The Directed Search Method for Pareto Front Approximations with Maximum Dominated Hypervolume

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Abstract. In many applications one is faced with the problem that multiple objectives have to be optimized at the same time. Since typically the solution set of such multi-objective optimization problems forms a manifold which cannot be computed analytically, one is in many cases interested in a suitable finite size approximation of this set. One widely used approach is to find a representative set that maximizes the dominated hypervolume that is defined by the images in objective space of these solutions and a given reference point.

In this paper, we propose a new point-wise iterative search procedure, Hypervolume Directed Search (HVDS), that aims to increase the hypervolume of a given point in an archive for bi-objective unconstrained optimization problems. We present the HVDS both as a standalone algorithm and as a local searcher within a specialized evolutionary algorithm. Numerical results confirm the strength of the novel approach.

Keywords: multi-objective optimization, evolutionary computation, dominated hypervolume, local search, directed search.

1 Introduction

In many real-world applications the problem arises that several objectives have to be optimized concurrently leading to a *multi-objective optimization problem* (MOP). The solution set of a MOP, the so-called Pareto set, typically forms a $(k-1)$-dimensional manifold, where k is the number of objectives involved in the problem [7]. For the treatment of MOPs specialized evolutionary algorithms, multi-objective evolutionary algorithms (EMOAs), have caught the interest of many researchers (see, e.g., [4] and references therein). Reasons for this include

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that EMOAs are applicable to a wide range of problems, are of global nature and hence in principle not dependent on the initial candidate set, and allow to compute a finite size representation of the Pareto set in a single run of the algorithm. On the other hand, it is known that EMOAs tend to converge slowly resulting in a relatively high number of function evaluations needed to obtain a suitable representation of the set of interest. As a possible remedy, researchers have proposed *memetic strategies* in the recent past (e.g., [8]). Algorithms of that type hybridize local search strategies mainly coming from mathematical programming with EMOAs in order to obtain fast and reliable global search procedures.

In this paper, we derive an algorithm that fits into the last category. To be more precise, we present a local search mechanism, HVDS, that aims to improve the dominated hypervolume [21] of a point or set for a given MOP. The new search procedure is based on the Directed Search Method [15,9] that is able to steer the search into any direction given in objective space $\mathcal{O}$ and which is hence well-suited for the problem at hand since the hypervolume is defined in $\mathcal{O}$. We present the HVDS both as standalone algorithm and as local search engine within SMS-EMOA [2] which is a state-of-the-art EMOA for approximations w.r.t. maximum dominated hypervolume. Numerical experiments show the benefit of the new approach.

The remainder of this paper is organized as follows: In Section 2, we state the background required for the understanding of the sequel. In Section 3, we present the algorithm HVDS which aims to improve the hypervolume as standalone algorithm and propose a possible integration of it into an EMOA in Section 4. In Section 5, we present some numerical results, and finally, we conclude in Section 6.

2 Background

A general multi-objective optimization problem (MOP) can be stated as follows:

$$\min_{x \in Q} \{F(x)\}, \tag{1}$$

where F is defined as the vector of the objective functions $F : Q \to \mathbb{R}^k$, $F(x) = (f_1(x), \ldots, f_k(x))$, and where each objective is given by $f_i : Q \to \mathbb{R}$. In this study we will focus on unconstrained bi-objective problems, i.e., problems of form (1) with $k = 2$ and $Q = \mathbb{R}^n$. The optimality of a MOP is defined by the concept of *dominance*.

Definition 1

(a) Let $v, w \in \mathbb{R}^k$. Then the vector v is less than *w ($v <_p w$), if $v_i < w_i$ for all $i \in \{1, \ldots, k\}$. The relation $\leq_p$ is defined analogously.*

(b) A vector $y \in Q$ is dominated *by a vector $x \in Q$ ($x \prec y$) with respect to (1) if $F(x) \leq_p F(y)$ and $F(x) \neq F(y)$, else y is called non-dominated by x.*

(c) *A point $x \in Q$ is called* (Pareto) optimal *or a* Pareto point *if there is no $y \in Q$ which dominates x.*
(d) *The set of all Pareto optimal solutions is called the* Pareto set *and its image the* Pareto front.

Recently, the Directed Search (DS) Method has been proposed that allows to steer the search from a given point into a desired direction $d \in \mathbb{R}^k$ in objective space [15]. To be more precise, given a point $x \in \mathbb{R}^n$, a search direction $\nu \in \mathbb{R}^n$ is sought such that

$$\lim_{t \searrow 0} \frac{f_i(x_0 + t\nu) - f_i(x_0)}{t} = d_i, \quad i = 1, \ldots, k. \tag{2}$$

Such a direction vector ν solves the following system of linear equations:

$$J(x_0)\nu = d, \tag{3}$$

where $J(x)$ denotes the Jacobian of F at x. Since typically $k \ll n$, we can assume that the system in Equation (3) is (highly) underdetermined. Among the solutions of Equation (3), the one with the least 2-norm can be viewed as the greedy direction for the given context. This solution is given by

$$\nu_+ := J(x)^+ d, \tag{4}$$

where $J(x)^+$ denotes the pseudo inverse of $J(x)$. Since there is no restriction on d the search can be steered in any direction, e.g., toward and along the Pareto set. See [15,14] for a Pareto descent method and a continuation method based on DS. In [9] a modification of the DS is presented that does not require gradient information.

A commonly accepted measure [20] for assessing the quality of an approximation is the so-called *dominated hypervolume* of a population.

Definition 2. *Let $v^{(1)}, v^{(2)}, \ldots v^{(\mu)} \in \mathbb{R}^k$ be a nondominated set and $R \in \mathbb{R}^k$ such that $v^{(i)} \prec R$ for all $i = 1, \ldots, \mu$. The value*

$$H(v^{(1)}, \ldots, v^{(\mu)}; R) = \Lambda_d \left(\bigcup_{i=1}^{\mu} [v^{(i)}, R] \right) \tag{5}$$

is termed the dominated hypervolume *with respect to* reference point R, *where $\Lambda_d(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^k$.*

This measure has a number of appealing properties but determining its value is getting the more tedious the larger the number of objectives is considered [1]. In case of two objectives ($k = 2$) and lexicographically ordered nondominated set $v^{(1)}, v^{(2)}, \ldots v^{(\mu)}$ the calculation of (5) reduces to

$$H(v^{(1)}, \ldots, v^{(\mu)}; R) = \left[r_1 - v_1^{(1)}\right] \cdot \left[r_2 - v_2^{(1)}\right] + \sum_{i=2}^{\mu} \left[r_1 - v_1^{(i)}\right] \cdot \left[v_2^{(i-1)} - v_2^{(i)}\right].$$

3 The Algorithm

Here we describe the adaption of the DS to the context of hypervolume approximations. For this, we will first consider the simplest case that the archive only consists of one element that has to be improved. In a next step we consider archives of general size. The reason for this is that we will reduce the general case to the one element problem.

3.1 One Element Archives

We assume that we are given the archive $A = \{x\}$, i.e., we are given one point $x \in Q$ that is assigned for local search. Further, we are given a reference point $R = (r_1, r_2)^T \in \mathbb{R}^2$ for the hypervolume calculations.

In the following, we divide the objective space into three different regions, and will propose a different movement in each of these regions (compare to Figure 1):

- **Region** I**.** The objective vector $F(x)$ is 'far away' from the Pareto front (denoted by '$F(x) \in I$'). In that case, a greedy search toward the rough location of the Pareto front is desired.
- **Region** II**.** $F(x)$ is 'in between', i.e., not far away nor near the Pareto front. In that case, a descent direction has to be selected such that a movement in that direction maximizes the hypervolume.
- **Region** III**.** $F(x)$ is 'near' to the Pareto front. In that case, a movement toward the Pareto front will lead to non-significant improvements of the dominated hypervolume. Instead, a search *along* the Pareto front will be performed.

To assign the objective vector $F(x)$ into one of these regions, we can utilize some properties of the descent cone of a MOP: If x is 'far away from the Pareto set, then the objectives gradients nearly point into the same direction, and if x is

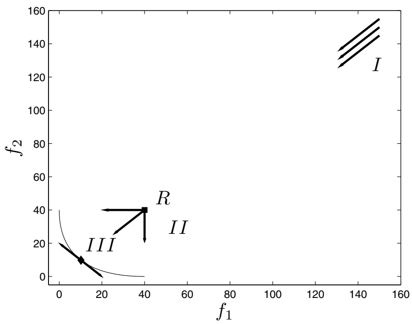


Fig. 1. Division of the objective space into distance regions

'close' then they point in opposite directions [3]. Hence, we can decide if $F(x)$ is in Region I, II or III by considering the angle between the gradients. Let

$$g_i := \nabla f_i(x), \quad i = 1, 2, \tag{6}$$

then the angle between g_1 and g_2 is defined by

$$cos\alpha = \frac{g_1^T g_2}{||g_1|| \, ||g_2||} \in [-1, 1]. \tag{7}$$

If $cos\ \alpha = 1$, both gradients point into the same direction (↓↓) which happens, roughly speaking, if x is infinitely far from the Pareto set. If $cos\ \alpha = 0$, the gradients are orthogonal to each other (←↓). Finally, when $cos\ \alpha = -1$, the grandients point into opposite directions (↓↑) which happens if x is on the Pareto set (i.e., zero distance). In order to divide the search space into three distance regions that can be numerically detected, we choose two values $a, b \in (-1, 1)$ with $b < a$ and define:

$$\begin{array}{lll} F(x) \in I & :\Leftrightarrow cos\ \alpha & \geq\ a, \\ F(x) \in II & :\Leftrightarrow cos\ \alpha & \in\ (b, a), \\ F(x) \in III & :\Leftrightarrow cos\ \alpha & \leq\ b, \end{array}$$

For the computations made in Section 5 we tested our approach using different values for a and b due to the problems behavior, finally the values taken to perform the experiments were $a = 0.8$ and $b = -0.8$, since they were the values that achieved better results. In a general case these values depend on how the cone (built by the gradient) behaves when is near the Pareto front. Now we describe the local search within each region.

Local Search in Region I. As shown in [14], large improvements in objective space can only be obtained when choosing

$$d_I = \begin{pmatrix} -1 \\ -|\lambda| \end{pmatrix}, \tag{8}$$

where $\|\nabla f_2(x)\|_2 = |\lambda| \|\nabla f_1(x)\|_2$, which defines a movement toward the rough location of the Pareto front. Hence, d_I can be chosen together with the DS approach. Alternatively, one can use Pareto descent methods since they define similar movements in Region I. For our computations we have used the method proposed in [10], namely the descent direction

$$\nu = \frac{1}{2} \left(\frac{g_1}{||g_1||} + \frac{g_2}{||g_2||} \right) \tag{9}$$

coupled with an Armijo-like step size control as used in [10].

Local Search in Region II. Given x such that $F(x) \in II$, the task is to find a search direction $d_{II} <_p 0$ such that a movement in that direction maximizes the hypervolume. Using DS, we can write the image of the new iteration x_{new} as

$$y_{new} = F(x) + td_{II}, \tag{10}$$

where $t \in \mathbb{R}R$ is a given (fixed) step size and d_{II} is to be chosen such that it solves the two-dimensional problem

$$\max_{d \in \mathbb{R}^2} \nu(d) = (r_1 - f_1(x) - td_1) \times (r_2 - f_2(x) - td_2), \tag{11}$$

$$\text{s.t.} ||d||_2^2 = 1$$

If one replaces the 2-norm by the infinity norm in the constraint of Equation (11) (which drops the assumption that the movement is done with an equal step in objective space) a straightforward computation shows that

$$d_{II,\infty} = F(x) - R. \tag{12}$$

solves the modified problem. We have used this direction for our implementations since it is easier to calculate and yields no difference in the performance of the algorithm.

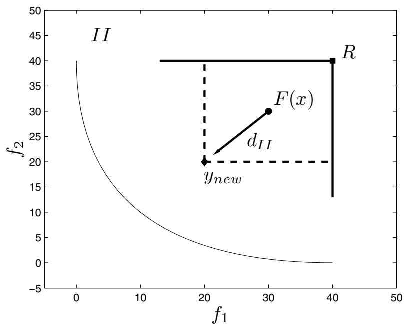


Fig. 2. Local search in Region II

Local Search in Region III. Finally, in the case $F(x)$ is in Region III, a movement along the Pareto front is desired to increase the dominated hypervolume. Here, we propose to linearize the Pareto front at $y = F(x)$ and to compute the optimal step size along direction d_{III} that describes the linearization (compare to Figure 3). Direction d_{III} can be computed as follows: Let x be a Karush-Kuhn Tucker (KKT) point. It is known that the corresponding weight vector α s.t. $\sum_{i=1}^{2} \alpha_i \nabla f_i(x) = 0$ is orthogonal to the linearized Pareto front at $F(x)$, and α solves the following quadratic optimization problem (see [12]):

$$\min_{\alpha \in \mathbb{R}^2} \left\{ \|\alpha_1 \nabla f_1(x) + \alpha_2 \nabla f_2(x)\|_2^2 : \ \alpha_i \geq 0, \ i = 1, 2, \ \alpha_1 + \alpha_2 = 1 \right\} \tag{13}$$

Hence, one can compute a solution $\tilde{\alpha}$ of (13) and set

$$d_{III} = \begin{pmatrix} -\tilde{\alpha}_1 \\ \tilde{\alpha}_2 \end{pmatrix} \tag{14}$$

The maximization of the hypervolume leads thus to the one-dimensional problem

$$\max_{t \in \mathbb{R}} \tilde{\nu}(t) = (r_1 - f_1(x) - td_1) \times (r_2 - f_2(x) - td_2), \tag{15}$$

where $d_{III} = (d_1, d_2)^T$, which has an analytic solution in case the weight vector α has no entries equal to zero.

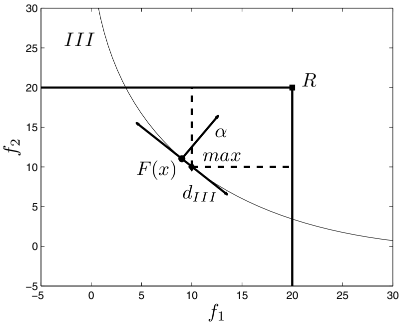


Fig. 3. Movement along linearized Pareto front in order to improve the hypervolume in Region *III*

Proposition 1. *Let $\alpha >_p 0$, then the global maximizer of Problem 15 is given by*

$$t^* = \frac{d_1 r_2 + d_2 r_1 - d_1 f_2(x) - d_2 f_1(x)}{2 d_1 d_2}. \tag{16}$$

Proof. If $\alpha_p >_p 0$, then it follows by (14) that $d_1, d_2 \neq 0$. The first derivative of $\tilde{\nu}$ is given by

$$\tilde{v}\,'(t) = 2td_1d_2 + d_2 f_1(x) - r_1 d_2 + d_1 f_2(x) - d_1 r_2 \tag{17}$$

Setting this to zero leads to

$$t^* = \frac{d_1 r_2 + d_2 r_1 - d_1 f_2(x) - d_2 f_1(x)}{2 d_1 d_2}. \tag{18}$$

Further, the second derivative at t^* is given by

$$\tilde{v}\,''(t^*) = 2d_1 d_2 < 0. \tag{19}$$

The negativity holds since $\alpha >_p 0$ and by construction of d_{III}, and the claim follows. □

We stress that the above solution holds for the linearized problem which is of course a simplification of the problem at hand. We have observed that the step size t^* leads to satisfying results in particular if (i) the Pareto front is almost linear, and (ii) if the reference point R and the current objective vector $F(x)$ are not too far away from each other. For practical implementations, it is advisable to define a maximal step size t_{max} to bound the search. Also note that the step size t^* is defined for a search in *objective* space while the new iterate $x_{new} = x + t_x \nu$ is obtained via a line search in *parameter* space. For this, we follow the suggestion made in [11] to make the match $t_x = t^*$ that works particularly well for small values of t^*. Finally, we note that the above consideration is made for KKT points. However, these computations work also well if the candidate solution x is near to the Pareto set. In particular, d_{III} points along the Pareto front.

Algorithm 1 summarizes the above discussion and presents the HVDS as standalone algorithm.

Algorithm 1. HVDS as standalone algorithm for one element archives

Require: x_0: starting point, a, b: values for region assignment; R: reference point
  $i := 0$
  repeat
    compute the angle θ of $\nabla f_j(x_i)$, $j = 1, 2$ as in Eq. (7)
    if $\theta > a$ **then** ▷ $F(x_i) \in I$
      Compute ν_I as in Eq. (9)
      Compute $t_I \in \mathbb{R}_+$
      $x_{i+1} = x_i + t_I \nu_I$
    else if $\theta \in (b, a)$ **then** ▷ $F(x_i) \in II$
      $d_{II} = F(x_i) - R$
      $\nu_{II} = J(x_i)^+ d_{II}$
      Compute $t_{II} \in \mathbb{R}_+$
      $x_{i+1} = x_i + t_{II} \nu_{II}$
    else ▷ $F(x_i) \in III$
      get the convex weight α according to Eq. (13)
      $d_{III} = (-\alpha[2], \alpha[1])^T$
      $\nu_{III} = J(x_i)^+ d_{III}$
      Compute t_{III} as in Eq. (16)
      $x_{i+1} = x_i + t_{III} * \nu_{III}$
    end if
    $i := i + 1$
  until $t_{III} = 0$ or a maximum number of iterations is reached

3.2 General Archives

We now consider the general case where the archive contains l elements, i.e., $A = \{x_1, \ldots, x_l\}$. As we will see (and which fits our intuition) the 'optimal' search direction for a given point $x \in A$ depends in some cases on the location of the other elements of A. However, we can reduce all cases to the one element case with appropriate adjustments to the reference point.

In the following we consider the local search in all three distance regions.

Local Search in Region I. If a point $x \in A$ that is chosen for local search is far away from the Pareto front, a movement into its direction is desired regardless of the location of the other elements of A. Hence, we propose to proceed as for the one element case.

Local Search in Region II. Since we consider two-objective problems, the images in objective space of A can be sorted by one of the objective values which we assume in the following. Let $x_i \in A$ be given that is assigned for local search. Figure 4 that shows such a scenario suggests that the hypervolume contribution of x_{new} that is obtained via a modification of x_i is restricted to the region between $F(x_{i-1})$ and $F(x_{i-1})$. Hence, for $i \in \{2, \ldots, l-1\}$ we propose to choose the new reference point

$$R_{F(x_i)} = \begin{pmatrix} f_1(x_{i+1}) \\ f_1(x_{i-1}) \end{pmatrix} \tag{20}$$

and to proceed analog to the one element case using the direction

$$d_{II,x_i} = F(x_i) - R_{F(x_i)}. \tag{21}$$

For the extreme points (i.e., $i \in \{1, l\}$) we proceed again with $R_{F(x_i)} = F(x_i) - R$.

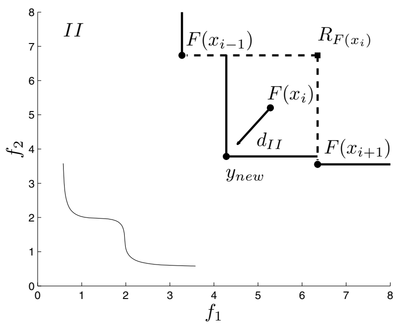


Fig. 4. Local search in Region II for multiple archive entries

Local Search in Region III. Analog to the above discussion we can proceed with points in the third distance region. To be more precise, we propose to use the reference point $R_{F(x_i)}$ for intermediate points (i.e., $i \in \{2, \ldots, l-1\}$) and the original point R for the extreme archive entries.

4 Integrating HVDS into SMS-EMOA

Here we make a first attempt to integrate the local search mechanism HVDS into an EMOA in order to obtain a fast and reliable algorithm to obtain hypervolume approximations of a given MOP. We have chosen to take the state-of-the-art algorithm SMS-EMOA ([2]), however, we stress that HVDS can in principle be hybridized with any other hypervolume based EMOA.

Each iteration step in SMS-EMOA is divided into three sections: First, one new element is generated through an evolutionary process that is inserted into the current population. In the second section, the population is partitioned into h separate groups $(S_1, \ldots, S_h)$ with respect to the degree of nondominance. Finally, the algorithm computes the contributions of the points according to hypervolume and the element with the least hypervolume contribution is discarded from the archive.

We propose to integrate the new local search mechanism as follows: After the update of the archive in iteration step i, m_i elements of the population P_i are chosen for local improvement via HVDS, X_{LS} will represents the set of the elements taken. Since it is assumed that HVDS actually improves the hypervolume value of a given element, no consideration of the hypervolume contributions is nesessary (which is a time-consuming task), but the new iterates replace the initial points. Algorithm 2 shows the pseudo-code of the new hybrid SMS-EMOA-HVDS.

Algorithm 2. SMS-EMOA-HVDS

Initialize a population $P \subset Q$ with μ elements at random
repeat
  generate offspring $x \in Q$ from P by variation
  $P := P \cup \{x\}$
  build ranking $S_1, \ldots, S_h$ from P
  compute the hypervolume contribution for each $x \in S_h$
  denote by x^* the element with the least hypervolume contribution
  $P := P \setminus \{x^*\}$
  choose the set $X_{LS} \subset P$ with $|X_{LS}| = m$
  for all $i = 1, \ldots, m$ **do**
    $x_{i,0} = i$th element of X_{LS}
    $\tilde{x}_i =$HVDS($x_{i,0}$, a, b, R)
    $P := P \cup \{\tilde{x}_i\} \setminus \{x_{i,0}\}$
  end for
until stopping criterion fulfilled
return P

5 Numerical Results

5.1 HVDS as Standalone Algorithm

First we test the ability of the HVDS as standalone algorithm. For this, we will use the following two uni-modal problems:

$$F_1 : \mathbb{R}^{10} \to \mathbb{R}^2 \tag{22}$$

$$f_1(x) = ||x - a_1||_2^2, \quad f_2(x) = ||x - a_2||_2^2,$$

where $a_1 = (1, \dots, 1)^T$, $a_2 = (-1, \dots, -1)^T$, and

$$F_2 : \mathbb{R}^2 \to \mathbb{R}^2 \tag{23}$$

$$f_1(x) = \frac{1}{2}(\sqrt{1 + (x_1 + x_2)^2} + \sqrt{1 + (x_1 - x_2)^2} + x_1 - x_2) + \lambda \cdot e^{-(x_1 - x_2)^2}$$

$$f_2(x) = \frac{1}{2}(\sqrt{1 + (x_1 + x_2)^2} + \sqrt{1 + (x_1 - x_2)^2} - x_1 + x_2) + \lambda \cdot e^{-(x_1 - x_2)^2},$$

where $\lambda = 0.85$. MOP (22) ([16], denoted by 'Convex') has a convex Pareto front, and the front of MOP (23) ([18], 'Dent') is convex-concave (see Figure 5).

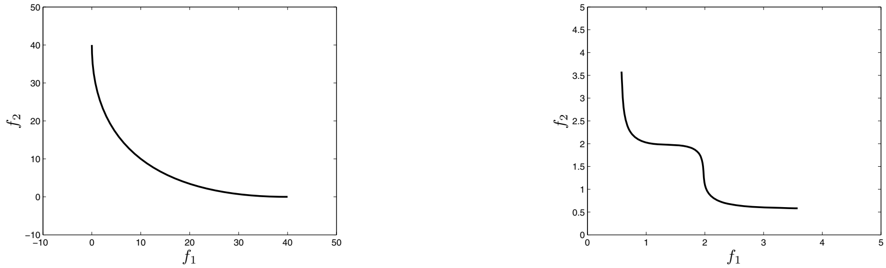


Fig. 5. Pareto fronts of MOP (22) (left) and MOP (23) (right)

First we test the HVDS for one element archives. For the sake of a comparison, we define a simple hill climber as follows: For a given point x, a further candidate solution y is taken from a neighborhood of x. As next iterate, the solution with the highest hypervolume value is taken and the search is continued in the same manner. We have chosen this strategy since it relates to a stochastic local search procedure within hypervolume-based MOEAs. Figures 6 and 7 show exemplary runs for both methods on each problem. Figure 8 shows the hypervolume against the number of function evaluations for both problems and methods. Here we count five function evaluations for the cost of one gradient evaluation which would be the case when using automatic differentiation [6]. In both cases, HVDS is able to get higher hypervolume values in the early stage of the algorithm. For Dent, the algorithm is even able to terminate after 130 function evaluations at the optimal hypervolume value.

Next, we make a first attempt to investigate the ability of the HVDS within set based search. For this, we have made the following adaption of the standalone HVDS as presented in Algorithm 1: Instead of one starting point x_0 we choose an initial population $X = \{x_0^{(1)}, \dots, x_0^{(5)}\}$ consisting of five elements. The iteration step is then performed individually for all elements (i.e., $x_{i+1}^{(j)} = x_i^{(j)} + t\nu$ as described in Algorithm 1) using the choice of the reference point as proposed in

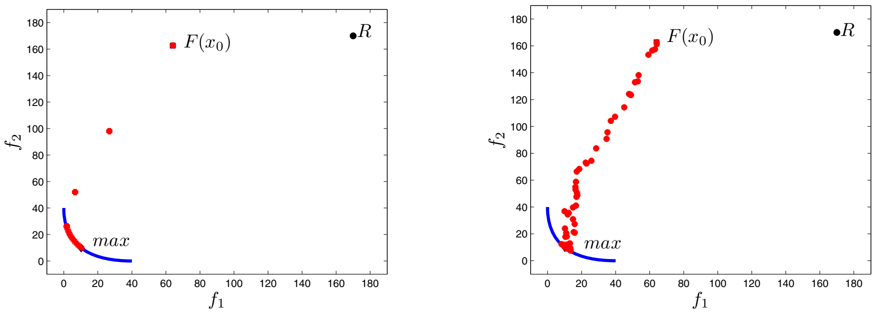


Fig. 6. Result of the HVDS and the hypervolume hill climber on Convex

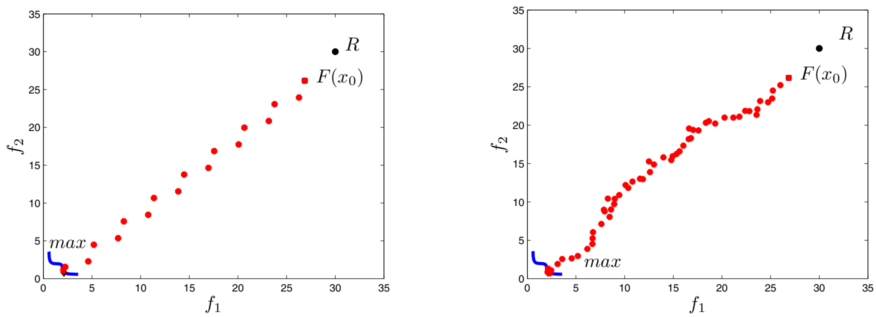


Fig. 7. Result of the HVDS and the hypervolume hill climber on Dent

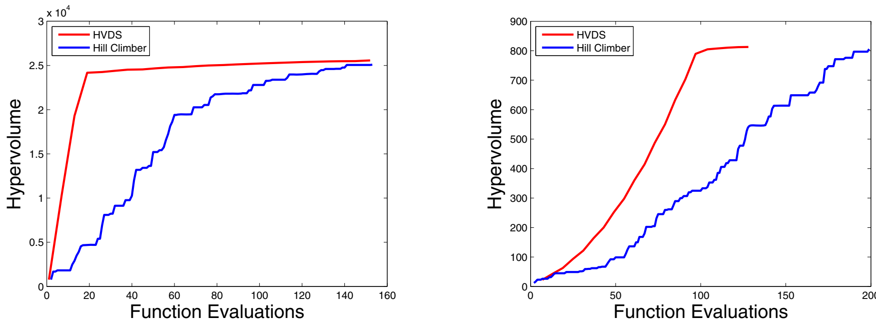


Fig. 8. Comparison the hypervolume hill climber on Convex (left) and Dent (right). The results are averaged over 20 test runs.

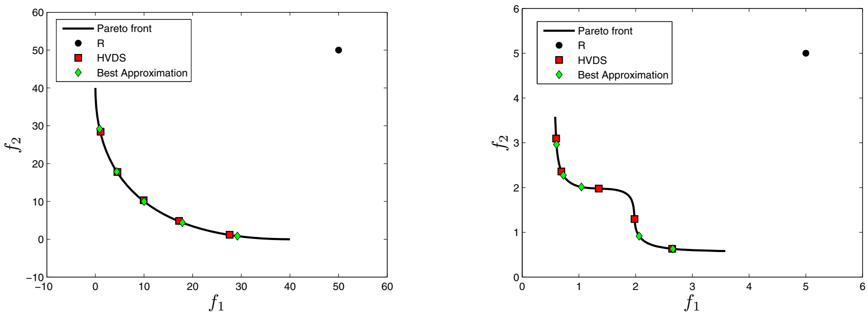


Fig. 9. Numerical results of the 5 element HVDS on Convex (left) and Dent (right)

Table 1. Comparison of the 5 element HVDS and the SMS-EMOA with $\mu = 5$. The results are averaged over 20 test runs.

	HVDS		SMS-EMOA		
	# Iterations	Hypervolume	# Iterations	Hypervolume	Best Value
Convex	1400	2100.1424	1400	1992.9788	2107.6523
Dent	885	16.6941	900	16.5721	16.8225

Table 2. HV results of SMS-EMOA with and without HVDS as local searcher after 2500 iterations of the algorithm (using the same number of function evaluations). The values are obtained from 20 test runs.

	SMS-EMOA			SMS-EMOA-HVDS		
	Average	Deviation	Median	Average	Deviation	Median
Convex	2003.867	68.956	2021.200	2161.668	18.039	2164.803
Dent	17.234	0.031	17.241	17.245	0.023	17.248
ZDT1	105.015	0.948	105.002	108.965	1.654	109.512
ZDT2	97.592	2.965	96.176	107.463	3.563	109.207
ZDT3	113.771	1.857	114.330	116.097	1.948	117.576
ZDT4	76.536	13.485	82.107	71.552	15.770	71.352

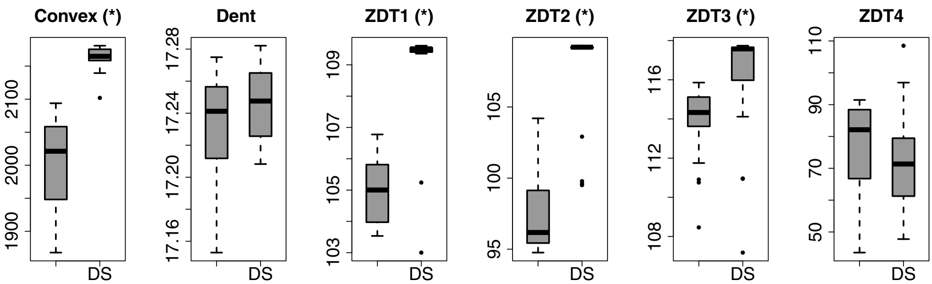
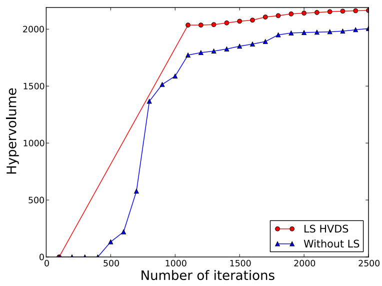
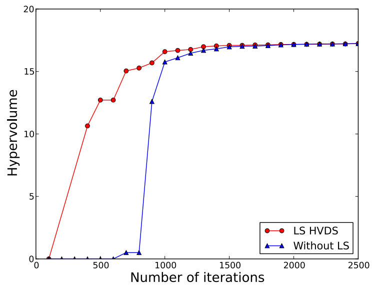


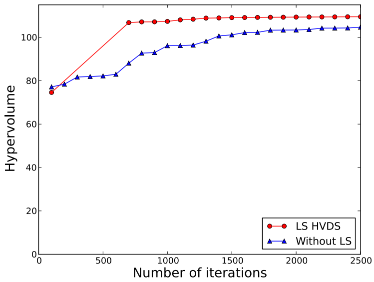
Fig. 10. Boxplots of the HV at the final iteration of the SMS-EMOA and its hybrid variant (DS) on the considered test problems. Statistically significant differences due to the Wilcoxon-Rank-Sum Test with $\alpha = 0.05$ are marked with (*).



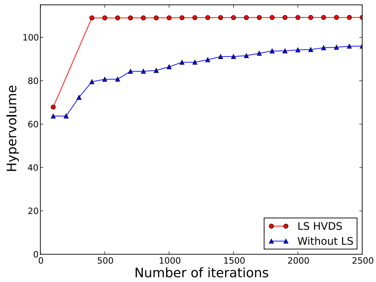
(a) MOP (22)



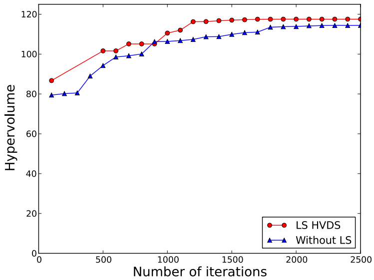
(b) MOP (23)



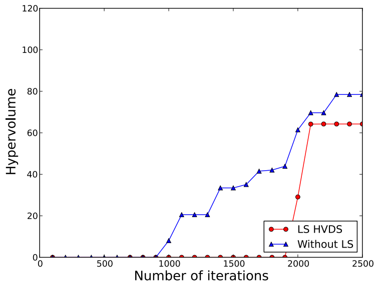
(c) ZDT1



(d) ZDT2



(e) ZDT3



(f) ZDT4

Fig. 11. Numerical results of SMS-EMOA and its hybrid variant on some benchmark models

Section 3.2. Figure 9 shows some numerical results and Table 1 a comparison to SMS-EMOA with population size $\mu = 5$. Also in this case, HVDS is able to get better hypervolume approximations. However, it has to be noted that for problem Dent none of the methods converge toward the optimal archive but the values get stuck on the value shown in Table 1 even for a higher budget of function evaluations. This might be due to the fact that only *one* point is iterated at each step. A possible remedy would be to modify *all* points in each iteration, however, to the sacrifice of a much higher computational burden.

5.2 HVDS within SMS-EMOA

Finally, we investigate the potential of HVDS as local searcher within SMS-EMOA. For the computations we have realized SMS-EMOA-HVDS as follows: To pull the current archive to the Pareto set, we have chosen to run two HVDS runs in the beginning of the search. Due to the relatively high cost of this search, we have omitted further HVDS calls in subsequent iterations (i.e., we have taken $m_1 = 2$ together with a budget of 50 iterations and $m_i = 0$ for $i > 1$).

Table 2 and Figure 11 show some numerical results on the above MOPs as well as on ZDT1-4 from [19]. Boxplots of the respective HV values after the final iteration are given in Figure 10. In 4 out of 6 cases the new hybrid is superior to its base EMOA while the differences in location of the HV values are not statistically significant for Dent and ZDT4. The latter is certainly due to the choice of the local search since the two runs got stuck in local minima, and hence, the effort was lost. Further variants of local search, e.g., the application of more but shorter HVDS runs, have to be tested which we leave for future research.

6 Conclusions and Future Work

In this paper, we have presented a new local search procedure for hypervolume approximations of a given multi-objective optimization problem. The new method, HVDS, is based on the Directed Search Method which is able to steer the search into any direction given in objective space and has been adapted to the given context. We have presented the HVDS both as standalone algorithm and as local search engine within the state-of-the-art hypervolume based algorithm SMS-EMOA. The benefit of the novel method has been shown on several numerical experiments.

For future work, there are many aspects that have to be considered. For instance, the current study was restricted to unconstrained bi-objective problems which has to be generalized for sake of a broader applicability. Further, by the same reason, it would be desirable to use the gradient free version of the Directed Search Method in the hybrid which needs a careful consideration of the neighborhood structure of the base EMOA [9]. Finally, it might be interesting to adapt the method to other indicators, e.g., to obtain Hausdorff approximations of the set of interest [5,13,17].

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