Chapter 10
A Quasi Ramsey Theorem

The basic problem of (combinatorial) discrepancy theory is how to color a set with two colors as uniformly as possible with respect to a given family of subsets. The aim is to achieve that each of the two colors meets each subset under consideration in approximately the same number of elements. From the finite Ramsey theorem (cf. Corollary 7.2) we know already that if the set of all 2-subsets of n is 2-colored, and the family of all ℓ-subsets for some $\ell < \frac{1}{2}\log n$ is considered, the situation is as bad as possible: for any 2-coloring we will find a monochromatic ℓ-set. As ℓ gets larger one can color more uniformly though one still has the preponderance phenomenon.

Let k and n be positive integers and let $\chi_k : [n]^k \to \{-1, +1\}$ be a 2-coloring of the k-subsets of n. For $T \subseteq n$ let

$$\chi_k(T) = \sum_{X \in [T]^k} \chi_k(X).$$

Then $\chi_k(T) = 0$ means that T is colored as uniformly as possible, i.e., the color '-1' and the color '$+1$' occur equally often. The discrepancy of χ_k is defined by

$$\mathrm{disc}(\chi_k) = \max_{T \subseteq n} |\chi_k(T)|.$$

and the discrepancy of n with respect to colorings of k-subsets is given by

$$\mathrm{disc}(k, n) = \min \mathrm{disc}(\chi_k),$$

where the minimum is taken over all 2-colorings $\chi_k : [n]^k \to \{-1, +1\}$. Trivially, $\mathrm{disc}(1, n) = \lceil \frac{n}{2} \rceil$ for every n. From Corollary 7.2 we also get that $\mathrm{disc}(2, n) > \frac{1}{2}\log n$.

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Extending earlier results of Erdős (1963) and Erdős and Spencer (1972) proved:

Theorem 10.1 (Erdős, Spencer). *Let k be a positive integer. Then there exist constants $c_0 = c_0(k)$ and $c_1 = c_1(k)$ such that for every n*

$$c_0 n^{\frac{k+1}{2}} \leq \operatorname{disc}(k, n) \leq c_1 n^{\frac{k+1}{2}}$$

In this section we will focus on the discrepancy problem for finite sets, i.e., on Theorem 10.1. For an excellent surveys on discrepancy results in general see e.g. Sós (1983) and Beck and Sós (1995) or the book by Chazelle (2000).

10.1 The Upper Bound

It is not surprising that the upper bound in Theorem 10.1 is given by probabilistic means. The basic tool in proving this upper bound is the inequality of Chernoff (1952). Here we use it in a version given by Spencer (1985, p. 362).

Lemma 10.2 (Chernoff). *Let X_i, $i < n$, be mutually independent random variables with* $\operatorname{Prob}[X_i = -1] = \operatorname{Prob}[X_i = +1] = \frac{1}{2}$ *for $i < n$ and put $S_n = \sum_{i<n} X_i$. Let $a > 0$ be some constant. Then*

$$\operatorname{Prob}[S_n > a] < e^{-\frac{1}{2}\frac{a^2}{n}}.$$

□

Now fix some $k \geq 1$ and let $\chi_k : [n]^k \to \{-1, +1\}$ be a random mapping, taking the values -1 and $+1$ each with probability $\frac{1}{2}$ and independently. For each $T \subseteq n$ the distribution of $\chi_k(T)$ is the same as that of $S_{\binom{|T|}{k}}$ and therefore, by Chernoff's lemma,

$$\operatorname{Prob}[|\chi_k(T)| > cn^{\frac{k+1}{2}}] < 2\exp\left(\frac{-c^2 n^{k+1}}{2\binom{|T|}{k}}\right) < 2\ \exp\left(\frac{-c^2 n^{k+1}}{2n^k}\right) = 2e^{-\frac{c^2}{2}n}.$$

Since there are 2^n choices for T we get

$$\operatorname{Prob}[\max_{T \subseteq n} |\chi_k(T)| > cn^{\frac{k+1}{2}}] \leq 2^{n+1} e^{-\frac{c^2}{2}n} < 1,$$

choosing, e.g., $c = c_1 < \sqrt{2\ln 2} + 1$. So there exists $\chi_k : [n]^k \to \{-1, +1\}$ such that $\max_{T \subseteq n} |\chi_k(T)| \leq c_1 n^{\frac{k+1}{2}}$ and, hence, $\operatorname{disc}(k, n) \leq c_1 n^{\frac{k+1}{2}}$. □

10.2 A Lemma of Erdős

In connection with his investigations on a lemma of Littlewood and Offord (1943) and Erdős (1945) proved the following result.

Lemma 10.3. *Let $x_0, \ldots, x_{n-1}$ be reals satisfying $|x_i| \geq 1$ for every $i < n$. Then for every $r \in \mathbb{R}$ the number of sums $\sum_{i<n} \epsilon_i x_i$, where $\epsilon_i \in \{0, +1\}$, which fall into the (halfopen) interval $[r, r+1[$ does not exceed $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. We first show that it suffices to consider the case that the x_i are all non-negative. Indeed, assume that $x_i < 0$ for some $i < n$. If we replace x_i by $-x_i$ and each ϵ_i by $(\epsilon_i + 1) \bmod 2$, then all sums are shifted by exactly $-x_i$. The lemma thus follows by considering the case $r - x_i$.

So assume that $x_i \geq 1$ for every i. Now for every sum $\sum_{i<n} \epsilon_i x_i$, the ϵ_i can be viewed as the characteristic function of a subset of n. If $\sum_{i<n} \epsilon x_i$ and $\sum_{i<n} \eta_i x_i$ are both in $[r, r+1[$, for some $r \in \mathbb{R}$, then neither of the corresponding subsets contains the other. Hence, by Sperner's lemma (Sperner 1928), the number of sums which fall in the interval $[r, r+1[$ does not exceed $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. □

What we actually need in order to prove Theorem 10.1 is the following corollary of Lemma 10.3:

Corollary 10.4. *There exists a positive integer n_0 such that for every $n \geq n_0$, for every $0 < c \leq 1$ and for every sequence $x_0, \ldots, x_{n-1}$ of reals satisfying $|x_i| \geq 1$ for at least cn many $i < n$ we have that*

$$|\sum_{j \in J} x_j| \geq c\frac{\sqrt{n}}{2}, \tag{10.1}$$

for at least $\frac{1}{5}2^n$ choices of $J \subseteq n$.

Proof. Let $I \subseteq n$ be such that $|x_i| \geq 1$ for every $i \in I$ and such that $|I| \geq cn$. Let $J \subseteq n$. If (10.1) does not hold then

$$-\sum_{j \in J \setminus I} x_j - c\frac{\sqrt{n}}{2} < \sum_{j \in I \cap J} x_j < -\sum_{j \in J \setminus I} x_j + c\frac{\sqrt{n}}{2}.$$

Now we think this open interval to be covered with $\lceil c\sqrt{n} \rceil$ halfopen intervals of length 1. Then assuming $J \setminus I$ to be fixed for the moment, by Erdős' lemma the assertion (10.1) is not fulfilled for at most

$$\lceil c\sqrt{n} \rceil \binom{cn}{\lfloor \frac{cn}{2} \rfloor} < \frac{4}{5} 2^{cn}$$

choices of $I \cap J$. (The inequality follows from $\binom{x}{x/2} = (1 + o(1))\sqrt{2/(\pi x)}2^x$.) Summing over all possible $J \setminus I$ (at most $2^{(1-c)n}$ many) yields the corollary. □

Note that Erdős (1945) proved already that for any sequence of reals $x_0, \dots, x_{n-1}$ with $|x_i| \geq 1$ the number of sums $\sum_{i<n} \epsilon_i x_i$ which fall into the interior of any interval of length 2 m, for some positive integer m, is not greater than the sum of the m greatest binomial coefficients. This, of course, allows to strengthen Corollary 10.4 considerably, but this is not of use for our purposes.

10.3 The Lower Bound: The Graph Case

Because of its particular interest and since its proof becomes considerably easier, we separate the graph case, i.e., the case $k = 2$.

Proposition 10.5. *There exist constants c_0 and c_1 so that for every n*

$$c_0 n^{3/2} \leq \operatorname{disc}(2, n) \leq c_1 n^{3/2}.$$

Proof. The upper bound was proven in Sect. 10.1, so we concentrate on the lower bound. Interpreting the lower bound in terms of graphs, Proposition 10.5 says that for every graph $G = (n, E)$ there exists an (induced) subgraph which has considerably more edges, viz. $c_0 n^{3/2}$, than non-edges, or vice versa. Assume that every edge has weight $+1$ and every non-edge has weight -1, which defines some $\chi : [n]^2 \to \{-1, +1\}$. Let $A_0, A_1 \subseteq n$ be disjoint subsets of n. Then, by abuse of language, we put

$$\chi(A_0, A_1) = \sum \chi(e),$$

where the summation is taken over all edges having one endpoint in A_0 and the other endpoint in A_1. Now we prove the lower bound proceeding in two steps. First we show:

There exists $\epsilon > 0$ such that for every $n \geq 2n_0$ (without loss of generality n is even), for every $\chi : [n]^2 \to \{-1, +1\}$ and every pair $A_0, A_1 \subseteq n$ of disjoint sets satisfying $|A_0| = |A_1| = \frac{n}{2}$, there exist $B_0 \subseteq A_0$, and $B_1 \subseteq A_1$ so that

$$|\chi(B_0, B_1)| \geq \epsilon n^{3/2}.$$

In order to prove this fix some $a \in A_1$. By Corollary 10.4 for $c = 1$, we have that

$$\left| \{B \subseteq A_0 \mid |\chi(B, a)| \geq \frac{\sqrt{n}}{2\sqrt{2}}\} \right| \geq \frac{1}{5} 2^{n/2}.$$

Thus putting $\delta = \frac{1}{20}$ we obtain the existence of $B_0 \subseteq A_0$ satisfying

$$|\{a \in A_1 \mid |\chi(B_0, a)| \geq \frac{\sqrt{n}}{2\sqrt{2}}\}| \ \geq \ 2\delta n.$$

By symmetry we can assume that

$$\left|\{a \in A_1 \mid \chi(B_0, a) \geq \frac{\sqrt{n}}{2\sqrt{2}}\}\right| \ \geq \ \delta n.$$

Now let $B_1 = \{a \in A_1 \mid |\chi(B_0, a) \geq \frac{\sqrt{n}}{2\sqrt{2}}\}$. Then

$$\chi(B_0, B_1) \ = \ \sum_{a \in B_1} \chi(B_0, a) \ \geq \ \delta n \frac{\sqrt{n}}{2\sqrt{2}} = \epsilon n^{3/2},$$

choosing $\epsilon = \frac{\delta}{2\sqrt{2}}$, thus our claim.

In a second step we have to transfer the imbalance of the bipartite graph into an imbalance of some subgraph. For this purpose let $\hat{c}_0 = \frac{\epsilon}{3}$ and observe that

$$\chi(B_0, B_1) = \chi(B_0 \cup B_1) - \chi(B_0) - \chi(B_1).$$

Thus, by the pigeonhole principle, either B_0, or B_1, or $B_0 \cup B_1$, has a discrepancy of size at least $\hat{c}_0 n^{3/2}$. Choosing $c_0 \leq \hat{c}_0$ to take care of the n's smaller than n_0 completes the proof of Proposition 10.5. □

10.4 The Lower Bound: The General Case

The general approach for the case $k \geq 2$ is similar as in the graph case. First we aim at finding k pairwise disjoint subsets $A_0, \ldots, A_k$ such that the collection of all k-subsets that meat each of the A_i exactly once have a high discrepancy. In a second step we then argue that this implies the existence of a set A' that has a high discrepancy. The main idea is similar to the graph case. Differences arise mainly from the fact that given pairwise disjoint sets $A_0, \ldots, A_{k-1}$ there are many more ways to form a k-subset in $A_1 \cup \ldots \cup A_k$ than just transversals and subsets of some A_i. This motivates the following definition.

Let $k \geq 2$ and let $\chi_k : [n]^k \to \{-1, +1\}$ be a coloring and $(A_i)_{i<j}$, for some $j \leq k$, be a family of pairwise disjoint subsets of n. Then we define

$$\chi_k(A_0, \ldots, A_{j-1}) = \sum \chi_k(A),$$

where the summation is taken over all sets $A \in [n]^k$ satisfying $A \subseteq \bigcup_{i<j} A_i$ and $A \cap A_i \neq \emptyset$ for every $i < j$. In particular, for $j = k$, the summation goes over all transversals of $A_0, \ldots, A_{k-1}$.

In the following assume that $k \geq 2$ and a coloring $\chi_k : [n]^k \to \{-1, +1\}$ is fixed and let n_0 be the constant from Corollary 10.4. First we show:

Lemma 10.6. *There exists $\epsilon > 0$ such that for every $\chi_k : [n]^k \to \{-1, +1\}$ and for every family $(A_i)_{i<k}$ of pairwise disjoint subsets of n satisfying $|A_0| = \ldots = |A_{k-1}| = t$ for some $t \geq n_0$ there exist $B_0 \subseteq A_0, \ldots, B_{k-1} \subseteq A_{k-1}$ so that*

$$|\chi_k(B_0, \ldots, B_{k-1})| \geq \epsilon t^{\frac{k+1}{2}}.$$

To prove Lemma 10.6 we show

Lemma 10.7. *There exist positive constants $c_1, \ldots, c_{k-1}$ and $d_1, \ldots, d_{k-1}$ so that for every positive integer $j < k$ and every family $(A_i)_{i<j}$ of pairwise disjoint subsets of n satisfying $|A_0| = \ldots = |A_{j-1}| = t$, for some $t \geq n_0$, and for every $\chi_j : [n]^j \to \{-1, +1\}$ we have*

$$|\{(C_0, \ldots, C_{j-1}) \mid \forall i < j\text{: } C_i \subseteq A_i \text{ and } |\chi_i(C_0, \ldots, C_{j-1})| \geq c \cdot t^{j/2}\}| \geq d_j \cdot 2^{t \cdot j}.$$

We mimic the argument used to prove the first assertion in the graph-case to show how Lemma 10.7 implies Lemma 10.6.

Proof of Lemma 10.6. Fix some $a \in A_{k-1}$. Then by Lemma 10.7 (for $j = k - 1$ and defining χ_{k-1} by $\chi_{k-1}(C_0, \ldots, C_{k-2}) = \chi_k(C_0, \ldots, C_{k-2}, \{a\})$) we have that

$$|\{(C_0, \ldots, C_{k-2}) \mid |\chi_k(C_0, \ldots, C_{k-2}, \{a\})| \geq c_{k-1} t^{\frac{k-1}{2}}\}| \geq d_{k-1} 2^{t(k-1)}.$$

Put $\delta = \frac{d_{k-1}}{2}$. Then we get the existence of a family $(B_i)_{i<k-1}$, where $B_i \subseteq A_i$, so that

$$|\{a \in A_{k-1} \mid |\chi_k(B_0, \ldots, B_{k-2}, \{a\})| \geq c_{k-1} t^{\frac{k-1}{2}}\}| \geq 2\delta t.$$

Again by symmetry we can assume that

$$|\{a \in A_{k-1} \mid \chi_k(B_0, \ldots, B_{k-2}, \{a\}) \geq c_{k-1} t^{\frac{k-1}{2}}\}| \geq \delta t.$$

Let $B_{k-1} = \{a \in A_{k-1} \mid \chi_k(B_0, \ldots, B_{k-2}, \{a\}) \geq c_{k-1} t^{\frac{k-1}{2}}\}$. Then

$$\chi_k(B_0, \ldots, B_{k-1}) = \sum_{a \in B_{k-1}} \chi_k(B_0, \ldots, B_{k-2}, \{a\}) \geq \delta t \cdot c_{k-1} t^{\frac{k-1}{2}} = \epsilon t^{\frac{k+1}{2}},$$

choosing $\epsilon = \delta \cdot c_{k-1}$, thus proving Lemma 10.6. □

Proof of Lemma 10.7. We proceed by induction on j. Observe in the case $j = 1$ we are given a function χ_1 that assigns values to points and we are interested in certain subsets of A_0. This is exactly the situation of Corollary 10.4. The base case of the induction thus follows from Corollary 10.4 (applied for $c = 1$) by choosing $c_1 = 1/2$ and $d_1 = \frac{1}{5}$.

So assume the validity of Lemma 10.7 for some $j \in [1, k-2]$ and fix some $\chi_{j+1} : [n]^{j+1} \to \{-1, +1\}$. Note that for every fixed $a \in A_j$ the function χ_{j+1} naturally gives rise to a function $\chi_j : [n]^j \to \{-1, +1\}$ via $\chi_j(X) := \chi_{j+1}(X \cup \{a\})$. To these function we can then apply the induction hypothesis. Let

$$\mathcal{M} = \{(C_0, \ldots, C_{j-1}, \{a\}) \mid \\ C_i \subseteq A_i, i < j, a \in A_j \text{ s.t. } |\chi_{j+1}(C_0, \ldots, C_{j-1}, \{a\})| \geq c_j t^{j/2}\}.$$

Then the induction hypothesis implies that we have for every a at least $d_j 2^{tj}$ subsets $(C_0, \ldots, C_{j-1})$ so that $(C_0, \ldots, C_{j-1}, \{a\}) \in \mathcal{M}$. Thus, we know

$$|\mathcal{M}| \geq t \cdot d_j 2^{tj}.$$

On the other hand we have:

$$|\mathcal{M}| = \sum_{\substack{(C_0,\ldots,C_{j-1}) \\ C_i \subseteq A_i}} |\{a \in A_j \mid (C_0, \ldots, C_{j-1}, \{a\}) \in \mathcal{M}\}|.$$

Here we have 2^{tj} summands, each of which has size (at most) t, that together sum up to at least $d_j t 2^{tj}$. An easy calculation thus gives: there are at least $\frac{d_j}{2} 2^{tj}$ summands which are larger than $\frac{d_j}{2} t$.

Fix such a $(C_0, \ldots, C_{j-1})$. Then there are $\frac{d_j}{2} t$ many $a \in A_j$ such that $(C_0, \ldots, C_{j-1}, \{a\}) \in \mathcal{M}$ meaning that $|\chi_{j+1}(C_0, \ldots, C_{j-1}, \{a\})| \geq c_j t^{j/2}$. If we thus let

$$x_a = \frac{1}{c_j} t^{-j/2} \chi_{j+1}(C_0, \ldots, C_{j-1}, \{a\}).$$

for every $a \in A_j$, then $|x_a| \geq 1$ for at least $\frac{d_j}{2} t$ many $a \in A_j$.

Apply Corollary 10.4 with respect to $c = \frac{d_j}{2}$. Then we have for at least $\frac{1}{5} 2^t$ choices $C_j \subseteq A_j$ that

$$|\chi_{j+1}(C_0, \ldots, C_j)| = \left| \sum_{a \in C_j} \chi_{j+1}(C_0, \ldots, C_{j-1}, \{a\}) \right| = c_j t^{j/2} \left| \sum_{a \in C_j} x_a \right| \\ \geq c_j t^{j/2} \cdot \frac{d_j}{4} t^{1/2} = \frac{c_j d_j}{4} t^{(j+1)/2}.$$

As this is true for at least $\frac{d_j}{2}2^{tj}$ choices of $(C_0, \dots, C_{j-1})$, choosing $c_{j+1} = \frac{c_j d_j}{4}$ and $d_{j+1} = \frac{d_j}{10}$ completes the proof of Lemma 10.7. □

In the next step we transform the imbalance of a product into an imbalance for a set:

Lemma 10.8. *Let $\eta > 0$ and $j \leq k$ be a positive integer. Then there exists an $\xi = \xi(\eta, j) > 0$ such that for every $\chi_k : [n]^k \to \{-1, +1\}$ and every family $(B_i)_{i<j}$ of pairwise disjoint subsets of n, we have that $|\chi_k(B_0, \dots, B_{j-1})| \geq \eta n^{\frac{k+1}{2}}$ implies the existence of some $I \subseteq j$ satisfying*

$$|\chi_k(\bigcup_{i \in I} B_i)| \geq \xi n^{\frac{k+1}{2}}.$$

Proof. For $j = 1$ and every $\eta > 0$ the lemma is trivial choosing $\xi = \eta$. So assume the validity of Lemma 10.8 for some $j < k$ and all $\eta > 0$, and let $(B_i)_{i \leq j}$ be a family of pairwise disjoint subsets of n such that $|\chi_k(B_0, \dots, B_j)| \geq \eta n^{\frac{k+1}{2}}$, for some $\eta > 0$.

Observe that

$$\chi_k(B_0, \dots, B_j) = \chi_k(\bigcup_{i \leq j} B_i) - \sum \chi_k(B_{i_1}, \dots, B_{i_\ell}),$$

where the summation is taken over all proper (and nonempty) subfamilies of $B_0, \dots, B_j$. Hence, at least one of the summands of the right hand side has absolute value at least $\frac{\eta}{2^{j+1}} n^{\frac{k+1}{2}}$. If $\chi_k(\bigcup_{i \leq j} B_i)$ has this size, we are done. Otherwise, applying the inductive hypothesis to the appropriate summand replacing η by $\frac{\eta}{2^{j+1}}$ proves Lemma 10.8. □

Now the proof of Theorem 10.1 is easily finished. Without loss of generality we can assume that $n = k \cdot t$. Let $\chi_k : [n]^k \to \{-1, +1\}$ be a coloring and $A_0 \cup \dots \cup A_{k-1} = n$ be a partition of n into k disjoint sets each of size $t \geq n_0$. Then, by Lemma 10.6, there exist $B_0 \subseteq A_0, \dots, B_{k-1} \subseteq A_{k-1}$ so that

$$|\chi_k(B_0, \dots, B_{k-1})| \geq \epsilon t^{\frac{k+1}{2}} = \epsilon k^{-\frac{k+1}{2}} n^{\frac{k+1}{2}}.$$

Applying Lemma 10.8 for $\eta = \epsilon k^{-\frac{k+1}{2}}$ and $j = k$ yields a constant $\xi = \xi(\eta, k)$ and a subset $I \subseteq k$ satisfying

$$|\chi_k(\bigcup_{i \in I} B_i)| \geq \xi n^{\frac{k+1}{2}}.$$

Choosing $c_0 \leq \xi$ in such a way that c_0 takes care of all small n completes the proof of Theorem 10.1. □