Chapter 7
The Melan and Mindlin Problems

This chapter is devoted to solve the equilibrium problem of a linearly elastic isotropic half-space, subject to a load concentrated at an interior point. The two-dimensional version is named after Ernst Melan (1890–1963), who solved it in 1932 [1]; the three-dimensional version was studied and solved in 1936 [2] by Raymond D. Mindlin (1906–1987), who returned to it some years later [3, 4].

We concentrate of the case of paramount interest in geomechanics, when the load is directed orthogonally to the boundary plane and the Mindlin elastic state is used to compute stresses and soil settlements due to one or more foundation piles. As we shall see, the stress field depends on constitutive choices; no doubt, ordinary soil is far from being elastic and isotropic, and yet Mindlin solution is widely used to estimate footing settlements [5].

7.1 Solution by Superposition

The method we use to solve Melan's and Mindlin's problems is the same, and differs from the methods used by those authors: essentially, as exemplified in the last section of Chap. 1, we proceed by superposition/restriction/super-position.

Our first and main concern is to determine the stress field. This we do in four steps. Preliminarily, we consider a space $\mathcal{S}$ (two-dimensional in Melan's case, three-dimensional in Mindlin's) and we choose an origin $o \in \mathcal{S}$ and a direction $\boldsymbol{e}_1$, so that it makes sense to consider the half-spaces $\mathcal{HS}^{\pm} = \{x \in \mathcal{S} \mid \pm (x - o) \cdot \boldsymbol{e}_1 > 0\}$. Then,

(i) we determine the Kelvin stress $\breve{\boldsymbol{S}}$ induced in $\mathcal{S}$ by a concentrated load $\boldsymbol{f}$ applied at $x = o + a\boldsymbol{e}_1,\ a > 0$;

(ii) we determine the Kelvin stress $\widehat{\boldsymbol{S}}$ in the same space, this time due to a load $-\boldsymbol{f}$ concentrated at $x = o - a\boldsymbol{e}_1$;

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(iii) we consider the restriction $\widetilde{\boldsymbol{S}}$ to $\mathcal{HS}^+$ of the point-wise superposition of the stress fields $\breve{\boldsymbol{S}}$ and $\widehat{\boldsymbol{S}}$, and compute the traction vector $\tilde{\boldsymbol{s}} = -\widetilde{\boldsymbol{S}}\boldsymbol{e}_1$ on the plane boundary of $\mathcal{HS}^+$;

(iv) we superpose to $\widetilde{\boldsymbol{S}}$ a stress field $\overline{\boldsymbol{S}}$ in $\mathcal{HS}^+$ such that the resulting boundary traction $-(\widetilde{\boldsymbol{S}} + \overline{\boldsymbol{S}})\boldsymbol{e}_1$ is null. The stress field solving the M problem at hand is $\boldsymbol{S}^M := \widetilde{\boldsymbol{S}} + \overline{\boldsymbol{S}}$.

We shall go through this sequence of four steps twice, in Sect. 7.2.1 for Melan's problem and in Sect. 7.3.1 for Mindlin's.

Since each of the stress fields we consider is compatible, such is the field $\boldsymbol{S}^M$. Having found $\boldsymbol{S}^M$, finding the strain and displacement fields is the matter of routine computations, completely similar to those we made in Chaps. 5 and 6 for the same purposes.

7.2 The Melan Problem

7.2.1 The Stress Field

Preliminarily, we use relations (6.16)–(6.17) to write the components of the stress field for the plane Kelvin problem in a Cartesian frame with the same origin (Fig. 7.1). These components are:

$$
\begin{aligned}
S_{11} &= -\frac{f}{4\pi}\frac{x_1}{(x_1^2+x_2^2)^2}\big((3+\nu_0)x_1^2 + (1-\nu_0)x_2^2\big),\\
S_{22} &= \frac{f}{4\pi}\frac{x_1}{(x_1^2+x_2^2)^2}\big((1-\nu_0)x_1^2 - (1+3\nu_0)x_2^2\big),\\
S_{12} &= -\frac{f}{4\pi}\frac{x_2}{(x_1^2+x_2^2)^2}\big((3+\nu_0)x_1^2 + (1-\nu_0)x_2^2\big).
\end{aligned}
\tag{7.1}
$$

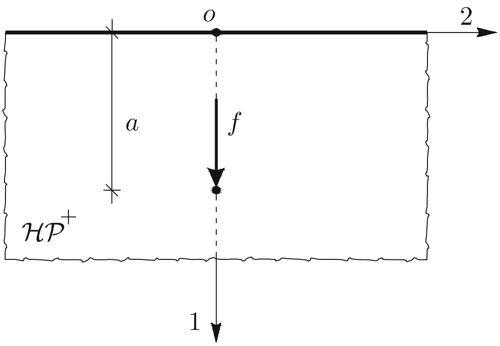

Fig. 7.1 The Melan Problem

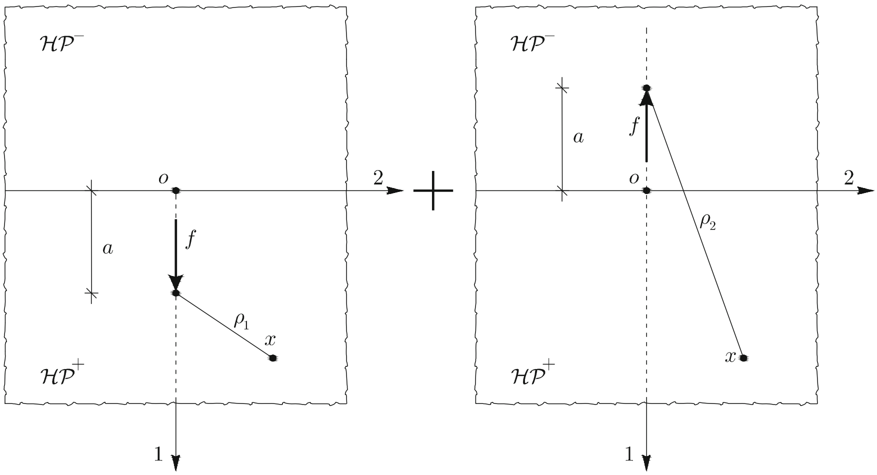

Fig. 7.2 Superposition of mirror-symmetric concentrated loads applied at mirror-symmetric points of half-planes $\mathcal{H}P^+$ and $\mathcal{H}P^-$

7.2.1.1 Steps (i) and (ii)

We use formulas (7.1) twice, to determine the stress fields $\breve{\boldsymbol{S}}$ and $\widehat{\boldsymbol{S}}$ induced in $\mathcal{HS}$ by, respectively, a load $\boldsymbol{f} = f\boldsymbol{e}_1$ applied $x = o + a\boldsymbol{e}_1$ and a load $\boldsymbol{f} = -f\boldsymbol{e}_1$ applied $x = o - a\boldsymbol{e}_1$ (Fig. 7.2); all we have to do are two changes in origin. We find:

$$\begin{aligned}
\breve{S}_{11}(x_1, x_2) &= -\frac{f}{4\pi}\frac{x_1 - a}{((x_1 - a)^2 + x_2^2)^2}\big((3 + \nu_0)(x_1 - a)^2 + (1 - \nu_0)x_2^2\big),\\
\breve{S}_{22}(x_1, x_2) &= \frac{f}{4\pi}\frac{x_1 - a}{((x_1 - a)^2 + x_2^2)^2}\big((1 - \nu_0)(x_1 - a)^2 - (1 + 3\nu_0)x_2^2\big),\\
\breve{S}_{12}(x_1, x_2) &= -\frac{f}{4\pi}\frac{x_2}{((x_1 - a)^2 + x_2^2)^2}\big((3 + \nu_0)(x_1 - a)^2 + (1 - \nu_0)x_2^2\big), \qquad (7.2)
\end{aligned}$$

and

$$\begin{aligned}
\widehat{S}_{11}(x_1, x_2) &= \frac{f}{4\pi}\frac{x_1 + a}{((x_1 + a)^2 + x_2^2)^2}\big((3 + \nu_0)(x_1 + a)^2 + (1 - \nu_0)x_2^2\big),\\
\widehat{S}_{22}(x_1, x_2) &= -\frac{f}{4\pi}\frac{x_1 + a}{((x_1 + a)^2 + x_2^2)^2}\big((1 - \nu_0)(x_1 + a)^2 - (1 + 3\nu_0)x_2^2\big),\\
\widehat{S}_{12}(x_1, x_2) &= \frac{f}{4\pi}\frac{x_2}{((x_1 + a)^2 + x_2^2)^2}\big((3 + \nu_0)(x_1 + a)^2 + (1 - \nu_0)x_2^2\big). \qquad (7.3)
\end{aligned}$$

7.2.1.2 Steps (iii) and (iv)

Component-wise summation of (7.2) and (7.3), followed by restriction to $x_1 \geq 0$, yields the stress field $\widetilde{\boldsymbol{S}}$ over the closure of $\mathcal{HP}^+$. We quickly see that the traction vector $\tilde{\boldsymbol{s}} = -\widetilde{\boldsymbol{S}}(0, x_2)\boldsymbol{e}_1$ is not null, contrary to Melan's prescription that the traction vector be zero all over the boundary of $\mathcal{HP}^+$. Instead, we have:

$$\begin{aligned}\tilde{\boldsymbol{s}}(x_2) &= -\widetilde{S}_{11}(0, x_2)\boldsymbol{e}_1, \\ \widetilde{S}_{11}(0, x_2) &= -\frac{f}{4\pi}\frac{2a\big((3+\nu_0)a^2 + (1-\nu_0)x_2^2\big)}{(a^2+x_2^2)^2}.\end{aligned} \tag{7.4}$$

Therefore, the issue is to find another stress field $\overline{\boldsymbol{S}}$ over the closure of $\mathcal{HP}^+$, such that

$$(\widetilde{\boldsymbol{S}}(0, x_2) + \overline{\boldsymbol{S}}(0, x_2))\boldsymbol{e}_1 \equiv \boldsymbol{0}.$$

We construct $\overline{\boldsymbol{S}}$ by using the Boussinesq-Flamant stress field as a *stress Green function* (a notion we introduced in the simpler context of Sect. 1.2).

The Cartesian components of the Boussinesq-Flamant plane stress field can be easily deduced from (4.14); they are:

$$\begin{aligned}S_{11}^{BF}(x_1, x_2) &= -\frac{2f}{\pi}\frac{x_1^3}{(x_1^2+x_2^2)^2}, \\ S_{22}^{BF}(x_1, x_2) &= -\frac{2f}{\pi}\frac{x_1 x_2^2}{(x_1^2+x_2^2)^2} \\ S_{12}^{BF}(x_1, x_2) &= -\frac{2f}{\pi}\frac{x_1^2 x_2}{(x_1^2+x_2^2)^2}.\end{aligned}$$

The components of the *Green tensor* $\boldsymbol{G}^{BF}$ are obtained from those of $\boldsymbol{S}^{BF}$ by setting $f = 1$ and replacing x_2 by $(x_2 - \xi)$, that is, relocating the origin on the plane $x_1 = 0$ (see Fig. 7.3). These measures yield:

$$\begin{aligned}\widehat{G}_{11}^{BF}(x_1, x_2; \xi) &= -\frac{2}{\pi}\frac{x_1^3}{(x_1^2+(x_2-\xi)^2)^2}, \\ \widehat{G}_{22}^{BF}(x_1, x_2; \xi) &= -\frac{2}{\pi}\frac{x_1 x_2^2}{(x_1^2+(x_2-\xi)^2)^2}, \\ \widehat{G}_{12}^{BF}(x_1, x_2; \xi) &= -\frac{2}{\pi}\frac{x_1^2 x_2}{(x_1^2+(x_2-\xi)^2)^2}.\end{aligned}$$

We are now in position to determine the tensor $\overline{\boldsymbol{S}}$:

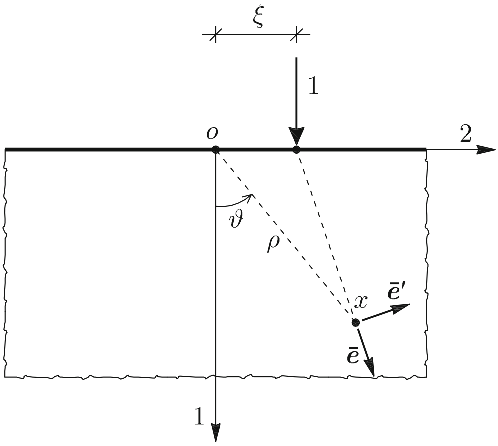

Fig. 7.3 The origin relocation that permits to deduce $\boldsymbol{G}^{BF}$ from $\boldsymbol{S}^{BF}$

$$\overline{\boldsymbol{S}}(x_1, x_2) = \int_{-\infty}^{+\infty} \hat{p}(x_1, \xi)\widehat{\boldsymbol{G}}^{BF}(x_1, x_2; \xi)\, d\xi,$$

where the load function $\hat{p}$ is the negative of the surface traction (7.4) that we want to eliminate:

$$\hat{p}(x_1, \xi) := \frac{f}{4\pi}\frac{2a}{(a^2+\xi^2)^2}\big((3+\nu_0)a^2 + (1-\nu_0)\xi^2\big).$$

Finding $\overline{\boldsymbol{S}}$ is the matter of a nontrivial computation, whose development is the same for all components; we here sketch it for the first component, details are found in Appendix A.7.

To begin with, we have that

$$\begin{aligned}\overline{S}_{11}(x_1, x_2) &= \int_{-\infty}^{+\infty} \hat{p}(x_1, \xi)\widehat{G}_{11}(x_1, x_2; \xi)\, d\xi \\ &= -\frac{afx_1^3}{\pi^2}\Big(a^2(3+\nu_0)I_1 + (1-\nu_0)I_2\Big),\end{aligned} \tag{7.5}$$

with

$$I_1 := \int_{-\infty}^{+\infty} \frac{1}{(a^2+\xi^2)^2(x_1^2+(x_2-\xi)^2)^2}\,,$$

$$I_2 := \int_{-\infty}^{+\infty} \frac{\xi^2}{(a^2+\xi^2)^2(x_1^2+(x_2-\xi)^2)^2}\,.$$

A lengthy computation based on the methods of residues yields:

$$I_1 = \frac{2\pi}{4a^3x_1^3}\frac{(x_1+a)^3(x_1^2+3ax_1+a^2)+(a^3+x_1^3)x_2^2}{\big((x_1+a)^2+x_2^2\big)^3},$$
$$I_2 = \frac{\pi}{2ax_1^3}\frac{x_1^2(x_1+a)^3+(x_1+a)(x_1^2+5ax_1+a^2)x_2^2+ax_2^2}{\big((x_1+a)^2+x_2^2\big)^3}. \tag{7.6}$$

Substituting (7.6) into (7.5) we arrive at:

$$\begin{aligned}\overline{S}_{11}(x_1,x_2) = &-\frac{f}{2\pi\big((x_1+a)^2+x_2^2\big)^3}\Big((3+\nu_0)\big((x_1+a)^3(x_1^2+3ax_1+a^2)\\ &+(a^3+x_1^3)x_2^2\big)+(1-\nu_0)\big(x_1^2(x_1+a)^3\\ &+(x_1+a)(x_1^2+5ax_1+a^2)x_2^2+ax_2^4\big)\Big).\end{aligned}$$

With this, we are ready to write the first component of the Melan stress tensor:

$$\begin{aligned}\widehat{S}^{Me}_{11}(x_1,x_1) &= \widetilde{S}_{11}+\overline{S}_{11}\\ &= -\frac{f}{2\pi}\Bigg((1+\nu_0)\Bigg(\frac{(x_1-a)^3}{\rho_1^4}+\frac{(x_1+a)\big((x_1+a)^2+2ax_1\big)}{\rho_2^4}-\frac{8ax_1(a+x_1)x_2^2}{\rho_2^6}\Bigg)\\ &+\frac{1-\nu_0}{2}\Bigg(\frac{x_1-a}{\rho_1^2}+\frac{3x_1+a}{\rho_2^2}-\frac{4x_1x_2^2}{\rho_2^4}\Bigg)\Bigg),\end{aligned}$$

where

$$\rho_1 := \sqrt{(x_1-a)^2+x_2^2}\,,\quad \rho_2 := \sqrt{(x_1+a)^2+x_2^2}\,.$$

The other two components are found in a completely analogous manner. Their expressions are:

$$\begin{aligned}(2\pi f^{-1})\widehat{S}^{Me}_{22}(x_1,x_2) = &-\Bigg((1+\nu_0)\Bigg(\frac{(x_1-a)x_2^2}{\rho_1^4}+\frac{(x_1+a)(x_2^2+2a^2)-2ax_2^2}{\rho_2^4}\\ &+\frac{8ax_1(a+x_1)x_2^2}{\rho_2^6}\Bigg)+\frac{1-\nu_0}{2}\Bigg(-\frac{x_1-a}{\rho_1^2}+\frac{x_1+3a}{\rho_2^2}+\frac{4x_1x_2^2}{\rho_2^4}\Bigg)\Bigg),\end{aligned}$$

$$\begin{aligned}(2\pi f^{-1})\widehat{S}^{Me}_{12}(x_1,x_2) = &-x_2\Bigg((1+\nu_0)\Bigg(\frac{(x_1-a)^2}{\rho_1^4}+\frac{x_1^2-2ax_1-a^2}{\rho_2^4}\\ &+\frac{8ax_1(a+x_1)^2}{\rho_2^6}\Bigg)+\frac{1-\nu_0}{2}\Bigg(\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}-\frac{4x_1(a+x_1)}{\rho_2^4}\Bigg)\Bigg).\end{aligned}$$

7.2.2 The Strain and Displacement Fields

As we have done systematically so far, we obtain the Melan strain field by inserting the stress field we just obtained in the inverse constitutive law (2.57). After some manipulations, we have:

$$
\begin{aligned}
(2\pi E_0 f^{-1})\, \widehat{E}^{Me}_{11}(x_1, x_2) \\
&= -(1+\nu_0)\Bigg(\frac{(x_1-a)^3 - \nu_0(x_1-a)x_2^2}{\rho_1^4} \\
&\quad + \frac{(x_1+a)\big((x_1+a)^2+2ax_1\big) - \nu_0\big((x_1+a)(x_2^2+2a^2)-2ax_2^2\big)}{\rho_2^4} \\
&\quad - \frac{(1+\nu_0)8ax_1(a+x_1)x_2^2}{\rho_2^6}\Bigg) + \frac{1-\nu_0}{2}\Bigg(\frac{(1+\nu_0)(x_1-a)}{\rho_1^2} \\
&\quad + \frac{3x_1+a-\nu_0(x_1+3a)}{\rho_2^2} - \frac{(1-\nu_0)4x_1x_2^2}{\rho_2^4}\Bigg),
\end{aligned}
\tag{7.7}
$$

$$
\begin{aligned}
(2\pi E_0 f^{-1})\, \widehat{E}^{Me}_{22}(x_1, x_2) \\
&= -(1+\nu_0)\Bigg(\frac{(x_1-a)x_2^2 - \nu_0(x_1-a)^3}{\rho_1^4} \\
&\quad + \frac{(x_1+a)(x_2^2+2a^2)-2ax_2^2 - \nu_0(x_1+a)\big((x_1+a)^2+2ax_1\big)}{\rho_2^4} \\
&\quad (1+\nu_0)\frac{8ax_1(a+x_1)x_2^2}{\rho_2^6}\Bigg) + \frac{1-\nu_0}{2}\Bigg(-(1+\nu_0)\frac{x_1-a}{\rho_1^2} \\
&\quad + \frac{x_1+3a-\nu_0(3x_1+a)}{\rho_2^2} - (1-\nu_0)\frac{4x_1x_2^2}{\rho_2^4}\Bigg)\Bigg),
\end{aligned}
\tag{7.8}
$$

$$
\begin{aligned}
(2\pi E_0 f^{-1})\, \widehat{E}^{Me}_{12}(x_1, x_2) &= -(1+\nu_0)x_2\Bigg((1+\nu_0)\Bigg(\frac{(x_1-a)^2}{\rho_1^4} + \frac{x_1^2-2ax_1-a^2}{\rho_2^4} \\
&\quad + \frac{8ax_1(a+x_1)^2}{\rho_2^6}\Bigg) + \frac{1-\nu_0}{2}\Bigg(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} - \frac{4x_1(a+x_1)}{\rho_2^4}\Bigg)\Bigg).
\end{aligned}
$$

In order to determine the displacement field, we have to solve the following system of PDEs:

$$
u_{1,1} = E^{Me}_{11}, \quad u_{2,2} = E^{Me}_{22}, \quad u_{1,2} + u_{2,1} = 2\, E^{Me}_{12},
\tag{7.9}
$$

subject to the symmetry conditions:

$$
\hat{u}_1(x_1, x_2) = \hat{u}_1(x_1, -x_2), \quad \hat{u}_2(x_1, x_2) = -\hat{u}_2(x_1, -x_2).
\tag{7.10}
$$

With the use of (7.7) and (7.8), integration of 7.9_1 and 7.9_2 yields:

$$\begin{aligned}\hat{u}_1^{Me}(x_1, x_2) &= \int_{\bar{x}_1}^{x_1} \widehat{E}_{11}^{Me}(s, x_2)\, ds + \hat{g}_1(x_2) \\ &= -\frac{f}{8\pi E_0}\Bigg(\frac{2(1+\nu_0)x_2^2}{\rho_1^2} - \frac{2(1+\nu_0)\big(2ax_1(1+\nu_0) - (3-\nu_0)x_2^2\big)}{\rho_2^2} \\ &\quad + \frac{8a(1+\nu_0)^2 x_1 x_2^2}{\rho_2^4} + (3-\nu_0)(1+\nu_0)\log\rho_1 \\ &\quad + \big(5 - (2-\nu_0)\nu_0\big)\log\rho_2\Bigg) + \hat{g}_1(x_2),\end{aligned} \tag{7.11}$$

$$\begin{aligned}\hat{u}_2^{Me}(x_1, x_2) &= \int_{\bar{x}_2}^{x_2} \widehat{E}_{22}^{Me}(x_1, s)\, ds + \hat{g}_2(x_1) \\ &= \frac{f}{4\pi E_0}\Bigg((1+\nu_0)(x_1-a)x_2\left(\frac{1+\nu_0}{\rho_1^2} + \frac{3-\nu_0}{\rho_2^2}\right) \\ &\quad + \frac{4a(1+\nu_0)^2 x_1 x_2 (x_1+a)}{\rho_2^4} - 4(1-\nu_0)\arctan\left(\frac{x_2}{x_1+a}\right)\Bigg) + \hat{g}_2(x_1).\end{aligned} \tag{7.12}$$

Note that the symmetry condition $(7.10)_2$ implies that

$$\hat{g}_2(x_1) \equiv 0. \tag{7.13}$$

To determine function $\hat{g}_1$, we insert in $(7.9)_3$ relations (7.11) and (7.12) (with (7.13) taken into account), and find out that:

$$\left(\int_{\bar{x}_1}^{x_1} \widehat{E}_{11}^{Me}(s, x_2)\, ds\right)_{,2} + \left(\int_{\bar{x}_2}^{x_2} \widehat{E}_{22}^{Me}(x_1, s)\, ds\right)_{,1} - 2\,\widehat{E}_{12}^{Me}(x_1, x_2) = 0 = \hat{g}_1'(x_2).$$

Hence, function $\hat{g}_1$ must be constant-valued; we dispose of the residual irrelevant indeterminacy by taking the relative constant null.

7.3 The Mindlin Problem

To solve this problem (Fig. 7.4), we take once more the four steps listed sequentially in Sect. 7.1, this time with considerable analytical complications.

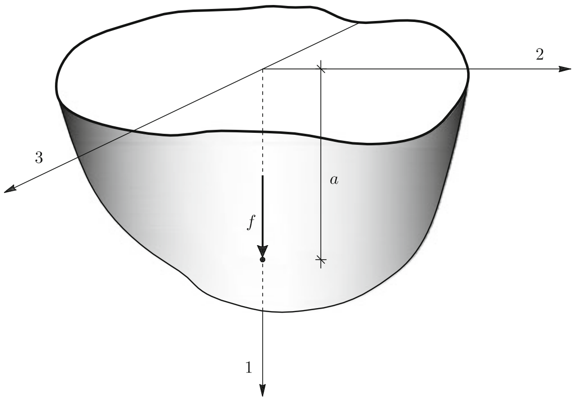

Fig. 7.4 The Mindlin Problem

7.3.1 The Stress Field

7.3.1.1 Steps (i) and (ii)

To determine the fields $\check{S}$ and $\widehat{S}$ over the whole space $\mathcal{S}$, we make use of the solution of the 3-D Kelvin Problem. From (6.25), with two appropriate changes in origin, we deduce that

$$\check{S}_{zz}(z,r) = -\frac{f}{8\pi(1-\nu)}\left(3\frac{(z-a)^3}{\rho_1^5} - (1-2\nu)\frac{z-a}{\rho_1^3}\right),$$

$$\check{S}_{rr}(z,r) = -\frac{f}{8\pi(1-\nu)}\left(3\frac{(z-a)r^2}{\rho_1^5} + (1-2\nu)\frac{z-a}{\rho_1^3}\right),$$

$$\check{S}_{\varphi\varphi}(z,r) = \frac{f(1-2\nu)}{8\pi(1-\nu)}\frac{z-a}{\rho_1^3},$$

$$\check{S}_{zr}(z,r) = -\frac{f}{8\pi(1-\nu)}\left(3\frac{(z-a)^2 r}{\rho_1^5} - (1-2\nu)\frac{z-a}{\rho_1^3}\right),$$

and

$$\widehat{S}_{zz}(z,r) = \frac{f}{8\pi(1-\nu)}\left(3\frac{(z+a)^3}{\rho_1^5} - (1-2\nu)\frac{z+a}{\rho_2^3}\right),$$

$$\widehat{S}_{rr}(z,r) = \frac{f}{8\pi(1-\nu)}\left(3\frac{(z+a)r^2}{\rho_2^5} + (1-2\nu)\frac{z+a}{\rho_2^3}\right),$$

$$\widehat{S}_{\varphi\varphi}(z,r) = -\frac{f(1-2\nu)}{8\pi(1-\nu)}\frac{z+a}{\rho_2^3},$$

$$\widehat{S}_{zr}(z,r) = \frac{f}{8\pi(1-\nu)}\left(3\frac{(z+a)^2 r}{\rho_2^5} - (1-2\nu)\frac{z+a}{\rho_2^3}\right),$$

where

$$\rho_1 := \sqrt{(z-a)^2 + r^2}, \qquad \rho_2 := \sqrt{(z+a)^2 + r^2}. \tag{7.14}$$

Component-wise summation plus restriction to the half-space $\mathcal{HS}^+$ yield the field $\widetilde{\boldsymbol{S}}$; at the boundary of $\mathcal{HS}^+$, the associated traction vector is:

$$\begin{aligned} \tilde{\boldsymbol{s}} &= -\widetilde{S}_{11}(0,x_2,x_3)\boldsymbol{e}_1, \\ \widetilde{S}_{11}(0,x_2,x_3) &= -\frac{f}{4\pi}\frac{a\big(2a^2(2-\nu) + \big(x_2^2 + x_3^2\big)(1-2\nu)\big)}{(1-\nu)\big(a^2 + x_2^2 + x_3^2\big)^{\frac{5}{2}}} \end{aligned} \tag{7.15}$$

(cf. (7.4)).

7.3.1.2 Steps (iii) and (iv)

To eliminate the effect of the undesired surface traction (7.15), we have to superimpose to $\widetilde{\boldsymbol{S}}$ a stress field

$$\overline{\boldsymbol{S}}(x_1,x_2,x_3) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{p}(\eta,\zeta)\widehat{\boldsymbol{G}}^B(x_1,x_1,x_3;\eta,\zeta)\,d\eta d\zeta,$$

where $\boldsymbol{G}^B$ is the stress Green function associated with the Boussinesq stress $\boldsymbol{S}^B$, and where

$$\hat{p}(\eta,\zeta) := \frac{f}{4\pi}\frac{a\big(2a^2(2-\nu) + \big(\eta^2 + \zeta^2\big)(1-2\nu)\big)}{(1-\nu)\big(a^2 + \eta^2 + \zeta^2\big)^{\frac{5}{2}}}, \tag{7.16}$$

(cf. (7.15)). Hereafter, we exemplify the construction of $\overline{\boldsymbol{S}}$, a cumbersome task indeed, by undertaking it for the component $\overline{S}_{11}$.

The integral in question is:

$$\overline{S}_{11}(x_1,x_2,x_3) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{p}(\eta,\zeta)\,\widehat{G}_{11}^B(x_1,x_1,x_3;\eta,\zeta)\,d\eta d\zeta. \tag{7.17}$$

where, in view of $(5.62)_1$,

$$\widehat{G}_{11}^B(x_1,x_1,x_3;\eta,\zeta) = -\frac{3}{2\pi}\frac{x_1^3}{\big(x_1^2 + (x_2-\eta)^2 + (x_3-\zeta)^2\big)^{\frac{5}{2}}}.$$

We have been unable to come up with an explicit evaluation for long, until we found the following circuitous route.[1]

We recall from Sect. 5.8 that the Boussinesq stress field can be given the Boussinesq-Papkovitch-Neuber representation (A.16) in terms of two harmonic functions ψ^B and φ^B. In particular, component S_{11}^B admits the representation (5.77), that we here recall for the reader's convenience:

$$S_{11}^B = \frac{1}{1-\nu}\big(2(1-\nu)\psi_{,1}^B - \varphi_{,11}^B - x_1\psi_{,11}^B\big),$$

where

$$\psi^B = \frac{1}{2\pi\rho}, \quad \varphi^B = \frac{1-2\nu}{2\pi}\log(x_1+\rho), \quad \rho^2 := x_1^2 + x_2^2 + x_3^2.$$

Accordingly, the associated stress Green function G_{11}^B turns out to be:

$$G_{11}^B = \frac{1}{1-\nu}\big(2(1-\nu)\gamma_{1,1} - \gamma_{2,11} - x_1\gamma_{1,11}\big),$$

where

$$\gamma_1 = \widehat{\gamma}_1(x_1, x_2, x_3; \eta, \zeta) := \widehat{\psi}^B(x_1, x_2 - \eta, x_3 - \zeta),$$
$$\gamma_2 = \widehat{\gamma}_2(x_1, x_2, x_3; \eta, \zeta) := \widehat{\varphi}^B(x_1, x_2 - \eta, x_3 - \zeta).$$

And, the stress component $\overline{S}_{11}$ we are looking for can be given the following form:

$$\overline{S}_{11} = \frac{1}{1-\nu}\big(2(1-\nu)\overline{\psi}_{,1} - \overline{\varphi}_{,11} - x_1\overline{\psi}_{,11}\big),$$

where the harmonic functions $\overline{\psi}$ and $\overline{\varphi}$ have the following expressions in terms of the harmonic functions γ_1 and γ_2:

$$\overline{\psi} = \int_{\partial\mathcal{H}^+} \hat{p}(\eta,\zeta)\,\widehat{\gamma}_1(x_1, x_1, x_3; \eta, \zeta)\, d\eta d\zeta,$$
$$\overline{\varphi} = \int_{\partial\mathcal{H}^+} \hat{p}(\eta,\zeta)\,\widehat{\gamma}_2(x_1, x_1, x_3; \eta, \zeta)\, d\eta d\zeta. \quad (7.18)$$

Thus, in place of the awkward integral (7.17), our task is to compute the integrals (7.18). This is doable, with the use of certain well-known properties of harmonic functions.

[1] We are indebted to Professor G. Tarantello for many useful conversations on the matters; our techniques are akin to those used in [4] and [6].

To begin with, recall (from [7], say) *Green's second identity*:

$$\int_{\Omega} u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}, \tag{7.19}$$

where u and v are scalar fields defined over region Ω, whose boundary $\partial\Omega$ has an a.e. well-defined outward normal $\mathbf{n}$. We apply this identity for $\Omega \equiv \mathcal{HS}^+$, u a harmonic function, and v the solution Γ of the boundary-value problem:

$$\begin{cases} \Delta\widehat{\Gamma}(x_1, x_2, x_3) = \delta(x_1 + a, x_2, x_3) & \text{in } \mathcal{HS}^+, \\ \widehat{\Gamma}(x_1, x_1, x_3) = 0 & \text{on } \partial\mathcal{HS}^+, \end{cases} \tag{7.20}$$

where $\delta(x_1 + a, x_2, x_3)$ is the Dirac delta function (see Sect. A.1) centered at point $x = o + a\boldsymbol{e}_1$, namely,

$$\Gamma = \widehat{\Gamma}(x_1, x_2, x_3) := \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x_1 - a)^2 + x_2^2 + x_3^2}} - \frac{1}{\sqrt{(x_1 + a)^2 + x_2^2 + x_3^2}}\right).$$

we find:

$$\widehat{u}(x_1 + a, x_2, x_3) = \int_{\partial\mathcal{HS}^+} \widehat{u}(x_1, \eta, \zeta)\frac{\partial\widehat{\Gamma}}{\partial x_1}(x_1, \eta, \zeta). \tag{7.21}$$

Moreover, function $\hat{p}$ in (7.16) can be written as follows in terms of the normal derivative of Γ:

$$\hat{p}(\eta, \zeta) = \frac{f}{2(1-\nu)}\left(2(1-\nu)\frac{\partial\widehat{\Gamma}}{\partial x_1}(x_1, x_2, x_3) - a\frac{\partial^2\widehat{\Gamma}}{\partial a\partial x_1}(x_1, x_2, x_3)\right)\Big|_{(0,\eta,\zeta)}.$$

With this, integrals (7.18) take the convenient form:

$$\overline{\psi}(x_1, x_2, x_3) = -\frac{f}{2(1-\nu)}\left(2(1-\nu)\int_{\partial\mathcal{H}^+} \widehat{\gamma}_1\frac{\partial\widehat{\Gamma}}{\partial x_1} - a\frac{\partial}{\partial a}\int_{\partial\mathcal{H}^+} \widehat{\gamma}_1\frac{\partial\widehat{\Gamma}}{\partial x_1}\right) \tag{7.22}$$

and

$$\overline{\varphi}(x_1, x_2, x_3) = -\frac{f}{8(1-\nu)}\left(2(1-\nu)\int_{\partial\mathcal{H}^+} \widehat{\gamma}_2\frac{\partial\widehat{\Gamma}}{\partial x_1} - a\frac{\partial}{\partial a}\int_{\partial\mathcal{H}^+} \widehat{\gamma}_2\frac{\partial\widehat{\Gamma}}{\partial x_1}\right). \tag{7.23}$$

To evaluate the integrals in the right sides of (7.22) and (7.23), we make use of (7.21) and find:

$$\int_{\partial\mathcal{H}^+} \widehat{\gamma}_1 \frac{\partial \widehat{\Gamma}}{\partial x_1} = \widehat{\psi}^B(x_1 + a, x_2, x_3),$$

$$\int_{\partial\mathcal{H}^+} \widehat{\gamma}_2 \frac{\partial \widehat{\Gamma}}{\partial x_1} = \widehat{\varphi}^B(x_1 + a, x_2, x_3),$$

whence

$$\overline{\psi} = -\frac{f}{2\pi}\left(\frac{a(x_1+a)}{2\rho_2^3} + \frac{1-\nu}{\rho_2}\right),$$

$$\overline{\varphi} = -\frac{f}{2\pi}\left(\frac{a(x_1+a)}{2\rho_2^3} + (1-2\nu)\log(x_1 + a + \rho_2)\right).$$

All in all, the first component of the stress tensor field solving the Mindlin Problem is:

$$S_{11}^{Mi} = \frac{f}{8\pi(1-\nu)}\Bigg(-\frac{(1-2\nu)(x_1-a)}{\rho_1^3} + \frac{(1-2\nu)(x_1-a)}{\rho_2^3} - \frac{3(x_1-a)^2}{\rho_1^5} - \frac{3(3-4\nu)x_1(x_1+a)^2 - 3a(x_1+a)(5x_1-a)}{\rho_2^5} - \frac{30ax_1(x_1+a)^3}{\rho_2^7}\Bigg) \tag{7.24}$$

(ρ_1 and ρ_2 are defined in (7.14)).

At the expenses of completely similar long computations, the remaining stress components are found to be:

$$S_{rr}^{Mi} = \frac{f}{8\pi(1-\nu)}\Bigg(\frac{(1-2\nu)(z-a)}{\rho_1^3} - \frac{(1-2\nu)(z+7a)}{\rho_2^3} + \frac{4(1-\nu)(1-2\nu)}{\rho_2(\rho_2+z+a)} - \frac{3r^2(z-a)}{\rho_1^5} + \frac{6a(1-2\nu)(z+a)^2 - 6a^2(z+a) - 3(3-4\nu)r^2(z-a)}{\rho_2^5} - \frac{30ar^2z(z+a)}{\rho_2^7}\Bigg),$$

$$S_{\varphi\varphi}^{Mi} = \frac{f(1-2\nu)}{8\pi(1-\nu)}\Bigg(\frac{(z-a)}{\rho_1^3} + \frac{(3-4\nu)(z+a) - 6a}{\rho_2^3} - \frac{4(1-\nu)}{\rho_2(\rho_2+z+a)} + \frac{6a(z+a)^2}{\rho_2^5} - \frac{6a^2(z+a)}{(1-2\nu)\rho_2^5}\Bigg),$$

$$S_{zr}^{Mi} = \frac{fr}{8\pi(1-\nu)}\left(-\frac{1-2\nu}{\rho_1^3} + \frac{1-2\nu}{\rho_2^3} - \frac{3(z-a)^2}{\rho_1^5} - \frac{30az(z+a)^2}{\rho_2^7} - \frac{3(3-4\nu)z(z+a) - 3a(3z+a)}{\rho_2^5}\right). \tag{7.25}$$

Note that, the Boussinesq stress (5.62) is recovered for $a = 0$.

7.3.2 The Strain and Displacement Fields

The Mindlin displacement strain field is found by insertion of the stress representations (7.24) and (7.25) into the inverse constitutive equation (2.45). After some algebraic manipulations, one finds:

$$\begin{aligned}(-16\pi G(1-\nu)(1+\nu)f^{-1})E_{zz}^{Mi} &= \frac{3(z-a)\left(a^2-2az-\nu r^2+z^2\right)}{\rho_1^5} \\ &+ \left(\frac{30az(a+z)\left(a^2+2az-\nu r^2+z^2\right)}{\rho_2^7} - \frac{3}{\rho_2^5}\left(a^3\left(4\nu^2-1\right) + a^2\left(12\nu^2 z+z\right)\right.\right. \\ &\left. + a\left(\nu(4\nu-3)r^2 + \left(8\nu^2+4\nu-1\right)z^2\right) + (4\nu-3)z\left(z^2-\nu r^2\right)\right) \\ &\left. + (2\nu-1)\frac{\left(a\left(4\nu^2+10\nu-1\right)+\left(4\nu^2-2\nu+1\right)z\right)}{\rho_2^3}\right) - \frac{\left(4\nu^2-1\right)(z-a)}{\rho_1^3},\end{aligned}$$

$$\begin{aligned}(16\pi(1-\nu)Gf^{-1})E_{rr}^{Mi} &= \frac{-6a^2(a+z) - 3(4\nu-3)r^2(a-z) + 6a(1-2\nu)(a+z)^2}{\rho_2^5} \\ &+ \frac{3r^2(a-z)}{\rho_1^5} - \frac{30ar^2z(a+z)}{\rho_2^7} + \frac{4(\nu-1)(2\nu-1)}{\rho_2(a+\rho_2+z)} + \frac{(2\nu-1)(a-z)}{\rho_1^3} \\ &+ \frac{(2\nu-1)(7a+z)}{\rho_2^3} - \frac{\nu\left(3(a-z)\left(a^2-2az+r^2+z^2\right)\right)}{(\nu+1)\rho_1^5} \\ &+ \frac{\nu\left(30az(a+z)\left(a^2+2az+r^2+z^2\right)\right)}{(\nu+1)\rho_2^7} \\ &+ \frac{\nu\left(3\left(a^3(4\nu+1)+a^2(8\nu-5)z+a\left((4\nu-3)r^2-3z^2\right)-(4\nu-3)z\left(r^2+z^2\right)\right)\right)}{(\nu+1)\rho_2^5} \\ &- \frac{\nu\left((2\nu-1)(a(4\nu+11)+(4\nu-3)z)\right)}{(\nu+1)\rho_2^3} + \frac{(2\nu-1)(a-z)}{(\nu+1)\rho_1^3},\end{aligned}$$

$$
\begin{aligned}
&(16\pi(1-\nu)Gf^{-1})E^{Mi}_{\varphi\varphi}\\
&=(1-2\nu)\Bigg(-\frac{6a^2(a+z)}{\rho_2^5}+\frac{4(\nu-1)}{\rho_2\,(a+\rho_2+z)}\\
&\quad+\frac{(3-4\nu)(a+z)-6a}{\rho_2^3}+\frac{6a(a+z)^2}{\rho_2^5}+\frac{z-a}{\rho_1^3}\Bigg)\\
&\quad-\frac{3\nu(a-z)\left(a^2-2az+r^2+z^2\right)}{(\nu+1)\rho_1^5}\\
&\quad+\frac{30\nu az(a+z)\left(a^2+2az+r^2+z^2\right)}{(\nu+1)\rho_2^7}\\
&\quad+\frac{3\nu\left(a^3(4\nu+1)+a^2(8\nu-5)z+a\left((4\nu-3)r^2-3z^2\right)-(4\nu-3)z\left(r^2+z^2\right)\right)}{(\nu+1)\rho_2^5}\\
&\quad-\frac{3\nu(2\nu-1)(a(4\nu+11)+(4\nu-3)z)}{(\nu+1)\rho_2^3}-\frac{\nu(2\nu-1)(a-z)}{(\nu+1)\rho_1^3},
\end{aligned}
$$

and

$$
\begin{aligned}
&(16\pi G(1-\nu)(1+\nu)f^{-1})E^{Mi}_{zr}\\
&=-3\frac{(a-z)\left(a^2\nu+a(\nu r+r-2\nu z)+\nu r^2-(\nu+1)rz+\nu z^2\right)}{\rho_1^5}\\
&\quad+\frac{3\left(a^3\nu(4\nu+1)+a^2((\nu+1)r+\nu(8\nu-5)z)+a\nu\left((4\nu-3)r^2+4(\nu+1)rz-3z^2\right)\right)}{\rho_2^5}\\
&\quad+\frac{3\left(-(4\nu-3)z\left(\nu r^2-(\nu+1)rz+\nu z^2\right)\right)}{\rho_2^5}\\
&\quad+\frac{30az(a+z)\left(a^2\nu-a(\nu r+r-2\nu z)+\nu r^2-(\nu+1)rz+\nu z^2\right)}{\rho_2^7}\\
&\quad-\frac{(2\nu-1)(\nu(a(4\nu+11)+(4\nu-3)z)+(\nu+1)r)}{\rho_2^3}\\
&\quad+\frac{(2\nu-1)(-a\nu+\nu r+r+\nu z)}{\rho_1^3}.
\end{aligned}
$$

Note that, if $a=0$, the deformation field reduces to Boussinesq's, as given by (5.65).

To find the displacement field, we could follow a by now familiar course, and exploit the compatibility equation (2.9) as we did in SubSect. 7.2.2. However, we prefer to perform this task by employing the same procedure we adopted for the stress field, namely,

(i) we superimpose the Kelvin displacements $\breve{\boldsymbol{u}}$ and $\widehat{\boldsymbol{u}}$ corresponding to the stress fields $\breve{\boldsymbol{S}}$ and $\widehat{\boldsymbol{S}}$ defined in Sect. 7.3.1;

(ii) we consider the restriction $\widetilde{\boldsymbol{u}}$ to $\mathcal{HS}^+$ of the above point-wise superposition, and we further superimpose to it the displacement field $\overline{\boldsymbol{u}}$, that we determine by means of (7.18) and (5.76).

The outcome is the Mindlin displacement field, in cylindrical coordinates:

$$u_z^{Mi} = \frac{fr}{16\pi G(1-\nu)}\left(\frac{z-a}{\rho_1^3} + \frac{(3-4\nu)(z-a)}{\rho_2^3} - \frac{4(1-\nu)(1-2\nu)}{\rho_2(\rho_2+z+a)} + \frac{6az(z+a)}{\rho_2^5}\right),$$

$$u_r^{Mi} = \frac{f}{16\pi G(1-\nu)}\left(\frac{3-4\nu}{\rho_1} + \frac{8(1-\nu)^2-(3-4\nu)}{\rho_2} + \frac{(z-a)^2}{\rho_1^3} + \frac{(3-4\nu)(z+a)^2-2az}{\rho_2^3} + \frac{6az(z+a)^2}{\rho_2^5}\right).$$

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