# Chapter 11
# Partition Relations for Cardinal Numbers

Recall the infinite version of Ramsey's theorem: $\omega \rightarrow (\omega)^k_r$, whenever $k, r$ are positive integers. The aim of this section is to discuss some extensions of this relation to larger cardinals. Our treatment will be far from complete. For $\omega$ more results on this topic we refer the reader to the book of Erdős et al. (1984).

We start with a negative result, proved by Erdős and Rado (1952) which shows that the exponent $k$ may not be replaced by an infinite cardinal without conflicting with the axiom of choice.

**Proposition 11.1.** *Let $\kappa \geq \omega$ be a cardinal. Then*

$$\kappa \not\rightarrow (\omega)^\omega_2,$$

*where $\not\rightarrow$ denotes the negation of $\rightarrow$.*

*Proof.* Let $<_{\text{well}}$ be a well-ordering of $[\kappa]^\omega$, the set of countable subsets of $\kappa$. We define a coloring $\Delta : [\kappa]^\omega \rightarrow 2$ witnessing to $\kappa \not\rightarrow (\omega)^\omega_2$ as follows:

$$\Delta(A) = \begin{cases} 0, & \text{if there exists } B \subset A \text{ such that } B <_{\text{well}} A \\ 1, & \text{otherwise.} \end{cases}$$

Now let $F \in [\kappa]^\omega$ and let $A = \{a_i \mid i < \omega\} \in [F]^\omega$ be the first $\omega$-subset with respect to $<_{\text{well}}$ in $F$. Take any proper $\omega$-subset $B \subset A$, then $A <_{\text{well}} B$ and therefore $\Delta(A) = 1$. On the other hand, let $A^* = \{a_{2i+1} \mid i < \omega\}$ and for each $m < \omega$ let $A_m = \{a_0, a_2, \ldots, a_{2m}\} \cup A^*$. Put $A_{m_0} = \min\{A_m \mid m < \omega\}$, where the minimum is taken with respect to the well-ordering $<_{\text{well}}$ . Then $A_{m_0} <_{\text{well}} A_{m_0+1}$ and $A_{m_0} \subset A_{m_0+1}$. Hence, $\Delta(A_{m_0+1}) = 0$ which proves Proposition 11.1. □

This result prevents us from considering colorings of infinite subsets in this chapter. But observe that the proof given above uses essentially the Axiom of Choice, i.e., Zermelo's well-ordering theorem. If one drops the Axiom of Choice, even the relation $\omega \rightarrow (\omega)^\omega_2$ may be consistent, cf., e.g., Mathias (1969) and Kleinberg (1970).

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Throughout this chapter we assume the Axiom of Choice. All set-theoretic notions used are standard and can be found, e.g., in Jech (1978).

Following the convention introduced by John von Neumann we identify ordinals with the set of their predecessors and cardinals with their initial ordinals. For every cardinal $\kappa$ let $\kappa^+$ denote the least cardinal greater than $\kappa$, i.e., the (cardinal) successor of $\kappa$. A cardinal $\kappa$ is called regular, if for every $\lambda < \kappa$ and any choice of subsets $A_\nu \subseteq \kappa$ for $\nu < \lambda$ with $|A_\nu| < \kappa$ it follows that $|\bigcup_{\nu<\lambda} A_\nu| < \kappa$, in other words, $\kappa$ cannot be written as the union of less than $\kappa$ many sets of cardinality less than $\kappa$. It can easily be shown, using the Axiom of Choice, that every successor cardinal is regular.

Addition and multiplication of infinite cardinals $\kappa$ and $\lambda$ is easy:

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

Exponentiation, in general, is more difficult, but for our purposes it is enough to know that $\lambda \le \kappa$ implies that $\lambda^\kappa = 2^\kappa$ and $\lambda^n = \lambda$ for every finite $n$.

As usual we denote the first infinite cardinal also by $\aleph_0$, i.e., $\aleph_0 = \omega$, and the second one, the first uncountable cardinal, by $\aleph_1$.

Section 11.1 is devoted to the proof of the Erdős-Rado partition theorem for cardinals, in Sect. 11.2 some negative partition relations are given essentially showing that the Erdős-Rado theorem is best possible in the sense that the Ramsey numbers are correctly estimated. In Sect. 11.3 Dushnik-Miller's theorem (for regular cardinals) is discussed. In Sect. 11.4 we consider the question for which cardinals $\kappa$ other than $\omega$ the relation $\kappa \to (\kappa)^2_2$ might be true. Finally, in Sect. 11.5 we glance briefly at canonical partition relations for cardinals.

## 11.1 Erdős-Rado's Partition Theorem for Cardinals

The following quite general partition relation for cardinals is due to Erdős and Rado (1956). Let $\exp_0(\kappa) = \kappa$ and $\exp_{k+1}(\kappa) = 2^{\exp_k(\kappa)}$.

**Theorem 11.2 (Erdős, Rado).** *Let $\kappa \ge \omega$ be a cardinal and $k$ be a positive integer. Then*

$$\exp_{k-1}(\kappa)^+ \to (\kappa^+)^k_\kappa.$$

*Proof.* We proceed by induction on $k$, the case $k = 1$, i.e., $\kappa^+ \to \left(\kappa^+\right)^1_\kappa$, reduces to the pigeonhole principle. So assume that the theorem is valid for some $k \ge 1$, put $\lambda = \exp_{k-1}(\kappa)$ and let $\Delta : \binom{(2^\lambda)^+}{k+1} \to \kappa$ be a coloring. We want to find a monochromatic set $F \subset (2^\lambda)^+$ of size $\kappa^+$. For each $x < (2^\lambda)^+$ let $\Delta_x : \binom{(2^\lambda)^+ \setminus \{x\}}{k} \to \kappa$ be defined by $\Delta_x(A) := \Delta(A \cup \{x\})$. We claim:

1. There exists a set $S \subset (2^\lambda)^+$ of cardinality $|S| = 2^\lambda$ such that for every $M \subset S$ with $|M| \leq \lambda$ and every $x \in (2^\lambda)^+ \setminus S$ there exists $y = y(M, x) \in S \setminus M$ so that $\Delta_x \rceil [M]^k = \Delta_y \rceil [M]^k$.

Before proving (1) we show how this implies the theorem. Fix some $x \in (2^\lambda)^+ \setminus S$. By transfinite induction we define a set $Y = \{y_\mu \mid \mu < \lambda^+\} \subseteq S$ as follows. Let $y_0 \in S$ be arbitrary and assume that $\{y_\mu \mid \mu < \nu\} = M$ has been defined for some $\nu < \lambda^+$. Then let $y_\nu = y(M, x)$ be according to (1). Observe that for every $A \in \binom{Y}{k+1}$, where $A = \{y_{i_0}, \ldots, y_{i_{k-1}}, y_\nu\}$ with $i_0 < \ldots < i_{k-1} < \nu$, we have that

$$\Delta(\{y_{i_0}, \ldots, y_{i_{k-1}}, y_\nu\}) = \Delta_{y_\nu}(\{y_{i_0}, \ldots, y_{i_{k-1}}\}) = \Delta_x(\{y_{i_0}, \ldots, y_{i_{k-1}}\}). \quad (11.1)$$

Now consider $\Delta_x : [Y]^k \to \kappa$. Since $|Y| = \lambda^+$ and according to the inductive hypothesis there exists an $F \subseteq Y$ with $|F| = \kappa^+$ so that $\Delta_x(A) = \Delta_x(B)$ for all $A, B \in [F]^k$. Thus, the theorem follows from (11.1).

It remains to prove (1). We define an ascending sequence $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_\mu \subseteq \ldots, \mu < \lambda$, of subsets of $(2^\lambda)^+$, each of size $2^\lambda$, as follows.

Choose $S_0 \subset (2^\lambda)^+$ with $|S_0| = 2^\lambda$ arbitrarily and for each limit ordinal $\nu$ let $S_\nu = \bigcup_{\mu<\nu} S_\mu$. Now assume that $S_\mu$ with $|S_\mu| = 2^\lambda$ has been defined. We now define $S_{\mu+1}$.

Observe that there exist at most $(2^\lambda)^\lambda = 2^\lambda$ subsets of $S_\mu$ of size $\lambda$ and therefore there exist at most $\lambda \cdot 2^\lambda = 2^\lambda$ subsets $M$ of $S_\mu$ of size at most $\lambda$. Fix such an $M \subseteq S_\mu$. Then there exist at most $2^\lambda$ mappings $f : [M]^k \to \kappa$ (recall that $\kappa \leq \lambda$). This shows that

$$|\{\Delta_x \rceil [M]^k \mid x \in (2^\lambda)^+ \setminus M\}| \leq 2^\lambda.$$

Assume a well-ordering on $(2^\lambda)^+$ to be given and for every $x \in (2^\lambda)^+ \setminus S_\mu$ let $y(M, x)$ be the smallest $y \in (2^\lambda)^+ \setminus M$ such that $\Delta_x \rceil [M]^k = \Delta_y \rceil [M]^k$. Denote the set of those $y$ by $Y(M)$. Then $|Y(M)| \leq 2^\lambda$. Now put

$$S_{\mu+1} = S_\mu \cup \bigcup Y(M),$$

where the union is taken over all $M \subseteq S_\mu$ with $|M| \leq \lambda$. Then $|S_{\mu+1}| \leq 2^\lambda + 2^\lambda \cdot 2^\lambda = 2^\lambda$. Finally let $S = \bigcup_{\mu<\lambda} S_\mu$. Then $|S| \leq \lambda^+ \cdot 2^\lambda = 2^\lambda$ and, by construction, $S$ has the desired properties. $\square$

It seems to be worth while to state the following special case explicitly:

**Corollary 11.3.** *For every $\kappa \geq \omega$ it follows that $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$, and, even more special, $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$.* $\square$

## 11.2 Negative Partition Relations

Next we are going to show that Theorem 11.2 is, in a sense, best possible. Before we will do this in general, we prove that Corollary 11.3 is the best we can expect.

Let $\kappa$ be a cardinal. Then we denote by $2^\kappa$ the set of all sequences of length $\kappa$ over the alphabet $2 = \{0, 1\}$. Hence, every $x \in 2^\kappa$ can be written as $x = (x(0), \dots, x(\nu), \dots)$ where $x(\nu) < 2$ for every $\nu < \kappa$. The natural order on 2, i.e., $0 < 1$, gives a lexicographic order on $2^\kappa$ which will be denoted by $\prec$. So $x \prec y$ if and only if $x(\nu) < y(\nu)$ where $\nu$ is the least $\nu$ such that $x(\nu) \neq y(\nu)$. In fact, we know that then $x(\nu) = 0$ and $y(\nu) = 1$.

**Lemma 11.4.** *For every $\kappa \geq \omega$ it follows that $2^\kappa \not\rightarrow (3)^2_\kappa$, and hence $2^{\aleph_0} \not\rightarrow (3)^2_{\aleph_0}$.*

*Proof.* Let $\Delta : [2^\kappa]^2 \to \kappa$ be defined by $\Delta(\{x, y\})$ being the least position $\nu < \kappa$ such that $x(\nu) \neq y(\nu)$. Obviously it is impossible to have pairwise distinct $x, y, z \in 2^\kappa$, such that $\Delta(\{x, y\}) = \Delta(\{x, z\}) = \Delta(\{y, z\})$. □

The following result of Sierpiński (1933) shows in particular that the straightforward generalization of Ramsey's theorem, viz. $\aleph_1 \rightarrow (\aleph_1)^2_2$, is false.

**Theorem 11.5 (Sierpiński).** *For every $\kappa \geq \omega$ it follows that $2^\kappa \not\rightarrow (\kappa^+)^2_2$, and hence $2^{\aleph_0} \not\rightarrow (\aleph_1)^2_2$.*

*Proof.* We will derive Theorem 11.5 from the following fact:

1. There does not exist any increasing or decreasing $\kappa^+$-sequence in $2^\kappa$ with respect to $\prec$.

We show that $2^\kappa$ has no increasing $\kappa^+$-sequence. The decreasing case can be handled analogously. To derive a contradiction assume that $X = \{x_\nu : \nu < \kappa^+\} \subseteq 2^\kappa$ is an increasing $\kappa^+$-sequence, i.e., $x_\mu \prec x_\nu$ whenever $\mu < \nu$. For each $\nu < \kappa^+$ and each $\mu < \kappa$ let $x_\nu \rceil \mu = (x_\nu(0), \dots, x_\nu(\mu'), \dots)$, $\mu' < \mu$ be the initial segment of length $\mu$ of $x_\nu$. Now let $\eta \leq \kappa$ be the least ordinal such that $|\{x_\nu \rceil \eta \mid \nu < \kappa^+\}| = \kappa^+$. Without loss of generality we can assume that $x_\mu \rceil \eta \neq x_\nu \rceil \eta$ for all $x_\mu$ and $x_\nu$ in $X$. Otherwise one could choose an appropriate subset of $X$ which is still of size $\kappa^+$. Define a sequence $d_\nu, \nu < \kappa^+$, where $d_\nu$ gives the least position at which $x_\nu$ and $x_{\nu+1}$ differ. By our assumption on $X$ we know that $d_\nu < \eta \leq \kappa$ for every $\nu < \kappa^+$. Thus there exists $\gamma < \eta$ such that $d_\nu = \gamma$ for $\kappa^+$ many $\nu$. Observe that $|\{x_\nu \rceil \gamma \mid \nu < \kappa^+\}| \leq \kappa$. So let $\mu' < \mu$ be such that $d_{\mu'} = d_\mu = \gamma$ and $x_{\mu'} \rceil \gamma = x_\mu \rceil \gamma$. Then $x_\mu \prec x_{\mu'+1}$. But on the other hand, since $\mu' < \mu' + 1 \leq \mu$, we have $x_{\mu'+1} \preceq x_\mu$, a contradiction which proves (1).

We now prove Theorem 11.5 by defining a 2-coloring of $[2^\kappa]^2$ which does not admit a monochromatic $\kappa^+$-set. So let $x_\nu, \nu < 2^\kappa$, be any enumeration of $2^\kappa$ and let $\Delta : [2^\kappa]^2 \to 2$ be given by

$$\Delta(x_\mu, x_\nu) = \begin{cases} 0, & \text{if } \mu < \nu \text{ and } x_\mu \prec x_\nu \\ 1, & \text{if } \mu < \nu \text{ and } x_\nu \prec x_\mu. \end{cases}$$

Then a monochromatic set of size $\kappa^+$ would contradict (1). $\square$

The following more general results are contained in Erdős et al. (1965).

**Theorem 11.6.** *Let $\kappa \geq \omega$ be a cardinal and $k \geq 2$ be a positive integer. Then*

$$\exp_{k-1}(\kappa) \not\to (\kappa^+)^k_2.$$

**Theorem 11.7.** *Let $\kappa \geq \omega$ be a cardinal and $k \geq 3$ be a positive integer. Then*

$$\exp_{k-1}(\kappa) \not\to (k+1)^k_\kappa.$$

## 11.3 Dushnik-Miller's Theorem

By Theorem 11.6 we have that $\aleph_1 \not\to (\aleph_1)^2_2$. On the other hand, Ramsey's theorem trivially implies $\aleph_1 \to (\omega)^2_2$. In this section we prove a partition relation which is, in a sense, halfway between these two relations.

For cardinals $\kappa$, $\lambda_0$ and $\lambda_1$ let $\kappa \to (\lambda_0, \lambda_1)^2_2$ denote the assertion that for every 2-coloring of $[\kappa]^2$ there exists either a set of size $\lambda_0$ which is monochromatic in color 0 or a set of size $\lambda_1$ which is monochromatic in color 1. Hence, in particular, $\aleph_1 \to (\omega, \omega)^2_2$. The following result is due to Dushnik and Miller (1941):

**Theorem 11.8.** *Let $\kappa \geq \omega$ be a regular cardinal. Then $\kappa \to (\kappa, \omega)^2_2$ and, in particular, $\aleph_1 \to (\aleph_1, \omega)^2_2$.*

We should mention that Theorem 11.8 is also true for singular (i.e., non-regular) cardinals, cf. Dushnik and Miller (1941).

*Proof of Theorem 11.8.* Let $\Delta : [\kappa]^2 \to 2$ be a coloring. First, we show:

1. If for every $S \subseteq \kappa$ of size $\kappa$ there exists an $x \in S$ such that $|\{y \in S \mid \Delta(\{x, y\}) = 1\}| = \kappa$ then there exists a countable set $D \subseteq \kappa$ such that $D$ is monochromatic with color 1.

Let $\Gamma(x) = \{y < \kappa \mid \Delta(\{x, y\}) = 1\}$. Choose $d_0 < \kappa$ arbitrarily such that $|\Gamma(d_0)| = \kappa$. Now assume that $D_n = \{d_0, \ldots, d_n\}$ is defined such that $\Delta\rceil[D_n]^2 \equiv 1$ and such that $|S_n| = \kappa$, where $S_n = \bigcap\{\Gamma(d_i) \mid d_i \in D_n\}$. Then choose $d_{n+1} \in S_n$ such that $|\{y \in S_n \mid \Delta(\{d_{n+1}, y\}) = 1\}| = \kappa$. Clearly, $D = \bigcup_{n<\omega} D_n$ satisfies (1).

So we assume that there is no countable set $D \subseteq \kappa$ with $\Delta\rceil[D]^2 \equiv 1$. Then let $S \subseteq \kappa$ be of size $\kappa$ so that for every element $x \in S$ if follows that $|\{y \in S \mid \Delta(\{x, y\}) = 1\}| < \kappa$. We construct recursively a sequence $x_\nu$, such that $\Delta(\{x_\mu, x_\nu\}) = 0$ whenever $\mu < \nu < \kappa$. Assume that $(x_\nu \in S \mid \nu < \nu')$ have been constructed for some $\nu' < \kappa$. Then $|S \cap (\bigcup_{\nu<\nu'} \Gamma(x_\nu))| < \kappa$. Notice that here the regularity of $\kappa$ is needed. Now choose $x_{\nu'} \in S \setminus \bigcup_{\nu<\nu'} \Gamma(x_\nu)$, completing the proof of Theorem 11.8. $\square$

## 11.4 Weakly Compact Cardinals

As shown in Theorem 11.2, for every pair $\lambda, \rho$ of cardinals, where $\rho < \lambda$, and every positive integer $k$ there exists a cardinal $\kappa$ such that $\kappa \to (\lambda)^k_\rho$. Moreover, in case $\lambda$ is a successor cardinal we have determined the smallest $\kappa$ satisfying this relation. But it seems to be a natural question to ask whether $\kappa$ can be $\lambda$, in particular whether the relation $\kappa \to (\kappa)^2_2$ can hold for cardinals other than $\kappa = \omega$. The answer to this question leads immediately to large cardinals.

Let $\kappa$ be an uncountable cardinal, i.e., $\kappa > \omega$. Then $\kappa$ is called inaccessible if $\kappa$ is regular and $2^\lambda < \kappa$ for every $\lambda < \kappa$. Inaccessible cardinals were introduced by Sierpiński and Tarski (1930). In particular they have the property that $|X| < \kappa$ implies $|\mathcal{P}(X)| < \kappa$. This and some other properties inaccessible cardinals share with $\omega$. So, in a sense, one can say that an inaccessible cardinal is related to smaller cardinals as $\omega$ is related to finite cardinals. But it is not at all clear whether inaccessible cardinals do exist. To be more precise: One can show that the existence of such cardinals cannot be proved in ZF + Axiom of Choice. Erdős et al. (1965) showed that the requirement $\kappa \to (\kappa)^2_2$ leads at least to inaccessible cardinals.

**Theorem 11.9.** *If $\kappa > \omega$ and $\kappa \to (\kappa)^2_2$, then $\kappa$ is inaccessible.*

*Proof.* We have to show that $\kappa$ is regular and that $2^\lambda < \kappa$ for every $\lambda < \kappa$. The second assertion follows immediately from Sierpiński's Theorem 11.5. Assume that $\kappa \leq 2^\lambda$ for some $\lambda < \kappa$. Then $2^\lambda \not\to (\lambda^+)^2_2$ implies $\kappa \not\to (\lambda^+)^2_2$ and hence $\kappa \not\to (\kappa)^2_2$.

So it remains to show that $\kappa$ is regular. Suppose not. Then there exists a family $X_\nu$, $\nu < \lambda$, for some $\lambda < \kappa$ of pairwise disjoint sets such that $|X_\nu| < \kappa$ for each $\nu < \lambda$ and $\kappa = |\bigcup\{X_\nu \mid \nu < \lambda\}|$. Define $\Delta : [\kappa]^2 \to 2$ by $\Delta(\{x, y\}) = 0$ if $\{x, y\} \subseteq X_\nu$ for some $\nu < \lambda$, $\Delta(\{x, y\}) = 1$, otherwise. Obviously, there does not exist $M \in [\kappa]^\kappa$ which is monochromatic with respect to $\Delta$, thus contradicting $\kappa \to (\kappa)^2_2$. $\square$

Cardinals $\kappa > \omega$ satisfying $\kappa \to (\kappa)^2_2$ are called weakly compact. From what is said before it follows that their existence cannot be proved in ZF + Axiom of Choice. In a sense, the situation is even worse. One can show that $\kappa \to (\kappa)^2_2$ fails for many inaccessible cardinals including the first one provided such numbers exist at all. Moreover, even if the existence of an inaccessible cardinal is assumed it cannot be proved in ZF + Axiom of Choice that there is a weakly compact cardinal. For a detailed discussion and an extensive bibliography on this topic, compare Erdős et al. (1984).

We close this paragraph with stating a result which shows that if there exists a weakly compact cardinal it has indeed quite strong partition properties.

**Theorem 11.10.** *If $\kappa \to (\kappa)^2_2$, then $\kappa \to (\kappa)^k_\rho$, for every $k < \omega$ and every $\rho < \kappa$.*

This result can be shown using similar arguments as in the proof of Ramsey's theorem. We omit the proof.

## 11.5 Canonical Partition Relations for Cardinals

Finally, we briefly review some canonical partition results for infinite cardinals. The canonical Ramsey arrow extends naturally to arbitrary cardinals, $\kappa \xrightarrow{\text{can}} (\lambda)^k$ meaning that for every coloring $\Delta$ of the $k$-subsets of $\kappa$ with arbitrary many colors there exists a $\lambda$-subset $F \in [\kappa]^\lambda$ of $\kappa$ so that $\Delta\rceil[F]^k$ is canonical. For $\kappa = \lambda = \omega$, $k < \omega$, this relation was shown in the Erdős-Rado canonization theorem (Theorem 1.4). The argument given there to prove $\omega \xrightarrow{\text{can}} (\omega)^k$ actually shows that if $\lambda$, $\kappa$ and $k$ are cardinals with $\lambda > 2k$ such $\kappa \to (\lambda)^{2k}_k$ then $\kappa \xrightarrow{\text{can}} (\lambda)^k$. Combining this observation with Theorem 11.10 we obtain immediately:

**Theorem 11.11.** *If $\kappa \to (\kappa)^2_2$, then $\kappa \xrightarrow{\text{can}} (\kappa)^k$ for every $k < \omega$.* □

Moreover, applying the relation $\exp_{k-1}(\kappa)^+ \to (\kappa^+)^k_k$ (instead of Ramsey's theorem) in the proof of Theorem 1.4 yields that

$$\exp_{2k-1}(\kappa)^+ \xrightarrow{\text{can}} (\kappa^+)^\kappa.$$

However, this is far from best possible. Baumgartner (1975) showed that the same cardinal which satisfies the Erdős-Rado partition relation is already large enough for the canonical partition relation:

**Theorem 11.12.** *Let $\kappa \geq \omega$ be a cardinal and $k$ be a positive integer. Then*

$$\exp_{k-1}(\kappa)^+ \xrightarrow{\text{can}} (\kappa^+)^k.$$

We omit the proof of this result which combines ideas behind a theorem of Fodor (1956) on regressive mappings and the Erdős-Rado canonization theorem.