Chapter 1
One-Dimensional Paradigms

In this introductory chapter, we work in a one-dimensional (1-D) setting. Firstly, we exemplify the nonstandard integration method we are going to use systematically in Part II. Secondly, we exemplify the Green-kernel integration method to be exploited, in particular, for the problems collected in Part III. Finally, we use these two integration methods to solve the 1-D versions of Kelvin's and Mindlin's problems. We invite the reader to return to this chapter after studying the developments in Part II and III, so as to experience the subtle pleasure of looking at a mathematically elementary subject from a superior point of view.

Our notations are those widely used in engineering mechanics, but we make a systematic effort to suggest, through the use of the standard terminology of continuum mechanics, the kinship between the 1-D objects we here manipulate and their 3-D counterparts to be introduced in the next chapter.

1.1 Integration Methods, Standard and Not

Consider a linearly elastic straight beam, of length l, cross-section area A, and Young modulus E, whose axial stiffness EA may depend on the axial coordinate z: see Fig. 1.1, where both ends are shown hinged, a choice irrelevant to the substance of our developments to come. The applied axial load q (a *distance force*, in the terminology of continuum mechanics) is diffused; it induces an internal stress measured by the normal force N, the 1-D counterpart of the *stress tensor* $\boldsymbol{S}$. The axial deformation is denoted by ε, the axial displacement by w; their 3-D counterparts are the *strain tensor* $\boldsymbol{E}$ and the *displacement vector* $\boldsymbol{u}$, respectively.

The beam's equilibrium problem is ruled by the following field equations:

$$N' + q = 0, \qquad (\textit{balance}\text{ eq.}) \tag{1.1}$$

$$\varepsilon = w', \qquad (\textit{compatibility}\text{ eq.}) \tag{1.2}$$

$$N = (EA)\varepsilon, \qquad (\textit{constitutive}\text{ eq.}) \tag{1.3}$$

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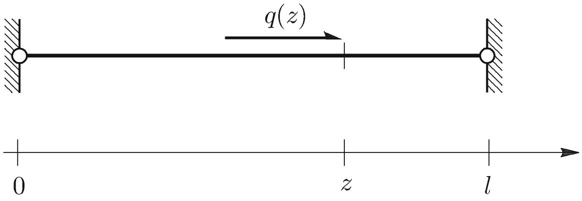

Fig. 1.1 A doubly-hinged beam subject to a diffused axial load

(a prime signifies differentiation); the following boundary equations reflect the presence of the end hinges:

$$w(0) = 0 = w(l), \qquad (\textit{boundary} \text{ eq.s}) \tag{1.4}$$

According to the standard method of solution, the field equations are combined so as to arrive at a single equation for w; since this equation is differential, one speaks of an *integration method*. Here is the flow chart:
firstly, the constitutive and compatibility equations are combined;

$$N = (EA)w'; \tag{1.5}$$

secondly, this last relation is substituted in the balance equation, yielding:

$$((EA)w')' + q = 0; \tag{1.6}$$

two successive integrations introduce two constants, that are determined by the use of the boundary conditions.

The integration method we are going to use is different, in that it produces a solution by taking into account the equations of system (1.1)–(1.4) in a different order.

The first step consists in integrating the balance Eq. (1.1), whose general solution:

$$N(z) = N(0) - \int_0^z q(s)ds \tag{1.7}$$

depends on the parameter $N(0)$. Now, a normal force field is *compatible* if, by making use of the inverse of the constitutive relation (1.3):

$$\varepsilon = (EA)^{-1}N,$$

it can be associated with an axial deformation field from which a displacement field w can be constructed according to the compatibility Eq. (1.2); that is to say, a normal force field is compatible if it induces a displacement field of the following form:

$$w(z)=w(0)+\int_0^z \varepsilon(\zeta)d\zeta = w(0)+\int_0^z \frac{N}{EA}(\zeta)d\zeta\,. \tag{1.8}$$

Hence, the minimal compatibility requirement for a field N having the representation (1.7) is that the function

$$\zeta \mapsto \frac{N}{EA}(\zeta)$$

be integrable over the interval $(0, l)$; this requirement is easy to guarantee, whenever the axial stiffness is reasonably smooth, given that relation (1.7) implies the continuity of the normal force.

Once the needed smoothness is ascertained, the second step consists simply in combining (1.8) with (1.7), so as to arrive at

$$w(z)=w(0)+\int_0^z \frac{1}{EA(\zeta)}\Big(N(0)-\int_0^\zeta q(s)ds\Big)d\zeta;$$

finally, the imposition of the boundary conditions (1.4) determines uniquely the values of parameters $w(0)$ and $N(0)$.

Remark 1.1 The normal force field corresponding to the solution of problem (1.1)–(1.4) is obtained by substituting into (1.7) the value of $N(0)$ satisfying the condition that both ends of the beam under study do not move, namely,

$$0=\Big(\int_0^l \frac{1}{EA(\zeta)}d\zeta\Big)N(0)-\int_0^l \Big(\frac{1}{EA(\zeta)}\int_0^\zeta q(s)ds\Big)d\zeta.$$

For general assignments of the data, this is the best we can do. But, it is not rare that information of this sort can be read off from the posing of the problem at hand: for example, if both axial stiffness and load are constant-valued, then it is easy to see, by symmetry, that

$$N(0)=\frac{1}{2}ql.$$

Throughout this book, to ease the solution process of all 2- and 3-D equilibrium problems we deal with, we shall be making a systematic use of symmetries in the data assignment.

Remark 1.2 The success of our integration method hinged on the invertibility of the constitutive equation (which is induced by assuming that $EA(z)>0$ for $z\in[0,l]$, so as to guarantee the model's physical plausibility) and on the assumed smoothness of the data (both functions q and $(EA)^{-1}$ must be integrable over the interval $(0, 1)$). Such conditions have direct counterparts in the case of elasticity problems in dimension greater than 1. However, when we exploit our integration method for anyone of those problem, it so happens that the compatibility requirement for an equilibrium stress field $\boldsymbol{S}$ does not consist only in asking that it is conveniently smooth, but also that it satisfies a second-order partial differential equation. A 1-D context is too poor to keep track of all the complications we are going to face in dimension 2 or 3. Similarly, in such a context there is a scarce or null chance of illustrating in an adequate manner another characteristic trait of our method, that is, the just mentioned preliminary examination of intrinsic symmetries that often allows for an explicit parametric representation of the unknown fields.

Remark 1.3 Formulation (1.6) is the so-called *strong formulation* of the problem we considered. With such a formulation, very regular solutions of the problem are sought: e.g., in the case of the boundary-value problem depicted in Fig. 1.1, we seek $w \in C^2(]0,l[) \cap C([0,l])$; this condition becomes even stricter, namely, $w \in C^2(]0,l[) \cap C^1(]0,l])$, if the hinge at the right end is replaced by a simple support where a force P in the axial direction is applied, because in this case the homogeneous *Dirichlet*-type boundary condition $w(l) = 0$ is replaced by the inhomogeneous *Neumann*-type boundary condition $N(l) = P$, or rather, in view of (1.5), by $(EA\,w')(l) = P$.

1.2 Green-Kernel Integration

Let us consider again the beam we studied in Sect. 1.1, this time subject to an axial force $\boldsymbol{f} = f\boldsymbol{e}$ applied in the section of abscissa z_0, with $0 < z_0 < l$ (Fig. 1.2), and assume that the axial stiffness EA is constant. Our purpose is to find an analytic expression for the axial displacement $w = w(z, z_0; f)$.

To this end, we observe that the boundary reactions ($-a\boldsymbol{f}$ at $z = 0$ and $-b\boldsymbol{f}$ at $z = l$, with both a and b positive) are determined by the balance equation:

$$a + b = f \tag{1.9}$$

and the continuity condition at $z = z_0$ for the displacement:

$$a\,z_0 = b(l - z_0); \tag{1.10}$$

the solution of system (1.9)–(1.10) is:

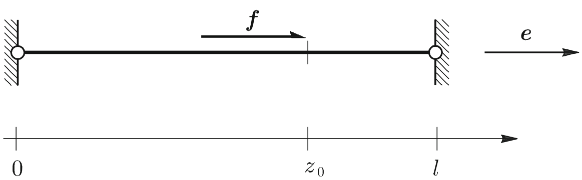

Fig. 1.2 A doubly-hinged beam subject to a concentrated axial load

$$a = f\left(1 - \frac{z_0}{l}\right), \quad b = f\,\frac{z_0}{l}\,.$$

The left part of the beam, where the strain is $(EA)^{-1}a$, lengthens, by $(EA)^{-1}az_0$ units; the right part shortens, by $(EA)^{-1}b(l - z_0)$ units; the total length cannot change. Consequently, the normal force is piece-wise constant:

$$\begin{aligned} N(z) &= f\left(1 - \frac{z_0}{l}\right) \qquad \text{for } \; 0 \le z < z_0, \\ &= -f\,\frac{z_0}{l} \qquad \text{for } \; z_0 < z \le l\,, \end{aligned}$$

and the same is true for the axial strain

$$\varepsilon = \frac{N}{EA}\,.$$

Hence, to satisfy the compatibility condition

$$\varepsilon = w',$$

we find that

$$\begin{aligned} w(z, z_0; f) &= \frac{f}{EA}\left(1 - \frac{z_0}{l}\right)z \qquad \text{for } \; 0 \le z < z_0, \\ &= \frac{f}{EA}\left(1 - \frac{z}{l}\right)z_0 \qquad \text{for } \; z_0 \le z \le l\,. \end{aligned} \tag{1.11}$$

Remark 1.4 Note that

$$\frac{f}{EA}\left(1 - \frac{z}{l}\right)z_0 = \frac{f}{EA}\left(1 - \frac{z_0}{l}\right)z_0 - \frac{f}{EA}\frac{z_0}{l}(z - z_0),$$

an expression in terms of (1.11) of the identity of

$$w(z, z_0; f) = w(z_0, z_0; f) - \big(w(z, z_0; f) - w(z_0, z_0; f)\big).$$

Note also that

$$w(z, z_0; f) = w(z_0, z; f), \tag{1.12}$$

i.e., that the displacement induced at z by a force applied at z_0 equals the displacement induced at the latter point when the same force is applied in the former, an elementary manifestation of a basic result in 3-D linear elasticity, the *Reciprocity Theorem* of Enrico Betti (1823–1892).

Let us now rewrite the expression (1.11) for the axial displacement in a compact form:

$$w(z, z_0; f) = \frac{f}{EA}\Big(\Big(1 - \frac{z_0}{l}\Big)z - H(z - z_0)(z - z_0)\Big), \quad 0 \le z \le l,$$

where H is the restriction to the interval $[0, l]$ of *Heaviside's step function* (Sect. A.1):

$$\begin{aligned} H(z - z_0) &= 0 \qquad \text{per } \ 0 \le z \le z_0, \\ &= 1 \qquad \text{per } \ z_0 < z \le l\,. \end{aligned}$$

By definition, the *displacement Green function* for the boundary-value problem under examination is:

$$G_w(z, z_0) := w(z, z_0; 1) = \frac{1}{EA}\Big(\Big(1 - \frac{z_0}{l}\Big)z - H(z - z_0)(z - z_0)\Big). \tag{1.13}$$

This function can be regarded as the solution of the linear differential problem

$$L[w] = \delta(z, z_0)\, f, \quad L[w] := -(EA)w'',$$

where δ is the *Dirac delta* (Sect. A.1), when the boundary conditions are:

$$w(0) = 0 = w(l). \tag{1.14}$$

If the beam is subject to a diffuse axial load q, the corresponding displacement field is given by the formula:

$$w(z; q) = \int_0^l G_w(z, \zeta)q(\zeta)\, d\zeta.$$

We can interpret formally this result as an implicit definition of the *linear integral operator* L^{-1} that inverts the *linear differential operator* L under the boundary conditions (1.14); and we can write:

$$w = L^{-1}[q] = \int_0^l G_w(z, \zeta)q(\zeta)\, d\zeta, \tag{1.15}$$

where the Green function is the so-called *kernel* of the integral representation for L^{-1}. Moreover, the normal force field due to the diffuse load q can be written as follows:

$$N(z) = \int_0^l G_N(z, \zeta)q(\zeta)\, d\zeta, \quad G_N(z, \zeta) = 1 - \frac{\zeta}{l} - H(z, \zeta),$$

where the *stress Green function* G_N is the kernel of the integral operator that inverts formally the equilibrium differential operator

$$\widetilde{L}[N] := -N'.$$

Remark 1.5 The displacement Green function is symmetric:

$$G_w(z_1, z_2) = G_w(z_2, z_1),$$

because the operator L is *self-adjoint*, that is, satisfies the following condition:

$$< u, L[v] > := \int_0^l u(\zeta)L[v(\zeta)]\, d\zeta = \int_0^l v(\zeta)L[u(\zeta)]\, d\zeta = < v, L[u] >,$$

for all pairs of regular fields u, v defined over $[0, l]$ that obey the prescribed boundary conditions.

Self-adjointness of operator L is the crucial property for the validity of Betti's Reciprocity Theorem. In fact, let the beam in Fig. 1.2 be loaded by a unit force applied at z_2 first, then at z_1, with $z_1 < z_2$. With a use of Green representation (1.15) for the inverse operator, we find:

$$w(z_1, z_2; 1) = \int_0^l G_w(z_1, \zeta)\delta(z_2, \zeta)\, d\zeta = G_w(z_1, z_2) = \Big(1 - \frac{z_2}{l}\Big)z_1,$$

$$w(z_2, z_1; 1) = \int_0^l G_w(z_2, \zeta)\delta(z_1, \zeta)\, d\zeta = G_w(z_2, z_1) = \Big(1 - \frac{z_2}{l}\Big)z_1$$

(cf. (1.12) and (1.13)).

1.3 The One-Dimensional Kelvin Problem

In this section, we deal with a problem which can be regarded as the 1-D version of the Kelvin problem to be tackled in Chap. 5, i.e., the equilibrium problem of an infinite linearly elastic rod, subject to an axial load applied at a point, which we identify with the origin of the coordinates (Fig. 1.3).

Our intention is to solve this problem by the same sequence of operations we shall use for its 2- and 3-D versions.

In order to underline this similitude as much as possible, we give the elastic state in the beam a redundant representation, beginning with the normal force, that we represent in tensorial form as a dyad[1]:

Fig. 1.3 An infinite beam subject to a concentrated axial load

[1] The *dyadic product* of two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is defined as follows:

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{v} := (\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a}, \quad \text{for all vectors } \boldsymbol{v}.$$

$$\boldsymbol{N}(z) = N(z)\boldsymbol{e} \otimes \boldsymbol{e}\,.$$

We require the field $\boldsymbol{N}$ to satisfy, *in the sense of distributions on the real line* $\mathcal{R}$, the differential equation:

$$\operatorname{div}\boldsymbol{N} + \boldsymbol{f} = \boldsymbol{0}, \quad \boldsymbol{f} = f\delta(0)\boldsymbol{e}, \tag{1.16}$$

an equation that can be rewritten in the form:

$$\big(N' + f\delta(0)\big)\boldsymbol{e} = \boldsymbol{0},$$

where the divergence operator is reduced to the differentiation operator. For $\boldsymbol{v} = v(z)\boldsymbol{e}$ a test function with compact support, Eq. (1.16) assumes a more meaningful form:

$$\int_{\mathcal{R}} \big(N' + f\delta(0)\big)\boldsymbol{e}\cdot\boldsymbol{v}\,dz = \int_{\mathcal{R}} \big(N' + f\delta(0)\big)v\,dz = 0,$$

or rather, and better, the distributional form:

$$\int_{\mathcal{R}} N v' dz = f\,v(0) \quad \text{if } 0 \in \operatorname{supp}(v), \qquad N' = 0 \quad \text{otherwise.} \tag{1.17}$$

On denoting with $N(0\pm)$ the values that the piece-wise constant function $z \mapsto N(z)$ assumes on the half-lines $\mathcal{R}^{\pm}$, the first of equations (1.17) can be written in the final form:

$$[\![N]\!]_0 + f = 0, \tag{1.18}$$

where the divergence operator in (1.16) is replaced by the *jump operator*:

$$[\![N]\!]_z := N(z+) - N(z-), \tag{1.19}$$

evaluated at $z = 0$.[2]

In conclusion, here is how the normal force field looks like:

$$\boldsymbol{N}(z) = -\frac{1}{2}\,f\,\operatorname{sgn}(z)\,\boldsymbol{e}\otimes\boldsymbol{e}, \quad z \neq 0 \tag{1.20}$$

(the sgn function is defined in Sect. A.1.) the relative Green function is:

$$G_N(z,\zeta) = -\frac{1}{2}\operatorname{sgn}(z-\zeta)\,.$$

[2] To obtain (1.18), it is sufficient to note that

$$\int_{\mathcal{R}} N v' dz = N(0-)\int_{\mathcal{R}^-} v' dz + N(0+)\int_{\mathcal{R}^+} v' dz = -v(0)\big(N(0+) - N(0-)\big),$$

The normal-force field plays here the role of Kelvin's 3-D stress field. It is interesting to see that a concentrated load induces at the point where it is applied a *discontinuous* normal force in dimension 1, and a *singular* stress in dimension 2 or 3.

We complete the determination of the elastic state in the beam by oberving that the inverse constitutive equation yields:

$$\boldsymbol{E}(z) = (EA)^{-1}\boldsymbol{N}(z);$$

the displacement field $\boldsymbol{u} = u\,\boldsymbol{e}$ can be found by integration of the differential equation:

$$\left(\boldsymbol{u}' \otimes \boldsymbol{e} = \boldsymbol{E} \quad \Leftrightarrow\right) \;\; u' = -\frac{f}{2\,EA}\,\mathrm{sgn}\,(z), \quad z \neq 0,$$

with the transmission condition $[\![u]\!]_0 = 0$. Integration yields:

$$u(z) = u(0) - \frac{f}{2\,EA}\,|z|\,; \tag{1.21}$$

$u(0)$ denotes a rigid translation which remains arbitrary, since no kinematic constraint is available to determine it.

Remark 1.6 Solving the Kelvin problem is important not only in itself but also because it allows to construct by differentiation or integration an arbitrary number of *strain nuclei*, that is to say, of singular solutions associated with load systems consisting of more than one concentrated force. By a suitable combination of deformation nuclei, followed at times by an operation of restriction to the domain of interest, many classical problems in linear elasticity can be solved, where the applied loads are not necessarily concentrated, as for example the problem considered by Lamé of a pressurized cavity in an infinite medium.

In a 1-D context, there is only one type of deformation nucleus, to which both 3-D nuclei called *double force* and *compression/dilatation center* reduce; we now proceed to construct it.

Let us apply to the elastic beam of infinite length we have been considering in this section both a force $h^{-1} f\boldsymbol{e}$ at the point of abscissa $z = +h/2$ and the force $-h^{-1} f\boldsymbol{e}$ at $z = -h/2$. By linearity, anyone of the fields $f\,c(z, h)$ of which the elastic state of the beam consists can be evaluated at any typical point z in terms of the corresponding field $f\,c_K(z)$ obtained by solving the Kelvin problem, by means of the following formula:

$$c(z, h) = h^{-1}c_K(z - h/2) - h^{-1}c_K(z + h/2) = -\frac{c_K(z + h/2) - c_K(z - h/2)}{h},$$

whence

$$\lim_{h\to 0} c(z, h) = -c_K'(z).$$

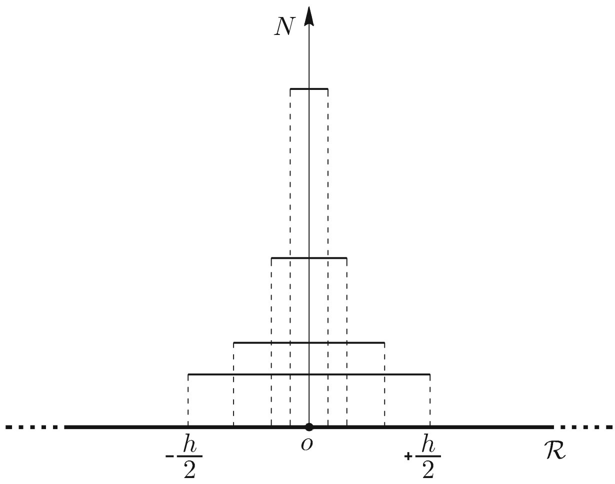

Fig. 1.4 A visualization of the graph of the mapping $h \mapsto (\text{sgn}\,(z + h/2) - \text{sgn}\,(z - h/2))/2h$ for h smaller and smaller

In case we take c_K to be the field $f^{-1}\boldsymbol{N}$ specified by (1.20), we have:

$$h^{-1}N(z-h/2)-h^{-1}N(z+h/2) = \frac{1}{2}\,\frac{\text{sgn}\,(z+h/2)-\text{sgn}\,(z-h/2)}{h} \underset{h\to 0}{\longrightarrow} \delta(z),$$

where the limit process is the well-known one suggested by Fig. 1.4. In case c_K is identified with the displacement field (1.21), we have:

$$h^{-1}u(z-h/2) - h^{-1}u(z+h/2) = \frac{1}{2\,EA}\,\frac{|z+h/2|-|z-h/2|}{h} \underset{h\to 0}{\longrightarrow} \text{sgn}\,(z)\,.$$

All in all, we say that a *dilatation center* is found at the origin when the normal force and displacement fields have the following form:

$$\boldsymbol{N}_{DC}(z) = f\,\delta(z)\,\boldsymbol{e}\otimes\boldsymbol{e}, \quad \boldsymbol{u}_{DC}(z) = f\,\text{sgn}\,z\,\boldsymbol{e}$$

(needless to say, the first of these relations must be interpreted in the sense of distributions on $\mathcal{R}$).

1.4 The One-Dimensional Mindlin Problem

In this section, we consider a problem that can be regarded as the 1-D version of the Mindlin problem (Sect. 7.3): a semi-infinite linearly elastic beam, subject to an axial load, applied at an internal point; the reader is referred to Fig. 1.5 for notations.

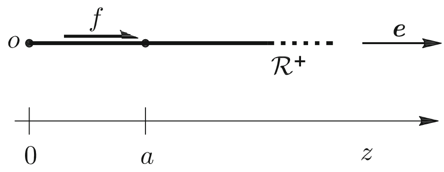

Fig. 1.5 A semi-infinite beam subject to an axial load applied at an internal point

We look for a normal force field

$$\boldsymbol{N} = N(z)\boldsymbol{e} \otimes \boldsymbol{e}$$

on the half-line $\mathcal{R}^+$, such as to satisfy the equilibrium equation

$$\big(N' + f\delta(a)\big)\boldsymbol{e} = \boldsymbol{0}$$

and the boundary condition

$$\Big(\boldsymbol{N}(0)\boldsymbol{e} = \boldsymbol{0}\Big) \quad \Leftrightarrow \quad N(0) = 0. \tag{1.22}$$

A weak formulation of this problem is: to find a piece-wise continuously differentiable function N defined over $\mathcal{R}^+$, such that, for all test functions $\boldsymbol{v} = v(z)\boldsymbol{e}$ with compact support in $\mathcal{R}^+$,

$$[\![N]\!]_a + f = 0 \quad \text{if } a \in \text{supp}(v), \qquad N' = 0 \quad \text{otherwise.}$$

It is not difficult to see that the solution is:

$$\boldsymbol{N}(z) = -f\, H(z-a)\, \boldsymbol{e} \otimes \boldsymbol{e},$$

where H is the restriction to $[0, +\infty)$ of the Heaviside function. The associated strain field is:

$$\boldsymbol{E}(z) = -\frac{f}{EA}\, H(z-a)\, \boldsymbol{e} \otimes \boldsymbol{e}.$$

With a view toward constructing the relative displacement field, we note that

$$\boldsymbol{u}'(z) = u'(z)\boldsymbol{e} = \boldsymbol{E}(z)\boldsymbol{e} = -\frac{f}{EA}\, H(z-a)\, \boldsymbol{e} \quad \Rightarrow \quad u'(z) = -\frac{f}{EA}\, H(z-a).$$

To integrate the last relation, we recall that

$$H(t) = \frac{1}{2}\,\big(1 + \text{sgn}\, t\big),$$

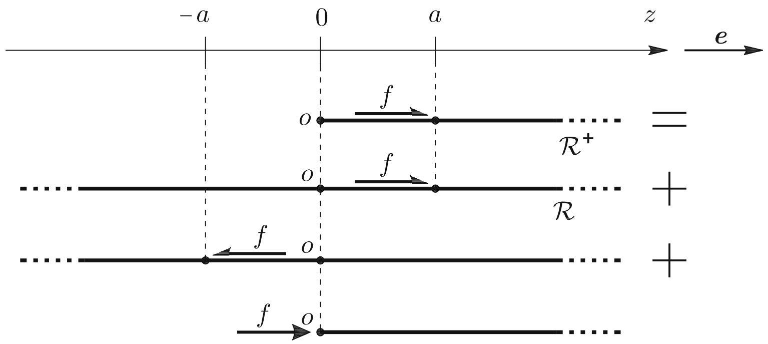

Fig. 1.6 The 1-D Mindlin's problem as a superposition of problems

and that the sign function is the distributional derivative of the modulus function. With the use of these facts, we deduce that

$$\boldsymbol{u}(z) = \Big(u(a) - \frac{f}{EA}\,\big(H(z-a)|z-a|\big)\Big)\boldsymbol{e}\,.$$

Remark 1.7 In preparation for the study of the higher-dimensional situation to be dealt with in Chap. 7, it is instructive to arrive at the solution of the 1-D Mindlin problem by superposition, with an iterated use of Kelvin solution (1.20). The procedure we propose is visualized in Fig. 1.6. Here are the steps:

(i) in a infinite beam subject to a load $f\boldsymbol{e}$ applied at $z=a$, the normal-force field is, to within an origin translation, Kelvin's:

$$\breve{\boldsymbol{N}}(z) = -\frac{1}{2}\,f\,\mathrm{sgn}\,(z-a)\,\boldsymbol{e}\otimes\boldsymbol{e},\quad z\neq a; \tag{1.23}$$

(ii) likewise, when the load $-f\boldsymbol{e}$ is applied at $z=-a$, Kelvin's normal-force field is:

$$\widehat{\boldsymbol{N}}(z) = \frac{1}{2}\,f\,\mathrm{sgn}\,(z+a)\,\boldsymbol{e}\otimes\boldsymbol{e},\quad z\neq -a; \tag{1.24}$$

(iii) superposing these two fields on $\mathcal{R}$ yields the field:

$$\breve{\boldsymbol{N}}(z) + \widehat{\boldsymbol{N}}(z) = -\frac{1}{2}\,f\,\big(\mathrm{sgn}\,(z-a) - \mathrm{sgn}\,(z+a)\big)\,\boldsymbol{e}\otimes\boldsymbol{e},$$

whose restriction to the half-line $\mathcal{R}^+$ is:

$$\widetilde{\boldsymbol{N}}(z) = f\big(1 - H(z-a)\big)\,\boldsymbol{e}\otimes\boldsymbol{e},\quad z\geq 0;$$

(iv) finally, to satisfy the boundary condition (1.22), we add to $\widetilde{\boldsymbol{N}}$ the constant field

$$\overline{\boldsymbol{N}}(z) = -f\,\boldsymbol{e} \otimes \boldsymbol{e},$$

that is, the normal-force field in the lowest beam in Fig. 1.6.

We anticipate that the last step turns out to be fairly more difficult to take in the 3-D case, when the construction of an adscititious field guaranteeing fulfilment of the free-boundary condition will be performed by having recourse to a Green-kernel integration.