Chapter 1
Fractional Brownian Motion and Related Processes

Fractional Brownian motion (fBm) is the only Gaussian self-similar process with stationary increments. It was introduced in [102] in 1940 and the first study dedicated to it [117] appeared in 1968. The stochastic analysis of this process has been intensively developed, starting in the nineties, due to its various practical applications. Later, other processes related to fBm came to attention: bifractional Brownian motion, sub-fractional Brownian motion, multifractional Brownian motion, mixed fractional Brownian motion, etc. The purpose of this chapter is to review the basic properties of some of these fractional processes.

1.1 Fractional Brownian Motion

Fractional Brownian motion constitutes the main motivation for the theory of stochastic integration beyond the world of semi-martingales. The applications of this process in practice are significant and therefore a stochastic calculus for it was needed. There already exists a vast literature that describes different aspects of this stochastic process. We refer to the monographs [75, 95, 121, 136, 160] among others. Here we provide only a succinct review of the basic properties of this process.

Definition 1.1 Let $H \in (0,1)$. Fractional Brownian motion is defined as a centered Gaussian process $(B_t^H)_{t\geq 0}$ with covariance function

$$R_H(t,s) := \mathbf{E}\big(B_t^H B_s^H\big) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \quad s,t \geq 0. \tag{1.1}$$

The index H is called the Hurst parameter and it determines the main properties of the process B^H, such as self-similarity, regularity of the sample paths and long memory.

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1.1.1 Basic Properties

Proposition 1.1 *Fractional Brownian motion is an H-self-similar process and it has self-similarity. It is actually the unique H-self-similar Gaussian process with stationary increments.*

Proof For any $c > 0$ the process $(B_{ct})_{t\geq 0}$ is a centered Gaussian process with covariance

$$\mathbf{E}\big(B^H_{cs}B^H_{ct}\big) = \frac{1}{2}\big((ct)^{2H} + (cs)^{2H} - c^{2H}|t-s|^{2H}\big), \quad s,t \geq 0.$$

The same holds for the process $(c^H B^H_t)_{t\geq 0}$. Being Gaussian with the same covariance, the two stochastic processes therefore have the same finite dimensional distributions. It can also easily be seen that for every $h \geq 0$ the covariance of the Gaussian process $(B^H_{t+h} - B^H_h)_{t\geq 0}$ satisfies

$$\mathbf{E}\big(B^H_{t+h} - B^H_h\big)\big(B^H_{s+h} - B^H_h\big) = R_H(t,s)$$

so it is constant with respect to h. This proves that the process B^H has stationary increments.

The fact that fBm is the only Gaussian self-similar process with stationary increments follows from Theorem A.1. □

Proposition 1.2 *For any $s,t \geq 0$ we have*

$$\mathbf{E}\big|B^H_t - B^H_s\big|^2 = |t-s|^{2H}.$$

In particular, the process B^H has δ-Hölder continuous paths for any $\delta < H$.

Proof Fix $s,t \geq 0$. Then

$$\begin{aligned}\mathbf{E}\big|B^H_t - B^H_s\big|^2 &= \mathbf{E}\big|B^H_t\big|^2 - 2\mathbf{E}B^H_t B^H_s + \mathbf{E}\big|B^H_s\big|^2\\ &= t^{2H} - 2R_H(t,s) + s^{2H}\\ &= |t-s|^{2H}.\end{aligned}$$

Since for any $s \leq t$ the random variable $B_t - B_s$ has the distribution $\sqrt{\mathbf{E}|B^H_t - B^H_s|^2} \times Z = |t-s|^H Z$ where Z denotes a standard normal random variable, we obtain that for any $p \geq 1$

$$\mathbf{E}\big|B^H_t - B^H_s\big|^p = \mathbf{E}|Z|^p|t-s|^{Hp}.$$

The Hölder continuity follows from the Kolmogorov continuity theorem (see Theorem B.1). □

Proposition 1.3 *Fractional Brownian motion is not a Markov process except in the case* $H = \frac{1}{2}$.

Proof Recall that ([155]) a Gaussian process with covariance R is Markovian if and only if

$$R(s,u)R(t,t) = R(s,t)R(t,u)$$

for every $s \le t \le u$. One can see that B^H does not satisfy this condition if $H \neq \frac{1}{2}$. □

We defined in Definition A.3 the concepts of long-memory and short-memory processes.

Proposition 1.4 *If* $H > \frac{1}{2}$ *the fractional Brownian motion exhibits long-range dependence. If* $H < \frac{1}{2}$ *the fractional Brownian motion is a short-memory process.*

Proof We have

$$r(n) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right)$$

for any $n \ge 1$ and the function $r(n)$ behaves as $H(2H-1)n^{2H-2}$ for large n. See Proposition A.2. □

Let us note that

Proposition 1.5 *The fBm is not a semimartingale except when* $H = 1/2$.

Proof Again, several proofs, based in general on the expression of the quadratic variation of the fBm (see Exercise 1.1), have been presented previously. We refer, for example, to [75, 136] for recent references. □

1.1.2 Stochastic Integral Representation

Fractional Brownian motion can be expressed as a Wiener integral with respect to the Wiener process in several ways. Let us recall two of them.

Wiener Integral Representation on a Finite Interval Let B^H be a fractional Brownian motion with parameter $H \in (0,1)$. The fBm admits a representation as a Wiener integral of the form

$$B^H = \int_0^t K_H(t,s)dW_s, \tag{1.2}$$

where $W = \{W_t, t \in T\}$ is a Wiener process, and $K_H(t,s)$ is the kernel

$$K_H(t,s) = d_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right), \tag{1.3}$$

d_H being a constant and

$$F_1(z) = d_H\left(\frac{1}{2} - H\right) \int_0^{z-1} \theta^{H-\frac{3}{2}}\left(1 - (\theta+1)^{H-\frac{1}{2}}\right) d\theta.$$

If $H > \frac{1}{2}$, the kernel K_H has the simpler expression

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \tag{1.4}$$

where $t > s$ and $c_H = \left(\frac{H(H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$. The fact that the process defined by (1.2) is a fBm follows from the equality

$$\int_0^{t \wedge s} K_H(t,u) K_H(s,u) du = R_H(t,s). \tag{1.5}$$

The kernel K_H satisfies the condition

$$\frac{\partial K_H}{\partial t}(t,s) = d_H\left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \tag{1.6}$$

Moving Average Representation fBm can be represented as an integral with respect to a standard Brownian motion on the whole real line. Let $(B_s)_{s\in\mathbb{R}}$ be a standard Brownian motion. Then

$$B_t^H = C(H)^{-1} \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] dB_s, \tag{1.7}$$

with $C(H) > 0$ an explicit normalizing constant, is a fractional Brownian motion.

1.1.3 The Canonical Hilbert Space

Consider $(B_t^H)_{t\in[0,T]}$ a fractional Brownian motion with Hurst parameter $H \in (0,1)$ and denote by $\mathcal{H}$ its canonical Hilbert space. If $H = \frac{1}{2}$ then $B^{\frac{1}{2}}$ is the standard Brownian motion (Wiener process) W and in this case $\mathcal{H} = L^2([0,T])$. Otherwise $\mathcal{H}$ is the Hilbert space on $[0,T]$ extending the set of indicator function $\mathbf{1}_{[0,T]}$, $t \in [0,T]$ (by linearity and closure under the inner product) the rule

$$\langle \mathbf{1}_{[0,s]}; \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} = R_H(s,t) := 2^{-1}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

The followings facts will be needed in the sequel (we refer to [147] or [136] for their proofs):

- If $H > \frac{1}{2}$, the elements of $\mathcal{H}$ may be not functions but distributions; it is therefore more practical to work with subspaces of $\mathcal{H}$ that are sets of functions. Such a subspace is

$$|\mathcal{H}| = \left\{ f : [0, T] \to \mathbb{R} \Big| \int_0^T \int_0^T |f(u)||f(v)||u - v|^{2H-2} dv du < \infty \right\}.$$

  Then $|\mathcal{H}|$ is a strict subspace of $\mathcal{H}$ and we actually have the inclusions

$$L^2([0, T]) \subset L^{\frac{1}{H}}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{1.8}$$

- The space $|\mathcal{H}|$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|f\|^2_{|\mathcal{H}|} = \int_0^T \int_0^T |f(u)||f(v)||u - v|^{2H-2} dv du.$$

- If $H > \frac{1}{2}$ and f, g are two elements in the space $|\mathcal{H}|$, their scalar product in $\mathcal{H}$ can be expressed as

$$\langle f, g \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T du dv |u - v|^{2H-2} f(u) g(v) \tag{1.9}$$

  where $\alpha_H = H(2H - 1)$.
- For $H > \frac{1}{2}$, define the "transfer" operator

$$K_H^* \varphi(s) = \int_s^T \varphi(t) \partial_1 K_H(t, s) dt \tag{1.10}$$

  where $\partial_1 K_H(t, s) = \frac{\partial K_H}{\partial t}(t, s)$. This operator provides an isometry between the space $\mathcal{H}$ and $L^2([0, T])$ in the sense that

$$\left\| K^* \varphi \right\|_{L^2([0,T])} = \|\varphi\|_{\mathcal{H}}.$$

  As a consequence, $\varphi \in \mathcal{H}$ if and only if $K^* \varphi \in L^2([0, T])$.
- If $H < \frac{1}{2}$ then the canonical Hilbert space is a space of functions. It can be defined as the class of function $\varphi : [0, T] \to \mathbb{R}$ such that

$$K_H^* \varphi \in L^2([0, T])$$

  where the transfer operator K_H^* is defined by

$$K_H^* \varphi(s) = K_H(T, s) + \int_s^T (\varphi(t) - \varphi(s)) \partial_1 (t, s) dt. \tag{1.11}$$

The family $(B^H(\varphi), \varphi \in \mathcal{H})$ is an isonormal process in the sense of Appendix C. Therefore it is possible to construct multiple stochastic integrals and Malliavin derivatives with respect to this process. We will intensively use these techniques later in this book. If $\varphi \in \mathcal{H}$, we define $B^H(\varphi) = \int_0^T \varphi_s dB_s^H$ and we call this object the Wiener integral with respect to B^H. This Wiener integral can be expressed as a Wiener integral with respect to the Brownian motion by the transfer formula

$$\int_0^T \varphi_s dB_s^H = \int_0^T K_H^* \varphi(s) dW_s \tag{1.12}$$

where K_H^* is given by (1.11) if $H < \frac{1}{2}$ and by (1.10) when $H > \frac{1}{2}$.

1.2 Bifractional Brownian Motion

We will now focus our attention on a Gaussian process that generalizes fractional Brownian motion, called *bifractional Brownian motion* and introduced in [90]. Recall that fBm is the only self-similar Gaussian process with stationary increments starting from zero. For small increments, in models such as turbulence, fBm seems a good model but it is sometimes inadequate for large increments. For this reason, in [90] the authors introduced an extension of fBm which retained some of the properties (self-similarity, Gaussianity, stationarity for small increments) but enlarged the modeling tool kit. Moreover, it happens that this process is a quasi-helix, as defined, for example, in [98, 99].

Definition 1.2 The *bifractional Brownian motion* $(B_t^{H,K})_{t\geq 0}$ is a centered Gaussian process, starting from zero, with covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right) \tag{1.13}$$

with $H \in (0,1)$ and $K \in (0,1]$.

Note that, $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$.

1.2.1 Basic Properties

Proposition 1.6 *The process is HK-self-similar.*

Proof For every $c > 0$ and $s, t \geq 0$ the following holds

$$R^{H,K}(ct, cs) = c^{2HK} R^{H,K}(t,s).$$

Indeed,

$$\begin{aligned} R^{H,K}(ct, cs) &= \frac{1}{2^K}\left(\left((ct)^{2H} + (cs)^{2H}\right)^K - |ct - cs|^{2HK}\right) \\ &= c^{2HK} R^{H,K}(t, s) \end{aligned}$$

and this implies that $(B_{ct}^{H,K})_{t\ge 0}$ and $(c^{HK} B_t^{H,K})_{t\ge 0}$ have the same finite dimensional distributions. □

The following inequality plays an important role in the stochastic analysis of bifractional Brownian motion.

Proposition 1.7 *Let $T > 0$. For every $s, t \in [0, T]$, we have*

$$2^{-K}|t-s|^{2HK} \le \mathbf{E}\big(B_t^{H,K} - B_s^{H,K}\big)^2 \le 2^{1-K}|t-s|^{2HK}. \tag{1.14}$$

As a consequence, the process $B^{H,K}$ is Hölder continuous of order δ for any $0 < \delta < H$.

Proof The bound (1.14) has been proved in [90]. Since for any $s, t \ge 0$ the variable $B_t^{H,K} - B_s^{H,K}$ has the same law as $\sqrt{\mathbf{E}(B_t^{H,K} - B_s^{H,K})^2}Z$ with $Z \sim N(0,1)$ it follows that for any $p \ge 1$

$$\mathbf{E}\big(B_t^{H,K} - B_s^{H,K}\big)^p = \mathbf{E}|Z|^p \mathbf{E}\big(B_t^{H,K} - B_s^{H,K}\big)^{p/2} \le c|t-s|^{HKp}$$

with $c = 2^{\frac{(1-K)p}{2}}$. It remains to apply the Kolmogorov continuity theorem (Theorem B.1). □

Inequality (1.14) shows that the process $B^{H,K}$ is a quasi-helix in the sense of J.P. Kahane (see [98] and [99] for various properties and applications of quasi-helices).

The increments of the process $B^{H,K}$ are not stationary, except when $K = 1$; this can easily be seen since for every $s, t \ge 0$

$$\mathbf{E}\big|B_t^{H,K} - B_s^{H,K}\big|^2 = t^{2HK} + s^{2HK} - 2^{1-K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right).$$

But they do satisfy the following.

Proposition 1.8 *If $\sigma_\varepsilon^2(t) := \mathbf{E}(B_{t+\varepsilon}^{H,K} - B_t^{H,K})^2$, then*

$$\lim_{\varepsilon\to 0}\frac{\sigma_\varepsilon^2(t)}{\varepsilon^{2HK}} = 2^{1-K}. \tag{1.15}$$

Proof For every $t \in [0, T]$

$$\sigma_\varepsilon^2(t) = 2^{1-K}\varepsilon^{2HK} + (t+\varepsilon)^{2HK} + t^{2HK} - 2^{1-K}\left((t+\varepsilon)^{2H} + t^{2H}\right)^K.$$

Then clearly

$$\lim_{\varepsilon \to 0} \varepsilon^{-2HK} \sigma_\varepsilon^2(t) = 2^{1-K}. \qquad \square$$

The above property will be interpreted by saying that, for small increments, the process $B^{H,K}$ is 'almost' with stationary increments.

Unlike fractional Brownian motion, bifractional Brownian motion does not have a Wiener integral representation. However, it does admit the following decomposition (see [109]). Define, for $0 < K < 1$, the process

$$X_t^K = \int_0^\infty \left(1 - e^{-\theta t}\right) \theta^{-\frac{1+K}{2}} dW_\theta \tag{1.16}$$

where $(W_\theta, \theta \in \mathbb{R}_+)$ is a Wiener process. Then X^K is a centered Gaussian process with covariance

$$\begin{aligned}
\mathbf{E} X_t^K X_s^K := R^X(t,s) &= \int_0^\infty \left(1 - e^{-\theta t}\right)\left(1 - e^{-\theta s}\right) \theta^{-1-K} d\theta \\
&= \frac{\Gamma(1-K)}{K} \left(t^K + s^K - (t+s)^K\right). \qquad (1.17)
\end{aligned}$$

Proposition 1.9 *Let $(B_t^{H,K})_{t \geq 0}$ be a bi-fBm and consider $(W_\theta, \theta \geq 0)$ a Wiener process independent of $B^{H,K}$. Define for every $t \geq 0$*

$$X_t^{H,K} := X_{t^{2H}}^K.$$

Then the processes $(C_1 X_t^{H,K} + B_t^{H,K})_{t \geq 0}$ and $(C_2 B_t^{HK})_{t \geq 0}$ have the same law, where $C_1 = \sqrt{\frac{K 2^{-K}}{\Gamma(1-K)}}$ and $C_2 = 2^{\frac{1-K}{2}}$.

Proof Let

$$Y_t^{H,K} = C_1 X_t^{H,K} + B_t^{H,K}$$

for every $t \geq 0$. Then by (1.17), for every $s, t \geq 0$

$$\begin{aligned}
\mathbf{E} Y_t^{H,K} Y_s^{H,K} &= C_1^2 \mathbf{E} X_t^{H,K} X_s^{H,K} + \mathbf{E} B_t^{H,K} B_s^{H,K} \\
&= 2^{-K} \left(t^{2HK} + s^{2HK} - \left(t^{2H} + s^{2H}\right)^K\right) \\
&\quad + 2^{-K} \left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right) \\
&= 2^{-K} \left(t^{2HK} + s^{2HK} - |t-s|^{2HK}\right). \qquad \square
\end{aligned}$$

1.2.2 Quadratic Variations when $2HK = 1$

The case $2HK = 1$ is very interesting. First note that the process $B^{H,K}$ with $2HK = 1$ has the same order of self-similarity as the standard Wiener process. But it also has the same quadratic variations as Brownian motion, modulo a constant. Let us discuss the asymptotic behavior of the quadratic variations of the bifractional Brownian motion in the case $2HK = 1$. A general result on variations of bi-fBm can be found in Exercise 1.7.

We start with the following technical lemma.

Lemma 1.1 *Let us consider the following function on* $[1, \infty)$

$$h(y) = y^{2HK} + (y-1)^{2HK} - \frac{2}{2^K}\big(y^{2H} + (y-1)^{2H}\big)^K \tag{1.18}$$

where $H \in (0, 1)$ *and* $K \in (0, 1)$. *Then,*

$$h(y) \text{ converges to } 0 \text{ as } y \text{ goes to } \infty. \tag{1.19}$$

Moreover if $2HK = 1$,

$$\lim_{y\to+\infty} yh(y) = \frac{1}{4}(1-2H). \tag{1.20}$$

Proof Let $y = \frac{1}{\varepsilon}$, then

$$h(y) = h\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon^{2HK}}\left[1 + (1-\varepsilon)^{2HK} - \frac{2}{2^K}\big(1 + (1-\varepsilon)^{2H}\big)^K\right].$$

Using Taylor's expansion, for ε close to 0, we obtain

$$h\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon^{2HK}}\big(H^2K(K-1)\varepsilon^2 + o(\varepsilon^2)\big). \tag{1.21}$$

Thus

$$\lim_{y\to+\infty} h(y) = \lim_{\varepsilon\to 0} h(1/\varepsilon) = 0.$$

For the case $2HK = 1$, by (1.21) we have

$$\frac{1}{\varepsilon}h\left(\frac{1}{\varepsilon}\right) = \frac{1}{4}(1-2H) + \frac{1}{\varepsilon^2}o\big(\varepsilon^2\big).$$

Thus (1.20) is satisfied. This completes the proof. □

Using the above lemma, we can prove that, for $2HK = 1$, the bi-fBm has, modulo a multiplicative constant, the same quadratic variation as Brownian motion.

Proposition 1.10 *Suppose that* $2HK = 1$, *fix* $t \geq 0$ *and let* $0 = t_0 < t_1 < \cdots < t_n = t$ *be a partition of the interval* $[0, t]$ *with* $t_i = \frac{it}{n}$ *for* $i = 0, \ldots, n$. *Then*

$$V_t^n := \sum_{j=1}^{n}\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)^2 \xrightarrow[n\to\infty]{} \frac{1}{2^{K-1}}t \quad \text{in } L^2(\Omega).$$

Proof Let h be the function given by (1.18). A straightforward calculation shows that, using Lemma 1.1,

$$\mathbf{E}V_t^n = \frac{t}{n}\sum_{j=1}^{n} h(j) + \frac{t}{2^{K-1}} \underset{n\to\infty}{\longrightarrow} \frac{t}{2^{K-1}}.$$

To obtain the conclusion it suffices to show that

$$\lim_{n\to\infty} \mathbf{E}\big(V_t^n\big)^2 = \left(\frac{t}{2^{K-1}}\right)^2.$$

In fact we have,

$$\mathbf{E}\big(V_t^n\big)^2 = \sum_{i,j=1}^{n} \mathbf{E}\big(\big(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K}\big)\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)\big)^2.$$

Let

$$\mu_n(i,j) = \mathbf{E}\big(\big(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K}\big)\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)\big)^2.$$

It follows by linear regression that

$$\mu_n(i,j) = \mathbf{E}\Big(N_1^2\Big|\theta_n(i,j)N_1 + \sqrt{\delta_n(i,j) - \big(\theta_n(i,j)\big)^2}N_2\Big|^2\Big)$$

where N_1 and N_2 are two independent normal random variables,

$$\begin{aligned}\theta_n(i,j) &:= \mathbf{E}\big(\big(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K}\big)\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)\big)\\ &= \frac{t}{2^K n}\Big[\big(i^{2H} + j^{2H}\big)^K - 2|j-i| - \big(i^{2H} + (j-1)^{2H}\big)^K + |j-i-1|\\ &\quad - \big((i-1)^{2H} + j^{2H}\big)^K + |j-i+1| + \big((i-1)^{2H} + (j-1)^{2H}\big)^K\Big]\end{aligned}$$

and

$$\delta_n(i,j) := \mathbf{E}\big(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K}\big)^2\mathbf{E}\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)^2.$$

Hence

$$\mu_n(i,j) = 2\big(\theta_n(i,j)\big)^2 + \delta_n(i,j).$$

For $1 \le i < j$, we define a function $f_j : (1,\infty) \to \mathbb{R}$, by

$$\begin{aligned}f_j(x) = \big((x-1)^{2H} + j^{2H}\big)^K - \big((x-1)^{2H} + (j-1)^{2H}\big)^K\\ - \big(x^{2H} + j^{2H}\big)^K + \big(x^{2H} + (j-1)^{2H}\big)^K.\end{aligned}$$

We compute

$$\begin{aligned} f_j'(x) &= \left(\frac{(x-1)^{2H}+j^{2H}}{(x-1)^{2H}}\right)^{K-1} - \left(\frac{(x-1)^{2H}+(j-1)^{2H}}{(x-1)^{2H}}\right)^{K-1} \\ &\quad - \left(\frac{x^{2H}+j^{2H}}{x^{2H}}\right)^{K-1} + \left(\frac{x^{2H}+(j-1)^{2H}}{x^{2H}}\right)^{K-1} \\ &:= g(x-1) - g(x) \geq 0. \end{aligned}$$

Hence f_j is increasing and positive, since the function

$$g(x) = \left(1 + \frac{j^{2H}}{x^{2H}}\right)^{K-1} - \left(1 + \frac{(j-1)^{2H}}{x^{2H}}\right)^{K-1}$$

is decreasing on $(1, \infty)$. This implies that for every $1 \leq i < j$

$$\left|\theta_n(i,j)\right| = \frac{t}{2^K n} f_j(i) \leq \frac{t}{2^K n} f_j(j) \leq \frac{t}{n}\left|h(j)\right|$$

and $|\theta_n(i,i)| = \frac{t}{n}|h(i)+2|$ for any $i \geq 1$.

Thus

$$\sum_{i,j=1}^{n} \theta_n(i,j)^2 \leq \frac{2t^2}{n^2} \sum_{\substack{i<j \\ i,j=1}}^{n} h(j)^2 + \frac{t^2}{n^2} \sum_{i=1}^{n} \big(h(i)+2\big)^2.$$

Combining this with (1.20), we obtain that $\sum_{i,j=1}^{n} \theta_n(i,j)^2$ converges to 0 as $n \to \infty$. On the other hand, by (1.20)

$$\sum_{i,j=1}^{n} \delta_n(i,j) = \frac{t^2}{n^2} \sum_{i,j=1}^{n} \left(h(i) + \frac{1}{2^{K-1}}\right)\left(h(j) + \frac{1}{2^{K-1}}\right) \underset{n\to\infty}{\longrightarrow} \left(\frac{t}{2^{K-1}}\right)^2.$$

Consequently, $\mathbf{E}(V_t^n)^2$ converges to $(\frac{t}{2^{K-1}})^2$ as $n \to \infty$, and the conclusion follows. □

Proposition 1.11 *If $2HK = 1$ and $K \neq 1$, the process $B^{H,K}$ is a short-memory process. If $HK > \frac{1}{2}$ the process $B^{H,K}$ has long memory.*

Proof Recall Definition A.3. We can write

$$\begin{aligned} r(n) &= \mathbf{E}\big(B_1^{H,K}\big(B_{n+1}^{H,K} - B_n^{H,K}\big)\big) \\ &= \frac{1}{2^K}\big(\big((n+1)^{2H}+1\big)^K - n^{2HK}\big) - \frac{1}{2^K}\big(\big((n)^{2H}+1\big)^K - (n-1)^{2HK}\big) \\ &= \frac{1}{2^K} n^{2HK} f\left(\frac{1}{n}\right) \end{aligned}$$

where

$$f(x) = \left((1+x)^{2H} + x^{2H}\right)^K - 1 - \left(1 + x^{2H}\right)^K + (1-x)^{2HK}$$

with

$$f'(x) = 2HKx^{2H-1}G_1(x) - 2HKG_2(x)$$

where $((1 + x)^{2H} + x^{2H})^{K-1} - (1 + x^{2H})^{K-1}$ and $G_2(x) = ((1 + x)^{2H} + x^{2H})^{K-1}(1 + x)^{2H-1} - (1 - x)^{2HK-1}$. Note that $G_1(0) = 0$ and $G_1'(0) = 2H(K-1)$ and $G_2(0) = 0$ with

$$G_2'(0) = 2H(K-1) + (2H-1) + (2HK-1).$$

Note that $G_2'(0) = 0$ if $2HK = 1$! Therefore $f(x)$ behaves as $cst.x^{2H+1}$ if $2HK = 1$ for x close to zero and $f(x)$ behaves as $cst.x^2$ if $2HK > 1$. □

Remark 1.1 Consider $K = 1$ in Proposition 1.10. Then $H = \frac{1}{2}$ and we retrieve a well-known result concerning Brownian motion.

1.2.3 The Extended Bifractional Brownian Motion

An extension of bi-fBm has been introduced in [21] as follows. Define the process X^K by (1.16) with $K \in (1, 2)$.

Proposition 1.12 *For every $K \in (1, 2)$ the covariance of the process X is given by*

$$\mathbf{E} X_t^K X_s^K = \frac{\Gamma(2-K)}{K(K-1)}\left((t+s)^k - t^K - s^K\right)$$

for every $s, t \geq 0$.

Proposition 1.13 *Assume $H \in (0, 1)$ and $K \in (1, 2)$ with $HK \in (0, 1)$. Consider a fBmB^{HK} and an independent Wiener process W. Define X^K by (1.16) as a Wiener integral with respect to W.*

$$X_t^{H,K} := X_{t^{2H}}^K$$

for every $t \geq 0$. Then the process

$$B_t^{H,K} = aB_t^{HK} + bX_t^{H,K}$$

with $a = \sqrt{2^{1-K}}$ and $b = \sqrt{\frac{K(K-1)}{2^K \Gamma(2-K)}}$ is a centered Gaussian process with covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right)$$

and hence is a bi-fBm.

Proof One can follow the lines of Proposition 1.9. □

The extended bi-fBm shares the properties of the bi-fBm with $K \in (0,1)$: it has the quasi-helix property (see Exercise 1.4), it has long memory for $HK > \frac{1}{2}$ and short-memory for $HK < \frac{1}{2}$ (see Exercise 1.5). On the other hand, it is a semimartingale for $HK = \frac{1}{2}$ (see Exercise 1.6).

1.3 Sub-fractional Brownian Motion

This process was introduced in [33].

Definition 1.3 Sub-fractional Brownian motion (sub-fBm) is defined as a centered Gaussian process $(S_t^H)_{t\geq 0}$ with covariance

$$R(t,s) = s^{2H} + t^{2H} - \frac{1}{2}\big((s+t)^{2H} + |t-s|^{2H}\big), \quad s,t \geq 0$$

with $H \in (0,1)$.

Sub-fractional Brownian motion arises from occupation time fluctuations of branching particle systems (see [33]). It has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths, variation and renormalized variation and it is neither a Markov processes nor a semimartingale). Moreover, sub-fBm has non-stationary increments and the increments over non-overlapping intervals are more weakly correlated and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason, in [33] it is called sub-fBm). The above mentioned properties make sub-fBm a possible candidate for models which involve long-dependence, self-similarity and nonstationarity.

Remark 1.2 Trivially, for $H = \frac{1}{2}$ the sub-fBm reduces to the standard Brownian motion.

Proposition 1.14 *The process S^H is self-similar of order H.*

Proof Let $c > 0$. It is immediate that for every $s, t \geq 0$

$$R(ct, cs) = c^{2H} R(t,s)$$

holds and this implies the H-self-similarity of the process. □

The increments of the process S^H behave in the following way.

Proposition 1.15

$$\big(2 - 2^{2H-1}\big)|t-s|^{2H} \leq \mathbf{E}\big(S_t^H - S_s^H\big)^2 \leq |t-s|^{2H}, \quad \text{if } H > 1/2$$

and

$$|t-s|^{2H} \le \mathbf{E}\big(S_t^H - S_s^H\big)^2 \le \big(2-2^{2H-1}\big)|t-s|^{H}, \quad \text{if } H < 1/2.$$

Consequently, the process S^H has order continuous paths of order $0 < \delta < H$.

Proof See [33] or [182]. □

This means that sub-fBm is, like bi-fBm, a quasi-helix.

Proposition 1.16 *For every $s, t \ge 0$,*

$$\mathbf{E}\big|S_t^H - S_s^H\big|^2 = -2^{2H-1}\big(t^{2H}+s^{2H}\big) + (t+s)^{2H} + (t-s)^{2H}$$

and in particular for every $t \ge 0$

$$\mathbf{E}\big(S_t^H\big)^2 = \big(2-2^{2H-1}\big)t^{2H}.$$

From Proposition 1.16 we deduce that sub-fBm is not a process with stationary increments.

Sub-fBm can also be defined in terms of the sum of the odd part and of the even part of a fractional Brownian motion on the whole real line. Actually, we have

Proposition 1.17 *Let $(B_t^H)_{t\in\mathbb{R}}$ be a fBmon the whole real line, that is, a centered Gaussian process with covariance*

$$\mathbf{E}B_t^H B_s^H = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \quad s, t \in \mathbb{R}.$$

Define for $t \ge 0$

$$S_t^H = \frac{1}{\sqrt{2}}\big(B_t^H + B_{-t}^H\big).$$

Then S^H is a sub-fBm.

Proof It suffices to compute the covariance of S^H and to verify that it coincides with the covariance of sub-fBm. □

See also Exercises 1.9 and 1.10 for other properties of subfractional Brownian motion.

1.4 Bibliographical Notes

The study of fractional Brownian motion has a long history. As mentioned earlier, this stochastic process was introduced in [102] and first analyzed in [117].

The original motivation to analyze this process came from empirical studies carried out by the hydrologist Hurst, published in [96], that attested the presence of long-range dependence in hydrology. The last two decades has seen intensive development with regard to the stochastic analysis of this process. Various types of stochastic integrals with respect to fBm have been introduced and various stochastic equations driven by this process have been considered. We refer to the monographs [31, 95, 121, 125, 136] and the references therein for a detailed exposition of this theory. Simultaneously with the development of the stochastic analysis for fBm, new fractional-type processes have been introduced and studied. Bifractional Brownian motion was defined in [90] and first analyzed in [159]. Subsequently, various properties of this stochastic process were revealed in, among other references, [8, 21, 26, 72, 77, 104, 109, 113, 177]. Sub-fractional Brownian motion first appeared as a limit of branching processes in [33] and has since been studied in many works, such as [32, 44, 182, 183, 186] and [151] among others. There exist other self-similar processes related to fractional Brownian motion. We refer, for example, to mixed fractional Brownian motion which has been used as a model in financial models (see [46]) or to multifractional Brownian motion (see e.g. [11]). Several examples of Gaussian self-similar processes related to fractional Brownian motion are presented in [32].

1.5 Exercises

Exercise 1.1 Let B^H be a fBm. Prove that for each $T > 0$ the following convergences hold in $L^2(\Omega)$.

$$\sum_{i=0}^{n-1} \left| B^H_{\frac{(i+1)T}{n}} - B^H_{\frac{iT}{n}} \right|^p \to 0 \quad \text{if } p > \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \left| B^H_{\frac{(i+1)T}{n}} - B^H_{\frac{iT}{n}} \right|^p \to \rho_{\frac{1}{H}} t \quad \text{if } p = \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \left| B^H_{\frac{(i+1)T}{n}} - B^H_{\frac{iT}{n}} \right|^p \to \infty \quad \text{if } p < \frac{1}{H}$$

with $\rho_p = \mathbf{E}|N(0,1)|^p$.

Exercise 1.2 ([90]) Prove that the right-hand side of (1.13) is a covariance function.

Exercise 1.3 ([90], Proposition 2.3) Assume $(B_t^{K,H})_{t\geq 0}$ is a bi-fBm. For every $H \in (0,1)$ and $K \in (0,1]$,

$$\lim_{\varepsilon\to 0} \sup_{t\in[t_0-\varepsilon,t_0+\varepsilon]} \left| \frac{B_t^{H,K} - B_{t_0}^{H,K}}{t-t_0} \right| = +\infty$$

with probability one for every t_0. Deduce that the trajectories of the bi-fBm (and hence those of the fBm) are not differentiable.

Exercise 1.4 ([21]) Let $B^{H,K}$ be a bi-fBm with $H \in (0,1)$, $K \in (1,2)$ and $HK \in (0,1)$. Prove that for every s, t

$$2^{1-K}|t-s|^{2HK} \leq \mathbf{E}\big(B_t^{H,K} - B_s^{H,K}\big)^2 \leq |t-s|^{2HK} \quad \text{if } 0 < H \leq \frac{1}{2}$$

and

$$2^{1-K}|t-s|^{2HK} \leq \mathbf{E}\big(B_t^{H,K} - B_s^{H,K}\big)^2 \leq 2^{2-K}|t-s|^{2HK} \quad \text{if } H \geq \frac{1}{2}.$$

Exercise 1.5 ([21]) Let $B^{H,K}$ be a bi-fBm with $H \in (0,1)$, $K \in (1,2)$ and $HK \in (0,1)$. Prove that this process has short-memory if $HK < \frac{1}{2}$ and it has long memory if $HK > \frac{1}{2}$.

Exercise 1.6 ([21]) Let $B^{H,K}$ be a bi-fBm with $H \in (0,1)$, $K \in (1,2)$ and $HK \in (0,1)$. Prove that it is a semimartingale when $2HK = 1$.

Exercise 1.7 Let $B^{H,K}$ a bi-fBm. Prove that for each $T > 0$ the following convergences hold in $L^2(\Omega)$.

$$\sum_{i=0}^{n-1} \big| B^{H,K}_{\frac{(i+1)T}{n}} - B^{H,K}_{\frac{iT}{n}} \big|^p \to 0 \quad \text{if } p > \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \big| B^{H,K}_{\frac{(i+1)T}{n}} - B^{H,K}_{\frac{iT}{n}} \big|^p \to \rho_{\frac{1}{H}} t \quad \text{if } p = \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \big| B^{H,K}_{\frac{(i+1)T}{n}} - B^{H,K}_{\frac{iT}{n}} \big|^p \to \infty \quad \text{if } p < \frac{1}{H}$$

with $\rho_p = \mathbf{E}|N(0,1)|^p$. Deduce that the bi-fBm is not a semimartingale if $2HK \neq 1$.

Exercise 1.8 For every $K \in (0,1]$ and $H \in (0,1)$, the process $B^{H,K}$ is not a Markov process.

Hint The argument is the same as in the fBm case. Recall that (see [155]) a Gaussian process with covariance R is Markovian if and only if

$$R(s,u)R(t,t) = R(s,t)R(t,u)$$

for every $s \le t \le u$. It is straightforward to check that $B^{H,K}$ does not satisfy this condition.

Exercise 1.9 Let S^H be a sub-fBm and B^H be a fBm. Denote by R^{S^H} and R^{B^H} their covariance functions respectively. Prove that for every $s, t \ge 0$

$$R^{S^H}(t,s) > R^{B^H}(t,s) \quad \text{if } H < \frac{1}{2}$$

and

$$R^{S^H}(t,s) < R^{B^H}(t,s) \quad \text{if } H > \frac{1}{2}.$$

Exercise 1.10 Let S^H be a sub-fBm. Prove that for each $T > 0$ the following convergences hold in $L^2(\Omega)$.

$$\sum_{i=0}^{n-1} |S^H_{\frac{(i+1)T}{n}} - S^H_{\frac{iT}{n}}|^p \to 0 \quad \text{if } p > \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} |S^H_{\frac{(i+1)T}{n}} - S^H_{\frac{iT}{n}}|^p \to \rho_{\frac{1}{H}} T \quad \text{if } p = \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} |S^H_{\frac{(i+1)T}{n}} - S^H_{\frac{iT}{n}}|^p \to \infty \quad \text{if } p < \frac{1}{H}$$

with $\rho_p = \mathbf{E}|N(0,1)|^p$.

Exercise 1.11 (See [163]) Define for $s < t$ and $n \ge 1$

$$K^n(t,s) := n \int_{s-\frac{1}{n}}^{s} K\left(\frac{[nt]}{n}, u\right) du$$

where K is the kernel of the fractional Brownian motion (1.3) and put

$$B_t^n = \int_0^t K^n(t,s) dW_s^n = \sum_{i=1}^{[nt]} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{[nt]}{n}, s\right) ds \frac{\xi_i}{\sqrt{n}}$$

where $[\cdot]$ denotes the integer part. Prove that the disturbed random walk B^n converges weakly, as $n \to \infty$, to the fractional Brownian motion in the Skorohod topology.

Exercise 1.12 Let $\mathcal{H}$ be the canonical Hilbert space associated to the fBm on $[0, T]$. Show that

$$\|\varphi\|^2_{\mathcal{H}} \leq b^2_H t^{2H-1} \|\varphi\|^2_{L^2[0,T]}.$$

Exercise 1.13 ([51]) Let $\mathcal{H}$ be the canonical Hilbert space associated to the fBm with $H > \frac{1}{2}$. Let $f(x) = \cos(x)$ and $g(x) = \sin x$ for $x \in \mathbb{R}$. Then for every $a, b \in \mathbb{R}$, $a < b$

$$\|f 1_{(a,b)}\|^2_{\mathcal{H}} = \alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v) \\ + \alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2} \sin(b-a-v)$$

and

$$\|g 1_{(a,b)}\|^2_{\mathcal{H}} = \alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v) \\ - \alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2} \sin(b-a-v).$$

Exercise 1.14 ([51]) For every $a, b \in \mathbb{R}$ with $a < b$,

$$\int_a^b \int_a^b du dv \sin(u-v)|u-v|^{2H-2} = 0$$

for every $H > \frac{1}{2}$.

Exercise 1.15 ([14]) Let $\varphi(t) = \sin t$, $t \in [0, T]$ and denote by $\mathcal{H}(0, t)$ the canonical space of the fBm on $(0, t)$. Then show that

$$\|\varphi\|^2_{\mathcal{H}(0,T)} = c_H \int_{\mathbb{R}} \frac{(\sin \tau T - \tau \sin T)^2 + (\cos \tau T - \cos T)^2}{(\tau^2 - 1)^2} |\tau|^{-(2H-1)} d\tau,$$

where $c_H = \Gamma(2H+1) \sin(\pi H)/(2\pi)$.

Exercise 1.16 Let B^{H_1}, B^{H_2} be two fractional Brownian motions with Hurst parameters H_1, H_2 respectively. We will assume that the self-similar parameters H_1 and H_2 are both bigger than $\frac{1}{2}$. We will also assume that the two fractional Brownian motions can be expressed as Wiener integrals with respect to the same Wiener process B as

$$\begin{aligned} B_t^{H_1} &= c(H_1) \int_{\mathbb{R}} dB_y \int_0^t (u-y)_+^{H_1 - \frac{3}{2}} du, \\ B_t^{H_2} &= c(H_2) \int_{\mathbb{R}} dB_y \int_0^t (u-y)_+^{H_2 - \frac{3}{2}} du \end{aligned} \tag{1.22}$$

where the constants $c(H_1), c(H_2)$ are such that $\mathbf{E}[(B_1^{H_1})^2] = \mathbf{E}[(B_1^{H_2})^2] = 1$.

1. Prove that

$$c(H_1)^2 = \frac{H_1(2H_1-1)}{\beta(2-2H_1, H_1-\frac{1}{2})}. \tag{1.23}$$

2. Let $t > s$. Then show that

$$\mathbf{E}\big[\big(B_t^{H_1} - B_s^{H_1}\big)\big(B_t^{H_2} - B_s^{H_2}\big)\big] = b(H_1, H_2)|t-s|^{2H}$$

   where

$$b(H_1, H_2) = \frac{c(H_1)c(H_2)}{2H(2H-1)}\left(\beta\left(2-2H, H_1 - \frac{1}{2}\right) + \beta\left(2-2H, H_2 - \frac{1}{2}\right)\right)$$

   where $c(H_1), c(H_2)$ are given by (1.23).

Exercise 1.17 Another type of variation for a stochastic process has been defined by Russo and Vallois in [158]. These variations are mainly used in the context of stochastic calculus via regularization.

We will use the concept of α-*strong variation*: that is, we say that the continuous process X has an α-variation ($\alpha > 0$) if

$$ucp - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t |X_{s+\varepsilon} - X_s|^\alpha ds \quad \text{exists.} \tag{1.24}$$

Here ucp stands for the uniform limit in probability. The limit is denoted by $[\![X]\!]_t^{(\alpha)}$.

Let B be a fBm. Prove that for every $t \in [0, T]$

$$[\![B]\!]_t^{(p)} = 0 \quad \text{if } pH > 1$$

and

$$[\![B]\!]_t^{(p)} = \infty \quad \text{if } pH < 1.$$

Exercise 1.18 (See [159]) Let $(B_t^{H,K})_{t\in[0,T]}$ be a bifractional Brownian motion with parameters $H \in (0, 1)$ and $K \in (0, 1]$. Then

$$\big[\!\big[B^{H,K}\big]\!\big]_t^{(\alpha)} = 0, \quad \text{if } \alpha > \frac{1}{HK}$$

and

$$\big[\!\big[B^{H,K}\big]\!\big]_t^{(\alpha)} = 2^{\frac{1-K}{HK}} \rho_{HK} t \quad \text{if } \alpha = \frac{1}{HK},$$

where $\rho_{HK} = \mathbf{E}|N|^{1/HK}$, N being a standard normal random variable. Discuss the case $2HK = 1$.

Exercise 1.19 ([19]) Consider the family of stochastic processes $(\eta_\varepsilon)_{\varepsilon>0}$ defined by

$$\eta_\varepsilon(t) = \int_0^t K(t,s)\theta_\varepsilon(s)ds \tag{1.25}$$

where

$$\theta_\varepsilon(s) = \frac{1}{\varepsilon}(-1)^{N(\frac{s}{\varepsilon^2})}$$

(these are called the Stroock kernels) or

$$\theta_\varepsilon(s) = \frac{1}{\varepsilon}\sum_{k=1}^{\infty}\xi_k 1_{[k-1,k[}\left(\frac{s}{\varepsilon^2}\right)$$

(these are called the Donsker kernels) where $\xi_k, k \geq 1$ are independent with zero mean and variance one. Prove that the family η_ε converges weakly in the space $C_0(0,1)$ (the space of continuous functions on $[0,1]$ vanishing at zero) to the fBm.

Exercise 1.20 For every $\varepsilon > 0$, $H \in (0,1)$, $K \in (1,2)$ with $HK \in (0,1)$ and $t \in [0,T]$ define

$$B_\varepsilon^{H,K}(t) = \frac{2}{\varepsilon}\int_0^T K^{HK}(t,s)\sin\left(\theta N^{\frac{2}{\varepsilon^2}}\right)ds$$

and

$$X_\varepsilon^{H,K}(t) = \frac{2}{\varepsilon}\int_0^\infty \left(1-e^{-st^{2H}}\right)s^{-\frac{1+K}{2}}\cos\left(\theta N^{\frac{2}{\varepsilon^2}}\right)ds.$$

Then prove that the family of stochastic processes Y_ε given by

$$Y_\varepsilon(t) = aB_\varepsilon^{HK} + bX_\varepsilon^{H,K}$$

converges weakly in the space $C[0,T]$ (the space of continuous functions on $[0,T]$) to the extended bi-fBm.

Exercise 1.21 Let $X^{H,K}$ be the process defined in Proposition 1.9. Prove that, as $h \to \infty$

$$\mathbf{E}\left[\left(X_{h+t}^{H,K} - X_h^{H,K}\right)^2\right] = \Gamma(1-K)K^{-1}2^K H^2 K(1-K)t^2 h^{2(HK-1)}\left(1+o(1)\right).$$

Exercise 1.22 Let $B^{H,K}$ be a bi-fBm. Then show that

$$\left(B_{h+t}^{H,K} - B_h^{H,K}, t \geq 0\right) =^{(d)} \left(2^{(1-K)/2}B_t^{HK}, t \geq 0\right) \quad \text{as } h \to \infty,$$

where $=^{(d)}$ means convergence of all finite dimensional distributions.

Exercise 1.23 ([113]) Let us denote by

$$r(0,n)=\mathbf{E}\big[B_1^{H,K}\big(B_{n+1}^{H,K}-B_n^{H,K}\big)\big]$$

and for every $a\in\mathbb{N}$

$$r(a,a+n)=\mathbf{E}\big[\big(B_{a+1}^{H,K}-B_a^{H,K}\big)\big(B_{a+n+1}^{H,K}-B_{a+n}^{H,K}\big)\big]. \tag{1.26}$$

1. Show that for every $n\geq 1$

$$r(a,a+n)=:2^{-K}\big(f_a(n)+g(n)\big), \tag{1.27}$$

where

$$\begin{aligned} f_a(n) &= \big((a+1)^{2H}+(a+n+1)^{2H}\big)^K-\big((a+1)^{2H}+(a+n)^{2H}\big)^K \\ &\quad -\big(a^{2H}+(a+n+1)^{2H}\big)^K+\big(a^{2H}+(a+n)^{2H}\big)^K \end{aligned}$$

and for every $n\geq 1$

$$g(n)=(n+1)^{2HK}+(n-1)^{2HK}-2n^{2HK}.$$

2. Show that:
   (i) The function g is, modulo a constant, the autocorrelation function of the fractional noise with Hurst index HK. Indeed, for $n\geq 1$

   $$g(n)=2\mathbf{E}\big[B_1^{HK}\big(B_{n+1}^{HK}-B_n^{HK}\big)\big].$$

   (ii) g vanishes if $2HK=1$.
   (iii)

   $$\begin{aligned} f_a(n) &= -2^K C_1^2\mathbf{E}\big[\big(X_{a+1}^{H,K}-X_a^{H,K}\big)\big(X_{a+n+1}^{H,K}-X_{a+n}^{H,K}\big)\big] \\ &=: r^{X^{H,K}}(a,a+n) \end{aligned}$$

   for every a and n, where $X^{H,K}$ is given in Proposition 1.9.

3. Analyze the function f_a to understand “how far” bifractional Brownian noise is from “fractional Brownian noise”. In other words, how far is bifractional Brownian motion from a process with stationary increments.

   Concretely, show that for each n the following holds as $a\to\infty$

   $$f_a(n)=2H^2K(K-1)a^{2(HK-1)}\big(1+o(1)\big).$$

   Conclude that $\lim_{a\to\infty} f_a(n)=0$ for each n.

Exercise 1.24 ([113]) For $a, n \geq 0$, let $r(a, a+n)$ be given by (1.26). Then prove that for large n

$$r(a, a+n) = 2^{-K}\big[2HK(2HK-1)n^{2(HK-1)} \\ + HK(K-1)\big((a+1)^{2H} - a^{2H}\big)n^{2(HK-1)+(1-2H)} + \cdots\big].$$

Deduce that for every $a \in \mathbb{N}$ we have

$$\sum_{n\geq 0} r(a, a+n) = \infty \quad \text{if } 2HK > 1$$

and

$$\sum_{n\geq 0} r(a, a+n) < \infty \quad \text{if } 2HK \leq 1.$$

Exercise 1.25 (See [44]) Let $0 < H < 1$ and define

$$X_t^H = \int_0^\infty \big(1 - e^{-\theta t}\big)\theta^{\frac{3}{2}-H} dW_\theta$$

where $(W_\theta)_{\theta\geq 0}$ is a Wiener process. Let B^H be a fBm independent from W. Prove that:

1. If $H < \frac{1}{2}$ the process

$$S_t^H = \sqrt{-\frac{H(2H-1)}{2\Gamma(2-2H)}} X_t^H + B_t^H$$

   is a sub-fBm.
2. If $H > \frac{1}{2}$ the process

$$S_t^H = \sqrt{\frac{H(2H-1)}{2\Gamma(2-2H)}} X_t^H + B_t^H$$

   is a sub-fBm.

Exercise 1.26 ([29]) Consider a fBm $(B_t^H)_{t\geq 0}$ with $H > \frac{1}{2}$ and let

$$Y_t = at + B_t^H$$

with $a \in \mathbb{R}$. Define

$$\hat{a}_N = N \frac{\sum_{i,j=1}^{N^\alpha} j\Gamma_{i,j}^{-1} Y_i}{\sum_{i,j=1}^{N^\alpha} ij\Gamma_{i,j}^{-1}}. \tag{1.28}$$

If $t_j = \frac{j}{N}$, we let $Y_{t_j} = Y_j$ and

$$\Gamma_{i,j} = \mathrm{Cov}\big(B^H_{\frac{i}{N}}, B^H_{\frac{j}{N}}\big).$$

1. Show that

$$\hat{a}_N - a = N \frac{\sum_{i,j=1}^{N^\alpha} j \Gamma^{-1}_{i,j} B^H_{\frac{i}{N}}}{\sum_{i,j=1}^{N^\alpha} ij \Gamma^{-1}_{i,j}}, \tag{1.29}$$

   where the $\Gamma^{-1}_{i,j}$ are the entries of the matrix Γ^{-1}.
2. Deduce from (1.29) that $\hat{a}_N$ converges to a almost surely and in L^p, $p \geq 1$. (Actually $\hat{a}_N$ is a consistent estimator for the drift parameter a.)