Chapter 2
Elements of Linear Elasticity

In this chapter we give a short and yet fairly complete exposition of the elemental features of classic elasticity having relevance to our subject matters. This archetypal theory, probably the most successful and best well-known theory of continuum mechanics, has been given many excellent and exhaustive expositions. Among the textbooks including an ample coverage of the problems we deal with in this book we cite those by Love [8], Sokolnikoff [17], Malvern [9], Gladwell [5]; we also take from the Handbuch article by Gurtin [6], whose use of direct notation we find appropriate to avoid encumbering conceptual developments with component-wise expressions, and from [11]. Interestingly, no matter how early in the history of elasticity the consequences of concentrated loads were studied, some of those, namely, the occurrence of concentrated contact interactions between adjacent body parte, went overlooked until recently [12–16].

2.1 Displacement, Strain, Compatibility

The problems in linear elasticity we are interested in are formulated over an unbounded region R of an Euclidean space $\mathcal{E}^N$ of dimension $N = 2$ or 3, R being either a half-space or the whole of $\mathcal{E}^N$; as a rule, in the following we take $N = 3$. Points x of R have a position vector

$$\boldsymbol{x} := x - o$$

with respect to a chosen point of $\mathcal{E}^N$, the *origin* o; the components of $\boldsymbol{x}$ in an orthonormal Cartesian basis $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ are the Cartesian coordinates x_i:

$$\boldsymbol{x} = x_i \boldsymbol{e}_i\,.$$

P. Podio-Guidugli and A. Favata, *Elasticity for Geotechnicians*, Solid Mechanics and Its Applications 204, DOI: 10.1007/978-3-319-01258-2_2,

In this formula, we used *Einstein's convention*, consisting in leaving tacit the summation operation over the index range whenever in a monomial term an index is repeated twice: here, for example, this convention allows us to avoid the use of the more cumbersome notation

$$\boldsymbol{x} = \sum_{i=1}^{3} x_i \boldsymbol{e}_i .$$

Here and henceforth in this book we drop the qualifier 'orthogonal' for the only type of Cartesian coordinates we use.

In a deformation, a typical point $x \in R$ is displaced to a position

$$y = x + \boldsymbol{u}(x);$$

here, $\boldsymbol{u}$ is the vector field that describes the *displacement* from x to $y \in \mathcal{E}^N$. The *displacement gradient* is the tensor field whose value at x is by definition the outcome of taking the following limit:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}(\boldsymbol{u}(x + \varepsilon \boldsymbol{h}) - \boldsymbol{u}(x)) =: (\nabla \boldsymbol{u}(x))\boldsymbol{h}, \quad \forall\, \boldsymbol{h} \in \mathcal{V}, \tag{2.1}$$

where $\mathcal{V}$ is the N-dimensional vector space associated with $\mathcal{E}^N$. If $\boldsymbol{h}$ is a unit vector (that is, if $|\boldsymbol{h}| = 1$), the left side of the last relation defines the *directional derivative* of $\boldsymbol{u}$ in the direction $\boldsymbol{h}$:

$$\partial_{\boldsymbol{h}} \boldsymbol{u} := (\nabla \boldsymbol{u})\boldsymbol{h}.$$

On representing vector $\boldsymbol{u}$ in the chosen basis:

$$\boldsymbol{u} = u_i \boldsymbol{e}_i ,$$

an application of definition (2.1) yields the cartesian components of $\nabla \boldsymbol{u}$:

$$(\nabla \boldsymbol{u})_{ij} = u_{i,j},$$

where $,$'$_j$' denotes differentiation with respect to coordinate x_j:

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}.$$

Just as every other second-order tensor, $\nabla \boldsymbol{u}$ can be uniquely decomposed into the sum of its *symmetric part* $\boldsymbol{E}$ and its *skew-symmetric part* $\boldsymbol{W}$:

$$\begin{aligned}\nabla \boldsymbol{u} &= \boldsymbol{E}(\boldsymbol{u}) + \boldsymbol{W}(\boldsymbol{u}),\\ \boldsymbol{E}(\boldsymbol{u}) &:= \tfrac{1}{2}\big(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\big),\\ \boldsymbol{W}(\boldsymbol{u}) &:= \tfrac{1}{2}\big(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^T\big).\end{aligned} \tag{2.2}$$

The *strain tensor* $\boldsymbol{E}$ is a linear measure of the strain field associated with a given displacement field: its equal-index components ($E_{11} = u_{1,1}$ etc.) measure dilatation of fibers aligned with the Cartesian axes; the other components ($E_{12} = 1/2(u_{1,2} + u_{2,1})$ etc.) measure changes in the angle between fibers aligned along different axes; more generally, if $\boldsymbol{a}$ are $\boldsymbol{b}$ two mutually orthogonal unit vectors, $\boldsymbol{E}\boldsymbol{a}\cdot\boldsymbol{a}$ measures the dilatation of a fiber aligned with $\boldsymbol{a}$, and $\boldsymbol{E}\boldsymbol{a}\cdot\boldsymbol{b}(=\boldsymbol{E}\boldsymbol{b}\cdot\boldsymbol{a})$ measures the change in angle between fibers in the directions $\boldsymbol{a}$ and $\boldsymbol{b}$.[1]

The *rotation tensor* $\boldsymbol{W}$ furnishes a linear measure of the vorticity field associated with a given displacement field. The role of $\boldsymbol{W}$ is made clearer if the operation of taking the *curl* of $\boldsymbol{u}$ is introduced: this operation defines a vector field, denoted by $\operatorname{curl}\boldsymbol{u}$, such that

$$\boldsymbol{W}(\boldsymbol{u})\boldsymbol{a} =: \frac{1}{2}\operatorname{curl}\boldsymbol{u}\times\boldsymbol{a}, \quad \forall\,\boldsymbol{a}\in\mathcal{V}. \tag{2.3}$$

It follows from this definition that the Cartesian components of $\operatorname{curl}\boldsymbol{u}$ are:

$$(\operatorname{curl}\boldsymbol{u})_i = e_{ijk}u_{k,j}, \tag{2.4}$$

where e_{ijk} is *Ricci's symbol*.[2] We set:

[1] For more information about the role of $\boldsymbol{E}$ and, more generally, about the local analysis, both exact and approximate, of a deformation see [11], Chap. I

[2] In terms of the vectors composing the orthonormal Cartesian basis we chose, *Kronecker's symbol* δ_{ij} is given by

$$\delta_{ij} := \boldsymbol{e}_i\cdot\boldsymbol{e}_j,$$

whence

$$\delta_{ij} = \begin{cases}1 \text{ if } & i=j\\ 0 \text{ if } & i\neq j\end{cases};$$

moreover, relation

$$e_{ijk} := \boldsymbol{e}_i\times\boldsymbol{e}_j\cdot\boldsymbol{e}_k$$

defines *Ricci's symbol*, so that

$$e_{ijk} = \begin{cases}+1 & \text{if all indices i, j, k are different and, in addition,}\\ & \text{their sequence is an even-class permutation of 1, 2, 3;}\\ 0 & \text{if at least two of the indices i, j, k are equal;}\\ -1 & \text{if all indices i, j, k are different and, in addition,}\\ & \text{their sequence is an odd-class permutation of 1, 2, 3.}\end{cases}$$

Ricci's and Kronecker's symbols are linked by the following relation:

$$e_{ijk}e_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \tag{2.5}$$

By repeated saturation of pairs of free indices, two easy and often useful consequences of (2.5) are obtained:

$$\boldsymbol{w}(\boldsymbol{u}) := \frac{1}{2}\,\mathrm{curl}\,\boldsymbol{u};$$

this definition identifies $\boldsymbol{w}$ as the vector associated with $\boldsymbol{W}$ by the well-known one-to-one correspondence between $\mathcal{V}$ and Skw, the collection of all skew-symmetric second-order tensors, namely,

$$\mathcal{V} \ni \boldsymbol{v} \leftrightarrow \boldsymbol{V} \in \mathrm{Skw} \quad \Leftrightarrow \quad \boldsymbol{V}\boldsymbol{a} = \boldsymbol{v} \times \boldsymbol{a}, \quad \forall \boldsymbol{a} \in \mathcal{V}. \tag{2.6}$$

It is not difficult to show that

$$V_{ik} = \boldsymbol{e}_{ijk} v_j, \quad v_i = \frac{1}{2} e_{ijk} V_{kj}.$$

In view of definition (2.1), we write:

$$\begin{aligned}\boldsymbol{u}(x) &= \boldsymbol{u}(x_0) + \boldsymbol{E}(x_0)(x - x_0) + \boldsymbol{W}(x_0)(x - x_0) + \boldsymbol{O}^2(|x - x_0|)\\ &= \boldsymbol{u}(x_0) + \boldsymbol{w}(x_0) \times (x - x_0) + \boldsymbol{E}(x_0)(x - x_0) + \boldsymbol{O}^2(|x - x_0|),\end{aligned}$$

where $\boldsymbol{E}(x_0) = \boldsymbol{E}(\boldsymbol{u}(x_0))$ etc. The last equality makes clear what is meant by *local linear approximation* of a given displacement field $\boldsymbol{u}$, that is, by the approximation of $\boldsymbol{u}$ to within terms of order $\boldsymbol{O}^2(|x - x_0|)$ in a neighbourhood of an arbitrarily chosen interior point x_0 of R): it consists of the sum of a *rigid displacement*

$$\boldsymbol{u}_{rig}(x) := \boldsymbol{u}(x_0) + \boldsymbol{w}(x_0) \times (x - x_0),$$

made up of a *translation* $\boldsymbol{u}(x_0)$ and of a *rotation* about x_0 of vector $\boldsymbol{w}(x_0)$, and of a nonrigid displacement

$$\boldsymbol{u}_{def}(x) = \boldsymbol{E}(x_0)(x - x_0),$$

the only part of $\boldsymbol{u}$ inducing what in everyday language is called a 'small deformation'. In fact, $\boldsymbol{E}$ is often called the *infinitesimal strain tensor*, the modifier 'strain' being an alternative to 'deformation' and the modifier 'infinitesimal' being used to distinguish $\boldsymbol{E} = \mathrm{sym}(\nabla \boldsymbol{u})$ from other local measures of deformation that, being exact, depend nonlinearly on $\nabla \boldsymbol{u}$.

We introduce here some more notions to be used in what follows.

Lin is the space of all second-order tensors, regarded as linear transformations of $\mathcal{V}$ into itself; Sym and Skw are two complementary subspaces of Lin, respectively, the

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(i) formal multiplication of both sides by δ_{jm} yields:

$$e_{ijk} e_{ljk} = 2\,\delta_{il}\,;$$

(ii) one more saturation gives:

$$e_{ijk} e_{ijk} = 6\,.$$

subspace of symmetric ($\boldsymbol{A} = \boldsymbol{A}^T$) and skew-symmetric ($\boldsymbol{A} = -\boldsymbol{A}^T$) tensors. When $\dim(\mathcal{V}) = 3$, $\dim(\text{Lin}) = 9$, $\dim(\text{Sym}) = 6$ and $\dim(\text{Skw}) = 3$; when $\dim(\mathcal{V}) = 2$, $\dim(\text{Lin}_{(2)}) = 4$, $\dim(\text{Sym}_{(2)}) = 3$ and $\dim(\text{Skw}_{(2)}) = 1$.

Remark 2.1 With the use of (2.6), it can be shown that the vector associated with the skew-symmetric tensor $(\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a})$ is $\boldsymbol{b} \times \boldsymbol{a}$.[3] Every skew-symmetric tensor can be represented as the linear combination of the following tensors:

$$\begin{aligned} \boldsymbol{W}_1 &= -\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_2, \\ \boldsymbol{W}_2 &= -\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_3, \\ \boldsymbol{W}_3 &= -\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1, \end{aligned} \tag{2.7}$$

where

$$\boldsymbol{W}_i \leftrightarrow \boldsymbol{e}_i .$$

Remark 2.2 The *divergence* of a vector field $\boldsymbol{u}$ is the scalar field

$$\operatorname{div} \boldsymbol{u} := \operatorname{tr}(\nabla \boldsymbol{u});$$

it follows from this definition that

$$\operatorname{div} \boldsymbol{u} = \operatorname{tr} \boldsymbol{E}(\boldsymbol{u}) = E_{ii} = u_{i,i} .$$

Note that

$$\operatorname{div} \operatorname{curl} \boldsymbol{u} = 0,$$

and that, for φ a scalar field,

$$\operatorname{curl} \nabla\varphi = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \nabla\varphi = \Delta\varphi. \tag{2.8}$$

These two identities help to interpret a classical result in vector calculus, *Helmholtz's Decomposition Theorem*:

> given any sufficiently smooth field $\boldsymbol{u}$ over a bounded regular region R, there are a scalar field φ and a divergenceless vector field $\boldsymbol{w}$ over R such that
>
> $$\boldsymbol{u} = \nabla\varphi + \operatorname{curl} \boldsymbol{w};$$
>
> if $\boldsymbol{u} \in C(\bar{R}) \cap C^M(R)$, $M \geq 1$, then both φ and $\boldsymbol{w}$ are of class $C^M(R)$.

Note that a straightforward application of (2.8) yields:

$$\operatorname{curl} \boldsymbol{u} = \operatorname{curl} \operatorname{curl} \boldsymbol{w} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \Delta\varphi.$$

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[3] Recall that the symbol $\otimes$ signifies dyadic product, a notion introduced in the first footnote of Sect. 1.3; the second-order tensor $\boldsymbol{a} \otimes \boldsymbol{b}$ is defined by specifying its linear action on vectors.

2.1.1 Compatibility

With each displacement field $\boldsymbol{u}$ of class $C^1(R)$ we can always associate a continuous deformation field $\boldsymbol{E}$ such that

$$2\,\boldsymbol{E} = \nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T; \tag{2.9}$$

in components,

$$2E_{ij} = u_{i,j} + u_{j,i}. \tag{2.10}$$

This relation can also be regarded as the tensorial equation ruling the problem of finding a displacement field $\boldsymbol{u}$ associated with a given strain field $\boldsymbol{E}$. This problem is *overdetermined*, because the three unknown fields u_i are restricted by the six scalar equations (2.10). Not that problems of this type have necessarily no solution. However, for them the *well-posedness* issue (**a**. Are there solutions? **b**. If answer to **a** is yes, how many are they? **c**. Do solutions depend continuously on data?) can be discussed only after having checked that the assigned data satisfy certain a priori solvability conditions called *compatibility conditions*. We now deduce such conditions for the case of our current interest.

To begin with, we have to put together a curl notion for tensor-valued fields. We do so by exploiting the definition given in (2.3) for vector-valued fields:

$$(\operatorname{curl}\boldsymbol{A})\boldsymbol{a} := \operatorname{curl}(\boldsymbol{A}^T\boldsymbol{a}), \quad \forall\,\boldsymbol{a}\in\mathcal{V};$$

in components,

$$(\operatorname{curl}\boldsymbol{A})_{ij} = e_{ipq}A_{jq,p}.$$

If we now apply formally the operator curl on both sides of (2.9), we find[4]:

$$2\operatorname{curl}\boldsymbol{E} = \operatorname{curl}(\nabla\boldsymbol{u}) + \operatorname{curl}(\nabla\boldsymbol{u}^T) = \operatorname{curl}(\nabla\boldsymbol{u}^T) = 2\nabla\boldsymbol{w}. \tag{2.11}$$

Taking the curl of (2.11), we arrive at the sought-for compatibility condition:

$$\operatorname{curl}\operatorname{curl}\boldsymbol{E} = \boldsymbol{0}; \tag{2.12}$$

in components,

[4] That $\operatorname{curl}(\nabla\boldsymbol{u}) = \boldsymbol{0}$ follows from the definitions of (the two involved operators and) Ricci symbol:

$$(\operatorname{curl}(\nabla\boldsymbol{u}))_{ij} = e_{ipq}(\nabla\boldsymbol{u})_{jq,p} = e_{ipq}(u_{j,q})_{,p} = e_{ipq}u_{j,qp} = 0.$$

Furthermore, in view of (2.4),

$$(\operatorname{curl}(\nabla\boldsymbol{u}^T))_{ij} = e_{ipq}(u_{q,j})_{,p} = e_{ipq}u_{q,jp} = (e_{ipq}u_{q,p})_{,j} = 2\,w_{i,j}.$$

$$e_{ijk}e_{lmn}E_{jm,kn} = 0. \tag{2.13}$$

If region R is simply connected, for each given symmetric-valued field $\boldsymbol{E}$ of class $C^K(R)$, $K \geq 2$ there is a class $C^{K+1}(R)$ displacement field $\boldsymbol{u}$, which satisfies (2.9).[5] The field $\boldsymbol{u}$ can be constructed by means of *Cesàro's formula*:

$$u_i(x) = \int_{x_0}^{x} U_{ij}(y,x)dy_j, \quad U_{ij}(y,x) := E_{ij}(y) + (x_k - y_k)(E_{ij,k}(y) - E_{kj,i}(y)), \tag{2.14}$$

where the integral does not depend on the path that has been chosen in R to connect a given point x_0 with the typical point x. Needless to say, this formula determines $\boldsymbol{u}$ to within an arbitrary rigid displacement.

Remark 2.3 The representation (1.8) for the displacement field in an elastic beam subject solely to axial loads can be regarded as a minimal version of this general formula: for $\boldsymbol{e}$ a unit vector parallel to the axis, the strain field is

$$\boldsymbol{E}(z) = w'(z)\boldsymbol{e} \otimes \boldsymbol{e},$$

whence, by (2.14), $\boldsymbol{U}(z,\zeta) \equiv \boldsymbol{E}(\zeta)$ and

$$\boldsymbol{u}(z) = \int_{z_0}^{z} (\boldsymbol{U}(\zeta)\boldsymbol{e})\, d\zeta = \left(\int_{z_0}^{z} w'(\zeta)d\zeta\right)\boldsymbol{e}.$$

2.1.2 Plane Displacement Fields

A displacement field $\boldsymbol{u}$ is called *plane* whenever there is a Cartesian reference with respect to which $\boldsymbol{u}$ admits the representation:

$$u_\alpha = u_\alpha(x_1, x_2),\ \alpha = 1, 2, \quad u_3 \equiv 0, \tag{2.15}$$

at any point $x \in R$.[6] The corresponding strain state is:

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \quad E_{3i} \equiv 0$$

(compare with (2.10)).

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[5] For a proof of this result, which is due to the great Italian elasticist Eugenio Beltrami (1835–1900), who established it in 1889, see [6], Sect. 14, where various other results included in this section are also proved.

[6] When Greek indices are used, it is understood that they take the values 1 and 2; the range of Latin indices is the set $\{1, 2, 3\}$.

Remark 2.4 For an example of plane displacement field, consider the rigid displacement:

$$\boldsymbol{v} = \boldsymbol{t} + \alpha \boldsymbol{W}_3 \boldsymbol{x} = \boldsymbol{t} + \alpha\, \boldsymbol{e}_3 \times \boldsymbol{x}, \qquad \boldsymbol{t} = t_\alpha \boldsymbol{e}_\alpha, \quad \boldsymbol{x} = x_\alpha \boldsymbol{e}_\alpha, \tag{2.16}$$

consisting of a rotation of α radians about an axis of unit vector $\boldsymbol{e}_3$ and of a translation $\boldsymbol{t}$ in the plane perpendicular to $\boldsymbol{e}_3$. It is easy to see that each field $\boldsymbol{v}$ of type (2.16) solves the following differential system:

$$v_{1,1} = 0, \quad v_{2,2} = 0, \quad v_{1,2} + v_{2,1} = 0; \tag{2.17}$$

as a matter of fact, in components relations (2.16) read:

$$v_1 = t_1 - \alpha\, x_2, \quad v_2 = t_2 + \alpha\, x_1. \tag{2.18}$$

If a rigid plane field whatsoever is added to any plane deformation field, the relative strain state stays the same.

2.1.3 Plane Strain Fields

A strain field $\boldsymbol{E}$ is called *plane* whenever its component representation in a suitable Cartesian reference is:

$$E_{\alpha\beta} = E_{\alpha\beta}(x_1, x_2), \quad E_{3i} \equiv 0. \tag{2.19}$$

For such a field, the tensorial compatibility condition (2.12) shrinks to one scalar relation:

$$2\, E_{12,12} = E_{11,22} + E_{22,11}; \tag{2.20}$$

interestingly, of the six conditions (2.13) this is the one obtained when both free indices are taken equal to 3.

Remark 2.5 For plane strain fields, Cesàro's formula gives:

$$u_\alpha(x) = \int_{x_0}^{x} U_{\alpha\beta}(y, x) dy_\beta,$$
$$U_{\alpha\beta}(y, x) = E_{\alpha\beta}(y) + (x_\gamma - y_\gamma)(E_{\alpha\beta,\gamma}(y) - E_{\gamma\beta,\alpha}(y)).$$

The strain field associated with a plane displacement field is plane. We proceed to give a direct proof of the converse statement. To begin with, a displacement field $\boldsymbol{u}$ satisfying the last three relations (2.19) must be such that

$$u_{3,\alpha} + u_{\alpha,3} = 0 \tag{2.21}$$

and that

$$u_{3,3} = 0,$$

that is, such that u_3 be independent of x_3:

$$u_3 = u_3(x_1, x_2). \tag{2.22}$$

Relations (2.21) and (2.22) imply that

$$u_{\alpha,33} = 0,$$

or rather, equivalently, that

$$u_\alpha = \widehat{u}_\alpha(x_1, x_2) + x_3\, \widehat{v}_\alpha(x_1, x_2). \tag{2.23}$$

On combining this preliminary representation for u_α with what the first three relations (2.19) require (namely, that each of the components $E_{\alpha\beta}$ of $\boldsymbol{E}$ be indipendent of x_3), we infer that the vector field $\boldsymbol{v}$ must obey the differential relations (2.17), and hence that it must have the form (2.18); we then set:

$$v_1 = a_1 - b\,x_2, \quad v_2 = a_2 + b\,x_1. \tag{2.24}$$

At this point, we insert representations (2.22), (2.23) and (2.24) into relations (2.21), so as to obtain:

$$u_{3,1} + a_1 - b\,x_2 = 0, \quad u_{3,2} + a_2 + b\,x_1 = 0, \tag{2.25}$$

whence by differentiation we deduce that

$$u_{3,12} - b = 0, \quad u_{3,21} + b = 0,$$

that is,

$$b = 0, \quad u_{3,12} = 0.$$

With the use of the first result, we achieve a preliminary representation, more precise than (2.23), for functions u_α:

$$u_\alpha = \widehat{u}_\alpha(x_1, x_2) + a_\alpha x_3;$$

The definitive form we choose for such representation is:

$$\begin{aligned} u_1 &= \breve{u}_1(x_1, x_2) + t_1 - a_3\,x_2 + a_1 x_3, \\ u_2 &= \breve{u}_2(x_1, x_2) + t_2 + a_3\,x_1 + a_2 x_3, \end{aligned} \tag{2.26}$$

where $\breve{\boldsymbol{u}}$, the part of $\widehat{\boldsymbol{u}}$ responsible for shape and/or volume changes, is distinguished from the rigid part, the latter being of type (2.18). It is easy to check that the plane field $\boldsymbol{E}(\breve{\boldsymbol{u}})$ satisfies (2.20).

Now, given that $b = 0$, relations (2.25) have the following consequences:

$$(u_{3,1} + a_1 = 0 \;\Rightarrow)\;\; -a_1 x_1 + c_1(x_2) = u_3 = -a_2 x_2 + c_2(x_1) \;\; (\Leftarrow \; u_{3,2} + a_2 = 0).$$

This double expression for u_3 holds true for arbitrary values of the independent variables x_1, x_2 provided

$$a_1 x_1 + c_2(x_1) = a_2 x_2 + c_1(x_2) = t_3,$$

with t_3 an arbitrary constant; hence,

$$u_3(x_1, x_2) = t_3 - (a_1 x_1 + a_2 x_2).$$

This expression is found compatible with (2.21) and (2.26) if $a_1 = a_2 = 0$.

In conclusion, given a plane strain field as in (2.19), the corresponding displacement field consists of a plane field $\breve{\boldsymbol{u}}$ such that

$$\breve{u}_{\alpha,\beta} + \breve{u}_{\beta,\alpha} = 2E_{\alpha\beta}$$

and of a rigid displacement field featuring an arbitrary translation and an arbitrary small rotation about the third axis:

$$\boldsymbol{r} = \boldsymbol{t} + \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{A} = -\boldsymbol{A}^T, \quad \boldsymbol{x} = x_\alpha \boldsymbol{e}_\alpha, \tag{2.27}$$

where, on recalling $(2.7)_3$, $\boldsymbol{A} = -a_3(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) = a_3 \boldsymbol{W}$.

2.2 Forces, Stress, Equilibrium

In continuum mechanics, a body is generally thought of as subject to *distance* and *contact actions* on the part of its environment. No matter in what placement in physical space a body is observed, both types of actions are customarily modeled as *diffuse*: those at a distance as forces per unit volume, just as is done in the familiar case of gravity; contact actions as forces per unit surface, on the basis of examples like the pressure exerted by a fluid on a body immersed into it (the wind on a sail) or containing it (the water on a glass).

In most cases, distance actions between disjoint parts of the same body are neglected, as are the distance actions of a part on itself (e.g., self-gravitation). Distance actions at a typical interior body point x are specified by the value taken at that point by an assigned vector field $\widehat{\boldsymbol{d}}$; they are customarily split into *inertial* and *noninertial* parts:

$$\widehat{\boldsymbol{d}}(x) = \widehat{\boldsymbol{d}}^{in}(x) + \widehat{\boldsymbol{d}}^{ni}(x), \quad \widehat{\boldsymbol{d}}^{in}(x) := -\rho(x)\ddot{\boldsymbol{x}},$$

where $\rho(x)$ is the current *mass density*, and $\ddot{\boldsymbol{x}}$ the *acceleration*, at x. In this book, we shall never consider bodies in motion, and hence there will be no need to worry about inertial forces.

In all cases, in addition to contact interactions of a body with its environment, adjacent body parts are presumed to have diffuse *contact interactions*, which are thought of as accounting for the short-range forces between neighboring particles envisaged by discrete mechanics. Mathematically, such contact interactions are described by a vector field $\widehat{\boldsymbol{c}}(\cdot,\cdot)$ defined over the Cartesian product of the body's closure times the sphere of unit vectors: when evaluated at a point x of a common boundary surface oriented by the unit normal $\widehat{\boldsymbol{n}}(x)$, such so-called *stress-vector* field is interpreted as delivering the force $\widehat{\boldsymbol{c}}(x, \widehat{\boldsymbol{n}}(x))$ per unit area exerted either by the environment over the body or by the part lying on the positive side of the boundary surface over the adjacent part.[7]

Concentrated external actions, under form of forces applied at interior or boundary points, have also been considered; their mechanical effects are of central interest in this book. As we shall see, when applied at a boundary point—as is the case with the Flamant Problem we study in Chap. 4—they were regarded as limits of distributions of contact actions localized in a surface neighborhood of that point, which was made to shrink to null; similarly, when applied to an interior point, as in the case of Kelvin Problem to be studied in Chap. 5, they were regarded as limits of distributions of distance actions localized in a volume neighborhood of that point. Surprisingly enough, the occurrence of *concentrated contact interactions between adjacent body parts* went noticed until recently, when Flamant's and other problems of the same type were re-examined [12] (see also [13]).[8]

2.2.1 Cauchy's Notion of Stress

A body acted upon by a force system $(\boldsymbol{d}, \boldsymbol{c})$ is said to occupy an *equilibrium placement* $\mathcal{B}$ when it so happens that

[7] It appears that the concept of diffused contact interactions between internal adjacent body parts begun to condensate in Cauchy's mind on the basis of a similarity with standard examples of diffused contact loads exerted on a body by an environment of a different nature, such as the hydrostatic pressure of a fluid on an immersed solid [3]. Cauchy's model of internal contact interactions has been applied without changes to contact interactions of a body with its exterior, with the stress-vector mapping accounting for both. An implicit drawback of this practice is that no difference is made between geometrical surfaces obtained by ideal cuttings and fabricated surfaces obtained by actual cuttings [4]; moreover, the issue of boundary compatibility of a (body,environment) pair is completely overlooked [1, 2].

[8] The construction of an interaction theory general enough to allow for concentrated contact interactions between adjacent body parts has been undertaken by Schuricht [15, 16]; among the intriguing features of such a theory is the rethinking it involves of the body-part notion. In [14], examples are given of interactions in cuspidate bodies that concentrate at the cusp point, regarded as a body part.

$$\int_{\mathcal{P}} \boldsymbol{d}\cdot\boldsymbol{r} + \int_{\partial\mathcal{P}} \boldsymbol{c}\cdot\boldsymbol{r} = \boldsymbol{0}, \tag{2.28}$$

for all parts $\mathcal{P}$ of $\mathcal{B}$ and for all rigid fields $\boldsymbol{r}$ as in (2.27) (here, as anticipated, $\boldsymbol{d}$ stands for the noninertial distance force). By virtue of *Cauchy's Stress Theorem* (see, e.g., [7], Sect. 14), it follows from (2.28), when written for an arbitrary translation $\boldsymbol{t}$, that the stress-vector mapping can be represented as follows:

$$\widehat{\boldsymbol{c}}(x,\boldsymbol{n}) = \widehat{\boldsymbol{S}}(x)\boldsymbol{n}. \tag{2.29}$$

in terms of a *stress-tensor* field $\widehat{\boldsymbol{S}}$ defined over the closure of $\mathcal{B}$: the affine action of $\widehat{\boldsymbol{S}}(x)$ over the sphere of unit vectors yields the stress vector on the triple infinity of oriented planes through x. Conversely, given the stress-vector mapping $\widehat{\boldsymbol{c}}(x,\cdot)$ at a typical body point x and three mutually orthogonal unit vectors $\boldsymbol{n}_i$, the construct

$$\widehat{\boldsymbol{S}}(x) = \sum_{i=1}^{3} \widehat{\boldsymbol{c}}(x,\boldsymbol{n}_i)\otimes\boldsymbol{n}_i \tag{2.30}$$

defines the value at x of the stress-tensor field. Thus—and this is the main thrust of Cauchy's result—*the information carried by the stress-vector and stress-tensor mappings* $\widehat{\boldsymbol{c}}$ *and* $\widehat{\boldsymbol{S}}$ textitare essentially equivalent.

It follows from (2.28) and (2.29) that

$$\int_{\mathcal{P}} \boldsymbol{d} + \int_{\partial\mathcal{P}} \boldsymbol{S}\boldsymbol{n} = \boldsymbol{0}, \quad \forall\,\mathcal{P}\subset\mathcal{B},$$

whence, granted regularity,

$$\operatorname{div}\boldsymbol{S} + \boldsymbol{d} = \boldsymbol{0} \quad \text{in } \mathcal{B}. \tag{2.31}$$

Moreover, it follows from (2.31) and (2.28), when written for an arbitrary rotation $\boldsymbol{A}$, that the stress field is symmetric-valued:

$$\boldsymbol{S} = \boldsymbol{S}^T.$$

2.2.2 Free-Body Diagrams, Diffuse and Concentrated Forces

A feature of the equilibrium statement (2.28)—namely, that whatever part of an equilibrated body must be in equilibrium as well—would be hardly contended by anybody. The widespread and fruitful use of *free-body diagrams* in mechanics is based on this assumption, and on the accompanying presumption that a body part, when ideally isolated from the rest by a so-called *Euler cut*, would be in equilibrium if it were acted upon by external forces reproducing faithfully the forces, both external and internal, it directly experiences in reality. Usually, the subbodies whose equilibrium is characterized in this manner are imagined to have an everywhere smooth boundary. Not always so in this book, where consideration of sharp-cornered parts is at times necessary to exhibit the concentrations of contact forces that at times may occur (see e.g. Fig. 4.8).

Concentrated forces, regarded as convenient idealizations of diffused loads applied to a small part of a body's boundary, are of common use in engineering mechanics. To quote from a popular textbook, "the free-body diagram is the most important single step in the solution of problems in mechanics" ([10], p. 104); "modeling the action of forces" "exerted *on* the body to be *isolated*, *by* the body to be *removed*" (*ibid.*, p. 105; italics as in the original text) is a mandatory, preliminary step; and those forces, especially but not exclusively in statics, are for most practical purposes modeled as concentrated.

Strictly speaking, the equivalence in information content of (2.30) and (2.29) holds true for *diffused* contact force and *regular* stress fields. In the next chapters, we display and discuss situations when *concentrated* contact forces and *singular* stress fields are in order. Precisely, first by inspection of a problem of pure statics, which is the two-dimensional counterpart of the Flamant problem, then by inspection of the three-dimensional problem Flamant solved, as well as those solved by Boussinesq, Cerruti and Kelvin, we demonstrate *per exempla* that *partwise equilibrium of a simple continuous body may require that adjacent body parts exchange concentrated contact forces*.

We have seen that diffused contact loads are germane to contact interactions between adjacent body parts, so much so that they are customarily described by one and the same vector-valued mapping. Concentrated loads, applied at interior and boundary points, have been often considered in continuum mechanics, and carefully modeled mathematically (for the class of linearly elastic bodies, see [6], Sect. 52). We see no reason why the germane notion of concentrated contact interactions should not be introduced. They are *not* ubiquitous; in fact, they are a rather rare necessity. Let us revert for a moment to engineering mechanics for guidance. A judicious practice there is to make sure that the free-body diagram features *all* possible forces applied to the isolated body; at times, we find out that balance and/or symmetry conditions require that some of those forces be null. Likewise, in continuum mechanics, we should contemplate concentrated contact interactions by default, because there are cases, no matter how few, when they turn out to be crucial to guarantee partwise equilibrium.

If concentrated contact interactions are considered, an interesting problem to tackle is the conjectural equivalence in information of contact forces, regular and singular, and the accompanying, somewhere singular, stress field. Luckily, *concentrated forces occur 'naturally' in weak formulations of force-balance laws*, be they idealizations of applied loads or of contact interactions. In fact, in such formulations, concentrated loads are as 'natural' as edges and vertices in the domain where a boundary value problem is formulated. There is no need today to justify consideration of concentrated forces, as was done over a century ago, by thinking of them as limits of smooth distributions of volume or surface forces, just as there is no need to round off a domain's corners. In addition, weak formulations relieve us from dealing with a delicate issue arising when sequences of approximating problems are employed, namely, to investigate under what hypotheses an associated sequence of smooth solutions has a unique limit.

2.3 The Stress Response to Strain

In classical elasticity, the stress response to strain is described by a linear mapping of the collection of all symmetric tensors into itself:

$$\mathrm{Sym} \ni \boldsymbol{E} \mapsto \boldsymbol{S} \in \mathrm{Sym}, \quad \boldsymbol{S} = \mathbb{C}\boldsymbol{E} \quad (S_{ij} = \mathbb{C}_{ijhk} E_{hk}), \tag{2.32}$$

where the *elasticity tensor* $\mathbb{C}$ has the following index-pair symmetries:

$$\mathbb{C}_{ijhk} = \mathbb{C}_{jihk} = \mathbb{C}_{ijkh}, \quad \mathbb{C}_{ijhk} = \mathbb{C}_{hkij}.$$

Collectively, these symmetries guarantee that

(i) all of the $3^4 = 81$ Cartesian components of $\mathbb{C}$ are expressible in terms of only 21 of them, in general mutually independent;
(ii) there is a quadratic scalar-valued function defined over Sym:

$$\mathrm{Sym} \ni \boldsymbol{E} \mapsto \sigma \in \mathbb{R}, \quad \sigma = \sigma(\boldsymbol{E}) = \frac{1}{2}\boldsymbol{E} \cdot \mathbb{C}\boldsymbol{E} = \frac{1}{2}\mathbb{C}_{ijhk} E_{ij} E_{hk}, \tag{2.33}$$

referred to as the *strain energy* per unit referential volume, such that

$$\partial_{\boldsymbol{E}} \sigma(\boldsymbol{E}) = \mathbb{C}\boldsymbol{E}.$$

It follows from (2.32) and, respectively, (2.33) that, $\boldsymbol{S} = \boldsymbol{0}$ and $\sigma = 0$ for $\boldsymbol{E} = \boldsymbol{0}$. It is when both the stress and the strain energy are null at a point—that is, when the material is in a *natural state* at that point—that classical elasticity studies the local response of a linearly elastic material to the various causes of deformation. For reasons of physical plausibility, the strain energy is assumed to be *positive definite*, i.e., such that

$$\sigma(\boldsymbol{E}) \geq 0, \quad \sigma(\boldsymbol{E}) = 0 \;\Leftrightarrow\; \boldsymbol{E} = \boldsymbol{0}. \tag{2.34}$$

This assumption is more than sufficient to guarantee that the constitutive mapping (2.32) be invertible:

$$\boldsymbol{E} = \mathbb{C}^{-1}\boldsymbol{S}. \tag{2.35}$$

2.3.1 Isotropic Materials

When a material's response is "the same in whatever direction", that material is said *isotropic*. The elasticity tensor of an isotropic linearly elastic material is completely determined by two parameters only, the so-called *Lamé's moduli* λ and μ; the stress-strain law has the following form:

$$\boldsymbol{S} = 2\mu\boldsymbol{E} + \lambda(\mathrm{tr}\,\boldsymbol{E})\boldsymbol{I}, \quad S_{ij} = 2\mu E_{ij} + \lambda(E_{hh})\delta_{ij} \tag{2.36}$$

(here $\boldsymbol{I}$ denotes the identity tensor), while the strain energy reads:

$$\sigma(\boldsymbol{E}) = \mu\,|\boldsymbol{E}|^2 + \frac{1}{2}\lambda(\mathrm{tr}\,\boldsymbol{E})^2 = \mu\,E_{ij}E_{ij} + \frac{1}{2}\lambda(E_{hh})^2;$$

for (2.34) to hold, it is necessary and sufficient that

$$\mu > 0, \quad 3\lambda + 2\mu > 0. \tag{2.37}$$

It is not difficult to determine the form taken by the inverse constitutive equation (2.35). Firstly, on taking the trace of (2.36), one obtains that

$$\mathrm{tr}\,\boldsymbol{S} = (3\lambda + 2\mu)\mathrm{tr}\,\boldsymbol{E}; \tag{2.38}$$

next, in view also of (2.37), one arrives at:

$$\boldsymbol{E} = \frac{1}{2\mu}\Big(\boldsymbol{S} - \frac{\lambda}{3\lambda + 2\mu}(\mathrm{tr}\,\boldsymbol{S})\boldsymbol{I}\Big). \tag{2.39}$$

Remark 2.6 For isotropic materials, the equilibrium equation (2.31) can be written in terms of displacement as Louis Navier (1785–1836) did first:

$$\mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla(\mathrm{div}\,\boldsymbol{u}) + \boldsymbol{d} = \boldsymbol{0}. \tag{2.40}$$

In this equation, three differential operators appear: laplacian and divergence of a vector field, and gradient of a scalar field. On recalling how these operators look like in Cartesian components[9]:

$$(\Delta\boldsymbol{v})_i = v_{i,jj}, \quad \mathrm{div}\,\boldsymbol{v} = v_{i,i}, \quad \text{and} \quad (\nabla\varphi)_i = \varphi,_i,$$

[9] The laplacian of a vector field $\boldsymbol{v}$ is the vector field that obtains by taking the divergence of the gradient of $\boldsymbol{v}$:

$$\Delta\boldsymbol{v} = \mathrm{div}\,(\nabla\boldsymbol{v});$$

its Cartesian components have the form just shown because $(\nabla\boldsymbol{v})_{ij} = v_{i,j}$ and because, for $\boldsymbol{V}$ a second-order tensor field, $(\mathrm{div}\,\boldsymbol{V})_i = V_{ij,j}$.

the component version of Navier equation is easy to write:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + d_i = 0.$$

Remark 2.7 Let the distance forces be null. Then, on taking the divergence of Navier equation, one finds:

$$(\lambda + 2\mu)\Delta(\operatorname{div} \boldsymbol{u}) = 0,$$

whence, given that

$$\operatorname{div} \boldsymbol{u} = \operatorname{tr} \boldsymbol{E}(\boldsymbol{u})$$

and that it follows from (2.37) that

$$\lambda + 2\mu > 0,$$

one obtains

$$\Delta(\operatorname{tr} \boldsymbol{E}(\boldsymbol{u})) = 0.$$

But, if tr $\boldsymbol{E}(\boldsymbol{u})$ has to be a *harmonic* function (that is, a function whose laplacian is null), then tr $\boldsymbol{S}(\boldsymbol{u})$ must be harmonic as well, because of (2.38).[10] We shall deduce this condition again, in a different manner, in Sect. 2.4, where we study the compatibility issue in terms of stresses.

2.3.2 Mechanical Interpretation of the Elastic Moduli

The role of the elastic moduli is clarified when one imagines to perform some typical experiments, in each of which the one or the other modulus enters in a perspicuous manner. In the first two experiments we are going to consider, we record what stress accompanies a given strain according to the constitutive relation (2.36); in the third one, the stress is assigned, and the corresponding strain is computed with the use of (2.39).

(a) Simple shearing
For $\boldsymbol{a}, \boldsymbol{b}$ two orthogonal vectors,

$$\boldsymbol{E} = \tau(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) \quad \Rightarrow \quad \boldsymbol{S} = \tau\, 2\mu(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a});$$

therefore,

$$2\mu := \frac{\boldsymbol{b} \cdot \boldsymbol{S}\boldsymbol{a}}{\boldsymbol{b} \cdot \boldsymbol{E}\boldsymbol{a}},$$

[10] Here, $\boldsymbol{S}(\boldsymbol{u}) := 2\mu\boldsymbol{E}(\boldsymbol{u}) + \lambda(\operatorname{tr} \boldsymbol{E}(\boldsymbol{u}))\boldsymbol{I}$.

the *shear modulus*, measures the shear stress necessary to sustain a unit shearing strain.

(b) Uniform dilatation

$$\boldsymbol{E} = \tau \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{S} = \tau(3\lambda + 2\mu)\boldsymbol{I};$$

hence, the *dilatation modulus*:

$$3\lambda + 2\mu := \frac{\boldsymbol{S} \cdot \boldsymbol{I}}{\boldsymbol{E} \cdot \boldsymbol{I}} \tag{2.41}$$

is proportional to the *pressure* $1/3(\boldsymbol{S} \cdot \boldsymbol{I})$ accompanying the *volume change* $\boldsymbol{E} \cdot \boldsymbol{I}$.

(c) Uniaxial stress

Again, let $\boldsymbol{a}$ e $\boldsymbol{b}$ be two orthogonal vectors. Then,

$$\boldsymbol{S} = \tau \boldsymbol{a} \otimes \boldsymbol{a} \quad \Rightarrow \quad \boldsymbol{E} = \tau \frac{1}{2\mu} \left(\boldsymbol{a} \otimes \boldsymbol{a} - \frac{\lambda}{3\lambda + 2\mu} \boldsymbol{I} \right).$$

The *Young's modulus*

$$E := \frac{\boldsymbol{a} \cdot \boldsymbol{S}\boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{E}\boldsymbol{a}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

measures the axial stress necessary to cause a unit axial strain. The *Poisson's modulus* (also known as the *lateral-contraction modulus*)

$$\nu := -\frac{\boldsymbol{b} \cdot \boldsymbol{E}\boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{E}\boldsymbol{a}} = \frac{\lambda}{2(\lambda + \mu)}$$

measures the transverse-to-axial strain ratio in an experiment where an axial stress state is induced. The moduli E, ν and

$$G := \mu$$

are those currently used in the (geo)technical applications of linear and isotropic elasticity. We also note for later reference another expression for the dependence of volume changes on pressure:

$$\operatorname{tr} \boldsymbol{E} = \frac{1 - 2\nu}{E} \operatorname{tr} \boldsymbol{S} \tag{2.42}$$

(cf. (2.41)).

Remark 2.8 As Lamé's constitutive equation shows, two moduli characterize completely the response of an isotropic material. In fact, it is not difficult to see that the three technical moduli are linked by the consistency condition

$$E = 2(1+\nu)G. \tag{2.43}$$

Remark 2.9 The positivity inequalities (2.37) imply that

$$E,\ G > 0, \quad -1 < \nu < 1/2\,. \tag{2.44}$$

Therefore, linearly elastic and isotropic materials that contract transversely when axially extended (that is, materials for which $0 < \nu < 1/2$) have an E/G ratio strictly included between 2 and 3; and, for those whose $\nu \in (-1, -1/2)$, to have a Young's modulus smaller (even much smaller) that their shear modulus does not forbid the strain energy to be positive definite.

Remark 2.10 It is easy to express the Lamé moduli in terms of the technical moduli:

$$\lambda = \frac{2\nu}{1-2\nu}\,G = \frac{\nu}{(1-2\nu)(1+\nu)}\,E, \quad \mu = G = \frac{1}{2(1+\nu)}\,E.$$

In particular, it follows from these relations that

$$3\lambda + 2\mu = \frac{1}{1-2\nu}\,E.$$

With the use of the technical moduli, the inverse constitutive equation (2.39) reads:

$$\boldsymbol{E} = \frac{1}{E}\Big((1+\nu)\boldsymbol{S} - \nu(\operatorname{tr}\boldsymbol{S})\boldsymbol{I}\Big) = \frac{1}{2G}\Big(\boldsymbol{S} - \frac{\nu}{1+\nu}(\operatorname{tr}\boldsymbol{S})\boldsymbol{I}\Big). \tag{2.45}$$

Consequently, the equal-index components of $\boldsymbol{E}$ are exemplified by

$$E_{11} = \frac{1}{E}\Big(S_{11} - \nu(S_{22} + S_{33})\Big),$$

and the components with different indices by

$$E_{12} = \frac{1}{2G}S_{12},$$

all the other components being obtained via a cyclic permutation of indices. Continuing to use the technical moduli, the direct constitutive equation (2.36) and the strain energy read, respectively,

$$\boldsymbol{S} = \frac{E}{1+\nu}\Big(\boldsymbol{E} + \frac{\nu}{1-2\nu}(\operatorname{tr}\boldsymbol{E})\boldsymbol{I}\Big) = 2G\Big(\boldsymbol{E} + \frac{\nu}{1-2\nu}(\operatorname{tr}\boldsymbol{E})\boldsymbol{I}\Big) \tag{2.46}$$

and

$$\widetilde{\sigma}(\boldsymbol{E}) = \frac{E}{2(1+\nu)}\Big(|\boldsymbol{E}|^2 + \frac{\nu}{1-2\nu}(\operatorname{tr}\boldsymbol{E})^2\Big). \tag{2.47}$$

Remark 2.11 When the extensional rigidity is constant, the differential Eq. (1.6) for the axial deformations of a beam is:

$$w'' + \frac{q}{EA} = 0.$$

It is instructive to demonstrate the mutual consistency of the 3- and 1-D theories of elasticity by 'deducing' (1.6) from Navier equation. This can be done as follows. As in Remark 2.3, restrict attention to displacement fields of the form:

$$\boldsymbol{u}(x) = w(x_3)\boldsymbol{e}_3. \tag{2.48}$$

Then,

$$\Delta\boldsymbol{u} = w''\boldsymbol{e}_3, \quad \operatorname{div}\boldsymbol{u} = w' \quad \Rightarrow \quad \nabla(\operatorname{div}\boldsymbol{u}) = w''\boldsymbol{e}_3,$$

and hence Eq. (2.40) reduces to

$$(\lambda + 2\mu)\, w''\boldsymbol{e}_3 + \boldsymbol{d} = \boldsymbol{0}.$$

At this point, to conclude the announced deduction, it is enough to choose

$$\boldsymbol{d} = \frac{q}{A}\boldsymbol{e}_3$$

and to set

$$\lambda + 2\mu = E. \tag{2.49}$$

It remains for us to convince ourselves that the last position makes sense. Now, it is easy to see that, whenever the strain state

$$\boldsymbol{E} = \tau\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

corresponding to a displacement field (2.48) is induced in a linearly elastic isotropic material, the stress state is

$$\boldsymbol{S} = \tau\Big((\lambda + 2\mu)\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \lambda(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)\Big).$$

Thus, the relative Young'a modulus:

$$E = \frac{\boldsymbol{e}_3 \cdot \boldsymbol{S}\boldsymbol{e}_3}{\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3}$$

has just the expression (2.49).

2.3.3 Plane Stress Fields

A stress field $\boldsymbol{S}$ is said *plane* if there is a Cartesian frame where its representation fulfills a set of conditions formally identical to the conditions (2.19) defining a plane strain field, namely,

$$S_{\alpha\beta} = S_{\alpha\beta}(x_1, x_2), \quad S_{3i} = 0; \tag{2.50}$$

therefore, it has the form:

$$\boldsymbol{S} = S_{11}(x_1, x_2)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + S_{12}(x_1, x_2)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + S_{22}(x_1, x_2)\boldsymbol{e}_2 \otimes \boldsymbol{e}_2.$$

A plane stress field is *balanced for null distance forces* if its divergence is null:

$$\operatorname{div} \boldsymbol{S} = \boldsymbol{0}, \quad (\operatorname{div} \boldsymbol{S})_\alpha = S_{\alpha\beta},_\beta = 0. \tag{2.51}$$

In Sect. 4.2, we shall construct a general representation for those fields $\boldsymbol{S}$ that solve (2.51).

In a linearly elastic isotropic body, a plane stress field induces a strain field that is not plane in general, as an application of the response law (2.45) shows:

$$E_{11} = \frac{1}{E}\Big(S_{11} - \nu S_{22}\Big), \quad E_{22} = \frac{1}{E}\Big(S_{22} - \nu S_{11}\Big), \quad E_{12} = \frac{1}{2G} S_{12},$$
$$E_{3\alpha} = 0, \quad E_{33} = -\frac{\nu}{E}\Big(S_{11} + S_{22}\Big).$$

Quite similarly, a plane strain field does not induce a plane stress field in general, because relations (2.36) and (2.33) imply not only that

$$S_{3\alpha} = 0, \tag{2.52}$$

but also that

$$S_{33} = \lambda(E_{11} + E_{22}) = \frac{\nu E}{(1+\nu)(1-2\nu)}(E_{11} + E_{22}) \neq 0, \text{ in general.}$$

Note for later use that the last relation, when written in terms of stress components and technical moduli, reads:

$$S_{33} = \nu(S_{11} + S_{22}). \tag{2.53}$$

2.3.4 Plane Strain Fields Associated with Plane Stress Fields

Given a plane strain field, it is at times convenient to write the inverse constitutive relation delivering its nonnull components in a fashion formally identical to (2.3.3):

$$E_{11} = \frac{1}{E_0}\Big(S_{11} - \nu_0 S_{22}\Big), \quad E_{22} = \frac{1}{E_0}\Big(S_{22} - \nu_0 S_{11}\Big), \quad E_{12} = \frac{1}{2G}S_{12}, \tag{2.54}$$

where

$$E_0 := \frac{E}{1-\nu^2}\,, \quad \nu_0 := \frac{\nu}{1-\nu} \tag{2.55}$$

and hence[11]

$$E_0 = 2(1+\nu_0)G. \tag{2.56}$$

A comparison with (2.54) permits to regard the plane strain state (2.3.3) as a part of the strain state induced by a plane stress state in a body made of an isotropic material whose technical moduli are E_0, ν_0, and G.

Given the plane stress $\{S_{11}, S_{22}, S_{12}\}$ and the component S_{33} associated with it by the use of recipe (2.53), the corresponding plane strain is delivered by formulas (2.54)–(2.55). Such a construction is going to be of the essence to solve the 2-D version of Flamant problem with the method we propose.

Remark 2.12 Relation (2.54) can be given a version free from the specialty inherent to the use of Cartesian components and formally identical to $(2.45)_1$:

$$\boldsymbol{E} = \frac{1}{E_0}\Big((1+\nu_0)\boldsymbol{S} - \nu_0(\operatorname{tr}\boldsymbol{S})\boldsymbol{I}_{(2)}\Big), \tag{2.57}$$

where $\boldsymbol{S}$ is, as anticipated, a plane stress field and $\boldsymbol{I}_{(2)}$ denotes the two-dimensional identity tensor.

[11] For example, let us show how the first of (2.54) is arrived at: from $(2.33)_{1,2}$ we have that

$$S_{11} = \frac{E}{1+\nu}\Big(E_{11} + \frac{\nu}{1-2\nu}(E_{11}+E_{22})\Big), \quad S_{11}+S_{22} = \frac{E}{(1+\nu)(1-2\nu)}(E_{11}+E_{22});$$

consequently,

$$E_{11} = \frac{1+\nu}{E}S_{11} - \frac{\nu}{1-2\nu}\frac{(1+\nu)(1-2\nu)}{E}(S_{11}+S_{22}) = \frac{1+\nu}{E}\big(S_{11} - \nu(S_{11}+S_{22})\big) \text{ etc.}$$

2.4 Compatibility in Stress Terms

2.4.1 The Three-Dimensional Case

When the response mapping of a linearly elastic material is invertible, the compatibility condition (2.12) is written in stress terms in a straightforward manner, with the use of (2.35):

$$\operatorname{curl}\operatorname{curl}(\mathbb{C}^{-1}\boldsymbol{S}) = \boldsymbol{0}. \tag{2.58}$$

When an equilibrium problem is formulated in stress terms, the symmetric-valued fields $\boldsymbol{S}$ to be inserted in (2.58) must satisfy the equilibrium equation (2.31). In the applications we are interested in, three conditions hold, which make special and easy to handle the compatibility condition (2.58):

(i) the material is supposed to be isotropic, and hence, in view of $(2.45)_2$,

$$\mathbb{C}^{-1}\boldsymbol{S} = \frac{1}{2G}\Big(\boldsymbol{S} - \frac{\nu}{1+\nu}(\operatorname{tr}\boldsymbol{S})\boldsymbol{I}\Big); \tag{2.59}$$

(ii) the bodies under examination are supposed homogeneous, hence the elastic moduli are spatially constant;

(iii) distance actions are supposed to be null, and hence

$$\operatorname{div}\boldsymbol{S} = \boldsymbol{0}. \tag{2.60}$$

We now proceed to determine the form of condition (2.58) under these circumstances.

Firstly, it follows from (2.58), (2.59), and assumption (ii), that

$$\operatorname{curl}\operatorname{curl}\boldsymbol{S} - \frac{\nu}{1+\nu}\operatorname{curl}\operatorname{curl}\big((\operatorname{tr}\boldsymbol{S})\boldsymbol{I}\big) = \boldsymbol{0}. \tag{2.61}$$

To move further, we observe that each smooth symmetric-valued tensor field $\boldsymbol{A}$ satisfies identically the differential condition:

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\boldsymbol{A} = &-\Delta\boldsymbol{A} - \nabla(\nabla(\operatorname{tr}\boldsymbol{A})) + \nabla(\operatorname{div}\boldsymbol{A})\\ &+ (\nabla(\operatorname{div}\boldsymbol{A}))^T + (\Delta(\operatorname{tr}\boldsymbol{A}) - \operatorname{div}(\operatorname{div}\boldsymbol{A}))\boldsymbol{I}\end{aligned} \tag{2.62}$$

(cf. [6], Sect. 14); in components,

$$e_{ijk}e_{lmn}A_{jm},_{kn} = -A_{il},_{jj} - A_{jj},_{il} + A_{ij},_{jl} + A_{lj},_{ji} + (A_{jj},_{kk} - A_{jk},_{jk})\delta_{il}. \tag{2.63}$$

Consequently,

$$\operatorname{tr}\big(\operatorname{curl}\operatorname{curl}\boldsymbol{A}\big) = \Delta(\operatorname{tr}\boldsymbol{A}) - \operatorname{div}(\operatorname{div}\boldsymbol{A}); \tag{2.64}$$

moreover, for $\boldsymbol{A} = \alpha\boldsymbol{I}$, (2.62) yields:

$$\operatorname{curl}\operatorname{curl}\left(\alpha \boldsymbol{I}\right) = (\Delta\alpha)\boldsymbol{I} - \nabla(\nabla\alpha), \tag{2.65}$$

hence, in particular,

$$\operatorname{tr}\left(\operatorname{curl}\operatorname{curl}\left(\alpha \boldsymbol{I}\right)\right) = 2\Delta\alpha. \tag{2.66}$$

Thus, if a field $\boldsymbol{S}$ satisfying (2.60) is compatible, then necessarily it must be such that

$$\Delta(\operatorname{tr}\boldsymbol{S}) = 0, \tag{2.67}$$

a relation that is arrived at by taking the trace of (2.61), with the use of (2.64) and (2.66) and of the constitutive inequalities restricting the admissible values of ν (recall Remark 2.7). Due to this partial result, we deduce from (2.62) that

$$\operatorname{curl}\operatorname{curl}\boldsymbol{S} = -\Delta\boldsymbol{S} - \nabla\nabla(\operatorname{tr}\boldsymbol{S}),$$

and from (2.65) that

$$\operatorname{curl}\operatorname{curl}\left((\operatorname{tr}\boldsymbol{S})\boldsymbol{I}\right) = -\nabla\nabla(\operatorname{tr}\boldsymbol{S});$$

On taking the two last relations into account, (2.61) becomes the sought-for *compatibility condition in stress terms*:

$$\Delta\boldsymbol{S} + \frac{1}{1+\nu}\nabla\nabla(\operatorname{tr}\boldsymbol{S}) = \boldsymbol{0}. \tag{2.68}$$

Remark 2.13 Once a general representation has been found for all solutions of the equilibrium equation (2.60), we are going to use condition (2.68) to select those associable with strain and stress fields consistent with the constitutive behavior of the material under consideration. Remarkably, this behavior affects (2.68) only through the Poisson's modulus. A *universal* stress field—that is, a stress field being balanced and compatible for whatever isotropic material—must satisfy, in addition to (2.60), a system even more stringent than (2.68), namely,

$$\Delta\boldsymbol{S} = \boldsymbol{0}, \quad \nabla\nabla(\operatorname{tr}\boldsymbol{S}) = \boldsymbol{0}.$$

2.4.2 The Two-Dimensional Case

An assigned plane strain field whose Cartesian components are E_{11}, E_{22}, E_{12} is compatible if condition (2.20) holds; we repeat it here for the reader's convenience:

$$2\,E_{12,12} = E_{11,22} + E_{22,11}.$$

This condition can be written in terms of stresses with the use of the constitutive relations (2.54)–(2.56). One begins by finding:

$$\frac{1}{G}S_{12,12} = \frac{1}{E_0}\Big(S_{11,22} + S_{22,11} - \nu_0\big(S_{11,11} + S_{22,22}\big)\Big),$$

a relation that can be given the intermediate form

$$S_{12,12} = \frac{1}{2(1 + \nu_0)}\Big((S_{11} + S_{22}),_{11} + (S_{11} + S_{22}),_{22} - (1+\nu_o)(S_{11},_{11} + S_{22},_{22})\Big),$$

and then the final form

$$S_{\alpha\alpha},_{\beta\beta} = (1 + \nu_0)S_{\alpha\beta},_{\alpha\beta}.$$

When the field $\boldsymbol{S}$ is plane, the last condition can be witten more compactly:

$$\Delta(\mathrm{tr}\,\boldsymbol{S}) = (1 + \nu_0)\,\mathrm{div}\,(\mathrm{div}\,\boldsymbol{S}).$$

A consequence of this result, of paramount importance in certain developments to come, is the condition that a plane stress field, balanced for null distance forces, must satisfy to be compatible. In view of (2.50) and (2.51), that condition is:

$$\Delta(\mathrm{tr}\,\boldsymbol{S}) = 0. \tag{2.69}$$

It is not difficult to check that the same condition guarantees the compatibility of the three-dimensional stress field

$$\widetilde{\boldsymbol{S}} = \boldsymbol{S} + S_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \quad S_{33} = \nu S_{\alpha\alpha}. \tag{2.70}$$

Such a stress field, by way of the constitutive relations (2.54), is associable with a compatible plane strain field, which in turn is associated with a plane displacement field.

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