Chapter 6
The Kelvin Problem

Lord Kelvin (William Thompson, 1824–1907) solved the problem that was later named after him in 1848 [6].[1] The problem consists in finding the equilibrium state of a linearly elastic, isotropic material body occupying the whole space and being subject to a point load (Fig. 6.1).

6.1 Solution by Juxtaposition

The plane version of Kelvin's problem we study in the next section is a problem formulated on a plane orthogonal to a uniform line load (Fig. 6.2). As far as the applied loads are concerned, both the Kelvin Problem and its plane version can be regarded as the *juxtaposition of two anti-mirror symmetric problems*: two Boussinesq problems in the case of the 3-D Kelvin Problem, either two Boussinesq-Flamant or two plane Cerruti problems in the case of the 2-D Kelvin problem (Fig. 6.3; the Cerruti Problem is treated in Chap. 8).

6.1.1 Continuity Conditions at Sutures

Unfortunately, superposition of elastic states does not yield the desired Kelvin state, because it does not guarantee a 'seamless suture' over the common boundary. For this, two continuity conditions should be satisfied pointwise, the one for the *traction* field, the other for the *displacement* field[2]:

$$[\![\boldsymbol{Sn}]\!] = \boldsymbol{0}, \quad [\![\boldsymbol{u}]\!] = \boldsymbol{0}^2.$$

[1] An exposition of Kelvin's solution tailored after Love's [2] is found in the Appendix, Sect. A.6.

[2] Consistent with definition (1.19), here $[\![\Psi]\!] := \Psi^+ - \Psi^-$ denotes the *jump* of the field Ψ at a suture plane, in terms of the limits $\Psi^\pm$ of Ψ when the point of interest is attained from one or the other part of that plane.

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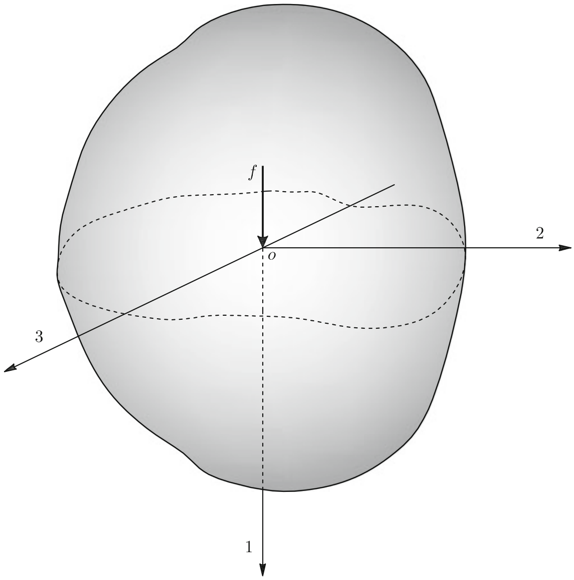

Fig. 6.1 The Kelvin Problem (this figure is taken from [1])

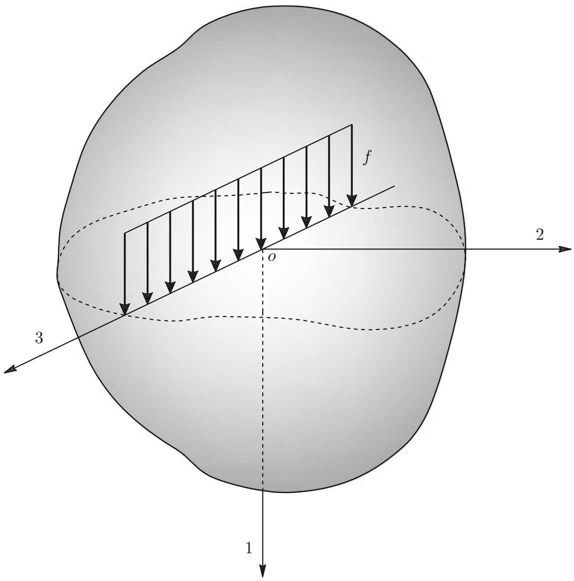

Fig. 6.2 The plane Kelvin problem

Juxtaposition of anti-mirror symmetric elastic states complies with the first condition trivially, because tractions are null all over the common boundary. On recalling the form of Flamant and Boussinesq displacement fields at $z = 0$, specified by, respectively, (4.35) and (5.72), we see that, while in both cases continuity of vertical displacements is gratis, horizontal components do jump:

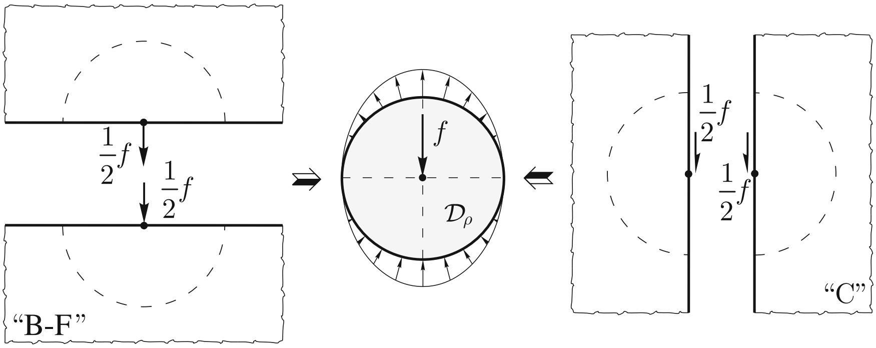

Fig. 6.3 Juxtaposition of loads and stress fields for two plane anti-mirror symmetric Boussinesq-Flamant and Cerruti Problems (this figure has been adapted from [4])

$$\left[\!\left[\boldsymbol{u}^F \cdot \boldsymbol{e}_2\right]\!\right]_{z=0} = -(1-2\nu)\frac{f}{E_0(1-\nu)}\operatorname{sgn} x_2\,,$$
$$\left[\!\left[\boldsymbol{u}^B \cdot \boldsymbol{h}\right]\!\right]_{z=0} = -(1-2\nu)\frac{f}{2\pi G}r^{-1};$$

thus, *solving the Kelvin Problem by juxtaposition is impossible*.

Nevertheless, we notice that, in both cases, continuity of horizontal components could be achieved for $\nu = 1/2$. This limit situation is excluded by the third of the inequalities (2.44), guaranteeing positivity of the elastic energy density stored by a *compressible* linearly elastic isotropic material. We see from (2.42) that, given the stress field and then $\operatorname{tr}\boldsymbol{S}$, the corresponding volume dilatation, measured by $\operatorname{tr}\boldsymbol{E}$, approaches zero when $\nu \to 1/2$, i.e., in the so-called *incompressibility limit*.[3] This fact prompts the expectation that, for incompressible linearly elastic materials, the Kelvin Problem be solvable by juxtaposition of two anti-mirror symmetric Boussinesq Problems for materials in the same class. We leave for the reader a task that is easy, after we solve the Boussinesq Problem for incompressible materials in the next subsection.

6.1.2 Conditional Solvability: The Boussinesq Problem for Incompressible Materials

An elasticity problem is solved when the relative *elastic state*—that is, the triplet $(\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{S})$ of displacement, deformation and stress fields—is known. When an *internal constraint* prevails—that is, an *a priori* limitation on admissible deformations is posed—it would be desirable to deduce the elastic state from the elastic state of the

[3] We also see from (2.47) that, under the same circumstances, for the stored energy to stay finite the volume changes must become smaller and smaller as ν approaches $1/2$.

corresponding unconstrained problem. In the present case, incompressibility is the internal constraint we deal with, and we would like to give a precise meaning to the following formal writing:

$$(\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{S})^{inc} = \lim_{\nu \to 1/2} (\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{S}).$$

Now, the solution of a linear elasticity problem *depends with continuity on data*, that is, on the information we have about: (i) the geometry of the region on which the problem is formulated; (ii) the nature of the material filling that region; (iii) the applied loads; (iv) the boundary conditions. The value of the Poisson modulus is a datum, on which the solution depends in general with continuity, as it is possible to see, for instance, in (4.37) and (5.70). Therefore, it makes sense to expect that the displacement field for the incompressible Boussinesq Problem be obtained by taking the limit for $\nu \to 1/2$ of the same field in the compressible case, which is:

$$\boldsymbol{u}^B_{inc} = \frac{f}{4\pi G}\rho^{-1}((\cos^2\vartheta + 1)\boldsymbol{e}_1 + |\sin\vartheta|\cos\vartheta\,\boldsymbol{h}). \tag{6.1}$$

Moreover, given that the operations of taking a spatial gradient and the incompressibility limit commute, we have from (5.71) that

$$\begin{aligned}
(E^B_{inc})_{zz} &= \frac{f}{4\pi G}\rho^{-2}\cos\vartheta(-3\cos^2\vartheta + 1),\\
(E^B_{inc})_{rr} &= \frac{f}{4\pi G}\rho^{-2}\cos\vartheta(-3\sin^2\vartheta + 1),\\
(E^B_{inc})_{\varphi\varphi} &= \frac{f}{4\pi G}\rho^{-2}\cos\vartheta,\\
(E^B_{inc})_{zr} &= -\frac{3f}{4\pi G}\rho^{-2}\cos^2\vartheta|\sin\vartheta|.
\end{aligned}$$

It is easily checked that

$$\operatorname{tr}\boldsymbol{E}^B_{inc} = 0;$$

thus, the strain field $\boldsymbol{E}^B_{inc}$ is deviatoric:[4]

$$\boldsymbol{E}^B_{inc} = \operatorname{dev}\boldsymbol{E}^B_{inc}.$$

On recalling that $\cos\vartheta = \rho^{-1}z$, it is equally easy to see that, for $z = 0$,

[4] Needless to say, the same developments follow by an application of definition $(2.2)_2$ to the field (6.1). Recall that each symmetric tensor $\boldsymbol{A}$ can be additively split into uniquely defined *deviatoric* and *spheric* parts:

$$\boldsymbol{A} = \operatorname{dev}\boldsymbol{A} + \operatorname{sph}\boldsymbol{A}, \quad \operatorname{sph}\boldsymbol{A} := \frac{1}{3}\operatorname{tr}\boldsymbol{A}, \quad \operatorname{dev}\boldsymbol{A} := \boldsymbol{A} - \operatorname{sph}\boldsymbol{A}.$$

$$\boldsymbol{E}^B_{inc}(0, r) \equiv \boldsymbol{0}, \tag{6.2}$$

Finding $\boldsymbol{S}^{inc}$ requires something more than taking a limit: an *ad hoc* modeling assumption is needed.

The constitutive Eq. (2.46)$_2$ for a compressible isotropic material can be written as follows:

$$\boldsymbol{S} = 2G\Big(\text{dev}\,\boldsymbol{E} + \frac{1+\nu}{1-2\nu}\,\text{sph}\,\boldsymbol{E}\Big). \tag{6.3}$$

In the incompressibility limit, the value of G is kept fixed, while both $(1-2\nu)$ and $\text{sph}\,\boldsymbol{E}$ tend to null; it is then necessary to give the limit of $(1-2\nu)^{-1}\text{sph}\,\boldsymbol{E}$ a meaning. We assume that a finite limit exists:

$$\lim_{\nu\to 1/2} \frac{1+\nu}{1-2\nu}\text{sph}\,\boldsymbol{E} = \pi\boldsymbol{I},$$

with the scalar-valued field π *constitutively indetermined*. Accordingly, we replace (6.3) by the constitutive equation:

$$\boldsymbol{S} = 2G\,\text{dev}\,\boldsymbol{E} + \pi\boldsymbol{I},$$

describing the mechanical response of a incompressible isotropic material, and we write, provisionally,

$$\boldsymbol{S}^B_{inc} = 2G\,\text{dev}\,\boldsymbol{E}^B_{inc} + \pi\boldsymbol{I}.$$

The equilibrium pressure field is determined by requiring that the stress field $\boldsymbol{S}^B_{inc}$ be divergenceless in the interior of $\mathcal{HS}^+$, a condition that reads:

$$\nabla\widehat{\pi}(z, r) = -2G\,\text{div}\,\widehat{\boldsymbol{E}}^B_{inc}(z, r) \quad \text{for } z, r > 0,$$

and by satisfying the boundary condition (5.15), which, in view of (6.2), reduces to:

$$\pi(0, r) = 0, \quad r > 0.$$

Remark 6.1 A material is *constrained* whenever some deformations are deemed constitutively impossible by requesting that the strain measure $\boldsymbol{E}$ satisfy an algebraic limitation of the following type:

$$\boldsymbol{V}\cdot\boldsymbol{E} = 0, \tag{6.4}$$

for a given *constraint tensor* $\boldsymbol{V} \in \text{Sym}$. For a constrained material, it is customary to decompose the stress tensor additively:

$$\boldsymbol{S} = \boldsymbol{S}^{(A)} + \boldsymbol{S}^{(R)},$$

with the *active stress* $\mathbf{S}^{(A)}$ determined by a tensor-valued constitutive function, defined on $\mathcal{A} := \{\mathbf{E} \,|\, \mathbf{V}\cdot\mathbf{E} = 0\}$, the subspace of Sym composed by all admissible deformations, and with the *reactive stress* $\mathbf{S}^{(R)}$ (i.e., the stress necessary to maintain the stipulated kinematic constraint), characterized by the condition that the work spent on whatever admissible deformation be null:

$$\mathbf{S}^{(R)} \cdot \mathbf{E} = 0, \quad \forall \mathbf{E} \in \mathcal{A}.$$

This last condition is equivalent to the following representation of the reactive stress:

$$\mathbf{S}^{(R)} = \sigma^{(R)}\mathbf{V},$$

where $\sigma^{(R)}$ a constitutively indeterminate scalar multiplier (e.g., for $\mathbf{V} = \mathbf{I}$, $\sigma^{(R)} = \pi$).[5]

Remark 6.2 The response symmetry of a constrained material is affected by the nature of the internal constraints, if any. The internal constraints compatible with isotropy are three: two are nontrivial, incompressibility and *shape preservation*, for which it is required that $\operatorname{dev}\mathbf{E} = \mathbf{0}$; one is trivial, *rigidity*, in which case the choice of a constraint tensor in (6.4) is arbitrary, and hence $\mathbf{E} = \mathbf{0}$; for a rigid material, the active stress is null, all stress is of reactive nature.

6.2 The 2-D Kelvin Problem

Suppose that a constant line load $\mathbf{f} = f\mathbf{e}_1$ (with $\dim(f) = FL^{-1}$) is applied along the x_3-axis (Fig. 6.2). To find the relative equilibrium state, our plan is:
(i) to individuate a large class of two-dimensional balanced stress fields, that is to say, stress fields of the form (4.8) that solve the *distributional equilibrium equation*

$$\operatorname{div}\mathbf{S}(x) + f\delta(o)\mathbf{e}_1 = \mathbf{0} \quad \text{for } x \in \mathcal{H}; \tag{6.5}$$

(ii) to add to each of such stress fields an auxiliary stress field:

$$\mathbf{S}^{(aux)} = S_{33}\mathbf{e}_3 \otimes \mathbf{e}_3, \quad S_{33} = \nu(S_{11} + S_{22}), \tag{6.6}$$

so as to obtain a family of three-dimensional stress fields, among which to choose, by means of condition (2.69), those compatible with the existence of a state of plane strain and deformation in the whole space;
(iii) to construct such strain and deformation states.

[5] More about internal constraint in linear elasticity is found in [3], Chapter III, Sections 17 and 18.

6.2.1 Balanced Stress Fields

We recall that a locally integrable stress field $\boldsymbol{S}$ being divergenceless over $\mathcal{H}\setminus o$ is said to solve equation (6.5) *in the sense of distributions over* $\mathcal{H}$ if

$$\int_{\mathcal{H}} \boldsymbol{S}\cdot\nabla\boldsymbol{v} = f\,\boldsymbol{v}(o)\cdot\boldsymbol{e}_1 \quad \text{for all test vector fields } \boldsymbol{v}\in C_c^\infty(\mathcal{H},\mathcal{V}^{(2)}); \tag{6.7}$$

here, as the notation suggests, a test vector field is a C^∞ field with compact support, defined over $\mathcal{H}$ and taking its values in the 2-D vector space $\mathcal{V}^{(2)}$, the translation space of $\mathcal{H}$. The direct mechanical interpretation of a condition of this type is that, for a stress field to balance the applied loads, *the stress working must equal the load working*, *for whatever test velocity field*.[6] We shall now derive a version of this condition that allows for a different and more specific mechanical interpretation.

For each fixed test field $\boldsymbol{v}$, let $\mathcal{D}_\rho$ be a disk of radius ρ centered at o and containing the support of $\boldsymbol{v}$, and let $\mathcal{D}_\varepsilon$ be a smaller disk, also centered at o. Then,

$$\int_{\mathcal{H}} \boldsymbol{S}\cdot\nabla\boldsymbol{v} = \int_{\mathcal{D}_\rho} \boldsymbol{S}\cdot\nabla\boldsymbol{v} = \int_{\mathcal{D}_\varepsilon} \boldsymbol{S}\cdot\nabla\boldsymbol{v} + \int_{\mathcal{D}_\rho\setminus\mathcal{D}_\varepsilon} \boldsymbol{S}\cdot\nabla\boldsymbol{v}.$$

Given that $\boldsymbol{S}$ is integrable and $\nabla\boldsymbol{v}$ is smooth,

$$\lim_{\varepsilon\to 0}\int_{\mathcal{D}_\varepsilon} \boldsymbol{S}\cdot\nabla\boldsymbol{v} = 0.$$

Moreover, in view of the identity

$$\boldsymbol{S}\cdot\nabla\boldsymbol{v} = \operatorname{div}(\boldsymbol{S}^T\boldsymbol{v}) - \boldsymbol{v}\cdot\operatorname{div}\boldsymbol{S},$$

the divergence theorem, the fact that $\operatorname{supp}(\boldsymbol{v})\subset\mathcal{D}_\rho$, and the fact that $\operatorname{div}\boldsymbol{S}$ is null over $\mathcal{H}\setminus o$, we have that

$$\int_{\mathcal{D}_\rho\setminus\mathcal{D}_\varepsilon} \boldsymbol{S}\cdot\nabla\boldsymbol{v} = \int_{\partial(\mathcal{D}_\rho\setminus\mathcal{D}_\varepsilon)} \boldsymbol{S}\boldsymbol{n}\cdot\boldsymbol{v} - \int_{\mathcal{D}_\rho\setminus\mathcal{D}_\varepsilon} \boldsymbol{v}\cdot\operatorname{div}\boldsymbol{S} = -\int_{\partial\mathcal{D}_\varepsilon} \boldsymbol{S}\boldsymbol{n}\cdot\boldsymbol{v}.$$

Therefore, for each admissible test field, condition (6.7) can be given the provisional form

$$\lim_{\varepsilon\to 0}\left(\int_{\partial\mathcal{D}_\varepsilon} \boldsymbol{S}\boldsymbol{n}\cdot\boldsymbol{v}\right) + f\,\boldsymbol{v}(o)\cdot\boldsymbol{e}_1 = 0.$$

Note that

[6] Alternative terminological choices are 'power' (or 'power expenditure') for 'working' and 'virtual' for 'test'; an alternative version of the italicized sentence above would read: *the stress power equals the load power for whatever virtual velocity field.*

$$\int_{\partial\mathcal{D}_\varepsilon} \boldsymbol{S}\boldsymbol{n}\cdot\boldsymbol{v} = \int_{-\pi}^{+\pi} \varepsilon\boldsymbol{S}(o+\varepsilon\widehat{\boldsymbol{e}}(\vartheta))\widehat{\boldsymbol{e}}(\vartheta)\cdot\boldsymbol{v}(o+\varepsilon\widehat{\boldsymbol{e}}(\vartheta))\,d\vartheta.$$

Thus, if the following condition holds:
(A) the vector field $\vartheta \mapsto \varepsilon\widehat{\boldsymbol{S}}(o+\varepsilon\widehat{\boldsymbol{e}}(\vartheta))\widehat{\boldsymbol{e}}(\vartheta)$ is independent of ε,
then

$$\lim_{\varepsilon\to 0}\Big(\int_{\partial\mathcal{D}_\varepsilon} \boldsymbol{S}\boldsymbol{n}\cdot\boldsymbol{v}\Big) = \Big(\int_{-\pi}^{+\pi} \varepsilon\widehat{\boldsymbol{S}}(\varepsilon,\vartheta)\widehat{\boldsymbol{e}}(\vartheta)\,d\vartheta\Big)\cdot\boldsymbol{v}(o),$$

and (6.7) can be given the final form

$$\int_{\partial\mathcal{D}_\varepsilon} \boldsymbol{S}\boldsymbol{n} + f\boldsymbol{e}_1 = \boldsymbol{0}.$$

The mechanical interpretation of this condition on the stress field—that the diffused contact force over the periphery of any disk balances the concentrated force applied at its center—can be seen as a counterpart of the mathematical interpretation of condition (6.5)—that the corresponding balanced stress field has *divergence measure* supported at the point where the concentrated force is applied.

It is not difficult to see that each stress field of the one-parameter family

$$\begin{aligned}\boldsymbol{S} = \widehat{\boldsymbol{S}}(\rho,\vartheta;\boldsymbol{e}_1) = \rho^{-1}(&\alpha_0\cos\vartheta\,\widehat{\boldsymbol{e}}(\vartheta)\otimes\widehat{\boldsymbol{e}}(\vartheta) + \gamma_0\cos\vartheta\,\widehat{\boldsymbol{e}}'(\vartheta)\otimes\widehat{\boldsymbol{e}}'(\vartheta)\\ &+\gamma_0\sin\vartheta(\widehat{\boldsymbol{e}}(\vartheta)\otimes\widehat{\boldsymbol{e}}'(\vartheta)+\widehat{\boldsymbol{e}}'(\vartheta)\otimes\widehat{\boldsymbol{e}}(\vartheta))),\quad \alpha_0-\gamma_0 = -\frac{f}{\pi},\end{aligned} \tag{6.8}$$

fulfills condition (A) and balances the applied load. In particular, the second of (6.8) follows from the balance of a body part in the form of a disk centered at the origin, of arbitrary radius ρ: since

$$\rho\boldsymbol{S}\boldsymbol{e} = \alpha_0\cos\vartheta\,\boldsymbol{e} + \gamma_0\sin\vartheta\,\boldsymbol{e}', \tag{6.9}$$

an easy calculation shows that

$$\int_{\partial\mathcal{D}_\rho} \boldsymbol{S}\boldsymbol{e} = -f\boldsymbol{e}_1 \tag{6.10}$$

(cf. e.g., [5], Section 78).

Remark 6.3 With the use of $(6.8)_2$, it is not difficult to transform $(6.9)_2$ into

$$\widehat{\boldsymbol{S}}(\rho,\vartheta;\boldsymbol{e}_1)\boldsymbol{e}(\vartheta) = \rho^{-1}\Big(\alpha_0\widehat{\boldsymbol{e}}(2\vartheta) + \frac{f}{\pi}\sin\vartheta\,\widehat{\boldsymbol{e}}'(\vartheta)\Big),$$

which allows for an easier visualization of the stress vector at any point of $\partial\mathcal{D}_\rho$; note that the first addendum does not contribute to the integral in (6.10).

Remark 6.4 The stress fields (6.8) have the form (4.8). Condition (4.9) has been dropped, because it makes no sense for the full-plane domain where Kelvin problem is formulated. The choices of $\widehat{a}$ and $\widehat{c}$ reflect the expected parities of these two functions. Choosing $\widehat{a}(\vartheta) = \sin\vartheta = \widehat{c}(\vartheta)$ leads to the Kelvin stress fields for the load $\boldsymbol{f} = f\boldsymbol{e}_2$, namely,

$$\widehat{\boldsymbol{S}}(\rho, \vartheta; \boldsymbol{e}_2) = \rho^{-1}(\alpha_0 \sin\vartheta\, \widehat{\boldsymbol{e}}(\vartheta) \otimes \widehat{\boldsymbol{e}}(\vartheta) - \gamma_0 \sin\vartheta\, \widehat{\boldsymbol{e}}'(\vartheta) \otimes \widehat{\boldsymbol{e}}'(\vartheta) \\ + \gamma_0 \cos\vartheta (\widehat{\boldsymbol{e}}(\vartheta) \otimes \widehat{\boldsymbol{e}}'(\vartheta) + \widehat{\boldsymbol{e}}'(\vartheta) \otimes \widehat{\boldsymbol{e}}(\vartheta))), \quad \alpha_0 + \gamma_0 = -\frac{f}{\pi}.$$

6.2.2 Compatible Stress Fields

As anticipated, we now seek what stress fields of the type (6.8) satisfy the compatibility condition (2.69). This is quickly done. Firstly, from (6.8) we deduce that

$$\mathrm{tr}\, \boldsymbol{S} = (\alpha_0 + \gamma_0)\rho^{-1} \cos\vartheta.$$

Then, with the use of the last of (3.19), we find that

$$\Delta(\rho^{-1} \cos\vartheta) = 0.$$

We then conclude, by taking (6.6) into account, that each of the stress fields of the one-parameter family

$$\widetilde{\boldsymbol{S}} = \boldsymbol{S} + \nu(\mathrm{tr}\, \boldsymbol{S})\boldsymbol{e}_3 \otimes \boldsymbol{e}_3,$$

is compatible with a state of plane strain and plane displacement, to be determined in the next subsection.

6.2.3 Strain and Displacements Fields

The strain field solving the plane Kelvin problem is obtained by inserting the stress field (6.8) into the inverse constitutive equation (2.57). One finds:

$$\boldsymbol{E} = \frac{1}{E_0}\rho^{-1}((\alpha_0 - \nu_0\gamma_0) \cos\vartheta\, \boldsymbol{e} \otimes \boldsymbol{e} + (\gamma_0 - \nu_0\alpha_0) \cos\vartheta\, \boldsymbol{e}' \otimes \boldsymbol{e}' \\ + (1 + \nu_0)\gamma_0 \sin\vartheta (\boldsymbol{e} \otimes \boldsymbol{e}' + \boldsymbol{e}' \otimes \boldsymbol{e})).$$

To determine the displacement field, one has to solve the following system of PDEs:

$$
\begin{aligned}
&u_{\rho,\rho} = \rho^{-1}\frac{\alpha_0 - \nu_0\gamma_0}{E_0}\cos\vartheta,\\
&u_{\vartheta,\vartheta} + u_\rho = \frac{\gamma_0 - \nu_0\alpha_0}{E_0}\cos\vartheta,\\
&u_{\vartheta,\rho} + \rho^{-1}(u_{\rho,\vartheta} - u_\vartheta) = \rho^{-1}\frac{2(1+\nu_0)}{E_0}\gamma_0\sin\vartheta,
\end{aligned}
\tag{6.11}
$$

where the unknown fields

$$
u_\rho := \boldsymbol{u}\cdot\boldsymbol{e} = \hat{u}_\rho(\rho,\vartheta), \qquad u_\vartheta := \boldsymbol{u}\cdot\boldsymbol{e}' = \hat{u}_\vartheta(\rho,\vartheta),
$$

must satisfy the intrinsic symmetry conditions of the plane Kelvin problem and therefore be such that

$$
\hat{u}_\rho(\rho,\vartheta) = \hat{u}_\rho(\rho,-\vartheta), \qquad \hat{u}_\vartheta(\rho,\vartheta) = -\hat{u}_\vartheta(\rho,-\vartheta). \tag{6.12}
$$

The integration of $(6.11)_1$ yields:

$$
\hat{u}_\rho(\rho,\vartheta) = \frac{\alpha_0 - \nu_0\gamma_0}{E_0}\log\rho\cos\vartheta + \hat{v}(\vartheta), \tag{6.13}
$$

with $\hat{v}$ an arbitrary even function, so as to satisfy condition $(6.12)_1$. With this provisional representation for $\hat{u}_\rho$, integration of $(6.11)_2$ yields:

$$
\hat{u}_\vartheta(\rho,\vartheta) = -\frac{\alpha_0 - \nu_0\gamma_0}{E_0}\log\rho\sin\vartheta - \widehat{V}(\vartheta) + \frac{\gamma_0 - \nu_0\alpha_0}{E_0}\sin\vartheta, \tag{6.14}
$$

where $\widehat{V}$ is a primitive of $\hat{v}$, and hence is odd. The addition of an arbitrary function of ρ to this expression of $\hat{u}_\vartheta$ is forbidden by condition $(6.12)_2$. Moreover, the third of (6.11) determines $\hat{v}$: on inserting (6.13) and (6.14) into it, we find that

$$
-\frac{\alpha_0 - \nu_0\gamma_0}{E_0}\sin\vartheta + \hat{v}'(\vartheta) + \widehat{V}(\vartheta) - \frac{\gamma_0 - \nu_0\alpha_0}{E_0}\sin\vartheta = \frac{2(1+\nu_0)}{E_0}\gamma_0\sin\vartheta,
$$

or rather, after differentiation and term rearrangement,

$$
\hat{v}''(\vartheta) + \hat{v}(\vartheta) = \frac{(3+\nu_0)\gamma_0 + (1-\nu_0)\alpha_0}{E_0}\cos\vartheta.
$$

The even solutions of this equation are:

$$
\hat{v}(\vartheta) = v_0\cos\vartheta + \frac{1}{2E_0}((3+\nu_0)\gamma_0 + (1-\nu_0)\alpha_0)\vartheta\sin\vartheta;
$$

their primitives are:

$$\widehat{V}(\vartheta) = v_0 \sin\vartheta - \frac{1}{2E_0}((3+\nu_0)\gamma_0 + (1-\nu_0)\alpha_0)(\vartheta\cos\vartheta - \sin\vartheta). \tag{6.15}$$

In addition to the parity requirements specified by (6.12), the displacement field must obey the ‘glueing condition’:

$$\boldsymbol{u}(\rho, -\pi) = \boldsymbol{u}(\rho, +\pi),$$

which, upon fiddling a bit with relations (6.13)–(6.15), is found equivalent to the scalar condition $\hat{V}(\pi) = 0$, or rather:

$$(3+\nu_0)\gamma_0 + (1-\nu_0)\alpha_0 = 0;$$

together with $(6.8)_2$, this condition allows to determine the two constants α_0 and γ_0:

$$\alpha_0 = -\frac{f}{4\pi}(3+\nu_0), \quad \gamma_0 = \frac{f}{4\pi}(1-\nu_0).$$

In conclusion, the plane Kelvin problem is solved by the displacement field:

$$\boldsymbol{u} = \hat{u}_\rho(\rho,\vartheta)\boldsymbol{e}(\vartheta) + \hat{u}_\vartheta(\rho,\vartheta)\boldsymbol{e}'(\vartheta),$$

with

$$\begin{aligned}\hat{u}_\rho(\rho,\vartheta) &= \frac{f}{4\pi\, E_0}\,(3+\nu_0^2)\,\log\rho\cos\vartheta,\\ \hat{u}_\vartheta(\rho,\vartheta) &= \frac{f}{4\pi\, E_0}\big(-(3+\nu_0^2)\log\rho + 1 + \nu_0 + 3\nu_0^2\big)\sin\vartheta\end{aligned}$$

(we have disposed of the rigid displacement:

$$\boldsymbol{u}_{rig} = v_0\left(\cos\vartheta\boldsymbol{e}(\vartheta) - \sin\vartheta\boldsymbol{e}'(\vartheta)\right) = v_0\,\boldsymbol{e}_1$$

by setting to null the constant v_0); the corresponding stress field is:

$$\begin{aligned}\boldsymbol{S} = {} & S_{\rho\rho}(\rho,\vartheta)\widehat{\boldsymbol{e}}(\vartheta)\otimes\widehat{\boldsymbol{e}}(\vartheta) + S_{\vartheta\vartheta}(\rho,\vartheta)\widehat{\boldsymbol{e}}'(\vartheta)\otimes\widehat{\boldsymbol{e}}'(\vartheta)\\ & + S_{\rho\vartheta}(\rho,\vartheta)\big(\widehat{\boldsymbol{e}}(\vartheta)\otimes\widehat{\boldsymbol{e}}'(\vartheta) + \widehat{\boldsymbol{e}}'(\vartheta)\otimes\widehat{\boldsymbol{e}}(\vartheta)\big),\end{aligned} \tag{6.16}$$

with

$$\begin{aligned}S_{\rho\rho} &= -\frac{f}{4\pi}(3+\nu_0)\rho^{-1}\cos\vartheta,\\ S_{\vartheta\vartheta} &= \frac{f}{4\pi}(1-\nu_0)\rho^{-1}\cos\vartheta,\\ S_{\rho\vartheta} &= \frac{f}{4\pi}(1-\nu_0)\rho^{-1}\sin\vartheta.\end{aligned} \tag{6.17}$$

Remark 6.5 In the expression (4.37) for the displacement field of the Flamant Problem, there is a term proportional to $\vartheta \cos\vartheta$, that is, of the same kind of the term we just eliminated by imposing the 'glueing condition'. Actually, in that problem, this condition does not apply, because all (displacement, strain, stress) fields are defined for ϑ variable in the interval $[-\pi/2, +\pi/2]$. This remark prompts us to underline a relevant difference in the posing of the Boussinesq-Flamant Problem and the plane Kelvin Problem. Although the equilibrium equations are the same, the domains on which the two problems are formulated are different. On the one hand, the need to satisfy the boundary conditions prevailing on the plane $z = 0$ reduces the class of balanced and compatible stress fields for the Flamant Problem to a subclass of that for the Kelvin Problem; on the other hand, in the latter problem, the larger freedom in the choice of stress fields is compensated by an additional kinematic constraint, the glueing condition, allowing for the determination of the unique solution.

6.3 The Kelvin Elastic State

6.3.1 The Stress Field

The Kelvin Problem is similar to Boussinesq's in that it enjoys the same cylindrical symmetry. Once again system (5.31) must be solved for a compatible stress field of the form (5.12):

$$\boldsymbol{S} = \sigma_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_2 \boldsymbol{h} \otimes \boldsymbol{h} + \sigma_3 \boldsymbol{h}' \otimes \boldsymbol{h}' + \sigma_4 (\boldsymbol{e}_1 \otimes \boldsymbol{h} + \boldsymbol{h} \otimes \boldsymbol{e}_1),$$

with the sequential procedure introduced on Sect. 5.3.3; in particular, the first three steps of that procedure allow to determine the expressions for the stress trace and the stress components σ_1, and σ_4, that we here recall for the reader's convenience:

$$\begin{aligned}
&\widetilde{\alpha}(\rho, \vartheta) = \alpha_0\, \rho^{-2} \cos\vartheta, \\
&\widetilde{\sigma}_1(\rho, \vartheta) = \rho^{-2}\,(\widetilde{\tau}_1(\vartheta) + \beta_0 \cos\vartheta)\,, \quad \widetilde{\tau}_1(\vartheta) = \frac{3}{2}\alpha_0 \cos^3\vartheta \\
&\widetilde{\sigma}_4(\rho, \vartheta) = \rho^{-2}\,(\widetilde{\tau}_4(\vartheta) + \beta_0 |\sin\vartheta|)\,, \quad \widetilde{\tau}_4(\vartheta) = \frac{3}{2}\alpha_0 \cos^2\vartheta\, |\sin\vartheta|.
\end{aligned} \tag{6.18}$$

What makes the difference are the values to assign to constants α_0, β_0. We begin to gain information on this point by imposing that a ball centered at the origin be in equilibrium:

$$f = -2\pi \int_0^{\pi} (\cos\vartheta\, \widetilde{\tau}_1(\vartheta) + |\sin\vartheta|\, \widetilde{\tau}_4(\vartheta))|\sin\vartheta|\, d\vartheta,$$

whence

$$\alpha_0 + 2\beta_0 = -\frac{f}{2\pi}\,. \tag{6.19}$$

We also record an alternative way of writing (6.18):

$$\begin{aligned}
\widehat{\alpha}(z,r) &= \alpha_0\frac{z}{\rho^3},\\
\widehat{\sigma}_1(z,r) &= \frac{3}{2}\alpha_0\frac{z^3}{\rho^5} + \beta_0\frac{z}{\rho^3},\\
\widehat{\sigma}_4(z,r) &= \frac{3}{2}\alpha_0\frac{z^2 r}{\rho^5} + \beta_0\frac{r}{\rho^3}.
\end{aligned} \tag{6.20}$$

The stress components σ_2 and σ_3 can be determined in the same way as for the Boussinesq Problem. To take step 4 (Sect. 5.4.3), we replace (5.56) by

$$\sigma_2 = -\sigma_3 - \sigma_1 + (1+\nu)\alpha = -\sigma_3 - \frac{3}{2}\alpha_0\frac{z^3}{\rho^5} + (\alpha_0(1+\nu) - \beta_0)\frac{z}{\rho^3}, \tag{6.21}$$

with which the differential Eq. (5.57) is replaced by:

$$\sigma_3 + (r\sigma_3)_{,r} = -\frac{3}{2}\alpha_0\frac{z^3}{\rho^5} + (\alpha_0(1+\nu) - \beta_0)\frac{z}{\rho^3} + 3\alpha_0\,\nu\,\frac{r^2 z}{\rho^5},$$

whose solution is:

$$\widehat{\sigma}_3(z,r) = -\big(\alpha_0(1-2\nu) - \beta_0\big)\frac{z}{\rho^3} - \big(\alpha_0(1-2\nu) - 2\beta_0\big)\frac{z^3}{2r^2\rho^3} + \frac{g(z)}{r^2}\,;$$

combining this with $(6.21)_2$, we also have that

$$\begin{aligned}
\widehat{\sigma}_2(z,r) = &-\frac{3}{2}\alpha_0\frac{z^3}{\rho^5} + (\alpha_0(1+\nu) - \beta_0)\frac{z}{\rho^3}\\
&+ \big(\alpha_0(1-2\nu) - \beta_0\big)\frac{z}{\rho^3} + \big(\alpha_0(1-2\nu) - 2\beta_0\big)\frac{z^3}{2r^2\rho^3} - \frac{g(z)}{r^2}
\end{aligned}$$

(cf. the last two equations in Sect. 3.4 of [1]). It remains for us to complete the determination of constants α_0, β_0, and to find the form of function g. We do it in a manner completely similar to what we did for the same purpose in Sect. 5.6.

Firstly, by using the inverse constitutive law $(2.45)_2$ and (6.20), we find that:

$$E_{\varphi\varphi} = -\frac{1}{2G}\left(\big(\alpha_0(1-\nu) - \beta_0\big)\frac{z}{\rho^3} + \big(\alpha_0(1-2\nu) - 2\beta_0\big)\frac{z^3}{2r^2\rho^3} - \frac{g(z)}{r^2}\right).$$

Secondly, with this and (5.66)$_3$, we obtain the following provisional expression for the radial displacement of points on any chosen horizontal plane:

$$u_r = r E_{\varphi\varphi} = -\frac{1}{2G}\left(\big(\alpha_0(1-\nu)-\beta_0\big)\frac{zr}{\rho^3} + \big(\alpha_0(1-2\nu)-2\beta_0\big)\frac{z^3}{2r\rho^3} - \frac{g(z)}{r}\right). \tag{6.22}$$

Thirdly, we impose again the kinematic symmetry condition (5.60):

$$\lim_{r\to 0^+} u_r(z,r) = 0,$$

and deduce from it that:

$$\alpha_0(1-2\nu)-2\beta_0 = 0, \quad g(z)=0; \tag{6.23}$$

relations (6.19) and (6.23)$_1$ imply that:

$$\alpha_0 = -\frac{f}{4\pi(1-\nu)}, \qquad \beta_0 = -\frac{f(1-2\nu)}{8\pi(1-\nu)}. \tag{6.24}$$

In conclusion, the Kelvin stress components turn out to have the following expressions:

$$\begin{aligned}
\widehat{\sigma}_1^K(z,r) &= -\frac{f}{8\pi(1-\nu)}\left(3\frac{z^3}{\rho^5} - (1-2\nu)\frac{z}{\rho^3}\right),\\
\widehat{\sigma}_2^K(z,r) &= -\frac{f}{8\pi(1-\nu)}\left(3\frac{zr^2}{\rho^5} + (1-2\nu)\frac{z}{\rho^3}\right),\\
\widehat{\sigma}_3^K(z,r) &= \frac{f(1-2\nu)}{8\pi(1-\nu)}\frac{z}{\rho^3},\\
\widehat{\sigma}_4^K(z,r) &= -\frac{f}{8\pi(1-\nu)}\left(3\frac{z^2 r}{\rho^5} - (1-2\nu)\frac{z}{\rho^3}\right)
\end{aligned} \tag{6.25}$$

(cf. Equations (40) in [1]).

6.3.2 The Strain and Displacement Fields

To deduce the strain field in Kelvin's problem, we combine the inverse constitutive Eq. (2.45) with (6.25), and find:

$$\begin{aligned}
\widehat{E}^K_{zz}(z,r) &= -\frac{f}{16\pi G(1-\nu)\rho^5}\big(4(1+\nu)z^3+(1-4\nu)zr^2\big),\\
\widehat{E}^K_{rr}(z,r) &= \frac{f}{16\pi G(1-\nu)\rho^5}\big(z^3-2zr^2\big),\\
\widehat{E}^K_{\varphi\varphi}(z,r) &= \frac{f}{16\pi G(1-\nu)}\frac{z}{\rho^3},\\
\widehat{E}^K_{zr}(z,r) &= -\frac{f}{16\pi G(1-\nu)\rho^5}\big(2(2-\nu)z^2r+(1-2\nu)r^3\big)
\end{aligned} \tag{6.26}$$

(cf. equations (41) in [1]). As to the displacement field, it is the matter of a straightforward calculation to substitute (6.24) into (6.22), to obtain, in view also of $(6.23)_2$, that

$$\hat{u}^K_r(z,r) = \frac{f}{16\pi G(1-\nu)}\frac{zr}{\rho^3}. \tag{6.27}$$

Moreover, $(5.66)_1$ and $(6.26)_1$ imply that

$$u_z = \frac{f}{16\pi G(1-\nu)}\left(\frac{2(1-2\nu)}{\rho}+\frac{1}{\rho}+\frac{z^2}{\rho}\right)+h(r).$$

To determine function h, we turn to $(5.66)_4$, rewrite it in the form:

$$u_{r,z} = 2E_{zr} - u_{z,r}\,,$$

and observe that, for this relation to be consistent with both $(6.26)_4$ and (6.27), function h must have constant value. We take it null. In fact, vector $\boldsymbol{h}_0 = h_0\boldsymbol{e}_1$ would represent an arbitrary translation of the whole space in the vertical direction, the only rigid displacement compatible with the symmetries of the problem and an inevitable indeterminacy, in the absence of Dirichlet boundary conditions, that we lightheartedly dispose of. In conclusion,

$$\hat{u}^K_z(z,r) = \frac{f}{16\pi G(1-\nu)}\left(\frac{2(1-2\nu)}{\rho}+\frac{1}{\rho}+\frac{z^2}{\rho}\right). \tag{6.28}$$

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