset{ind}{\nrightarrow} (K_{\omega,\omega})^e_2$$

Proof. We first have to define a 2-coloring of the edges of $R = (\omega, \mathcal{R})$. The idea is to play with two different orders on $\mathcal{R}$. The first one, denoted by $\leq$, is the usual order of nonnegative integers. To define the second one we need some preparation. Recall that nonnegative integers $k < m$ are joined by an edge if and only if 2^k occurs in the binary expansion of m. That is, if consider then binary expansion of m, i.e.

$$m = \sum_{i \geq 0} m_i 2^i \qquad \text{with } m_i \in \{0, 1\},$$

then all but finitely many of the m_i's are zero and we have $\{k, m\} \in \mathcal{R}$ if and only if $k < m$ and $m_k = 1$.

The second order, denoted by $\preceq$, is the lexicographic order of the binary expansions, from left to right with $0 < 1$. So for $m = \sum_{i \geq 0} 2^i m_i$ and $n = \sum_{i \geq 0} 2^i n_i$ we have $m \preceq n$ if and only if there exists j such that $m_i = n_i$ for all $i < j$ and $m_j < n_j$. Observe that this implies, for example, that all even nonnegative integers precede the odd ones. In general, $\preceq$ measures which of the two numbers is more 'odd' than the other, and this is, then, the larger one.

Now color an edge $\{k, m\} \in \mathcal{R}$ with color 0 if $\leq$ and $\preceq$ coincide on this edge, i.e., $k \leq m$ and $k \preceq m$, and color it with color 1 otherwise.

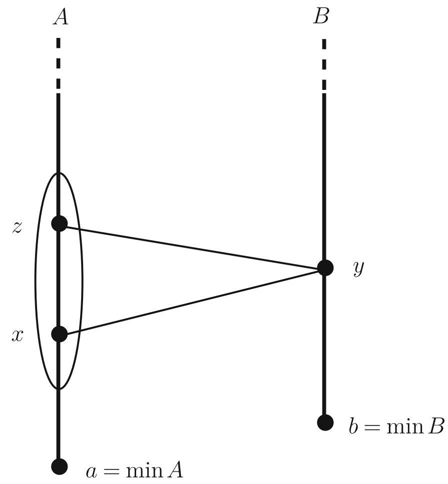

Fig. 13.1 There is no monochromatic $K_{\omega,\omega}$

Assume that R contains an induced $K_{\omega,\omega}$-subgraph which is monochromatic. Say, A and B are the two stable parts with $a = \min A < b = \min B$ (Fig. 13.1).

Let A' be an infinite subset of A such that $m_i = n_i$ for all $m, n \in A'$ and all $i \leq a$. Then $m_a = n_a = 0$ as a and m, resp., a and n are not joined by an edge. As A' and B are both infinite there exist $x, z \in A'$ and $y \in B$ such that $x < y < z$. Note that $y_a = 1 > 0 = x_a = z_a$ as a and y are joined by an edge and $a < y$.

If the subgraph is monochromatic in color 0, then the two orders coincide and we have $x \preceq y \preceq z$. As $y \preceq z$ there exists an i such that $y_j = z_j$ for all $j < i$ and $y_i < z_i$. As $y_a > z_a$ this implies $i < a$ and thus, by the definition of A', $y \preceq x$, a contradiction. If the subgraph is monochromatic in color 1, then $z \preceq y \preceq x$. In this case we obtain the desired contradiction similarly as above, with the rôles of x and z interchanged. □

Corollary 13.12.

$$R \overset{ind}{\nrightarrow} (R)^e_2$$

□

Let us call a graph G locally finite if each vertex of G is joined by an edge only to finitely many vertices of G or, alternatively, it is joined to almost all vertices of G (both kinds of vertices are allowed to occur). Clearly, $K_{\omega,\omega}$ is not locally finite. In contrast to Proposition 13.11, Erdős et al. (1975) prove the following positive partition theorem:

Theorem 13.13. *Let r be a positive integer and let G be a countable locally finite graph. Then for every r-coloring of the edges of Rado's graph there exists* *a monochromatic induced G-subgraph, in other words, $R \overset{ind}{\to} (G)^e_r$ for all positive integers r and countable locally finite graphs G.* □

There is still a gap between Proposition 13.11 and Theorem 13.13. A characterization of all those countable graphs G satisfying $R \to (G)^e_r$ for every positive integer r is not known. Clearly, such a G must not contain an infinite stable set which is completely joined to another infinite set.

We do not prove Theorem 13.13 here, but refer the reader to Erdős et al. (1975).