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Ciprian A. Tudor

Analysis of Variations for Self-similar Processes

A Stochastic Calculus Approach

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To the memory of my father Constantin Tudor
To my daughter Anna-Maria

Preface

This monograph is an introduction to the stochastic analysis of self-similar processes both in the Gaussian and non-Gaussian case.

The text is mostly self-contained and should be accessible to graduate students and researchers with a basic background in probability theory and stochastic processes. Although Part II of the monograph is based on the Malliavin calculus, the tools used are basic and consequently readers who are not familiar with the theory will nevertheless be able to follow the exposition.

The majority of these notes were completed during my research visits to several university and research centers such as Purdue University, Keio University, Universidad de Valparaíso, Humboldt Universität zu Berlin, Centre Interfacultaire Bernoulli at Lausanne, Ritsumeikan University, University of Trento, Charles University, University of Sydney and Centre de Recerca Matemàtica in Barcelona. I would like to thank my colleagues Frederi Viens, Makoto Maejima, Soledad Torres, Peter Imkeller, Robert Dalang, Marco Dozzi, Francesco Russo, Arturo-Kohatsu-Higa, Stefano Bonaccorsi, Bohdan Maslowski, Qiying Wang, Xavier Bardina and Marta Sanz-Solé for their kind invitations.

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Introduction

Self-similar processes are stochastic processes that are invariant in distribution under a suitable scaling of time and space. This property is crucial in applications such as network traffic analysis, mathematical finance, astrophysics, hydrology and image processing. For this reason, their analysis has long constituted an important research direction in probability theory. Several monographs, such as [75] or [160], provide a complete analysis of the properties of this class of stochastic processes and many other research papers and monographs focus on the practical aspects of self-similarity. A bibliographical guide to the applications of self-similar processes is provided in [191]. In the last few decades, new developments in self-similarity have been obtained, including the appearance of new classes of (Gaussian or non-Gaussian) self-similar processes and new techniques to study their behavior, related to the stochastic calculus (especially the Malliavin calculus). The aim of this text is to survey these new developments.

This monograph comprises two parts, each of them divided into several chapters, and Appendices A, B, C.

In Part I we discuss the basic properties of several classes of (Gaussian or non-Gaussian) self-similar stochastic processes. This part is divided into four chapters. Chapter 1 focuses on fractional Brownian motion and related processes. Fractional Brownian motion is the most well known self-similar process with stationary increments. It includes standard Brownian motion as a particular case. The applications of this process are now widely recognized. We survey the basic properties of the process and several other related processes that have recently emerged in scientific research, such as bifractional Brownian motion and subfractional Brownian motion. Chapter 2 treats the Gaussian solutions to stochastic heat and wave equations and in Chap. 3 we introduce some non-Gaussian self-similar processes which are known as *Hermite processes*. Chapter 4 contains some examples of multi-parameter self-similar processes and their basic properties.

Part II is dedicated to the study of quadratic (and other) variations of several self-similar processes. The variations of a stochastic process play a crucial role in its probabilistic and statistical analysis. Best known is the quadratic variation of a semi-martingale, which is crucial for its Itô formula; quadratic variation also has a direct utility in practice, in estimating unknown parameters, such as volatility in financial models, in the so-called "historical" context. For self-similar stochastic processes, the study of their variations constitutes a fundamental tool in constructing good estimators of their self-similarity parameters. These processes are well suited to modeling various phenomena where scaling and long memory are important factors (internet traffic, hydrology, econometrics, among others, see [191]). The most important modeling task is then to determine or estimate the self-similarity parameter, because it is also typically responsible for the process's long memory and its regularity properties. Studying such processes is thus an important research direction both in theory and in practice. The approach we use is based on the so-called Malliavin calculus and multiple Wiener-Itô integrals. Part II comprises two chapters. In the first we study the asymptotic behavior of various types of variations of fractional Brownian motion, of the Hermite process and of the solution to the linear heat equation. In the second chapter we study other types of variations for stochastic processes, including Hermite-type variations for self-similar processes and fields and so-called Spitzer's and Hsu-Robbins type results.

Each chapter concludes with a collection of exercises.

Paris
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Ciprian A. Tudor

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Part I
Examples of Self-similar Processes

Chapter 1
Fractional Brownian Motion and Related Processes

Fractional Brownian motion (fBm) is the only Gaussian self-similar process with stationary increments. It was introduced in [102] in 1940 and the first study dedicated to it [117] appeared in 1968. The stochastic analysis of this process has been intensively developed, starting in the nineties, due to its various practical applications. Later, other processes related to fBm came to attention: bifractional Brownian motion, sub-fractional Brownian motion, multifractional Brownian motion, mixed fractional Brownian motion, etc. The purpose of this chapter is to review the basic properties of some of these fractional processes.

1.1 Fractional Brownian Motion

Fractional Brownian motion constitutes the main motivation for the theory of stochastic integration beyond the world of semi-martingales. The applications of this process in practice are significant and therefore a stochastic calculus for it was needed. There already exists a vast literature that describes different aspects of this stochastic process. We refer to the monographs [75, 95, 121, 136, 160] among others. Here we provide only a succinct review of the basic properties of this process.

Definition 1.1 Let $H \in (0,1)$. Fractional Brownian motion is defined as a centered Gaussian process $(B_t^H)_{t\geq 0}$ with covariance function

$$R_H(t,s) := \mathbf{E}\big(B_t^H B_s^H\big) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \quad s,t \geq 0. \tag{1.1}$$

The index H is called the Hurst parameter and it determines the main properties of the process B^H, such as self-similarity, regularity of the sample paths and long memory.

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1.1.1 Basic Properties

Proposition 1.1 *Fractional Brownian motion is an H-self-similar process and it has self-similarity. It is actually the unique H-self-similar Gaussian process with stationary increments.*

Proof For any $c > 0$ the process $(B_{ct})_{t\geq 0}$ is a centered Gaussian process with covariance

$$\mathbf{E}\big(B^H_{cs}B^H_{ct}\big) = \frac{1}{2}\big((ct)^{2H} + (cs)^{2H} - c^{2H}|t-s|^{2H}\big), \quad s,t \geq 0.$$

The same holds for the process $(c^H B^H_t)_{t\geq 0}$. Being Gaussian with the same covariance, the two stochastic processes therefore have the same finite dimensional distributions. It can also easily be seen that for every $h \geq 0$ the covariance of the Gaussian process $(B^H_{t+h} - B^H_h)_{t\geq 0}$ satisfies

$$\mathbf{E}\big(B^H_{t+h} - B^H_h\big)\big(B^H_{s+h} - B^H_h\big) = R_H(t,s)$$

so it is constant with respect to h. This proves that the process B^H has stationary increments.

The fact that fBm is the only Gaussian self-similar process with stationary increments follows from Theorem A.1. □

Proposition 1.2 *For any $s,t \geq 0$ we have*

$$\mathbf{E}\big|B^H_t - B^H_s\big|^2 = |t-s|^{2H}.$$

In particular, the process B^H has δ-Hölder continuous paths for any $\delta < H$.

Proof Fix $s,t \geq 0$. Then

$$\begin{aligned}\mathbf{E}\big|B^H_t - B^H_s\big|^2 &= \mathbf{E}\big|B^H_t\big|^2 - 2\mathbf{E}B^H_t B^H_s + \mathbf{E}\big|B^H_s\big|^2 \\ &= t^{2H} - 2R_H(t,s) + s^{2H} \\ &= |t-s|^{2H}.\end{aligned}$$

Since for any $s \leq t$ the random variable $B_t - B_s$ has the distribution $\sqrt{\mathbf{E}|B^H_t - B^H_s|^2} \times Z = |t-s|^H Z$ where Z denotes a standard normal random variable, we obtain that for any $p \geq 1$

$$\mathbf{E}\big|B^H_t - B^H_s\big|^p = \mathbf{E}|Z|^p|t-s|^{Hp}.$$

The Hölder continuity follows from the Kolmogorov continuity theorem (see Theorem B.1). □

Proposition 1.3 *Fractional Brownian motion is not a Markov process except in the case* $H = \frac{1}{2}$.

Proof Recall that ([155]) a Gaussian process with covariance R is Markovian if and only if

$$R(s,u)R(t,t) = R(s,t)R(t,u)$$

for every $s \le t \le u$. One can see that B^H does not satisfy this condition if $H \neq \frac{1}{2}$. □

We defined in Definition A.3 the concepts of long-memory and short-memory processes.

Proposition 1.4 *If* $H > \frac{1}{2}$ *the fractional Brownian motion exhibits long-range dependence. If* $H < \frac{1}{2}$ *the fractional Brownian motion is a short-memory process.*

Proof We have

$$r(n) = \frac{1}{2}\big((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\big)$$

for any $n \ge 1$ and the function $r(n)$ behaves as $H(2H-1)n^{2H-2}$ for large n. See Proposition A.2. □

Let us note that

Proposition 1.5 *The fBm is not a semimartingale except when* $H = 1/2$.

Proof Again, several proofs, based in general on the expression of the quadratic variation of the fBm (see Exercise 1.1), have been presented previously. We refer, for example, to [75, 136] for recent references. □

1.1.2 Stochastic Integral Representation

Fractional Brownian motion can be expressed as a Wiener integral with respect to the Wiener process in several ways. Let us recall two of them.

Wiener Integral Representation on a Finite Interval Let B^H be a fractional Brownian motion with parameter $H \in (0,1)$. The fBm admits a representation as a Wiener integral of the form

$$B^H = \int_0^t K_H(t,s)dW_s, \tag{1.2}$$

where $W = \{W_t, t \in T\}$ is a Wiener process, and $K_H(t,s)$ is the kernel

$$K_H(t,s) = d_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right), \tag{1.3}$$

d_H being a constant and

$$F_1(z) = d_H\left(\frac{1}{2} - H\right) \int_0^{z-1} \theta^{H-\frac{3}{2}}\left(1 - (\theta+1)^{H-\frac{1}{2}}\right) d\theta.$$

If $H > \frac{1}{2}$, the kernel K_H has the simpler expression

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \tag{1.4}$$

where $t > s$ and $c_H = (\frac{H(H-1)}{\beta(2-2H, H-\frac{1}{2})})^{\frac{1}{2}}$. The fact that the process defined by (1.2) is a fBm follows from the equality

$$\int_0^{t\wedge s} K_H(t,u) K_H(s,u) du = R_H(t,s). \tag{1.5}$$

The kernel K_H satisfies the condition

$$\frac{\partial K_H}{\partial t}(t,s) = d_H\left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \tag{1.6}$$

Moving Average Representation fBm can be represented as an integral with respect to a standard Brownian motion on the whole real line. Let $(B_s)_{s\in\mathbb{R}}$ be a standard Brownian motion. Then

$$B_t^H = C(H)^{-1} \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] dB_s, \tag{1.7}$$

with $C(H) > 0$ an explicit normalizing constant, is a fractional Brownian motion.

1.1.3 The Canonical Hilbert Space

Consider $(B_t^H)_{t\in[0,T]}$ a fractional Brownian motion with Hurst parameter $H \in (0,1)$ and denote by $\mathcal{H}$ its canonical Hilbert space. If $H = \frac{1}{2}$ then $B^{\frac{1}{2}}$ is the standard Brownian motion (Wiener process) W and in this case $\mathcal{H} = L^2([0,T])$. Otherwise $\mathcal{H}$ is the Hilbert space on $[0,T]$ extending the set of indicator function $\mathbf{1}_{[0,T]}$, $t \in [0,T]$ (by linearity and closure under the inner product) the rule

$$\langle \mathbf{1}_{[0,s]}; \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} = R_H(s,t) := 2^{-1}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

The followings facts will be needed in the sequel (we refer to [147] or [136] for their proofs):

- If $H > \frac{1}{2}$, the elements of $\mathcal{H}$ may be not functions but distributions; it is therefore more practical to work with subspaces of $\mathcal{H}$ that are sets of functions. Such a subspace is

$$|\mathcal{H}| = \left\{ f : [0,T] \to \mathbb{R} \Big| \int_0^T \int_0^T |f(u)||f(v)||u-v|^{2H-2} dv du < \infty \right\}.$$

  Then $|\mathcal{H}|$ is a strict subspace of $\mathcal{H}$ and we actually have the inclusions

$$L^2\big([0,T]\big) \subset L^{\frac{1}{H}}\big([0,T]\big) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{1.8}$$

- The space $|\mathcal{H}|$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|f\|^2_{|\mathcal{H}|} = \int_0^T \int_0^T |f(u)||f(v)||u-v|^{2H-2} dv du.$$

- If $H > \frac{1}{2}$ and f, g are two elements in the space $|\mathcal{H}|$, their scalar product in $\mathcal{H}$ can be expressed as

$$\langle f, g \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T du dv |u-v|^{2H-2} f(u) g(v) \tag{1.9}$$

  where $\alpha_H = H(2H-1)$.
- For $H > \frac{1}{2}$, define the "transfer" operator

$$K_H^* \varphi(s) = \int_s^T \varphi(t) \partial_1 K_H(t,s) dt \tag{1.10}$$

  where $\partial_1 K_H(t,s) = \frac{\partial K_H}{\partial t}(t,s)$. This operator provides an isometry between the space $\mathcal{H}$ and $L^2([0,T])$ in the sense that

$$\|K^* \varphi\|_{L^2([0,T])} = \|\varphi\|_{\mathcal{H}}.$$

  As a consequence, $\varphi \in \mathcal{H}$ if and only if $K^* \varphi \in L^2([0,T])$.
- If $H < \frac{1}{2}$ then the canonical Hilbert space is a space of functions. It can be defined as the class of function $\varphi : [0,T] \to \mathbb{R}$ such that

$$K_H^* \varphi \in L^2\big([0,T]\big)$$

  where the transfer operator K_H^* is defined by

$$K_H^* \varphi(s) = K_H(T,s) + \int_s^T \big(\varphi(t) - \varphi(s)\big) \partial_1(t,s) dt. \tag{1.11}$$

The family $(B^H(\varphi), \varphi \in \mathcal{H})$ is an isonormal process in the sense of Appendix C. Therefore it is possible to construct multiple stochastic integrals and Malliavin derivatives with respect to this process. We will intensively use these techniques later in this book. If $\varphi \in \mathcal{H}$, we define $B^H(\varphi) = \int_0^T \varphi_s dB_s^H$ and we call this object the Wiener integral with respect to B^H. This Wiener integral can be expressed as a Wiener integral with respect to the Brownian motion by the transfer formula

$$\int_0^T \varphi_s dB_s^H = \int_0^T K_H^* \varphi(s) dW_s \tag{1.12}$$

where K_H^* is given by (1.11) if $H < \frac{1}{2}$ and by (1.10) when $H > \frac{1}{2}$.

1.2 Bifractional Brownian Motion

We will now focus our attention on a Gaussian process that generalizes fractional Brownian motion, called *bifractional Brownian motion* and introduced in [90]. Recall that fBm is the only self-similar Gaussian process with stationary increments starting from zero. For small increments, in models such as turbulence, fBm seems a good model but it is sometimes inadequate for large increments. For this reason, in [90] the authors introduced an extension of fBm which retained some of the properties (self-similarity, Gaussianity, stationarity for small increments) but enlarged the modeling tool kit. Moreover, it happens that this process is a quasi-helix, as defined, for example, in [98, 99].

Definition 1.2 The *bifractional Brownian motion* $(B_t^{H,K})_{t\geq 0}$ is a centered Gaussian process, starting from zero, with covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right) \tag{1.13}$$

with $H \in (0, 1)$ and $K \in (0, 1]$.

Note that, $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

1.2.1 Basic Properties

Proposition 1.6 *The process is HK-self-similar.*

Proof For every $c > 0$ and $s, t \geq 0$ the following holds

$$R^{H,K}(ct, cs) = c^{2HK} R^{H,K}(t, s).$$

Indeed,

$$R^{H,K}(ct, cs) = \frac{1}{2^K}\left(\left((ct)^{2H} + (cs)^{2H}\right)^K - |ct - cs|^{2HK}\right)$$
$$= c^{2HK} R^{H,K}(t, s)$$

and this implies that $(B^{H,K}_{ct})_{t\geq 0}$ and $(c^{HK} B^{H,K}_t)_{t\geq 0}$ have the same finite dimensional distributions. □

The following inequality plays an important role in the stochastic analysis of bifractional Brownian motion.

Proposition 1.7 *Let* $T > 0$. *For every* $s, t \in [0, T]$, *we have*

$$2^{-K}|t - s|^{2HK} \leq \mathbf{E}\left(B^{H,K}_t - B^{H,K}_s\right)^2 \leq 2^{1-K}|t - s|^{2HK}. \tag{1.14}$$

As a consequence, the process $B^{H,K}$ *is Hölder continuous of order* δ *for any* $0 < \delta < H$.

Proof The bound (1.14) has been proved in [90]. Since for any $s, t \geq 0$ the variable $B^{H,K}_t - B^{H,K}_s$ has the same law as $\sqrt{\mathbf{E}(B^{H,K}_t - B^{H,K}_s)^2} Z$ with $Z \sim N(0, 1)$ it follows that for any $p \geq 1$

$$\mathbf{E}\left(B^{H,K}_t - B^{H,K}_s\right)^p = \mathbf{E}|Z|^p \mathbf{E}\left(B^{H,K}_t - B^{H,K}_s\right)^{p/2} \leq c|t - s|^{HKp}$$

with $c = 2^{\frac{(1-K)p}{2}}$. It remains to apply the Kolmogorov continuity theorem (Theorem B.1). □

Inequality (1.14) shows that the process $B^{H,K}$ is a quasi-helix in the sense of J.P. Kahane (see [98] and [99] for various properties and applications of quasi-helices).

The increments of the process $B^{H,K}$ are not stationary, except when $K = 1$; this can easily be seen since for every $s, t \geq 0$

$$\mathbf{E}\left|B^{H,K}_t - B^{H,K}_s\right|^2 = t^{2HK} + s^{2HK} - 2^{1-K}\left(\left(t^{2H} + s^{2H}\right)^K - |t - s|^{2HK}\right).$$

But they do satisfy the following.

Proposition 1.8 *If* $\sigma^2_\varepsilon(t) := \mathbf{E}(B^{H,K}_{t+\varepsilon} - B^{H,K}_t)^2$, *then*

$$\lim_{\varepsilon\to 0} \frac{\sigma^2_\varepsilon(t)}{\varepsilon^{2HK}} = 2^{1-K}. \tag{1.15}$$

Proof For every $t \in [0, T]$

$$\sigma^2_\varepsilon(t) = 2^{1-K}\varepsilon^{2HK} + (t + \varepsilon)^{2HK} + t^{2HK} - 2^{1-K}\left((t + \varepsilon)^{2H} + t^{2H}\right)^K.$$

Then clearly

$$\lim_{\varepsilon \to 0} \varepsilon^{-2HK} \sigma_\varepsilon^2(t) = 2^{1-K}. \qquad \square$$

The above property will be interpreted by saying that, for small increments, the process $B^{H,K}$ is 'almost' with stationary increments.

Unlike fractional Brownian motion, bifractional Brownian motion does not have a Wiener integral representation. However, it does admit the following decomposition (see [109]). Define, for $0 < K < 1$, the process

$$X_t^K = \int_0^\infty \left(1 - e^{-\theta t}\right)\theta^{-\frac{1+K}{2}} dW_\theta \tag{1.16}$$

where $(W_\theta, \theta \in \mathbb{R}_+)$ is a Wiener process. Then X^K is a centered Gaussian process with covariance

$$\begin{aligned} \mathbf{E} X_t^K X_s^K := R^X(t,s) &= \int_0^\infty \left(1 - e^{-\theta t}\right)\left(1 - e^{-\theta s}\right)\theta^{-1-K} d\theta \\ &= \frac{\Gamma(1-K)}{K}\left(t^K + s^K - (t+s)^K\right). \end{aligned} \tag{1.17}$$

Proposition 1.9 *Let $(B_t^{H,K})_{t \geq 0}$ be a bi-fBm and consider $(W_\theta, \theta \geq 0)$ a Wiener process independent of $B^{H,K}$. Define for every $t \geq 0$*

$$X_t^{H,K} := X_{t^{2H}}^K.$$

Then the processes $(C_1 X_t^{H,K} + B_t^{H,K})_{t \geq 0}$ and $(C_2 B_t^{HK})_{t \geq 0}$ have the same law, where $C_1 = \sqrt{\frac{K 2^{-K}}{\Gamma(1-K)}}$ and $C_2 = 2^{\frac{1-K}{2}}$.

Proof Let

$$Y_t^{H,K} = C_1 X_t^{H,K} + B_t^{H,K}$$

for every $t \geq 0$. Then by (1.17), for every $s, t \geq 0$

$$\begin{aligned} \mathbf{E} Y_t^{H,K} Y_s^{H,K} &= C_1^2 \mathbf{E} X_t^{H,K} X_s^{H,K} + \mathbf{E} B_t^{H,K} B_s^{H,K} \\ &= 2^{-K}\left(t^{2HK} + s^{2HK} - \left(t^{2H} + s^{2H}\right)^K\right) \\ &\quad + 2^{-K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right) \\ &= 2^{-K}\left(t^{2HK} + s^{2HK} - |t-s|^{2HK}\right). \end{aligned} \qquad \square$$

1.2.2 Quadratic Variations when $2HK = 1$

The case $2HK = 1$ is very interesting. First note that the process $B^{H,K}$ with $2HK = 1$ has the same order of self-similarity as the standard Wiener process. But it also has the same quadratic variations as Brownian motion, modulo a constant. Let us discuss the asymptotic behavior of the quadratic variations of the bifractional Brownian motion in the case $2HK = 1$. A general result on variations of bi-fBm can be found in Exercise 1.7.

We start with the following technical lemma.

Lemma 1.1 *Let us consider the following function on* $[1, \infty)$

$$h(y) = y^{2HK} + (y-1)^{2HK} - \frac{2}{2^K}\big(y^{2H} + (y-1)^{2H}\big)^K \tag{1.18}$$

where $H \in (0,1)$ *and* $K \in (0,1)$. *Then,*

$$h(y) \text{ converges to } 0 \text{ as } y \text{ goes to } \infty. \tag{1.19}$$

Moreover if $2HK = 1$,

$$\lim_{y\to+\infty} yh(y) = \frac{1}{4}(1-2H). \tag{1.20}$$

Proof Let $y = \frac{1}{\varepsilon}$, then

$$h(y) = h\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon^{2HK}}\left[1 + (1-\varepsilon)^{2HK} - \frac{2}{2^K}\big(1 + (1-\varepsilon)^{2H}\big)^K\right].$$

Using Taylor's expansion, for ε close to 0, we obtain

$$h\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon^{2HK}}\big(H^2K(K-1)\varepsilon^2 + o\big(\varepsilon^2\big)\big). \tag{1.21}$$

Thus

$$\lim_{y\to+\infty} h(y) = \lim_{\varepsilon\to 0} h(1/\varepsilon) = 0.$$

For the case $2HK = 1$, by (1.21) we have

$$\frac{1}{\varepsilon}h\left(\frac{1}{\varepsilon}\right) = \frac{1}{4}(1-2H) + \frac{1}{\varepsilon^2}o\big(\varepsilon^2\big).$$

Thus (1.20) is satisfied. This completes the proof. □

Using the above lemma, we can prove that, for $2HK = 1$, the bi-fBm has, modulo a multiplicative constant, the same quadratic variation as Brownian motion.

Proposition 1.10 *Suppose that* $2HK = 1$, *fix* $t \geq 0$ *and let* $0 = t_0 < t_1 < \cdots < t_n = t$ *be a partition of the interval* $[0, t]$ *with* $t_i = \frac{it}{n}$ *for* $i = 0, \ldots, n$. *Then*

$$V_t^n := \sum_{j=1}^{n}\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)^2 \underset{n\to\infty}{\longrightarrow} \frac{1}{2^{K-1}}t \quad \text{in } L^2(\Omega).$$

Proof Let h be the function given by (1.18). A straightforward calculation shows that, using Lemma 1.1,

$$\mathbf{E}V_t^n = \frac{t}{n}\sum_{j=1}^{n} h(j) + \frac{t}{2^{K-1}} \underset{n\to\infty}{\longrightarrow} \frac{t}{2^{K-1}}.$$

To obtain the conclusion it suffices to show that

$$\lim_{n\to\infty} \mathbf{E}\big(V_t^n\big)^2 = \left(\frac{t}{2^{K-1}}\right)^2.$$

In fact we have,

$$\mathbf{E}\big(V_t^n\big)^2 = \sum_{i,j=1}^{n} \mathbf{E}\big(\big(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K}\big)\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)\big)^2.$$

Let

$$\mu_n(i,j) = \mathbf{E}\big(\big(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K}\big)\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)\big)^2.$$

It follows by linear regression that

$$\mu_n(i,j) = \mathbf{E}\Big(N_1^2\Big|\theta_n(i,j)N_1 + \sqrt{\delta_n(i,j) - \big(\theta_n(i,j)\big)^2}N_2\Big|^2\Big)$$

where N_1 and N_2 are two independent normal random variables,

$$\begin{aligned}\theta_n(i,j) &:= \mathbf{E}\big(\big(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K}\big)\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)\big)\\ &= \frac{t}{2^K n}\Big[\big(i^{2H} + j^{2H}\big)^K - 2|j-i| - \big(i^{2H} + (j-1)^{2H}\big)^K + |j-i-1|\\ &\qquad - \big((i-1)^{2H} + j^{2H}\big)^K + |j-i+1| + \big((i-1)^{2H} + (j-1)^{2H}\big)^K\Big]\end{aligned}$$

and

$$\delta_n(i,j) := \mathbf{E}\big(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K}\big)^2\mathbf{E}\big(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\big)^2.$$

Hence

$$\mu_n(i,j) = 2\big(\theta_n(i,j)\big)^2 + \delta_n(i,j).$$

For $1 \le i < j$, we define a function $f_j : (1,\infty) \to \mathbb{R}$, by

$$\begin{aligned}f_j(x) = {}& \big((x-1)^{2H} + j^{2H}\big)^K - \big((x-1)^{2H} + (j-1)^{2H}\big)^K\\ &- \big(x^{2H} + j^{2H}\big)^K + \big(x^{2H} + (j-1)^{2H}\big)^K.\end{aligned}$$

We compute

$$\begin{aligned} f_j'(x) &= \left(\frac{(x-1)^{2H}+j^{2H}}{(x-1)^{2H}}\right)^{K-1} - \left(\frac{(x-1)^{2H}+(j-1)^{2H}}{(x-1)^{2H}}\right)^{K-1} \\ &\quad - \left(\frac{x^{2H}+j^{2H}}{x^{2H}}\right)^{K-1} + \left(\frac{x^{2H}+(j-1)^{2H}}{x^{2H}}\right)^{K-1} \\ &:= g(x-1)-g(x) \geq 0. \end{aligned}$$

Hence f_j is increasing and positive, since the function

$$g(x) = \left(1+\frac{j^{2H}}{x^{2H}}\right)^{K-1} - \left(1+\frac{(j-1)^{2H}}{x^{2H}}\right)^{K-1}$$

is decreasing on $(1,\infty)$. This implies that for every $1 \leq i < j$

$$\left|\theta_n(i,j)\right| = \frac{t}{2^K n} f_j(i) \leq \frac{t}{2^K n} f_j(j) \leq \frac{t}{n}\left|h(j)\right|$$

and $|\theta_n(i,i)| = \frac{t}{n}|h(i)+2|$ for any $i \geq 1$.

Thus

$$\sum_{i,j=1}^{n} \theta_n(i,j)^2 \leq \frac{2t^2}{n^2} \sum_{\substack{i<j \\ i,j=1}}^{n} h(j)^2 + \frac{t^2}{n^2} \sum_{i=1}^{n} \big(h(i)+2\big)^2.$$

Combining this with (1.20), we obtain that $\sum_{i,j=1}^{n} \theta_n(i,j)^2$ converges to 0 as $n \to \infty$. On the other hand, by (1.20)

$$\sum_{i,j=1}^{n} \delta_n(i,j) = \frac{t^2}{n^2} \sum_{i,j=1}^{n} \left(h(i)+\frac{1}{2^{K-1}}\right)\left(h(j)+\frac{1}{2^{K-1}}\right) \underset{n\to\infty}{\longrightarrow} \left(\frac{t}{2^{K-1}}\right)^2.$$

Consequently, $\mathbf{E}(V_t^n)^2$ converges to $(\frac{t}{2^{K-1}})^2$ as $n \to \infty$, and the conclusion follows. □

Proposition 1.11 *If* $2HK = 1$ *and* $K \neq 1$, *the process* $B^{H,K}$ *is a short-memory process. If* $HK > \frac{1}{2}$ *the process* $B^{H,K}$ *has long memory.*

Proof Recall Definition A.3. We can write

$$\begin{aligned} r(n) &= \mathbf{E}\big(B_1^{H,K}\big(B_{n+1}^{H,K} - B_n^{H,K}\big)\big) \\ &= \frac{1}{2^K}\Big(\big((n+1)^{2H}+1\big)^K - n^{2HK}\Big) - \frac{1}{2^K}\Big(\big((n)^{2H}+1\big)^K - (n-1)^{2HK}\Big) \\ &= \frac{1}{2^K} n^{2HK} f\left(\frac{1}{n}\right) \end{aligned}$$

where

$$f(x) = \big((1+x)^{2H} + x^{2H}\big)^K - 1 - \big(1 + x^{2H}\big)^K + (1-x)^{2HK}$$

with

$$f'(x) = 2HKx^{2H-1}G_1(x) - 2HKG_2(x)$$

where $((1+x)^{2H} + x^{2H})^{K-1} - (1+x^{2H})^{K-1}$ and $G_2(x) = ((1+x)^{2H} + x^{2H})^{K-1}(1+x)^{2H-1} - (1-x)^{2HK-1}$. Note that $G_1(0) = 0$ and $G_1'(0) = 2H(K-1)$ and $G_2(0) = 0$ with

$$G_2'(0) = 2H(K-1) + (2H-1) + (2HK-1).$$

Note that $G_2'(0) = 0$ if $2HK = 1$! Therefore $f(x)$ behaves as $cst.x^{2H+1}$ if $2HK = 1$ for x close to zero and $f(x)$ behaves as $cst.x^2$ if $2HK > 1$. □

Remark 1.1 Consider $K = 1$ in Proposition 1.10. Then $H = \frac{1}{2}$ and we retrieve a well-known result concerning Brownian motion.

1.2.3 The Extended Bifractional Brownian Motion

An extension of bi-fBm has been introduced in [21] as follows. Define the process X^K by (1.16) with $K \in (1, 2)$.

Proposition 1.12 *For every* $K \in (1, 2)$ *the covariance of the process* X *is given by*

$$\mathbf{E}X_t^K X_s^K = \frac{\Gamma(2-K)}{K(K-1)}\big((t+s)^k - t^K - s^K\big)$$

for every $s, t \geq 0$.

Proposition 1.13 *Assume* $H \in (0, 1)$ *and* $K \in (1, 2)$ *with* $HK \in (0, 1)$. *Consider a fBm* B^{HK} *and an independent Wiener process* W. *Define* X^K *by* (1.16) *as a Wiener integral with respect to* W.

$$X_t^{H,K} := X_{t^{2H}}^K$$

for every $t \geq 0$. *Then the process*

$$B_t^{H,K} = aB_t^{HK} + bX_t^{H,K}$$

with $a = \sqrt{2^{1-K}}$ *and* $b = \sqrt{\frac{K(K-1)}{2^K\Gamma(2-K)}}$ *is a centered Gaussian process with covariance*

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\Big(\big(t^{2H} + s^{2H}\big)^K - |t-s|^{2HK}\Big)$$

and hence is a bi-fBm.

Proof One can follow the lines of Proposition 1.9. □

The extended bi-fBm shares the properties of the bi-fBm with $K \in (0,1)$: it has the quasi-helix property (see Exercise 1.4), it has long memory for $HK > \frac{1}{2}$ and short-memory for $HK < \frac{1}{2}$ (see Exercise 1.5). On the other hand, it is a semimartingale for $HK = \frac{1}{2}$ (see Exercise 1.6).

1.3 Sub-fractional Brownian Motion

This process was introduced in [33].

Definition 1.3 Sub-fractional Brownian motion (sub-fBm) is defined as a centered Gaussian process $(S_t^H)_{t\geq 0}$ with covariance

$$R(t,s) = s^{2H} + t^{2H} - \frac{1}{2}\big((s+t)^{2H} + |t-s|^{2H}\big), \quad s,t \geq 0$$

with $H \in (0,1)$.

Sub-fractional Brownian motion arises from occupation time fluctuations of branching particle systems (see [33]). It has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths, variation and renormalized variation and it is neither a Markov processes nor a semimartingale). Moreover, sub-fBm has non-stationary increments and the increments over non-overlapping intervals are more weakly correlated and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason, in [33] it is called sub-fBm). The above mentioned properties make sub-fBm a possible candidate for models which involve long-dependence, self-similarity and nonstationarity.

Remark 1.2 Trivially, for $H = \frac{1}{2}$ the sub-fBm reduces to the standard Brownian motion.

Proposition 1.14 *The process S^H is self-similar of order H.*

Proof Let $c > 0$. It is immediate that for every $s,t \geq 0$

$$R(ct, cs) = c^{2H} R(t,s)$$

holds and this implies the H-self-similarity of the process. □

The increments of the process S^H behave in the following way.

Proposition 1.15

$$\big(2 - 2^{2H-1}\big)|t-s|^{2H} \leq \mathbf{E}\big(S_t^H - S_s^H\big)^2 \leq |t-s|^{2H}, \quad \textit{if } H > 1/2$$

and

$$|t-s|^{2H} \le \mathbf{E}\big(S_t^H - S_s^H\big)^2 \le \big(2-2^{2H-1}\big)|t-s|^H, \quad \text{if } H < 1/2.$$

Consequently, the process S^H *has order continuous paths of order* $0 < \delta < H$.

Proof See [33] or [182]. □

This means that sub-fBm is, like bi-fBm, a quasi-helix.

Proposition 1.16 *For every* $s, t \ge 0$,

$$\mathbf{E}\big|S_t^H - S_s^H\big|^2 = -2^{2H-1}\big(t^{2H} + s^{2H}\big) + (t+s)^{2H} + (t-s)^{2H}$$

and in particular for every $t \ge 0$

$$\mathbf{E}\big(S_t^H\big)^2 = \big(2-2^{2H-1}\big)t^{2H}.$$

From Proposition 1.16 we deduce that sub-fBm is not a process with stationary increments.

Sub-fBm can also be defined in terms of the sum of the odd part and of the even part of a fractional Brownian motion on the whole real line. Actually, we have

Proposition 1.17 *Let* $(B_t^H)_{t\in\mathbb{R}}$ *be a fBmon the whole real line, that is, a centered Gaussian process with covariance*

$$\mathbf{E}B_t^H B_s^H = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \quad s, t \in \mathbb{R}.$$

Define for $t \ge 0$

$$S_t^H = \frac{1}{\sqrt{2}}\big(B_t^H + B_{-t}^H\big).$$

Then S^H *is a sub-fBm.*

Proof It suffices to compute the covariance of S^H and to verify that it coincides with the covariance of sub-fBm. □

See also Exercises 1.9 and 1.10 for other properties of subfractional Brownian motion.

1.4 Bibliographical Notes

The study of fractional Brownian motion has a long history. As mentioned earlier, this stochastic process was introduced in [102] and first analyzed in [117].

The original motivation to analyze this process came from empirical studies carried out by the hydrologist Hurst, published in [96], that attested the presence of long-range dependence in hydrology. The last two decades has seen intensive development with regard to the stochastic analysis of this process. Various types of stochastic integrals with respect to fBm have been introduced and various stochastic equations driven by this process have been considered. We refer to the monographs [31, 95, 121, 125, 136] and the references therein for a detailed exposition of this theory. Simultaneously with the development of the stochastic analysis for fBm, new fractional-type processes have been introduced and studied. Bifractional Brownian motion was defined in [90] and first analyzed in [159]. Subsequently, various properties of this stochastic process were revealed in, among other references, [8, 21, 26, 72, 77, 104, 109, 113, 177]. Sub-fractional Brownian motion first appeared as a limit of branching processes in [33] and has since been studied in many works, such as [32, 44, 182, 183, 186] and [151] among others. There exist other self-similar processes related to fractional Brownian motion. We refer, for example, to mixed fractional Brownian motion which has been used as a model in financial models (see [46]) or to multifractional Brownian motion (see e.g. [11]). Several examples of Gaussian self-similar processes related to fractional Brownian motion are presented in [32].

1.5 Exercises

Exercise 1.1 Let B^H be a fBm. Prove that for each $T > 0$ the following convergences hold in $L^2(\Omega)$.

$$\sum_{i=0}^{n-1} \left| B^H_{\frac{(i+1)T}{n}} - B^H_{\frac{iT}{n}} \right|^p \to 0 \quad \text{if } p > \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \left| B^H_{\frac{(i+1)T}{n}} - B^H_{\frac{iT}{n}} \right|^p \to \rho_{\frac{1}{H}} t \quad \text{if } p = \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \left| B^H_{\frac{(i+1)T}{n}} - B^H_{\frac{iT}{n}} \right|^p \to \infty \quad \text{if } p < \frac{1}{H}$$

with $\rho_p = \mathbf{E}|N(0, 1)|^p$.

Exercise 1.2 ([90]) Prove that the right-hand side of (1.13) is a covariance function.

Exercise 1.3 ([90], Proposition 2.3) Assume $(B_t^{K,H})_{t\geq 0}$ is a bi-fBm. For every $H \in (0,1)$ and $K \in (0,1]$,

$$\lim_{\varepsilon \to 0} \sup_{t \in [t_0-\varepsilon, t_0+\varepsilon]} \left| \frac{B_t^{H,K} - B_{t_0}^{H,K}}{t-t_0} \right| = +\infty$$

with probability one for every t_0. Deduce that the trajectories of the bi-fBm (and hence those of the fBm) are not differentiable.

Exercise 1.4 ([21]) Let $B^{H,K}$ be a bi-fBm with $H \in (0,1)$, $K \in (1,2)$ and $HK \in (0,1)$. Prove that for every s, t

$$2^{1-K}|t-s|^{2HK} \leq \mathbf{E}\big(B_t^{H,K} - B_s^{H,K}\big)^2 \leq |t-s|^{2HK} \quad \text{if } 0 < H \leq \frac{1}{2}$$

and

$$2^{1-K}|t-s|^{2HK} \leq \mathbf{E}\big(B_t^{H,K} - B_s^{H,K}\big)^2 \leq 2^{2-K}|t-s|^{2HK} \quad \text{if } H \geq \frac{1}{2}.$$

Exercise 1.5 ([21]) Let $B^{H,K}$ be a bi-fBm with $H \in (0,1)$, $K \in (1,2)$ and $HK \in (0,1)$. Prove that this process has short-memory if $HK < \frac{1}{2}$ and it has long memory if $HK > \frac{1}{2}$.

Exercise 1.6 ([21]) Let $B^{H,K}$ be a bi-fBm with $H \in (0,1)$, $K \in (1,2)$ and $HK \in (0,1)$. Prove that it is a semimartingale when $2HK = 1$.

Exercise 1.7 Let $B^{H,K}$ a bi-fBm. Prove that for each $T > 0$ the following convergences hold in $L^2(\Omega)$.

$$\sum_{i=0}^{n-1} \big|B^{H,K}_{\frac{(i+1)T}{n}} - B^{H,K}_{\frac{iT}{n}}\big|^p \to 0 \quad \text{if } p > \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \big|B^{H,K}_{\frac{(i+1)T}{n}} - B^{H,K}_{\frac{iT}{n}}\big|^p \to \rho_{\frac{1}{H}} t \quad \text{if } p = \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \big|B^{H,K}_{\frac{(i+1)T}{n}} - B^{H,K}_{\frac{iT}{n}}\big|^p \to \infty \quad \text{if } p < \frac{1}{H}$$

with $\rho_p = \mathbf{E}|N(0,1)|^p$. Deduce that the bi-fBm is not a semimartingale if $2HK \neq 1$.

Exercise 1.8 For every $K \in (0,1]$ and $H \in (0,1)$, the process $B^{H,K}$ is not a Markov process.

Hint The argument is the same as in the fBm case. Recall that (see [155]) a Gaussian process with covariance R is Markovian if and only if

$$R(s,u)R(t,t) = R(s,t)R(t,u)$$

for every $s \le t \le u$. It is straightforward to check that $B^{H,K}$ does not satisfy this condition.

Exercise 1.9 Let S^H be a sub-fBm and B^H be a fBm. Denote by R^{S^H} and R^{B^H} their covariance functions respectively. Prove that for every $s, t \ge 0$

$$R^{S^H}(t,s) > R^{B^H}(t,s) \quad \text{if } H < \frac{1}{2}$$

and

$$R^{S^H}(t,s) < R^{B^H}(t,s) \quad \text{if } H > \frac{1}{2}.$$

Exercise 1.10 Let S^H be a sub-fBm. Prove that for each $T > 0$ the following convergences hold in $L^2(\Omega)$.

$$\sum_{i=0}^{n-1} \left| S^H_{\frac{(i+1)T}{n}} - S^H_{\frac{iT}{n}} \right|^p \to 0 \quad \text{if } p > \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \left| S^H_{\frac{(i+1)T}{n}} - S^H_{\frac{iT}{n}} \right|^p \to \rho_{\frac{1}{H}} T \quad \text{if } p = \frac{1}{H}$$

and

$$\sum_{i=0}^{n-1} \left| S^H_{\frac{(i+1)T}{n}} - S^H_{\frac{iT}{n}} \right|^p \to \infty \quad \text{if } p < \frac{1}{H}$$

with $\rho_p = \mathbf{E}|N(0,1)|^p$.

Exercise 1.11 (See [163]) Define for $s < t$ and $n \ge 1$

$$K^n(t,s) := n \int_{s-\frac{1}{n}}^{s} K\left(\frac{[nt]}{n}, u\right) du$$

where K is the kernel of the fractional Brownian motion (1.3) and put

$$B^n_t = \int_0^t K^n(t,s)\, dW^n_s = \sum_{i=1}^{[nt]} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{[nt]}{n}, s\right) ds \frac{\xi_i}{\sqrt{n}}$$

where $[\cdot]$ denotes the integer part. Prove that the disturbed random walk B^n converges weakly, as $n \to \infty$, to the fractional Brownian motion in the Skorohod topology.

Exercise 1.12 Let $\mathcal{H}$ be the canonical Hilbert space associated to the fBm on $[0,T]$. Show that

$$\|\varphi\|_{\mathcal{H}}^2 \le b_H^2 t^{2H-1} \|\varphi\|_{L^2[0,T]}^2.$$

Exercise 1.13 ([51]) Let $\mathcal{H}$ be the canonical Hilbert space associated to the fBm with $H > \frac{1}{2}$. Let $f(x) = \cos(x)$ and $g(x) = \sin x$ for $x \in \mathbb{R}$. Then for every $a, b \in \mathbb{R}$, $a < b$

$$\begin{aligned}\|f 1_{(a,b)}\|_{\mathcal{H}}^2 = \alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v) \\ + \alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2} \sin(b-a-v)\end{aligned}$$

and

$$\begin{aligned}\|g 1_{(a,b)}\|_{\mathcal{H}}^2 = \alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v) \\ - \alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2} \sin(b-a-v).\end{aligned}$$

Exercise 1.14 ([51]) For every $a, b \in \mathbb{R}$ with $a < b$,

$$\int_a^b \int_a^b du dv \sin(u-v)|u-v|^{2H-2} = 0$$

for every $H > \frac{1}{2}$.

Exercise 1.15 ([14]) Let $\varphi(t) = \sin t$, $t \in [0,T]$ and denote by $\mathcal{H}(0,t)$ the canonical space of the fBm on $(0,t)$. Then show that

$$\|\varphi\|_{\mathcal{H}(0,T)}^2 = c_H \int_{\mathbb{R}} \frac{(\sin \tau T - \tau \sin T)^2 + (\cos \tau T - \cos T)^2}{(\tau^2-1)^2} |\tau|^{-(2H-1)} d\tau,$$

where $c_H = \Gamma(2H+1)\sin(\pi H)/(2\pi)$.

Exercise 1.16 Let B^{H_1}, B^{H_2} be two fractional Brownian motions with Hurst parameters H_1, H_2 respectively. We will assume that the self-similar parameters H_1 and H_2 are both bigger than $\frac{1}{2}$. We will also assume that the two fractional Brownian motions can be expressed as Wiener integrals with respect to the same Wiener process B as

$$\begin{aligned} B_t^{H_1} &= c(H_1) \int_{\mathbb{R}} dB_y \int_0^t (u-y)_+^{H_1-\frac{3}{2}} du, \\ B_t^{H_2} &= c(H_2) \int_{\mathbb{R}} dB_y \int_0^t (u-y)_+^{H_2-\frac{3}{2}} du \end{aligned} \tag{1.22}$$

where the constants $c(H_1), c(H_2)$ are such that $\mathbf{E}[(B_1^{H_1})^2] = \mathbf{E}[(B_1^{H_2})^2] = 1$.

1. Prove that

$$c(H_1)^2 = \frac{H_1(2H_1 - 1)}{\beta(2 - 2H_1, H_1 - \frac{1}{2})}. \tag{1.23}$$

2. Let $t > s$. Then show that

$$\mathbf{E}\big[\big(B_t^{H_1} - B_s^{H_1}\big)\big(B_t^{H_2} - B_s^{H_2}\big)\big] = b(H_1, H_2)|t - s|^{2H}$$

   where

$$b(H_1, H_2) = \frac{c(H_1)c(H_2)}{2H(2H-1)}\left(\beta\left(2 - 2H, H_1 - \frac{1}{2}\right) + \beta\left(2 - 2H, H_2 - \frac{1}{2}\right)\right)$$

   where $c(H_1), c(H_2)$ are given by (1.23).

Exercise 1.17 Another type of variation for a stochastic process has been defined by Russo and Vallois in [158]. These variations are mainly used in the context of stochastic calculus via regularization.

We will use the concept of α-*strong variation*: that is, we say that the continuous process X has an α-variation ($\alpha > 0$) if

$$ucp - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t |X_{s+\varepsilon} - X_s|^\alpha ds \quad \text{exists.} \tag{1.24}$$

Here ucp stands for the uniform limit in probability. The limit is denoted by $[\![X]\!]_t^{(\alpha)}$.

Let B be a fBm. Prove that for every $t \in [0, T]$

$$[\![B]\!]_t^{(p)} = 0 \quad \text{if } pH > 1$$

and

$$[\![B]\!]_t^{(p)} = \infty \quad \text{if } pH < 1.$$

Exercise 1.18 (See [159]) Let $(B_t^{H,K})_{t \in [0,T]}$ be a bifractional Brownian motion with parameters $H \in (0, 1)$ and $K \in (0, 1]$. Then

$$[\![B^{H,K}]\!]_t^{(\alpha)} = 0, \quad \text{if } \alpha > \frac{1}{HK}$$

and

$$[\![B^{H,K}]\!]_t^{(\alpha)} = 2^{\frac{1-K}{HK}} \rho_{HK} t \quad \text{if } \alpha = \frac{1}{HK},$$

where $\rho_{HK} = \mathbf{E}|N|^{1/HK}$, N being a standard normal random variable. Discuss the case $2HK = 1$.

Exercise 1.19 ([19]) Consider the family of stochastic processes $(\eta_\varepsilon)_{\varepsilon>0}$ defined by

$$\eta_\varepsilon(t) = \int_0^t K(t,s)\theta_\varepsilon(s)ds \tag{1.25}$$

where

$$\theta_\varepsilon(s) = \frac{1}{\varepsilon}(-1)^{N(\frac{s}{\varepsilon^2})}$$

(these are called the Stroock kernels) or

$$\theta_\varepsilon(s) = \frac{1}{\varepsilon}\sum_{k=1}^{\infty}\xi_k 1_{[k-1,k[}\left(\frac{s}{\varepsilon^2}\right)$$

(these are called the Donsker kernels) where $\xi_k, k \geq 1$ are independent with zero mean and variance one. Prove that the family η_ε converges weakly in the space $C_0(0,1)$ (the space of continuous functions on $[0,1]$ vanishing at zero) to the fBm.

Exercise 1.20 For every $\varepsilon > 0$, $H \in (0,1)$, $K \in (1,2)$ with $HK \in (0,1)$ and $t \in [0,T]$ define

$$B_\varepsilon^{H,K}(t) = \frac{2}{\varepsilon}\int_0^T K^{HK}(t,s)\sin\left(\theta N^{\frac{2}{\varepsilon^2}}\right)ds$$

and

$$X_\varepsilon^{H,K}(t) = \frac{2}{\varepsilon}\int_0^\infty \left(1-e^{-st^{2H}}\right)s^{-\frac{1+K}{2}}\cos\left(\theta N^{\frac{2}{\varepsilon^2}}\right)ds.$$

Then prove that the family of stochastic processes Y_ε given by

$$Y_\varepsilon(t) = aB_\varepsilon^{HK} + bX_\varepsilon^{H,K}$$

converges weakly in the space $C[0,T]$ (the space of continuous functions on $[0,T]$) to the extended bi-fBm.

Exercise 1.21 Let $X^{H,K}$ be the process defined in Proposition 1.9. Prove that, as $h \to \infty$

$$\mathbf{E}\left[\left(X_{h+t}^{H,K} - X_h^{H,K}\right)^2\right] = \Gamma(1-K)K^{-1}2^K H^2 K(1-K)t^2 h^{2(HK-1)}\left(1+o(1)\right).$$

Exercise 1.22 Let $B^{H,K}$ be a bi-fBm. Then show that

$$\left(B_{h+t}^{H,K} - B_h^{H,K}, t \geq 0\right) =^{(d)} \left(2^{(1-K)/2}B_t^{HK}, t \geq 0\right) \quad \text{as } h \to \infty,$$

where $=^{(d)}$ means convergence of all finite dimensional distributions.

Exercise 1.23 ([113]) Let us denote by

$$r(0,n) = \mathbf{E}\big[B_1^{H,K}\big(B_{n+1}^{H,K} - B_n^{H,K}\big)\big]$$

and for every $a \in \mathbb{N}$

$$r(a,a+n) = \mathbf{E}\big[\big(B_{a+1}^{H,K} - B_a^{H,K}\big)\big(B_{a+n+1}^{H,K} - B_{a+n}^{H,K}\big)\big]. \tag{1.26}$$

1. Show that for every $n \geq 1$

$$r(a,a+n) =: 2^{-K}\big(f_a(n) + g(n)\big), \tag{1.27}$$

   where

$$\begin{aligned} f_a(n) = {} & \big((a+1)^{2H} + (a+n+1)^{2H}\big)^K - \big((a+1)^{2H} + (a+n)^{2H}\big)^K \\ & - \big(a^{2H} + (a+n+1)^{2H}\big)^K + \big(a^{2H} + (a+n)^{2H}\big)^K \end{aligned}$$

   and for every $n \geq 1$

$$g(n) = (n+1)^{2HK} + (n-1)^{2HK} - 2n^{2HK}.$$

2. Show that:
   (i) The function g is, modulo a constant, the autocorrelation function of the fractional noise with Hurst index HK. Indeed, for $n \geq 1$

$$g(n) = 2\mathbf{E}\big[B_1^{HK}\big(B_{n+1}^{HK} - B_n^{HK}\big)\big].$$

   (ii) g vanishes if $2HK = 1$.
   (iii)

$$\begin{aligned} f_a(n) &= -2^K C_1^2 \mathbf{E}\big[\big(X_{a+1}^{H,K} - X_a^{H,K}\big)\big(X_{a+n+1}^{H,K} - X_{a+n}^{H,K}\big)\big] \\ &=: r^{X^{H,K}}(a,a+n) \end{aligned}$$

   for every a and n, where $X^{H,K}$ is given in Proposition 1.9.

3. Analyze the function f_a to understand "how far" bifractional Brownian noise is from "fractional Brownian noise". In other words, how far is bifractional Brownian motion from a process with stationary increments.
   Concretely, show that for each n the following holds as $a \to \infty$

$$f_a(n) = 2H^2K(K-1)a^{2(HK-1)}\big(1 + o(1)\big).$$

   Conclude that $\lim_{a\to\infty} f_a(n) = 0$ for each n.

Exercise 1.24 ([113]) For $a, n \geq 0$, let $r(a, a+n)$ be given by (1.26). Then prove that for large n

$$r(a, a+n) = 2^{-K}\big[2HK(2HK-1)n^{2(HK-1)} \\ + HK(K-1)\big((a+1)^{2H} - a^{2H}\big)n^{2(HK-1)+(1-2H)} + \cdots\big].$$

Deduce that for every $a \in \mathbb{N}$ we have

$$\sum_{n\geq 0} r(a, a+n) = \infty \quad \text{if } 2HK > 1$$

and

$$\sum_{n\geq 0} r(a, a+n) < \infty \quad \text{if } 2HK \leq 1.$$

Exercise 1.25 (See [44]) Let $0 < H < 1$ and define

$$X_t^H = \int_0^\infty \big(1 - e^{-\theta t}\big)\theta^{\frac{3}{2}-H} dW_\theta$$

where $(W_\theta)_{\theta\geq 0}$ is a Wiener process. Let B^H be a fBm independent from W. Prove that:

1. If $H < \frac{1}{2}$ the process

$$S_t^H = \sqrt{-\frac{H(2H-1)}{2\Gamma(2-2H)}} X_t^H + B_t^H$$

   is a sub-fBm.
2. If $H > \frac{1}{2}$ the process

$$S_t^H = \sqrt{\frac{H(2H-1)}{2\Gamma(2-2H)}} X_t^H + B_t^H$$

   is a sub-fBm.

Exercise 1.26 ([29]) Consider a fBm $(B_t^H)_{t\geq 0}$ with $H > \frac{1}{2}$ and let

$$Y_t = at + B_t^H$$

with $a \in \mathbb{R}$. Define

$$\hat{a}_N = N \frac{\sum_{i,j=1}^{N^\alpha} j\Gamma_{i,j}^{-1} Y_i}{\sum_{i,j=1}^{N^\alpha} ij\Gamma_{i,j}^{-1}}. \tag{1.28}$$

If $t_j = \frac{j}{N}$, we let $Y_{t_j} = Y_j$ and

$$\Gamma_{i,j} = \mathrm{Cov}\big(B^H_{\frac{i}{N}}, B^H_{\frac{j}{N}}\big).$$

1. Show that

$$\hat{a}_N - a = N \frac{\sum_{i,j=1}^{N^\alpha} j \Gamma^{-1}_{i,j} B^H_{\frac{i}{N}}}{\sum_{i,j=1}^{N^\alpha} ij \Gamma^{-1}_{i,j}}, \tag{1.29}$$

   where the $\Gamma^{-1}_{i,j}$ are the entries of the matrix Γ^{-1}.
2. Deduce from (1.29) that $\hat{a}_N$ converges to a almost surely and in L^p, $p \geq 1$. (Actually $\hat{a}_N$ is a consistent estimator for the drift parameter a.)

Chapter 2
Solutions to the Linear Stochastic Heat and Wave Equation

In this chapter we analyze the basic properties of some self-similar Gaussian processes that are solutions to stochastic partial differential equations with additive Gaussian noise. We will see that some of these processes are closely related to the fractional-type processes discussed in Chap. 1. The noise of the equation will be defined in various ways: white (meaning that is behaves as a Brownian motion) or corelared ("colored") in time and/or in space. The general context is as follows: consider the equation

$$Lu(t,x) = \Delta u(t,x) + \dot{W}(t,x) \tag{2.1}$$

with $t \in [0,T]$ and $x \in \mathbb{R}^d$ and with vanishing initial conditions. Here Δ is the Laplacian on $\mathbb{R}^d$

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2},$$

W is the noise of the equation and L is a first or second order operator with constant coefficients. In our analysis, we will consider the heat equation and then

$$Lu(t,x) = \frac{\partial u}{\partial t}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d$$

or the wave equation and in this case

$$Lu(t,x) = \frac{\partial^2 u}{\partial t^2}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d.$$

Usually, the solution to (2.1) is defined through its mild form

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\, dW(s,y)$$

where G is the solution of $Lu - \Delta u = 0$ and the above integral is a Wiener integral with respect to W. This Wiener integral can be understood in the sense of

C.A. Tudor, *Analysis of Variations for Self-similar Processes*, Probability and Its Applications, DOI 10.1007/978-3-319-00936-0_2,

Appendix C. Essentially the solution to (2.1) exists when this Wiener integral is well-defined and this happens when the integrand G belongs to the Hilbert space associated to the Gaussian noise W. In order to study the existence and the properties of the solution to (2.1), an important fact is the structure of the canonical Hilbert spaces associated with the noise and this depends on the covariance structure of the noise.

We denote by $C_0^\infty(\mathbb{R}^{d+1})$ the space of infinitely differentiable functions on $\mathbb{R}^{d+1}$ with compact support, and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ its dual. For $\varphi \in L^1(\mathbb{R}^d)$, we let $\mathcal{F}\varphi$ be the Fourier transform of φ:

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\varphi(x)dx.$$

2.1 The Solution to the Stochastic Heat Equation with Space-Time White Noise

We will first discuss the properties of the solution to the stochastic heat equation with additive Gaussian noise that behaves as a Wiener process both in time and in space.

2.1.1 The Noise

Let us first introduce the noise of the equation. Consider a centered Gaussian field $W = \{W(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance

$$\mathbf{E}W(t, A)W(s, B) = (t \wedge s)\lambda(A \cap B), \quad t \in [0, T], A \in \mathcal{B}_b\big(\mathbb{R}^d\big) \tag{2.2}$$

where λ denotes the Lebesgue measure. Also consider the stochastic partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + \dot{W}, \quad t \in [0, T],\ x \in \mathbb{R}^d \\ u(0, x) &= 0, \quad x \in \mathbb{R}^d, \end{aligned} \tag{2.3}$$

where the noise W is defined by (2.2). The noise W is usually referred to as a *space-time white noise* because it behaves as a Brownian motion both with respect to both the time and the space variable.

The canonical Hilbert space associated with the Gaussian process W is defined as the closure of the linear span generated by the indicator functions $1_{[0,t]\times A}$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ with respect to the inner product

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}} = (t \wedge s)\lambda(A \cap B).$$

In our case the space $\mathcal{H}$ is $L^2([0, T] \times \mathbb{R}^d)$.

2.1.2 The Solution

This mild solution is defined as

$$u(t,x)=\int_0^T\int_{\mathbb{R}^d} G(t-s,x-y)W(ds,dy),\quad t\in[0,T], x\in\mathbb{R}^d \tag{2.4}$$

where the above integral is a Wiener integral with respect to the Gaussian process W (see e.g. [13] for details) and G is the Green kernel of the heat equation given by

$$G(t,x)=\begin{cases}(2\pi t)^{-d/2}\exp(-\frac{|x|^2}{2t}) & \text{if } t>0, x\in\mathbb{R}^d\\ 0 & \text{if } t\le 0, x\in\mathbb{R}^d.\end{cases} \tag{2.5}$$

The Wiener integral in (2.4) is well-defined whenever the function $(s,y)\to G(t-s,x-y)$ belongs to $L^2([0,T]\times\mathbb{R}^d)$. As we will see in the sequel, this is not always the case and it depends on the spatial dimension d. Consequently the process $(u(t,x), t\in[0,T], x\in\mathbb{R})$, when it exists, is a centered Gaussian process. We also need the following expression of the Fourier transform of the Green kernel

$$\mathcal{F}G(t,\cdot)(\xi)=\exp\left(-\frac{t|\xi|^2}{2}\right),\quad t>0,\xi\in\mathbb{R}^d \tag{2.6}$$

where $\mathcal{F}G(t,\cdot)$ denotes the Fourier transform of the function $y\to G(t,y)$.

Proposition 2.1 *The solution* (2.4) *exists if and only if* $d=1$. *Moreover, the covariance of the solution* (2.4) *satisfies the following*: *for every* $x\in\mathbb{R}$ *we have*

$$\mathbf{E}\big(u(t,x)u(s,x)\big)=\frac{1}{\sqrt{2\pi}}(\sqrt{t+s}-\sqrt{|t-s|}),\quad \textit{for every } s,t\in[0,T]. \tag{2.7}$$

Proof Fix $x\in\mathbb{R}^d$. For every $s,t\in[0,T]$, using that for any $\varphi,\psi\in\mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\varphi(x)\psi(x)dx=(2\pi)^{-d}\int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}d\xi \tag{2.8}$$

we get

$$\begin{aligned}\mathbf{E}u(t,x)u(s,x)&=\int_0^{t\wedge s}du\int_{\mathbb{R}^d}G(t-u,x-y)^2dy\\&=(2\pi)^{-d}\int_0^s du\int_{\mathbb{R}^d}d\xi\mathcal{F}G(t-u,x-\cdot)(\xi)\overline{\mathcal{F}G(s-u,x-\cdot)}(\xi)\\&=(2\pi)^{-d}\int_0^s du\int_{\mathbb{R}^d}d\xi e^{-\frac{1}{2}(t-u)|\xi|^2}e^{-\frac{1}{2}(s-u)|\xi|^2}\end{aligned} \tag{2.9}$$

and then, if $s\le t$

$$\mathbf{E}u(t,x)u(s,x)=(2\pi)^{-d}\int_0^s du(t+s-2u)^{-\frac{d}{2}}\int_{\mathbb{R}^d}d\xi e^{-\frac{1}{2}|\xi|^2}$$

$$= (2\pi)^{-d/2} \int_0^s du(t+s-2u)^{-\frac{d}{2}}.$$

Take $t = s$. Then

$$\mathbf{E}u(t,x)^2 = (2\pi)^{-d/2} \int_0^t du(t-u)^{-\frac{d}{2}}$$

and it is obvious that the integral above is finite if and only if $d = 1$. In that case, from (2.9)

$$\mathbf{E}u(t,x)u(s,x) = (2\pi)^{-1/2}\big((t+s)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}\big). \qquad \square$$

This fact establishes an interesting connection between the law of the solution (2.4) and the bifractional Brownian motion from Sect. 1.2.

Corollary 2.1 *Let $(u(t,x), t \in [0,T], x \in \mathbb{R}^d)$ be given by (2.4). Then for every $x \in \mathbb{R}$*

$$\big(u(t,x), t \in [0,T]\big) =^{(d)} \big(\sqrt{C} B_t^{\frac{1}{2},\frac{1}{2}}, t \in [0,T]\big)$$

where $B^{\frac{1}{2},\frac{1}{2}}$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$ and $C := 2^{-K} \frac{1}{\sqrt{2\pi}}$. Here $=^{(d)}$ means equivalence of finite dimensional distributions.

Proof The assertion follows from relation (2.7) and Definition 1.2. $\square$

Remark 2.1 From (2.7), it follows that the stochastic process defined by (2.4) is self-similar of order $\frac{1}{4}$ with respect to the variable t.

2.2 The Spatial Covariance

The restriction $d = 1$ for the existence of the solution with space-time white noise is not convenient because we need to consider such models in higher dimensions. This has led researchers in the last few decades to investigate other types of noise that would allow such consideration of higher dimensions.

We begin by introducing the framework. Let μ be a non-negative tempered measure on $\mathbb{R}^d$, i.e. a non-negative measure which satisfies:

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^l \mu(d\xi) < \infty, \quad \text{for some } l > 0.$$

Since the integrand is non-increasing in l, we may assume that $l \geq 1$ is an integer. Note that $1 + |\xi|^2$ behaves as a constant near 0, and as $|\xi|^2$ at ∞, and hence (2.10) is equivalent to:

$$\int_{|\xi|\le 1} \mu(d\xi) < \infty, \quad \text{and} \quad \int_{|\xi|\ge 1} \mu(d\xi)\frac{1}{|\xi|^{2l}} < \infty, \quad \text{for some integer } l \ge 1. \tag{2.10}$$

Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be the Fourier transform of μ in $\mathcal{S}'(\mathbb{R}^d)$, i.e.

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi), \quad \forall \varphi \in \mathcal{S}\big(\mathbb{R}^d\big).$$

Simple properties of the Fourier transform show that for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x) f(x-y)\psi(y)dxdy = (2\pi)^{-d}\int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi). \tag{2.11}$$

2.3 The Solution to the Linear Heat Equation with White-Colored Noise

2.3.1 The Noise

Consider the so-called *white-colored noise*, meaning a Gaussian process $W = \{W(t,A); t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with zero mean and covariance

$$\mathbf{E}W(t,A)W(t,B) = (t \wedge s)\int_A\int_B f(x-y)dxdy. \tag{2.12}$$

The noise W behaves as a Brownian motion with respect to the time variable and it has a correlated spatial covariance. Here the kernel f should be the Fourier transform of a tempered non-negative measure μ on $\mathbb{R}^d$ as described in the previous paragraph.

Under this assumption the right-hand side of (2.12) is a covariance function. There are several examples of such kernels f.

Example 2.1 The Riesz kernel of order α:

$$f(x) = R_\alpha(x) := \gamma_{\alpha,d}|x|^{-d+\alpha}, \quad 0 < \alpha < d,$$

where $\gamma_{\alpha,d} = 2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = |\xi|^{-\alpha}d\xi$.

Example 2.2 The Bessel kernel of order α:

$$f(x) = B_\alpha(x) := \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}e^{-|x|^2/(4w)}dw, \quad \alpha > 0,$$

where $\gamma'_\alpha = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2}d\xi$.

Example 2.3 The Poisson kernel

$$f(x) = P_\alpha(x) := \gamma'''_{\alpha,d}\big(|x|^2 + \alpha^2\big)^{-(d+1)/2}, \quad \alpha > 0,$$

where $\gamma'''_{\alpha,d} = \pi^{-(d+1)/2}\Gamma((d+1)/2)\alpha$. In this case, $\mu(d\xi) = e^{-4\pi^2\alpha|\xi|}d\xi$.

Example 2.4 The heat kernel

$$f(x) = G_\alpha(x) := \gamma''_{\alpha,d}e^{-|x|^2/(4\alpha)}, \quad \alpha > 0,$$

where $\gamma''_{\alpha,d} = (4\pi\alpha)^{-d/2}$. In this case, $\mu(d\xi) = e^{-\pi^2\alpha|\xi|^2}d\xi$.

With the Gaussian process W we can associated a canonical Hilbert space $\mathcal{P}$. The space $\mathcal{P}$ defined as the completion of $\mathcal{D}((0,T)\times\mathbb{R}^d)$ (or the completion of $\mathcal{E}$, the linear space generated by the indicator functions $1_{[0,t]\times A}$, $t \in [0,T]$, $A \subset \mathcal{B}(\mathbb{R}^d)$) with respect to the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{P}} = \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(t,x)f(x-y)\psi(t,y)dydxdt$$

has been studied by several authors in connection with a Gaussian noise which is white in time and colored in space. In particular this space may contain distributions.

2.3.2 The Solution

The solution is defined again by (2.4) with W given by (2.12). The necessary and sufficient condition for (2.4) to exist has been proven in [59].

Proposition 2.2 *The stochastic heat equation with white-colored noise given by* (2.12) *admits a unique solution if and only if*

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty.$$

Proof For every $t \in [0,T]$ and $x \in \mathbb{R}^d$, using (2.6) and (2.11)

$$\begin{aligned}
\mathbf{E}u(t,x)^2 &= \int_0^t du \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-u,x-y)G\big(t-u,x-y'\big)f\big(y-y'\big)dydy' \\
&= (2\pi)^{-d}\int_0^t du \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}G(t-u,x-\cdot)(\xi)\overline{\mathcal{F}G(t-u,x-\cdot)}(\xi) \\
&= (2\pi)^{-d}\int_0^t du \int_{\mathbb{R}^d} \mu(d\xi)e^{-\frac{1}{2}(t-u)|\xi|^2}e^{-\frac{1}{2}(t-u)|\xi|^2}
\end{aligned}$$

$$= (2\pi)^{-d} \int_0^t du \int_{\mathbb{R}^d} \mu(d\xi) e^{-(t-u)|\xi|^2}$$
$$= (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(d\xi) \frac{1}{|\xi|^2} \big(1 - e^{-t|\xi|^2}\big).$$

One can prove that

$$c_{1,t} \frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2} \big(1 - e^{-t|\xi|^2}\big) \leq c_{2,t} \frac{1}{1+|\xi|^2}$$

with $c_{1,t}, c_{2,t}$ strictly positive constants that may be dependent on t. It can also be checked that the Green kernel belongs to the space $\mathcal{P}$ and the desired result is obtained. □

Remark 2.2 It has been proved in [59] that even in the non-linear case the stochastic heat equation $u_t = \frac{1}{2}\Delta u + g(u)\dot{W}$ (with standard assumptions on g) with white-colored noise admits a unique solution if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right) \mu(d\xi) < \infty.$$

Obviously, this condition is also meaningful in higher dimensions. For example in the case of the Riesz or Bessel kernels, we have the following.

Corollary 2.2 *Suppose that the spatial covariance is given by the Riesz kernel* (*Example* 2.1) *or by the Bessel kernel* (*Example* 2.2). *Then the stochastic heat equation with white-colored noise admits a unique solution if and only if*

$$d < 2 + \alpha.$$

This implies that one can consider every dimension $d \geq 1$.

It is possible to compute the covariance of the solution with respect to the time variable; actually for fixed $x \in \mathbb{R}^d$, $d \neq 2$ and for every $s \leq t$ we have

$$\begin{aligned}
&\mathbf{E}u(t,x)u(s,x) \\
&= \int_0^{t \wedge s} du \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-u, x-y) G\big(s-u, x-y'\big) f\big(y-y'\big) dy dy' \\
&= (2\pi)^{-d} \int_0^s du \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}G(t-u, x-\cdot)(\xi) \overline{\mathcal{F}G(s-u, x-\cdot)}(\xi) \\
&= (2\pi)^{-d} \int_0^s du \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{1}{2}(t-u)|\xi|^2} e^{-\frac{1}{2}(s-u)|\xi|^2}. \qquad (2.13)
\end{aligned}$$

For the Riesz kernel, this gives

Proposition 2.3 *Suppose we are in the case of the Riesz kernel f of order α (see Example 2.1). Then for every $x \in \mathbb{R}^d$ and for every $s, t \in [0, T]$*

$$\mathbf{E}u(t,x)u(s,x) = C_0^2\left((t+s)^{-\frac{d-\alpha}{2}+1} - (t-s)^{-\frac{d-\alpha}{2}+1}\right)$$

where

$$C_0 = \left[(2\pi)^{-d}\int_{\mathbb{R}^d}\mu(d\xi)e^{-\frac{1}{2}|\xi|^2}\frac{1}{-\frac{d-\alpha}{2}+1}2^{-K}\right]^{\frac{1}{2}}. \tag{2.14}$$

Proof Consider $s \le t$. From (2.13), by the change of variables $\tilde{\xi} = \sqrt{t+s-2u}\xi$

$$\begin{aligned}\mathbf{E}u(t,x)u(s,x) &= (2\pi)^{-d}\int_0^s du(t+s-2u)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\mu(d\xi)e^{-\frac{1}{2}|\xi|^2}\\ &= (2\pi)^{-d}\int_{\mathbb{R}^d}\mu(d\xi)e^{-\frac{1}{2}|\xi|^2}\frac{1}{-\frac{d-\alpha}{2}+1}\\ &\quad\times\left((t+s)^{-\frac{d-\alpha}{2}+1} - (t-s)^{-\frac{d-\alpha}{2}+1}\right).\end{aligned}$$ □

As a consequence, in the case of the spatial covariance given by the Riesz kernel, the solution of the heat equation with white noise in time coincides in distribution with, modulo a constant, a bifractional Brownian motion.

Corollary 2.3 *For fixed $x \in \mathbb{R}^d$, the solution to the white-colored heat equation coincides in distribution with*

$$\left(C_0 B_t^{H,K}\right)_{t\in[0,T]}$$

where $B^{H,K}$ is a bifractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{d-\alpha}{2}$ and C_0 is defined in (2.14).

Proof This follows from Proposition 2.3 and the expression of the covariance of the bi-fBm in Definition 1.2. □

Remark 2.3 In the case $\alpha = 0$ and $d = 1$ (corresponding to the space-time white noise case) we retrieve the formula (2.7) because $\int_{\mathbb{R}}\mu(d\xi)e^{-\frac{1}{2}|\xi|^2} = \sqrt{2\pi}$.

Corollary 2.4 *The solution of the heat equation with additive white-colored noise and with the spatial covariance given by the Riesz kernel of order α is self-similar of order $\frac{1}{2}(1 - \frac{d-\alpha}{2})$.*

Proof This is a consequence of Corollary 2.3 and of the self-similarity property of the bi-fBm (Proposition 1.6). □

Remark 2.4 Note that $1 - \frac{d-\alpha}{2} > 0$ because $d < \alpha + 2$ and $1 - \frac{d-\alpha}{2} < 1$ because $\alpha < d$. When $\alpha = 0$ and $d = 1$ (the space-time white noise case), the self-similarity order is $\frac{1}{4}$.

2.4 The Solution to the Fractional-White Heat Equation

In the sequel, the driving noise of the equation will behave as a fractional Brownian motion with respect to its time variable.

2.4.1 The Noise

On a complete probability space $(\Omega, \mathcal{F}, P)$, we consider a zero-mean Gaussian process $W^H = \{W^H(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbf{E}(W^H(t, A)W^H(s, B) = R_H(t, s)\lambda(A \cap B) =: \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}} \tag{2.15}$$

where λ is the Lebesgue measure. This noise is usually called “fractional-white” because it behaves as a fBm in time and as a Wiener process (“white”) in space.

We will assume throughout that the Hurst parameter H is contained in the interval $(\frac{1}{2}, 1)$.

We introduce now the canonical Hilbert space associated to the noise. Let $\mathcal{E}$ be the set of linear combinations of elementary functions $1_{[0,t]\times A}$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, and $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$.

We have, for $f, h \in \mathcal{H}$ smooth enough

$$\langle f, g\rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T dudv \int_{\mathbb{R}^d} dy|u - v|^{2H-2} f(y, u)g(y, v) \tag{2.16}$$

where $\alpha_H = H(2H - 1)$.

The map $1_{[0,t]\times A} \mapsto W_t^H(A)$ is an isometry between $\mathcal{E}$ and the Gaussian space of W^H, which can be extended by density to $\mathcal{H}$. We denote this extension by:

$$\varphi \mapsto W(\varphi) = \int_0^T \int_{\mathbb{R}^d} \varphi(t, x)W^H(dt, dx).$$

The above integral is a Wiener integral with respect to the Gaussian process W^H. This Wiener integral can be expressed as a Wiener integral with respect to the space-time white noise W which has a covariance given by (2.15). Actually, we will use the following transfer formula (see [112]).

Proposition 2.4 *If $f \in \mathcal{H}$ then*

$$\int_0^T \int_{\mathbb{R}^d} f(s,y)dW^H(s,y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} 1_{(0,T)}(u) f(u,x)(u-s)_+^{H-\frac{3}{2}} du \right) dW(s,y) \tag{2.17}$$

where W is a space-time white noise with covariance (2.2).

The representation (2.17) is obtained using the moving average expression of the fractional Brownian motion (1.7). See also Sect. 3.1.3 in the next chapter. Notice that a similar transfer formula can be written using the representation of the fractional Brownian motion as a Wiener integral on a finite interval (see e.g. [136]).

2.4.2 The Solution

Let us consider the linear stochastic heat equation

$$u_t = \frac{1}{2}\Delta u + \dot{W}^H, \quad t \in [0,T], x \in \mathbb{R}^d \tag{2.18}$$

with $u(.,0) = 0$, where $(W^H(t,x))_{t\in[0,T], x\in\mathbb{R}^d}$ is a centered Gaussian noise with covariance (2.15). The solution of (2.18) can be written in mild form as

$$U(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W^H(ds, dy), \quad t \in [0,T], x \in \mathbb{R}^d \tag{2.19}$$

where the above integral is a Wiener integral with respect to the noise W^H and G is given by (2.5).

Theorem 2.1 *The process $(U(t,x))_{t\in[0,T], x\in\mathbb{R}^d}$ exists and satisfies*

$$\sup_{t\in[0,T], x\in\mathbb{R}^d} \mathbf{E}\big(U(t,x)^2\big) < +\infty$$

if and only if $d < 4H$.

Proof We have, as in the case of white noise, using (2.16) and using the expression of the Fourier transform of the Green kernel (2.6),

$$\begin{aligned}
\mathbf{E}|U(t,x)|^2 &= (2\pi)^{-d}\alpha_H \int_0^t \int_0^t dudv|u-v|^{2H-2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi|^2(2t-u-v)} \\
&= (2\pi)^{-d/2}\alpha_H \int_0^t \int_0^t dudv|u-v|^{2H-2}(2t-u-v)^{-\frac{d}{2}} \\
&= (2\pi)^{-d/2}\alpha_H \int_0^t \int_0^t dudv|u-v|^{2H-2}(u+v)^{-\frac{d}{2}}
\end{aligned}$$

and the last integral is finite if and only if $2H > \frac{d}{2}$. □

Remark 2.5 This implies that, in contrast to the white-noise case, we are allowed to consider the spatial dimension d to be 1, 2 or 3.

Suppose that $s, t \in [0, T]$ and let

$$R(t, s) = \mathbf{E}\big(U(t, x)U(s, x)\big)$$

where $x \in \mathbb{R}^d$ is fixed. We will see that R does not depend on x.

Proposition 2.5 *For* $s, t \in [0, T]$

$$R(t, s) = \alpha_H (2\pi)^{-d/2} \int_0^t \int_0^s |u - v|^{2H-2} \big((t + s) - (u + v)\big)^{-\frac{d}{2}} dv du. \tag{2.20}$$

Proof The following holds

$$\begin{aligned} R(t, s) &= (2\pi)^{-d} \alpha_H \int_0^t \int_0^s du dv |u - v|^{2H-2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi|^2(t+s-u-v)} \\ &= \alpha_H (2\pi)^{-d/2} \int_0^t \int_0^s |u - v|^{2H-2} \big((t + s) - (u + v)\big)^{-\frac{d}{2}} dv du. \end{aligned}$$ □

Proposition 2.6 *The process* U *is self-similar* (*with respect to* t) *of order* $H - \frac{d}{4}$.

Proof This is an immediate consequence of relation (2.20). Indeed, for every $c > 0$,

$$\begin{aligned} R(ct, cs) &= \alpha_H (2\pi)^{-d/2} \int_0^{ct} \int_0^{cs} |u - v|^{2H-2} \big((ct + cs) - (u + v)\big)^{-\frac{d}{2}} dv du \\ &= c^{2H-\frac{d}{2}} R(t, s) \end{aligned}$$

by the change of variables $\tilde{u} = \frac{u}{c}, \tilde{v} = \frac{v}{c}$. □

In this part we will focus our attention on the behavior of the increments of the solution $U(t, x)$ to (2.18) with respect to the variable t. We will give sharp upper and lower bounds for the L^2-norm of this increment. We will assume in the sequel that $T = 1$. Concretely, we prove the following result.

Theorem 2.2 *There exists two strictly positive constants* C_1, C_2 *such that for any* $t, s \in [0, 1]$ *and for any* $x \in \mathbb{R}^d$

$$C_1 |t - s|^{2H-\frac{d}{2}} \le \mathbf{E}\big|U(t, x) - U(s, x)\big|^2 \le C_2 |t - s|^{2H-\frac{d}{2}}. \tag{2.21}$$

Proof By $c, c(H)\dots$ we will denote generic constants. We can write, for every $x \in \mathbb{R}^d$, $s, t \in [0, 1]$

$$\begin{aligned}
&\mathbf{E}\big|U(t,x) - U(s,x)\big|^2 \\
&= R(t,t) - 2R(t,s) + R(s,s) \\
&= c(H)\int_0^s\int_0^s dvdu|u-v|^{2H-2}\big[\big(2t-(u+v)\big)^{-\frac{d}{2}} \\
&\quad - 2\big((t+s)-(u+v)\big)^{-\frac{d}{2}} + \big(2s-(u+v)\big)^{-\frac{d}{2}}\big] \\
&\quad + \int_s^t\int_s^t dvdu|u-v|^{2H-2}\big(2t-(u+v)\big)^{-\frac{d}{2}} \\
&\quad - 2\int_s^t du\int_0^s dv|u-v|^{2H-2}\big[\big((t+s)-(u+v)\big)^{-\frac{d}{2}} - \big(2t-(u+v)\big)^{-\frac{d}{2}}\big] \\
&= A + B - C.
\end{aligned}$$

Since the term C is positive, we clearly have

$$\mathbf{E}\big|U(t,x) - U(s,x)\big|^2 \le A + B.$$

The term B can easily be estimated. Indeed, by the change of variables $\tilde{u} = s - u$, $\tilde{v} = v - s$ and then $\tilde{u} = \frac{u}{t-s}$, $\tilde{v} = \frac{v}{t-s}$,

$$B = c(H)(t-s)^{2H-\frac{d}{2}}. \tag{2.22}$$

Let us now consider the term A. By the change of variables $\tilde{u} = s - u$, $\tilde{v} = v - s$ and then $\tilde{u} = \frac{u}{t-s}$, $\tilde{v} = \frac{v}{t-s}$ we have

$$\begin{aligned}
A &= \int_0^s\int_0^s dudv|u-v|^{2H-2}\big[(2t-2s+u+v)^{-\frac{d}{2}} \\
&\quad - 2(t-s+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\big] \\
&= (t-s)^{2H-\frac{d}{2}}\int_0^{\frac{s}{t-s}}\int_0^{\frac{s}{t-s}} dudv|u-v|^{2H-2}\big[(2+u+v)^{-\frac{d}{2}} \\
&\quad - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\big] \\
&\le (t-s)^{2H-\frac{d}{2}}\int_0^{\infty}\int_0^{\infty} dudv|u-v|^{2H-2}\big[(2+u+v)^{-\frac{d}{2}} \\
&\quad - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\big].
\end{aligned} \tag{2.23}$$

Note that the integral $\int_0^\infty\int_0^\infty dudv|u-v|^{2H-2}[(2+u+v)^{-\frac{d}{2}} - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}]$ is finite: it is finite for u, v close to zero since $2H - \frac{d}{2} > 0$ and it is also finite for u, v close to infinity because

$$\left[(2+u+v)^{-\frac{d}{2}} - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\right] \leq c(u+v)^{-\frac{d}{2}-2}$$

(this can be seen by analyzing the asymptotic behavior of the function $(2+x)^{-\frac{d}{2}} - 2(1+x)^{-\frac{d}{2}} + x^{-\frac{d}{2}}$). By (2.22) and (2.23) we obtain the right-hand side of (2.21).

Let us now consider the lower bound. Using the Wiener integral representation (2.19) of the solution $U(t,x)$, we can write, for every $x \in \mathbb{R}^d$

$$U(t,x) - U(s,y) = \int_0^1 \int_{\mathbb{R}^d} \big(G(t-a, x-y) 1_{(0,t)}(a) \\ - G(s-a, x-y) 1_{(0,s)}(a)\big) dW^H(s,y)$$

and by the transfer rule (2.17)

$$U(t,x) - U(s,y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} dW(a,y) \bigg(\int_{\mathbb{R}} du G(t-u, x-y) 1_{(0,t)}(u)(u-a)_+^{H-\frac{3}{2}} \\ - \int_{\mathbb{R}} du G(s-u, x-y) 1_{(0,s)}(u)(u-a)_+^{H-\frac{3}{2}} \bigg)$$

where W is a space time white noise given by (2.2).

Now, by the isometry of the Brownian motion W we get

$$\mathbf{E}\big|U(t,x) - U(s,x)\big|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^d} dady \bigg(\int_{\mathbb{R}} du G(t-u, x-y) 1_{(0,t)}(u)(u-a)_+^{H-\frac{3}{2}} \\ - \int_{\mathbb{R}} du G(s-u, x-y) 1_{(0,s)}(u)(u-a)_+^{H-\frac{3}{2}} \bigg)^2 \\ \geq \int_s^t \int_{\mathbb{R}^d} dady \bigg(\int_{\mathbb{R}} du G(t-u, x-y) 1_{(0,t)}(u)(u-a)_+^{H-\frac{3}{2}} \\ - \int_{\mathbb{R}} du G(s-u, x-y) 1_{(0,s)}(u)(u-a)_+^{H-\frac{3}{2}} \bigg)^2 \\ = \int_s^t da \int_{\mathbb{R}^d} dy \bigg(\int_a^t du G(t-u, x-y)(u-a)^{H-\frac{3}{2}} \bigg)^2$$

because the part on the interval $(0,s)$ vanishes. By interchanging the order of integration,

$$\mathbf{E}\big|U(t,x) - U(s,x)\big|^2 \\ \geq \int_s^t da \int_{\mathbb{R}^d} dy \int_a^t \int_a^t dvdu \\ \times G(t-u, x-y)(u-a)^{H-\frac{3}{2}} G(t-v, x-y)(v-a)^{H-\frac{3}{2}}$$

$$= \int_s^t du \int_s^t dv \int_{\mathbb{R}^d} dy G(t-u, x-y) G(t-v, x-y) \\ \times \int_s^{u\wedge v} (u-a)^{H-\frac{3}{2}} (v-a)^{H-\frac{3}{2}} da.$$

We recall that (see e.g. [13]), for every $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} dy G(t-u, x-y) G(t-v, x-y) = c\big(2t-(u+v)\big)^{-\frac{d}{2}} \tag{2.24}$$

and, when $v < u$, by the change of variable $z = \frac{v-a}{u-a}$, we have

$$\int_s^{u\wedge v} da (u-a)^{H-\frac{3}{2}} (v-a)^{H-\frac{3}{2}} = (u-v)^{2H-2} \int_0^{\frac{v-s}{u-s}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz. \tag{2.25}$$

Therefore, by (2.24) and (2.25)

$$\begin{aligned} &\mathbf{E}\big|U(t,x) - U(s,x)\big|^2 \\ &\geq \int_s^t du \int_s^t dv \big(2t-(u+v)\big)^{-\frac{d}{2}} |u-v|^{2H-2} \int_0^{\frac{(v-s)\wedge(u-s)}{(u-s)\vee(v-s)}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz \\ &= \int_0^{t-s} du dv (u+v)^{-\frac{d}{2}} |u-v|^{2H-2} \int_0^{\frac{v\wedge u}{u\vee v}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz \\ &= (t-s)^{2H-\frac{d}{2}} \int_0^1 \int_0^1 du dv (u+v)^{-\frac{d}{2}} |u-v|^{2H-2} \int_0^{\frac{v\wedge u}{u\vee v}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz \\ &= C(t-s)^{2H-\frac{d}{2}}, \end{aligned}$$

where in the third and fourth lines we used successively the change of variables $u-s = \tilde{u}$, $v-s = \tilde{v}$ and $\frac{u}{t-s} = \tilde{u}$, $\frac{v}{t-s} = \tilde{v}$. The proof of the lower bound follows since the integral $\int_0^1 \int_0^1 du dv (u+v)^{-\frac{d}{2}} |u-v|^{2H-2} \int_0^{\frac{v\wedge u}{u\vee v}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz$ is clearly finite when $H > \frac{1}{2}$. □

Remark 2.6 The above result implies that the process U is Hölder continuous of order $H - \frac{d}{4}$ in time (this coincides with the self-similarity order, see Proposition 2.6). This extends the case of the space-time white noise in dimension $d = 1$ (recall that the solution of the heat equation with space-time time white noise is Hölder continuous of order $\frac{1}{4}$). Note also that in the case $d = 1$ the upper bound has also been obtained in [108] or [36].

2.4.3 On the Law of the Solution

Consider the process U given by (2.19). Suppose that $s \leq t$ and recall the notation

$$R(t,s) = \mathbf{E}\big(U(t,x)U(s,x)\big)$$

where $x \in \mathbb{R}^d$ is fixed. Also recall the formula (2.20)

$$R(t,s) = \alpha_H (2\pi)^{-\frac{d}{2}} \int_0^t \int_0^s |u-v|^{2H-2} \big((t+s)-(u+v)\big)^{-\frac{d}{2}} dv du$$

with $\alpha_H = H(2H-1)$.

The purpose of this section is to analyze the covariance of the solution $U(t,x)$ and to understand its relation with bifractional Brownian motion. Corollaries 2.1 and 2.3 say that, when the noise is white in time, the solution coincides in distribution with a bi-fBm. Proposition 2.2 shows that its increments have a similar behavior as those of the bi-fBm. But we will see that the situation is different if the noise is no longer white in time.

The following proposition gives a decomposition of the covariance function of $U(t,.)$ in the case $d \neq 2$ i.e. $d=1$ or $d=3$ since the solution exists for $d < 4H$. The lines of the below proof will explain why the case $d=2$ has to be excluded.

Proposition 2.7 *Suppose $d \neq 2$. The covariance function $R(t,s)$ can be decomposed as follows*

$$R(t,s) = (2\pi)^{-\frac{d}{2}} \alpha_H C_d \beta\left(2H-1, -\frac{d}{2}+2\right)\left[(t+s)^{2H-\frac{d}{2}} - (t-s)^{2H-\frac{d}{2}}\right] + R_1^{(d)}(t,s)$$

where $C_d = \frac{2}{2-d}$, $\beta(x,y)$ is the Beta function defined for $x,y>0$ by $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ and

$$R_1^{(d)}(t,s) = (2\pi)^{-\frac{d}{2}} \alpha_H C_d \left[\int_0^s da a^{2H-2}\left[\big((t+s)-a\big)^{-\frac{d}{2}+1} - \big((t-s)+a\big)^{-\frac{d}{2}+1}\right] - \int_0^s da (s-a)^{-\frac{d}{2}+1}\left[(t-a)^{2H-2} + (t+a)^{2H-2}\right]\right].$$

Proof Fix $t > s$. By performing the change of variables $u-v=a$ and $u+v=b$ with $a+b=2u \in (0,2t)$ and $b-a=2v \in (0,2s)$ in (2.20), we get

$$\begin{aligned} R(t,s) &= (2\pi)^{-\frac{d}{2}} \alpha_H \int_{-s}^t |a|^{2H-2} \int_{a\vee(-a)}^{(2t-a)\wedge(2s+a)} \big((t+s)-b\big)^{-\frac{d}{2}} dbda \\ &= (2\pi)^{-\frac{d}{2}} \alpha_H \left[\int_{-s}^0 (-a)^{2H-2} \int_{-a}^{2s+a} \big((t+s)-b\big)^{-\frac{d}{2}} dbda\right. \end{aligned}$$

$$+\int_0^{t-s} a^{2H-2} \int_a^{2s+a} \big((t+s)-b\big)^{-\frac{d}{2}} dbda$$
$$+\int_{t-s}^{t} a^{2H-2} \int_a^{2t-a} \big((t+s)-b\big)^{-\frac{d}{2}} dbda\Bigg].$$

By performing the change of variables $a \mapsto (-a)$ in the first summand, we get

$$R(t,s) = (2\pi)^{-\frac{d}{2}} \alpha_H \Bigg[\int_0^{s} a^{2H-2} \int_a^{2s-a} \big((t+s)-b\big)^{-\frac{d}{2}} dbda$$
$$+\int_0^{t-s} a^{2H-2} \int_a^{2s+a} \big((t+s)-b\big)^{-\frac{d}{2}} dbda$$
$$+\int_{t-s}^{t} a^{2H-2} \int_a^{2t-a} \big((t+s)-b\big)^{-\frac{d}{2}} dbda\Bigg]. \qquad \square$$

Remark 2.7 We can see why the case $d = 2$ must be treated separately in the latter equation. The integral with respect to db involves logarithms and it cannot lead to the covariance of the bifractional Brownian motion.

By explicitly computing the inner integrals, we obtain

$$R(t,s) = (2\pi)^{-\frac{d}{2}} \alpha_H C_d \Bigg[\int_0^{s} a^{2H-2} \Big[-\big((t+s)-b\big)^{-\frac{d}{2}+1}\Big]_{b=a}^{b=2s-a} da$$
$$+\int_0^{t-s} a^{2H-2} \Big[-\big((t+s)-b\big)^{-\frac{d}{2}+1}\Big]_{b=a}^{b=2s+a} da$$
$$+\int_{t-s}^{t} a^{2H-2} \Big[-\big((t+s)-b\big)^{-\frac{d}{2}+1}\Big]_{b=a}^{b=2t-a} da\Bigg]$$
$$= (2\pi)^{-\frac{d}{2}} \alpha_H C_d \Bigg[\int_0^{s} a^{2H-2} \big((t+s)-a\big)^{-\frac{d}{2}+1} da$$
$$-\int_0^{s} a^{2H-2} \big((t-s)+a\big)^{-\frac{d}{2}+1} da\Bigg]$$
$$+\alpha_H C_d \Bigg[\int_0^{t-s} a^{2H-2} \big((t+s)-a\big)^{-\frac{d}{2}+1} da$$
$$-\int_0^{t-s} a^{2H-2} \big((t-s)-a\big)^{-\frac{d}{2}+1} da\Bigg]$$
$$+\alpha_H C_d \Bigg[\int_{t-s}^{t} a^{2H-2} \big((t+s)-a\big)^{-\frac{d}{2}+1} da$$
$$-\int_{t-s}^{t} a^{2H-2} \big(a-(t-s)\big)^{-\frac{d}{2}+1} da\Bigg]$$

$$= (2\pi)^{-\frac{d}{2}}\alpha_H C_d\left[\int_0^{t+s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da \right.$$
$$\left. - \int_0^{t-s} a^{2H-2}\big((t-s)-a\big)^{-\frac{d}{2}+1}da\right]$$
$$+ R_1^{(d)}(t,s)$$

where

$$\begin{aligned} R_1^{(d)}&(t,s)\\ &= (2\pi)^{-\frac{d}{2}}\alpha_H C_d\left[\int_0^{s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da\right.\\ &- \int_0^{s} a^{2H-2}\big((t-s)+a\big)^{-\frac{d}{2}+1}da\\ &\left.- \int_{t-s}^{t} a^{2H-2}\big(a-(t-s)\big)^{-\frac{d}{2}+1}da - \int_t^{t+s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da\right]. \end{aligned} \tag{2.26}$$

At this point, we perform the change of variable $a \mapsto \frac{a}{t+s}$ and we obtain

$$\begin{aligned} \int_0^{t+s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da &= (t+s)^{2H-\frac{d}{2}}\int_0^1 a^{2H-2}(1-a)^{-\frac{d}{2}+1}da\\ &= \beta\left(2H-1, -\frac{d}{2}+2\right)(t+s)^{2H-\frac{d}{2}} \end{aligned}$$

and in the same way, with the change of variable $a \mapsto \frac{a}{t-s}$, we obtain

$$\begin{aligned} \int_0^{t-s} a^{2H-2}\big((t-s)-a\big)^{-\frac{d}{2}+1}da &= (t-s)^{2H-\frac{d}{2}}\int_0^1 a^{2H-2}(1-a)^{-\frac{d}{2}+1}da\\ &= \beta\left(2H-1, -\frac{d}{2}+2\right)(t-s)^{2H-\frac{d}{2}}. \end{aligned}$$

As a consequence, we obtain

$$R(t,s) = \alpha_H(2\pi)^{-\frac{d}{2}}C_d\beta\left(2H-1, -\frac{d}{2}+2\right)\left[(t+s)^{2H-\frac{d}{2}} - (t-s)^{2H-\frac{d}{2}}\right] + R_1^{(d)}(t,s)$$

with $R_1^{(d)}$ given by (2.26). Let us further analyze the function denoted by $R_1^{(d)}(t,s)$. Note that for every $s,t \in [0,T]$

$$(2\pi)^{\frac{d}{2}}R_1^{(d)}(t,s) = A(t,s) + B(t,s)$$

where

$$A(t,s)=\alpha_H C_d\Big[\int_0^s a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da \\ -\int_0^s a^{2H-2}\big((t-s)+a\big)^{-\frac{d}{2}+1}da\Big]$$

and

$$B(t,s)=\alpha_H C_d\Big[-\int_{t-s}^t a^{2H-2}\big(a-(t-s)\big)^{-\frac{d}{2}+1}da \\ -\int_t^{t+s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da\Big].$$

By the change of variables $a-t=\tilde{a}$, we can express B as

$$\begin{aligned}B(t,s)&=\alpha_H C_d\Big[-\int_{-s}^0 (a+t)^{2H-2}(a+s)^{-\frac{d}{2}+1}da \\ &\quad -\int_0^s (a+t)^{2H-2}(s-a)^{-\frac{d}{2}+1}da\Big] \\ &=-\alpha_H C_d\int_0^s da(s-a)^{-\frac{d}{2}+1}\big[(t-a)^{2H-2}+(t+a)^{2H-2}\big]\end{aligned}$$

and the desired conclusion is obtained.

Let us point out that the constant C_d is positive for $d=1$ and negative for $d=3$. This partially explains why different decompositions holds in these two cases. Thanks to the decomposition in Proposition 2.7, we have the following.

Theorem 2.3 *Assume $d=1$ and let U be the solution to the heat equation (2.18) with fractional-white noise (2.15). Let $B^{\frac{1}{2},2H-\frac{1}{2}}$ be a bifractional Brownian motion with parameters $H=\frac{1}{2}$ and $K=2H-\frac{1}{2}$. Let $(X_t^H)_{t\in[0,T]}$ be a centered Gaussian process with covariance, for $s,t\in[0,T]$*

$$\begin{aligned}R^{X^H}(t,s)&=2\frac{1}{\sqrt{2\pi}}\alpha_H\int_0^s (s-a)^{2H-2}\big[(t+a)^{\frac{1}{2}}-(t-a)^{\frac{1}{2}}\big]da \\ &=H\frac{1}{\sqrt{2\pi}}\int_0^s (s-a)^{2H-1}\big[(t+a)^{-\frac{1}{2}}+(t-a)^{-\frac{1}{2}}\big]da,\end{aligned} \tag{2.27}$$

and let $(Y_t^H)_{t\in[0,T]}$ be a centered Gaussian process with covariance

$$\begin{aligned}R^{Y^H}(t,s)&=2\frac{1}{\sqrt{2\pi}}\alpha_H\int_0^s (s-a)^{\frac{1}{2}}\big[(t+a)^{2H-2}+(t-a)^{2H-2}\big]da \\ &=H\frac{1}{\sqrt{2\pi}}\int_0^s (s-a)^{-\frac{1}{2}}\big[(t+a)^{2H-1}-(t-a)^{2H-1}\big]da.\end{aligned} \tag{2.28}$$

Suppose that U, X^H *and* Y^H *are independent. Then for every* $x \in \mathbb{R}^d$,

$$\left(U(t,x) + Y^H, t \in [0,T]\right) \overset{\text{Law}}{=} \left(C_0 B_t^{\frac{1}{2},\frac{1}{2}} + X_t^H, t \in [0,T]\right),$$

where $C_0^2 = \frac{2}{\sqrt{2\pi}} \alpha_H \beta(2H-1, -\frac{d}{2}+2)$.

Remark 2.8 As it is assumed that $H > 1/2$, the function R^{Y^H} always remains positive.

Proof Let us first verify that R^{X^H} is a covariance function. Clearly, it is symmetric and it can be written, for every $s, t \in [0,T]$, as

$$\begin{aligned}
&\sqrt{2\pi} R^{X^H}(t,s) \\
&\quad = H \int_0^{s\wedge t} (t \wedge s - a)^{2H-1} \left[\left((t\vee s)+a\right)^{-\frac{1}{2}} + \left((t \vee s)-a\right)^{-\frac{1}{2}}\right] \\
&\quad = H \int_0^{\infty} 1_{[0,t]}(a) 1_{[0,s]}(a)(t\wedge s - a)^{2H-1}\left((t+a)^{-\frac{1}{2}} \wedge (s+a)^{-\frac{1}{2}}\right) da \\
&\qquad + H \int_0^{\infty} 1_{[0,t]}(a) 1_{[0,s]}(a)(t\wedge s - a)^{2H-1}\left((t-a)^{-\frac{1}{2}} \wedge (s-a)^{-\frac{1}{2}}\right) da
\end{aligned}$$

and both summands above are positive definite (the same argument is used in [32], in the proof of Theorem 2.1). Similarly, the function R^{Y^H} is a covariance. If $d = 1$, we have $C_d = 2$ and

$$R(t,s) = 2\alpha_H (2\pi)^{-\frac{1}{2}} \beta\left(2H-1, \frac{3}{2}\right)\left[(t+s)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}\right] + R_1^{(1)}(t,s)$$

with

$$\begin{aligned}
\sqrt{2\pi} R_1^{(1)}(t,s) &= 2\alpha_H \int_0^s (s-a)^{2H-2}\left[(t+a)^{\frac{1}{2}} - (t-a)^{\frac{1}{2}}\right] da \\
&\quad - 2\alpha_H \int_0^s (s-a)^{\frac{1}{2}}\left[(t+a)^{2H-2} + (t-a)^{2H-2}\right] da \\
&= H \int_0^s (s-a)^{2H-1}\left[(t+a)^{-\frac{1}{2}} + (t-a)^{-\frac{1}{2}}\right] da \\
&\quad - 2\alpha_H \int_0^s (s-a)^{\frac{1}{2}}\left[(t+a)^{2H-2} + (t-a)^{2H-2}\right] da
\end{aligned}$$

where we used integration by parts in the first integral. □

In the case $d = 3$ we have the following.

Theorem 2.4 *Assume $d=3$. Let $B^{\frac{1}{2},2H-\frac{3}{2}}$ be a bifractional Brownian motion with $H=\frac{1}{2}$ and $K=2H-\frac{3}{2}$ and let $(Z_t^H)_{t\in[0,T]}$ be a centered Gaussian process with covariance $R_1^{(3)}(t,s)$. Then*

$$\left(U(t,x)+C_0B^{\frac{1}{2},2H-\frac{3}{2}}, t\in[0,T]\right) \overset{\text{Law}}{=} \left(Z_t^H, t\in[0,T]\right),$$

with C_0 defined as in Theorem 2.3.

Proof We have $C_3=-2$. In this case we can write

$$R(t,s)+2\alpha_H\beta\left(2H-1,\frac{1}{2}\right)\left[(t+s)^{2H-\frac{3}{2}}-(t-s)^{2H-\frac{3}{2}}\right]=R_1^{(3)}(t,s)$$

with

$$\begin{aligned}(2\pi)^{\frac{3}{2}}R_1^{(3)}(t,s) &= -2\alpha_H\int_0^s(s-a)^{2H-2}\left[(t+a)^{-\frac{1}{2}}-(t-a)^{-\frac{1}{2}}\right]da\\ &\quad +2\alpha_H\int_0^s(s-a)^{-\frac{1}{2}}\left[(t+a)^{2H-2}+(t-a)^{2H-2}\right].\end{aligned}$$

Note that $R_1^{(3)}$ is a covariance function because it is the sum of two covariance functions. □

Remark 2.9 Let us understand what happens with the decompositions in Theorems 2.3 and 2.4 when H is close to $\frac{1}{2}$. We focus on the case $d=1$. The phenomenon is interesting. We first notice that the process Y^H vanishes in this case. The covariance of the process X^H becomes

$$R^{X^{\frac{1}{2}}}(t,s)=\frac{1}{2\sqrt{2\pi}}\left((t+s)^{\frac{1}{2}}-(t-s)^{\frac{1}{2}}\right).$$

The constant $C_0^2=\frac{2}{\sqrt{2\pi}}\alpha_H\beta(2H-1,\frac{3}{2})$ is not defined for $H=\frac{1}{2}$ because of the presence of $2H-1$ in the argument of the beta function. But the following happens: since $2\alpha_H 1_{(0,1)}(u)(1-u)^{2H-2}$ is an approximation of unity, it follows that $\alpha_H\beta(2H-1,\frac{3}{2})$ converges to $\frac{1}{2}$ as H tends to $\frac{1}{2}$. Therefore C_0^2 becomes $\frac{1}{2\sqrt{2\pi}}$. Therefore we retrieve the result established in [166] and recalled in relation (2.7). In other words, in the fractional case $H\neq\frac{1}{2}$ the solution "retains" half of the bifractional Brownian motion $B^{\frac{1}{2},\frac{1}{2}}$ while the other half "spreads" into two parts.

2.5 The Solution to the Heat Equation with Fractional-Colored Noise

2.5.1 The Noise

The next step is to consider a noise with correlation structure both in time and in space. Consider the so-called *fractional-colored noise*, meaning a centered Gaussian process $W^H = \{W^H(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbf{E}\big(W^H(t, A)W^H(s, B)\big) = R_H(t, s) \int_A \int_B f\big(y - y'\big)dydy' =: \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{HP}} \tag{2.29}$$

where f is the spatial covariance kernel and R_H denotes the covariance of the fractional Brownian motion (1.1). Recall that f is the Fourier of a tempered nonnegative measure μ on $\mathbb{R}^d$.

To this Gaussian process we will associate a canonical Hilbert space whose structure is important in obtaining the existence and the properties of the solution. Let $\mathcal{E}$ be the set of linear combinations of elementary functions $1_{[0,t]\times A}$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, and $\mathcal{HP}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$. (Alternatively, $\mathcal{HP}$ can be defined as the completion of $C_0^\infty(\mathbb{R}^{d+1})$, with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$; see [13].)

The map $1_{[0,t]\times A} \mapsto W(t, A)$ is an isometry between $\mathcal{E}$ and the Gaussian space H^W of W, which can be extended to $\mathcal{HP}$. We denote this extension by:

$$\varphi \mapsto W(\varphi) = \int_0^T \int_{\mathbb{R}^d} \varphi(t, x)W(dt, dx).$$

We assume that $H > 1/2$. From (1.9) and (2.29), it follows that for any $\varphi, \psi \in \mathcal{E}$,

$$\begin{aligned}\langle \varphi, \psi \rangle_{\mathcal{HP}} &= \alpha_H \int_0^T \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(u, x)\psi(v, y) f(x - y)|u - v|^{2H-2}dxdydudv \\ &= \alpha_H (2\pi)^{-d} \int_0^T \int_0^T \int_{\mathbb{R}^d} \mathcal{F}\varphi(u, \cdot)(\xi)\overline{\mathcal{F}\psi(v, \cdot)(\xi)}|u - v|^{2H-2}\mu(d\xi)dudv.\end{aligned}$$

Moreover, we can interchange the order of the integrals $dudv$ and $\mu(d\xi)$, since for indicator functions φ and ψ, the integrand is a product of a function of (u, v) and a function of ξ. Hence, for $\varphi, \psi \in \mathcal{E}$, we have:

$$\langle \varphi, \psi \rangle_{\mathcal{HP}} = \alpha_H (2\pi)^{-d} \times \int_{\mathbb{R}^d} \int_0^T \int_0^T \mathcal{F}\varphi(u, \cdot)(\xi)\overline{\mathcal{F}\psi(v, \cdot)(\xi)}|u - v|^{2H-2}dudv\mu(d\xi). \tag{2.30}$$

The space $\mathcal{HP}$ may contain distributions, but contains the space $|\mathcal{HP}|$ of measurable functions $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\|\varphi\|^2_{|\mathcal{HP}|} := \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\varphi(u,x)\big|\big|\varphi(v,y)\big| f(x-y) \times |u-v|^{2H-2} dx dy du dv < \infty.$$

2.5.2 The Solution

Let us consider the equation (2.18) with the covariance of the noise W^H given by (2.12) and recall that the solution can be written in the mild form

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-u, x-y) W^H(ds, dy), \quad t \in [0,T], x \in \mathbb{R}^d.$$

We have the transfer formula

$$u(t,x) = \int_{-\infty}^t \int_{\mathbb{R}^d} \left(\int_a^t G(t-u, x-y)(u-a)^{H-\frac{3}{2}} \right) dW(a,y) \qquad (2.31)$$

where W is a centered Gaussian process with covariance given by (2.12).

Relation (2.31) follows from relation (2.17) using the moving average representation of the fBm (1.7). See also Sect. 3.1.3.

Remark 2.10 The process W behaves as a Wiener process with respect to the time variable and it has spatial covariance given by the Riesz kernel. In particular the increments of W with respect to the time variable are independent, meaning that $W(t,x) - W(s,x)$ is independent.

We have

Theorem 2.5 *The process* $(u(t,x))_{t\in[0,T], x\in\mathbb{R}^d}$ *given by* (2.31) *exists and satisfies*

$$\sup_{t\in[0,T], x\in\mathbb{R}^d} \mathbf{E}\big(u(t,x)^2\big) < +\infty$$

if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{2H} \mu(d\xi) < \infty.$$

Proof Note that $g_{tx} = G(t-\cdot, x-\cdot)$ is non-negative. Hence, $g_{tx} \in \mathcal{HP}$ if and only if $g_{tx} \in |\mathcal{HP}|$. This is equivalent to saying that $J_t := \|g_{tx}\|^2_{|\mathcal{HP}|} < \infty$ for all $t > 0$.

Note that

$$\begin{aligned}J_t &= \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_{tx}(u,y) g_{tx}(v,z) f(y-z)|u-v|^{2H-2} dydzdudv \\ &= (2\pi)^{-d}\alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \mathcal{F}g_{tx}(u,\cdot)(\xi)\overline{\mathcal{F}g_{tx}(v,\cdot)(\xi)}|u-v|^{2H-2}\mu(d\xi)dudv \\ &= (2\pi)^{-d}\alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \mathcal{F}G(t-u,\cdot)(\xi) \\ &\quad \times \overline{\mathcal{F}G(t-v,\cdot)(\xi)}|u-v|^{2H-2}\mu(d\xi)dudv.\end{aligned}$$

Using (2.6) and Fubini's theorem (whose application is justified since the integrand is non-negative), we obtain:

$$J_t = \alpha_H (2\pi)^{-d} \int_{\mathbb{R}^d} \int_0^t \int_0^t \exp\left(-\frac{u|\xi|^2}{2}\right) \exp\left(-\frac{v|\xi|^2}{2}\right)|u-v|^{2H-2} dudv\mu(d\xi).$$

The existence of the solution follows from Proposition 2.8 below, which also gives estimates for $J_t = \mathbf{E}|u(t,x)|^2$. □

Let $\mathcal{H}(0,t)$ denote the canonical Hilbert space of the fBm on the interval $(0,t)$ and let

$$B_t(\xi) = \int_0^t \int_0^t \exp\left(-\frac{u|\xi|^2}{2}\right) \exp\left(-\frac{v|\xi|^2}{2}\right)|u-v|^{2H-2} dudv.$$

Proposition 2.8 *For any* $t > 0, \xi \in \mathbb{R}^d$,

$$\frac{1}{4}\left(t^{2H} \wedge 1\right)\left(\frac{1}{1+|\xi|^2}\right)^{2H} \le B_t(\xi) \le C'_H\left(t^{2H}+1\right)\left(\frac{1}{1+|\xi|^2}\right)^{2H},$$

where $C'_H = b_H^2 (4H)^{2H}$.

Proof Suppose that $|\xi| \le 1$. Using the fact that $\|\varphi\|^2_{\mathcal{H}(0,t)} \le b_H^2 t^{2H-1}\|\varphi\|^2_{L^2(0,t)}$ (see Exercise 1.12) for all $\varphi \in L^2(0,t)$, $e^{-x} \le 1$ for any $x > 0$, and $\frac{1}{2} \le \frac{1}{1+|\xi|^2}$ if $|\xi| \le 1$,

$$B_t(\xi) \le b_H^2 t^{2H-1} \int_0^t \exp\left(-u|\xi|^2\right) du \le b_H^2 t^{2H} \le b_H^2 2^{2H} t^{2H}\left(\frac{1}{1+|\xi|^2}\right)^{2H}.$$

Suppose that $|\xi| \ge 1$. Using the fact that

$$\|\varphi\|^2_{\mathcal{H}(0,t)} \le b_H^2 \|\varphi\|^2_{L^{1/H}(0,t)}$$

for any $\varphi \in L^{1/H}(0,t)$ (see Chap. 1), $1-e^{-x} \leq 1$ for all $x>0$, and $\frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}$, we obtain:

$$B_t(\xi) \leq b_H^2 \left[\int_0^t \exp\left(-\frac{u|\xi|^2}{2H}\right)du\right]^{2H} = b_H^2 \left(\frac{2H}{|\xi|^2}\right)^{2H} \left[1-\exp\left(-\frac{t|\xi|^2}{2H}\right)\right]^{2H}$$

$$\leq b_H^2 (4H)^{2H} \left(\frac{1}{1+|\xi|^2}\right)^{2H}.$$

This proves the upper bound.

Next, we establish the lower bound. Suppose that $t|\xi|^2 \leq 1$. For any $u \in [0,t]$, $\frac{u|\xi|^2}{2} \leq \frac{t|\xi|^2}{2} \leq \frac{1}{2}$. Using the fact that $e^{-x} \geq 1-x$ for all $x>0$, we conclude that:

$$\exp\left(-\frac{u|\xi|^2}{2}\right) \geq 1-\frac{u|\xi|^2}{2} \geq \frac{1}{2}, \quad \forall u \in [0,t].$$

Hence

$$B_t(\xi) \geq \alpha_H \left(\frac{1}{2}\right)^2 \int_0^t \int_0^t |u-v|^{2H-2} du dv = \frac{1}{4} t^{2H} \geq \frac{1}{4} t^{2H} \left(\frac{1}{1+|\xi|^2}\right)^{2H}.$$

For the last inequality, we used the fact that $1 \geq \frac{1}{1+|\xi|^2}$.

Suppose that $t|\xi|^2 \geq 1$. Using the change of variables $u' = u|\xi|^2/2$, $v' = v|\xi|^2/2$, we obtain:

$$B_t(\xi) = \alpha_H \frac{2^{2H}}{|\xi|^{4H}} \int_0^{t|\xi|^2/2} \int_0^{t|\xi|^2/2} e^{-u'} e^{-v'} \left|u'-v'\right|^{2H-2} du' dv'.$$

Since the integrand is non-negative,

$$B_t(\xi) \geq \alpha_H \frac{2^{2H}}{|\xi|^{4H}} \int_0^{1/2} \int_0^{1/2} e^{-u} e^{-v} |u-v|^{2H-2} du dv$$

$$= 2^{2H} \|e^{-u}\|^2_{\mathcal{H}(0,1/2)} \frac{1}{|\xi|^{4H}} \geq 2^{2H} \left(\frac{1}{2}\right)^{2H+2} \left(\frac{1}{1+|\xi|^2}\right)^{2H},$$

where for the last inequality we used the fact that $\frac{1}{|\xi|^2} \geq \frac{1}{1+|\xi|^2}$, and $\|e^{-u}\|^2_{\mathcal{H}(0,1/2)} \geq (\frac{1}{2})^{2H+2}$. This follows since $e^{-u} \geq 1-u \geq \frac{1}{2}$ for all $u \in [0, \frac{1}{2}]$. □

Corollary 2.5 *For the covariance given by Riesz kernels* (*Example* 2.1) *and Bessel kernels* (*Example* 2.2) *of order* α, *the solution exists if and only if*

$$d < 4H + \alpha.$$

Proof This follows from Theorem 2.5 using the fact that the integral

$$\int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} \left(\frac{1}{1+|\xi|^2} \right)^{2H}$$

converges at zero if $\alpha < d$ and at infinity if $\alpha + 4H > d$. □

The covariance of the process can be written as (here $x \in \mathbb{R}^d$ is fixed)

$$\begin{aligned} \mathbf{E}u(t,x)u(s,y) = \alpha_H (2\pi)^{-d} \int_0^t \int_0^s dudv |u-v|^{2H-2} \\ \times \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} \end{aligned} \tag{2.32}$$

with $\alpha_H = H(2H-1)$.

The particular case of the Riesz kernel leads to some nice scaling properties.

Proposition 2.9 *Assume f is the Riesz kernel from Example 2.1. Then*

$$\begin{aligned} \mathbf{E}u(t,x)u(s,y) = \alpha_H (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{|\xi|^2}{2}} \\ \times \int_0^t \int_0^s dudv |u-v|^{2H-2} \big((t+s)-(u+v)\big)^{-\frac{d-\alpha}{2}}. \end{aligned}$$

Proof It suffices to make the change of variables $\tilde{\xi} = \sqrt{t+s-u-v}\,\xi$ in (2.32). □

Proposition 2.10 *When the spatial covariance is given by the Riesz kernel, the process u is self-similar with parameter*

$$H - \frac{d-\alpha}{2}.$$

Proof Taking into account the expression of the measure μ in Example 2.1, for every s,t

$$\begin{aligned} R(t,s) = \mathbf{E}u(t,x)u(s,y) = \alpha_H (2\pi)^{-d} \int_0^t \int_0^s dudv |u-v|^{2H-2} \\ \times \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}}. \end{aligned}$$

Let $c > 0$. By the change of variables $\tilde{u} = \frac{u}{c}, \tilde{v} = \frac{v}{c}$ in the integral $dudv$ and then by the change of variables $\tilde{\xi} = \sqrt{c}\xi$ in the integral $d\xi$ we get

$$R(ct, cs) = c^{2H - \frac{d-\alpha}{2}} R(t,s).$$ □

Let us analyze the behavior of the square mean of the increment of the solution to (2.18), that is,

$$\mathbf{E}\big|u(t,x) - u(s,y)\big|^2.$$

We will make the following assumption:

$$\mu(\xi) \sim c|\xi|^{-\alpha} d\xi, \quad \text{with } 0 < \alpha < d. \tag{2.33}$$

This means that for every function h such that the below integrals are finite, there exists two strictly positive constants c and c' such that

$$c' \int_{\mathbb{R}^d} h(\xi)|\xi|^{-\alpha} d\xi \le \int_{\mathbb{R}^d} h(\xi)\mu(d\xi) \le c \int_{\mathbb{R}^d} h(\xi)|\xi|^{-\alpha} d\xi.$$

Remark 2.11 The Riesz kernel and the Bessel kernel (with $\alpha < d$) satisfy (2.33).

Theorem 2.6 *Assume* (2.33). *There exists two strictly positive constants* C_1, C_2 *such that for any* $t, s \in [0, 1]$ *and for any* $x \in \mathbb{R}^d$

$$C_1|t-s|^{2H-\frac{d-\alpha}{2}} \le \mathbf{E}\big|u(t,x) - u(s,x)\big|^2 \le C_2|t-s|^{2H-\frac{d-\alpha}{2}}. \tag{2.34}$$

Remark 2.12 In the case $\alpha = 0$ (corresponding to fractional-white noise) we retrieve the result in Theorem 2.2.

Proof We will first prove the upper bound. Take $s \le t$, $s, t \in [0, 1]$.

$$\begin{aligned}
&\mathbf{E}\big|u(t,x) - u(s,x)\big|^2 \\
&= \alpha_H (2\pi)^{-d} \int_0^t \int_0^t dudv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} \\
&\quad - 2\alpha_H (2\pi)^{-d} \int_0^t \int_0^s dudv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} \\
&\quad + \alpha_H (2\pi)^{-d} \int_0^s \int_0^s dudv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(s-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} \\
&= \alpha_H (2\pi)^{-d} \int_s^t du \int_s^t dv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} \\
&\quad + \alpha_H (2\pi)^{-d} \int_0^s du \int_0^s dv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) \\
&\quad \times \big(e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} - 2e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} + e^{-\frac{(s-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}}\big) \\
&\quad + 2\alpha_H (2\pi)^{-d} \int_s^t du \int_0^s dv|u-v|^{2H-2} \\
&\quad \times \int_{\mathbb{R}^d} \mu(d\xi)\big(e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} - 2e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}}\big) \\
&:= A(t,s) + B(t,s) + C(t,s).
\end{aligned}$$

Let us first note that

$$C(t,s) = 2\alpha_H (2\pi)^{-d} \int_s^t du \int_0^s dv |u-v|^{2H-2} \times \int_{\mathbb{R}^d} d\xi e^{-\frac{(t-u)|\xi|^2}{2}} \left(e^{-\frac{(t-v)|\xi|^2}{2}} - e^{-\frac{(s-v)|\xi|^2}{2}}\right)$$

is negative and therefore it can be neglected for the proof of the upper bound.

Concerning the first term above (denoted by $A(t,s)$) we can write, by the change of variables $\tilde{u} = u - s$, $\tilde{v} = v - s$ and then $\tilde{u} = \frac{u}{t-s}$, $\tilde{v} = \frac{v}{t-s}$ and using (2.33)

$$A(t,s) \le c\alpha_H (2\pi)^{-d} |t-s|^{2H} \int_0^1 \int_0^1 du dv |u-v|^{2H-2} \times \int_{\mathbb{R}^d} de\xi |\xi|^{-\alpha} e^{-\frac{1}{2}(t-s)u|\xi|^2} e^{-\frac{1}{2}(t-s)v|\xi|^2}$$

and then, by the change of variables $\tilde{\xi} = \sqrt{t-s}\xi$ (meaning that $\tilde{\xi}_i = \sqrt{t-s}\xi_i$ for every $i = 1, \ldots, d$) we obtain

$$A(t,s) \le |t-s|^{2H - \frac{d-\alpha}{2}} C_0$$

with

$$C_0 = c\alpha_H (2\pi)^{-d} \int_0^1 \int_0^1 du dv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{1}{2}u|\xi|^2} e^{-\frac{1}{2}v|\xi|^2}$$
$$= c\alpha_H (2\pi)^{-d} \int_0^1 \int_0^1 du dv |u-v|^{2H-2} (u+v)^{-\frac{d-\alpha}{2}} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{1}{2}|\xi|^2}.$$

Note that the integral above is finite since $d < 4H + \alpha$.

It remains to analyze the term $B(t,s)$. Recall that

$$B(t,s) = \alpha_H (2\pi)^{-d} \int_0^s du \int_0^s dv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\mu(\xi) \times \left(e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} - 2e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(sv)|\xi|^2}{2}} + e^{-\frac{(s-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}}\right)$$

and with the change of variables $\tilde{u} = \frac{s-u}{t-s}$, $\tilde{v} = \frac{s-v}{t-s}$ and (2.33)

$$B(t,s) \le c\alpha_H (2\pi)^{-d} (t-s)^{2H} \int_0^s du \int_0^s dv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} \times \left(e^{-\frac{(t-s)(2+u+v)|\xi|^2}{2}} - 2e^{-\frac{(t-s)(1+u+v)|\xi|^2}{2}} + e^{-\frac{(t-s)(u+v)|\xi|^2}{2}}\right)$$

and using $\tilde{\xi} = \sqrt{t-s}\xi$

$$B(t,s) \le c\alpha_H (2\pi)^{-d} (t-s)^{2H - \frac{d-\alpha}{2}} \int_0^{\frac{s}{t-s}} du \int_0^{\frac{s}{t-s}} dv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha}$$

$$\times \big(e^{-\frac{(2+u+v)|\xi|^2}{2}} - 2e^{-\frac{(1+u+v)|\xi|^2}{2}} + e^{-\frac{(u+v)|\xi|^2}{2}}\big)$$

$$\leq c\alpha_H (2\pi)^{-d}(t-s)^{2H-\frac{d-\alpha}{2}} \int_0^\infty du \int_0^\infty dv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha}$$

$$\times \big(e^{-\frac{(2+u+v)|\xi|^2}{2}} - 2e^{-\frac{(1+u+v)|\xi|^2}{2}} + e^{-\frac{(u+v)|\xi|^2}{2}}\big).$$

Now, using the changes of variables $\tilde{\xi} = (2+u+v)\xi, \tilde{\xi} = (1+u+v)\xi$ and $\tilde{\xi} = (u+v)\xi$ respectively, we can write (with C_H a generic positive constant)

$$B(t,s) \leq C_H (t-s)^{2H-\frac{d-\alpha}{2}} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{|\xi|^2}{2}} \int_0^\infty du \int_0^\infty dv |u-v|^{2H-2}$$

$$\times \big[(2+u+v)^{-\frac{d-\alpha}{2}} - 2(1+u+v)^{-\frac{d-\alpha}{2}} + (u+v)^{-\frac{d-\alpha}{2}}\big].$$

The integral $\int_0^\infty \int_0^\infty du dv |uv|^{2H-2}[(2+u+v)^{-\frac{d}{2}} - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}]$ is finite: it is finite for u, v close to zero since $2H - \frac{d}{2} > 0$ and it is also finite for u, v close to infinitely because

$$\big[(2+u+v)^{-\frac{d}{2}} - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\big] \leq c(u+v)^{-\frac{d}{2}-2}$$

(this can be seen by analyzing the asymptotic behavior of the function $(2+x)^{-\frac{d}{2}} - 2(1+x)^{-\frac{d}{2}} + x^{-\frac{d}{2}}$). The proof of the lower bound follows the lines of the proof of the lower bound in Theorem 2.2, using the transfer formula (2.31) and the lower bound in (2.33). □

2.6 The Solution to the Wave Equation with White Noise in Time

The solutions to the linear wave equation with additive Gaussian noise constitute another interesting class of self-similar processes. In contrast to the other examples treated earlier in this monograph, they have the following interesting property: the self-similarity order is not the same as the Hölder regularity order. We first analyze the case of noise white in time and then we will discuss the situation when the noise behaves as a fractional Brownian motion with respect to the time variable.

2.6.1 The Equation

Consider the linear stochastic wave equation driven by a white-colored noise W. That is,

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d \\ u(0,x) &= 0, \quad x \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.35}$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $W = \{W(t,A); t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a centered Gaussian field with covariance

$$\mathbf{E}\big(W(t,A)W(s,B)\big) = (t \wedge s) \int_A \int_B f(x-y)dxdy \tag{2.36}$$

where f is the Fourier transform of a tempered measure μ on $\mathbb{R}^d$ (see Sect. 2.2). This is the so-called white-colored noise defined in Sect. 2.3.

Let G_1 be the fundamental solution of $u_{tt} - \Delta u = 0$. It is known that $G_1(t,\cdot)$ is a distribution in $\mathcal{S}'(\mathbb{R}^d)$ with rapid decrease. The easiest way to define G_1 is via its Fourier transform

$$\mathcal{F}G_1(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \tag{2.37}$$

for any $\xi \in \mathbb{R}^d, t > 0, d \geq 1$ (see e.g. [173]). In particular,

$$\begin{aligned} G_1(t,x) &= \frac{1}{2} 1_{\{|x|<t\}}, \quad \text{if } d = 1 \\ G_1(t,x) &= \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x|<t\}}, \quad \text{if } d = 2 \\ G_1(t,x) &= c_d \frac{1}{t} \sigma_t, \quad \text{if } d = 3, \end{aligned}$$

where σ_t denotes the surface measure on the 3-dimensional sphere of radius t.

2.6.2 The Solution

The solution of (2.35) is a square-integrable process $u = (u(t,x); t \in [0,T], x \in \mathbb{R}^d)$ defined by the Wiener integral representation with respect to the noise (2.36)

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W(ds, dy). \tag{2.38}$$

The solution exists when the above integral is well-defined. As for the heat equation, it depends on the dimension d and on the spatial covariance of the noise. For example, when the noise is white both in time and in space the solution exists if and only if $d = 1$.

The necessary and sufficient condition for the existence of the solution follows from [59].

Theorem 2.7 *The stochastic wave equation* (2.35) *admits a unique mild solution* $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ *if and only if*

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)\mu(d\xi)<\infty. \tag{2.39}$$

Remark 2.13 Recall that the same condition holds in the case of the heat equation with white-colored noise (Proposition 2.2).

Remark 2.14 When f is the Riesz kernel, condition (2.39) is equivalent to

$$d<2+\alpha.$$

When the noise is space-time white noise (corresponding to the case $\alpha=0$) the solution exists if and only if $d=1$.

Fix $x\in\mathbb{R}^d$. Then the covariance of the solution u (viewed as a process with respect to t) is

$$\begin{aligned}
&\mathbf{E}u(t,x)u(s,x)\\
&=(2\pi)^{-d}\int_0^{t\wedge s}du\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}dy'G_1(t-u,x-y)G_1\big(t-u,x-y'\big)f\big(y-y'\big)\\
&=\int_0^{t\wedge s}du\int_{\mathbb{R}^d}d\xi\mathcal{F}G_1(t-u,\cdot)(\xi)\overline{\mathcal{F}G_1(s-u,\cdot)}(\xi)\mu(d\xi)
\end{aligned}$$

where we used (2.11) and (2.37).

We will assume from now on that the spatial covariance of the noise W is given by the Riesz kernel. We make the change of notation $\alpha=d-\beta$ in the expression given in Example 2.1 and assume that the measure μ is

$$d\mu(\xi)=|\xi|^{-d+\beta}d\xi \quad \text{with } \beta\in(0,d).$$

In this case the kernel f is given by

$$f(\xi)=c_{\beta,d}|\xi|^{-\beta} \quad \text{with } \beta\in(0,d). \tag{2.40}$$

If f is as above, then

$$\begin{aligned}
\mathbf{E}u(t,x)u(s,x)=(2\pi)-d\int_0^{t\wedge s}du\\
\times\int_{\mathbb{R}^d}d\xi\frac{\sin((t-u)|\xi|)}{|\xi|}\frac{\sin((s-u)|\xi|)}{|\xi|}|\xi|^{-d+\beta}d\xi.
\end{aligned}$$

Proposition 2.11 *Suppose f is defined by* (2.40). *Then the process* $(u(t,x), t \geq 0)$ *given by* (2.38) *is self-similar of order* $\frac{3-\beta}{2}$.

Proof Let $c > 0$ and let R be the covariance of the process $t \to u(t,x)$. Then, with $a = (2\pi)^{-d}$, for every $s, t \geq 0$

$$\begin{aligned} R(ct, cs) &= a \int_0^{ct \wedge cs} du \int_{\mathbb{R}^d} d\xi \frac{\sin((ct-u)|\xi|)}{|\xi|} \frac{\sin((cs-u)|\xi|)}{|\xi|} |\xi|^{-d+\beta} d\xi \\ &= ac \int_0^{t \wedge s} du \int_{\mathbb{R}^d} d\xi \frac{\sin((ct-cu)|\xi|)}{|\xi|} \frac{\sin((cs-cu)|\xi|)}{|\xi|} |\xi|^{-d+\beta} d\xi \\ &= c^{3-\beta} R(t,s) \end{aligned}$$

where we made successively the change of variable $\tilde{u} = \frac{u}{c}$ and $\tilde{\xi} = c\xi$. □

The solution has the following time regularity (see [63, 64]).

Proposition 2.12 *Assume that*

$$\beta \in (0, d \wedge 2). \tag{2.41}$$

Let $t_0, M > 0$ and fix $x \in [-M, M]^d$. Then there exist positive constants c_1, c_2 such that for every $s, t \in [t_0, T]$

$$c_1 |t-s|^{2-\beta} \leq \mathbf{E}\big|u(t,x) - u(s,x)\big|^2 \leq c_2 |t-s|^{2-\beta}.$$

Remark 2.15 Let us highlight an interesting fact: the order of self-similarity and the order of Hölder continuity do not coincide in this case. This is the first example among the Gaussian processes discussed in this chapter when this phenomenon occurs.

Proposition 2.12 implies the following Hölder property for the solution to (2.35).

Corollary 2.6 *Assume* (2.41). *Then for every $x \in \mathbb{R}^d$ the application*

$$t \to u(t,x)$$

is almost surely Hölder continuous of order $\delta \in (0, \frac{2-\beta}{2})$.

Proof This follows easily from Proposition 2.12 and from the fact that u is Gaussian. □

Remark 2.16 The proof of Theorem 5.1 in [63] implies that the mapping $t \to u(t,x)$ is not Hölder continuous of order $\frac{2-\beta}{2}$.

2.7 The Stochastic Wave Equation with Linear Fractional-Colored Noise

For an interval $(a, b) \subset \mathbb{R}$, we define the restricted Fourier transform of a function $\varphi \in L^1(a, b)$:

$$\mathcal{F}_{a,b}\varphi(\tau) := \int_a^b e^{-i\tau x}\varphi(x)dx = \mathcal{F}(\varphi 1_{[a,b]})(\tau).$$

One can prove that $\mathcal{F}\varphi \in L^2(\mathbb{R})$, for any $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. By Plancharel's identity (2.8), for any $\varphi, \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have:

$$\int_{\mathbb{R}} \varphi(x)\psi(x)dx = (2\pi)^{-1} \int_{\mathbb{R}} \mathcal{F}\varphi(\tau)\overline{\mathcal{F}\psi(\tau)}d\xi.$$

In particular, for any $\varphi, \psi \in L^2(a, b)$, we have:

$$\int_a^b \varphi(x)\psi(x)dx = (2\pi)^{-1} \int_{\mathbb{R}} \mathcal{F}_{a,b}\varphi(\tau)\overline{\mathcal{F}_{a,b}\psi(\tau)}d\xi. \tag{2.42}$$

2.7.1 *The Equation*

Consider the linear stochastic wave equation driven by a fractional colored noise W with Hurst parameter $H \in (\frac{1}{2}, 1)$. That is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t, x) &= \Delta u(t, x) + \dot{W}(t, x), \quad t \in [0, T], x \in \mathbb{R}^d \\ u(0, x) &= 0, \quad x \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0, x) &= 0, \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.43}$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $W = \{W(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a centered Gaussian field with covariance

$$\mathbf{E}\big(W(t, A)W(s, B)\big) = R_H(t, s) \int_A \int_B f(x - y)dxdy \tag{2.44}$$

where R_H is the covariance of the fractional Brownian motion (1.1) and f is the Fourier transform of a tempered measure μ on $\mathbb{R}^d$ (see Sect. 2.2).

2.7.2 The Solution

The solution of (2.35) is a square-integrable process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ defined by:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W(ds, dy). \tag{2.45}$$

By definition, $u(t,x)$ exists if and only if the stochastic integral above is well-defined, i.e. $g_{tx} := G_1(t-\cdot, x-\cdot) \in \mathcal{HP}$ (this space was introduced in Sect. 2.5). In this case, $\mathbf{E}|u(t,x)|^2 = \|g_{tx}\|^2_{\mathcal{HP}}$.

We begin with an auxiliary result. To simplify the notation, we introduce the following functions: for $\lambda > 0$, $\tau > 0$, let

$$f_t(\lambda,\tau) = \sin \tau\lambda t - \tau \sin \lambda t, \qquad g_t(\lambda,\tau) = \cos \tau\lambda t - \cos \lambda t. \tag{2.46}$$

Lemma 2.1 *For any $\lambda > 0$ and $t > 0$,*

$$c_t \frac{\lambda^3}{1+\lambda^2} \leq \int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2} \big[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau) \big] d\tau \leq C_t \frac{\lambda^3}{1+\lambda^2},$$

where c_t, C_t are some positive constants.

Proof Using Exercise 1.15, we have

$$\frac{1}{(\tau^2-1)^2} \big[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau) \big] = \big| \mathcal{F}_{0,\lambda t} \varphi(\tau) \big|^2,$$

where $\varphi(x) = \sin x$. Using Plancharel's identity (2.42), we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2} \big[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau) \big] d\tau \\
&\quad = \int_{\mathbb{R}} \big| \mathcal{F}_{0,\lambda t} \varphi(\tau) \big|^2 d\tau = 2\pi \int_0^{\lambda t} |\sin x|^2 dx \\
&\quad = 2\pi\lambda \int_0^t |\sin \lambda s|^2 ds = 2\pi\lambda^3 \int_0^t \frac{|\sin \lambda s|^2}{\lambda^2} ds.
\end{aligned}$$

It now suffices to use the bound (see e.g. Lemma 6.1.2 of [161])

$$c_t \frac{1}{1+\lambda^2} \leq \int_0^t \frac{|\sin \lambda s|^2}{\lambda^2} ds \leq C_t \frac{1}{1+\lambda^2}.$$

□

We denote by $N_t(\xi)$ the $\mathcal{H}(0,t)$-norm of $u \mapsto \mathcal{F}G_1(u,\cdot)(\xi)$, i.e.

$$N_t(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^t \int_0^t \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv.$$

We also recall that (see Exercise 1.12) there exists a constant $b_H > 0$ such that

$$\|\varphi\|^2_{\mathcal{H}(0,t)} \le b_H^2 \|\varphi\|^2_{L^{1/H}(0,t)} \le b_H^2 t^{2H-1} \|\varphi\|^2_{L^2(0,t)} \tag{2.47}$$

for any $\varphi \in L^2(0,t)$.

Proposition 2.13 *For any $t > 0, \xi \in \mathbb{R}^d$*

$$N_t(\xi) \le C_{H,t} t^{2H+2} \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2}, \quad \text{if } |\xi| \le 1$$

$$N_t(\xi) \le c_{H,t} \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2}, \quad \text{if } |\xi| \ge 1.$$

Proof (a) Suppose that $|\xi| \le 1$. We use (2.47) and $|\sin x| \le x$ for any $x > 0$. Hence,

$$N_t(\xi) \le b_H^2 t^{2H-1} \frac{1}{|\xi|^2} \int_0^t \sin^2\big(u|\xi|\big) du \le b_H^2 t^{2H-1} \int_0^t u^2 du$$
$$= b_H^2 t^{2H-1} \frac{t^3}{3} \le \frac{1}{3} b_H^2 t^{2H+2} 2^{H+1/2} \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2},$$

where for the last inequality we used the fact that $\frac{1}{2} \le \frac{1}{1+|\xi|^2}$ if $|\xi| \le 1$.

(b) Suppose that $|\xi| \ge 1$. Using the change of variables $u' = u|\xi|$, $v' = v|\xi|$,

$$N_t(\xi) = \frac{\alpha_H}{|\xi|^{2H+2}} \int_0^{t|\xi|} \int_0^{t|\xi|} \sin\big(u'\big) \sin\big(v'\big) \big|u' - v'\big|^{2H-2} du' dv'$$
$$= \frac{1}{|\xi|^{2H+2}} \big\| \sin(\cdot) \big\|^2_{\mathcal{H}(0,t|\xi|)}.$$

Using the expression of the $\mathcal{H}(0,t|\xi|)$-norm of $\sin(\cdot)$ given in Exercise 1.15, we obtain:

$$N_t(\xi) = \frac{c_H}{|\xi|^{2H+2}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} \big[f_t^2\big(|\xi|, \tau\big) + g_t^2\big(|\xi|, \tau\big) \big] d\tau. \tag{2.48}$$

We split the integral into the regions $|\tau| \le 1/2$ and $|\tau| \ge 1/2$, and we denote the two integrals by $N_t^{(1)}(\xi)$ and $N_t^{(2)}(\xi)$.

Since $|f_t(\lambda, \tau)| \le 1 + |\tau|$ and $|g_t(\lambda, \tau)| \le 2$ for any $\lambda > 0, \tau > 0$, we have:

$$N_t^{(1)}(\xi) \le c_H \frac{1}{|\xi|^{2H+2}} \int_{|\tau| \le 1/2} \frac{|\tau|^{-(2H-1)}}{(1-\tau^2)^2} \big[\big(1 + |\tau|\big)^2 + 4 \big] d\tau$$
$$\le c_H \frac{1}{|\xi|^{2H+1}} \int_{|\tau| \le 1/2} C |\tau|^{-(2H-1)} d\tau$$

$$= C\frac{c_H}{1-H}\left(\frac{1}{2}\right)^{2-2H}\frac{1}{|\xi|^{2H+1}}.$$

We used the fact that $|\xi|^{2H+2} \geq |\xi|^{2H+1}$ if $|\xi| \geq 1$, and $\frac{1}{(1-\tau^2)^2}[(1+|\tau|)^2+4] \leq \frac{1}{(3/4)^2}[(3/2)^2+4] := C$ if $|\tau| \leq 1/2$.

Using the fact that $|\tau|^{-(2H-1)} \leq (\frac{1}{2})^{-(2H-1)}$ if $|\tau| \geq \frac{1}{2}$, Lemma 2.1, and the fact that $|\xi|^2/(1+|\xi|^2) \leq 1$, we obtain:

$$\begin{aligned}
N_t^{(2)}(\xi) &\leq \frac{c_H}{2^{-(2H-1)}}\frac{1}{|\xi|^{2H+2}}\int_{|\tau|\geq 1/2}\frac{1}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau \\
&\leq \frac{c_H}{2^{-(2H-1)}}\frac{1}{|\xi|^{2H+2}}\int_{\mathbb{R}}\frac{1}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau \\
&\leq \frac{c_H}{2^{-(2H-1)}}c_t^{(2)}\frac{1}{|\xi|^{2H+2}}\cdot|\xi|\frac{|\xi|^2}{1+|\xi|^2} \\
&\leq \frac{c_H}{2^{-(2H-1)}}c_t^{(2)}\frac{1}{|\xi|^{2H+1}}.
\end{aligned}$$

□

Proposition 2.14

(a) *If* $I_t^{(1)} < \infty$ *for* $t = 1$, *then* $\int_{|\xi|\leq 1}\mu(d\xi) < \infty$.
(b) *Let* $l \geq 1$ *be the integer from* (2.10) *and* $m = 2l-2$. *For any* $t > 0$,

$$\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+1}} \leq a_{H,t}\left(\sum_{i=0}^{m}b_t^i\right)I_t^{(2)} + b_t^{m+1}\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+m}}, \tag{2.49}$$

where $a_{H,t}, b_t, c_t$ *are positive constants.*

In particular, if $I_t^{(2)} < \infty$ *for some* $t > 0$, *then* $\int_{|\xi|\geq 1}|\xi|^{-(2H+1)}\mu(d\xi) < \infty$.

Proof (a) Using the fact that $\sin x/x \geq \sin 1$ for all $x \in [0,1]$, we have:

$$\begin{aligned}
I_1^{(1)} &= \int_{|\xi|\leq 1}\frac{\mu(d\xi)}{|\xi|^2}\int_0^1\int_0^1 \sin\big(u|\xi|\big)\sin\big(v|\xi|\big)|u-v|^{2H-2}dudv \\
&\geq \sin^2 1\int_{|\xi|\leq 1}\mu(d\xi)\int_0^1\int_0^1 uv|u-v|^{2H-2}dudv.
\end{aligned}$$

(b) According to (2.48),

$$I_t^{(2)} = c_H\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2}}\int_{\mathbb{R}}\frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau. \tag{2.50}$$

For any $k \in \{-1,0,\dots,m\}$, let

$$I(k) := \int_{|\xi|\geq 1}\frac{1}{|\xi|^{2H+2+k}}\mu(d\xi).$$

By (2.10), $I(m)=\int_{|\xi|\geq 1}|\xi|^{-(2H+2+m)}\mu(d\xi)\leq \int_{|\xi|\geq 1}|\xi|^{-2l}\mu(d\xi)<\infty$.

We will prove that the integrals $I(k)$ satisfy a certain recursive relation. By reverse induction, this will imply that all integrals $I(k)$ with $k\in\{-1,0,\ldots,m\}$ are finite. For this, for $k\in\{0,1\ldots,m\}$, we let

$$A_t(k):=\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\int_{\mathbb{R}}\frac{1}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau. \tag{2.51}$$

We consider separately the regions $\{|\tau|\leq 2\}$ and $\{|\tau|\geq 2\}$ and we denote the corresponding integrals by $A_t'(k)$ and $A_t''(k)$. For the region $\{|\tau|\leq 2\}$, we use the expression (2.50) of $I_t^{(2)}$. Using the fact that $|\xi|^{2H+2+k}\geq|\xi|^{2H+2}$ (since $k\geq 0$), and $|\tau|^{-(2H-1)}\geq 2^{-(2H-1)}$ if $|\tau|\leq 2$, we obtain:

$$\begin{aligned}
A_t'(k)&:=\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\int_{|\tau|\leq 2}\frac{1}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau\\
&\leq 2^{2H-1}\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2}}\int_{|\tau|\leq 2}\frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau\\
&\leq 2^{2H-1}\frac{1}{c_H}I_t^{(2)},\quad \text{by (2.50)}.
\end{aligned}$$

For the region $\{|\tau|\geq 2\}$, we use the fact $|f_t(\lambda,\tau)|\leq 1+|\tau|$ and $|g_t(\lambda,\tau)|\leq 2$ for all $\lambda>0$, $\tau>0$. Hence,

$$\begin{aligned}
A_t''(k)&:=\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\int_{|\tau|\geq 2}\frac{1}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau\\
&\leq \int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\int_{|\tau|\geq 2}\frac{1}{(\tau^2-1)^2}\big[\big(1+|\tau|\big)^2+4\big]d\tau=CI(k).
\end{aligned}$$

Hence, for any $k\in\{0,1,\ldots,m\}$

$$A_t(k)\leq 2^{2H-1}\frac{1}{c_H}I_t^{(2)}+CI(k).$$

Using Lemma 2.1, and the fact that $\frac{|\xi|^2}{1+|\xi|^2}\geq\frac{1}{2}$ if $|\xi|\geq 1$, we obtain:

$$A_t(k)\geq c_t^{(1)}\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\cdot\frac{|\xi|^3}{1+|\xi|^2}\geq\frac{1}{2}c_t^{(1)}I(k-1),$$

for all $k\in\{0,1,\ldots,m\}$. From the last two relations, we conclude that:

$$\frac{1}{2}c_t^{(1)}I(k-1)\leq 2^{2H-1}\frac{1}{c_H}I_t^{(2)}+CI(k),\quad \forall k\in\{0,1,\ldots,m\}, \tag{2.52}$$

or equivalently, $I(k-1)\leq a_{H,t}I_t^{(2)}+b_tI(k)$, for all $k\in\{0,1,\ldots,m\}$. Relation (2.49) follows by recursion. □

Remark 2.17 In the previous argument, the recursion relation (2.52) uses the fact that k is *non-negative* (see the estimate of $A'_t(k)$). Therefore, the "last" index k for which this relation remains true (counting downwards from m) is $k = 0$, leading us to the conclusion that $\int_{|\xi|\geq 1} |\xi|^{-(2H+1)}\mu(d\xi) < \infty$, if $I_t^{(2)} < \infty$.

Theorem 2.8 *The stochastic wave equation* (2.35) *admits a unique mild solution* $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ *if and only if*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{H+\frac{1}{2}} \mu(d\xi) < \infty. \tag{2.53}$$

Proof To have that $g_{tx} \in \mathcal{HP}$ we need in particular to have $I_t < \infty$ for all $t > 0$ (see [14] for more details), where

$$I_t := \alpha_H \int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}g_{tx}(u,\cdot)(\xi)\overline{\mathcal{F}g_{tx}(v,\cdot)(\xi)}|u-v|^{2H-2}dudv\mu(d\xi),$$

and $\mathbf{E}|u(t,x)|^2 = \|g_{tx}\|^2_{\mathcal{HP}} = I_t$. Since $\mathcal{F}g_{tx}(u,\cdot)(\xi) = e^{-i\xi\cdot x}\overline{\mathcal{F}G_1(t-u,\cdot)(\xi)}$,

$$I_t = \alpha_H \int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}G_1(u,\cdot)(\xi)\overline{\mathcal{F}G_1(v,\cdot)(\xi)}|u-v|^{2H-2}dudv\mu(d\xi).$$

Using (2.37), we obtain:

$$I_t = \alpha_H \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^t \int_0^t \sin\big(u|\xi|\big)\sin\big(v|\xi|\big)|u-v|^{2H-2}dudv.$$

We split the integral $\mu(d\xi)$ into two parts, corresponding to the regions $\{|\xi| \leq 1\}$ and $\{|\xi| \geq 1\}$. We denote the respective integrals by $I_t^{(1)}$ and $I_t^{(2)}$. Since the integrand is non-negative $I_t < \infty$ if and only if $I_t^{(1)} < \infty$ and $I_t^{(2)} < \infty$.

The fact that condition (2.53) is sufficient for $I_t < \infty$ follows by Proposition 2.13. The necessity follows by Proposition 2.14 (using Remark 2.18). □

Remark 2.18 Condition (2.53) is equivalent to

$$\int_{|\xi|\leq 1} \mu(d\xi) < \infty \quad \text{and} \quad \int_{|\xi|\geq 1} \frac{1}{|\xi|^{2H+1}}\mu(d\xi) < \infty.$$

Corollary 2.7

(i) *Let* $f(x) = \gamma_{\alpha,d}|x|^{-(d-\alpha)}$ *be the Riesz kernel of order* $\alpha \in (0,d)$. *Then* $\mu(d\xi) = |\xi|^{-\alpha}d\xi$ *and* (2.53) *is equivalent to* $\alpha > d - 2H - 1$.

(ii) *Let* $f(x) = \gamma_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}e^{-|x|^2/(4w)}dw$ *be the Bessel kernel of order* $\alpha > 0$. *Then* $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2}$ *and* (2.53) *is equivalent to* $\alpha > d - 2H - 1$.

The solution to the wave equation is also self-similar.

Proposition 2.15 *Fix $x \in \mathbb{R}^d$ and assume* (2.53). *Then the process* $(u(t,x), t \geq 0)$ *defined by* (2.45) *is self-similar of order*

$$H + 1 - \frac{d-\alpha}{2}.$$

Proof The covariance of u can be expressed as

$$\mathbf{E}u(t,x)u(s,x) = a(H)\int_0^t du \int_0^s dv|u-v|^{2H-2} \times \int_{\mathbb{R}^d} d\xi \frac{\sin((t-u)|\xi|)}{|\xi|}\frac{\sin((s-v)|\xi|)}{|\xi|}|\xi|^{-d}d\xi.$$

This easily implies the conclusion by a standard change of variables. □

Remark 2.19 Note that the self-similarity index

$$H + 1 - \frac{d-\alpha}{2}$$

is positive under condition (2.53).

Assume in the sequel that the spatial covariance of the noise W is given by the Riesz kernel under the form (2.40). Note that in this case condition (2.53) is equivalent to

$$\beta \in \big(0, d \wedge (2H+1)\big). \tag{2.54}$$

Remark 2.20 Since $H > \frac{1}{2}$ and so $2H+1 \in (2,3)$, for dimension $d = 1, 2$ we have $\beta \in (0, d)$ while for $d \geq 3$ we have $\beta \in (0, 2H+1)$.

Remark 2.21 As a consequence of Exercise 1.13 we deduce the following:

(i) For any $x > 0$ the quantity $\int_0^x v^{2H-2}\cos(v)(x-v)dv$ is positive (it is the sum of two norms).

(ii) For every $a, b \in \mathbb{R}$, $a < b$

$$\|f1_{(a,b)}\|^2_{\mathcal{H}} \leq 2\alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v).$$

(iii) For every $a, b \in \mathbb{R}$, $a < b$

$$\|f1_{(a,b)}\|^2_{\mathcal{H}} \geq 2\alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2}\sin(b-a-v).$$

Proposition 2.16 *Assume that*

$$\beta \in \big(2H-1, d \wedge (2H+1)\big). \tag{2.55}$$

Let $t_0, M > 0$ and fix $x \in [-M, M]^d$. Then there exist positive constants c_1, c_2 such that for every $s, t \in [t_0, T]$

$$c_1|t-s|^{2H+1-\beta} \le \mathbf{E}\big|u(t,x)-u(s,x)\big|^2 \le c_2|t-s|^{2H+1-\beta}.$$

Proof Let $h > 0$ and let us estimate the $L^2(\Omega)$-norm of the increment $u(t+h,x) - u(t,x)$. Splitting the interval $[0, t+h]$ into the intervals $[0,t]$ and $[t, t+h]$, and using the inequality $|a+b|^2 \le 2(a^2+b^2)$, we obtain:

$$\begin{aligned}\mathbf{E}\big|u(t+h,x)-u(t,x)\big|^2 &\le 2\big\{\big\|(g_{t+h,x}-g_{t,x})1_{[0,t]}\big\|^2_{\mathcal{HP}} + \|g_{t+h,x}1_{[t,t+h]}\|^2_{\mathcal{HP}}\big\} \\ &=: 2\big[E_{1,t}(h)+E_2(h)\big]. \qquad (2.56)\end{aligned}$$

The first summand can be handled in the following way.

$$\begin{aligned}E_{1,t}(h) &= \alpha_H(2\pi)^{-d}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t dvdv|u-v|^{2H-2}\mathcal{F}(g_{t+h,x}-g_{tx})(u,\cdot)(\xi) \\ &\quad\times\overline{\mathcal{F}(g_{t+h,x}-g_{tx})(v,\cdot)(\xi)} \\ &= \alpha_H(2\pi)^{-d}\int_{\mathbb{R}^d}\mu(d\xi) \\ &\quad\times\int_0^t\int_0^t dudv|u-v|^{2H-2}\big[\mathcal{F}G_1(u+h,\cdot)(\xi)-\mathcal{F}G_1(u,\cdot)(\xi)\big] \\ &\quad\times\overline{\mathcal{F}G_1(v+h,\cdot)(\xi)-\mathcal{F}G_1(v,\cdot)(\xi)} \\ &= \alpha_H(2\pi)^{-d}\int_0^t\int_0^t dudv|u-v|^{2H-2}I_h,\end{aligned}$$

where

$$\begin{aligned}I_h &= \int_{\mathbb{R}^d}\mu(d\xi)\big[\mathcal{F}G_1(u+h,\cdot)(\xi)-\mathcal{F}G_1(u,\cdot)(\xi)\big] \\ &\quad\times\big[\overline{\mathcal{F}G_1(v+h,\cdot)(\xi)-\mathcal{F}G_1(v,\cdot)(\xi)}\big] \\ &= \int_{\mathbb{R}^d}\mu(d\xi)\frac{(\sin((u+h)|\xi|)-\sin(u|\xi|))}{|\xi|}\frac{(\sin((v+h)|\xi|)-\sin(v|\xi|))}{|\xi|}.\end{aligned}$$

Using trigonometric identities we obtain

$$\begin{aligned}E_{1,t}(h) &= \alpha_H\int_0^t\int_0^t dudv|u-v|^{2H-2}\int_{\mathbb{R}^d}\mu(d\xi)\frac{\sin(\frac{h|\xi|}{2})^2}{|\xi|^2}\cos\bigg(\frac{(2u+h)|\xi|}{2}\bigg) \\ &\quad\times\cos\bigg(\frac{(2v+h)|\xi|}{2}\bigg) \\ &= c\cdot\alpha_H\int_0^t\int_0^t dudv|u-v|^{2H-2}\int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^{d-\beta+2}}\sin\big(h|\xi|\big)^2\end{aligned}$$

$$\times \cos\big((2u+h)|\xi|\big)\cos\big((2v+h)|\xi|\big),$$

and by making the change of variables $\tilde{u} = (2u+h)|\xi|$, $\tilde{v} = (2v+h)|\xi|$,

$$\begin{aligned} E_{1,t}(h) &= c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\ &\quad\times \int_{h|\xi|}^{(2t+h)|\xi|}\int_{h|\xi|}^{(2t+h)|\xi|} dudv|u-v|^{2H-2}\cos u\cos v \\ &= c\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \big\|\cos(\cdot)1_{(h|\xi|,(2t+h)|\xi|)}(\cdot)\big\|_{\mathcal{H}}^2, \quad (2.57) \end{aligned}$$

and using Exercise 1.13,

$$\begin{aligned} E_{1,t}(h) &= c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \times \bigg[\int_0^{2t|\xi|} \cos(v)v^{2H-2}\big(2t|\xi|-v\big)dv \\ &\quad + \cos\big(2t|\xi|+2h|\xi|\big)\int_0^{2t|\xi|} v^{2H-2}\big(\sin\big(2t|\xi|-v\big)\big)\bigg] \\ &= c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \times \bigg[2t|\xi|\int_0^{2t|\xi|} \cos(v)v^{2H-2}dv \\ &\quad - \sin\big(2t|\xi|\big)\big(2t|\xi|\big)^{2H-1} + (2H-1)\int_0^{2t|\xi|}\sin(v)v^{2H-2}dv \\ &\quad + \cos\big(2t|\xi|+2h|\xi|\big)\int_0^{2t|\xi|} v^{2H-2}\big(\sin\big(2t|\xi|-v\big)\big)\bigg] \quad (2.58) \end{aligned}$$

where we use integration by parts. By Remark 2.21, point (ii) we have the upper bound

$$\begin{aligned} E_{1,t}(h) &\le c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\ &\quad\times\bigg[2t|\xi|\int_0^{2t|\xi|}\cos(v)v^{2H-2}dv - \sin\big(2t|\xi|\big)\big(2t|\xi|\big)^{2H-1} \\ &\quad + (2H-1)\int_0^{2t|\xi|}\sin(v)v^{2H-2}dv\bigg]. \end{aligned}$$

We will treat the three summands above separately. For the first one,

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 2t|\xi|\int_0^{2t|\xi|}\cos(v)v^{2H-2}dv \\ &= c_{t,H}h^{2H+1-\beta}\bigg|\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+1}}\sin\big(|\xi|\big)^2\int_0^{\frac{2t|\xi|}{h}}\cos(v)v^{2H-2}dv\bigg| \end{aligned}$$

$$\leq c_{t,H} h^{2H+1-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin(|\xi|)^2 \left| \int_0^{\frac{2t|\xi|}{h}} \cos(v) v^{2H-2} dv \right|$$
$$\leq c_{t,H} h^{2H+1-\beta}$$

using condition (2.54) and the fact that the integral $\int_0^\infty \cos(v)v^{2H-2}dv$ is convergent (this implies that the function $x \in [0,\infty) \to \int_0^x \cos(v)v^{2H-2}dv$ admits a limit at infinity and is therefore bounded). On the other hand

$$\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \sin(2t|\xi|)(2t|\xi|)^{2H-1}$$
$$= c_t h^{3-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin(|\xi|)^2 \sin\left(\frac{2t|\xi|}{h}\right)$$
$$= c_t h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin(|\xi|)^2 \sin\left(\frac{2t|\xi|}{h}\right)$$
$$+ c_t h^{3-\beta} \int_{|\xi|>1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin(|\xi|)^2 \sin\left(\frac{2t|\xi|}{h}\right).$$

The second part over the region $|\xi| \geq 1$ is bounded by $ch^{3-\beta}$ simply by majorizing sine by one. The second integral has a singularity for $|\xi|$ close to zero. Using the fact that $\sin(x) \leq x$ for all $x \geq 0$, we will bound it above by

$$h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin(|\xi|)^2 \sin\left(\frac{2t|\xi|}{h}\right)$$
$$\leq c_t h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} |\xi|^2 \left|\sin\left(\frac{2t|\xi|}{h}\right)\right|^{2-2H} \left|\sin\left(\frac{2t|\xi|}{h}\right)\right|^{2H-1}$$
$$\leq c_t h^{2H+1-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H-1}}$$

where we bounded $|\sin(\frac{2t|\xi|}{h})|^{2-2H}$ by $c_t(|\xi|h^{-1})^{2-2H}$ and $|\sin(\frac{2t|\xi|}{h})|^{2H-1}$ by 1. The last integral is finite since $\beta > 2H-1$ (assumption (2.55)).

Finally

$$\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \int_0^{2t|\xi|} \sin(v) v^{2H-2} dv$$
$$= h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(|\xi|)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv$$
$$= h^{2H+2-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(|\xi|)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv$$

$$+ h^{2H+2-\beta} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(|\xi|\big)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv$$

$$\leq h^{2H+2-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} |\xi|^2 \int_0^{\frac{2t|\xi|}{h}} |\sin v| v^{2H-2} dv$$

$$+ h^{2H+2-\beta} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv. \tag{2.59}$$

Again using the fact that $\int_0^\infty \sin(v) v^{2H-2} dv$ it is convergent it is easy to see that the integral over the region $|\xi| \geq 1$ is bounded by $c_t h^{2H+2-\beta}$. For the integral over $|\xi| \leq 1$ we make the change of variables $\tilde{v} = \frac{vh}{\xi}$ and we get

$$h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+1}} \int_0^{2t} \left|\sin\left(\frac{v|\xi|}{h}\right)\right| v^{2H-2} dv$$

$$= h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+1}} \int_0^{2t} \left|\sin\left(\frac{v|\xi|}{h}\right)\right|^{2-2H} \left|\sin\left(\frac{v|\xi|}{h}\right)\right|^{2H-1} v^{2H-2} dv$$

$$\leq c_t h^{2h+1-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H-1}},$$

where we have made the same considerations as for the second summand in the decomposition of $E_{1,t}(h)$. In this way, we obtain the upper bound for the summand $E_{1,t}(h)$ in (2.56)

$$E_{1,t}(h) \leq C h^{2H+1-\beta}. \tag{2.60}$$

We now study the term $E_2(h)$ in (2.56) (the notation $E_2(h)$ instead of $E_{2,t}(h)$ is due to the fact that it does not depend on t, see below). Using successively the change of variables $\tilde{u} = \frac{u}{h}$, $\tilde{v} = \frac{v}{h}$ in the integral $dudv$ and $\tilde{\xi} = h\xi$ in the integral $d\xi$, the summand $E_2(h)$ can be written as

$$E_2(h) = \alpha_H \int_{\mathbb{R}^d} \int_t^{t+h} \int_t^{t+h} \mathcal{F}G_1(t+h-u,\cdot)(\xi)$$

$$\times \overline{\mathcal{F}G_1(t+h-v,\cdot)(\xi)} |u-v|^{2H-2} dudv \mu(d\xi)$$

$$= \alpha_H \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^h \int_0^h \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} dudv$$

$$= \alpha_H h^{2H} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^1 \int_0^1 \sin\big(u|\xi|h\big) \sin\big(v|\xi|h\big) |u-v|^{2H-2} dudv$$

$$= \alpha_H h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^1 \int_0^1 \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} dudv.$$

Let us use the following notation:

$$N_t(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^t \int_0^t \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv, \quad t \in [0,T], \xi \in \mathbb{R}^d. \tag{2.61}$$

By Proposition 2.13 the term

$$N_1(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^1 \int_0^1 \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv$$

satisfies the inequality

$$N_1(\xi) \le C_H \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2},$$

with C_H a positive constant not depending on h. Consequently the term $E_2(h)$ is bounded by

$$E_2(h) \le C h^{2H+2-\beta} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{H+\frac{1}{2}} \mu(d\xi) \tag{2.62}$$

and this is clearly finite due to (2.53). Relations (2.60) and (2.62) give the first part of the conclusion.

Let us analyze now the lower bound of the increments of $u(t,x)$ with respect to the variable t. Let $h > 0, x \in [-M,M]^d$ and $t \in [t_0, T]$ such that $t+h \in [t_0, T]$. From the decomposition

$$\begin{aligned}\mathbf{E}\big|u(t+h,x) - u(t,x)\big|^2 &= \big\|(g_{t+h,x} - g_{t,x}) 1_{[0,t]}\big\|^2_{\mathcal{HP}} + \|g_{t+h,x} 1_{[t,t+h]}\|^2_{\mathcal{HP}} \\ &\quad + 2\big\langle (g_{t+h,x} - g_{t,x}) 1_{[0,t]}, g_{t+h,x} 1_{[t,t+h]} \big\rangle_{\mathcal{HP}}\end{aligned}$$

we immediately obtain, since the second summand on the right-hand side is positive,

$$\begin{aligned}\mathbf{E}\big|u(t+h,x) - u(t,x)\big|^2 \ge\ & \big\|(g_{t+h,x} - g_{t,x}) 1_{[0,t]}\big\|^2_{\mathcal{HP}} \\ & + 2\big\langle (g_{t+h,x} - g_{t,x}) 1_{[0,t]}, g_{t+h,x} 1_{[t,t+h]} \big\rangle_{\mathcal{HP}} \\ :=\ & E_{1,t}(h) + E_{3,t}(h).\end{aligned}$$

We can assume, without any loss of the generality, that $t = \frac{1}{2}$. Let $E_{1,\frac{1}{2}}(h) := E_1(h)$. We first prove that

$$E_1(h) \ge c h^{2H+1-\beta} - c' h^{2H+2-\beta} \tag{2.63}$$

for h small enough. Recall that we have an exact expression for $E_1(h)$ (see (2.58)). Indeed,

$$E_1(h) = \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \big\|\cos(\cdot) 1_{(h|\xi|, h|\xi| + |\xi|)}\big\|^2_{\mathcal{H}}$$

$$
\begin{aligned}
&= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\
&\quad \times \int_{h|\xi|}^{(1+h)|\xi|} \int_{h|\xi|}^{(1+h)|\xi|} dudv |u-v|^{2H-2} \cos u \cos v \\
&= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\
&\quad \times \int_0^{|\xi|} \int_0^{|\xi|} dudv \cos\big(u+h|\xi|\big) \cos\big(v+h|\xi|\big) |u-v|^{2H-2}.
\end{aligned}
$$

By the trigonometric formula $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ we have

$$
\begin{aligned}
E_1(h) &= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\
&\quad \times \Big[\cos\big(h|\xi|\big)^2 \int_0^{|\xi|} \int_0^{|\xi|} dudv \cos u \cos v |u-v|^{2H-2} \\
&\quad - 2 \sin\big(h|\xi|\big) \cos\big(h|\xi|\big) \int_0^{|\xi|} \int_0^{|\xi|} dudv \sin u \cos v |u-v|^{2H-2} \\
&\quad + \sin\big(h|\xi|\big)^2 \int_0^{|\xi|} \int_0^{|\xi|} dudv \sin u \sin v |u-v|^{2H-2} \Big] \\
&:= A + B + C.
\end{aligned}
$$

We will neglect the first term since it is positive. We will bound the second term above by $ch^{2H+2-\beta}$. Again using trigonometric identities, Exercise 1.14 (used at the third line below), and the change of variables $\tilde{v} = u - v$ we have

$$
\begin{aligned}
&-2 \sin\big(h|\xi|\big) \cos\big(h|\xi|\big) \int_0^{|\xi|} \int_0^{|\xi|} dudv \sin u \cos v |u-v|^{2H-2} \\
&= - \sin\big(h|\xi|\big) \cos\big(h|\xi|\big) \int_0^{|\xi|} \int_0^{|\xi|} dudv \big(\sin(u+v) + \sin(u-v)\big) |u-v|^{2H-2} \\
&= - \sin\big(h|\xi|\big) \cos\big(h|\xi|\big) \int_0^{|\xi|} \int_0^{|\xi|} dudv \sin(u+v) |u-v|^{2H-2} \\
&= c \cdot \sin\big(h|\xi|\big) \cos\big(h|\xi|\big) \int_0^{|\xi|} v^{2H-2} \big(\cos\big(2|\xi| - v\big) - \cos(v)\big) dv
\end{aligned}
$$

and thus

$$
\begin{aligned}
B &= c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \sin\big(h|\xi|\big) \cos\big(h|\xi|\big) \\
&\quad \times \int_0^{|\xi|} v^{2H-2} \big(\cos\big(2|\xi| - v\big) - \cos(v)\big) dv
\end{aligned}
$$

$$
\begin{aligned}
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\bigl(h|\xi|\bigr)^3 \cos\bigl(h|\xi|\bigr) \sin\bigl(|\xi|\bigr) \\
&\quad \times \int_0^{|\xi|} v^{2H-2} \sin\bigl(|\xi| - v\bigr) dv \\
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\bigl(h|\xi|\bigr)^3 \cos\bigl(h|\xi|\bigr) \sin\bigl(|\xi|\bigr) \\
&\quad \times \int_0^{|\xi|} v^{2H-2} \bigl(\sin\bigl(|\xi|\bigr)\cos(v) - \cos\bigl(|\xi|\bigr)\sin(v)\bigr) dv \\
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\bigl(h|\xi|\bigr)^3 \cos\bigl(h|\xi|\bigr) \sin\bigl(|\xi|\bigr) \\
&\quad \times \left(\sin\bigl(|\xi|\bigr) \int_0^{|\xi|} v^{2H-2} \cos(v) dv - \cos\bigl(|\xi|\bigr) \int_0^{|\xi|} v^{2H-2} \sin(v) dv \right) \\
&= -c \cdot \alpha_H \int_{|\xi| \le 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\bigl(h|\xi|\bigr)^3 \cos\bigl(h|\xi|\bigr) \sin\bigl(|\xi|\bigr) \\
&\quad \times \left(\sin\bigl(|\xi|\bigr) \int_0^{|\xi|} v^{2H-2} \cos(v) dv - \cos\bigl(|\xi|\bigr) \int_0^{|\xi|} v^{2H-2} \sin(v) dv \right) \\
&\quad - c \cdot \alpha_H \int_{|\xi| \ge 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\bigl(h|\xi|\bigr)^3 \cos\bigl(h|\xi|\bigr) \sin\bigl(|\xi|\bigr) \\
&\quad \times \left(\sin\bigl(|\xi|\bigr) \int_0^{|\xi|} v^{2H-2} \cos(v) dv - \cos\bigl(|\xi|\bigr) \int_0^{|\xi|} v^{2H-2} \sin(v) dv \right).
\end{aligned}
$$

Taking the absolute value we see that the part over the set $|\xi| \le 1$ is bounded by ch^3 simply by majorizing $\sin(h|\xi|)$ by $h|\xi|$, $\cos(h|\xi|)\sin(|\xi|)$ by one, and

$$
\left| \sin\bigl(|\xi|\bigr) \int_0^{|\xi|} v^{2H-2} \cos(v) dv - \cos\bigl(|\xi|\bigr) \int_0^{|\xi|} v^{2H-2} \sin(v) dv \right|
$$

by a constant. For the part over the region $|\xi| \ge 1$ we again bound the last expression by a constant and we use the change of variables $\tilde{\xi} = h\xi$. This part will by bounded by

$$
\begin{aligned}
&h^{2H+2-\beta} \int_{|\xi| \ge h} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \bigl|\sin\bigl(|\xi|\bigr)^3 \cos\bigl(|\xi|\bigr) \sin\bigl(|\xi|/h\bigr)\bigr| \\
&\quad \le h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \bigl|\sin\bigl(|\xi|\bigr)^3\bigr| \\
&\quad \le ch^{2H+2-\beta}
\end{aligned}
$$

since the last integral is convergent at infinity by bounding sine by one and at zero by bounding $\sin(x)$ by x and using the assumption $\beta > 2H - 1$. Therefore

$$B \le ch^{2H+2-\beta}. \tag{2.64}$$

We now bound the summand C below. In this summand the $\mathcal{H}$ norm of the sine function appears and this has been analyzed in [14]. We have, after the change of variables $\tilde{u} = \frac{u}{|\xi|}$, $\tilde{v} = \frac{v}{|\xi|}$,

$$\begin{aligned} C &= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2}} \sin\big(h|\xi|\big)^4 \int_0^1 \int_0^1 \sin\big(u|\xi|\big)\sin\big(v|\xi|\big)|u-v|^{2H-2} du dv \\ &\ge \alpha_H \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta+2}} \sin\big(h|\xi|\big)^4 \int_0^1 \int_0^1 \sin\big(u|\xi|\big)\sin\big(v|\xi|\big)|u-v|^{2H-2} du dv. \end{aligned}$$

We will use the proof of Proposition 2.14. For h small, we will have that

$$\begin{aligned} C &\ge \alpha_H \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta}} \sin\big(h|\xi|\big)^4 \frac{1}{|\xi|^2} \int_0^1 \int_0^1 \sin\big(u|\xi|\big)\sin\big(v|\xi|\big)|u-v|^{2H-2} du dv \\ &\ge \alpha_H \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta}} \sin\big(h|\xi|\big)^4 \frac{1}{|\xi|^{2H+1}} \\ &= \alpha_H h^{2H+1-\beta} \int_{|\xi|\ge h} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin\big(|\xi|\big)^4 \\ &\ge \alpha_H h^{2H+1-\beta} \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin\big(|\xi|\big)^4 \\ &= c\cdot \alpha_H h^{2H+1-\beta}. \end{aligned} \tag{2.65}$$

Relations (2.64) and (2.65) imply (2.63). Now, from relation (2.63), for every $t_0 \le s < t < T$ with s, t close enough

$$E_1(t-s) \ge c(t-s)^{2H+1-\beta} - c'(t-s)^{2H+2-\beta} \ge \frac{c}{2}(t-s)^{2H+1-\beta}$$

if $|t-s| \le \frac{c}{2c'}$. To extend the above inequality to arbitrary values of $|t-s|$, we proceed as in [64], proof of Proposition 4.1. Notice that the function $g(t,s,x,y) := \mathbf{E}|u(t,x) - u(s,x)|^2$ is positive and continuous with respect to all its arguments and therefore it is bounded below on the set $\{(t,s,x,y) \in [t_0,T]^2 \times [-M,M]^{2d};$ $|t-s| \ge \varepsilon\}$ by a constant depending on $\varepsilon > 0$. Hence for $|t-s| \ge \frac{c}{2c'}$ it also holds that

$$E_1(t-s) \ge c_1 |t-s|^{2H+1-\beta}.$$

On the other hand, from (2.57) and (2.62) and the Cauchy-Schwarz inequality, we obtain

$$E_{3,t}(h) = \big\langle (g_{t+h,x} - g_{t,x}) 1_{[0,t]}, g_{t+h,x} 1_{[t,t+h]} \big\rangle_{\mathcal{HP}}$$

$$\leq \left\| (g_{t+h,x} - g_{t,x}) 1_{[0,t]} \right\|_{\mathcal{HP}} \| g_{t+h,x} 1_{[t,t+h]} \|_{\mathcal{HP}}$$
$$\leq c h^{\frac{2H+1-\beta}{2} + \frac{2H+2-\beta}{2}}.$$

Consequently,

$$\mathbf{E} \left| u(t+h,x) - u(t,x) \right|^2 \geq C h^{2H+1-\beta} - C' h^{\frac{2H+1-\beta}{2} + \frac{2H+2-\beta}{2}}$$

and this implies that for every $s, t \in [t_0, T]$ and $x \in [-M, M]^d$

$$\mathbf{E} \left| u(t,x) - u(s,x) \right|^2 \geq \frac{C}{2} |t-s|^{2H+1-\beta} \quad \text{if } |t-s| \leq \left(\frac{C}{2C'} \right)^{\frac{1}{2}}.$$

Similarly as above, the previous inequality can be extended to arbitrary values of $s, t \in [t_0, T]$. □

Proposition 2.16 implies the following Hölder property for the solution to (2.35).

Corollary 2.8 *Assume* (2.55). *Then for every* $x \in \mathbb{R}^d$ *the application*

$$t \to u(t,x)$$

is almost surely Hölder continuous of order $\delta \in (0, \frac{2H+1-\beta}{2})$.

Proof This is consequence of the relations (2.57) and (2.62) in the proof of Proposition 2.16 and of the fact that u is Gaussian. □

Let us make some remarks on the result in Proposition 2.16.

Remark 2.22

- Following the proof of Theorem 5.1 in [63] we can show that the mapping $t \to u(t,x)$ is not Hölder continuous of order $\frac{2H+1-\beta}{2}$.
- When H is close to $\frac{1}{2}$ we retrieve the regularity in time of the solution to the wave equation with white noise in time (see [63, 64]).

2.8 Bibliographical Notes

The study of stochastic partial differential equations (SPDEs) driven by a Gaussian noise which is white in time and has a non-trivial correlation structure in space (called "color") now constitutes a classical line of research. These equations represent an alternative to the standard SPDEs driven by a space-time white noise. A first step in this direction was made in [60], where the authors identify the necessary and sufficient conditions for the existence of the solution to the stochastic wave equation (in spatial dimension $d = 2$), in the space of real-valued stochastic processes.

The fundamental reference in this area is Dalang's seminal article [59], in which the author gives the necessary and sufficient conditions under which various SPDEs with a white-colored noise (e.g. the wave equation, the damped heat equation, the heat equation) have a process solution, in arbitrary spatial dimension. The methods used in this article exploit the temporal martingale structure of the noise, and cannot be applied when the noise is "colored" in time. Other related references are, among others: [61, 120, 143, 190] and [62]. The development of stochastic calculus with respect to fractional Brownian motion naturally led to the study of SPDEs driven by this Gaussian process. The motivation comes from the wide area of applications of fBm. We refer, among other references, to [84, 119, 139, 150] and [170] for theoretical studies of SPDEs driven by fBm and to [51] or [140] for the sample paths properties of the solution. To list only a few examples of the appearance of fractional noises in practical situations, we mention [103] for biophysics, [25] for financial time series, [66] for electrical engineering, and [42] for physics.

2.9 Exercises

Exercise 2.1 Let u be the solution of the heat equation with space-time white noise. Show that there exist two positive constants C_1, C_2 such that for every $s, t \in [0, T]$

$$C_1|t-s|^{\frac{1}{2}} \leq \mathbf{E}\big|u(t,x)-u(s,y)\big|^2 \leq C_2|t-s|^{\frac{1}{2}}.$$

Study the variations of this process.

Hint Use the fact that u has the same law as a bi-fBm, modulo a constant.

Exercise 2.2 ([51]) Let u be the solution of the fractional-(Riesz) colored wave equation. Let us denote by Δ the following metric on $[0, T] \times \mathbb{R}^d$

$$\Delta\big((t,x);(s,y)\big) = |t-s|^{2H+1-\beta} + |x-y|^{2H+1-\beta}. \tag{2.66}$$

Fix $M > 0$ and assume (2.55). Prove that for every $t, s \in [t_0, T]$ and $x, y \in [-M, M]^d$ there exist positive constants C_1, C_2 such that

$$C_1\Delta\big((t,x);(s,y)\big) \leq \mathbf{E}\big|u(t,x)-u(s,y)\big|^2 \leq C_2\Delta\big((t,x);(s,y)\big).$$

Exercise 2.3 ([13]) Consider the linear heat equation with white-colored noise where the spatial covariance is given by the heat kernel (Example 2.4) or by the Poisson kernel (Example 2.3). Give the necessary and sufficient conditions in terms of d and α for the existence of the solution.

Exercise 2.4 Consider the linear heat equation with white-colored noise where the spatial covariance is given by the Riesz or Bessel kernel. Prove that the solution is Hölder continuous with respect to time of order $0 < \delta < \frac{1}{2} - \frac{d-\alpha}{2}$.

Exercise 2.5 Consider the Gaussian processes with covariances given by (2.27) and (2.28) respectively. Prove that these processes are self-similar and give the self-similarity order.

Exercise 2.6 Consider the heat equation with fractional-colored noise and spatial covariance given by a kernel f. If f is the heat kernel of order α, or the Poisson kernel of order α, then prove that the solution exists for any $H > 1/2$ and $d \geq 1$.

Exercise 2.7 Let f be the Riesz kernel of order $\alpha \in (0, d)$, and set

$$I_t = \alpha_H \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha-2H-2} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} \left[f_t^2\big(|\xi|, \tau\big) + g_t^2\big(|\xi|, \tau\big) \right] d\tau$$

with f_t, g_t given by (2.46).

1. Show that

$$I_t = 2\alpha_H c_d \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} \left(\int_0^\infty \frac{(\sin \tau\lambda t - \tau \sin \lambda t)^2}{\lambda^2} \lambda^{-\theta} d\lambda \right. \\ \left. + \int_0^\infty \frac{(\cos \tau\lambda t - \cos \lambda t)^2}{\lambda^2} \lambda^{-\theta} d\lambda \right),$$

   where $\theta = \alpha + 1 - d + 2H > 0$.
2. If $\theta < 1$, show that the two integrals $d\lambda$ can be expressed in terms of the covariance functions of the odd and even parts of the fBm (see [70]).

Exercise 2.8 Consider $f(x) = \prod_{i=1}^d (\alpha_{H_i} |x_i|^{2H_i-2})$ with $H_i > 1/2$ for all $i = 1, \ldots, d$.

1. Prove that f is the Fourier transform of the measure $\mu(d\xi) = \prod_{i=1}^d (c_{H_i} \times |\xi_i|^{-(2H_i-1)})$.
2. Prove that (2.53) is equivalent to $\sum_{i=1}^d (2H_i - 1) > d - 2H - 1$.

Hint This can be seen by using the change of variables to polar coordinates.

Exercise 2.9 Prove that the solutions to the heat and wave equation with white or fractional noise in time and with white or colored noise in space are all continuous with respect to the space variable.

Chapter 3
Non-Gaussian Self-similar Processes

An interesting class of self-similar processes can be defined as limits that appear in the so-called *Non-Central Limit Theorem* (see e.g. [168] or [67]). We briefly recall the context.

Let us recall the notion of Hermite rank. Denote by $H_m(x)$ the Hermite polynomial of degree m given by $H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}$ and let g be a function on $\mathbb{R}$ such that $\mathbf{E}[g(\xi_0)] = 0$ and $\mathbf{E}[g(\xi_0)^2] < \infty$. Assume that g has the following expansion in Hermite polynomials

$$g(x) = \sum_{l=0}^{\infty} c_l H_l(x)$$

where $c_l = \frac{1}{l!}\mathbf{E}[g(\xi_0) H_l(\xi_0)]$. *The Hermite rank of* g is defined by

$$k = \min\{l \mid c_l \neq 0\}.$$

Since $\mathbf{E}[g(\xi_0)] = 0$, we have $k \geq 1$.

Let g be a function of Hermite rank k and let $(\xi_n)_{n\in\mathbb{Z}}$ be a stationary Gaussian sequence with mean 0 and variance 1 which exhibits long range dependence in the sense that the correlation function satisfies

$$r(n) := \mathbf{E}(\xi_0 \xi_n) = n^{\frac{2H-2}{k}} L(n)$$

where $H \in (\frac{1}{2}, 1)$, $k \geq 1$ and L is a slowly varying function at infinity (see e.g. [75] for the definition). Then the following family of stochastic processes

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j) \tag{3.1}$$

converges as $n \to \infty$, in the sense of finite dimensional distributions, to a self-similar stochastic process with stationary increments that lives in the kth Wiener

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chaos (as presented in Appendix C). This process is called the Hermite process of order k. The class of Hermite processes includes fractional Brownian motion which is the only Gaussian process in this class. Their practical aspects are striking: they provide a wide class of processes from which to model long memory, self-similarity, and Hölder-regularity, allowing significant deviation from fBm and other Gaussian processes. Since they are non-Gaussian and self-similar with stationary increments, the Hermite processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian. The need for non-Gaussian self-similar processes in practice (for example in hydrology) is mentioned in the paper [169] based on the study of stochastic modeling for river-flow time series in [107]. This chapter contains an analysis of the basic properties of the Hermite process with a special focus on the Rosenblatt process which is, after fBm, the most well known Hermite process.

3.1 The Hermite Process

3.1.1 Definition and Basic Properties

We will adopt the following definition of the Hermite process.

Definition 3.1 Let $(B(t))_{t\in\mathbb{R}}$ be a Wiener process. The Hermite process $(Z_H^k(t))_{t\geq 0}$ of order k and with self-similarity index $H\in(\frac{1}{2},1)$ is defined as

$$Z_H^k(t)=c(H,k)\int_{\mathbb{R}^k}\int_0^t\left(\prod_{j=1}^k(s-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{k}\right)}\right)ds\,dB(y_1)\cdots dB(y_k),\qquad(3.2)$$

where $x_+=\max(x,0)$. The above integral is a multiple integral of order k with respect to the Brownian motion W (in the sense of Appendix C) and the constant $c(H,k)$ is a normalizing constant that ensures $\mathbf{E}(Z_H^k(t))^2=1$.

Remark 3.1 Throughout, a random variable which has the same law as $Z_H^k(1)$ will be called a *Hermite random variable*.

The most studied Hermite process is of course *fractional Brownian motion* (which is obtained in (3.2) for $k=1$; compare (3.2) with (1.7)) due to its large range of applications. The process obtained in (3.2) for $k=2$ is known as *the Rosenblatt process*. It was introduced by Rosenblatt in [157] and was given its name by M. Taqqu in [167].

Let us first compute the covariance of the Hermite process and the constant $c(H,k)$ from (3.2).

We need the following lemma.

Lemma 3.1 *For* $a+b<-1$

$$\int_{-\infty}^{u\wedge v}(u-y)^a(v-y)^b dy=\beta(-1-a-b,b+1)|u-v|^{a+b+1}. \tag{3.3}$$

Proof Suppose $u>v$. Then use the change of variables $z=\frac{v-y}{z-y}$. □

In the sequel we will denote Z_H^k by Z^k.

Proposition 3.1 *The constant* $c(H,k)$ *in* (3.2) *is given by*

$$c(H,k)^2=\left(\frac{\beta(\frac{1}{2}-\frac{1-H}{k},\frac{2H-2}{k})^k}{k!H(2H-1)}\right)^{-1}. \tag{3.4}$$

Moreover, for every $k\geq 1$, *the process* Z_H^k *satisfies, for every* $s,t\geq 0$,

$$R(t,s):=\mathbf{E}Z^k(t)Z^k(s)=\frac{1}{2}\big(t^{2H}+s^{2H}-|t-s|^{2H}\big).$$

Proof By Fubini and the isometry of multiple Wiener-Itô integrals, one has

$$\begin{aligned}
R(t,s)&=k!c(H,k)^2\int_{\mathbb{R}^k}\left(\int_0^t\int_0^s\prod_{j=1}^k(u-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\right.\\
&\qquad\left.\times(v-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}dvdu\right)dy_1\cdots dy_k\\
&=k!c(H,k)^2\int_0^t\int_0^s\int_{\mathbb{R}^k}\left[\prod_{j=1}^k(u-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\right.\\
&\qquad\left.\times(v-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}dy_1\cdots dy_k\right]dvdu\\
&=k!c(H,k)^2\int_0^t\int_0^s\left[\int_{\mathbb{R}}(u-y)_+^{-(\frac{1}{2}+\frac{1-H}{k})}(v-y)_+^{-(\frac{1}{2}+\frac{1-H}{k})}dy\right]^k dvdu.
\end{aligned}$$

Let $\beta(p,q)=\int_0^1 z^{p-1}(1-z)^{p-1}dz$, $p,q>0$ be the beta function. By using the identity (see (3.3))

$$\int_{\mathbb{R}}(u-y)_+^{a-1}(v-y)_+^{a-1}dy=\beta(a,1-2a)|u-v|^{2a-1}$$

we get

$$R(t,s) = k!c(H,k)^2\beta\left(\frac{1}{2}-\frac{1-H}{k},\frac{2H-2}{k}\right)^k\int_0^t\int_0^s\left(|u-v|^{\frac{2H-2}{k}}\right)^k dvdu$$

$$= k!c(H,k)^2\frac{\beta(\frac{1}{2}-\frac{1-H}{k},\frac{2H-2}{k})^k}{H(2H-1)}\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

In order to obtain $\mathbf{E}(Z_H^k(t))^2 = 1$, we will choose $c(H,k)$ to be given by (3.4) and we will have

$$R(t,s) = \frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right), \quad s,t \geq 0.$$

□

Remark 3.2 Note that the covariance does not suffice to deduce the law of the process, since the Hermite process is not Gaussian (except when $k = 1$).

As mentioned earlier, the Hermite process shares many of the properties of fBm.

Proposition 3.2 *The process Z_H^k given by (3.2) is H-self-similar.*

Proof Let $c > 0$. We will use the self-similarity property of the Wiener process. We have

$$Z^k(ct) = c(H,k)\int_{\mathbb{R}^k}\int_0^{ct}\left(\prod_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\right)dsdB(y_1)\cdots dB(y_k)$$

$$= cc(H,k)\int_{\mathbb{R}^k}\int_0^{t}\left(\prod_{j=1}^k (cs-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\right)dsdB(y_1)\cdots dB(y_k)$$

$$= cc(H,k)\int_{\mathbb{R}^k}\int_0^{t}\left(\prod_{j=1}^k (cs-cy_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\right)dsdB\left(c^{-1}y_1\right)\cdots dB\left(c^{-1}y_k\right)$$

$$= cc^{-(\frac{1}{2}+\frac{1-H}{k})}c(H,k)$$

$$\times\int_{\mathbb{R}^k}\int_0^{t}\left(\prod_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\right)dsdB\left(c^{-1}y_1\right)\cdots dB\left(c^{-1}y_k\right)$$

$$=^{(d)} cc^{-(\frac{1}{2}+\frac{1-H}{k})}c^{-\frac{k}{2}}c(H,k)$$

$$\times\int_{\mathbb{R}^k}\int_0^{t}\left(\prod_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\right)dsdB(y_1)\cdots dB(y_k)$$

$$= c^H Z^k(t).$$

Recall that $=^{(d)}$ denotes the equivalence of finite dimensional distributions. □

Proposition 3.3 *The increments of the process Z^k are stationary.*

Proof It follows immediately from (3.2) that for every $h > 0$ the processes $(Z^k(t+h) - Z^k(t))_{t\geq 0}$ and $(Z^k(t))_{t\geq 0}$ coincide in law. □

Remark 3.3 From Propositions 3.2 and 3.3 one can deduce the expression of the covariance of the Hermite process by using Proposition A.1.

Proposition 3.4 *All moments of the Hermite process are finite and for every $p \geq 1$*

$$\mathbf{E}\left|Z^k(t)\right|^p = \mathbf{E}\left|Z^k(1)\right|^p t^{2H}$$

for every $t \geq 0$.

Proof This is a consequence of Proposition 3.2. □

From the stationarity of increments and the self-similarity, it follows that

Proposition 3.5 *For any $p \geq 1$*

$$\mathbf{E}\left[\left|Z_H^k(t) - Z_H^k(s)\right|^p\right] = \mathbf{E}\left|Z^k(1)\right|^p |t-s|^{pH}.$$

As a consequence the Hermite process has Hölder continuous paths of order δ with $0 < \delta < H$.

Proof To get the Hölder continuity, one applies the Kolmogorov criterium (Proposition A.2). □

Proposition 3.6 *The Hermite process exhibits long-range dependence.*

Proof This follows from Proposition A.2 because the self-similarity index is $H > \frac{1}{2}$. □

We mention that different expressions of the exponent in (3.2) are used in the literature, but we have chosen this one so that the order of similarity is equal to H.

3.1.2 Other Representations

One can express the Hermite process as a multiple integral with respect to a Wiener process on a finite time interval. This representation uses the kernel K^H of the fractional Brownian motion (1.3).

Theorem 3.1 *Let $H \in (\frac{1}{2}, 1)$. Consider the process $(Y_t^{(q,H)})_{t\in[0,T]}$ with $q \geq 1$ given by*

$$Y_t^{(q,H)} = d(H) \int_0^t \cdots \int_0^t dW_{y_1} \cdots dW_{y_q} \times \left(\int_{y_1 \vee \cdots \vee y_q}^t \partial_1 K^{H'}(u, y_1) \cdots \partial_1 K^{H'}(u, y_q) du \right), \quad t \in [0,1] \tag{3.5}$$

where $K^{H'}$ is the usual kernel of the fractional Brownian motion (1.3), $(W_t)_{t\in[0,T]}$ is a Wiener process and

$$H' = 1 + \frac{H-1}{q} \iff (2H' - 2)q = 2H - 2. \tag{3.6}$$

Then, if $d(H)$ is such that $\mathbf{E}|Y_1^{(q,H)}|^2 = 1$, the process $(Y_t^{q,H})_{t\in[0,T]}$ has the same finite dimensional distributions as the Hermite process $(Z_H^k(t))_{t\in[0,T]}$ given by (3.2).

Proof See [148]. We will prove the result for the case $k = 2$ in Sect. 3.2. □

Remark 3.4 There is an alternative way to define $Z_t^{(q,H)}$ as a multiple integral with respect to fractional Brownian motion. Some details can be found in [133]. Let $B^{H'}$ be a fractional Brownian motion with Hurst parameter H' given by (3.6) and denote by $I_q^{B^H}$ the multiple integral of order q with respect to this process. We define

$$Z_t^{(q,H)} = c(H) I_q^{B^{H'}}(\mu_t), \quad t \in [0,1] \tag{3.7}$$

where μ_t denotes the uniform measure on the diagonal D_t of $[0,t]^q$. The constant $c(H)$ is chosen so that the covariance of $Z^{(q)}$ is equal to $\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$.

3.1.3 Wiener Integrals with Respect to the Hermite Process

In this section, we introduce Wiener integrals with respect to the Hermite process. Consider a Hermite process given by (3.2).

Let us denote by $\mathcal{E}$ the class of elementary functions on $\mathbb{R}$ of the form

$$f(u) = \sum_{l=1}^{n} a_l 1_{(t_l, t_{l+1}]}(u), \quad t_l < t_{l+1},\ a_l \in \mathbb{R},\ l = 1, \ldots, n. \tag{3.8}$$

For $f \in \mathcal{E}$ as above, it is natural to define its Wiener integral with respect to the Hermite process Z_H^k by

$$\int_{\mathbb{R}} f(u) dZ_H^k(u) = \sum_{l=1}^{n} a_l \big(Z_H^k(t_{l+1}) - Z_H^k(t_l) \big). \tag{3.9}$$

In order to extend the definition (3.9) to a larger class of integrands, let us first make some observations. By formula (3.2) we can write

$$Z_H^k(t) = \int_{\mathbb{R}^k} I(1_{[0,t]})(y_1, \dots, y_k) dB(y_1) \cdots dB(y_k),$$

where by I we denote the mapping on the set of functions $f : \mathbb{R} \to \mathbb{R}$ to the set of functions $f : \mathbb{R}^k \to \mathbb{R}$

$$I(f)(y_1, \dots, y_k) = c(H,k) \int_{\mathbb{R}} f(u) \prod_{j=1}^{k} (u - y_i)_+^{-(\frac{1}{2} + \frac{1-H}{k})} du.$$

Note that for $k = 1$ this operator can be expressed in terms of fractional integrals and derivatives (see [9, 147]). Thus the definition (3.9) can also be written in the following form, due to the obvious linearity of I

$$\begin{aligned}
\int_{\mathbb{R}} f(u) dZ_H^k(u) &= \sum_{l=1}^{n} a_l \big(Z_H^k(t_{l+1}) - Z_H^k(t_l) \big) \\
&= \sum_{l=1}^{n} a_l \int_{\mathbb{R}^k} I(1_{(t_l, t_{l+1}]})(y_1, \dots, y_k) dB(y_1) \cdots dB(y_k) \\
&= \int_{\mathbb{R}^k} I(f)(y_1, \dots, y_k) dB(y_1) \cdots dB(y_k). \qquad (3.10)
\end{aligned}$$

We now introduce the following space

$$\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{R} \,\Big|\, \int_{\mathbb{R}^k} \big(I(f)(y_1, \dots, y_k) \big)^2 dy_1 \cdots dy_k < \infty \right\}$$

endowed with the norm

$$\|f\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^k} \big(I(f)(y_1, \dots, y_k) \big)^2 dy_1 \cdots dy_k.$$

We have

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= c(H,k)^2 \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} f(u) f(v) \prod_{j=1}^{k} (u - y_i)_+^{-(\frac{1}{2} + \frac{1-H}{k})} \right. \\
&\qquad \left. \times (v - y_i)_+^{-(\frac{1}{2} + \frac{1-H}{k})} dv du \right) dy_1 \cdots dy_k \\
&= c(H,k)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) f(v) \left(\int_{\mathbb{R}} (u - y)_+^{-(\frac{1}{2} + \frac{1-H}{k})} \right.
\end{aligned}$$

$$\times (v-y)_+^{-(\frac{1}{2}+\frac{1-H}{k})} dy\Big)^k dv du$$

$$= H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2} dv du.$$

Hence

$$\mathcal{H} = \left\{ f:\mathbb{R}\to\mathbb{R} \,\Big|\, \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2} dv du < \infty \right\}$$

and

$$\|f\|_{\mathcal{H}}^2 = H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2} dv du.$$

Let us observe that the mapping

$$f \mapsto \int_{\mathbb{R}} f(u) dZ_H^k(u) \tag{3.11}$$

provides an isometry from $\mathcal{E}$ to $L^2(\Omega)$. Indeed, for f of the form (3.8), we have

$$\begin{aligned}
\mathbf{E}\big[I(f)^2\big] &= \sum_{i,j=0}^{n-1} a_i a_j \mathbf{E}\big[\big(Z_H(t_{i+1}) - Z_H(t_i)\big)\big(Z_H(t_{j+1}) - Z_H(t_j)\big)\big] \\
&= \sum_{i,j=0}^{n-1} a_i a_j \big(R(t_{i+1},t_{j+1}) - R(t_{i+1},t_j) - R(t_i,t_{j+1}) + R(t_i,t_j)\big) \\
&= \sum_{i,j=0}^{n-1} a_i a_j H(2H-1)\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} |u-v|^{2H-2} dv du \\
&= \sum_{i,j=0}^{n-1} a_i a_j \langle 1_{(t_i,t_{i+1}]}, 1_{(t_j,t_{j+1}]}\rangle_{\mathcal{H}} = \|f\|_{\mathcal{H}}^2 .
\end{aligned}$$

On the other hand, it has been proved in [147] that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{H}$. As a consequence the mapping (3.11) can be extended to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ and relation (3.10) still holds.

This extension will be called the Wiener integral with respect to the Hermite process.

The space $\mathcal{H}$ coincides with the canonical Hilbert space associated to the fBm (see Sect. 1.1.3). Therefore the followings facts hold (see [147] or [136]):

- The elements of $\mathcal{H}$ may be not functions but distributions; it is therefore more practical to work with subspaces of $\mathcal{H}$ that are sets of functions. Such a subspace is

$$|\mathcal{H}| = \left\{ f : \mathbb{R} \to \mathbb{R} \;\Big|\; \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} dv du < \infty \right\}.$$

Then $|\mathcal{H}|$ is a strict subspace of $\mathcal{H}$ and we actually have the inclusions

$$L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^{\frac{1}{H}}(\mathbb{R}) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{3.12}$$

- The space $|\mathcal{H}|$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|f\|^2_{|\mathcal{H}|} = \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} dv du.$$

- A spectral domain included in $\mathcal{H}$ can also be defined as

$$\widehat{\mathcal{H}} = \left\{ f \in L^2(\mathbb{R}) \;\Big|\; \int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2H+1} dx < \infty \right\}, \tag{3.13}$$

where $\widehat{f}$ denotes the Fourier transform of f. We have again that $\widehat{\mathcal{H}}$ is a strict subspace of $\mathcal{H}$ and the inclusion

$$L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^{\frac{1}{H}}(\mathbb{R}) \subset \widehat{\mathcal{H}} \subset \mathcal{H}$$

holds. We define

$$\|f\|^2_{\widehat{\mathcal{H}}} = \int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2H+1} dx.$$

- There are elements in $|\mathcal{H}|$ that are not in $\widehat{\mathcal{H}}$ and vice versa.

3.2 A Particular Case: The Rosenblatt Process

In this section we will analyze some basic properties of the Rosenblatt process; in particular we are interested in its representation as a stochastic integral on a finite interval. As mentioned earlier, this process is obtained by taking $k = 2$ in the relation (3.2), so

$$Z_2(t) := Z(t) = a(H) \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_0^t (s-y_1)_+^{-\frac{2-H}{2}} (s-y_2)_+^{-\frac{2-H}{2}} ds \right) dB(y_1) dB(y_2) \tag{3.14}$$

where $(B(y), y \in \mathbb{R})$ is a standard Brownian motion on $\mathbb{R}$. The constant $a(H)$ is a positive normalizing constant and it is chosen so that $\mathbf{E}(Z(1)^2) = 1$. Indeed, it follows from (3.4) that

$$a(H)^2 = \left(\frac{\beta(\frac{H}{2}, H-1)^2}{2H(2H-1)} \right)^{-1}.$$

Recall that the process $(Z(t))_{t\in[0,T]}$ is self-similar of order H and it has stationary increments; it admits a Hölder continuous version of order $\delta < H$. Since $H \in (\frac{1}{2}, 1)$, it follows that the process Z exhibits long-range dependence.

3.2.1 Stochastic Integral Representation on a Finite Interval

As in the fBm case, we would like to represent Z_t as a stochastic integral with respect to a Brownian motion with time interval $[0, T]$. Recall that a fBm with $H > \frac{1}{2}$ can be written as (relation (1.2))

$$B_t^H = \int_0^t K^H(t,s)dW_s \tag{3.15}$$

with $(W_t, t \in [0, T])$ a standard Wiener process and

$$K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

where $t > s$ and $c_H = (\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})})^{\frac{1}{2}}$.

Note that to prove the representation (3.15) (at least in law) it suffices to see that the right-hand side has the same covariance R as the fBm; otherwise, it can easily be seen from the expression of the kernel K that the right-hand side of (3.15) is H-self-similar with stationary increments and as a consequence it cannot be anything else but a fractional Brownian motion with parameter H.

Since the Rosenblatt process is not Gaussian, the proof of a similar representation to (3.15) in this case needs a supplementary argument. We will use the concept of *cumulants*. The cumulants of a random variable X having all moments appear as the coefficients in the Maclaurin series of $g(t) = \log \mathbf{E}e^{tX}$, $t \in \mathbb{R}$. The first cumulant c_1 is the expectation of X while the second one is the variance of X. Generally, the nth cumulant is given by $g^{(n)}(0)$. The key fact is that for random variables in the second Wiener chaos the cumulants characterizes the law.

Let us consider a multiple integral $I_2(f)$ with $f \in L^2(\mathbb{R}^2)$ symmetric. Then the mth cumulant of the random variable $I_2(f)$ is given by (see [131] or [80])

$$\begin{aligned} c_m\big(I_2(f)\big) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} & f(y_1, y_2) f(y_2, y_3) \\ & \times \cdots \times f(y_{m-1}, y_m) f(y_m, y_1) dy_1 \cdots dy_m. \end{aligned} \tag{3.16}$$

In fact we have the following

Proposition 3.7 *Let K be the kernel (1.4) and let $(Z(t))_{t\in[0,T]}$ be a Rosenblatt process given by (3.14) with parameter H. Then it follows that*

$$Z(t) =^{(d)} d(H)\int_0^t\int_0^t\left[\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du\right]dB(y_1)dB(y_2) \tag{3.17}$$

where $(B_t, t\in[0,T])$ is a Brownian motion,

$$H' = \frac{H+1}{2} \tag{3.18}$$

and

$$d(H) = \frac{1}{H+1}\left(\frac{H}{2(2H-1)}\right)^{-\frac{1}{2}}. \tag{3.19}$$

Remark 3.5

(i) The constant $d(H)$ is a normalizing constant, it has been chosen so that $\mathbf{E}(Z(t)Z(s)) = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$. Indeed,

$$\begin{aligned}\mathbf{E}\big(Z(t)Z(s)\big) &= 2d(H)^2\int_0^{t\wedge s}\int_0^{t\wedge s} dy_1dy_2 \\ &\quad\times\left(\int_{y_1\vee y_2}^t\int_{y_1\vee y_2}^s \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)\frac{\partial K^{H'}}{\partial u}(v,y_1)\right. \\ &\quad\left.\times\frac{\partial K^{H'}}{\partial v}(v,y_2)dudv\right) \\ &= 2d(H)^2\int_0^t\int_0^s dvdu\left(\int_0^{u\wedge v}\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(v,y_1)dy_1\right)^2 \\ &= 2d(H)^2\big(H'(2H'-1)\big)^2\int_0^t\int_0^s |u-v|^{2H-2}dvdu = R(t,s).\end{aligned}$$

(ii) It can be seen without difficulty that the process

$$Z'(t) := d(H)\int_0^t\int_0^t\left[\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du\right]dB(y_1)dB(y_2)$$

defines an H-self-similar process. Indeed, for any $c>0$,

$$\begin{aligned}Z'(ct) &= \int_0^{ct}\int_0^{ct}\left[\int_{y_1\vee y_2}^{ct}\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du\right]dB(y_1)dB(y_2) \\ &= \int_0^{ct}\int_0^{ct}\left[\int_{\frac{y_1}{c}\vee\frac{y_2}{c}}^t \frac{\partial K^{H'}}{\partial u}(cu,y_1)\frac{\partial K^{H'}}{\partial u}(cu,y_2)cdu\right]dB(y_1)dB(y_2)\end{aligned}$$

$$= \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(cu, cy_1) \frac{\partial K^{H'}}{\partial u}(cu, cy_2) cdu \right] dB(cy_1) dB(cy_2)$$

and since $B(cy) =^{(d)} c^{\frac{1}{2}} B(y)$ and $\frac{\partial K^{H'}}{\partial u}(cu, cy_i) = c^{H' - \frac{3}{2}} \frac{\partial K^{H'}}{\partial u}(u, y_i)$ we obtain $Z(ct) =^{(d)} c^H Z(t)$.

Proof Let us denote by $Z'(t)$ the right-hand side of (3.17). Consider $b_1, \ldots, b_n \in \mathbb{R}$ and $t_1, \ldots, t_n \in [0, T]$. We need to show that the random variables

$$\sum_{l=1}^n b_l Z(t_l), \qquad \sum_{l=1}^n b_l Z'(t_l)$$

have the same distribution.

As mentioned above, the law of the multiple Wiener-Itô integral $I_2(f)$ is uniquely determined by its cumulants (3.16).

We will show that, for every $t, s \in [0, T]$, the random variables $Z_t + Z_s$ and $Z'_t + Z'_s$ have the same law; the general case will follow by a similar calculation. It follows that

$$Z'_t + Z'_s = I_2(f_{t,s})$$

where

$$f_{t,s}(y_1, y_2) = 1_{[0,t]}(y_1) 1_{[0,t]}(y_2) \int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du$$
$$+ 1_{[0,s]}(y_1) 1_{[0,s]}(y_2) \int_{y_1 \vee y_2}^s \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) duv. \quad (3.20)$$

Putting $a_m := \frac{(m-1)!}{2} 2^m d(H)^m$ we have

$$c_m(f_{s,t}) = a(m) \int_{\mathbb{R}^m} f_{t,s}(y_1, y_2) \cdots f_{t,s}(y_m, y_1) dy_1 \cdots dy_m$$
$$= a(m) \int_{\mathbb{R}^m} dy_1 \cdots dy_m \left(\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u_1, y_1) \frac{\partial K^{H'}}{\partial u}(u_1, y_2) du_1 \right.$$
$$\left. + \int_{y_1 \vee y_2}^s \frac{\partial K^{H'}}{\partial u}(u_1, y_1) \frac{\partial K^{H'}}{\partial u}(u_1, y_2) du_1 \right)$$
$$\times \left(\int_{y_2 \vee y_3}^t \frac{\partial K^{H'}}{\partial u}(u_2, y_2) \frac{\partial K^{H'}}{\partial u}(u_2, y_3) du_2 \right.$$
$$\left. + \int_{y_2 \vee y_3}^s \frac{\partial K^{H'}}{\partial u}(u_2, y_2) \frac{\partial K^{H'}}{\partial u}(u_2, y_3) du_2 \right)$$
$$\times \cdots$$

$$\times \left(\int_{y_m \vee y_1}^{t} \frac{\partial K^{H'}}{\partial u}(u_m, y_m) \frac{\partial K^{H'}}{\partial u}(u_m, y_1) du_m \right.$$
$$\left. + \int_{y_m \vee y_1}^{s} \frac{\partial K^{H'}}{\partial u}(u_m, y_1) \frac{\partial K^{H'}}{\partial u}(u_m, y_m) du_m \right)$$

and by the classical Fubini theorem

$$\begin{aligned} c_m(f_{s,t}) &= a(m) \sum_{t_j \in \{t,s\}} \int_0^{t_1} \cdots \int_0^{t_m} du_1 \cdots du_m \\ &\quad \times \left(\int_0^{u_1 \wedge u_m} \frac{\partial K^{H'}}{\partial u_1}(u_1, y_1) \frac{\partial K^{H'}}{\partial u_m}(u_m, y_1) dy_1 \right) \\ &\quad \times \left(\int_0^{u_1 \wedge u_2} \frac{\partial K^{H'}}{\partial u_1}(u_1, y_2) \frac{\partial K^{H'}}{\partial u_2}(u_2, y_2) dy_2 \right) \\ &\quad \times \cdots \\ &\quad \times \int_0^{u_{m-1} \wedge u_m} \frac{\partial K^{H'}}{\partial u_{m-1}}(u_m, y_m) \frac{\partial K^{H'}}{\partial u_m}(u_m, y_m) dy_m \\ &= a(m) \sum_{t_j \in \{t,s\}} \int_0^{t_1} \cdots \int_0^{t_m} du_1 \cdots du_m \\ &\quad \times |u_1 - u_2|^{2H'-2} |u_2 - u_3|^{2H'-2} \cdots |u_m - u_1|^{2H'-2} \end{aligned} \tag{3.21}$$

with $a'(m) = a(m)(H'(2H'-1))^m$.

The computation of the cumulant of $Z_t + Z_s$ is similar. Indeed, we can write, for $s, t \in [0, T]$,

$$Z(t) + Z(s) = I_2(g_{s,t})$$

where

$$g_{s,t} = a(H) \left(\int_0^t (u - y_1)_+^{\frac{H-2}{2}} (u - y_2)_+^{\frac{H-2}{2}} du + \int_0^s (u - y_1)_+^{\frac{H-2}{2}} (u - y_2)_+^{\frac{H-2}{2}} du \right)$$

and the mth cumulant of the kernel $g_{s,t}$ is given by (here $b(m) = \frac{(m-1)!}{2} 2^m a(H)^m$)

$$c_m(g_{s,t}) = b(m) \int_{\mathbb{R}^m} dy_1 \cdots dy_m \left(\int_0^t (u_1 - y_1)_+^{\frac{H-2}{2}} (u_1 - y_2)_+^{\frac{H-2}{2}} du_1 \right.$$
$$\left. + \int_0^s (u_1 - y_1)_+^{\frac{H-2}{2}} (u_1 - y_2)_+^{\frac{H-2}{2}} du_1 \right)$$
$$\times \left(\int_0^t (u_2 - y_2)_+^{\frac{H-2}{2}} (u_2 - y_3)_+^{\frac{H-2}{2}} du_2 \right.$$

$$
\begin{aligned}
&\quad + \int_0^s (u_2-y_2)_+^{\frac{H-2}{2}} (u_2-y_3)_+^{\frac{H-2}{2}} du_2 \Big) \\
&\quad \times \cdots \\
&\quad \times \left(\int_0^t (u_m-y_m)_+^{\frac{H-2}{2}} (u_m-y_1)_+^{\frac{H-2}{2}} du_1 \right. \\
&\quad \left. \times + \int_0^s (u_m-y_m)_+^{\frac{H-2}{2}} (u_m-y_1)_+^{\frac{H-2}{2}} du_m \right) \\
&= b(m) \sum_{t_j \in \{t,s\}} \int_0^{t_1} \cdots \int_0^{t_m} du_1 \cdots du_m \\
&= \int_{\mathbb{R}} (u_1-y_1)_+^{\frac{H-2}{2}} (u_m-y_1)_+^{\frac{H-2}{2}} dy_1 \int_{\mathbb{R}} (u_1-y_2)_+^{\frac{H-2}{2}} (u_2-y_2)_+^{\frac{H-2}{2}} dy_2 \\
&\quad \times \cdots \\
&\quad \times \int_{\mathbb{R}} (u_{m-1}-y_m)_+^{\frac{H-2}{2}} (u_m-y_m)_+^{\frac{H-2}{2}} dy_m.
\end{aligned}
$$

Using identity (3.3) we get

$$
\begin{aligned}
c_m(g_{s,t}) = b(m)\beta\left(\frac{H}{2}, H-1\right)^m \sum_{t_j \in \{t,s\}} \int_0^{t_1} \cdots \int_0^{t_m} du_1 \cdots du_m \\
\times |u_1-u_2|^{2H'-2} |u_2-u_3|^{2H'-2} \cdots |u_m-u_1|^{2H'-2} \qquad (3.22)
\end{aligned}
$$

and it remains to observe that $a'(m) = b(m)\beta(\frac{H}{2}, H-1)^m$ which implies that (3.21) equals (3.22). $\square$

We will conclude this section by proving that the Rosenblatt process possesses a similar property to fBm, that is, it can be approximated by a sequence of semi-martingales (in fact, since here we have $H > \frac{1}{2}$, by a sequence of bounded variation processes). See also [10] for related results.

The basic observation is that, if one formally interchanges the stochastic and Lebesgue integrals in (3.17), one gets

$$
Z(t) \text{``}=\text{''} \int_0^t \left(\int_0^u \int_0^u \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial u}(u,y_2) dB(y_1) dB(y_2) \right) du
$$

but the above expression cannot hold because the kernel $\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)$ does not belong to $L^2([0,T]^2)$ since the partial derivative $\frac{\partial K^{H'}}{\partial u}(u,y_1)$ behaves on the diagonal as $(u-y_1)^{\frac{H-2}{2}}$.

Let us define, for every $\varepsilon > 0$,

$$
Z^{\varepsilon}(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2) du \right]
$$

$$
\begin{aligned}
&\times dB(y_1)dB(y_2)\\
&= \int_0^t \left(\int_0^u \int_0^u \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2) dB(y_1)dB(y_2) \right) du\\
&:= \int_0^t A_\varepsilon(u)du.
\end{aligned}
$$

Since $A_\varepsilon \in L^2([0,T] \times \Omega)$ for every $\varepsilon > 0$ and it is adapted, it follows that the process Z^ε is a semimartingale.

Proposition 3.8 *For every* $t \in [0,T]$, $Z^\varepsilon(t) \to Z(t)$ *in* $L^2(\Omega)$.

Proof We have

$$
\begin{aligned}
Z^\varepsilon(t) - Z(t) = \int_0^t \int_0^t & dB(y_1)dB(y_2)\\
\times \Bigg(\int_{y_1 \vee y_2}^t & \left(\frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2) \right.\\
& \left. - \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \right) du \Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{E}\left|Z^\varepsilon(t) - Z(t)\right|^2\\
&= 2 \int_0^t \int_0^t dy_1 dy_2 \int_{y_1 \vee y_2}^t \int_{y_1 \vee y_2}^t dv du\\
&\times \left(\frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2) - \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \right)\\
&\times \left(\frac{\partial K^{H'}}{\partial v}(v+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial v}(v+\varepsilon, y_2) - \frac{\partial K^{H'}}{\partial v}(v, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) \right).
\end{aligned}
$$

Clearly the quantity $(\frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2) - \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2))$ converges to zero as $\varepsilon \to 0$ for every u, y_1, y_2 and the conclusion follows by the dominated convergence theorem. □

Remark 3.6 Another representation in law of the Rosenblatt process can be given as follows. Let W be a complex-valued Gaussian random measure on $\mathbb{R}$ such that for Borel sets in $\mathbb{R}$, A, B, A_j, $\mathbf{E}[W(A)] = 0, \mathbf{E}[W(A)\overline{W(B)}] =$ the Lebesgue measure of $A \cap B$, $W(\bigcup_{j=1}^n A_j) = \sum_{j=1}^n W(A_j)$ for mutually disjoint $A_1, \ldots, A_n$ and $W(A) = W(-A)$.

Let

$$\mathcal{H}_D = \Big\{ h : h \text{ is a complex-valued function on } \mathbb{R}, h(x) = \overline{h(-x)}, \int_{\mathbb{R}} h(x)^2 |x|^{D-1} dx < \infty \Big\}$$

with $D = 1 - H$ and for every $t \geq 0$ define an integral operator A_t by

$$A_t h(x) = C(D) \int_{-\infty}^{\infty} \frac{e^{it(x-y)-1}}{i(x-y)} h(y)|y|^{D-1} dy, \quad h \in \mathcal{H}_D. \tag{3.23}$$

Then A_t is a self-adjoint Hilbert-Schmidt operator (see [67]), all eigenvalues $\lambda_n(t), n = 1, 2, \ldots$, are real and satisfy $\sum_{n=1}^{\infty} \lambda_n^2(t) < \infty$.

Then for every $t_1, \ldots, t_d \geq 0$,

$$\big(Z_D(t_1), \ldots, Z_D(t_d)\big) =^{(d)} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)\big(\epsilon_n^2 - 1\big), \ldots, \sum_{n=1}^{\infty} \lambda_n(t_d)\big(\epsilon_n^2 - 1\big) \right),$$

where $\{\epsilon_n\}$ are i.i.d. $N(0, 1)$ random variables.

The case $d = 1$ was shown in Proposition 2 of [67] while the case $d \geq 1$ can be found in [115].

3.3 The Non-symmetric Rosenblatt Process

We will introduce here a variant of the Rosenblatt process called *the non-symmetric Rosenblatt process*. As we will see in this section, this process is also H-self-similar with stationary increments and it lives in the second Wiener chaos. This shows that in the second Wiener chaos we can have many self-similar processes with stationary increments. This does not happens in the first Wiener chaos, where fBm is the only self-similar process with stationary increments.

Let $H_1, H_2 \in (0, 1)$ such that

$$H_1 + H_2 > 1. \tag{3.24}$$

Consider the stochastic process $Y^{H_1,H_2} = (Y_t^{H_1,H_2})_{t \geq 0}$ given by, for every $t \geq 0$,

$$Y_t^{H_1,H_2} = c(H_1, H_2) \int_{\mathbb{R}^2} \left(\int_0^t (u - y_1)_+^{\frac{H_1}{2}-1} (u - y_2)_+^{\frac{H_2}{2}-1} du \right) dB_{y_1} dB_{y_2}, \tag{3.25}$$

where the integral above is a multiple Wiener-Itô stochastic integral of order 2.

Let us denote by f_t the kernel of $Y_t^{H_1,H_2}$, that is,

$$f_t(y_1, y_2) = c(H_1, H_2) \int_0^t (u - y_1)_+^{\frac{H_1}{2}-1} (u - y_2)_+^{\frac{H_2}{2}-1} du \tag{3.26}$$

for every $y_1, y_2 \in \mathbb{R}$. The kernel f_t is in general not symmetric with respect to the variables y_1, y_2 (except the case $H_1 = H_2$). We denote by $\tilde{f}_t$ its symmetrization

$$\tilde{f}_t(y_1, y_2) = \frac{1}{2}\big(f_t(y_1, y_2) + f_t(y_2, y_1)\big).$$

In this way, using the usual notation for multiple integrals, we can write $Y_t^{H_1,H_2} = I_2(f_t)$ for every $t \geq 0$. Condition (3.24) ensures that the kernel f_t belongs to $L^2([0,\infty)^2)$ for every t (this can be seen in the sequel of this section) and thus the double integral in (3.25) is well-defined.

The constant $c(H_1, H_2)$ will be chosen so that $\mathbf{E}[Y_1^2] = 1$. This constant, which will be explicitly calculated later, plays an important role.

Proposition 3.9 *Assume* (3.24). *Then the process* $(Y_t^{H_1,H_2})_{t\in[0,\infty)}$ *is* $\frac{1}{2}(H_1 + H_2)$ *self-similar and it has stationary increments.*

Proof Let $c > 0$. We have

$$\begin{aligned}
Y_{ct}^{H_1,H_2} &= c(H_1, H_2) \int_{\mathbb{R}^2} \left(\int_0^{ct} (u - y_1)_+^{\frac{H_1}{2}-1} (u - y_2)_+^{\frac{H_2}{2}-1} du \right) dB_{y_1} dB_{y_2} \\
&= c(H_1, H_2) c \int_{\mathbb{R}^2} \left(\int_0^{t} (cu - y_1)_+^{\frac{H_1}{2}-1} (cu - y_2)_+^{\frac{H_2}{2}-1} du \right) dB_{y_1} dB_{y_2} \\
&= c(H_1, H_2) c \int_{\mathbb{R}^2} \left(\int_0^{t} (cu - cy_1)_+^{\frac{H_1}{2}-1} (cu - cy_2)_+^{\frac{H_2}{2}-1} du \right) dB_{cy_1} dB_{cy_2} \\
&\stackrel{\mathrm{d}}{=} c^{\frac{H_1+H_2}{2}} Y_t
\end{aligned}$$

where we have used the $\frac{1}{2}$-self-similarity of the Wiener process B. Here $\stackrel{\mathrm{d}}{=}$ means equivalence of all finite dimensional distributions. It is obvious that the process $(Y_t^{H_1,H_2})$ has self-similarity since for every $h > 0$ and $t \geq 0$ we have $(Y_{t+h}^{H_1,H_2} - Y_h^{H_1,H_2}) \stackrel{\mathrm{d}}{=} (Y_t^{H_1,H_2})$. □

Remark 3.7 The particular case $H_1 = H_2 = H$ corresponds to the Rosenblatt process as defined in [67, 167]. We will call this process *the symmetric Rosenblatt process*. The process Y^{H_1,H_2} with $H_1 \neq H_2$ will be called the *non-symmetric Rosenblatt process*. Also note that the self-similar parameter of Y^{H_1,H_2} is always contained in the interval $(\frac{1}{2}, 1)$.

We will now compute the renormalizing constant appearing in (3.25).

Lemma 3.2 *Assume* $H_1, H_2 \in (0,1)$ *and* (3.24). *The normalizing constant* $c(H_1, H_2)$ *appearing in the definition of* Y^{H_1,H_2} *in* (3.25) *is given by*

$$c(H_1,H_2)^{-2} = \frac{1}{H(2H-1)}\left(\beta\left(1-H_1,\frac{H_1}{2}\right)\beta\left(1-H_2,\frac{H_2}{2}\right) + \beta\left(1-H,\frac{H_1}{2}\right)\beta\left(1-H,\frac{H_2}{2}\right)\right)$$

where $2H = H_1 + H_2$.

Remark 3.8 In the particular case $H_1 = H_2 = H$ we have

$$c(H,H)^{-2} := c(H) = \frac{2}{H(2H-1)}\beta\left(1-H,\frac{H}{2}\right)^2$$

and it coincides with the constant (3.4) with $k=2$.

Remark 3.9 Using again $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, the renormalizing constant $C(H_1, H_2)$ can be expressed in terms of gamma functions as follows:

$$\begin{aligned} c(H_1,H_2)^{-2} &= \frac{1}{H(2H-1)}\left[\frac{\Gamma(1-H_1)\Gamma(\frac{H_1}{2})}{\Gamma(1-\frac{H_1}{2})}\frac{\Gamma(1-H_2)\Gamma(\frac{H_2}{2})}{\Gamma(1-\frac{H_2}{2})} + \frac{\Gamma(1-H)\Gamma(\frac{H_1}{2})}{\Gamma(1-\frac{H_2}{2})}\frac{\Gamma(1-H)\Gamma(\frac{H_2}{2})}{\Gamma(1-\frac{H_1}{2})}\right] \\ &= \frac{1}{H(2H-1)}\frac{\Gamma(\frac{H_1}{2})\Gamma(\frac{H_2}{2})}{\Gamma(1-\frac{H_1}{2})\Gamma(1-\frac{H_2}{2})} \\ &\quad \times \left(\Gamma(1-H_1)\Gamma(1-H_2) + \Gamma(1-H)^2\right). \end{aligned}$$

We will prove in the next section that the processes Y^{H_1,H_2} given by (3.25) have different laws depending upon the values of the self-similarity parameters H_1 and H_2. We will use the cumulants (3.16).

Remark 3.10 Recall that the law of a multiple integral of order two is completely determined by its cumulants in the sense that, if two multiple integrals of order 2 have the same cumulants, then their distributions are the same (see [80]).

Let us compute the cumulants of the random variable $I_2(\tilde{f}_t)$ with fixed $t \geq 0$ and f_t given by (3.26). Using formula (3.16) and the expression of the kernel $\tilde{f}$, we get

$$\begin{aligned} c_m\big(I_2(\tilde{f}_t)\big) &= 2^{m-1}(m-1)!2^{-m}c(H_1,H_2)^m \\ &\quad \times \int_{\mathbb{R}^m}\left(\int_0^t (u_1-y_1)_+^{\frac{H_1}{2}-1}(u_1-y_2)_+^{\frac{H_2}{2}-1}du_1\right. \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t (u_1-y_2)_+^{\frac{H_1}{2}-1}(u_1-y_1)_+^{\frac{H_2}{2}-1}du_1\Big)\\
&\times\Big(\int_0^t (u_2-y_2)_+^{\frac{H_1}{2}-1}(u_2-y_3)_+^{\frac{H_2}{2}-1}du_2\\
&+\int_0^t (u_2-y_2)_+^{\frac{H_1}{2}-1}(u_2-y_3)_+^{\frac{H_2}{2}-1}du_2\Big)\\
&\times\cdots\\
&\times\Big(\int_0^t (u_{m-1}-y_{m-1})_+^{\frac{H_1}{2}-1}(u_{m-1}-y_m)_+^{\frac{H_2}{2}-1}du_{m-1}\\
&+\int_0^t (u_{m-1}-y_m)_+^{\frac{H_1}{2}-1}(u_{m-1}-y_{m-1})_+^{\frac{H_2}{2}-1}du_{m-1}\Big)\\
&\times\Big(\int_0^t (u_m-y_m)_+^{\frac{H_1}{2}-1}(u_m-y_1)_+^{\frac{H_2}{2}-1}du_m\\
&+\int_0^t (u_m-y_1)_+^{\frac{H_1}{2}-1}(u_m-y_m)_+^{\frac{H_2}{2}-1}du_m\Big)dy_1\cdots dy_m.
\end{aligned}
$$

We can state the main result of this section.

Proposition 3.10 *Let us consider the process* $(Y_t^{H_1,H_2})_{t\geq 0}$ *given by* (3.25). *There exist pairs* $(H_1,H_2),(H_1',H_2')\in(0,1)^2$ *with* $H_1+H_2=H_1'+H_2'=2H>1$ *such that* $(H_1,H_2)\neq(H_1',H_2')$ *and for any* $t>0$, *the laws of the random variables* $Y_t^{H_1,H_2}$ *and* $Y_t^{H_1',H_2'}$ *are different.*

Proof It suffices to show that for fixed t the two random variables $Y_t^{H_1,H_2}$ and $Y_t^{H_1',H_2'}$ have at least one different cumulant. The first two cumulants (that is, the expectation and the variance) of these two random variables are the same since Y^{H_1,H_2} is an H-self-similar process with stationary increments. Let us compute the third cumulant.

Let us consider the case $m=3$. Then, by changing the order of integration, we get

$$
\begin{aligned}
&c_3\big(I_2(\tilde{f}_t)\big)\\
&\quad=c(H_1,H_2)^3\int_0^t\int_0^t\int_0^t\Big[\Big(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\Big)\\
&\qquad\times\Big(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_2}{2}-1}(u_2-y)_+^{\frac{H_1}{2}-1}dy\Big)\\
&\qquad\times\Big(\int_{\mathbb{R}}(u_2-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\right)\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_2}{2}-1}(u_2-y)_+^{\frac{H_1}{2}-1}dy\right)\\
&\times\left(\int_{\mathbb{R}}(u_2-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\right)\\
&+\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\right)\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_2}{2}-1}(u_2-y)_+^{\frac{H_2}{2}-1}dy\right)\\
&\times\left(\int_{\mathbb{R}}(u_2-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\right)\\
&+\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\right)\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_2}{2}-1}(u_2-y)_+^{\frac{H_2}{2}-1}dy\right)\\
&\times\left(\int_{\mathbb{R}}(u_2-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\right)\\
&+\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\right)\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_1}{2}-1}(u_2-y)_+^{\frac{H_1}{2}-1}dy\right)\\
&\times\left(\int_{\mathbb{R}}(u_2-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\right)\\
&+\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\right)\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_1}{2}-1}(u_2-y)_+^{\frac{H_1}{2}-1}dy\right)\\
&\times\left(\int_{\mathbb{R}}(u_2-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\right)\\
&+\left(\int_{\mathbb{R}}dy(u_1-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}\right)\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_1}{2}-1}(u_2-y)_+^{\frac{H_2}{2}-1}dy\right)\\
&\times\left(\int_{\mathbb{R}}(u_2-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\right)\\
&+\left(\int_{\mathbb{R}}dy(u_1-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}\right)\left(\int_{\mathbb{R}}(u_1-y)_+^{\frac{H_1}{2}-1}(u_2-y)_+^{\frac{H_2}{2}-1}dy\right)\\
&\times\left(\int_{\mathbb{R}}(u_2-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\right)\Bigg]du_1du_2du_3.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
c_3\big(I_2(\tilde{f}_t)\big) =: c(H_1,H_2)^3\int_0^t\int_0^t\int_0^t\big[&g_{H_1,H_2}(u_1,u_2,u_3)+g_{H_1,H_2}(u_3,u_2,u_1)\\
&+f_{H_1,H_2}(u_1,u_2,u_3)\\
&+f_{H_1,H_2}(u_1,u_3,u_3)+f_{H_1,H_2}(u_2,u_1,u_3)+f_{H_1,H_2}(u_2,u_3,u_1)\\
&+f_{H_1,H_2}(u_3,u_1,u_2)+f_{H_1,H_2}(u_3,u_2,u_1)\big]du_1du_2du_3,
\end{aligned}
$$

where

$$g_{H_1,H_2}(u_1,u_2,u_3) = \left(\int_{\mathbb{R}} (u_1-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\right)$$
$$\times\left(\int_{\mathbb{R}} (u_1-y)_+^{\frac{H_2}{2}-1}(u_2-y)_+^{\frac{H_1}{2}-1}dy\right)$$
$$\times\left(\int_{\mathbb{R}} (u_2-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\right)$$

and

$$f_{H_1,H_2}(u_1,u_2,u_3) = \left(\int_{\mathbb{R}} (u_1-y)_+^{\frac{H_1}{2}-1}(u_3-y)_+^{\frac{H_1}{2}-1}dy\right)$$
$$\times\left(\int_{\mathbb{R}} (u_1-y)_+^{\frac{H_2}{2}-1}(u_2-y)_+^{\frac{H_1}{2}-1}dy\right)$$
$$\times\left(\int_{\mathbb{R}} (u_2-y)_+^{\frac{H_2}{2}-1}(u_3-y)_+^{\frac{H_2}{2}-1}dy\right).$$

Therefore, the function under the integral $du_1du_2du_3$ is symmetric with respect to the variables u_1,u_2,u_3. The integral $\int_0^t\int_0^t\int_0^t du_1du_2du_3$ is then equal to

$$3!\int_{u_3<u_2<u_1,u_1,u_2,u_3\in[0,t]} du_1du_3du_3.$$

Furthermore, it follows from Lemma 3.1 that, for $u_3<u_3<u_1$

$$g_{H_1,H_2}(u_1,u_2,u_3) = \beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)(u_1-u_3)^{\frac{H_1+H_2}{2}-1}$$
$$\times\beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)(u_1-u_2)^{\frac{H_1+H_2}{2}-1}$$
$$\times\beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)(u_2-u_3)^{\frac{H_1+H_2}{2}-1}$$

and

$$f_{H_1,H_2}(u_1,u_2,u_3) = \beta\left(1-H_1,\frac{H_1}{2}\right)(u_1-u_3)^{\frac{H_1+H_2}{2}-1}$$
$$\times\beta\left(1-H_2,\frac{H_1}{2}\right)(u_1-u_2)^{\frac{H_1+H_2}{2}-1}$$
$$\times\beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)(u_2-u_3)^{\frac{H_1+H_2}{2}-1}.$$

Thus we have

$$
\begin{aligned}
c_3\big(I_2(\tilde{f}_t)\big) &= 3!c(H_1,H_2)^3\bigg[\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\Big)\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\Big) \\
&\quad\times\bigg(\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\Big)+\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\Big)\bigg) \\
&\quad+2\beta\Big(1-H_1,\frac{H_1}{2}\Big)\beta\Big(1-H_2,\frac{H_2}{2}\Big)\bigg(\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\Big) \\
&\quad+\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\Big)\bigg)\bigg] \\
&\quad\times\int_{u_3<u_2<u_1,u_1,u_2,u_3\in[0,t]}(u_1-u_3)^{\frac{H_1+H_2}{2}-1}(u_1-u_2)^{\frac{H_1+H_2}{2}-1} \\
&\quad\times(u_2-u_3)^{\frac{H_1+H_2}{2}-1}du_1du_2du_3 \\
&= 3!c(H_1,H_2)^3\bigg(\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\Big)+\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\Big)\bigg) \\
&\quad\times\bigg(2\beta\Big(1-H_1,\frac{H_1}{2}\Big)\beta\Big(1-H_2,\frac{H_2}{2}\Big) \\
&\quad+\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\Big)\beta\Big(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\Big)\bigg) \\
&\quad\times\int_{u_3<u_2<u_1,u_1,u_2,u_3\in[0,t]}(u_1-u_3)^{\frac{H_1+H_2}{2}-1}(u_1-u_2)^{\frac{H_1+H_2}{2}-1} \\
&\quad\times(u_2-u_3)^{\frac{H_1+H_2}{2}-1}du_1du_2du_3 \\
&= 3!c(H_1,H_2)^3\bigg(\beta\Big(1-H,\frac{H_1}{2}\Big)+\beta\Big(1-H,\frac{H_2}{2}\Big)\bigg) \\
&\quad\times\bigg(2\beta\Big(1-H_1,\frac{H_1}{2}\Big)\beta\Big(1-H_2,\frac{H_2}{2}\Big) \\
&\quad+\beta\Big(1-H,\frac{H_1}{2}\Big)\beta\Big(1-H,\frac{H_2}{2}\Big)\bigg) \\
&\quad\times\int_{u_3<u_2<u_1,u_1,u_2,u_3\in[0,t]}(u_1-u_3)^{H-1}(u_1-u_2)^{H-1} \\
&\quad\times(u_2-u_3)^{H-1}du_1du_2du_3
\end{aligned}
$$

and using gamma integrals we get

$$
\begin{aligned}
&c_3\big(I_2(\tilde{f}_t)\big) \\
&\quad= 3!c(H_1,H_2)^3 \\
&\qquad\times\frac{\Gamma(1-H)\Gamma(\frac{H_1}{2})\Gamma(\frac{H_2}{2})}{(\Gamma(1-\frac{H_1}{2})\Gamma(1-\frac{H_2}{2}))^2}\bigg(\Gamma\Big(\frac{H_1}{2}\Big)\Gamma\Big(1-\frac{H_1}{2}\Big)+\Gamma\Big(\frac{H_2}{2}\Big)\Gamma\Big(1-\frac{H_2}{2}\Big)\bigg)
\end{aligned}
$$

$$\times \big(2\Gamma(1-H_1)\Gamma(1-H_2)+\Gamma(1-H)^2\big)$$
$$\times \int_{u_3<u_2<u_1,u_1,u_2,u_3\in[0,t]} (u_1-u_3)^{H-1}(u_1-u_2)^{H-1}$$
$$\times (u_2-u_3)^{H-1} du_1 du_2 du_3.$$

It is obvious, given the expression of the normalizing constant $c(H_1, H_2)$, that there exist $(H_1, H_2) \neq (H_1', H_2')$ with $c_3(I_2(f_{H_1,H_2})) \neq c_3(I_2(f_{H_1',H_2'}))$ (see also the following remark for a numerical example). □

Remark 3.11 Since the gamma function can be numerically calculated for any value of the parameter (see for example http://www.efunda.com/math/gamma/findgamma.cfm), the constant appearing in the expression of the third cumulant above can also be numerically computed and it can be seen that it has different values when $(H_1, H_2) \neq (H_1', H_2')$ and $H_1 + H_2 = H_1' + H_2'$. Take for example $H_1 = H_2 = 0.4$ and $H_1' = 0.3, H_2' = 0.5$. Then $\Gamma(0,4) = 2.21$, $\Gamma(0.2) = 4.49$, $\Gamma(0.8) = 1.16$, $\Gamma(0.6) = 1.48$, $\Gamma(0.3) = 2.99$, $\Gamma(0.5) = 1.77$, $\Gamma(0.15) = 6.22$, $\Gamma(0.25) = 3.32$, $\Gamma(0.75) = 1.22$, $\Gamma(0.85) = 1.11$ and this leads, after exact computation, to different values for the cumulants.

Remark 3.12 There are other classes of self-similar processes with stationary increments in the second Wiener chaos. See Exercise 3.5.

3.4 Bibliographical Notes

Although defined during the 60s and 70s ([67, 157, 168]) due to their appearance in the Non-Central Limit Theorem, the systematic analysis of Hermite processes has only been developed during the last ten years, motivated by their nice properties (self-similarity, stationarity of the increments, long-range dependence). As attested by the empirical data, these stochastic processes are well suited to applications featuring non-Gaussian long-range dependence. An example is provided in [169], which uses an empirical study from [107], see also the bibliographical guide [191]. Results on several aspects (stochastic integration, weak approximation, distributional properties, estimation) related to this class of stochastic processes can be found in [6, 7, 30, 34, 45, 81, 112, 144, 148, 174, 188], among others. The non-symmetric Rosenblatt process has been studied in [114].

3.5 Exercises

Exercise 3.1 ([171]) Define the following approximation for the Rosenblatt process

$$Z_t^n = \sum_{i,j=1;i\neq j}^{n} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} F\left(\frac{[nt]}{n}, u, v\right) dv du \frac{\xi_i}{\sqrt{n}} \frac{\xi_j}{\sqrt{n}}, \quad t \in [0, T] \tag{3.27}$$

where F is the kernel of the Rosenblatt process in the representation (3.17)

$$F(t, y_1, y_2) = 1_{[0,t]}(y_1)1_{[0,t]}(y_2) \int_{y_1 \vee y_2}^{t} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

Prove that the family of stochastic processes $(Z_t^n)_{t \in [0,T]}$ converges in the sense of finite dimensional distributions to the Rosenblatt process $(Z_t^H)_{t \in [0,T]}$ (with self-similarity index $H = 2H' - 1$). (Compare with Exercise 1.11.)

Remark 3.13 We eliminate the diagonal "$i = j$" because the Rosenblatt process is defined as a double Wiener-Itô integral and as a consequence it has zero mean. When the diagonal $i = j$ is included in (3.27) then the limit is in general a double Stratonovich integral (see [92] or [162]).

Exercise 3.2 Let $(Z_t^{(H)})_{t \in [0,1]}$ be a Rosenblatt process with self-similarity index $H \in (\frac{1}{2}, 1)$. Then, for every $s, t \in [0, 1]$, prove that

$$\mathbf{E}\big(Z_t^{(H)} - Z_s^{(H)}\big)^3 = C(H)|t - s|^{3H} \tag{3.28}$$

where

$$C(H) = 8a(H)^3 d(H)^3 \int_{[0,1]^3} \big(|u - v|\,|u - u'|\,|v - u'|\big)^{2H'-2} du du' dv. \tag{3.29}$$

(Observe a significant difference from the Gaussian case: here this cubic mean is not zero.)

Exercise 3.3 ([148]) Prove the following positive half-axis representation of the Hermite process:

$$c_{K,H} \int_{(0\infty)^k} \int_0^t \prod_{j=1}^{k} x_j^{\frac{1}{2}-H'} (1 - s x_j)^{H' - \frac{3}{2}} dB(x_1) \cdots dB(x_k).$$

Exercise 3.4 ([47, 112]) Let $(Z_H^k(t))_{t \in \mathbb{R}}$ be a Hermite process of order k and let $\xi \in L^0(\mathbb{R})$. Show that the following are true for almost all ω and for every $\lambda, \sigma > 0$.

(i) For all $t > a$, the integral $\int_a^t e^{\lambda u} dZ_H^k(u, \omega)$ exists in the Riemann-Stieltjes sense and it is equal to

$$e^{\lambda t} Z_H^k(t, \omega) - e^{\lambda a} Z_H^k(a, \omega) - \lambda \int_a^t Z_H^k(u, \omega) e^{\lambda u} du.$$

Moreover, the function $t \mapsto \int_a^t e^{\lambda u} dZ_H^k(u, \omega)$ is continuous.

(ii) The unique continuous solution of the equation

$$y^{\xi}(t) = \xi(\omega) - \lambda \int_0^t y^{\xi}(s) ds + \sigma Z_H^k(t, \omega), \quad t \geq 0$$

is given by

$$y^{\xi}(t) = e^{-\lambda t}\left(\xi(\omega) + \sigma \int_0^t e^{\lambda u}\, dZ_H^k(u,\omega)\right), \quad t \geq 0.$$

The above integral is a Wiener integral with respect to the Hermite process. In particular, if $\xi = \sigma \int_{-\infty}^0 e^{\lambda u}\, dZ_H^k(u,\omega)$, then

$$y(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)}\, dZ_H^k(u,\omega), \quad t \geq 0.$$

(iii) Prove that y exhibits long-range dependence for $H \in (\frac{1}{2}, 1)$ and that, when $t \in \mathbb{R}$, $N = 1, 2, \ldots$ and $s \to \infty$, its covariance function behaves as

$$\mathbf{E}\big[y(t)y(t+s)\big] = \frac{1}{2}\sigma^2 \sum_{m=1}^{N} \lambda^{-2n} \left(\prod_{j=0}^{2n-1} (2H - j)\right) s^{2H-2n} + O\big(s^{2H-2N-2}\big). \tag{3.30}$$

Exercise 3.5 ([157] and [122]) Consider α, β such that $\frac{1}{2} < \alpha < \frac{3}{4}$ and $0 < 2 - 2\alpha - \beta < 1$. Define for every $t \geq 0$ (B is a Brownian motion)

$$X_t = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_0^\infty (u - y_1)_+^{-\alpha} (u - y_2)_+^{-\alpha} \big(|u|^{-\beta} - |u - t|^{-\beta}\big) du\right) dB_{y_1} dB_{y_2}.$$

1. Prove that the process $X = (X_t)_{t \geq 0}$ defined above is H-self-similar with stationary increments where $H = 2 - \beta - 2\alpha$.
2. Prove that for suitable choices of α, β, the law of the process X defined above is different from the law of the process Y (3.25).

Chapter 4
Multiparameter Gaussian Processes

A two-parameter stochastic process is a stochastic process indexed by a time interval which is a subset of $\mathbb{R}^2$. The most studied cases is two-parameter Brownian motion (also called the Brownian sheet).

The stochastic analysis of two-parameter Brownian motion and more generally, of two-parameter martingales, was mostly developed in the eighties. There are several monographs (e.g. [69, 101]) that discuss various aspects of the multiparameter martingales. Later, some classes of stochastic processes that are not (semi)-martingales came to attention. One of them is the so-called *fractional Brownian sheet*. In this part of the monograph we survey its basic properties, as well as those of the two-parameter Hermite processes. We also generalize these processes to the multiparameter case (i.e. where the number of time parameters is greater than 2).

4.1 The Anisotropic Fractional Brownian Sheet

Let us define the anisotropic fractional Brownian sheet.

Definition 4.1 A fractional Brownian sheet $(W^{\alpha,\beta}_{s,t})_{(s,t)\in[0,\infty)^2}$ with Hurst indices $(\alpha,\beta)\in(0,1)^2$ is a centered two-parameter Gaussian process whose covariance function is given by

$$\begin{aligned}\mathbf{E}\big(W^{\alpha,\beta}_{t,s}W^{\alpha,\beta}_{u,v}\big) &= R_\alpha(t,u)R_\beta(s,v)\\ &= \frac{1}{2}\big(t^{2\alpha}+u^{2\alpha}-|t-u|^{2\alpha}\big)\frac{1}{2}\big(s^{2\beta}+v^{2\beta}-|s-v|^{2\beta}\big).\end{aligned}$$

Recall that R_α is the covariance function of the one-parameter fractional Brownian motion (1.1). The process was introduced in [100] and then studied in [12].

Remark 4.1 The process $W^{\alpha,\beta}$ is called anisotropic because the covariance is defined as the product of two-covariances of a one-parameter fBm. There exists an *isotropic* fractional Brownian sheet, see Remark 4.3.

C.A. Tudor, *Analysis of Variations for Self-similar Processes*,
Probability and Its Applications, DOI 10.1007/978-3-319-00936-0_4,

When $\alpha = \beta = \frac{1}{2}$ we obtain the Brownian sheet, which is a centered Gaussian process $(W_{s,t})_{s,t\geq 0}$ with covariance

$$\mathbf{E} W_{s,t} W_{u,v} = (t \wedge u)(s \wedge v)$$

for every $s, t, u, v \geq 0$.

Remark 4.2 The partial processes $t \to W^{\alpha,\beta}_{t,s}$ and $s \to W^{\alpha,\beta}_{t,s}$ are "weighted" fractional Brownian motions (see Exercise 4.1).

4.1.1 Basic Properties

The basic properties of $W^{\alpha,\beta}$ are listed below.

Proposition 4.1 *The process $W^{\alpha,\beta}$ is self-similar of order (α, β) (in the sense of Definition A.4 in Appendix A).*

Proof From Definition A.4, we need to prove that for every $h, k > 0$, the process $\hat{W}^{\alpha,\beta}$ defined by

$$\big(\hat{W}^{\alpha,\beta}_{s,t}\big)_{s,t} = \big(h^{\alpha} k^{\beta} W^{\alpha,\beta}_{\frac{s}{h},\frac{t}{k}}\big)_{s,t}$$

has the same law as $W^{\alpha,\beta}$. For every $s, t, u, v \geq$ and $h, k > 0$ we have

$$\begin{aligned}
\mathbf{E}\hat{W}^{\alpha,\beta}_{s,t}\hat{W}^{\alpha,\beta}_{u,v} &= h^{2\alpha}k^{2\beta}\mathbf{E}W^{\alpha,\beta}_{\frac{s}{h},\frac{t}{k}}W^{\alpha,\beta}_{\frac{u}{h},\frac{v}{k}} \\
&= h^{2\alpha}k^{2\beta}R_{\alpha}\left(\frac{s}{h},\frac{u}{h}\right)R_{\beta}\left(\frac{t}{k},\frac{v}{k}\right) \\
&= R_{\alpha}(s,u)R_{\beta}(t,v) \\
&= \mathbf{E}W^{\alpha,\beta}_{s,t}W^{\alpha,\beta}_{u,v}.
\end{aligned}$$

Since both $W^{\alpha,\beta}$ and $\hat{W}^{\alpha,\beta}$ are centered Gaussian processes, we obtain the equivalence of their finite dimensional distributions. □

Proposition 4.2 *The process $W^{\alpha,\beta}$ has stationary increments.*

Proof Recall Definition A.5. It suffices to check that for every $h, k > 0$ the process

$$\big(W^{\alpha,\beta}_{s+h,t+k} - W^{\alpha,\beta}_{h,t+k} - W^{\alpha,\beta}_{s+h,k} + W^{\alpha,\beta}_{h,k}, s, t \geq 0\big)$$

has the same law as $(W^{\alpha,\beta}_{s,t}, s, t \geq 0)$. To this end, it suffices to compute the covariance of the first process and, using the fact that the fBm has stationary increments, to conclude that it coincides with the covariance of $W^{\alpha,\beta}$. □

Proposition 4.3 *The fractional Brownian sheet* $(W^{\alpha,\beta}_{s,t})_{s,t\geq 0}$ *admits a version* $\tilde{W}^{\alpha,\beta}$ *with continuous trajectories. Moreover, its paths are Hölder continuous of order* (α', β') *with* $\alpha' < \alpha$ *and* $\beta' < \beta$ *in the sense that: for every* ω *there exists a* $C_\omega > 0$ *such that for every* s, s_1, t, t_1

$$\left|\tilde{W}^{\alpha,\beta}_{s,t} - \tilde{W}^{\alpha,\beta}_{s,t_1} - \tilde{W}^{\alpha,\beta}_{s_1,t} + \tilde{W}^{\alpha,\beta}_{s_1,t_1}\right| \leq C_\omega |s-s_1|^{\alpha'} |t-t_1|^{\beta'}.$$

Proof From the self-similarity and the stationarity of the increments of the fractional Brownian sheet $W^{\alpha,\beta}$, it follows that for every $p \geq 1$

$$\mathbf{E}\left|W^{\alpha,\beta}_{s+h,t+k} - W^{\alpha,\beta}_{h,t+k} - W^{\alpha,\beta}_{s+h,k} + W^{\alpha,\beta}_{h,k}\right|^p = C_p |h|^{p\alpha} |k|^{p\beta}$$

with $C_p > 0$ depending only on p. Then it suffices to use the Kolmogorov continuity criterium for two-parameter processes (see Theorem B.2). □

4.1.2 Stochastic Integral Representation

There exists a two-parameter version of the Wiener integral representation (1.2). The fractional Brownian sheet with Hurst parameters $\alpha, \beta \in (0,1)$ and with time parameters $s, t \in [0,T]$ can be defined as (see [20])

$$W^{\alpha,\beta}_{s,t} = \int_0^t \int_0^s K^\alpha(t,u) K^\beta(s,v) dW_{u,v}, \quad \text{for every } s,t \in [0,T], \tag{4.1}$$

where $(W_{u,v})_{u,v\in[0,T]}$ is the Brownian sheet and the deterministic kernels K_α, K_β are defined by (1.3). The stochastic integral in (4.1) is a Wiener integral with respect to the Wiener sheet W. There also exists a moving average representation (see formula (4.5) with $k=1$). It is immediate that the process given by (4.1) is a fractional Brownian sheet. Indeed, the isometry of the Wiener integral implies that for all s, t, u, v

$$\begin{aligned}\mathbf{E} W^{\alpha,\beta}_{s,t} W^{\alpha,\beta}_{u,v} &= \int_0^{t\wedge u} K_\alpha(t,a) K_\alpha(u,a) da \int_0^{s\wedge v} K_\beta(s,b) K_\beta(v,b) db \\ &= R_\alpha(t,u) R_\beta(s,v)\end{aligned}$$

by (1.5).

The canonical Hilbert Space $\mathcal{H}^{\alpha,\beta}$ of the fractional Brownian sheet is the closure of the linear space generated by the indicator functions on $[0,T]^2$ with respect to the scalar product

$$\langle 1_{[0,t]\times[0,s]}, 1_{[0,u]\times[0,v]} \rangle_{\mathcal{H}^{\alpha,\beta}} = R_\alpha(t,u) R_\beta(s,v).$$

Fix $\alpha, \beta > \frac{1}{2}$. Notice that in this case, by tensorization of the scalar product in the space $\mathcal{H}$ of the fBm, we have for every $f, g \in \mathcal{H}^{\alpha,\beta}$

$$\langle f, g\rangle_{\mathcal{H}^{\alpha,\beta}} = c(\alpha)c(\beta)\int_0^T\int_0^T\int_0^T\int_0^T f(a,b)g(m,n)|a-m|^{2\alpha-2} \times |b-n|^{2\beta-n}dadbdmdn \tag{4.2}$$

and $c(\alpha) = \alpha(2\alpha-1)$. Define the operator $K^{*,2}_{\alpha,\beta}$ on the space of step functions on $[0,T]^2$ to $L^2([0,T]^2)$ given by

$$\left(K^{*,2}_{\alpha,\beta}f\right)(s,t) = \int_t^T\int_s^T f\left(r,r'\right)\frac{\partial K^\alpha}{\partial r}(r,t)\frac{\partial K^\beta}{\partial r'}\left(r',s\right)drdr'. \tag{4.3}$$

We have

$$\left\langle K^{*,2}_{\alpha,\beta}f, K^{*,2}_{\alpha,\beta}g\right\rangle_{L^2([0,T]^2)} = \langle f, g\rangle_{\mathcal{H}^{\alpha,\beta}}. \tag{4.4}$$

Indeed,

$$\begin{aligned}
&\left\langle K^{*,2}_{\alpha,\beta}f, K^{*,2}_{\alpha,\beta}g\right\rangle_{L^2([0,T]^2)} \\
&= \int_0^T\int_0^T\left(\int_u^T\int_v^T f(a,b)\frac{\partial K^\alpha}{\partial a}(a,u)\frac{\partial K^\beta}{\partial b}(b,v)dadb\right) \\
&\quad \times\left(\int_u^T\int_v^T g(m,n)\frac{\partial K^\alpha}{\partial m}(m,u)\frac{\partial K^\beta}{\partial n}(n,v)dmdn\right)dudv \\
&= \int_0^T\int_0^T\int_0^T\int_0^T f(a,b)g(m,n) \\
&\quad \times\left(\int_0^{a\wedge m}\int_0^{b\wedge n}\frac{\partial K^\alpha}{\partial m}(m,u)\frac{\partial K^\beta}{\partial n}(n,v)dmdndudv\right)dadbdmdn \\
&= \int_0^T\int_0^T\int_0^T\int_0^T f(a,b)g(m,n)\frac{\partial^2 R^\alpha}{\partial a\partial m}(a,m)\frac{\partial^2 R^\beta}{\partial b\partial n}(b,n) \\
&= c(\alpha)c(\beta)\int_0^T\int_0^T\int_0^T\int_0^T f(a,b)g(m,n)|a-m|^{2\alpha-2} \\
&\quad \times|b-n|^{2\beta-n}dadbdmdn \\
&= \langle f, g\rangle_{\mathcal{H}^{\alpha,\beta}}.
\end{aligned}$$

Therefore, a function $f : [0,T]^2 \to \mathbb{R}$ belongs to $\mathcal{H}^{\alpha,\beta}$ if and only if $K^{*,2}f$ is in $L^2([0,T]^2)$. This also implies the transfer formula

$$\int_{[0,T]^2} f(x,y)dW^{\alpha,\beta}_{x,y} = \int_{[0,T]^2} K^{*,2}_{\alpha,\beta}f(x,y)dW_{x,y}$$

where W is the Wiener process from (4.1).

In the case when $\alpha < \frac{1}{2}$ and/or $\beta < \frac{1}{2}$ we can also define a transfer operator $K^{*,2}$ but the expression is more complicated.

4.2 Two-Parameter Hermite Processes

Let $(W(x,y), x, y \in \mathbb{R})$ be a two-parameter Brownian motion. For $k \geq 1$ and $H_1, H_2 \in (\frac{1}{2}, 1)$ define, for $s, t \geq 0$

$$Z_{t,s}^{k,H} = c_{H,k} \int_{(\mathbb{R}^2)^k} dW(x_1, y_1) \cdots dW(x_k, y_k) \times \left(\int_0^t da \int_0^s db \prod_{j=1}^k (a - x_j)_+^{-(\frac{1}{2} - \frac{1-H_1}{k})} (b - y_j)_+^{-(\frac{1}{2} - \frac{1-H_2}{k})} \right). \quad (4.5)$$

The above integral represents a multiple integral of order k with respect to the Wiener sheet W. The constant $c_{H,k}$ is given by

$$c_{H,k} = c_{H_1,k} c_{H_2,k}$$

with $c_{H_1,k}, c_{H_2,k}$ the constants from the integral representation of the Hermite process (3.4). This constant guarantees that $\mathbf{E}(Z_{t,s}^{k,H})^2 = 1$.

The two-parameter Hermite process has the same covariance as the fractional Brownian sheet with Hurst indices H_1, H_2.

Proposition 4.4 *For all $s, t, u, v \geq 0$*

$$\mathbf{E} Z_{t,s}^{k,H_1} Z_{u,v}^{k,H_2} = R_{H_1}(t,u) R_{H_2}(s,v)$$

with R_{H_1}, R_{H_2} given by (1.1).

Proof By the isometry of multiple integrals and using identity (3.3)

$$\begin{aligned}
\mathbf{E} Z_{t,s}^{k,H} Z_{u,v}^{k,H} &= c_{k,H}^2 k! \int_{\mathbb{R}^{2k}} dx_1 \cdots dx_k dy_1 \cdots dy_k \\
&\quad \times \left(\int_0^t da \int_0^s db \prod_{j=1}^k (a - x_j)_+^{-(\frac{1}{2} - \frac{1-H_1}{2})} (b - y_j)_+^{-(\frac{1}{2} - \frac{1-H_2}{2})} \right) \\
&\quad \times \left(\int_0^u da' \int_0^v db' \prod_{j=1}^k \left(a' - x_j\right)_+^{-(\frac{1}{2} - \frac{1-H_1}{2})} \left(b' - y_j\right)_+^{-(\frac{1}{2} - \frac{1-H_2}{2})} \right) \\
&= c_{k,H}^2 k! \int_0^t da \int_0^s db \int_0^u da' \int_0^v db'
\end{aligned}$$

$$\times \left(\int_{\mathbb{R}} (a-x)_+^{-(\frac{1}{2}-\frac{1-H_1}{2})} (a'-x)_+^{-(\frac{1}{2}-\frac{1-H_1}{2})} \right)^k$$
$$\times \left(\int_{\mathbb{R}} (b-y)_+^{-(\frac{1}{2}-\frac{1-H_2}{2})} (b'-y)_+^{-(\frac{1}{2}-\frac{1-H_2}{2})} \right)^k$$
$$= c_{k,H}^2 k! \beta\left(\frac{1}{2} - \frac{1-H_1}{k}, \frac{2H_1-2}{k}\right)^k \beta\left(\frac{1}{2} - \frac{1-H_2}{k}, \frac{2H_2-2}{k}\right)^k$$
$$\times \int_0^t da \int_0^s db \int_0^u da' \int_0^v db' |a-a'|^{2H_1-2} |b-b'|^{2H_2-2}$$
$$= R_{H_1}(t,u) R_{H_2}(s,v). \qquad \square$$

Proposition 4.5 *The process Z^{k,H_1,H_2} has stationary increments in the sense of Definition A.5.*

Proof From the representation (4.5) one can see that for every $a, b > 0$,

$$Z^{k,H_1,H_2}_{t+a,s+b} - Z^{k,H_1,H_2}_{t,s+b} - Z^{k,H_1,H_2}_{t+a,s} + Z^{k,H_1,H_2}_{a,b}$$
$$=^{(d)} c_{H,k} \int_{(\mathbb{R}^2)^k} dW(x_1,y_1) \cdots dW(x_k,y_k)$$
$$\times \left(\int_0^t da \int_0^s db \prod_{j=1}^k (a-x_j)_+^{-(\frac{1}{2}-\frac{1-H_1}{k})} (b-y_j)_+^{-(\frac{1}{2}-\frac{1-H_2}{k})} \right)$$
$$= Z^{k,H}_{t,s}. \qquad \square$$

Proposition 4.6 *The process Z^{k,H_1,H_2} is self-similar in the sense of Definition A.4.*

Proof For every $h_1, h_2 > 0$, consider the process

$$\hat{Z}^{k,H_1,H_2}_{t,s} = h_1^{H_1} h_2^{H_2} Z^{k,H_1,H_2}_{\frac{t}{h_1},\frac{s}{h_2}}.$$

We want to show that it has the same law as Z^{k,H_1,H_2}. We have

$$h_1^{H_1} h_2^{H_2} Z^{k,H_1,H_2}_{\frac{t}{h_1},\frac{s}{h_2}}$$
$$= h_1^{H_1} h_2^{H_2} c_{H,k} \int_{(\mathbb{R}^2)^k} dW(x_1,y_1) \cdots dW(x_k,y_k)$$
$$\times \left(\int_0^{\frac{t}{h_1}} da \int_0^{\frac{s}{h_2}} db \prod_{j=1}^k (a-x_j)_+^{-(\frac{1}{2}-\frac{1-H_1}{k})} (b-y_j)_+^{-(\frac{1}{2}-\frac{1-H_2}{k})} \right)$$
$$= h_1^{H_1-1} h_2^{H_2-1} c_{H,k} \int_{(\mathbb{R}^2)^k} dW(x_1,y_1) \cdots dW(x_k,y_k)$$

$$\times \left(\int_0^t da \int_0^s db \prod_{j=1}^k (ah_1 - x_j)_+^{-(\frac{1}{2} - \frac{1-H_1}{k})} (bh_2 - y_j)_+^{-(\frac{1}{2} - \frac{1-H_2}{k})} \right)$$

$$= h_1^{H_1-1} h_2^{H_2-1} h_1^{-k(\frac{1}{2} - \frac{1-H_1}{k})} h_2^{-k(\frac{1}{2} - \frac{1-H_2}{k})} c_{H,k}$$

$$\times \int_{(\mathbb{R}^2)^k} dW\big(x_1 h_1^{-1}, y_1 h_2^{-1}\big) \cdots dW\big(x_k h_1^{-1}, y_k h_2^{-2}\big)$$

$$\times \left(\int_0^t da \int_0^s db \prod_{j=1}^k (a - x_j)_+^{-(\frac{1}{2} - \frac{1-H_1}{k})} (b - y_j)_+^{-(\frac{1}{2} - \frac{1-H_2}{k})} \right)$$

$$=^{(d)} c_{H,k} \int_{(\mathbb{R}^2)^k} dW(x_1, y_1) \cdots dW(x_k, y_k)$$

$$\times \left(\int_0^t da \int_0^s db \prod_{j=1}^k (a - x_j)_+^{-(\frac{1}{2} - \frac{1-H_1}{k})} (b - y_j)_+^{-(\frac{1}{2} - \frac{1-H_2}{k})} \right)$$

where in the last line we use the scaling property of the Brownian sheet. □

Proposition 4.7 *The process Z^{k,H_1,H_2} admits a version with continuous trajectories. Moreover, its paths are Hölder continuous of order (α', β') with $\alpha' < \alpha$ and $\beta' < \beta$ in the sense that: for every ω there exists a $C_\omega > 0$ such that for every s, s_1, t, t_1*

$$\big| Z_{s,t}^{k,H_1,H_2} - Z_{s,t_1}^{k,H_1,H_2} - Z_{s_1,t}^{k,H_1,H_2} + Z^{k,H_1,H_2} \big| \le C_\omega |s - s_1|^{\alpha'} |t - t_1|^{\beta'}.$$

Proof This is a consequence of Propositions 4.5, 4.6 and of the Kolmogorov continuity criterium (Theorem B.2). □

As for the fractional Brownian sheet, it is possible to define the two-parameter Hermite process as a multiple integral with respect to the Wiener sheet on a finite time interval. Let $H_1, H_2 \in (\frac{1}{2}, 1)$, $k \ge 1$ and define

$$\begin{aligned} Y_{s,t}^{k,H_1,H_2} = b_{k,H_1,H_2} &\int_0^t \int_0^s dW_{u_1,v_1} \cdots \int_0^t \int_0^s dW_{u_k,v_k} \\ &\times \left(\int_{u_1 \vee \cdots \vee u_k}^t da \partial_1 K_{H_1'}(a, u_1) K_{H_1'}(a, u_2) \cdots K_{H_1'}(a, u_k) \right) \\ &\times \left(\int_{v_1 \vee \cdots \vee v_k}^t db \partial_1 K_{H_2'}(b, v_1) \partial_1 K_{H_2'}(b, v_2) \cdots \partial_1 K_{H_2'}(b, v_k) \right) \end{aligned} \quad (4.6)$$

with $K_{H_i'}$, $i = 1, 2$ given by (1.4) and

$$H' = 1 + \frac{H-1}{q} \iff \big(2H' - 2\big)q = 2H - 2.$$

The constant b_{k,H_1,H_2} is chosen so that $\mathbf{E}(Y_{s,t}^{k,H_1,H_2})^2 = 1$. It can be proven that the process $(Y_{s,t}^{k,H_1,H_2})_{s,t\in[0,T]}$ has the same covariance as the process defined through (4.5) and it is stationary increments of order (H_1, H_2) (in the sense of Definition A.4). This implies that when $k = 1$ both (4.5) and (4.6) are fractional Brownian sheets. For $k = 2$ one can show that the two representations (4.5) and (4.6) have the same distribution by using the cumulants of a multiple integral of order two (see (3.16)).

4.3 Multiparameter Hermite Processes

Throughout this section we use the following notation. Fix $d \in \mathbb{N}\backslash\{0\}$ and consider multi-parametric processes indexed in $\mathbb{R}^d$. We shall use bold notation for multi-indexed quantities, i.e., $\mathbf{a} = (a_1, a_2, \ldots, a_d)$, $\mathbf{ab} = (a_1b_1, a_2b_2, \ldots, a_db_d)$, $\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, \ldots, a_d/b_d)$, $[\mathbf{a}, \mathbf{b}] = \prod_i^d [a_i, b_i]$, $(\mathbf{a}, \mathbf{b}) = \prod_i^d (a_i, b_i)$, $\sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}]} a_{\mathbf{i}} = \sum_{i_1}^{N_1} \sum_{i_2}^{N_2} \ldots \sum_{i_d}^{N_d} a_{i_1,i_2,\ldots,i_d}$, $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}$, and $\mathbf{a} < \mathbf{b}$ if and only if $a_1 < b_1, a_2 < b_2, \ldots, a_d < b_d$ (analogously for the other inequalities).

Before introducing the *Hermite sheet* we briefly recall the *fractional Brownian sheet* and the *standard Brownian sheet*.

The *d-parametric anisotropic fractional Brownian sheet* is the centered Gaussian process $\{B_{\mathbf{t}}^{\mathbf{H}} : \mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d\}$ with Hurst multi-index $\mathbf{H} = (H_1, \ldots, H_d) \in (0, 1)^d$. It is equal to zero on the hyperplanes $\{\mathbf{t} : t_i = 0\}$, $1 \le i \le d$, and its covariance function is given by

$$R_{\mathbf{H}}(\mathbf{s}, \mathbf{t}) = \mathbf{E}\big[B_{\mathbf{s}}^{\mathbf{H}} B_{\mathbf{t}}^{\mathbf{H}}\big] = \prod_{i=1}^d R_{H_i}(s_i, t_i) = \prod_{i=1}^d \frac{s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}}{2}. \tag{4.7}$$

This extends Definition 4.1 for $d = 2$.

Remark 4.3 There also exists an isotropic version of the fractional Brownian sheet (see e.g. [5]). It is defined as a centered Gaussian process with covariance

$$\mathbf{E}X(\mathbf{t})X(\mathbf{s}) = \frac{1}{2}\big(\|\mathbf{t}\|^{2H} + \|\mathbf{s}\|^{2H} - \|\mathbf{t} - \mathbf{s}\|^{2H}\big), \quad \mathbf{t}, \mathbf{s} \in [0, \infty)^d$$

where $\|\cdot\|$ denotes the Euclidian norm.

The *d-parametric standard Brownian sheet* is the Gaussian process $\{W_{\mathbf{t}} : \mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d\}$ equal to zero on the hyperplanes $\{\mathbf{t} : t_i = 0\}$, $1 \le i \le d$, and with covariance function given by

$$R(\mathbf{s}, \mathbf{t}) = \mathbf{E}[W_{\mathbf{s}}, W_{\mathbf{t}}] = \prod_i^d R(s_i, t_i) = \prod_i^d s_i \wedge t_i. \tag{4.8}$$

Let $q \geq 1, q \in \mathbb{Z}$ and the Hurst multi-index $\mathbf{H} = (H_1, H_2, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. The *Hermite sheet of order* q is given by

$$
\begin{aligned}
Z_{\mathbf{H}}^q(\mathbf{t}) = c(\mathbf{H}, q) & \int_{\mathbb{R}^{d\cdot q}} \int_0^{t_1} \cdots \\
& \times \int_0^{t_d} \left(\prod_{j=1}^q (s_1 - y_{1,j})_+^{-(\frac{1}{2}+\frac{1-H_1}{q})} \cdots (s_d - y_{d,j})_+^{-(\frac{1}{2}+\frac{1-H_d}{q})} \right) \\
& \times ds_d \cdots ds_1 \, dW(y_{1,1}, \ldots, y_{d,1}) \cdots dW(y_{1,q}, \ldots, y_{d,q}) \\
= c(\mathbf{H}, q) & \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^q (\mathbf{s} - \mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{1-\mathbf{H}}{q})} d\mathbf{s} \, dW(\mathbf{y}_1) \cdots dW(\mathbf{y}_q) \qquad (4.9)
\end{aligned}
$$

where $x_+ = \max(x, 0)$. For a better understanding of multiple stochastic integrals we refer to [136]. As pointed out before, when $q = 1$, (4.9) is the fractional Brownian sheet with Hurst multi-index $\mathbf{H} = (H_1, H_2, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. For $q \geq 2$ the process $Z_{\mathbf{H}}^q(\mathbf{t})$ is not Gaussian and for $q = 2$ we denominate it as the *Rosenblatt sheet*.

Now let's calculate the covariance $R_{\mathbf{H}}^q(\mathbf{s}, \mathbf{t})$ of the Hermite sheet. Using the isometry of multiple Wiener-Itô integrals and Fubini's theorem one obtains

$$
\begin{aligned}
R_{\mathbf{H}}^q(\mathbf{s}, \mathbf{t}) &= \mathbf{E}\big[Z_{\mathbf{H}}^q(\mathbf{s}) Z_{\mathbf{H}}^q(\mathbf{t})\big] \\
&= \mathbf{E}\Bigg\{ c(\mathbf{H}, q)^2 \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{s}} \prod_{j=1}^q (\mathbf{u} - \mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{1-\mathbf{H}}{q})} d\mathbf{u} \, dW(\mathbf{y}_1) \cdots dW(\mathbf{y}_q) \\
&\quad \times \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^q (\mathbf{v} - \mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{1-\mathbf{H}}{q})} d\mathbf{v} \, dW(\mathbf{y}_1) \cdots dW(\mathbf{y}_q) \Bigg\} \\
&= c(\mathbf{H}, q)^2 \int_{\mathbb{R}^{d\cdot q}} \Bigg\{ \int_0^{s_1} \cdots \int_0^{s_d} \prod_{j=1}^q \prod_{i=1}^d (u_i - y_{i,j})_+^{-(\frac{1}{2}+\frac{1-H_i}{q})} du_d \cdots du_1 \\
&\quad \times \int_0^{t_1} \cdots \int_0^{t_d} \prod_{j=1}^q \prod_{i=1}^d (v_i - y_{i,j})_+^{-(\frac{1}{2}+\frac{1-H_i}{q})} dv_d \cdots dv_1 \Bigg\} \\
&\quad \times dy_{1,1} \cdots dy_{d,1} \cdots dy_{1,q} \cdots dy_{d,q} \\
&= c(\mathbf{H}, q)^2 \int_0^{t_1} \int_0^{s_1} \int_{\mathbb{R}^q} \prod_{j=1}^q (u_1 - y_{1,j})_+^{-(\frac{1}{2}+\frac{1-H_1}{q})} \\
&\quad \times (v_1 - y_{1,j})_+^{-(\frac{1}{2}+\frac{1-H_1}{q})} dy_{1,1} \cdots dy_{1,q} du_1 dv_1 \\
&\quad \vdots
\end{aligned}
$$

$$\times \int_0^{t_d} \int_0^{s_d} \int_{\mathbb{R}^q} \prod_{j=1}^{q} (u_d - y_{d,j})_+^{-(\frac{1}{2}+\frac{1-H_d}{q})}$$

$$\times (v_d - y_{d,j})_+^{-(\frac{1}{2}+\frac{1-H_d}{q})} dy_{d,1} \cdots dy_{d,q} du_d dv_d$$

but

$$\int_{\mathbb{R}^q} \prod_{j=1}^{q} (u - x_j)_+^{-(\frac{1}{2}+\frac{1-H}{q})} (v - x_j)_+^{-(\frac{1}{2}+\frac{1-H}{q})} dx_1 \cdots dx_q$$

$$= \left[\int_{\mathbb{R}} (u - x)_+^{-(\frac{1}{2}+\frac{1-H}{q})} (v - x)_+^{-(\frac{1}{2}+\frac{1-H}{q})} \right]^q, \tag{4.10}$$

so

$$R_{\mathbf{H}}^q(\mathbf{s}, \mathbf{t}) = c(\mathbf{H}, q)^2 \int_0^{t_1} \int_0^{s_1} \left[\int_{\mathbb{R}} (u_1 - y_1)_+^{-(\frac{1}{2}+\frac{1-H_1}{q})} (v_1 - y_1)_+^{-(\frac{1}{2}+\frac{1-H_1}{q})} \right]^q du_1 dv_1$$

$$\vdots$$

$$\times \int_0^{t_d} \int_0^{s_d} \left[\int_{\mathbb{R}} (u_d - y_d)_+^{-(\frac{1}{2}+\frac{1-H_d}{q})} (v_d - y_d)_+^{-(\frac{1}{2}+\frac{1-H_d}{q})} \right]^q du_d dv_d.$$

Recalling that the Beta function $\beta(p, q) = \int_0^1 z^{p-1}(1-z)^{p-1} dz$, $p, q > 0$, satisfies the following identity

$$\int_{\mathbb{R}} (u - y)_+^{a-1} (v - y)_+^{a-1} dy = \beta(a, 2a - 1)|u - v|^{2a-1} \tag{4.11}$$

we see that

$$R_{\mathbf{H}}^q(\mathbf{s}, \mathbf{t}) = c(\mathbf{H}, q)^2 \int_0^{t_1} \int_0^{s_1} \beta\left(\frac{1}{2} - \frac{1 - H_1}{q}, \frac{2(H_1 - 1)}{q}\right)^q$$

$$\times |u_1 - v_1|^{2(H_1-1)} du_1 dv_1$$

$$\times \cdots$$

$$\times \int_0^{t_d} \int_0^{s_d} \beta\left(\frac{1}{2} - \frac{1 - H_d}{q}, \frac{2(H_d - 1)}{q}\right)^q \cdot |u_d - v_d|^{2(H_d-1)} du_d dv_d$$

$$= c(\mathbf{H}, q)^2 \beta\left(\frac{1}{2} - \frac{1 - H_1}{q}, \frac{2(H_1 - 1)}{q}\right)^q \frac{1}{2H_1(2H_1 - 1)}$$

$$\times \left(s_1^{2H_1} + t_1^{2H_1} - |t_1 - s_1|^{2H_1}\right)$$

$$\times \cdots$$

$$\times \beta\left(\frac{1}{2} - \frac{1 - H_d}{q}, \frac{2(H_d - 1)}{q}\right)^q \frac{1}{2H_d(2H_d - 1)}$$

$$\times \left(s_d^{2H_d} + t_d^{2H_d} - |t_d - s_d|^{2H_d}\right).$$

So now we choose

$$c(\mathbf{H}, q)^2 = \left(\frac{\beta(\frac{1}{2} - \frac{1-H_1}{q}, \frac{2(H_1-1)}{q})^q}{H_1(2H_1 - 1)}\right)^{-1} \cdots \left(\frac{\beta(\frac{1}{2} - \frac{1-H_d}{q}, \frac{2(H_d-1)}{q})^q}{H_d(2H_d - 1)}\right)^{-1} \tag{4.12}$$

and in this way we obtain $\mathbf{E}(Z_{\mathbf{H}}^q(\mathbf{t})^2) = \mathbf{t}^{2\mathbf{H}} = t_1^{2H_1} \cdots t_d^{2H_d}$, and finally

$$\begin{aligned} R_{\mathbf{H}}^q(\mathbf{s}, \mathbf{t}) &= \frac{1}{2}\left(s_1^{2H_1} + t_1^{2H_1} - |t_1 - s_1|^{2H_1}\right) \cdots \left(s_d^{2H_d} + t_d^{2H_d} - |t_d - s_d|^{2H_d}\right) \\ &= \prod_i^d \frac{s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}}{2} \\ &= \prod_i^d R_{H_i}(s_i, t_i) = R_{\mathbf{H}}(\mathbf{s}, \mathbf{t}). \end{aligned} \tag{4.13}$$

Remark 4.4 As mentioned at the beginning, from the previous development we see that the covariance structure is the same for all $q \geq 1$, hence it coincides with the covariance of the fractional Brownian sheet.

We will next establish the basic properties of the Hermite sheet: self-similarity, stationarity of the increments and Hölder continuity.

The concept of self-similarity for multiparameter stochastic processes is introduced in Definition A.6.

Proposition 4.8 *The Hermite sheet is self-similar of order* $\mathbf{H} = (H_1, \ldots, H_d)$.

Proof The scaling property of the Wiener sheet implies that for every $0 < \mathbf{c} = (c_1, \ldots, c_d) \in \mathbb{R}^d$ the processes $(W(\mathbf{ct})_{\mathbf{t}\geq 0})$ and $(\sqrt{\mathbf{c}}W(\mathbf{t}))_{\mathbf{t}\geq 0}$ have the same finite dimensional distributions. Therefore, if $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^d$, using obvious changes of variables in the integrals $d\mathbf{s}$ and dW,

$$\begin{aligned} \hat{Z}_{\mathbf{H}}^q(t) &= \mathbf{h}^{\mathbf{H}} Z_{\frac{\mathbf{t}}{\mathbf{h}}}^q \\ &= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\frac{\mathbf{t}}{\mathbf{h}}} \prod_{j=1}^q (\mathbf{s} - \mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{1-\mathbf{H}}{q})} d\mathbf{s}\, dW(\mathbf{y}_1) \cdots dW(\mathbf{y}_q) \\ &= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-\mathbf{1}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^q \left(\frac{\mathbf{s}}{\mathbf{h}} - \mathbf{y}_j\right)_+^{-(\frac{1}{2}+\frac{1-\mathbf{H}}{q})} d\mathbf{s}\, dW(\mathbf{y}_1) \cdots dW(\mathbf{y}_q) \\ &= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-\mathbf{1}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^q \left(\frac{\mathbf{s}}{\mathbf{h}} - \frac{\mathbf{y}_j}{\mathbf{h}}\right)_+^{-(\frac{1}{2}+\frac{1-\mathbf{H}}{q})} \end{aligned}$$

$$
\begin{aligned}
&\times d\mathbf{s}\, dW\big(\mathbf{h}^{-1}\mathbf{y}_1\big)\cdots dW\big(\mathbf{h}^{-1}\mathbf{y}_q\big)\\
&= c(\mathbf{H},q)\mathbf{h}^{\mathbf{H}-\mathbf{1}}\mathbf{h}^{q(\frac{1}{2}+\frac{1-\mathbf{H}}{q})}\int_{\mathbb{R}^{d\cdot q}}\int_0^{\mathbf{t}}\prod_{j=1}^{q}(\mathbf{s}-\mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{1-\mathbf{H}}{q})}\\
&\quad\times d\mathbf{s}\, dW\big(\mathbf{h}^{-1}\mathbf{y}_1\big)\cdots dW\big(\mathbf{h}^{-1}\mathbf{y}_q\big)\\
&=^{(d)} c(\mathbf{H},q)\mathbf{h}^{\mathbf{H}-\mathbf{1}}\mathbf{h}^{q(\frac{1}{2}+\frac{1-\mathbf{H}}{q})}\mathbf{h}^{-\frac{q}{2}}\int_{\mathbb{R}^{d\cdot q}}\int_0^{\mathbf{t}}\prod_{j=1}^{q}(\mathbf{s}-\mathbf{y}_j)_+^{-(\frac{1}{2}+\frac{1-\mathbf{H}}{q})}\\
&\quad\times d\mathbf{s}\, dW(\mathbf{y}_1)\cdots dW(\mathbf{y}_q)\\
&= Z^q_{\mathbf{H}}(t)
\end{aligned}
$$

where $=^{(d)}$ means equivalence of finite dimensional distributions. □

Proposition 4.9 *The Hermite sheet* $(Z^q(\mathbf{t}))_{\mathbf{t}\geq 0}$ *has stationary increments (in the sense of Definition A.7).*

Proof Developing the increments of the process using the definition of the Hermite sheet and using the change of variables $\mathbf{s}' = \mathbf{s} - \mathbf{h}$, it is immediate that for every $\mathbf{h} > 0, \mathbf{h} \in \mathbb{R}^d$,

$$\Delta Z^q_{[\mathbf{h},\mathbf{h}+\mathbf{t}]} =^{(d)} \Delta Z^q_{[0,\mathbf{t}]}$$

for every $\mathbf{t}$. □

Proposition 4.10 *The trajectories of the Hermite sheet* $(Z^q(\mathbf{t}), \mathbf{t} \geq 0)$ *are Hölder continuous of any order* $\delta = (\delta_1, \ldots, \delta_d) \in [0, \mathbf{H}]$ *in the following sense: for every* $\omega \in \Omega$, *there exists a constant* $C_\omega > 0$ *such that for every* $\mathbf{s}, \mathbf{t} \in \mathbb{R}^d$, $\mathbf{s}, \mathbf{t} \geq 0$,

$$\big|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}\big| \leq C_\omega |t_1 - s_1|^{\delta_1} \cdots |t_d - s_d|^{\delta_d} = C_\omega |\mathbf{t} - \mathbf{s}|^{\delta}.$$

Proof Using Cencov's criteria (see [43]) and the fact that the process Z^q is almost surely equal to 0 when $t_i = 0$, it suffices to check that

$$\mathbf{E}\big|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}\big|^p \leq C\big(|t_1 - s_1| \cdots |t_d - s_d|\big)^{1+\gamma} \tag{4.14}$$

for some $p \geq 2$ and $\gamma > 0$. From the self-similarity and the stationarity of the increments of the process Z^q, we have for every $p \geq 2$

$$\mathbf{E}\big|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}\big|^p = \mathbf{E}|Z_{\mathbf{1}}|^p \big(|t_1 - s_1| \cdots |t_d - s_d|\big)^{p\mathbf{H}}$$

and this obviously implies (4.14). □

4.4 Bibliographical Notes

Several two-parameter (or multiparameter) processes related to fractional Brownian motion have been proposed in the literature. These include for example the *fractional Brownian field* ([35, 110]), *Lévy's fractional Brownian field* ([50]) and the *anisotropic fractional Brownian sheet* ([12, 185]). Each process has been intensively studied in the last few decades. Various aspects of these processes have been studied: stochastic integration ([178, 179]), sample path properties [189], chaos expansion and local times [71], stochastic equations ([76]), quadratic variations ([152–154]) etc., to cite only a few. The study of multiparameter non-Gaussian self-similar processes, including the Hermite class, is incipient at the time of writing. We refer to [40, 53], or [152].

4.5 Exercises

Exercise 4.1 Consider now the processes $s \to W_{s,t}^{\alpha,\beta}$ and $t \to W_{s,t}^{\alpha,\beta}$. These processes are real fractional Brownian motions with the same law as $t^{\beta} W^{\alpha}$ and $s^{\alpha} W^{\beta}$ respectively and with covariances

$$R_1(s_1, s_2) = t^{2\beta} \frac{1}{2} \left(s_1^{2\alpha} + s_2^{2\alpha} - |s_1 - s_2|^{2\alpha} \right)$$

and

$$R_2(t_1, t_2) = s^{2\alpha} \frac{1}{2} \left(t_1^{2\beta} + t_2^{2\beta} - |t_1 - t_2|^{2\beta} \right)$$

respectively.

Exercise 4.2 ([12]) Show that the fractional Brownian sheet $(W_{s,t}^{\alpha,\beta})_{s,t \geq 0}$ admits a continuous version with respect to $(\alpha, \beta) \in (0, 1)^2$.

Exercise 4.3 Find the constant C in Proposition 4.3.

Exercise 4.4 ([20]) Consider

$$y_{\varepsilon}(s, t) = \int_0^t \int_0^s \frac{1}{\varepsilon^2} \sqrt{xy} (-1)^{N(x/\varepsilon, y/\varepsilon)} dx f y$$

with N a Poisson process in the plane. Then y_{ε} converges weakly to the Brownian sheet as $\varepsilon \to 0$ (see [18]). Prove that the family of stochastic processes $(X_{\varepsilon})_{\varepsilon > 0}$ defined by

$$X_{\varepsilon}(s, t) = \int_0^t \int_0^s K_{\alpha}(s, u) K_{\beta}(t, v) \frac{1}{\varepsilon^2} \sqrt{xy} (-1)^{N(x/\varepsilon, y/\varepsilon)} du dv$$

with $(\alpha, \beta) \in (0, 1)^2$, converges weakly to the fBs with parameters α, β in the space $C_0([0, 1]^2)$ (the space of continuous functions on $[0, 1]^2$ vanishing at the origin).

Exercise 4.5 Show that the two-parameter Hermite process has moments of every order $p \geq 2$.

Exercise 4.6 Consider the process $(Y^{k,H_1,H_2}_{s,t})_{s,t\in[0,T]}$ given by (4.6).

1. Find the constant b_{k,H_1,H_2}.
2. Prove that the covariance of the process $(Y^{k,H_1,H_2}_{s,t})_{s,t\in[0,T]}$ is given by

$$\mathbf{E}Y^{k,H_1,H_2}_{s,t}Y^{k,H_1,H_2}_{u,v} = R_{H_1}(t,u)R_{H_2}(s,v)$$

   where R_{H_1}, R_{H_2} denotes the covariance of the fBm (1.1).
3. Prove that the process $(Y^{k,H_1,H_2}_{s,t})_{s,t\in[0,T]}$ is self-similar of order (H_1, H_2) (in the sense of Definition A.4).
4. For $k = 2$ prove that the process Y^{k,H_1,H_2} has the same finite dimensional distributions as the process (4.5).

Hint Use the cumulants and follow the lines of the proof of Proposition 3.7.

Exercise 4.7 ([52]) Consider *the two-parameter fractional Ornstein-Uhlenbeck process* defined as the solution of the stochastic equation

$$X_{t,s} = -\theta \int_0^t \int_0^s X_{v,u}dvdu + B^{\alpha,\beta}_{t,s}, \quad (t,s) \in [0,T] \times [0,S]. \tag{4.15}$$

Here $B^{\alpha,\beta}$ denotes a fractional Brownian sheet with Hurst parameters $\alpha, \beta \in (\frac{1}{2}, 1)$. We also suppose that $X_{0,0} = X_{t,0} = X_{0,s} = 0$ for every t, s.

1. Show that (4.15) admits a unique strong solution which can be expressed as

$$X_{t,s} = \int_0^T \int_0^S f(t,s,t_0,s_0)dB^{\alpha,\beta}_{t_0,s_0} \tag{4.16}$$

   where

$$f(t,s,t_0,s_0) = 1_{[0,t]}(t_0)1_{[0,s]}(s_0)\sum_{n\geq 0}(-1)^n\theta^n\frac{(t-t_0)^n(s-s_0)^n}{(n!)^2}. \tag{4.17}$$

   We will call the solution X to (4.15) the fractional Ornstein-Uhlenbeck sheet.
2. Let us consider the Bessel function of order 0 given, for every $x \in \mathbb{R}$, by

$$J_0(x) = \sum_{n\geq 0}\frac{(-1)^n}{n!^2}\left(\frac{x}{2}\right)^{2n}.$$

   This Bessel function admits the integral representation, for every $x \in \mathbb{R}$

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\rho)d\rho.$$

Prove that the kernel f (4.17) of the solution $(X_{t,s})_{t,s\in[0,T]\times[0,S]}$ can be expressed as

$$\begin{aligned} f(t,s,u,v) &= 1_{[0,t]}(u)1_{[0,s]}(v)J_0\big(2\sqrt{\theta(t-u)(s-v)}\big) \\ &= 1_{[0,t]}(u)1_{[0,s]}(v)\frac{1}{\pi}\int_0^\pi \cos\big(2\sqrt{\theta(t-u)(s-v)}\sin\rho\big)d\rho. \end{aligned} \tag{4.18}$$

Part II
Variations of Self-similar Processes: Central and Non-Central Limit Theorems

Chapter 5
First and Second Order Quadratic Variations. Wavelet-Type Variations

Let $(X_t)_{t\in[0,T]}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$. We will focus on the study of the *quadratic variation statistic* of the process X which is defined by

$$V_N(X) = \sum_{i=0}^{N-1} (X_{t_{i+1}} - X_{t_i})^2 \tag{5.1}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ denotes a partition of the interval $[0, T]$. We will discuss the asymptotic behavior, as $N \to \infty$, of the sequence $V_N(X)$ in several situations: when X is a fractional Brownian motion, a Rosenblatt process, a Hermite process or the solution to a linear heat equation with fractional noise in time.

Generally, the quadratic variations play an important role, in various aspects, in the analysis of a stochastic process. For example, in the case of Brownian motion, the limit of the sequence (5.1) is an important element in the Itô stochastic calculus with respect to Brownian motion. The same is true for martingales and also for several processes which are not semi-martingales (fractional Brownian motion, bifractional Brownian motion, etc.). Another field where the asymptotic behavior of (5.1) is important is in estimation theory: for self-similar processes the quadratic variations are used to construct consistent estimators for the self-similarity order. Understanding the limit in distribution of the sequence V_N directly gives the asymptotic behavior of the associated estimators. For a complete presentation of various estimators for the self-similarity index, see [28].

In this chapter our aim is to understand the limit in distribution of the sequence (5.1) when the mesh of the partition tends to zero. We will treat several examples of self-similar processes: fractional Brownian motion, the Rosenblatt process and more generally, a Hermite process with arbitrary Hermite rank, and the solution to a linear heat equation with fractional noise. We will also discuss other types of variations: quadratic variations with high order increments (that is, where one replaces the first order increment $X_{t_{i+1}} - X_{t_i}$ in (5.1) by the second order increment $X_{t_{i+1}} - 2X_{t_i} + X_{t_{i-1}}$) or the variations based on wavelet expansion.

The techniques that we used to prove the asymptotic behavior in distribution of the quadratic variations are based on the Malliavin calculus and multiple Wiener-Itô

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integrals. We recall in this chapter the criteria, in terms of the Malliavin derivatives, for a sequence of random variables to converge to the normal distribution. We will use only the basic tools of the Malliavin calculus described in Appendix C, so we believe that a reader who is not very familiar with it will nevertheless still be able to follow the presentation.

5.1 Quadratic Variations of Fractional Brownian Motion

Let $(B_t^H)_{t\in[0,1]}$ be a fractional Brownian motion with time interval [0, 1] and Hurst parameter $H \in (0,1)$. Let $0 = t_0 < t_1 \cdots < t_N = 1$ be a partition of the unit interval [0, 1] such that $t_i = \frac{i}{N}$ for $i = 0, \ldots, N$ and define, for $N \geq 1$,

$$V_N = \frac{1}{N}\sum_{i=0}^{N-1}\left[\frac{(B^H_{t_{i+1}} - B^H_{t_i})^2}{\mathbf{E}(B^H_{t_{i+1}} - B^H_{t_i})^2} - 1\right]. \tag{5.2}$$

Clearly,

$$\mathbf{E}\big(B^H_{t_{i+1}} - B^H_{t_i}\big)^2 = (t_{i+1} - t_i)^{2H} = N^{-2H}$$

and thus

$$V_N = \frac{1}{N}\sum_{i=0}^{N-1}\big[N^{2H}\big(B^H_{t_{i+1}} - B^H_{t_i}\big)^2 - 1\big].$$

The sequence (5.2) is usually called the centered quadratic variations statistic since $\mathbf{E}V_N = 0$ for every $N \geq 1$. The aim is to find the limit in distribution of the sequence V_N when $N \to \infty$.

5.1.1 Evaluation of the L^2-Norm of the Quadratic Variations

We first analyze the asymptotic behavior of the sequence $\mathbf{E}V_N^2$ as $N \to \infty$. We will use the properties of multiple stochastic integrals listed in Appendix C.

Lemma 5.1 *Let V_N be given by (5.2). Then*

$$V_N = N^{2H-1} I_2\left(\sum_{i=1}^{N} A_i \otimes_1 A_i\right) \tag{5.3}$$

where

$$A_{i,N} := A_i = \mathbf{1}_{((i-1)/N, i/N]} \tag{5.4}$$

for $i = 1, \ldots, N$. Here I_2 denotes the multiple integral of order 2 with respect to the fractional Brownian motion B^H.

Proof The basic and trivial observation is that $B^H_{t_i} - B^H_{t_{i-1}} = I_1(A_i)$ for $1 = 1, \dots, N$. The product formula of multiple integrals (see (C.4)) in our present case yields

$$V_N = N^{2H-1} I_2\left(\sum_{i=1}^{N} A_i \otimes_1 A_i\right).$$

□

Then, by the isometry of multiple integrals (C.1), we get from (5.3)

$$\mathbf{E}|V_N|^2 = 2N^{4H-2} \sum_{i=1}^{N}\sum_{j=1}^{N} \big|\langle A_i, A_j\rangle_{\mathcal{H}}\big|^2 \tag{5.5}$$

where $\mathcal{H}$ is the canonical Hilbert space of the fBm $B^H := B$. To calculate this quantity, we observe that

$$\begin{aligned}\langle A_i, A_j\rangle_{\mathcal{H}} &= \mathbf{E}\big[B(A_i)B(A_j)\big] \\ &= 2^{-1}\left(2\left|\frac{i-j}{N}\right|^{2H} - \left|\frac{i-j-1}{N}\right|^{2H} - \left|\frac{i-j+1}{N}\right|^{2H}\right)\end{aligned}$$

with $B(A_i) = B_{\frac{i}{n}} - B_{\frac{i-1}{n}}$, $i = 1, \dots, n$.

This expression is close to $H(2H-1)N^{-2}|(i-j)/N|^{2H-2}$, but we must take care whether the series $\sum_k k^{4H-4}$ converges or diverges; this will depend on whether H is less or greater than $\frac{3}{4}$.

Let

$$\rho_H(k) = \frac{1}{2}\big(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\big)$$

for every $k \in \mathbb{Z}$. Then, by Proposition 1.4,

$$\rho_H(k) \sim_{|k|\to\infty} H(2H-1)|k|^{2H-2}. \tag{5.6}$$

Recall that the symbol $\sim$ means that both sides have the same limit as $k \to \infty$.

Proposition 5.1 *If $H < 3/4$, then*

$$\lim_{N\to\infty} \mathbf{E}\big[|\sqrt{N} V_N|^2\big] = c_{1,H}$$

where

$$c_{1,H} = 2\sum_{k\in\mathbb{Z}} \rho_H\big(|k|\big)^2. \tag{5.7}$$

Proof We have, using the change of index $i-j=k$

$$\mathbf{E}|V_N|^2 = 2N^{-2}\sum_{i,j=0}^{N-1}\big(\rho_H\big(|i-j|\big)\big)^2 = 2N^{-2}\sum_{k=-(N-1)}^{N-1}\sum_{i=k}^{k+(N-1)}\big(\rho_H\big(|k|\big)\big)^2$$
$$= 2N^{-1}\sum_{k=-(N-1)}^{N-1}\rho_H\big(|k|\big)^2$$

and then

$$N\mathbf{E}V_N^2 \xrightarrow[N\to\infty]{} c_{1,H} = 2\sum_{k\in\mathbb{Z}}\big(\rho_H\big(|k|\big)\big)^2.$$

Note that the series $\sum_{k\in\mathbb{Z}}(\rho_H(|k|))^2$ is convergent if and only if $H<\frac{3}{4}$, by (5.6). □

When $H>\frac{3}{4}$ the series $\sum_{k\in\mathbb{Z}}|k|^{4H-4}$ does not converge and therefore $\mathbf{E}V_N^2$ will have a different asymptotic behavior. Actually,

Proposition 5.2 *If* $H>3/4$

$$\lim_{N\to\infty}\mathbf{E}\big[\big|N^{2-2H}V_N\big|^2\big] = c_{2,H}$$

with

$$c_{2,H} := 2H^2(2H-1)/(4H-3). \tag{5.8}$$

Proof In this case, we will instead compare the series in $\mathbf{E}[|V_N|^2]$ to an integral; in the sum defining this quantity (5.5), the diagonal term corresponding to $|i-j|=0$ can be ignored. Indeed,

$$N^{4-4H}N^{4H-2}\sum_{i=1}^{N}\big|\langle A_i, A_i\rangle_{\mathcal{H}}\big|^2 = N^{3-4H}$$

and this converges to zero when $H>\frac{3}{4}$. Thus, since

$$\big|\langle A_i, A_j\rangle\big| \sim N^{-2H}|i-j|^{2H-2}$$

for i, j large enough ($i\neq j$), we have that

$$N^{4-4H}\mathbf{E}V_N^2 \sim 2\big(H(2H-1)\big)^2 N^{-2}\sum_{i,j=1;i\neq j}^{N}\left|\frac{i}{j}-\frac{j}{N}\right|^{4H-4}$$
$$\xrightarrow[N]{} 2\big(H(2H-1)\big)^2\int_0^1\int_0^1|x-y|^{4H-4}dxdy.$$

Notice that the integral $\int_0^1 \int_0^1 |x-y|^{4H-4} dxdy$ is finite when $H > \frac{3}{4}$. Consequently

$$\lim_{N\to\infty} \mathbf{E}\big[\big|N^{2-2H} V_N\big|^2\big] = 4\big(H(2H-1)\big)^2 \int_0^1 \int_0^x (x-y)^{4H-4}$$

$$= \frac{4(H(2H-1))^2}{(4H-3)(4H-2)} = c_{2,H}$$

with $c_{2,H}$ given by (5.8). □

In the case $H = \frac{3}{4}$ we have the following renormalization of $\mathbf{E}V_N^2$.

Proposition 5.3 *If* $H = 3/4$, *let*

$$c'_{1,H} := \big(2H(2H-1)\big)^2 = 9/16 \tag{5.9}$$

then

$$\mathbf{E}\Bigg[\sqrt{\frac{N}{\log N}} V_N\Bigg]^2 \xrightarrow[N\to\infty]{} c'_{1,H}. \tag{5.10}$$

Proof In this case, we have from (5.5)

$$\mathbf{E}(V_N)^2 = \frac{2}{N} + \frac{1}{N}\sum_{k=0}^{N-1}\big(2k^{2H} - (k-1)^{2H} - (k+1)^{2H}\big)^2$$

$$- \frac{1}{N^2}\sum_{k=0}^{N-1} k\big(2k^{2H} - (k-1)^{2H} - (k+1)^{2H}\big)^2$$

and since $2k^{2H} - (k-1)^{2H} - (k+1)^{2H}$ behaves as $(3/4)k^{-1/2}$ we get

$$\mathbf{E}(V_N)^2 \sim c'_{1,H}(\log N)/N.$$

Thus, $\lim_{N\to\infty} \mathbf{E}[|\tilde{F}_N|^2] = 1$ where $c'_{1,H} = 9/16$ and

$$\tilde{F}_N := \left(\frac{N}{c'_{1,H} \log N}\right)^{\frac{1}{2}} V_N. \tag{5.11}$$

□

5.1.2 *The Malliavin Calculus and Stein's Method*

For $H < \frac{3}{4}$, let

$$F_N := c_{1,H}^{-\frac{1}{2}} N^{\frac{1}{2}} V_N \tag{5.12}$$

where V_N and $c_{1,H}$ are defined by (5.2) and (5.7) respectively. From Proposition 5.1 it follows that

$$\mathbf{E}F_N^2 \to_{N\to\infty} 1.$$

We prove that F_N converges in law to the standard normal law and we give the rate of convergence. Our approach is based on the now classical Stein's method combined with the Malliavin calculus. The reader is referred to Appendix C for the basic tools of the Malliavin calculus.

Let us recall the context. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t\geq 0}$ a one-dimensional Brownian motion on this space. Let F be a random variable defined on Ω which is differentiable in the sense of the Malliavin calculus. Then, using the so-called Stein method introduced by Nourdin and Peccati in [127] (see also [128] and [129]), it is possible to measure the distance between the law of F and the standard normal law $N(0,1)$. This distance, denoted by d, can be defined in several ways, for example as the Kolmogorov distance, the Wasserstein distance, the total variation distance or as the Fortet-Mourier distance.

Concretely, let X, Y be two random variables. The distance between the law of X and the law of Y is usually defined by (here $\mathcal{L}(F)$ denotes the law of F)

$$d\big(\mathcal{L}(X), \mathcal{L}(Y)\big) = \sup_{h\in\mathcal{A}} \big|\mathbf{E}h(X) - \mathbf{E}h(Y)\big|$$

where $\mathcal{A}$ is a suitable class of functions.

For example, if $\mathcal{A}$ is the set of indicator functions

$$1_{(-\infty,z]}, \quad z \in \mathbb{R}$$

we obtain the Kolmogorov distance

$$d_K\big(\mathcal{L}(X), \mathcal{L}(Y)\big) = \sup_{z\in\mathbb{R}} \big|P(X \leq z) - P(Y \leq z)\big|.$$

If $\mathcal{A}$ is the set of 1_B with B a Borel set, one has the total variation distance

$$d_{TV}\big(\mathcal{L}(X), \mathcal{L}(Y)\big) = \sup_{B\in\mathcal{B}(\mathbb{R})} \big|P(X \in B) - P(Y \in B)\big|.$$

If

$$\mathcal{A} = \big\{h; \|h\|_L \leq 1\big\}$$

($\|\cdot\|_L$ is the Lipschitz norm) one has the Wasserstein distance.

Other examples of distances between the distributions of random variables are the Fortet-Mourier and Kantorovich distances.

One can give a bound for the distance between the law of a random variable and the standard normal law in terms of the Malliavin derivatives. More precisely we have,

$$d\big(\mathcal{L}(F), N(0,1)\big) \leq c\sqrt{\mathbf{E}\big(1 - \langle DF, D(-L)^{-1}F\rangle_{L^2([0,1])}\big)^2}. \tag{5.13}$$

Here D denotes the Malliavin derivative with respect to W, and L is the generator of the Ornstein-Uhlenbeck semigroup defined in Appendix C. When the underlying Gaussian process is an arbitrary isonormal process X then the space $L^2([0,1])$ has to be replaced by the Hilbert space $\mathcal{H}$ associated to X. The constant c is equal to 1 in the case where d is the Kolmogorov distance as well as in the case where d is the Wasserstein distance, $c = 2$ for the case where d is the total variation distance and $c = 4$ in the case where d is the Fortet-Mourier distance. See also [126, Appendix C].

In the case when the random variable F in (5.13) belongs to a Wiener chaos of fixed order, we have the following result from [127] (see also the recent book [126]). In the sequel d will denote any of the distances defined above.

Theorem 5.1 *Let $I_q(f)$ be a multiple integral of order $q \geq 1$ with respect to an isonormal process X. Then*

$$d\big(\mathcal{L}\big(I_q(f)\big), N(0,1)\big) \leq c_q\big[\mathbf{E}\big(\|DI_q(f)\|^2_{\mathcal{H}} - q\big)^2\big]^{\frac{1}{2}}.$$

Here D is the Malliavin derivative with respect to X and $\mathcal{H}$ is the canonical Hilbert space associated to X.

We will also use the following result (see Theorem 4 in [137] and also [138]), known as the *Fourth Moment Theorem.*

Theorem 5.2 *Fix $n \geq 2$ and let $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (with $f_k \in \mathcal{H}^{\odot n}$ for every $k \geq 1$) be a sequence of square integrable random variables in the nth Wiener chaos of an isonormal process X such that $\mathbf{E}[F_k^2] \to 1$ as $k \to \infty$. Then the following are equivalent*:

(i) *The sequence $(F_k)_{k \geq 0}$ converges in distribution to the normal law $N(0,1)$.*
(ii) *One has $\mathbf{E}[F_k^4] \to 3$ as $k \to \infty$.*
(iii) *For all $1 \leq l \leq n-1$ it holds that $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{\mathcal{H}^{\otimes 2(n-l)}} = 0$.*
(iv) *$\|DF_k\|^2_{\mathcal{H}} \to n$ in $L^2(\Omega)$ as $k \to \infty$, where D is the Malliavin derivative with respect to X.*

Criterion (iv) is due to [137]; we will refer to it as the Nualart–Ortiz-Latorre criterion. It shows that the bound in Theorem 5.1 is "sharp" in the case of multiple stochastic integrals.

There also exists a multidimensional version of Theorem 5.2. This exhibits an interesting and useful result: for a vector of multiple stochastic integrals the convergence in distribution to the normal law of each component of the vector implies the convergence in distribution to the (multivariate) normal law of the vector. This result was proved in [142].

Theorem 5.3 *For every $n \geq 1$, let*

$$F_n = \big(F_n^1, \ldots, F_n^d\big)$$

where for every $i = 1, \dots, d$ the random variable F_n^i is a multiple integral of order $q_i \geq 1$. Assume that

$$\lim_{n\to\infty} \mathbf{E} F_n^i F_n^j = C_{i,j}$$

for every $1 \leq i, j \leq n$. Assume that $C = (C_{i,j})_{1\leq i,j\leq d}$ is a symmetric non-negative definite matrix. Then the following are equivalent:

(i) *The sequence $(F_n)_{n\geq 1}$ converges in distribution to the d-dimensional normal law $N(0, C)$.*
(ii) *For every $1 \leq i \leq d$ the sequence $(F_n^i)_{n\geq 1}$ converges in law to the normal law $N(0, C_{i,i})$.*

5.1.3 The Central Limit Theorem of the Quadratic Variations for $H \leq \frac{3}{4}$

Clearly Theorem 5.1 applies to F_N given by (5.12) since it is a multiple integral of order 2. We will use

Lemma 5.2 *With A_i given by* (5.4) *and if $H \in (0, 3/4)$,*

$$\langle A_i, A_j\rangle_{\mathcal{H}} = 2^{-1}\left(2\left|\frac{i-j}{N}\right|^{2H} - \left|\frac{i-j-1}{N}\right|^{2H} - \left|\frac{i-j+1}{N}\right|^{2H}\right),$$

as $N \to \infty$, we have

$$\sum_{i,i'=1}^{N}\sum_{j=1}^{i}\sum_{j'=1}^{i'} \langle A_i; A_{i'}\rangle_{\mathcal{H}}\langle A_i; A_j\rangle_{\mathcal{H}}\langle A_{i'}; A_{j'}\rangle_{\mathcal{H}}\langle A_j; A_{j'}\rangle_{\mathcal{H}} = o\big(N^{-4}\big).$$

Proof As a general rule that we will exemplify below, we have the following: if $i = i'$ or $i = i' \pm 1$ the term $\langle A_i; A_{i'}\rangle_{\mathcal{H}}$ will give a contribution of order $\frac{1}{N^{2H}}$ while if $|i - i'| \geq 2$ the same term will have a contribution less that $cst. \frac{|i-i'|^{2H-2}}{N^{2H-2}} N^{-2}$. Using this rule, although several cases appear, the main term will be obtained when all indices are separated by a distance of at least two.

We can deal with the diagonal terms first. With $i = i'$ and $j = j'$, the corresponding contribution is of order

$$N^{-4H}\left(\sum_{i,j=1}^{N} \big|\langle A_i; A_j\rangle_{\mathcal{H}}\big|\right)^2 \asymp N^{-8H} = \mathcal{O}\big(N^{-4}\big).$$

It is trivial to check that the terms with $i = i'$ and $j = j' \pm 1$, as well as the terms with $i = i' \pm 1$ and $j = j' \pm 1$, again yield the order N^{-1}. By changing the roles of the indices, we also treat all terms of the type $|i - i'| \leq 2$ and $|j - i| \leq 2$.

Now for the hyperplane terms with $i = i'$ and $|j - j'| \geq 2$, $|j - i| \geq 2$, $|j' - i| \geq 2$, we can use the relations of the form

$$\langle A_i; A_j \rangle_{\mathcal{H}} \leq 2^{2-2H} H(2H-1) N^{-2} \left|(i-j)/N\right|^{2H-2},$$

also holding for the pairs (i, j') and (j, j'), to obtain that the corresponding contribution is of the order

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{|j-j'|\geq 2; |j-i|\geq 2; |j'-i|\geq 2} N^{-2H} N^{-6} \left|(i-j)/N\right|^{2H-2} \\
&\qquad \times \left|\left(i-j'\right)/N\right|^{2H-2} \left|\left(j-j'\right)/N\right|^{2H-2} \\
&\quad = N^{-3-2H} \sum_{i=1}^{N} \sum_{|j-j'|\geq 2; |j-i|\geq 2; |j'-i|\geq 2} N^{-3} \left|(i-j)/N\right|^{2H-2} \left|\left(i-j'\right)/N\right|^{2H-2} \\
&\qquad \times \left|\left(j-j'\right)/N\right|^{2H-2} \\
&\quad \asymp N^{-3-2H} = \mathcal{O}\left(N^{-4}\right)
\end{aligned}$$

where we used the fact that the last summation above converges as a Riemann sum to the finite integral $\int_{[0,1]^3} |(x-y)(x-z)(y-z)|^{2H-2} dx dy dz$, and then the fact that $H < 3/4$. For the hyperplanes term of the form $i = i' \pm 1$ and $|j - j'| \geq 2$, $|j - i| \geq 2$, $|j' - i| \geq 2$, or $|i - i'| \geq 2$, $|i - j| \geq 2$, and $|j - j'| \geq 2$, the calculation is identical.

Lastly, and similarly to the case just treated, when all indices are distant by at least 2 units, we can again use the upper bound $N^{-2}|(i-j)/N|^{2H-2}$ for $\langle A_i; A_j \rangle_{\mathcal{H}}$ and the other three pairs, obtaining a contribution of the form

$$\begin{aligned}
&\sum_{|i-i'|\geq 2; |j-j'|\geq 2; |j-i|\geq 2; |j'-i|\geq 2} N^{-8} \left|\frac{i-i'}{N}\right|^{2H-2} \left|\frac{i-j}{N}\right|^{2H-2} \\
&\qquad \times \left|\frac{i-j'}{N}\right|^{2H-2} \left|\frac{j-j'}{N}\right|^{2H-2} \\
&\quad \asymp N^{-4} \int_{[0,1]^4} \left|\left(x-x'\right)(x-y)\left(x'-z\right)(y-z)\right|^{2H-2} dx' dx dy dz;
\end{aligned}$$

since $H < 3/4$, we have $8H - 6 < 0$, and the above also tends to 0 albeit more slowly than the other terms. □

Theorem 5.4 *Assume $H < \frac{3}{4}$ and let F_N be given by (5.12). Then as $N \to \infty$, the sequence $(F_N)_N$ converges in distribution to the standard normal law. Moreover*

$$d\left(\mathcal{L}(F_N), N(0,1)\right) \leq c N^{2H-\frac{3}{2}}. \tag{5.14}$$

Proof Using Theorem 5.1 one needs to show that $\|DF_N\|^2_{\mathcal{H}}$ converges in $L^2(\Omega)$ to $n = 2$ and to understand the rate of convergence to this limit.

We will see that this only works for $H \leq 3/4$. Using the rule

$$D_r I_2(f) = 2I_1\big(f(\cdot, r)\big)$$

when f is symmetric, we have

$$D_r V_N = 2N^{2H-1} \sum_{i=1}^{N} A_i(r) I_1(A_i).$$

Hence

$$\|DV_N\|^2_{\mathcal{H}} = 4N^{4H-2} \sum_{i,j=1}^{N} I_1(A_i) I_1(A_j) \langle A_i; A_j \rangle_{\mathcal{H}} \tag{5.15}$$

and therefore

$$\mathbf{E}\big[\|DV_N\|^2_{\mathcal{H}}\big] = 4N^{4H-2} \sum_{i,j=1}^{N} \big|\langle A_i; A_j \rangle_{\mathcal{H}}\big|^2.$$

We note immediately from (5.5) and Proposition 5.1 that

$$\mathbf{E}\big[\|DV_N\|^2_{\mathcal{H}}\big] = 2\mathbf{E}\big[V_N^2\big],$$

and from the results of the previous section, $\lim_{N\to\infty} \mathbf{E}[\|DF_N\|^2_{\mathcal{H}}] = 2$.

Thus it now suffices to show that

$$\|DF_N\|^2_{\mathcal{H}} - \mathbf{E}\big[\|DF_N\|^2_{\mathcal{H}}\big] + \mathbf{E}\big[\|DF_N\|^2_{\mathcal{H}}\big] - 2$$

converges to 0 in $L^2(\Omega)$.

A simple use of the product formula for multiple integrals gives

$$\|DV_N\|^2_{\mathcal{H}} - \mathbf{E}\big[\|DV_N\|^2_{\mathcal{H}}\big] = 4N^{4H-2} \sum_{i,j=1}^{N} \langle A_i; A_j \rangle_{\mathcal{H}} I_2(A_i \otimes A_j)$$

and thus

$$\begin{aligned}
&\mathbf{E}\big[\|DF_N\|^2_{\mathcal{H}} - \mathbf{E}\big[\|DF_N\|^2_{\mathcal{H}}\big]\big]^2 \\
&\quad = (c_{1,H})^{-2} N^2 \big(4N^{4H-2}\big)^2 4 \\
&\qquad \times \sum_{i,i'=1}^{N} \sum_{j=1}^{i} \sum_{j'=1}^{i'} \langle A_i; A_{i'} \rangle_{\mathcal{H}} \langle A_i; A_j \rangle_{\mathcal{H}} \langle A_{i'}; A_{j'} \rangle_{\mathcal{H}} \langle A_j; A_{j'} \rangle_{\mathcal{H}}.
\end{aligned}$$

Since $1/2 < H < 3/4$, by Lemma 5.2, the conclusion is that

$$\mathbf{E}\big[\|DF_N\|_{\mathcal{H}}^2 - \mathbf{E}\big[\|DF_N\|_{\mathcal{H}}^2\big]\big]^2$$

is asymptotically equivalent to a constant multiple of N^{8H-6} and thus it converges to zero for $H < \frac{3}{4}$.

On the other hand, the rate of convergence to zero of

$$\mathbf{E}\big[\|DV_N\|_{\mathcal{H}}^2\big] - 2 = 2\mathbf{E}F_N^2 - 2$$

is of order less than N^{4H-3} (the reader may consult the proof of Theorem 6.2). □

The convergence to a Gaussian distribution also holds in the limit case $H = \frac{3}{4}$.

Proposition 5.4 *Let $\tilde{F}_N$ be given by (5.11). Then, as $N \to \infty$, $(\tilde{F}_N)_N$ converges in law to the standard normal distribution $N(0,1)$. Moreover*

$$d\big(\mathcal{L}(\tilde{F}_N), N(0,1)\big) \leq c\frac{1}{\sqrt{\log N}}.$$

Proof This case is treated similarly to the previous one. We record the following for later use. As $N \to \infty$

$$\mathbf{E}\big[\|D\tilde{F}_N\|_{\mathcal{H}}^2\big] = \frac{N}{\log N}\big(c'_{1,H}\big)^{-1}4N^{4H-2}\sum_{i,j=1}^{N}\langle A_i, A_j\rangle_{\mathcal{H}}^2 \to 2. \tag{5.16}$$

For the rate of convergence in the base $H = \frac{3}{4}$, we refer to [39]. □

Note that Theorem 5.4 and Proposition 5.4 are particular cases of the more general results stated in Theorem 6.1.

5.1.4 The Non-Central Limit of the Quadratic Variations for $H > \frac{3}{4}$

We assume here that $H > 3/4$. In this case, using the scaling

$$\bar{F}_N = N^{2-2H}V_N/\sqrt{c_{2,H}}$$

one checks that

$$\|D\bar{F}_N\|_{\mathcal{H}}^2 - \mathbf{E}\|D\bar{F}_N\|_{\mathcal{H}}^2$$

does not converge to 0. Therefore, the Nualart–Ortiz-Latorre characterization (Theorem 5.2) says that the limit of F_N is not Gaussian. But we can also prove directly that the limit is not Gaussian. Indeed, we have

Theorem 5.5 *If $H \in (3/4, 1)$, let $c_{2,H}$ be defined by (5.8). Then*

$$\bar{F}_N := \sqrt{N^{4-4H}/c_{2,H}}\, V_N \tag{5.17}$$

converges in $L^2(\Omega)$ to a standard Rosenblatt random variable with (self-similarity) parameter $H_0 = 2H - 1$; this random variable is equal to by is equal to (see (3.17))

$$\frac{(4H-3)^{1/2}}{H(2(2H-1))^{1/2}} \iint_{[0,1]^2} \left(\int_{r \vee s}^1 \frac{\partial K^H}{\partial u}(u,s) \frac{\partial K^H}{\partial u}(u,r)du \right) dW(r)dW(s) \tag{5.18}$$

where W is the standard Brownian motion used in the representation $B_t = I_1(K^H(t,\cdot))$ (1.2).

Proof In order to find the limit of $\bar{F}_N$, let us return to the definition of this quantity in terms of the Wiener process W such that

$$B_t = \int_0^t K^H(t,s)dW(s)$$

(see Chap. 1, (1.2)). We then note that $\bar{F}_N$ can be written as

$$\bar{F}_N = \tilde{I}_2(f_N)$$

where $\tilde{I}_2$ is the double Wiener integral operator with respect to W, and $f_N = N c_{2,H}^{-1/2} \sum_{i=1}^N \tilde{A}_i \otimes \tilde{A}_i$ where

$$\tilde{A}_i(s) = \mathbf{1}_{[0,\frac{i+1}{N}]}(s) K^H\left(\frac{i+1}{N}, s\right) - \mathbf{1}_{[0,\frac{i}{N}]}(s) K^H\left(\frac{i}{N}, s\right). \tag{5.19}$$

Lemma 5.3 below shows that f_N converges in $L^2([0,1]^2)$ to $c_{2,H}^{-1/2} L_1^{2H-1}$ where L_1^{2H-1} is the function

$$(r,s) \mapsto L_1^{2H-1}(r,s) := \int_{r \vee s}^1 \frac{\partial K^H}{\partial u}(u,s) \frac{\partial K^H}{\partial u}(u,r)du. \tag{5.20}$$

Now define the random variable $Y := d(H_0)\tilde{I}_2(L_1^{2H-1})$ where

$$\begin{aligned} d(H_0) &= (H_0+1)^{-1}\big(2(2H_0-1)/H_0\big)^{1/2} \\ &= (4H-3)^{1/2}(2H-1)^{-1/2}\big/(\sqrt{2}H) = c_{2,H}^{-1/2}. \end{aligned}$$

This Y is a standard Rosenblatt random variable with parameter $H_0 = 2H - 1$, as can be seen for instance in Sect. 3.2. By the isometry property for stochastic integrals,

$$\mathbf{E}\big[|\bar{F}_N - Y|^2\big] = \| f_N - c_{2,H}^{-1/2} L_1^{2H-1} \|^2_{L^2([0,1]^2)},$$

which, by the convergence of Lemma 5.3, proves that $\bar{F}_N$ converges to the Rosenblatt random variable $Y = c_{2,H}^{-1/2}\tilde{I}_2(L_1^{2H-1})$ in $L^2(\Omega)$. □

Lemma 5.3 *With* $H \in (3/4, 1)$, *and*

$$\tilde{A}_i(s) = \mathbf{1}_{[0,\frac{i+1}{N}]}(s)K^H\left(\frac{i+1}{N}, s\right) - \mathbf{1}_{[0,\frac{i}{N}]}(s)K^H\left(\frac{i}{N}, s\right),$$

we have that $L_N(r,s) := N\sum_{i=1}^N \tilde{A}_i(r)\tilde{A}_i(s)$ *converges in* $L^2([0,1]^2)$ *to the function*

$$(r,s) \mapsto L_1^{2H-1}(r,s) := \int_{r\vee s}^1 \frac{\partial K^H}{\partial u}(u,s)\frac{\partial K^H}{\partial u}(u,r)du.$$

Proof $\tilde{A}_i(s)$ can be rewritten as

$$\begin{aligned}\tilde{A}_i(s) &= \mathbf{1}_{[0,\frac{i}{N}]}(s)\left(K^H\left(\frac{i+1}{N}, s\right) - K^H\left(\frac{i}{N}, s\right)\right) + \mathbf{1}_{[\frac{i}{N},\frac{i+1}{N}]}(s)K^H\left(\frac{i+1}{N}, s\right)\\ &= N^{-1}\mathbf{1}_{[0,\frac{i}{N}]}(s)\frac{\partial K^H}{\partial u}(\xi_i, s) + \mathbf{1}_{[\frac{i}{N},\frac{i+1}{N}]}(s)K^H\left(\frac{i+1}{N}, s\right)\\ &=: B_i(s) + C_i(s)\end{aligned}$$

where $\xi_i = \xi_i(s)$ depends on s but is nonetheless in the interval $[i/N, (i+1)/N]$. The product $\tilde{A}_i(r)\tilde{A}_i(s)$ yields square-type terms with $B_i(s)B_i(r)$ and $C_i(s)C_i(r)$, and a cross-product term. This last term is treated like the term involving $C_i(s)C_i(r)$, and we leave it to the reader. Now, using the fact that $K(t,s) \le c(t/s)^{H-1/2}(t-s)^{H-1/2}$ we write

$$\begin{aligned}&\iint_{[0,1]^2} drds\left|2Nc_{2,H}^{-1/2}\sum_{i=1}^N C_i(s)C_i(r)\right|^2\\ &\le 4N^2c_{2,H}^{-1}\iint_{[0,1]^2} drds\sum_{i=1}^N\sum_{j=1}^N \mathbf{1}_{[\frac{i}{N},\frac{i+1}{N}]}(s)\mathbf{1}_{[\frac{j}{N},\frac{j+1}{N}]}(r)\\ &\quad\times\left(\frac{i+1}{Ns}\right)^{H-1/2}\left(\frac{j+1}{Nr}\right)^{H-1/2} N^{2-4H}\\ &\le 4N^{2-4H}c_{2,H}^{-1}\iint_{[0,1]^2} dtdu\sum_{i=1}^N\sum_{j=1}^N N^{-2}\\ &\quad\times\left(1+\frac{1}{i}\right)^{H-1/2}\left(1+\frac{1}{j}\right)^{H-1/2}\\ &\le 8N^{2-4H}c_{2,H}^{-1}.\end{aligned}$$

Since $H > 1/2$, this proves that the portion of $\tilde{D}^2 \bar{F}_N$ corresponding to C_i tends to 0 in $L^2([0,1]^2)$. For the dominant term, we calculate

$$\begin{aligned}
&\left| 2Nc_{2,H}^{-1/2} \sum_{i=1}^{N} B_i(r)B_i(s) - 2c_{2,H}^{-1/2} L(r,s) \right| \\
&\quad = 2c_{2,H}^{-1/2} \left| \sum_{i=1}^{N} \mathbf{1}_{[0,\frac{i}{N}]}(r \vee s) \frac{\partial K^H}{\partial u}\big(\xi_i(r), r\big) \frac{\partial K^H}{\partial u}\big(\xi_i(s), s\big) \right. \\
&\qquad \left. - \int_{r\vee s}^{1} \frac{\partial K^H}{\partial u}(u,s) \frac{\partial K^H}{\partial u}(u,r)du \right|.
\end{aligned}$$

This converges to 0 pointwise as a limit of Riemann sums. At this point we can conclude that the sequence $L_N(y_1, y_2)$ converges (in probability for instance) to $L_1^{2H-1}(y_1, y_2)$ for every $y_1, y_2 \in [0,1]$. Our desired result will follow if we prove that the sequence $(L_N)_{N\geq 1}$ is a Cauchy sequence in $L^2([0,1]^2)$. We have

$$\begin{aligned}
&\|L_N - L_M\|^2_{L^2([0,1]^2)} \\
&= N^2 \sum_{i,j=0}^{N-1} \left[\mathbf{E}\left(B^H\left(\frac{i+1}{N}\right) - B^H\left(\frac{i}{N}\right)\right)\left(B^H\left(\frac{j+1}{N}\right) - B^H\left(\frac{j}{N}\right)\right)\right]^2 \\
&\quad + M^2 \sum_{i,j=0}^{M-1} \left[\mathbf{E}\left(B^H\left(\frac{i+1}{M}\right) - B^H\left(\frac{i}{M}\right)\right)\left(B^H\left(\frac{j+1}{M}\right) - B^H\left(\frac{j}{M}\right)\right)\right]^2 \\
&\quad - 2MN \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \left[\mathbf{E}\left(B^H\left(\frac{i+1}{N}\right) - B^H\left(\frac{i}{N}\right)\right)\right. \\
&\quad \left. \times \left(B^H\left(\frac{j+1}{M}\right) - B^H\left(\frac{j}{M}\right)\right)\right]^2
\end{aligned}$$

and we have already seen that

$$N^2 \sum_{i,j=0}^{N-1} \left[\mathbf{E}\left(B^H\left(\frac{i+1}{N}\right) - B^H\left(\frac{i}{N}\right)\right)\left(B^H\left(\frac{j+1}{N}\right) - B^H\left(\frac{j}{N}\right)\right)\right]^2$$

converges to the constant $H^2(2H-1)/(H-3/4)$.

We now consider the sum

$$MN \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \left[\mathbf{E}\left(B^H\left(\frac{i+1}{N}\right) - B^H\left(\frac{i}{N}\right)\right)\left(B^H\left(\frac{j+1}{M}\right) - B^H\left(\frac{j}{M}\right)\right)\right]^2$$

$$= MN \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \Bigg[\left| \frac{i+1}{N} - \frac{j+1}{M} \right|^{2H} + \left| \frac{i}{N} - \frac{j}{M} \right|^{2H} - \left| \frac{i+1}{N} - \frac{j}{M} \right|^{2H}$$

$$- \left| \frac{i}{N} - \frac{j+1}{M} \right|^{2H} \Bigg]^2.$$

For any two-variable function g such that $\frac{\partial g}{\partial x \partial y}(x, y)$ exists and belongs to $L^2([0, 1]^2)$ it can easily be shown (by a Riemann sum argument) that

$$MN \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \left[g\left(\frac{i+1}{N}, \frac{j+1}{M}\right) + g\left(\frac{i}{N}, \frac{j}{M}\right) - g\left(\frac{i+1}{N}, \frac{j}{M}\right) - g\left(\frac{i}{N}, \frac{j+1}{M}\right) \right]^2$$

can be written as

$$\frac{1}{MN} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \left(\frac{\partial g}{\partial x \partial y}(a_i, b_j) \right)^2$$

with a_i located between $\frac{i}{N}$ and $\frac{i+1}{N}$ and b_j located between $\frac{j}{M}$ and $\frac{j+1}{M}$ and consequently it converges to

$$\big(2H(2H-1)\big)^2 \int_0^1 \int_0^1 |x-y|^{4H-4} dx dy = H^2(2H-1)/(H-3/4) = c_{2,H}$$

with $c_{2,H}$ defined by (5.8). □

To obtain the error bound in the Non-Central Limit Theorem (for $H > \frac{3}{4}$) we will use a criterium proved in [65]. This criterium applies to the total variation distance. Recall that the total variation distance between the probability distributions of two real-valued random variables X and Y is defined by

$$d_{\mathrm{TV}}\big(\mathcal{L}(X), \mathcal{L}(Y)\big) = \sup_{A \in \mathcal{B}(\mathbb{R})} \big| P(Y \in A) - P(X \in A) \big| \tag{5.21}$$

where $\mathcal{B}(\mathbb{R})$ denotes the class of Borel sets of $\mathbb{R}$. We have the following result (see [65]) on the total variation distance between elements of a fixed Wiener chaos.

Theorem 5.6 *Fix an integer $q \geq 2$ and let $f \in \mathcal{H}^{\odot q} \setminus \{0\}$. Then, for any sequence $\{f_n\}_{n \geq 1} \subset \mathcal{H}^{\odot q}$ converging to f, there exists a constant $c_{q,f}$, depending only on q and f, such that*

$$d_{TV}\big(I_q(f_n), I_q(f)\big) \leq c_{q,f} \| f_n - f \|_{\mathcal{H}^{\odot q}}^{1/q}.$$

Using Theorem 5.6, the rate of convergence in the Non-Central Limit Theorem is as follows.

Theorem 5.7 *For* $H > \frac{3}{4}$,

$$d_{TV}\big(\mathcal{L}(\bar{F}_N), Z\big) \le cN^{\frac{3}{4}-H}.$$

Proof See Theorem 1.2 in [39]. □

5.1.5 Multidimensional Convergence of the 2-Variations

This section is devoted to the study of the vectorial convergence of the 2-variations statistics. We will restrict ourselves to the case $H \le \frac{3}{4}$ in which the limit of the components are Gaussian random variables. Our strategy is based on Theorem 5.3.

Define the following filters constructed from the filter $a = \{1, -1\}$:

$$a^1 = \{1, -1\}, \qquad a^2 = \{1, -2, 1\}, \qquad a^3 = \{1, 0, 0, -1\}, \quad \ldots$$
$$a^M = (1, 0, 0, \ldots, -1\}$$

where M is an integer and at each step p, the vector a^p has $p-1$ zeros. Note that for every $p = 1, \ldots, M$, the filter a^p is a $p+1$ dimensional vector.

Consider the statistics based on the above filters ($1 \le p \le M$)

$$\begin{aligned} V_N\big(2, a^p\big) &= \frac{1}{N-p+1}\sum_{i=p}^{N}\left[\frac{(B(\frac{i}{N}) - B(\frac{i-p}{N}))^2}{\mathbf{E}(B(\frac{i}{N}) - B(\frac{i-p}{N}))^2} - 1\right] \\ &= \frac{1}{N-p+1}\sum_{i=p}^{N}\left[\big(I_1(A_{i,p})\big)^2\left(\frac{p}{N}\right)^{-2H} - 1\right] \\ &= \frac{1}{N-p+1}\left(\frac{p}{N}\right)^{-2H}\sum_{i=p}^{N} I_2(A_{i,p}\otimes A_{i,p}) \end{aligned}$$

where

$$A_{i,p} = 1_{[\frac{i-p}{N}, \frac{i}{N}]}, \qquad 1 \le p \le M, \; p \le i \le N.$$

We have the following vectorial limit theorem.

Theorem 5.8 *Let* B *be a fBm with* $H \in (0, 3/4)$ *and let* $M \ge 1$. *For* $1 \le p, q \le M$ *define*

$$\begin{aligned} c_{p,q,H} &:= \frac{1}{(pq)^{2H}}\sum_{k\ge 1}\big(|k|^{2H} + |k-p+q|^{2H} - |k-p|^{2H} - |k+q|^{2H}\big)^2 \\ &\quad + c'_{p,q,H}, \quad \text{and} \quad c_{p,H} := c_{p,p,H} \end{aligned}$$

with $c'_{p,q,H} = \frac{(|p-q|^{2H}-p^{2H}-q^{2H})^2}{2(pq)^{2H}}$ *and*

$$\bar{F}_N\big(a^p\big) := \sqrt{N} c_{p,H}^{-1} V_N(2, a_p). \tag{5.22}$$

Then the vector $(\bar{F}_N(a^1), \dots, F_N(a^M))$ *converges, as* $N \to \infty$, *to a Gaussian vector with covariance matrix* $C = C_{i,j}$ *where* $C_{p,q} = \frac{c_{p,q,H}}{\sqrt{c_{p,H} c_{q,H}}}$.

If $H = \frac{3}{4}$, *define*

$$d_{p,q,H} := \frac{1}{(pq)^{2H}} \frac{3}{16}, \quad \text{and} \quad d_{p,H} := d_{p,p,H},$$

and

$$\tilde{F}_N\big(a^p\big) = \sqrt{\frac{N}{\log N}} d_{p,H}^{-1/2} V_N\big(2, a^p\big).$$

Then the vector $(F_N(a^1), \dots, F_N(a^M))$ *converges, as* $N \to \infty$, *to a Gaussian vector with covariance matrix* $D = D_{i,j}$ *where* $D_{p,q} = \frac{d_{p,q,H}}{\sqrt{d_{p,H} d_{q,H}}}$.

Proof Let us estimate the covariance of two such statistics

$$\begin{aligned}
&\mathbf{E}\big[V_N\big(2, a^p\big) V_N\big(2, a^q\big)\big] \\
&\quad = \frac{N^{4H}}{(N-p+1)(N-q+1)} \frac{1}{(pq)^{2H}} 2 \sum_{i=p}^{N} \sum_{j=q}^{N} \langle A_{i,p} \otimes A_{i,p}, A_{j,q} \otimes A_{j,q} \rangle_{\mathcal{H} \otimes \mathcal{H}} \\
&\quad = \frac{N^{4H}}{(N-p+1)(N-q+1)} \frac{2}{(pq)^{2H}} \sum_{i=p}^{N} \sum_{j=q}^{N} \langle A_{i,p} \otimes A_{j,q} \rangle_{\mathcal{H}}^2.
\end{aligned}$$

The next step is to compute the scalar product

$$\begin{aligned}
\langle A_{i,p} \otimes A_{j,q} \rangle_{\mathcal{H}} &= \langle 1_{[\frac{i-p}{N}, \frac{i}{N}]}, 1_{[\frac{j-q}{N}, \frac{j}{N}]} \rangle_{\mathcal{H}} \\
&= \frac{1}{2} \Bigg[\left| \frac{i-j}{N} \right|^{2H} + \left| \frac{i-j-p+q}{N} \right|^{2H} \\
&\qquad - \left| \frac{i-j-p}{N} \right|^{2H} - \left| \frac{i-j+q}{N} \right|^{2H} \Bigg].
\end{aligned}$$

Assume that $p \geq q$. We need to estimate the sum

$$\sum_{i=p}^{N} \sum_{j=q}^{N} \Bigg[\left| \frac{i-j}{N} \right|^{2H} + \left| \frac{i-j-p+q}{N} \right|^{2H} - \left| \frac{i-j-p}{N} \right|^{2H} - \left| \frac{i-j+q}{N} \right|^{2H} \Bigg]^2$$

$$= \frac{1}{N^{4H}} \sum_{j=q}^{p-1} \sum_{i=p}^{N} \big(|i-j|^{2H} + |i-j-p+q|^{2H}$$

$$- |i-j-p|^{2H} - |i-j+q|^{2H}\big)^2$$

$$+ \frac{1}{N^{4H}} \sum_{j=p}^{N} \sum_{i=p}^{N} \big(|i-j|^{2H} + |i-j-p+q|^{2H}$$

$$- |i-j-p|^{2H} - |i-j+q|^{2H}\big)^2 + c'_{p,q,H}$$

$$= \frac{2}{N^{4H}} \sum_{j=p}^{N} \sum_{k=1}^{N-j} \big(|k|^{2H} + |k-p+q|^{2H} - |k-p|^{2H} - |k+q|^{2H}\big)^2 + c'_{p,q,H}$$

$$= \frac{2}{N^{4H}} \sum_{k=1}^{N-p} (N-k-p) \big(|k|^{2H} + |k-p+q|^{2H} - |k-p|^{2H} - |k+q|^{2H}\big)^2$$

$$= \frac{2}{N^{4H}} \sum_{k=1}^{N-p} (N-k-p) k^{4H} g\left(\frac{1}{k}\right)^2 + c'_{p,q,H}$$

where

$$g(x) = 1 + \big(1-(p-q)x\big)^{2H} - (1-px)^{2H} - (1+qx)^{2H}.$$

By the asymptotic behavior of the function g near zero, we obtain for large k

$$g\left(\frac{1}{k}\right) \sim 2H(2H-1)pq\frac{1}{k^2}.$$

We distinguish again the cases $H < \frac{3}{4}$ and $k = \frac{3}{4}$ and we conclude that

$$\mathbf{E}\big[V_N\big(2,a^p\big)V_N\big(2,a^q\big)\big] \sim_{N\to\infty} c_{p,q,H}\frac{1}{N}, \quad \text{for } H < \frac{3}{4}$$

and

$$\mathbf{E}\big[V_N\big(2,a^p\big)V_N\big(2,a^q\big)\big] \sim_{N\to\infty} d_{p,q,H}\frac{\log N}{N} \quad \text{for } H = \frac{3}{4}$$

where the constants $c_{p,q,H}$ and $d_{p,q,H}$ are defined in the statement of the theorem. The conclusion then follows from Proposition 5.3. □

5.2 Quadratic Variations of the Rosenblatt Process

Our observed process is a Rosenblatt process $(Z(t))_{t\in[0,1]}$ with self-similarity parameter $H \in (\frac{1}{2}, 1)$. Recall from Sect. 3.2 that this centered process is self-similar with stationary increments, and lives in the second Wiener chaos. Its covariance is identical to that of fractional Brownian motion. Our goal is to study the asymptotic behavior of its quadratic variations. The classical techniques (e.g, those from [67, 167], or [168]) do not work well for this process. Therefore, the use of the Malliavin calculus and multiple stochastic integrals is of interest.

We will use the representation of the Rosenblatt process on a finite interval given in Sect. 3.2. The Rosenblatt process can be represented as follows (see (3.17)): for every $t \in [0, 1]$

$$\begin{aligned} Z^H(t) &:= Z(t) \\ &= d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(u, y_2) du \right] dW(y_1) dW(y_2) \end{aligned}$$

where $(W(t), t \in [0, 1])$ is some standard Brownian motion, $K^{H'}$ is the standard kernel of fractional Brownian motion (see (1.4)), $H' = \frac{H+1}{2}$ and the constant $d(H)$ is defined by (3.19).

For every $t \in [0, 1]$ we will denote the kernel of the Rosenblatt process with respect to W by

$$\begin{aligned} L_t^H(y_1, y_2) &:= L_t(y_1, y_2) \\ &:= d(H) \left[\int_{y_1 \vee y_2}^t \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(u, y_2) du \right] 1_{[0,t]^2}(y_1, y_2). \end{aligned} \tag{5.23}$$

In other words, in particular, for every t

$$Z(t) = I_2\big(L_t(\cdot)\big)$$

where I_2 denotes the multiple integral of order 2 introduced in Appendix C.

Consider now the 2-variations (or the quadratic variations) given by

$$\begin{aligned} V_N &= \frac{1}{N} \sum_{i=1}^N \frac{(Z(\frac{i}{N}) - Z(\frac{i-1}{N}))^2}{\mathbf{E}(Z(\frac{i}{N}) - Z(\frac{i-1}{N}))^2} - 1 \\ &= N^{2H-1} \sum_{i=1}^N \left[\left(Z\left(\frac{i}{N}\right) - Z\left(\frac{i-1}{N}\right) \right)^2 - N^{-2H} \right]. \end{aligned} \tag{5.24}$$

The first step is to decompose in chaos the random variable V_N (5.24).

Proposition 5.5 *For every* N,

$$V_N = N^{2H-1} \sum_{i=1}^N \big(I_4(A_i \otimes A_i) + 4 I_2(A_i \otimes_1 A_i) \big) := T_4 + T_2$$

where, for $i = 1, \dots, N$ *and for* $y_1, y_2 \in [0, 1]$

$$A_i(y_1, y_2) := L^H_{\frac{i}{N}}(y_1, y_2) - L^H_{\frac{i-1}{N}}(y_1, y_2) \tag{5.25}$$

and L^H *is defined by* (5.23).

Proof The product formula for multiple Wiener-Itô integrals (C.4) yields

$$I_2(f)^2 = I_4(f \otimes f) + 4I_2(f \otimes_1 f) + 2I_0(f \otimes_2 f).$$

With A_i defined by (5.25) we can thus write

$$\begin{aligned}\left(Z\left(\frac{i}{N}\right) - Z\left(\frac{i-1}{N}\right)\right)^2 &= \left(I_2(A_i)\right)^2 \\ &= I_4(A_i \otimes A_i) + 4I_2(A_i \otimes_1 A_i) + 2I_0(A_i \otimes_2 A_i)\end{aligned}$$

and this implies that the 2-variation decomposes into a 4th chaos term and a 2nd chaos term:

$$V_N = N^{2H-1} \sum_{i=1}^{N} \left(I_4(A_i \otimes A_i) + 4I_2(A_i \otimes_1 A_i)\right) := T_4 + T_2.$$

□

A detailed study of the two terms above will shed light on some interesting facts: if $H \le \frac{3}{4}$ the term T_4 continues to exhibit "normal" behavior (when renormalized, it converges in law to a Gaussian distribution), while the term T_2, which turns out to be dominant, never converges to a Gaussian law. One can say that the second Wiener chaos portion is "ill-behaved"; however, once it is subtracted, one obtains a sequence converging to $N(0, 1)$ (for $H \in (\frac{1}{2}, \frac{2}{3})$), which has an impact for statistical applications.

5.2.1 *Evaluation of the* L^2*-Norm*

We now analyze the asymptotic behavior of the sequence $\mathbf{E}V_N^2$. From Proposition 5.5, one needs to evaluate the two terms T_2 and T_4 appearing in the chaos expansion of V_N.

The Term in the Second Wiener Chaos Let us evaluate the mean square of the second term

$$T_2 := N^{2H-1} 4 \sum_{i=1}^{N} I_2(A_i \otimes_1 A_i).$$

Proposition 5.6 *Let $a(H), d(H)$ be given by* (5.27) *and* (3.19) *respectively. Then*

$$\lim_{N\to\infty} \mathbf{E}\big[T_2^2\big]N^{2-2H} = 64a(H)^2 d(H)^4\left(\frac{1}{2H-1}-\frac{1}{2H}\right) = 16d(H)^2 := c_{3,H}.$$

Proof We use the notation $I_i = (\frac{i-1}{N}, \frac{i}{N}]$ for $i = 1, \dots, N$. The contraction $A_i \otimes_1 A_i$ is given by

$$\begin{aligned}(A_i \otimes_1 A_i)(y_1, y_2) &= \int_0^1 A_i(x, y_1)A_i(x, y_2)dx \\ &= d(H)^2 \int_0^1 dx\ 1_{[0,\frac{i}{N}]}(y_1 \vee x)1_{[0,\frac{i}{N}]}(y_2 \vee x) \\ &\quad \times \left(\int_{x\vee y_1}^{\frac{i}{N}} \partial_1 K^{H'}(u,x)\partial_1 K^{H'}(u,y_1)du - 1_{[0,\frac{i-1}{N}]}(y_1\vee x)\right. \\ &\quad \left.\times \int_{x\vee y_1}^{\frac{i-1}{N}} \partial_1 K^{H'}(u,x)\partial_1 K^{H'}(u,y_1)du\right) \\ &\quad \times \left(\int_{x\vee y_2}^{\frac{i}{N}} \partial_1 K^{H'}(v,x)\partial_1 K^{H'}(v,y_2)dv - 1_{[0,\frac{i-1}{N}]}(y_2\vee x)\right. \\ &\quad \left.\times \int_{x\vee y_2}^{\frac{i-1}{N}} \partial_1 K^{H'}(v,x)\partial_1 K^{H'}(v,y_2)dv\right). \end{aligned} \tag{5.26}$$

With

$$a(H) := H'\big(2H'-1\big) = H(H+1)/2 \tag{5.27}$$

note the following fact (see [136], Chap. 5):

$$\int_0^{u\wedge v} \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(v,y_1)dy_1 = a(H)|u-v|^{2H'-2}; \tag{5.28}$$

in fact, this relation can be easily derived from $\int_0^{t\wedge s} K^{H'}(t,u)K^{H'}(s,u)du = R^{H'}(t,s)$ (relation (1.5)), and will be used repeatedly in the sequel.

To use this relation, we first expand the product in the expression for the contraction in (5.26), taking care to keep track of the indicator functions. The resulting initial expression for $(A_i \otimes_1 A_i)(y_1, y_2)$ contains four terms, which are all of the following form:

$$\begin{aligned} C_{a,b} := d(H)^2 &\int_0^1 dx 1_{[0,a]}(y_1\vee x)1_{[0,b]}(y_2\vee x) \\ &\times \int_{u=y_1\vee x}^{a} \partial_1 K^{H'}(u,x)\partial_1 K(u,y_1)du \\ &\times \int_{v=y_2\vee x}^{b} \partial_1 K^{H'}(v,x)\partial_1 K^{H'}(v,y_2)dv. \end{aligned}$$

Here in order to use Fubini's theorem, bringing the integral over x inside, we first note that $x < u \wedge v$ while $u \in [y_1, a]$ and $v \in [y_2, b]$. Also note that the conditions $x \le u$ and $u \le a$ imply $x \le a$, and thus $1_{[0,a]}(y_1 \vee x)$ can be replaced, after Fubini, by $1_{[0,a]}(y_1)$. Therefore, using (5.28), the above expression equals

$$\begin{aligned}
C_{a,b} &= d(H)^2 1_{[0,a]\times[0,b]}(y_1, y_2) \int_{y_1}^{a} \partial_1 K^{H'}(u, y_1) du \int_{y_2}^{b} \partial_1 K^{H'}(v, y_2) dv \\
&\quad \times \int_0^{u\wedge v} \partial_1 K^{H'}(u, x) \partial_1 K^{H'}(v, x) dx \\
&= d(H)^2 1_{[0,a]\times[0,b]}(y_1, y_2) \\
&\quad \times \int_{u=y_1}^{a} \int_{v=y_2}^{b} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} du dv \\
&= d(H)^2 \int_{u=y_1}^{a} \int_{v=y_2}^{b} \partial_1 K(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} du dv.
\end{aligned}$$

The last inequality above comes from the fact that the indicator functions in y_1, y_2 are redundant: they can be pulled back into the integral over $dudv$ and therein, the functions $\partial_1 K^{H'}(u, y_1)$ and $\partial_1 K^{H'}(v, y_2)$ are, by definition, as functions of y_1 and y_2, supported by smaller intervals than $[0, a]$ and $[0, b]$, namely $[0, u]$ and $[0, v]$ respectively.

Now, the contraction $(A_i \otimes_1 A_i)(y_1, y_2)$ equals $C_{i/N,i/N} + C_{(i-1)/N,(i-1)/N} - C_{(i-1)/N,i/N} - C_{i/N,(i-1)/N}$. Therefore, from the last expression above,

$$\begin{aligned}
&(A_i \otimes_1 A_i)(y_1, y_2) \\
&= a(H) d(H)^2 \Bigg(\int_{y_1}^{\frac{i}{N}} du \int_{y_2}^{\frac{i}{N}} dv \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} \\
&\quad - \int_{y_1}^{\frac{i}{N}} du \int_{y_2}^{\frac{i-1}{N}} dv \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} \\
&\quad - \int_{y_1}^{\frac{i-1}{N}} du \int_{y_2}^{\frac{i}{N}} dv \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} \\
&\quad + \int_{y_1}^{\frac{i-1}{N}} du \int_{y_2}^{\frac{i-1}{N}} dv \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} \Bigg). \qquad (5.29)
\end{aligned}$$

Since the integrands in the above four integrals are identical, we can simplify the above formula, grouping the first two terms, for instance, to obtain an integral of v over $I_i = (\frac{i-1}{N}, \frac{i}{N}]$, with integration over u in $[y_1, \frac{i}{n}]$. The same operation on the last two terms gives the negative of the same integral over v, with integration over u in $[y_1, \frac{i-1}{n}]$. Then grouping these two resulting terms yields a single term, which is an integral for (u, v) over $I_i \times I_i$. We obtain the following final expression for our contraction:

$$(A_i \otimes_1 A_i)(y_1, y_2) = a(H)d(H)^2 \times \iint_{I_i \times I_i} \partial_1 K^{H'}(u, y_1)\partial_1 K^{H'}(v, y_2)|u-v|^{2H'-2} du dv. \tag{5.30}$$

Now, since the integrands in the double Wiener integrals defining T_2 are symmetric, we get

$$\mathbf{E}\big[T_2^2\big] = N^{4H-2} 16 \cdot 2! \sum_{i,j=1}^{N} \langle A_i \otimes_1 A_i, A_j \otimes_1 A_j \rangle_{L^2([0,1]^2)}.$$

To evaluate the inner product of the two contractions, we first use Fubini's theorem with expression (5.30); by doing so, one must realize that the support of $\partial_1 K^{H'}(u, y_1)$ is $\{u > y_1\}$, which then makes the upper endpoint 1 for the integration in y_1 redundant; similar remarks hold with u', v, v', and y_2. In other words, we have

$$\begin{aligned}
&\langle A_i \otimes_1 A_i, A_j \otimes_1 A_j \rangle_{L^2([0,1])^2} \\
&= a(H)^2 d(H)^4 \int_0^1 \int_0^1 dy_1 dy_2 \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} du' dv' du dv |u-v|^{2H'-2} \\
&\quad \times \big|u'-v'\big|^{2H'-2} \partial_1 K^{H'}(u, y_1)\partial_1 K^{H'}(v, y_2)\partial_1 K^{H'}\big(u', y_1\big)\partial_1 K^{H'}\big(v', y_2\big) \\
&= a(H)^4 d(H)^4 \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} |u-v|^{2H'-2}\big|u'-v'\big|^{2H'-2} du' dv' dv du \\
&\quad \times \int_0^{u \wedge u'} \partial_1 K^{H'}(u, y_1)\partial_1 K^{H'}\big(u', y_1\big) dy_1 \\
&\quad \times \int_0^{v \wedge v'} \partial_1 K^{H'}(v, y_2)\partial_1 K^{H'}\big(v', y_2\big) dy_2 \\
&= a(H)^4 d(H)^4 \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} |u-v|^{2H'-2}\big|u'-v'\big|^{2H'-2} \\
&\quad \times \big|u-u'\big|^{2H'-2}\big|v-v'\big|^{2H'-2} du' dv' dv du
\end{aligned} \tag{5.31}$$

where we used the expression (5.28) in the last step. Therefore we have immediately

$$\mathbf{E}\big[T_2^2\big] = N^{4H-2} 32 a(H)^4 d(H)^4 \sum_{i,j=1}^{N} \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} du' dv' dv du \times |u-v|^{2H'-2}\big|u'-v'\big|^{2H'-2}\big|u-u'\big|^{2H'-2}\big|v-v'\big|^{2H'-2}. \tag{5.32}$$

By Lemma 5.4 below, we conclude that

$$\lim_{N\to\infty} \mathbf{E}\big[T_2^2\big]N^{2-2H} = 64a(H)^2 d(H)^4\left(\frac{1}{2H-1} - \frac{1}{2H}\right) = 16d(H)^2 := c_{3,H}. \tag{5.33}$$

□

Lemma 5.4 *For all $H > 1/2$, with $I_i = (\frac{i-1}{N}, \frac{i}{N}]$, $(i = 1, \dots, N)$*

$$\begin{aligned}\lim_{N\to\infty} N^{2H} \sum_{i,j=1}^{N} \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} |u-v|^{2H'-2}\left|u'-v'\right|^{2H'-2} \\ \times \left|u-u'\right|^{2H'-2}\left|v-v'\right|^{2H'-2} du'dv'dvdu \\ = 2a(H)^{-2}\left(\frac{1}{2H-1} - \frac{1}{2H}\right).\end{aligned} \tag{5.34}$$

Proof We make the change of variables

$$\bar{u} = \left(u - \frac{i-1}{N}\right)N$$

with $d\bar{u} = Ndu$ and we proceed similarly for the other variables u', v, v'. For the integral we need to calculate:

$$\begin{aligned}&\int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} |u-v|^{2H'-2}\left|u'-v'\right|^{2H'-2}\left|u-u'\right|^{2H'-2}\left|v-v'\right|^{2H'-2} du'dv'dvdu \\ &= \frac{1}{N^{4H}}\int_0^1\int_0^1\int_0^1\int_0^1 dudvdu'dv'|u-v|^{2H'-2}\left|u'-v'\right|^{2H'-2} \\ &\quad \times \left|u-u'+i-j\right|^{2H'-2}\left|v-v'+i-j\right|^{2H'-2},\end{aligned}$$

where we used the fact that $8H' - 8 = 4H - 4$. This needs to be summed over $\sum_{i,j=1}^{N}$; the sum can be divided into two parts: a diagonal part containing the terms $i = j$ and a non-diagonal part containing the terms $i \neq j$. As in the calculations contained in the previous sections, one can see that the non-diagonal part is dominant. Indeed, the diagonal part of (5.34) is equal to

$$\begin{aligned}N^{-2H}\sum_{i=1}^{N}\int_{[0,1]^4} dudvdu'dv'|u-v|^{2H'-2}\left|u'-v'\right|^{2H'-2}\left|u-u'\right|^{2H'-2} \\ \times \left|v-v'\right|^{2H'-2}\end{aligned}$$

$$= N^{1-2H} \int_{[0,1]^4} dudvdu'dv' |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \times \left|u-u'\right|^{2H'-2} \left|v-v'\right|^{2H'-2}$$

and this tends to zero because $H > \frac{1}{2}$.

Therefore the behavior of the quantity in the statement of the lemma will be given by that of

$$\begin{aligned} &\frac{2}{N^{2H}} \sum_{i>j} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dudvdu'dv' \\ &\times |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \left|u-u'+i-j\right|^{2H'-2} \left|v-v'+i-j\right|^{2H'-2} \\ &= \frac{2}{N^{2H}} \sum_{i=1}^{N} \sum_{k=1}^{N-i} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dudvdu'dv' \\ &\times |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \left|u-u'+k\right|^{2H'-2} \left|v-v'+k\right|^{2H'-2} \\ &= \frac{2}{N^{2H}} \sum_{k=1}^{N} (N-k) \int_0^1 \int_0^1 \int_0^1 \int_0^1 dudvdu'dv' \\ &\times |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \left|u-u'+k\right|^{2H'-2} \left|v-v'+k\right|^{2H'-2}. \end{aligned}$$

Note that

$$\begin{aligned} &\frac{1}{N^{2H}} \sum_{k=1}^{N} (N-k) \left|u-u'+k\right|^{2H'-2} \left|v-v'+k\right|^{2H'-2} \\ &= \frac{1}{N} \sum_{k=1}^{N} \left(1-\frac{k}{N}\right) \left|\frac{u-u'}{N}+\frac{k}{N}\right|^{2H'-2} \left|\frac{v-v'}{N}+\frac{k}{N}\right|^{2H'-2}. \end{aligned}$$

Because the terms of the form $(u-u')/N$ are negligible in comparison with k/N for all but the smallest k's, the above expression is asymptotically equivalent to the Riemann sum approximation of the Riemann integral

$$\int_0^1 (1-x) x^{4H'-4} dx = 1/(2H-1) - 1/(2H)$$

where $2H'-2 = H-1$. The lemma follows. □

The Term in the Fourth Wiener Chaos Now for the L^2-norm of the term denoted by

$$T_4 := N^{2H-1}\sum_{i=1}^{N} I_4(A_i\otimes A_i),$$

by the isometry formula for multiple stochastic integrals, and using a correction term to account for the fact that the integrand in T_4 is non-symmetric, we have

$$\begin{aligned}\mathbf{E}\big[T_4^2\big] &= 8N^{4H-2}\sum_{i,j=1}^{N}\langle A_i\otimes A_i; A_j\otimes A_j\rangle_{L^2([0,1]^4)}\\ &\quad + 4N^{4H-2}\sum_{i,j=1}^{N} 4\langle A_i\otimes_1 A_j; A_j\otimes_1 A_i\rangle_{L^2([0,1]^2)} =: \mathcal{T}_{4,0}+\mathcal{T}_{4,1}.\end{aligned}$$

We separate the calculation of the two terms $\mathcal{T}_{4,0}$ and $\mathcal{T}_{4,1}$ above. We will see that these two terms are of exactly the same magnitude, so both calculations have to be performed precisely.

The first term $\mathcal{T}_{4,0}$ can be written as

$$\mathcal{T}_{4,0} = 8N^{4H-2}\sum_{i,j=1}^{N}\big|\langle A_i, A_j\rangle_{L^2([0,1]^2)}\big|^2.$$

We calculate each individual scalar product $\langle A_i, A_j\rangle_{L^2([0,1]^2)}$ as

$$\begin{aligned}\langle A_i, A_j\rangle_{L^2([0,1]^2)} &= \int_0^1\int_0^1 A_i(y_1,y_2)A_j(y_1,y_2)dy_1dy_2\\ &= d(H)^2\int_0^1\int_0^1 dy_1dy_2 1_{[0,\frac{i}{N}\wedge\frac{j}{N}]}(y_1\vee y_2)\\ &\quad\times\bigg(\int_{y_1\vee y_2}^{\frac{i}{N}}\partial_1K^{H'}(u,y_1)\partial_1K^{H'}(u,y_2)du - 1_{[0,\frac{i-1}{N}]}(y_1\vee y_2)\\ &\quad\times\int_{y_1\vee y_2}^{\frac{i-1}{N}}\partial_1K^{H'}(u,y_1)\partial_1K^{H'}(u,y_2)du\bigg)\\ &\quad\times\bigg(\int_{y_1\vee y_2}^{\frac{j}{N}}\partial_1K^{H'}(v,y_1)\partial_1K^{H'}(v,y_2)dv - 1_{[0,\frac{j-1}{N}]}(y_1\vee y_2)\\ &\quad\times\int_{y_1\vee y_2}^{\frac{j-1}{N}}\partial_1K^{H'}(v,y_1)\partial_1K^{H'}(v,y_2)dv\bigg)\\ &= d(H)^2\int_{\frac{i-1}{N}}^{\frac{i}{N}}\int_{\frac{j-1}{N}}^{\frac{j}{N}} dudv\bigg[\int_0^{u\wedge v}\partial_1K^{H'}(u,y_1)\partial_1K^{H'}(v,y_1)dy_1\bigg]^2.\end{aligned}$$

Here (5.28) yields

$$\langle A_i, A_j\rangle_{L^2([0,1]^2)} = d(H)^2 a(H)^2 \int_{I_i}\int_{I_j} |u-v|^{2H-2} du dv$$

where again we have used the notation $I_i = (\frac{i-1}{N}, \frac{i}{N}]$ for $i = 1, \dots, N$. We finally obtain

$$\langle A_i, A_j\rangle_{L^2([0,1]^2)} = \frac{d(H)^2 a(H)^2}{H(2H-1)} \frac{1}{2}\left[2\left|\frac{i-j}{N}\right|^{2H} - \left|\frac{i-j+1}{N}\right|^{2H} - \left|\frac{i-j-1}{N}\right|^{2H}\right] \tag{5.35}$$

where, more precisely, $d(H)^2 a(H)^2 (H(2H-1))^{-1} = 2$. In particular, with some positive constants $c_{1,H}$, $c_{2,H}$, and $c'_{1,H}$ using the proofs of Propositions 5.1, 5.2 and 5.3, we get, asymptotically for large N,

$$\lim_{N\to\infty} N\mathcal{T}_{4,0} = c_{1,H}, \quad 1/2 < H < \frac{3}{4}, \tag{5.36}$$

$$\lim_{N\to\infty} N^{4-4H}\mathcal{T}_{4,0} = c_{2,H}, \quad H > \frac{3}{4}, \tag{5.37}$$

$$\lim_{N\to\infty} \frac{N}{\log N}\mathcal{T}_{4,0} = c'_{1,H} = 16, \quad H = \frac{3}{4}. \tag{5.38}$$

The second term $\mathcal{T}_{4,1}$ can be dealt with by obtaining an expression for

$$\langle A_i \otimes_1 A_j; A_j \otimes_1 A_i\rangle_{L^2([0,1]^2)}$$

in the same way as the expression obtained in (5.31). We get

$$\begin{aligned}
\mathcal{T}_{4,1} &= 16N^{4H-2} \sum_{i,j=1}^{N} \langle A_i \otimes_1 A_j; A_j \otimes_1 A_i\rangle_{L^2([0,1]^2)} \\
&= 16 d(H)^4 a(H)^4 N^{-2} \sum_{i,j=1}^{N} \int_0^1\int_0^1\int_0^1\int_0^1 dy dz dy' dz' \\
&\quad \times |y-z+i-j|^{2H'-2} \left|y'-z'+i-j\right|^{2H'-2} \\
&\quad \times \left|y-y'+i-j\right|^{2H'-2} \left|z-z'+i-j\right|^{2H'-2}.
\end{aligned}$$

Now similarly to the proof of Lemma 5.4, we find the following three asymptotic behaviors for the term $\mathcal{T}_{4,1}$ (here $\tau_{1,H}, \tau_{2,H}, \tau_{2,H}$ are positive constants):

- if $H \in (\frac{1}{2}, \frac{3}{4})$, then $\tau_{1,H}^{-1} N\mathcal{T}_{4,1}$ converges to 1;
- if $H > \frac{3}{4}$, then $\tau_{2,H}^{-1} N^{4-4H}\mathcal{T}_{4,1}$ converges to 1;
- if $H = \frac{3}{4}$ then $\tau_{3,H}^{-1}(N/\log N)\mathcal{T}_{4,1}$ converges to 1.

Combining these results for $\mathcal{T}_{4,1}$ with those for $\mathcal{T}_{4,0}$ in lines (5.36), (5.37) and (5.38), we obtain the asymptotics of $\mathbf{E}[T_4^2]$ as $N \to \infty$:

$$\lim_{N\to\infty} N\mathbf{E}\big[T_4^2\big] = e_{1,H}, \quad \text{if } H \in \left(\frac{1}{2}, \frac{3}{4}\right),$$

$$\lim_{N\to\infty} N^{4-4H}\mathbf{E}\big[T_4^2\big] = e_{2,H}, \quad \text{if } H \in \left(\frac{3}{4}, 1\right)$$

$$\lim_{N\to\infty} \frac{N}{\log N}\mathbf{E}\big[T_4^2\big] = e_{3,H}, \quad \text{if } H = \frac{3}{4}$$

where

$$e_{1,H} := (1/2)c_{1,H} + \tau_{1,H}, \qquad e_{2,H} := (1/2)c_{2,H} + \tau_{2,H}, \qquad e_{3,H} := c_{3,H} + \tau_{3,H}. \tag{5.39}$$

Taking into account the estimates (5.36), (5.37), (5.38), with $c_{3,H}$ in (5.33), we see that $\mathbf{E}[T_4^2]$ is always of smaller order than $\mathbf{E}[T_2^2]$; therefore the mean-square behavior of V_N is given by that of the term T_2 only.

Theorem 5.9 *For every $H > 1/2$*

$$\lim_{N\to\infty} \mathbf{E}\left[\left(N^{1-H} V_N \frac{1}{\sqrt{c_{3,H}}}\right)^2\right] = 1. \tag{5.40}$$

5.2.2 *The Limit of the Quadratic Variations of the Rosenblatt Process*

In this subsection we study the asymptotic behavior of the term denoted by T_2 which appears in the decomposition of V_N. Recall that this is the dominant term, given by

$$T_2 = 4N^{2H-1} I_2\left(\sum_{i=1}^{N} (A_i \otimes_1 A_i)\right)$$

and, with $\sqrt{c_{3,H}} = 4d(H)$ given in (5.33), we showed that

$$\lim_{N\to\infty} \mathbf{E}\big[\big(N^{1-H} T_2 c_{3,H}^{-1/2}\big)^2\big] = 1.$$

With $T_N := N^{1-H} T_2 c_{3,H}^{-1/2}$, one can show that in $L^2(\Omega)$, $\lim_{N\to\infty} \|DT_N\|^2_{L^2([0,1])} = 2 + c$ where c is a strictly positive constant. As a consequence the Nualart-Ortiz criterium (Theorem 5.2, point iv.) cannot be used. However, it is straightforward to find the limit of T_2, and thus of V_N, in $L^2(\Omega)$ in this case. We have the following result.

Theorem 5.10 *For all $H \in (1/2, 1)$, the normalized* 2*-variation $N^{1-H} V_N/(4d(H))$ converges in $L^2(\Omega)$ to the Rosenblatt random variable $Z(1)$. Note that this is the actual observed value of the Rosenblatt process at time* 1.

Proof Since we have already proved that $N^{1-H} T_4$ converges to 0 in $L^2(\Omega)$, it is sufficient to prove that $N^{1-H} T_2/(4d(H)) - Z(1)$ converges to 0 in $L^2(\Omega)$. Since T_2 is a second-chaos random variable, i.e. is of the form $I_2(f_N)$ where f_N is a symmetric function in $L^2([0,1]^2)$, it suffices to prove that

$$\frac{N^{1-H}}{4d(H)} f_N$$

converges to L_1 in $L^2([0,1]^2)$, where L_1 is given by (5.23). From (5.30) we get

$$f_N(y_1, y_2) = 4N^{2H-1} a(H) d(H)^2 \times \sum_{i=1}^{N} \left(\iint_{I_i \times I_i} |u-v|^{2H'-2} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) du dv \right). \tag{5.41}$$

We now show that $\frac{N^{1-H}}{4d(H)} f_N$ converges pointwise, for $y_1, y_2 \in [0,1]$, to the kernel of the Rosenblatt random variable. On the interval $I_i \times I_i$, we may replace the evaluation of $\partial_1 K^{H'}$ and $\partial_1 K^{H'}$ at u and v by setting $u = v = i/N$. We then get that $f_N(y_1, y_2)$ is asymptotically equivalent to

$$\begin{aligned} &4N^{2H-1} a(H) d(H)^2 \sum_{i=1}^{N} \mathbf{1}_{i/N \geq y_1 \vee y_2} \partial_1 K^{H'}(i/N, y_1) \partial_1 K^{H'}(i/N, y_2) \\ &\quad \times \iint_{I_i \times I_i} du dv |u-v|^{2H'-2} \\ &= 4N^{H-1} d(H)^2 \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{i/N \geq y_1 \vee y_2} \partial_1 K^{H'}(i/N, y_1) \partial_1 K^{H'}(i/N, y_2) \end{aligned}$$

where we used the identity $\iint_{I_i \times I_i} du dv |u-v|^{2H'-2} = a(H)^{-1} N^{-2H'} = a(H)^{-1} N^{-H-1}$. Therefore we can write for every $y_1, y_2 \in (0,1)^2$, by invoking a Riemann sum approximation,

$$\begin{aligned} &\lim_{N \to \infty} \frac{N^{1-H}}{4d(H)} f_N(y_1, y_2) \\ &\quad = d(H) \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{i/N \geq y_1 \vee y_2} \partial_1 K^{H'}(i/N, y_1) \partial_1 K^{H'}(i/N, y_2) \\ &\quad = d(H) \int_{y_1 \vee y_2}^{1} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(u, y_2) du = L_1(y_{1,} y_2). \end{aligned}$$

To finish the proof, it suffices to check that the sequence $N^{1-H} f_N$ is Cauchy in $L^2([0,1]^2)$. This can be checked by a straightforward calculation. Indeed, one has, with $C(H)$ a positive constant not depending on M and N,

$$\begin{aligned}
&\left\| N^{1-H} f_N - M^{1-H} f_M \right\|^2_{L^2([0,1]^2)} \\
&= C(H) N^{2H} \sum_{i,j=1}^{N} \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \\
&\quad \times \left|u-u'\right|^{2H'-2} \left|v-v'\right|^{2H'-2} du'dv'dudv \\
&\quad + C(H) M^{2H} \sum_{i,j=1}^{M} \int_{\frac{i-1}{M}}^{\frac{i}{M}}\int_{\frac{i-1}{M}}^{\frac{i}{M}}\int_{\frac{j-1}{M}}^{\frac{j}{M}}\int_{\frac{j-1}{M}}^{\frac{j}{M}} |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \\
&\quad \times \left|u-u'\right|^{2H'-2} \left|v-v'\right|^{2H'-2} du'dv'dudv \\
&\quad - 2C(H) M^{1-H} N^{1-H} M^{2H-1} N^{2H-1} \sum_{i=1}^{N}\sum_{j=1}^{M} \int_{I_i}\int_{I_i}\int_{\frac{j-1}{M}}^{\frac{j}{M}}\int_{\frac{j-1}{M}}^{\frac{j}{M}} du'dv'dudv \\
&\quad \times |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \left|u-u'\right|^{2H'-2} \left|v-v'\right|^{2H'-2}.
\end{aligned} \tag{5.42}$$

The first two terms have already been studied in Lemma 5.4. We have shown that

$$\begin{aligned}
N^{2H} \sum_{i,j=1}^{N} \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \\
\times \left|u-u'\right|^{2H'-2} \left|v-v'\right|^{2H'-2} du'dv'dudv
\end{aligned}$$

converges to $(a(H)^2 H(2H-1))^{-1}$. Thus each of the first two terms in (5.42) converge to $C(H)$ times that same constant as M, N go to infinity. By the change of variables already used several times $\bar{u} = (u - \frac{i}{N})N$, the last term in (5.42) is equal to

$$\begin{aligned}
&C(H)(MN)^H \frac{1}{N^2M^2} (NM)^{2H'-2} \sum_{i=1}^{N}\sum_{j=1}^{M} \int_{[0,1]^4} dudvdu'dv' |u-v|^{2H'-2} \\
&\quad \times \left|u'-v'\right|^{2H'-2} \left|\frac{u}{N} - \frac{u'}{M} + \frac{i}{N} - \frac{j}{M}\right|^{2H'-2} \left|\frac{v}{N} - \frac{v'}{M} + \frac{i}{N} - \frac{j}{M}\right|^{2H'-2} \\
&= \frac{C(H)}{MN} \sum_{i=1}^{N}\sum_{j=1}^{M} \int_{[0,1]^4} dudvdu'dv' |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \\
&\quad \times \left|\frac{u}{N} - \frac{u'}{M} + \frac{i}{N} - \frac{j}{M}\right|^{2H'-2} \left|\frac{v}{N} - \frac{v'}{M} + \frac{i}{N} - \frac{j}{M}\right|^{2H'-2}.
\end{aligned}$$

For large i, j the term $\frac{u}{N} - \frac{u'}{M}$ is negligible in comparison with $\frac{i}{N} - \frac{j}{M}$ and it can be ignored. Therefore, the last term in (5.42) is equivalent to a Riemann sum than tends as $M, N \to \infty$ to the constant $(\int_0^1 \int_0^1 |u - v|^{2H'-2} du dv)^2 \int_0^1 \int_0^1 |x - y|^{2(2H'-2)}$. This is precisely equal to $2(a(H)^2 H(2H-1))^{-1}$, i.e. the limit of the sum of the first two terms in (5.42). Since the last term has a leading negative sign, the announced Cauchy convergence is established, completing the proof of the theorem. □

Remark 5.1 One can show that the variations V_N (5.24) converge to zero almost surely as N goes to infinity. Indeed, the results in this section already show that V_N converges to 0 in $L^2(\Omega)$, and thus in probability, as $N \to \infty$; to obtain almost sure convergence we only need to use an argument in [172] based on the Borel-Cantelli lemma.

5.2.3 Normality of the 4th Chaos Term T_4 when $H \le 3/4$

The calculations for T_4 above prove that $\lim_{N\to\infty} \mathbf{E}[G_N^2] = 1$ for $H < 3/4$ where $e_{1,H}$ is given in (5.39) and

$$G_N := \sqrt{N} N^{2H-1} e_{1,H}^{-1/2} I_4\left(\sum_{i=1}^{N} A_i \otimes A_i\right). \tag{5.43}$$

Similarly, for $H = \frac{3}{4}$, we showed that $\lim_{N\to\infty} \mathbf{E}[\tilde{G}_N^2] = 1$ where $e_{3,H}$ is given in (5.39) and

$$\tilde{G}_N := \sqrt{\frac{N}{\log N}} N^{2H-1} e_{3,H}^{-1} I_4\left(\sum_{i=1}^{N} A_i \otimes A_i\right). \tag{5.44}$$

Using the criterion of Nualart and Ortiz-Latorre (Part (iv) in Theorem 5.2), we prove the following asymptotic normality for G_N and $\tilde{G}_N$.

Theorem 5.11 *If $H \in (1/2, 3/4)$, then G_N given by (5.43) converges in distribution as*

$$\lim_{N\to\infty} G_N = N(0, 1). \tag{5.45}$$

If $H = 3/4$ then $\tilde{G}_N$ given by (5.44) converges in distribution as

$$\lim_{N\to\infty} \tilde{G}_N = N(0, 1). \tag{5.46}$$

Proof See [181]. □

5.3 Quadratic Variations of the Hermite Process

5.3.1 Chaos Expansion and Evaluation of the L^2-Norm

Let $Z^{(q,H)}$ be a Hermite process of order q with self-similarity parameter $H \in (\frac{1}{2}, 1)$ as defined by (3.5) using the kernel K_H (1.4). Define the *centered quadratic variation statistic*

$$V_N := \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{(Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}})^2}{\mathbf{E}[(Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}})^2]} - 1 \right]. \tag{5.47}$$

Also for $H \in (1/2, 1)$, and $q \in \mathbf{N} \setminus \{0\}$, we define a constant which will recur throughout:

$$d(H,q) := \frac{(H(2H-1))^{1/2}}{(q!(H'(2H'-1))^q)^{1/2}}, \quad H' = 1 + \frac{(H-1)}{q}. \tag{5.48}$$

We prove that, under suitable normalization, this sequence converges in $L^2(\Omega)$ to a Rosenblatt random variable.

Taking into account the results in Sects. 5.1 and 5.2, this shows that fBm is the only Hermite process for which there exists a range of parameters allowing normal convergence of the quadratic variations, while for all other Hermite processes, convergence to a second chaos random variable is universal. Our proofs are again based on chaos expansions into multiple Wiener integrals and the Malliavin calculus. The main line of the proof is as follows: since the variable $Z_t^{(q,H)}$ is an element of the qth Wiener chaos, the product formula for multiple integrals (C.4) implies that the statistics V_N can be decomposed into a sum of multiple integrals from order 2 to order $2q$. The dominant term in this decomposition, which gives the final renormalization order $N^{(2-2H)/q}$, is the term which is a double Wiener integral, and one proves it *always* converges to a Rosenblatt random variable; all other terms are of much lower orders, which is why the only remaining term, after renormalization, converges to a second chaos random variable. The difference with the fBm case comes from the limit of the term of order 2, which in that case is sometimes Gaussian and sometimes Rosenblatt-distributed, depending on the value of H.

Since $\mathbf{E}(Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}})^2 = N^{-2H}$, the centered quadratic variation statistic V_N of the Hermite process can be written as

$$V_N = N^{2H-1} \sum_{i=0}^{N-1} \left[\left(Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}} \right)^2 - N^{-2H} \right].$$

Let $I_i := [\frac{i}{N}, \frac{i+1}{N}]$. In preparation for calculating the variance of V_N we will find an explicit expansion of V_N in Wiener chaos. We have $Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}} = I_q(f_{i,N})$

where we denoted by $f_{i,N}(y_1, \dots, y_q)$ the kernel of q variables

$$1_{[0,\frac{i+1}{N}]}(y_1 \vee \cdots \vee y_q) d(H,q) \int_{y_1 \vee \cdots \vee y_q}^{\frac{i+1}{N}} \partial_1 K^{H'}(u, y_1) \cdots \partial_1 K^{H'}(u, y_q) du$$
$$-1_{[0,\frac{i}{N}]}(y_1 \vee \cdots \vee y_q) d(H,q) \int_{y_1 \vee \cdots \vee y_q}^{\frac{i}{N}} \partial_1 K^{H'}(u, y_1) \cdots \partial_1 K^{H'}(u, y_q) du.$$

Using the product formula for multiple integrals (C.4), we obtain

$$I_q(f_{i,N}) I_q(f_{i,N}) = \sum_{l=0}^{q} l! \binom{q}{k}^2 I_{2q-2l}(f_{i,N} \otimes_l f_{i,N})$$

where $f \otimes_l g$ denotes the l-contraction of the functions f and g given by (C.5). Let us compute these contractions; for $l = q$ we have

$$\langle f_{i,N} \otimes_q f_{i,N} \rangle = q! \langle f_{i,N}, f_{i,N} \rangle_{L^2([0,1])^{\otimes q}} = \mathbf{E}\Big[\Big(Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}}\Big)^2\Big] = N^{-2H}.$$

In the following the notation $\partial_1 K(t,s)$ will be used for $\partial_1 K^{H'}(t,s)$. For $l = 0$ we have

$$\begin{aligned}\langle f_{i,N} \otimes_0 f_{i,N})(y_1, \dots y_q, z_1, \dots, z_q\rangle &= (f_{i,N} \otimes f_{i,N})(y_1, \dots y_q, z_1, \dots, z_q) \\ &= f_{i,N}(y_1, \dots, y_q) f_{i,N}(z_1, \dots, z_q)\end{aligned}$$

while for $1 \le k \le q-1$

$$\begin{aligned}
&\langle f_{i,N} \otimes_k f_{i,N})(y_1, \dots y_{q-k}, z_1, \dots, z_{q-k}\rangle \\
&\quad = d(H,q)^2 \int_{[0,1]^k} d\alpha_1 \cdots d\alpha_k \bigg(\mathbf{1}^{y_i}_{i+1,q-k} \mathbf{1}^{\alpha_i}_{i+1,k} \\
&\qquad \times \int_{\mathbf{I}^y_{i+1,k}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k}) \partial_1 K(u, \alpha_1) \cdots \partial_1 K(u, \alpha_k) \\
&\qquad - \mathbf{1}^{y_i}_{i,q-k} \mathbf{1}^{\alpha_i}_{i,k} \int_{\mathbf{I}^y_{i,k}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k}) \partial_1 K(u, \alpha_1) \cdots \partial_1 K(u, \alpha_k) \bigg) \\
&\qquad \times \bigg(\mathbf{1}^{z_i}_{i+1,q-k} \mathbf{1}^{\alpha_i}_{i+1,k} \\
&\qquad \times \int_{\mathbf{I}^z_{i+1,k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) \partial_1 K(v, \alpha_1) \cdots \partial_1 K(v, \alpha_k) \\
&\qquad - \mathbf{1}^{z_i}_{i,q-k} \mathbf{1}^{\alpha_i}_{i,k} \int_{\mathbf{I}^z_{i,k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) \partial_1 K(v, \alpha_1) \cdots \partial_1 K(v, \alpha_k) \bigg)
\end{aligned}$$

where $\mathbf{1}_{i,k}^{x_j}$ denotes the indicator function $1_{[0,\frac{i}{N}]^k}(x_j)$ with x being y, z, or α, and $\mathbf{I}_{i,k}^{x}$ denotes the interval $[x_1 \vee \cdots \vee x_{q-k} \vee \alpha_1 \cdots \vee \alpha_k; i/N]$, with x being y or z. By interchanging the order of the integration we get

$$\begin{aligned}
&\langle f_{i,N} \otimes_k f_{i,N})(y_1, \dots y_{q-k}, z_1, \dots, z_{q-k}\rangle \\
&= d(H,q)^2 \Bigg\{ 1_{[0,\frac{i+1}{N}]^{2q-2k}}(y_i, z_i) \int_{y_1 \vee \cdots \vee y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k}) \\
&\quad \times \int_{z_1 \vee \cdots \vee z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha \right)^k \\
&\quad - 1_{[0,\frac{i+1}{N}]^{q-k}}(y_i) 1_{[0,\frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \cdots \vee y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k}) \\
&\quad \times \int_{z_1 \vee \cdots \vee z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha \right)^k \\
&\quad - 1_{[0,\frac{i}{N}]^{q-k}}(y_i) 1_{[0,\frac{i+1}{N}]^{q-k}}(z_i) \int_{y_1 \vee \cdots \vee y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k}) \\
&\quad \times \int_{z_1 \vee \cdots \vee z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha \right)^k \\
&\quad + 1_{[0,\frac{i}{N}]^{q-k}}(y_i) 1_{[0,\frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \cdots \vee y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k}) \\
&\quad \times \int_{z_1 \vee \cdots \vee z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) \\
&\quad \times \left(\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha \right)^k \Bigg\}
\end{aligned}$$

and since

$$\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha = a(H')|u-v|^{2H'-2}$$

with $a(H') = H'(2H'-1)$, we obtain

$$\begin{aligned}
&\langle f_{i,N} \otimes_k f_{i,N})(y_1, \dots y_{q-k}, z_1, \dots, z_{q-k}) \\
&\quad = d(H,q)^2 a(H')^k \\
&\qquad \times \Bigg\{ 1_{[0,\frac{i+1}{N}]^{q-k}}(y_i) 1_{[0,\frac{i+1}{N}]^{q-k}}(z_i) \int_{y_1 \vee \cdots \vee y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k})
\end{aligned}$$

$$\times \int_{z_1 \vee \cdots \vee z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) |u - v|^{(2H'-2)k}$$

$$- 1_{[0, \frac{i+1}{N}]^{q-k}}(y_i) 1_{[0, \frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \cdots \vee y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k})$$

$$\times \int_{z_1 \vee \cdots \vee z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) |u - v|^{(2H'-2)k}$$

$$- 1_{[0, \frac{i}{N}]^{q-k}}(y_i) 1_{[0, \frac{i+1}{N}]^{q-k}}(z_i) \int_{y_1 \vee \cdots \vee y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k})$$

$$\times \int_{z_1 \vee \cdots \vee z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) |u - v|^{(2H'-2)k}$$

$$+ 1_{[0, \frac{i}{N}]^{q-k}}(y_i) 1_{[0, \frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \cdots \vee y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \cdots \partial_1 K(u, y_{q-k})$$

$$\times \int_{z_1 \vee \cdots \vee z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \cdots \partial_1 K(v, z_{q-k}) |u - v|^{(2H'-2)k} \Big\}. \tag{5.49}$$

As a consequence, we can write

Proposition 5.7

$$V_N = T_{2q} + c_{2q-2} T_{2q-2} + \cdots + c_4 T_4 + c_2 T_2 \tag{5.50}$$

where

$$c_{2q-2k} := k! \binom{q}{k}^2 \tag{5.51}$$

are the combinatorial constants from the product formula for $0 \le k \le q-1$, *and*

$$T_{2q-2k} := N^{2H-1} I_{2q-2k} \left(\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N} \right), \tag{5.52}$$

where the integrands in the last formula above are given explicitly in (5.49).

This Wiener chaos decomposition of V_N allows us to find V_N's precise order of magnitude via its variance's asymptotics.

Proposition 5.8 *With*

$$c_{1,H} = \frac{4d(H,q)^4 (H'(2H'-1))^{2q}}{(4H'-3)(4H'-2)[(2H'-2)(q-1)+1]^2[(H'-1)(q-1)+1]^2}, \tag{5.53}$$

we have

$$\lim_{N\to\infty} \mathbf{E}\big[c_{1,H}^{-1} N^{(2-2H')2} c_2^{-2} V_N^2\big] = 1.$$

Proof We only need to estimate the L^2-norm of each term appearing in the chaos decomposition (5.50) of V_N, since these terms are orthogonal in L^2. We can write, for $0 \le k \le q-1$,

$$\begin{aligned}\mathbf{E}\big[T_{2q-2k}^2\big] &= N^{4H-2}(2q-2k)!\left\|\left(\sum_{i=0}^{N-1} f_{i,N}\otimes_k f_{i,N}\right)^s\right\|^2_{L^2([0,1]^{2q-2k})}\\ &= N^{4H-2}(2q-2k)!\sum_{i,j=0}^{N-1}\langle f_{i,N}\tilde{\otimes}_k f_{i,N}, f_{j,N}\tilde{\otimes}_k f_{j,N}\rangle_{L^2([0,1]^{2q-2k})}\end{aligned}$$

where $(g)^s = \tilde{g}$ and $f_{i,N}\tilde{\otimes}_k f_{i,N}$ denotes the symmetrization of the function $f_{i,N}\otimes_k f_{i,N}$. We will consider first the term T_2 obtained for $k=q-1$. In this case, the kernel $\sum_{i=0}^{N-1} f_{i,N}\otimes_{q-1} f_{i,N}$ is symmetric and we can avoid its symmetrization. Therefore

$$\begin{aligned}\mathbf{E}\big[T_2^2\big] &= 2!N^{4H-2}\left\|\sum_{i=0}^{N-1} f_{i,N}\otimes_{q-1} f_{i,N}\right\|^2_{L^2([0,1]^2)}\\ &= 2!N^{4H-2}\sum_{i,j=0}^{N-1}\langle f_{i,N}\otimes_{q-1} f_{i,N}, f_{j,N}\otimes_{q-1} f_{j,N}\rangle_{L^2([0,1]^2)}.\end{aligned}$$

We compute now the scalar product in the above expression. By using Fubini's theorem, we end up with the following easier expression

$$\begin{aligned}&\langle f_{i,N}\otimes_{q-1} f_{i,N}, f_{j,N}\otimes_{q-1} f_{j,N}\rangle_{L^2([0,1]^2)}\\ &\quad= a\big(H'\big)^{2q} d(H,q)^4\int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j}|u-v|^{(2H'-2)(q-1)}\big|u'-v'\big|^{(2H'-2)(q-1)}\\ &\qquad\times\big|u-u'\big|^{2H'-2}\big|v-v'\big|^{2H'-2}dv'du'dvdu.\end{aligned}$$

Using the change of variables $y = (u-\frac{i}{N})N$ and similarly for the other variables, we now obtain

$$\begin{aligned}\mathbf{E}\big[T_2^2\big] &= 2d(H,q)^4\big(H'\big(2H'-1\big)\big)^{2q} N^{4H-2}N^{-4}N^{-(2H'-2)2q}\\ &\quad\times\sum_{i,j=0}^{N-1}\int_0^1\int_0^1\int_0^1\int_0^1 dydzdy'dz'|y-z|^{(2H'-2)(q-1)}\big|y'-z'\big|^{(2H'-2)(q-1)}\\ &\quad\times\big|y-y'+i-j\big|^{(2H'-2)}\big|z-z'+i-j\big|^{(2H'-2)}.\end{aligned}$$

This can be viewed as the sum of a diagonal part ($i = j$) and a non-diagonal part ($i \neq j$), where the non-diagonal part is dominant, as the reader will readily check. Therefore, the behavior of $\mathbf{E}[T_2^2]$ will be given by

$$\begin{aligned}\mathbf{E}\big[T_2'^2\big] &:= 2!d(H,q)^4\big(H'\big(2H'-1\big)\big)^{2q}N^{-2}2\\ &\quad\times\sum_{i>j}\int_0^1\int_0^1\int_0^1\int_0^1 dydzdy'dz'\big(|y-z|\big|y'-z'\big|\big)^{(2H'-2)(q-1)}\\ &\quad\times\big(\big|y-y'+i-j\big|\big|z-z'+i-j\big|\big)^{(2H'-2)}\\ &= 2!d(H,q)^4\big(H'\big(2H'-1\big)\big)^{2q}N^{-2}2\\ &\quad\times\sum_{i=0}^{N-2}\sum_{\ell=2}^{N-i}\int_0^1\int_0^1\int_0^1\int_0^1 dydzdy'dz'\big(|y-z|\big|y'-z'\big|\big)^{(2H'-2)(q-1)}\\ &\quad\times\big(\big|y-y'+\ell-1\big|\big|z-z'+\ell-1\big|\big)^{(2H'-2)}\\ &= 2!d(H,q)^4\big(H'\big(2H'-1\big)\big)^{2q}N^{-2}2\\ &\quad\times\sum_{\ell=2}^{N}(N-\ell+1)\int_0^1\int_0^1\int_0^1\int_0^1 dydzdy'dz'\\ &\quad\times\big(|y-z|\big|y'-z'\big|\big)^{(2H'-2)(q-1)}\\ &\quad\times\big(\big|y-y'+\ell-1\big|\big|z-z'+\ell-1\big|\big)^{(2H'-2)}.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{1}{N^2}\sum_{\ell=2}^{N}(N-\ell+1)\big|y-y'+\ell-1\big|^{(2H'-2)}\big|z-z'+\ell-1\big|^{(2H'-2)}\\ &\quad= N^{2(2H'-2)}\frac{1}{N}\sum_{\ell=2}^{N}\left(1-\frac{\ell-1}{N}\right)\left|\frac{y-y'}{N}+\frac{\ell-1}{N}\right|^{2H'-2}\left|\frac{z-z'}{N}+\frac{\ell-1}{N}\right|^{2H'-2}.\end{aligned}$$

Using a Riemann sum approximation argument we conclude that

$$\mathbf{E}\big[T_2'^2\big]\sim\frac{4d(H,q)^4(H'(2H'-1))^{2q}\times N^{2(2H'-2)}}{(4H'-3)(4H'-2)[((2H'-2)(q-1)+1)]^2[(H'-1)(q-1)+1]^2}.$$

Therefore, it follows that

$$\mathbf{E}\big[c_{1,H}^{-1}N^{2(2-2H')}T_2^2\big]\to_{N\to\infty}1, \tag{5.54}$$

with $c_{1,H}$ as in (5.53).

Let us now study the term $T_4, \dots, T_{2q}$ given by (5.52). Here the function $\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N}$ is no longer symmetric but we will show that the behavior of its L^2-norm is dominated by $\mathbf{E}[T_2^2]$. Since for any square integrable function g one has $\|\tilde{g}\|_{L^2} \le \|g\|_{L^2}$, we have for $k = 0, \dots, q-2$

$$\begin{aligned}\frac{1}{(2q-2k)!}\mathbf{E}\big[T_{2q-2k}^2\big] &= N^{4H-2}\left\|\sum_{i=0}^{N-1} f_{i,N}\tilde{\otimes}_k f_{i,N}\right\|^2_{L^2([0,1]^{2q-2k})}\\ &\le N^{4H-2}\left\|\sum_{i=0}^{N-1} f_{i,N}\otimes_k f_{i,N}\right\|^2_{L^2([0,1]^{2q-2k})}\\ &= N^{4H-2}\sum_{i,j=0}^{N-1}\langle f_{i,N}\otimes_k f_{i,N}, f_{j,N}\otimes_k f_{j,N}\rangle_{L^2([0,1]^{2q-2k})}\end{aligned} \tag{5.55}$$

and proceeding as above, with $e_{H,q,k} := (2q-2k)!(H'(2H'-1))^{2q} d(H,q)^4$, we can write

$$\begin{aligned}\mathbf{E}\big[T_{2q-2k}^2\big] \le e_{H,q,k} N^{4H-2}\sum_{i,j=0}^{N-1}\int_{I_i}\int_{I_i} dy_1 dz_1 \int_{I_j}\int_{I_j} dy_1' dz_1' |y_1 - z_1|^{(2H'-2)k}\\ \times \big|y_1' - z_1'\big|^{(2H'-2)k}\big|y_1 - y_1'\big|^{(2H'-2)(q-k)}\big|z_1 - z_1'\big|^{(2H'-2)(q-k)}\end{aligned}$$

and using a change of variables as before,

$$\begin{aligned}\mathbf{E}\big[T_{2q-2k}^2\big] &\le e_{H,q,k} N^{4H-2-4} N^{-(2H'-2)2q}\\ &\quad\times\sum_{i,j=0}^{N-1}\int_{[0,1]^4} dy dz dy' dz' \big(|y-z|\big|y'-z'\big|\big)^{(2H'-2)k}\\ &\quad\times\big|y-y'+i-j\big|^{(2H'-2)(q-k)}\big|z-z'+i-j\big|^{(2H'-2)(q-k)}\\ &= \frac{(2q-2k)!d(H,q)^4}{a(H')^{-2q}}\frac{N^{(2H'-2)(2q-2k)}}{N^2}\\ &\quad\times\sum_{\ell=2}^{N}\left(1-\left(\frac{\ell-1}{N}\right)\right)\int_{[0,1]^4} dy dz dy' dz'\big(|y-z|\big|y'-z'\big|\big)^{(2H'-2)k}\\ &\quad\times\left(\left|\frac{y-y'}{N}+\frac{\ell-1}{N}\right|\left|\frac{z-z'}{N}+\frac{\ell-1}{N}\right|\right)^{(2H'-2)(q-k)}.\end{aligned} \tag{5.56}$$

Since off a diagonal term (again of lower order), the terms $\frac{z-z'}{N}$ are dominated by $\frac{\ell}{N}$ for large l, N it follows that, for $1 \le k \le q-1$

$$\mathbf{E}\left[c_{q-k,H}^{-1} N^{(2-2H')(2q-2k)} T_{2q-2k}^2\right] = O(1) \tag{5.57}$$

when $N \to \infty$, with

$$c_{q-k,H} = 2\left(\int_0^1 (1-x)x^{(2H'-2)(2q-2k)}dx\right)a\left(H'\right)^{-2}(2q-2k)!d(H,q)^2 a\left(H'\right)^{2q}. \tag{5.58}$$

It is obvious that the dominant term in the decomposition of V_N is the term in the chaos of order 2. [The case $k=0$ is in the same situation for $H > \frac{3}{4}$ and for $H \in (\frac{1}{2}, \frac{3}{4})$ the term T_{2q} obtained for $k=0$ has to be renormalized by N; in any case it is dominated by the term T_2]. More specifically we have for any $k \le q-2$,

$$\mathbf{E}\left[N^{2(2-2H')} T_{2q-2k}^2\right] = O\left(N^{-2(2-2H')2(q-k-1)}\right). \tag{5.59}$$

Combining this with the orthogonality of chaos integrals, we immediately get that, up to terms that tend to 0, $N^{2-2H'}V_N$ and $N^{2-2H'}T_2$ have the same norm in $L^2(\Omega)$. This completes the proof of the proposition. □

Summarizing the spirit of the proof of Proposition 5.8, to understand the behavior of the renormalized sequence V_N it suffices to study the limit of the term

$$I_2\left(N^{2H-1}N^{(2-2H')}\sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N}\right) \tag{5.60}$$

with

$$\begin{aligned}
&(f_{i,N} \otimes_{q-1} f_{i,N})(y,z) \\
&\quad = d(H,q)^2 a\left(H'\right)^{q-1}\Bigg(1_{[0,\frac{i}{N}]}(y \vee z) \\
&\qquad \times \int_{I_i}\int_{I_i} dv du \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)} \\
&\qquad + 1_{[0,\frac{i}{N}]}(y)1_{I_i}(z)\int_{I_i}\int_z^{\frac{i+1}{N}} dv du \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)} \\
&\qquad + 1_{I_i}(y)1_{[0,\frac{i}{N}]}(z)\int_y^{\frac{i+1}{N}}\int_{I_i} dv du \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)} \\
&\qquad + 1_{I_i}(y)1_{I_i}(z)\int_y^{\frac{i+1}{N}}\int_z^{\frac{i+1}{N}} dv du \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)}\Bigg).
\end{aligned} \tag{5.61}$$

We will see in the proof of the next theorem that, of the contribution of the four terms on the right-hand side of (5.61), only the first one does not tend to 0 in $L^2(\Omega)$. Hence the following notation will be useful: f_2^N will denote the integrand of the contribution to (5.60) corresponding to that first term, and r_2 will be the remainder of the integrand in (5.60). In other words,

$$f_2^N + r_2 = N^{2H-1} N^{(2-2H')} \sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N} \tag{5.62}$$

and

$$\begin{aligned} f_2^N(y,z) := {} & N^{2H-1} N^{(2-2H')} d(H,q)^2 a\big(H'\big)^{q-1} \\ & \times \sum_{i=0}^{N-1} 1_{[0,\frac{i}{N}]}(y \vee z) \\ & \times \int_{I_i} \int_{I_i} dv du \partial_1 K(u,y) \partial_1 K(v,z) |u-v|^{(2H'-2)(q-1)}. \end{aligned} \tag{5.63}$$

Theorem 5.12 *The sequence given by* (5.60) *converges in* $L^2(\Omega)$ *as* $N \to \infty$ *to the constant* $c_{1,H}^{1/2}$ *times a standard Rosenblatt random variable* $Z_1^{(2,2H'-1)}$ *with self-similarity parameter* $2H'-1$ *and* H' *is given by* (3.6)*. Consequently, we also have that* $c_{1,H}^{-1/2} N^{(2-2H')} c_2^{-1} V_N$ *converges in* $L^2(\Omega)$ *as* $N \to \infty$ *to the same Rosenblatt random variable.*

Proof The first statement of the theorem is that $N^{2-2H'} T_2$ converges to

$$c_{1,H}^{1/2} Z_1^{(2,2H'-1)}$$

in $L^2(\Omega)$. From (5.60) it follows that T_2 is a second-chaos random variable, with kernel

$$N^{2H-1} \sum_{i=0}^{N-1} (f_{i,N} \otimes_{q-1} f_{i,N})$$

(see the expression in (5.61)), so we only need to prove this kernel converges in $L^2([0,1]^2)$. The first observation is that $r_2(y,z)$ defined in (5.62) converges to zero in $L^2([0,1]^2)$ as $N \to \infty$. The crucial fact is that the intervals I_i, which are disjoint, appear in the expression of this term and this implies that the non-diagonal terms vanish when we take the square norm of the sum; in fact it can easily be seen that the norm in L^2 of r_2 corresponds to the diagonal part in the evaluation in $\mathbf{E}T_2^2$ which is clearly dominated by the non-diagonal part, so this result comes as no surprise. The proof follows the lines of Sect. 5.2. This shows $N^{(2-2H')} T_2$ is the sum of $I_2(f_2^N)$ and a term which tends to 0 in $L^2(\Omega)$. Our next step is thus simply to calculate the limit in $L^2(\Omega)$, if any, of $I_2(f_2^N)$ where f_2^N has been defined in (5.63). By the isometry property (C.1), limits of second-chaos r.v.'s in $L^2(\Omega)$ are equivalent to limits of their symmetric kernels in $L^2([0,1]^2)$. Note that f_2^N is symmetric. Therefore, it is sufficient to prove that f_2^N converges to the kernel of the Rosenblatt process at time 1. We have by definition

$$f_2^N(y,z) = \big(H'(2H'-1)\big)^{(q-1)} d(H,q)^2 N^{2H-1} N^{2-2H'} \times \sum_{i=0}^{N-1} \int_{I_i}\int_{I_i} |u-v|^{(2H'-2)(q-1)} \partial_1 K^{H'}(u,y)\partial_1 K^{H'}(v,z).$$

Thus for every y, z,

$$\begin{aligned} f_2^N(y,z) &= d(H,q)^2\big(H'(2H'-1)\big)^{(q-1)} N^{2H-1} N^{2-2H'} \\ &\quad \times \sum_{i=0}^{N-1} \int_{I_i}\int_{I_i} |u-v|^{(2H'-2)(q-1)} \partial_1 K^{H'}(u,y)\partial_1 K^{H'}(v,z)dudv \\ &= d(H,q)^2\big(H'(2H'-1)\big)^{(q-1)} N^{2H-1} N^{2-2H'} \\ &\quad \times \sum_{i=0}^{N-1} \int_{I_i}\int_{I_i} |u-v|^{(2H'-2)(q-1)} \\ &\quad \times \big(\partial_1 K^{H'}(u,y)\partial_1 K^{H'}(v,z) - \partial_1 K^{H'}(i/N,z)\partial_1 K^{H'}(i/N,z)\big)dudv \\ &\quad + d(H,q)^2\big(H'(2H'-1)\big)^{(q-1)} N^{2H-1} N^{2-2H'} \\ &\quad \times \sum_{i=0} \int_{I_i}\int_{I_i} |u-v|^{(2H'-2)(q-1)} \partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z)dudv \\ &=: A_1^N(y,z) + A_2^N(y,z). \end{aligned}$$

As in the proof of Theorem 5.10, one can show that $\mathbf{E}[\|A_1^N\|^2_{L^2([0,1]^2)}] \to 0$ as $N \to \infty$. Regarding the second term $A_2^N(y,z)$, the summand is zero if $i/N < y \vee z$, therefore we get that f_2^N is equivalent to

$$\begin{aligned} &N^{2H-1} N^{2-2H'} d(H,q)^2\big(H'(2H'-1)\big)^{(q-1)} \sum_{i=0}^{N-1} \int_{I_i}\int_{I_i} |u-v|^{(2H'-2)(q-1)} \\ &\quad \times \partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z)dudv \\ &= \big(H'(2H'-1)\big)^{(q-1)} d(H,q)^2 N^{2H-1} N^{2-2H'} \\ &\quad \times \sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z) 1_{y\vee z<i/N} \int_{I_i}\int_{I_i} |u-v|^{(2H'-2)(q-1)} dudv \end{aligned}$$

$$
\begin{aligned}
&= \big(H'(2H'-1)\big)^{(q-1)}\big[\big((2H'-2)(q-1)+1\big)\big((H'-1)(q-1)+1\big)\big]^{-1}\\
&\quad \times \frac{N^{2H-1}N^{(2-2H')q}}{N^2}\sum_{i=0}^{N-1}\partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z)1_{y\vee z<i/N}\\
&= \frac{d(H,q)^2}{d(2H'-2,2)}\frac{(H'(2H'-1))^{(q-1)}}{((2H'-2)(q-1)+1)((H'-1)(q-1)+1)}\\
&\quad \times d\big(2H'-2,2\big)N^{-1}\sum_{i=0}^{N-1}\partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z)1_{y\vee z<i/N}.
\end{aligned}
$$

The sequence $d(2H'-2,2)N^{-1}\sum_{i=0}^{N-1}\partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z)1_{y\vee z<i/N}$ is a Riemann sum that converges pointwise on $[0,1]^2$ to the kernel of the Rosenblatt process $Z^{2H'-1,2}$ at time 1. To obtain the convergence in $L^2([0,1]^2)$ we will apply the dominated convergence theorem. Indeed,

$$
\begin{aligned}
&\int_0^1\int_0^1\left|\frac{1}{N}\sum_{i=0}^{N-1}\partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z)1_{y\vee z<i/N}\right|^2 dydz\\
&= \frac{1}{N^2}\sum_{i,j=0}^{N-1}\left|\int_0^1 \partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(j/N,y)1_{y<(i\wedge j)/N}dy\right|^2\\
&\le \frac{1}{N^2}\sum_{i,j=0}^{N-1}\big|\mathbf{E}[\Delta Z_{i/N}\Delta Z_{j/N}]\big|^2,
\end{aligned}
$$

where $\Delta Z_{i/N}$ is the difference $Z(\frac{i}{N})-Z(\frac{i-1}{N})$ for a Rosenblatt process Z. We now show that the above sum is always $\ll N^2$, which proves that the last expression, with the N^{-2} factor, is bounded. In fact for $H_1 = 2H'-1$

$$
\begin{aligned}
&\sum_{i,j=0}^{N-1}\big|\mathbf{E}[\Delta Z_{i/N}\Delta Z_{j/N}]\big|^2\\
&= \sum_{i,j=0}^{N-1}\left|\left|\frac{i-j+1}{N}\right|^{2H_1}+\left|\frac{i-j-1}{N}\right|^{2H_1}-2\left|\frac{i-j}{N}\right|^{2H_1}\right|^2\\
&= \frac{N^{-4H_1}}{4}\sum_{i,j=0}^{N-1}\big||i-j+1|^{2H_1}+|i-j-1|^{2H_1}-2|i-j|^{2H_1}\big|^2\\
&\le \frac{N^{-4H_1}}{4}2N\sum_{\ell=-N+1}^{N-1}\big||\ell+1|^{2H_1}+|\ell-1|^{2H_1}-2|\ell|^{2H_1}\big|^2.
\end{aligned}
$$

The function $g(\ell) = |\ell+1|^{2H_1} + |\ell-1|^{2H_1} - 2|\ell|^{2H_1}$ behaves like $H_1(2H_1-1)$ $|\ell|^{2H_1-2}$ for large ℓ. We need to separate the cases of convergence and divergence of the series $\sum_{-\infty}^{\infty} |g(\ell)|^2$. It is divergent as soon as $H_1 \geq 3/4$, or equivalently $H' \geq 7/8$, in which case we get for some constant c not dependent on N,

$$\sum_{i,j=0}^{N-1} \left|\mathbf{E}[\Delta Z_{i/N} \Delta Z_{j/N}]\right|^2 \leq cN^{-4H_1+1+4H_1-3} = cN^{-2} \ll N^2.$$

The series $\sum_{-\infty}^{\infty} |g(\ell)|^2$ is convergent if $H' < 7/8$, in which case we get

$$\sum_{i,j=0}^{N-1} \left|\mathbf{E}[\Delta Z_{i/N} \Delta Z_{j/N}]\right|^2 \leq cN^{-4H_1+1}.$$

For this to be $\ll N^2$, we simply need $-4H_1+1<2$, i.e. $H' > 5/8$. However, since $q \geq 2$ and $H > 1/2$ we always have $H' > 3/4$. Therefore in all cases, the sequence $A_2^N(y,z)$ is bounded in $L^2([0,1]^2)$ and in this way we obtain the L^2-convergence to the kernel of a Rosenblatt process of order 1. The first statement of the theorem is proved. In order to show that $c_{1,H}^{-1/2} N^{(2-2H')} c_2^{-1} V_N$ converges in $L^2(\Omega)$ to the same Rosenblatt random variable as the normalized version of the quantity in (5.60), it is sufficient to show that, after normalization by $N^{2-2H'}$, each of the remaining terms in the chaos expansion (5.50) of V_N converge to zero in $L^2(\Omega)$, i.e. that $N^{(2-2H')} T_{2q-2k}$ converges to zero in $L^2(\Omega)$, for all $1 \leq k < q-1$. From (5.59) we have

$$\mathbf{E}\big[N^{2(2-2H')} T_{2q-2k}^2\big] = O\big(N^{-2(2-2H')2(q-k-1)}\big)$$

which is all that is needed, concluding the proof of the theorem. □

5.3.2 The Reproduction Property for Hermite Processes

We also study the limits of the other terms in the decomposition (Proposition 5.7) of V_N, (5.47) those of order higher than 2, and we notice interesting facts: all these terms, except the term of highest order $2q$, have limits which are Hermite random variables of various orders and self-similarity parameters. We call this *the reproduction property* for Hermite processes, because from one Hermite process of order q, one can reconstruct other Hermite processes of all lower orders. The exception to this rule is that the normalized term of highest order $2q$ converges to a Hermite r.v. of order $2q$ if $H > 3/4$, but converges to Gaussian limit if $H \in (1/2, 3/4]$.

Theorem 5.13

- *For every $H \in (\frac{1}{2}, 1)$ and for every $k = 1, \ldots, q-2$ we have*

$$\lim_{N\to\infty} N^{(2-2H')(q-k)} T_{2q-2k} = z_{k,H} Z^{(2q-2k,(2q-2k)(H'-1)+1)}, \quad \textit{in } L^2(\Omega) \quad (5.64)$$

where $Z^{(2q-2k,(2q-2k)(H'-1)+1)}$ *denotes a Hermite random variable with self-similarity parameter* $(2q-2k)(H'-1)+1$ *and* $z_{k,H} = d(H,q)^2 a(H')^k \times ((H'-1)k+1)^{-1}(2(H'-1)+1)^{-1}$.

- *Moreover, if* $H \in (\frac{3}{4}, 1)$ *then*

$$\lim_{N\to\infty} N^{2-2H} x_{2,H}^{-1/2} T_{2q} = Z^{(2q,2H-1)}, \quad \text{in } L^2(\Omega). \tag{5.65}$$

5.4 Quadratic Variations of the Solution to the Stochastic Heat Equation

We analyze here the quadratic variations of the Gaussian process $(u(t,x), t \in [0,T], x \in \mathbb{R}^d)$ given by the solution (2.31) to the linear stochastic heat equation driven by a fractional-colored noise (meaning a centered Gaussian process with covariance (2.29)). We will assume that the spatial covariance is given by the Riesz kernel from Example 2.1. In this case, the covariance of the process u has been computed in Proposition 2.9. In particular, it follows that this covariance does not depend on $x \in \mathbb{R}^d$. Therefore we will consider a centered Gaussian process $(U_t)_{t\in[0,1]}$ with covariance

$$R(t,s) = d(\alpha, H) \int_0^t \int_0^s |u-v|^{2H-2}\big((t+s)-(u+v)\big)^{-\frac{d_\alpha}{2}} dv du \tag{5.66}$$

with $0 < d_\alpha \le d$ and $d_\alpha < 4H$ and

$$d(\alpha, H) = \alpha_H (2\pi)^{-d} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{|\xi|^2}{2}}.$$

We put $d_\alpha = d - \alpha$.

Define the centered quadratic variations of the process U

$$V_N := \sum_{i=0}^{N-1} \big[(U_{t_{i+1}} - U_{t_i})^2 - \mathbf{E}(U_{t_{i+1}} - U_{t_i})^2\big]. \tag{5.67}$$

We study the limit behavior of this sequence as $N \to \infty$. This will give the behavior of the associated estimators for the parameter H. Other works on statistical inference for fractional equations with the Malliavin calculus include [94, 180].

Since

$$U_{t_{i+1}} - U_{t_i} = I_1^U(1_{(t_i,t_{i+1})})$$

(I_n^U denotes the multiple integral with respect to U; in the sequel we will simply denote it by I_n) we can express the sequence V_N as a multiple integral of order 2

$$V_N = I_2\left(\sum_{i=0}^{N-1} 1_{(t_i,t_{i+1})}^{\otimes 2}\right)$$

by the product formula (C.4).

5.4.1 The L^2-Norm of the Sequence V_N

Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of the unit interval [0, 1] with $t_i = \frac{i}{N}$ for $i = 0, \ldots, N$. We have, using the isometry of multiple stochastic integrals (C.1)

$$\begin{aligned}\mathbf{E}V_N^2 &= 2! \sum_{i,j=0}^{N-1} \left\langle 1_{(t_i,t_{i+1})}^{\otimes 2}, 1_{(t_j,t_{j+1})}^{\otimes 2} \right\rangle \\ &= 2! \sum_{i,j=0}^{N-1} \langle 1_{(t_i,t_{i+1})}, 1_{(t_j,t_{j+1})} \rangle^2.\end{aligned}$$

Here $\langle \cdot, \cdot \rangle_{\mathcal{U}} := \langle \cdot, \cdot \rangle$ denotes the scalar product in the canonical Hilbert space $\mathcal{U}$ associated with the process U which is defined as the closure of the set of indicator functions $(1_{[0,t]}, t \in [0, T])$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle = R(t, s)$$

where $R(t, s)$ is given by (5.66). Then

$$\begin{aligned}&d(\alpha, H)^{-1} \langle 1_{(t_i,t_{i+1})}, 1_{(t_j,t_{j+1})} \rangle \\ &= \int_0^{\frac{i+1}{N}} du \int_0^{\frac{j+1}{N}} dv |u - v|^{2H-2} \left(\frac{i+1}{N} + \frac{j+1}{N} - (u + v) \right)^{-\frac{d\alpha}{2}} \\ &\quad - \int_0^{\frac{i+1}{N}} du \int_0^{\frac{j}{N}} dv |u - v|^{2H-2} \left(\frac{i+1}{N} + \frac{j}{N} - (u + v) \right)^{-\frac{d\alpha}{2}} \\ &\quad - \int_0^{\frac{i}{N}} du \int_0^{\frac{j+1}{N}} dv |u - v|^{2H-2} \left(\frac{i}{N} + \frac{j+1}{N} - (u + v) \right)^{-\frac{d\alpha}{2}} \\ &\quad + \int_0^{\frac{i}{N}} du \int_0^{\frac{j}{N}} dv |u - v|^{2H-2} \left(\frac{i}{N} + \frac{j}{N} - (u + v) \right)^{-\frac{d\alpha}{2}}.\end{aligned}$$

By the change of variables $\tilde{u} = uN$, $\tilde{v} = vN$ we get

$$\begin{aligned}&d(\alpha, H)^{-1} \langle 1_{(t_i,t_{i+1})}, 1_{(t_j,t_{j+1})} \rangle \\ &= N^{-2H+\frac{d\alpha}{2}} \left[\int_0^{i+1} du \int_0^{j+1} dv |u - v|^{2H-2} \big(i + 1 + j + 1 - (u + v)\big)^{-\frac{d\alpha}{2}} \right. \\ &\quad - \int_0^{i+1} du \int_0^{j} dv |u - v|^{2H-2} \big(i + 1 + j - (u + v)\big)^{-\frac{d\alpha}{2}} \\ &\quad - \int_0^{i} du \int_0^{j+1} dv |u - v|^{2H-2} \big(i + j + 1 - (u + v)\big)^{-\frac{d\alpha}{2}}\end{aligned}$$

$$+\int_0^i du \int_0^j dv|u-v|^{2H-2}\big(i+j-(u+v)\big)^{-\frac{d_\alpha}{2}}\Big]$$

$$:= N^{-2H+\frac{d_\alpha}{2}}\big[A(i,j)+B(i,j)+C(i,j)\big]$$

where

$$A(i,j)=\int_i^{i+1} du \int_j^{j+1} dv|u-v|^{2H-2}\big(i+j+2-(u+v)\big)^{-\frac{d_\alpha}{2}} \tag{5.68}$$

$$\begin{aligned}B(i,j)=&\int_i^{i+1} du \int_0^{j} dv|u-v|^{2H-2}\\ &\times\Big[\big(i+j+2-(u+v)\big)^{-\frac{d_\alpha}{2}}-\big(i+j+1-(u+v)\big)^{-\frac{d_\alpha}{2}}\Big]\\ &+\int_0^{i} du \int_j^{j+1} dv|u-v|^{2H-2}\\ &\times\Big[\big(i+j+2-(u+v)\big)^{-\frac{d_\alpha}{2}}-\big(i+j+1-(u+v)\big)^{-\frac{d_\alpha}{2}}\Big]\end{aligned} \tag{5.69}$$

and

$$\begin{aligned}C(i,j)=&\int_0^{i} du \int_0^{j} dv|u-v|^{2H-2}\Big[\big(i+j+2-(u+v)\big)^{-\frac{d_\alpha}{2}}\\ &-2\big(i+j+1-(u+v)\big)^{-\frac{d_\alpha}{2}}+\big(i+j-(u+v)\big)^{-\frac{d_\alpha}{2}}\Big].\end{aligned} \tag{5.70}$$

Therefore

$$\begin{aligned}\mathbf{E}V_N^2 &= 2d(\alpha,H)^2N^{-4H+d_\alpha}\sum_{i,j=0}^{N-1}\big[A(i,j)+B(i,j)+C(i,j)\big]^2\\ &= 2d(\alpha,H)^2N^{-4H+d_\alpha}\sum_{i,j=0}^{N-1}\big[A(i,j)^2+B(i,j)^2+C(i,j)^2\\ &\quad+2A_{i,j}B(i,j)+2A_{i,j}C(i,j)+2B_{i,j}C(i,j)\big]\\ &:= 2d(\alpha,H)^2(T_{1,N}+T_{2,N}+T_{3,N}+T_{4,N}+T_{5,N}+T_{6,N}).\end{aligned} \tag{5.71}$$

We will evaluate the asymptotic behavior, as $N\to\infty$, of the six terms above. Actually, it happens that the six summands that appear in the decomposition of $\mathbf{E}V_N^2$ are all of the same magnitude. There is no negligible part that can be ignored in the estimation of $\mathbf{E}V_N^2$.

The following renormalization result holds.

Theorem 5.14 *For $H<\frac{3}{4}$, and for $i=1,\ldots,6$*

$$N^{4H-d_\alpha-1}T_{i,N} \xrightarrow[N\to\infty]{} K_{i,1}$$

and for $H > \frac{3}{4}$ *and* $i = 1, \dots, 6$

$$N^{2-d_\alpha} T_{i,N} \xrightarrow[N\to\infty]{} K_{i,2}$$

with $K_{i,1}, K_{i,2}$ *strictly positive constants*, $i = 1, \dots, 6$. *Consequently, for* $H < \frac{3}{4}$,

$$N^{4H-d_\alpha-1}\mathbf{E}V_N^2 \xrightarrow[N\to\infty]{} 2d(\alpha, H)^2 \sum_{i=1}^{6} K_{i,1} := K_1$$

and for $H > \frac{3}{4}$,

$$N^{2-d_\alpha}\mathbf{E}V_N^2 \xrightarrow[N\to\infty]{} 2d(\alpha, H)^2 \sum_{i=1}^{6} K_{i,2} := K_2.$$

5.4.2 Limit Behavior of the Quadratic Variations

Suppose first that $H < \frac{3}{4}$. Let us denote by $\tilde{V}_N$ the sequence

$$\tilde{V}_N = \sum_{i=0}^{N-1}\left[\frac{(U_{t_{i+1}} - U_{t_i})^2}{\mathbf{E}(U_{t_{i+1}} - U_{t_i})^2} - 1\right].$$

Using the behavior of the increments of the process U (Theorem 2.6) and Theorem 5.14, we notice that $\mathbf{E}(\frac{1}{\sqrt{N}}\tilde{V}_N)^2$ converges to a constant. This suggests that V_N converges to a Gaussian distribution.

We will prove this claim in the sequel. Our approach is based on the Stein method (Theorem 5.1). Let

$$F_N := K_1^{-\frac{1}{2}} N^{2H-\frac{d_\alpha}{2}-\frac{1}{2}} V_N \tag{5.72}$$

where the constant K_1 is defined in Theorem 5.14. From Theorem 5.14 it follows that

$$\mathbf{E}F_N^2 \underset{N}{\to} 1.$$

Theorem 5.15 *For* $H < \frac{3}{4}$

$$d\big(F_N, N(0,1)\big)^2 \xrightarrow[N\to\infty]{} 0.$$

Proof We start by computing the Malliavin derivative of F_N and then we evaluate its norm. We have, for every s

$$D_s F_N = 2K_1^{-\frac{1}{2}} N^{2H-\frac{d_\alpha}{2}-\frac{1}{2}} \sum_{i=0}^{N-1} I_1(1_{(\frac{i}{N},\frac{i+1}{N})})1_{(\frac{i}{N},\frac{i+1}{N})}(s)$$

and

$$\|DF_N\|^2_{L^2([0,1])} = 4K_1^{-1}N^{4H-d_\alpha-1} \times \sum_{i,j=0}^{N-1} I_1(1_{(\frac{i}{N},\frac{i+1}{N})})I_1(1_{(\frac{j}{N},\frac{j+1}{N})})\langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])}.$$

Therefore

$$d\big(F_N, N(0,1)\big)^2 \leq cN^{8H-2d_\alpha-2}\mathbf{E}\left[\sum_{i,j=0}^{N-1}\langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])} I_2(1_{(\frac{j}{N},\frac{j+1}{N})}\otimes 1_{(\frac{j}{N},\frac{j+1}{N})})\right]^2 + \big(\mathbf{E}F_N^2 - 1\big).$$

We will first analyze the convergence of the multiple integral in the second Wiener chaos. We have, with c denoting a generic strictly positive constant,

$$\begin{aligned}
&\mathbf{E}\left[\sum_{i,j=0}^{N-1}\langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])} I_2(1_{(\frac{j}{N},\frac{j+1}{N})}\otimes 1_{(\frac{j}{N},\frac{j+1}{N})})\right]^2 \\
&= 2\sum_{i,j,i',j'=0}^{N-1}\langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])}\langle 1_{(\frac{i'}{N},\frac{i'+1}{N})}, 1_{(\frac{j'}{N},\frac{j'+1}{N})}\rangle_{L^2([0,1])} \\
&\quad \times \langle 1_{(\frac{i}{N},\frac{i+1}{N})}\otimes 1_{(\frac{j}{N},\frac{j+1}{N})}, 1_{(\frac{i'}{N},\frac{i'+1}{N})}\otimes 1_{(\frac{j'}{N},\frac{j'+1}{N})}\rangle \\
&= c\sum_{i,j,i',j'=0}^{N-1}\langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])}\langle 1_{(\frac{i'}{N},\frac{i'+1}{N})}, 1_{(\frac{j'}{N},\frac{j'+1}{N})}\rangle_{L^2([0,1])} \\
&\quad \times \langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{i'}{N},\frac{i'+1}{N})}\rangle_{L^2([0,1])}\langle 1_{(\frac{j}{N},\frac{j+1}{N})}, 1_{(\frac{j'}{N},\frac{j'+1}{N})}\rangle_{L^2([0,1])}.
\end{aligned}$$

One can show that

$$\langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])},$$

which equals $N^{-2H+\frac{d_\alpha}{2}}(A(i,j)+B(i,j)+C(i,j))$ (these terms are given by (5.68), (5.69) and (5.70)), "behaves" for large i, j as $N^{-2H+\frac{d_\alpha}{2}}|i-j|^{2H-2}$ and similar results hold for the other factors above. Thus

$$
\begin{aligned}
& N^{8H-2d_\alpha-2}\mathbf{E}\left[\sum_{i,j=0}^{N-1}\langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])} I_2(1_{(\frac{j}{N},\frac{j+1}{N})}\otimes 1_{(\frac{j}{N},\frac{j+1}{N})})\right]^2 \\
& \leq CN^{-2}\sum_{i,j,i',j'=0}^{N-1}\left|i'-j'\right|^{2H-2}\left|i-i'\right|^{2H-2}\left|j-j'\right|^{2H-2} \\
& \leq CN^{8H-6}\frac{1}{N^4}\sum_{i,j,i',j'=0}^{N-1}\left(\frac{|i-j|}{N}\right)^{2H-2}\left(\frac{|i'-j'|}{N}\right)^{2H-2} \\
& \quad \times\left(\frac{|i-i'|}{N}\right)^{2H-2}\left(\frac{|j-j'|}{N}\right)^{2H-2} \\
& \leq CN^{8H-6}\int_{[0,1]^4}|u-v|^{2H-2}\left|u'-v'\right|^{2H-2}\left|u-u'\right|^{2H-2}\left|v-v'\right|^{2H-2}.
\end{aligned}
$$

On the other hand from Theorem 5.14 we have that $\mathbf{E}F_N^2 - 1 \to_{N\to\infty} 0$. This concludes the proof of the theorem. □

When $H > \frac{3}{4}$, the quadratic variations of the solution to the fractional-colored heat equation satisfy a non-central limit theorem.

Theorem 5.16 *Suppose* $H > \frac{3}{4}$. *The renormalized quadratic variation* $\widehat{V}_N = K_2^{-\frac{1}{2}} N^{1-\frac{d}{2}} V_N$, *with* K_2 *defined in Theorem* 5.14, *converges in distribution, as* $N \to \infty$, *to a Rosenblatt random variable.*

Proof See Exercise 5.11 for the main lines of the proof. □

We refer to [149] and [166] for related results on the stochastic heat equation with time-space white noise.

5.5 Estimators for the Self-similarity Parameter

In this part we construct estimators for the self-similarity exponent of a Hermite process based on the discrete observations of the driving process at times $0, \frac{1}{N}, \ldots, 1$. It is known that the asymptotic behavior of the statistics V_N (5.2) is related to the asymptotic properties of a class of estimators for the Hurst parameter H. This is mentioned for instance in [57].

We recall how this is set up. Suppose that the observed process X is a Hermite process; it may be Gaussian (fractional Brownian motion) or non-Gaussian (the

Rosenblatt process or even a higher order Hermite process). Let

$$S_N = \frac{1}{N}\sum_{i=1}^{N}\left(X\left(\frac{i}{N}\right) - X\left(\frac{i-1}{N}\right)\right)^2. \tag{5.73}$$

Recall that $\mathbf{E}[S_N] = N^{-2H}$. By estimating $\mathbf{E}[S_N]$ by S_N we can construct the estimator

$$\hat{H}_N = -\frac{\log S_N}{2\log N}. \tag{5.74}$$

To prove that this is a strongly consistent estimator for H, we begin by writing

$$1 + V_N = S_N N^{2H}$$

where V_N is the original 2-variation, and thus

$$\begin{aligned}\log(1+V_N) &= \log S_N + 2H\log N\\ &= -2(\hat{H}_N - H)\log N.\end{aligned}$$

One can show that V_N converges almost surely to 0 (this can be done by using the Borel-Cantelli lemma and the hypercontractivity property of multiple stochastic integrals (2.33)), and thus $\log(1+V_N) = V_N(1+o(1))$ where $o(1)$ converges to 0 almost surely as $N \to \infty$. Hence we obtain

$$V_N = 2(H - \hat{H}_N)(\log N)\big(1 + o(1)\big). \tag{5.75}$$

Relation (5.75) means that V_N's behavior immediately gives the behavior of $\hat{H}_N - H$.

Specifically, we can now state our convergence results in the Gaussian case.

Theorem 5.17 *Suppose that $H > \frac{1}{2}$ and assume that the observed process is a fBm with Hurst parameter H. Then strong consistency holds for $\hat{H}_N$, i.e. almost surely,*

$$\lim_{N\to\infty} \hat{H}_N = H \tag{5.76}$$

and

- *if $H \in (\frac{1}{2}, \frac{3}{4})$, then, in distribution as $N \to \infty$,*

$$\sqrt{N}\log(N)\frac{2}{\sqrt{c_{1,H}}}(\hat{H}_N - H) \to N(0,1);$$

- *if $H \in (\frac{3}{4}, 1)$, then, in distribution as $N \to \infty$,*

$$N^{2-2H}\log(N)\frac{2}{\sqrt{c_{2,H}}}(\hat{H}_N - H) \to Z$$

where Z is the law of a standard Rosenblatt random variable (see (5.18));

- *if $H = \frac{3}{4}$, then, in distribution as $N \to \infty$,*

$$\sqrt{N \log N} \frac{2}{\sqrt{c'_{1,H}}} (\hat{H}_N - H) \to N(0,1).$$

The constants $c_{1,H}, c_{2,H}, c'_{1,H}$ are those from Sect. 5.1.

Proof This follows from Theorems 5.4, 5.5 and Proposition 5.4. □

See also Exercises 5.5 and 5.6.

5.6 Quadratic Variation with Higher Order Increments

In its simplest form, the kth power variation statistic of a process $(X_t : t \in [0,1])$, calculated using N data points, is defined as the following quantity (the absolute value of the increment may be used in the definition for non-even powers):

$$V_N := \frac{1}{N} \left[\sum_{i=0}^{N-1} \frac{(X_{\frac{i+1}{N}} - X_{\frac{i}{N}})^k}{\mathbf{E}(X_{\frac{i+1}{N}} - X_{\frac{i}{N}})^k} - 1 \right]. \tag{5.77}$$

There exists a direct connection between the behavior of the variations and the convergence of an estimator for the self-similarity order based on these variations (see Sect. 5.5 and also [28, 57, 68, 181]): if the renormalized variation satisfies a central limit theorem then so does the estimator, a desirable fact for statistical purposes.

We have seen in Sect. 5.2 that the quadratic variations of the Rosenblatt process Z (the V_N above with $k = 2$), exhibit the following facts: the normalized sequence $N^{1-H} V_N$ satisfies a Non-Central Limit Theorem, it converges in the mean square to the Rosenblatt random variable $Z(1)$ (the value of the process Z at time 1); from this, we can construct an estimator for H whose behavior is still non-normal. The same result is also obtained in the case of estimators based on the wavelet coefficients (see the next Chap. 5.7). In the simpler case of fBm, this situation still occurs when $H > 3/4$ (Sect. 5.1). For statistical applications, a situation in which asymptotic normality holds might be preferable. To achieve this we will use "longer filters" (i.e., we replace the increments $X_{\frac{i+1}{N}} - X_{\frac{i}{N}}$ by the second-order increments $X_{\frac{i+1}{N}} - 2X_{\frac{i}{N}} + X_{\frac{i-1}{N}}$), or higher order increments for instance. We will see that this approach leads to a Gaussian limit for the variations of fBm without any restriction on the Hurst parameter H. But in the case of the Rosenblatt method the asymptotic behavior of the quadratic variations based on longer filters is still non-normal.

5.6.1 Longer Filters

By a "filter" we mean the following:

Definition 5.1 A filter α of length $\ell \in \mathbb{N}$ and order $p \in \mathbb{N} \setminus 0$ is an $(\ell + 1)$-dimensional vector $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$ such that

$$\sum_{q=0}^{\ell} \alpha_q q^r = 0, \quad \text{for } 0 \le r \le p-1, r \in \mathbf{Z}$$

$$\sum_{q=0}^{\ell} \alpha_q q^p \neq 0$$

with the convention $0^0 = 1$.

If we associate such a filter α with the fbm or to the Rosenblatt process (both denoted by Z below) we get the filtered process V^α according to the following scheme:

$$V^\alpha\left(\frac{i}{N}\right) := \sum_{q=0}^{\ell} \alpha_q Z\left(\frac{i-q}{N}\right), \quad \text{for } i = \ell, \ldots, N-1. \tag{5.78}$$

Some examples are the following:

1. For $\alpha = \{1, -1\}$
$$V^\alpha\left(\frac{i}{N}\right) = Z\left(\frac{i}{N}\right) - Z\left(\frac{i-1}{N}\right).$$
This is a filter of length 1 and order 1.
2. For $\alpha = \{1, -2, 1\}$
$$V^\alpha\left(\frac{i}{N}\right) = Z\left(\frac{i}{N}\right) - 2Z\left(\frac{i-1}{N}\right) + Z\left(\frac{i-2}{N}\right).$$
This is a filter of length 2 and order 2.
3. More generally, longer filters produced by finite-differencing are such that the coefficients of the filter α are the binomial coefficients with alternating signs. Therefore, borrowing the notation ∇ from time series analysis, $\nabla Z(i/N) = Z(i/N) - Z((i-1)/N)$, we define $\nabla^j = \nabla \nabla^{j-1}$ and we may write the jth-order finite-difference-filtered process as follows
$$V^{\alpha_j}\left(\frac{i}{N}\right) := (\nabla^j Z)\left(\frac{i}{N}\right).$$

From Now on We Assume the Filter Order Is Strictly Greater than 1 ($p \geq 2$) For such a filter α the quadratic variation statistic is defined as

$$V_N := \frac{1}{N-\ell} \sum_{i=\ell}^{N-1} \left[\frac{|V^\alpha(\frac{i}{N})|^2}{\mathbf{E}|V^\alpha(\frac{i}{N})|^2} - 1 \right].$$

Using the definition of a filter, we can compute the covariance of the filtered process $V^\alpha(\frac{i}{N})$.

Proposition 5.9 *Consider the sequence* (5.78) *and let*

$$\pi_H^\alpha(j) := \mathbf{E}\left[V^\alpha\left(\frac{i}{N}\right) V^\alpha\left(\frac{i+j}{N}\right) \right].$$

Then

$$\pi_H^\alpha(j) = -\frac{N^{-2H}}{2} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |j+q-r|^{2H}. \tag{5.79}$$

Proof We have for every j

$$\begin{aligned}
\pi_H^\alpha(j) &:= \mathbf{E}\left[V^\alpha\left(\frac{i}{N}\right) V^\alpha\left(\frac{i+j}{N}\right) \right] \\
&= \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \mathbf{E}\left[Z\left(\frac{i-q}{N}\right) Z\left(\frac{i+j-r}{N}\right) \right] \\
&= \frac{N^{-2H}}{2} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left(|i-q|^{2H} + |i+j-r|^{2H} - |j+q-r|^{2H} \right) \\
&= -\frac{N^{-2H}}{2} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |j+q-r|^{2H} \\
&\quad + \frac{N^{-2H}}{2} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left(|i-q|^{2H} + |i+j-r|^{2H} \right).
\end{aligned}$$

Since the term $\sum_{q,r=0}^{\ell} \alpha_q \alpha_r (|i-q|^{2H} + |i+j-r|^{2H})$ vanishes we get that (5.79). □

Therefore, we can rewrite the variation statistic as follows

$$V_N = \frac{1}{N-\ell} \sum_{i=\ell}^{N-1} \left[\frac{|V^\alpha(\frac{i}{N})|^2}{\pi_H^\alpha(0)} - 1 \right]$$

$$= \frac{2N^{2H}}{N-\ell}\left(-\sum_{q,r=0}^{\ell}\alpha_r\alpha_q|q-r|^{2H}\right)^{-1}\sum_{i=\ell}^{N-1}\left[\left|V^{\alpha}\left(\frac{i}{N}\right)\right|^2 - \pi_H^{\alpha}(0)\right]$$

$$= \frac{2N^{2H}}{c(H)(N-\ell)}\sum_{i=\ell}^{N-1}\left[\left|V^{\alpha}\left(\frac{i}{N}\right)\right|^2 - \pi_H^{\alpha}(0)\right],$$

where

$$c(H) = -\sum_{q,r=0}^{\ell}\alpha_r\alpha_q|q-r|^{2H}. \tag{5.80}$$

The next lemma is informative, and will be useful in the sequel.

Lemma 5.5 *$c(H)$ is positive for all $H \in (0, 1]$. Also, $c(0) = 0$.*

Proof For $H < 1$, we may rewrite $c(H)$ by using the representation of the function $|q-r|^{2H}$ via the fBm B^H and its covariance function R_H given in (1.1). Indeed we have

$$\begin{aligned}
c(H) &= -\sum_{q,r=0}^{\ell}\alpha_r\alpha_q\mathbf{E}\left[\left(B^H(q) - B^H(r)\right)^2\right] \\
&= -\sum_{q,r=0}^{\ell}\alpha_r\alpha_q\left(R_H(q,q) + R_H(r,r) - 2R_H(q,r)\right) \\
&= -2\left(\sum_{q=0}^{\ell}\alpha_q\right)\left(\sum_{r=0}^{\ell}\alpha_r R_H(r,r)\right) + 2\sum_{q,r=0}^{\ell}\alpha_r\alpha_q R_H(q,r) \\
&= 0 + 2\sum_{q,r=0}^{\ell}\alpha_r\alpha_q R_H(q,r) = \mathbf{E}\left[\left(\sum_{q=0}^{\ell}\alpha_q B^H(q)\right)^2\right] > 0
\end{aligned}$$

where in the second-to-last line we used the filter property which implies $\sum_{q=0}^{\ell}\alpha_q = 0$, and the last inequality follows from the fact that $\sum_{q=0}^{\ell}\alpha_q B^H(q)$ is Gaussian and non-constant. When $H = 1$, the same argument as above holds because the Gaussian process X such that $X(0) = 0$ and $\mathbf{E}[(X(t) - X(s))^2] = |t-s|^2$ is evidently equal in law to $X(t) = tN$ where N is a fixed standard normal r.v. The assertion that $c(0) = 0$ comes from the filter property. □

5.6.2 *The Case of Fractional Brownian Motion*

Let Z be a fBm. Observe that we can write the filter ed process as an integral belonging to the first Wiener chaos: since for every t we have $Z(t) = I_1(L_t(\cdot))$ with L given by (1.4)

$$V^{\alpha}\left(\frac{i}{N}\right) = \sum_{q=0}^{\ell} \alpha_q Z\left(\frac{i-q}{N}\right) = I_1\left(\sum_{q=0}^{\ell} \alpha_q L_{\frac{i-q}{N}}\right) := I_1(C_i),$$

where

$$C_i := \sum_{q=0}^{\ell} \alpha_q L_{\frac{i-q}{N}}. \tag{5.81}$$

Proposition 5.10 *With C_i as in (5.81), the variation statistic V_N is given by*

$$V_N = \frac{2N^{2H}}{c(H)(N-l)} \sum_{i=\ell}^{N-1} \left[\left|I_1(C_i)\right|^2 - \pi_H^{\alpha}(0)\right]$$
$$= \frac{2N^{2H}}{c(H)(N-\ell)} \sum_{i=\ell}^{N-1} I_2(C_i \otimes C_i).$$

Then

Proposition 5.11

$$\mathbf{E}[\sqrt{N} V_N]^2 \underset{N}{\to} c_1(H)$$

with

$$c_1(H) = 2c(H)^{-2} \sum_{k=1}^{\infty} \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H} \right|^2 + 2. \tag{5.82}$$

and $c(H)$ is defined by (5.80).

Proof

$$\mathbf{E} V_N^2 = 2\left(\frac{2N^{2H}}{c(H)(N-l)}\right)^2 \sum_{i,j=}^{N-1} \langle C_i \otimes C_i, C_j \otimes C_j \rangle$$
$$= 2\left(\frac{2N^{2H}}{c(H)(N-l)}\right)^2 \sum_{i,j=l}^{N-1} \langle C_i, C_j \rangle^2$$

with

$$
\begin{aligned}
&\langle C_i, C_j\rangle_{L^2([0,1])}\\
&= \frac{\alpha(H)^2 d(H)^2}{H(2H-1)} \sum_{q,r=0}^{\ell} \alpha_q\alpha_r \left[\left|\frac{i-q}{N}\right|^{2H} + \left|\frac{j-r}{N}\right|^{2H} - \left|\frac{j-i+q-r}{N}\right|^{2H}\right]\\
&= \frac{1}{2} \sum_{q,r=0}^{\ell} \alpha_q\alpha_r \left[\left|\frac{i-q}{N}\right|^{2H} + \left|\frac{j-r}{N}\right|^{2H} - \left|\frac{j-i+q-r}{N}\right|^{2H}\right]\\
&= \frac{1}{2}\left[\left(\sum_{q=0}^{\ell}\alpha_q\left|\frac{i-q}{N}\right|^{2H}\right)\left(\sum_{r=0}^{\ell}\alpha_r\right) + \left(\sum_{r=0}^{\ell}\alpha_r\left|\frac{j-r}{N}\right|^{2H}\right)\left(\sum_{q=0}^{\ell}\alpha_q\right)\right.\\
&\quad \left. - \sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{i-j+q-r}{N}\right|^{2H}\right]\\
&= -\frac{1}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{i-j+q-r}{N}\right|^{2H} = \pi_H^{\alpha}(i-j).
\end{aligned}
$$

The last equality holds since $\sum_{q=0}^{\ell}\alpha_q = 0$ by the filter definition. Therefore, we have

$$
\begin{aligned}
&\sum_{i,j=\ell}^{N-1}\left|\langle C_i, C_j\rangle_{L^2([0,1]^2)}\right|^2\\
&= \frac{1}{4}\sum_{i,j=\ell}^{N-1}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{i-j+q-r}{N}\right|^{2H}\right|^2\\
&= \frac{1}{4}\sum_{i=\ell}^{N-1}\sum_{k=0}^{N-2}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{k+q-r}{N}\right|^{2H}\right|^2\\
&= \frac{N^{-4H}}{4}(N-\ell-1)\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|q-r|^{2H}\right|^2\\
&\quad + \frac{1}{4}\sum_{i=\ell}^{N-1}\sum_{k=1}^{N-2}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{k+q-r}{N}\right|^{2H}\right|^2\\
&= c(H)^2\frac{N^{-4H}(N-\ell-1)}{4} + \frac{1}{4}\sum_{k=0}^{N-2}(N-k-2)\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{k+q-r}{N}\right|^{2H}\right|^2
\end{aligned}
$$

$$= c(H)^2 \frac{(N-l-1)N^{-4H}}{4} + \frac{N^{-4H+1}}{4} \sum_{k=0}^{N-2} \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H} \right|^2$$

$$- 2\frac{N^{-4H}}{4} \sum_{k=0}^{N-2} \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H} \right|^2$$

$$- \frac{N^{-4H}}{4} \sum_{k=0}^{N-2} k \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H} \right|^2 .$$

At this point we need the next lemma to estimate the behavior of the above quantity. This lemma is the key point which implies the fact that the longer variation statistics has, in the case when the observed process is the fractional Brownian motion, a Gaussian limit without any restriction on H (this was first noticed in [85]).

Lemma 5.6 *For all* $H \in (0, 1)$, *we have that*

(i) $\sum_{k=1}^{\infty} | \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H} |^2 < +\infty$; *and*
(ii) $\sum_{k=1}^{\infty} k | \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H} |^2 < +\infty$.

Proof of (i) Let $f(x) = \sum_{q,r=0}^{\ell} \alpha_q \alpha_r (1+(q-r)x)^{2H}$, so the summand can be written as

$$\sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H} = k^{2H} f\left(\frac{1}{k}\right).$$

Using a Taylor expansion at $x_0 = 0$ for the function $f(x)$ we get that

$$\big(1+(q-r)x\big)^{2H} \approx 1 + 2H(q-r)x + \cdots$$
$$+ \frac{2H(2H-1)\cdots(2H-n+1)}{n!}(q-r)^n x^n .$$

For small x we observe that the function $f(x)$ is asymptotically equivalent to

$$2H(2H-1)\cdots\big(2H-(p-1)\big)x^{2p},$$

where p is the order of the filter. Therefore, the general term of the series is equivalent to

$$(2H)^2(2H-1)^2\cdots\big(2H-(p-1)\big)^2 k^{4H-4p}.$$

Therefore for all $H < p - \frac{1}{4}$ the series converges to a constant depending only on H. Due to our choice for the order of the filter $p \geq 2$, we obtain the desired result.

Proof of (ii) Similarly as before, we can write the general term of the series as

$$k\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|k+q-r|^{2H}\right|^2 = k\left|k^{2H}f\left(\frac{1}{k}\right)\right|^2$$
$$\approx (2H)^2(2H-1)^2\cdots\big(2H-(p-1)\big)^2k^{4H-4p-1}.$$

Therefore for all $H < p$ the series converges to a constant depending only on H. □

This concludes the proof of Proposition 5.11. □

Theorem 5.18 *Let*

$$G_N = c_1(H)^{-\frac{1}{2}}\sqrt{N}V_N \tag{5.83}$$

with $c_1(H)$ *from* (5.82). *For all* $H \in (1/2, 1)$ G_N *as defined above converges in distribution to the standard normal law.*

Proof Since

$$DG_N = cN^{2H+\frac{1}{2}}\frac{1}{N-l}\sum_{i=l}^{N-1}I_1(C_i)1_{C_i}$$

we get

$$\begin{aligned}\mathbf{E}\big(\|DG_N\|^2-2\big)^2 &= \mathbf{E}\big(\|DG_N\|^2-\mathbf{E}\|DG_N\|^2\big)^2+\mathbf{E}\|DG_N\|^2-2\\ &= cN^{8H+2}(N-l)^{-4}\\ &\quad\times\sum_{i,j,i',j'=l}^{N-1}\langle C_i,C_j\rangle\langle C_{i'},C_{j'}\rangle\langle C_i,C_{i'}\rangle\langle C_j,C_j\rangle+2\mathbf{E}V_N^2-2.\end{aligned}$$

The series $\sum_{i,j,i',j'=l}^{N-1}\langle C_i,C_j\rangle\langle C_{i'},C_{j'}\rangle\langle C_i,C_{i'}\rangle\langle C_j,C_j\rangle$ can be written as N^{-8H} multiplied by a convergent series, therefore the first summand above is of order N^{-2}. We have previously proved that the difference $2\mathbf{E}V_N^2-2$ converges to zero. □

5.6.3 The Case of the Rosenblatt Process

To ensure asymptotic normality in the case of the Rosenblatt process, it was shown in Sect. 5.2 (Exercise 5.3) that one may perform a compensation of the non-normal component of the quadratic variation. In fact, this is possible only in the case of the Rosenblatt process; it is not possible for higher-order Hermite processes, and is not possible for fBm with $H > 3/4$ (recall that the case of fBm with $H \le 3/4$ does not require any compensation). The compensation technique for the Rosenblatt process yields asymptotic variances which are difficult to calculate and may be very high.

The question then arises to find out whether using longer filters for the Rosenblatt process might yield asymptotically normal estimators, and/or might result in low asymptotic variances.

Let Z be a Rosenblatt process with $Z(t) = I_2(L_t)$ and L given by (5.23). Using the product formula (C.4) for multiple stochastic integrals now results in the Wiener chaos expansion of V_N.

Proposition 5.12 *With C_i as in (5.81), the variation statistic V_N is given by*

$$\begin{aligned} V_N &= \frac{2N^{2H}}{c(H)(N-l)} \sum_{i=\ell}^{N-1} \left[\left| I_2(C_i) \right|^2 - \pi_H^{\alpha}(0) \right] \\ &= \frac{2N^{2H}}{c(H)(N-\ell)} \left[\sum_{i=\ell}^{N-1} I_4(C_i \otimes C_i) + 4 \sum_{i=\ell}^{N-1} I_2(C_i \otimes_1 C_i) \right] \\ &:= T_4 + T_2, \end{aligned} \tag{5.84}$$

where T_4 is a term belonging to the 4th Wiener chaos and T_2 a term living in the 2nd Wiener chaos.

Evaluation of the L^2-Norm In order to determine the convergence of V_N, using the orthogonality of the integrals belonging in different chaos, we will study each term separately. This section begins by calculating the second moments of T_2 and T_4.

We use an alternative expression for the filtered process. More specifically, putting $b_q := \sum_{r=0}^{q} \alpha_r$, we rewrite C_i as follows, for any $i = \ell, \ldots, N-1$:

$$\begin{aligned} C_{i,\ell} := C_i = \sum_{q=0}^{\ell} \alpha_q L_{\frac{i-q}{N}} \\ &= \alpha_0 (L_{\frac{i}{N}} - L_{\frac{i-1}{N}}) + (\alpha_0 + \alpha_1)(L_{\frac{i-1}{N}} - L_{\frac{i-2}{N}}) \\ &\quad + \cdots + (\alpha_0 + \cdots + \alpha_{\ell-1})(L_{\frac{i-(\ell-1)}{N}} - L_{\frac{i-\ell}{N}}) \\ &= \sum_{q=0}^{\ell} b_q (L_{\frac{i-(q-1)}{N}} - L_{\frac{i-q}{N}}). \end{aligned} \tag{5.85}$$

Recall that the filter properties imply $\sum_{q=0}^{\ell} \alpha_q = 0$ and $\alpha_\ell = -\sum_{q=0}^{\ell-1} \alpha_q$.

The Term in the Second Wiener Chaos By Proposition 5.12, we can express $\mathbf{E}(T_2^2)$ as:

$$\mathbf{E}\big(T_2^2\big) = \frac{64N^{4H}}{c(H)^2(N-\ell)^2}\mathbf{E}\left[\left(\sum_{i=\ell}^{N-1} I_2(C_i \otimes_1 C_i)\right)^2\right]$$
$$= \frac{2!64N^{4H}}{c(H)^2(N-\ell)^2}\sum_{i,j=\ell}^{N-1}\langle C_i \otimes_1 C_i, C_j \otimes_1 C_j\rangle_{L^2([0,1]^2)}.$$

Proposition 5.13 *We have*

$$\lim_{N\to\infty} \mathbf{E}\big[\big|N^{1-H}T_2\big|^2\big] = c_{2,H},$$

where

$$c_{2,H} = \frac{64}{c(H)^2}\left(\frac{2H-1}{H(H+1)^2}\right)$$
$$\times\left\{\sum_{q,r=0}^{\ell} b_q b_r\big[|1+q-r|^{2H'} + |1-q+r|^{2H'} - 2|q-r|^{2H'}\big]\right\}^2. \quad (5.86)$$

Proof We start by computing the contraction term $C_i \otimes_1 C_i$:

$$(C_i \otimes_1 C_i)(y_1, y_2) = \int_0^1 C_i(x, y_1)C_i(x, y_2)dx$$
$$= \sum_{q,r=0}^{\ell} b_q b_r \int_0^1 \big(L_{\frac{i-(q-1)}{N}}(x, y_1) - L_{\frac{i-q}{N}}(x, y_1)\big)$$
$$\times \big(L_{\frac{i-(r-1)}{N}}(x, y_2) - L_{\frac{i-r}{N}}(x, y_2)\big)dx$$
$$= d(H)^2 \sum_{q,r=0}^{\ell} b_q b_r 1_{[0,\frac{i-q+1}{N}]}(y_1)1_{[0,\frac{i-r+1}{N}]}(y_2)\int_0^{\frac{i-q+1}{N}\wedge\frac{i-r+1}{N}} dx$$
$$\times\left(\int_{\frac{i-q}{N}}^{\frac{i-q+1}{N}} \frac{\partial K^{H'}}{\partial u}(u, x)\frac{\partial K^{H'}}{\partial u}(u, y_1)du\right)$$
$$\times\left(\int_{\frac{i-r}{N}}^{\frac{i-r+1}{N}} \frac{\partial K^{H'}}{\partial v}(v, x)\frac{\partial K^{H'}}{\partial v}(v, y_2)dv\right)$$
$$= d(H)^2 \sum_{q,r=0}^{\ell} b_q b_r 1_{[0,\frac{i-q+1}{N}]}(y_1)1_{[0,\frac{i-r+1}{N}]}(y_2)$$

$$\times \int_{I_{i_q}} \int_{I_{i_r}} dudv \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(v, y_2) dudv$$

$$\times \left(\int_0^{u \wedge v} dx \frac{\partial K^{H'}}{\partial u}(u, x) \frac{\partial K^{H'}}{\partial v}(v, x) \right)$$

$$= \alpha(H) d(H)^2 \sum_{q,r=0}^{\ell} b_q b_r 1_{[0, \frac{i-q+1}{N}]}(y_1) 1_{[0, \frac{i-r+1}{N}]}(y_2)$$

$$\times \int_{I_{i_q}} \int_{I_{i_r}} dudv |u-v|^{2H'-2} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) dudv,$$

where $I_{i_q} = (\frac{i-q}{N}, \frac{i-q+1}{N}]$.

Now, the inner product computes as

$$\langle C_i \otimes_1 C_i, C_j \otimes_1 C_j \rangle_{L^2[0,1]^2}$$

$$= \alpha(H)^2 d(H)^4 \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_0^1 \int_0^1 dy_1 dy_2$$

$$\times \int_{I_{i_{q_1}}} \int_{I_{i_{r_1}}} \int_{I_{j_{q_2}}} \int_{I_{j_{r_2}}} dudvdu'dv' |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2}$$

$$\times \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) \frac{\partial K^{H'}}{\partial u'}\left(u', y_1\right) \frac{\partial K^{H'}}{\partial v'}\left(v', y_2\right) dudvdu'dv'$$

$$= \alpha(H)^2 d(H)^4 \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2}$$

$$\times \int_{I_{i_{q_1}}} \int_{I_{i_{r_1}}} \int_{I_{j_{q_2}}} \int_{I_{j_{r_2}}} dudvdu'dv' |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2}$$

$$\times \left(\int_0^{u \wedge u'} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u'}\left(u', y_1\right) dy_1 \right)$$

$$\times \left(\int_0^{v \wedge v'} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial v'}\left(v', y_2\right) dy_2 \right)$$

$$= \alpha(H)^4 d(H)^4 \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{I_{i_{q_1}}} \int_{I_{i_{r_1}}} \int_{I_{j_{q_2}}} \int_{I_{j_{r_2}}} dudvdu'dv'$$

$$\times |u-v|^{2H'-2} \left|u'-v'\right|^{2H'-2} \left|u-u'\right|^{2H'-2} \left|v-v'\right|^{2H'-2}.$$

We make the following change of variables

$$\bar{u} = \left(u - \frac{i - q_1}{N}\right) N$$

and the second moment of T_2 becomes

$$\begin{aligned}
\mathbf{E}\big(T_2^2\big) &= \frac{128\alpha(H)^4 d(H)^4}{c(H)^2} \frac{N^{4H}}{(N-\ell)^2} \\
&\quad \times \sum_{i,j=\ell}^{N-1} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{I_{iq_1}} \int_{I_{ir_1}} \int_{I_{jq_2}} \int_{I_{jr_2}} du dv du' dv' \\
&\quad \times |u-v|^{2H'-2} \big|u'-v'\big|^{2H'-2} \big|u-u'\big|^{2H'-2} \big|v-v'\big|^{2H'-2} \\
&= \frac{128\alpha(H)^4 d(H)^4}{c(H)^2} \frac{N^{4H}}{(N-\ell)^2} \frac{1}{N^4 N^{8H'-8}} \sum_{i,j=\ell}^{N-1} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \\
&\quad \times \int_{[0,1]^4} du dv du' dv' |u-v-q_1+r_1|^{2H'-2} \big|u'-v'-q_2+r_2\big|^{2H'-2} \\
&\quad \times \big|u-u'+i-j-q_1+q_2\big|^{2H'-2} \big|v-v'+i-j-r_1+r_2\big|^{2H'-2} \\
&= \frac{128\alpha(H)^4 d(H)^4}{c(H)^2} \frac{1}{(N-\ell)^2} \sum_{i,j=\ell}^{N-1} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \\
&\quad \times \int_{[0,1]^4} du dv du' dv' |u-v-q_1+r_1|^{2H'-2} \big|u'-v'-q_2+r_2\big|^{2H'-2} \\
&\quad \times \big(\big|u-u'+i-j-q_1+q_2\big|^{2H'-2} \big|v-v'+i-j-r_1+r_2\big|^{2H'-2}\big).
\end{aligned}$$

Let $cst. = \frac{128\alpha(H)^4 d(H)^4}{c(H)^2}$. We study first the diagonal terms of the above double sum

$$\begin{aligned}
\mathbf{E}\big(T_{2-\mathrm{diag}}^2\big) = cst. \frac{N-\ell-1}{(N-\ell)^2} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{[0,1]^4} du dv du' dv' \\
\times |u-v-q_1+r_1|^{2H'-2} \big|u'-v'-q_2+r_2\big|^{2H'-2} \\
\times \big|u-u'-q_1+q_2\big|^{2H'-2} \big|v-v'-r_1+r_2\big|^{2H'-2}.
\end{aligned}$$

We conclude that

$$\mathbf{E}\big(T_{2-\mathrm{diag}}^2\big) = \mathcal{O}\big(N^{-1}\big).$$

Let us now consider the non-diagonal terms

$$\begin{aligned}\mathbf{E}\big(T_{2-\text{off}}^2\big) &= 2cst. \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \\ &\quad \times \int_{[0,1]^4} du dv du' dv' |u - v - q_1 + r_1|^{2H'-2} \big|u' - v' - q_2 + r_2\big|^{2H'-2} \\ &\quad \times \frac{1}{(N-\ell)^2} \Bigg(\sum_{i,j=\ell, i\neq j}^{N-1} \big|u - u' + i - j - q_1 + q_2\big|^{2H'-2} \\ &\quad \times \big|v - v' + i - j - r_1 + r_2\big|^{2H'-2} \Bigg). \end{aligned} \tag{5.87}$$

Observe that the term (5.87) can be calculated as follows:

$$\begin{aligned}&\frac{1}{(N-\ell)^2} \sum_{i,j=\ell i\neq j}^{N-1} \big|u - u' + i - j - q_1 + r_1\big|^{2H'-2} \big|v - v' + i - j - r_1 + r_2\big|^{2H'-2} \\ &= \frac{1}{(N-\ell)^2} \sum_{i=\ell}^{N-1} \sum_{k=1}^{N-i} \big|u - u' + k - q_1 + q_2\big|^{2H'-2} \big|v - v' + k - r_1 + r_2\big|^{2H'-2} \\ &= \frac{1}{(N-\ell)^2} \sum_{k=\ell}^{N-1} (N-k-1) \big|u - u' + k - q_1 + q_2\big|^{2H'-2} \\ &\quad \times \big|v - v' + k - r_1 + r_2\big|^{2H'-2} \\ &= N^{4H'-4} \frac{N}{(N-\ell)^2} \sum_{k=\ell}^{N-1} \left(1 - \frac{k+1}{N}\right) \left|\frac{u-u'}{N} + \frac{k}{N} - \frac{q_1 - q_2}{N}\right|^{2H'-2} \\ &\quad \times \left|\frac{v-v'}{N} + \frac{k}{N} - \frac{r_1 - r_2}{N}\right|^{2H'-2}. \end{aligned}$$

We may now use a Riemann sum approximation and the fact that $4H' - 4 = 2H - 2 > -1$. Since ℓ is fixed and q_1 and q_2 are less than ℓ, we get that the term in (5.87) is asymptotically equivalent to

$$\begin{aligned}\sum_{k=\ell}^{N-1} \left(1 - \frac{k}{N}\right) \left|\frac{k}{N}\right|^{2H'-2} \left|\frac{k}{N}\right|^{2H'-2} &= \int_0^1 (1-x) x^{2H-2} dx + o(1) \\ &= \frac{1}{2H(2H-1)} + o(1). \end{aligned}$$

We conclude that

$$\begin{aligned}
&\mathbf{E}\big(T_2^2\big)+o\big(N^{2H-2}\big)\\
&=\frac{cst.N^{2H-2}}{H(2H-1)}\sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1}b_{r_1}b_{q_2}b_{r_2}\\
&\quad\times\int_{[0,1]^4} dudvdu'dv'|u-v-q_1+r_1|^{2H'-2}\big|u'-v'-q_2+r_2\big|^{2H'-2}.
\end{aligned}$$

Using the fact that

$$\begin{aligned}
&\int_{[0,1]^2}|u-v-q+r|^{2H'-2}dudv\\
&=\frac{1}{2H'(2H'-1)}\big[|1+q-r|^{2H'}+|1-q+r|^{2H'}-2|q-r|^{2H'}\big]
\end{aligned}$$

the proposition follows. □

The Term in the Fourth Wiener Chaos In this paragraph we estimate the second moment of T_4, the fourth chaos term appearing in the decomposition of the variation V_N. Here the function $\sum_{i=\ell}^{N-1}(C_i\otimes C_i)$ is no longer symmetric and we need to symmetrize this kernel to calculate T_4's second moment. In other words, by Proposition 5.12, we have that

$$\begin{aligned}
\mathbf{E}\big(T_4^2\big)&=\frac{4N^{4H}}{c(H)^2(N-\ell)^2}\mathbf{E}\left[\left(\sum_{i=\ell}^{N-1}I_4(C_i\otimes C_i)\right)^2\right]\\
&=\frac{4N^{4H}}{c(H)^2(N-\ell)^2}4!\sum_{i,j=\ell}^{N-1}\langle C_i\tilde{\otimes}C_i, C_j\tilde{\otimes}C_j\rangle_{L^2([0,1]^4)}
\end{aligned}$$

where $C_i\tilde{\otimes}C_i:=\widetilde{C_i\otimes C_i}$. Thus, we can use the following combinatorial formula: If f and g are two symmetric functions in $L^2([0,1]^2)$, then

$$\begin{aligned}
4!\langle f\tilde{\otimes}f, g\tilde{\otimes}g\rangle_{L^2([0,1]^4)}&=(2!)^2\langle f\otimes f, g\otimes g\rangle_{L^2([0,1]^4)}\\
&\quad+(2!)^2\langle f\otimes_1 g, g\otimes_1 f\rangle_{L^2([0,1]^2)}.
\end{aligned}$$

This implies

$$\begin{aligned}
\mathbf{E}\big(T_4^2\big)&=\frac{4N^{4H}}{c(H)^2(N-\ell)^2}4!\sum_{i,j=\ell}^{N-1}\langle C_i\tilde{\otimes}C_i, C_j\tilde{\otimes}C_j\rangle_{L^2([0,1]^4)}\\
&=\frac{4N^{4H}}{c(H)^2(N-\ell)^2}4\sum_{i,j=\ell}^{N-1}\langle C_i\otimes C_i, C_j\otimes C_j\rangle_{L^2([0,1]^4)}
\end{aligned}$$

$$+\frac{4N^{4H}}{c(H)^2(N-\ell)^2}4\sum_{i,j=\ell}^{N-1}\langle C_i\otimes_1 C_j, C_j\otimes_1 C_i\rangle_{L^2([0,1]^2)}$$
$$:= T_{4,(1)} + T_{4,(2)}.$$

The next proposition shows that the two terms $T_{4,(1)}$ and $T_{4,(2)}$ have the same order of magnitude, with only the normalizing constant being different.

Proposition 5.14 *Recall the constant $c(H)$ defined in (5.80). Let*

$$\tau_{1,H} := \sum_{k=\ell}^{\infty}\sum_{q_1,q_2,r_1,r_1=0}^{\ell} b_{q_1}b_{q_2}b_{r_1}b_{r_2}\int_{[0,1]^4} dudvdu'dv'$$
$$\times \left[|u-v+k-q_1+r_1|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}\right.$$
$$\left.\times \left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\right]$$

and

$$\rho_H^{\alpha}(k) := \frac{\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|k+q-r|^{2H}}{c(H)}.$$

Then we have the following asymptotic variance for $\sqrt{N}T_4$:

$$\lim_{N\to\infty}\mathbf{E}\left[|\sqrt{N}T_4|^2\right] = c_{1,H} := 4!\left(1+\sum_{k=0}^{\infty}\left|\rho_H^{\alpha}(k)\right|^2\right)+\tau_{1,H}. \tag{5.88}$$

The proof is left as an exercise (Exercise 5.14). Observe that in the Wiener chaos decomposition of V_N the leading term is the term in the second Wiener chaos (i.e. T_2) since it is of order N^{H-1}, while T_4 is of the smaller order $N^{-1/2}$. We note that, in contrast to the case of filters of length 1 and power 1, the barrier $H = 3/4$ no longer appears in the estimate of the magnitude of T_4. Thus, the asymptotic behavior of V_N is determined by the behavior of T_2. In other words, the previous three propositions imply the following.

Theorem 5.19 *For all $H \in (1/2, 1)$ we have that*

$$\lim_{N\to\infty}\mathbf{E}\left[\left|N^{1-H}V_N\right|^2\right] = c_{2,H},$$

where $c_{2,H}$ is defined in (5.86).

5.6.4 The Asymptotic Distribution of the Quadratic Variations

For the asymptotic distribution of the variation statistic we have the following proposition.

Theorem 5.20 *For all* $H \in (1/2, 1)$, *both* $\frac{N^{1-H}}{\sqrt{c_{2,H}}} T_2$ *and the normalized quadratic variation* $\frac{N^{1-H}}{\sqrt{c_{2,H}}} V_N$ *converge in* $L^2(\Omega)$ *to the Rosenblatt random variable* $Z(1)$.

Proof The strategy for proving this theorem is simple. First of all Proposition 5.14 implies immediately that $N^{1-H} T_4$ converges to zero in $L^2(\Omega)$. Thus if we can prove the theorem's statement about T_2, the statement about V_N will follow immediately from Proposition 5.12.

Next, to show $\frac{N^{1-H}}{\sqrt{c_{2,H}}} T_2$ converges to the random variable $Z(1)$ in $L^2(\Omega)$, recall that T_2 is a second-chaos random variable of the form $I_2(f_N)$, where $f_N(y_1, y_2)$ is a symmetric function in $L^2([0,1]^2)$, and that this double Wiener-Itô integral is with respect to the Brownian motion W used to define $Z(1)$, i.e. that $Z(1) = I_2(L_1)$ where L_1 is the kernel of the Rosenblatt process at time 1, as defined in (5.23). Therefore, by the isometry property of Wiener-Itô integrals (see (C.1)), it is necessary and sufficient to show that $\frac{N^{1-H}}{\sqrt{c_{2,H}}} f_N$ converges in $L^2([0,1]^2)$ to L_1. This can be proved as in Theorem 5.10. □

5.7 Wavelet-Type Quadratic Variations

There are different types of variations of stochastic processes that are used in statistics. One of these is the wavelet-type variation. The context is as follows. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function with support included in the interval $[0, 1]$ (called the "mother wavelet"). Assume that there exists an integer $Q \geq 1$ such that

$$\int_{\mathbb{R}} t^p \psi(t) dt = 0 \quad \text{for } p = 0, 1, \ldots, Q-1 \tag{5.89}$$

and

$$\int_{\mathbb{R}} t^Q \psi(t) dt \neq 0.$$

We will call the integer $Q \geq 1$ the *number of vanishing moments*. For a stochastic process $(X_t)_{t \in [0,N]}$ and for a "scale" $a \in \mathbb{N}^*$ we define its wavelet coefficient by

$$d(a,i) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \psi\left(\frac{t}{a} - i\right) X_t dt = \sqrt{a} \int_0^1 \psi(x) X_{a(x+i)} dx \tag{5.90}$$

for $i = 1, 2, \ldots, N_a$ with $N_a := [N/a] - 1$. Let us set

$$\tilde{d}(a,i) = \frac{d(a,i)}{(\mathbf{E} d^2(a,i))^{\frac{1}{2}}}$$

and

$$V_N(a) = \frac{1}{N_a} \sum_{i=1}^{N_a} \left(\tilde{d}^2(a,i) - 1\right). \tag{5.91}$$

The wavelet analysis consists in studying the behavior of the sequence $V_N(a)$ when $N \to \infty$. But if X is respectively a stationary long memory or a self-similar second-order process, $\mathbf{E}d^2(a,i)$ is a power-law function of a with, respectively, an exponent $2H-1$ (when $a \to \infty$) or $2H+1$. Therefore, if $V_N(a)$ is proved to converge to 0, a log-log-regression of $\frac{1}{N_a}\sum_{i=1}^{N_a} d^2(a_j,i)$ onto a_j will provide an estimator of H (with an appropriate choice of $(a_j)_j$). Hence, the asymptotic behavior of $V_N(a)$ will completely give the behavior of the estimator. For examples of the applications of wavelets to parameter identification the reader is referred, among other references, to [2, 4, 15–17, 79, 123, 124] and [3].

Our purpose is to develop a wavelet-based analysis of fBm and the Rosenblatt process using multiple Wiener-Itô integrals.

5.7.1 Wavelet-Type Variations of Fractional Brownian Motion

A Presentation Using Chaos Expansion We will assume in this part that $X = B^H$ is a fBm with Hurst parameter $H \in (0,1)$. Recall also that the fBm $(B_t^H)_{t\in[0,N]}$ with Hurst parameter $H \in (0,1)$ can be written as (Chap. 1)

$$B_t^H = \int_0^t K^H(t,s)dW_s, \quad t \in [0,N]$$

where $(W_t, t \in [0,N])$ is a standard Wiener process and for $s < t$ and $H > \frac{1}{2}$, K^H is the kernel given by (1.4).

In this case it is trivial to decompose in chaos the wavelet coefficient $d(a,i)$. By the stochastic Fubini theorem we can write

$$\begin{aligned} d(a,i) &= \sqrt{a}\int_0^1 \psi(x) B^H_{a(x+i)}dx = \sqrt{a}\int_0^1 \psi(x)dx\left(\int_0^{a(x+i)} dB_u^H\right) \\ &= \sqrt{a}\int_0^1 \psi(x)dx\int_0^{a(x+i)} K^H\big(a(x+i),u\big)dW_u = I_1\big(f_{a,i}(\cdot)\big) \end{aligned}$$

where I_1 denotes the multiple integral of order one (actually, the Wiener integral with respect to W) and

$$f_{a,i}(u) := 1_{[0,a(i+1)]}(u)\sqrt{a}\int_{(\frac{u}{a}-i)\vee 0}^1 \psi(x)K^H\big(a(x+i),u\big)dx. \tag{5.92}$$

Lemma 5.7 *For all $a > 0$ and $i \in \mathbb{N}$,*

$$\mathbf{E}\big(d^2(a,i)\big) = \|f_{a,i}\|^2_{\mathcal{H}} = a^{2H+1}C_\psi(H)$$

$$\text{with } C_\psi(H) := -\frac{1}{2}\int_0^1\int_0^1 \psi(x)\psi\big(x'\big)\big|x-x'\big|^{2H}dxdx'. \tag{5.93}$$

Using the product formula (C.4)

$$I_1(f)I_1(g) = I_2(f \otimes g) + \langle f, g\rangle_{\mathcal{H}}$$

(here and in the sequel $\mathcal{H}$ denotes the space $L^2([0, N])$) and we get

$$V_N(a) = \frac{1}{N_a}\sum_{i=1}^{N_a}\left(\frac{I_2(f_{a,i}^{\otimes 2}) + \|f_{a,i}\|_{\mathcal{H}}^2}{(\mathbf{E}d(a,i))^2} - 1\right) = I_2\big(f_N^{(a)}\big)$$

where

$$f_N^{(a)} := a^{-2H-1}C_\psi(H)^{-1}\frac{1}{N_a}\sum_{i=1}^{N_a} f_{a,i}^{\otimes 2}. \tag{5.94}$$

A Multidimensional Central Limit Theorem Satisfied by $(V_N(a_i))_{1\le i\le m}$ When the observed process is the fBm with $H < 3/4$, the statistics $V_N(a)$ satisfy a Central Limit Theorem. Since $\mathbf{E}I_2^2(f) = 2!\|f\|_{\mathcal{H}}^2$ we have for $(a_i)_{1\le i\le m}$ a family of integers such that $a_i = ia$ for $i = 1, \dots, m$ and $a \in \mathbb{N}^*$,

$$\begin{aligned}
&\operatorname{Cov}\big(V_N(a_p), V_N(a_q)\big)\\
&\quad = 2!\big(pqa^2\big)^{-2H-1}C_\psi(H)^{-2}\frac{1}{N_{a_p}}\frac{1}{N_{a_q}}\sum_{j=1}^{N_{a_p}}\sum_{j'=1}^{N_{a_q}}\big\langle f_{a_p,j}^{\otimes 2}, f_{a_q,j'}^{\otimes 2}\big\rangle_{\mathcal{H}^{\otimes 2}}\\
&\quad = 2\big(pqa^2\big)^{-2H-1}C_\psi(H)^{-2}\frac{1}{N_{a_p}}\frac{1}{N_{a_q}}\sum_{j=1}^{N_{a_p}}\sum_{j'=1}^{N_{a_q}}\langle f_{a_p,i}, f_{a_q,j}\rangle_{\mathcal{H}}^2.
\end{aligned}$$

Lemma 5.8 *If $Q > 1$ and $H \in (0, 1)$ or if $Q = 1$ and $H \in (0, 3/4)$,*

$$\frac{N}{a}\operatorname{Cov}\big(V_N(a_p), V_N(a_q)\big) \xrightarrow[N\to\infty]{} \ell_1(p,q,H) \quad \textit{with}$$

$$\ell_1(p,q,H) = \frac{1}{2d_{pq}(pq)^{2H-1}}\sum_{k=-\infty}^{\infty}\left(\frac{1}{C_\psi(H)}\int_0^1\int_0^1 \psi(x)\psi\big(x'\big) \times \big|px - qx' + kd_{pq}\big|^{2H}dx\,dx'\right)^2, \tag{5.95}$$

where $d_{pq} = \mathrm{GCD}(p, q)$.

Proof We have

$$\begin{aligned}
&\langle f_{a_p,j}, f_{a_q,j'}\rangle_{\mathcal{H}}\\
&\quad = \mathbf{E}\big(d(a_p, j)d\big(a_q, j'\big)\big)
\end{aligned}$$

$$= -\frac{1}{2}\left(pqa^2\right)^{1/2} a^{2H} \int_0^1 \int_0^1 \psi(x)\psi\left(x'\right)\left|px - qx' + pj - qj'\right|^{2H} dx dx' \tag{5.96}$$

and from a Taylor expansion and using property (5.89) satisfied by ψ,

$$\langle f_{a_p,j}, f_{a_q,j'}\rangle_{\mathcal{H}}^2 = pqa^{4H+2}\mathcal{O}\left(1 + \left|pj - qj'\right|\right)^{4H-4Q}$$

$$\Longrightarrow \quad \left|\mathrm{Cov}\left(V_N(a_p), V_N(a_q)\right)\right| \le C\frac{1}{N_{a_q}^2}\sum_{j=1}^{N_{a_p}}\sum_{j'=1}^{N_{a_q}}\mathcal{O}\left(1 + \left|pj - qj'\right|\right)^{4H-4Q}. \qquad \square$$

Using Lemma 5.8 we can prove the following limit theorem.

Theorem 5.21 *Let $V_N(a)$ be defined by (5.91) and $L_1(H) = (\ell_1(p, q, H))_{1\le p,q\le m}$. Then if $Q > 1$ and $H \in (0, 1)$ or if $Q = 1$ and $H \in (0, 3/4)$, for all $a > 0$,*

$$\left(\sqrt{\frac{N}{a}}V_N(ia)\right)_{1\le i\le m} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_m\left(0, L_1(H)\right).$$

Proof See [15]. $\square$

Remark 5.2 To go from the one-dimensional limit theorem to the multidimensional one, one can use Lemma 5.8 and Theorem 5.3.

A Non-Central Limit Theorem Satisfied by $V_N(a)$ Suppose now $Q = 1$ and $H > \frac{3}{4}$. We obtain the following non-central limit theorem for the wavelet coefficient of the fBm with $H > \frac{3}{4}$. Define

$$\ell_2(H) := \left(\frac{2H^2(2H-1)}{4H-3}\right)^{1/2}\frac{\left(\int_0^1 x\psi(x)dx\right)^2}{C_\psi(H)}. \tag{5.97}$$

Then,

Theorem 5.22 *For fBm, if $Q = 1$ and $\frac{3}{4} < H < 1$ then*

$$\ell_2^{-1}(H)N_a^{2-2H}V_N(a) \underset{N}{\to} Z_1^{2H-1},$$

where Z_1^{2H-1} is a Rosenblatt random variable with self-similarity index $2H - 1$ given by (3.17).

Proof With $f_N^{(a)}$ defined as in (5.94), we can write

$$N_a^{2-2H}V_N(a) = N_a^{2-2H}I_2\left(f_N^{(a)}\right).$$

But using the expression of $f_{a,i}$ provided in (5.92),

$$
\begin{aligned}
f_N^{(a)}(y_1, y_2) := \frac{1}{a^{2H} C_\psi(H)} \frac{1}{N_a} \sum_{i=1}^{N_a} 1_{[0,a(i+1)]}(y_1) 1_{[0,a(i+1)]}(y_2) \\
\times \int_{(\frac{y_1}{a}-i)\vee 0}^{1} \int_{(\frac{y_2}{a}-i)\vee 0}^{1} \psi(x)\psi(z) \\
\times K^H\big(a(x+i), y_1\big) K^H\big(a(z+i), y_2\big) dx dz.
\end{aligned}
$$

To show that the sequence $\ell_2^{-1}(H) N_a^{2-2H} I_2(f_N^{(a)})$ converges in law to the Rosenblatt random variable Z^{2H-1} it suffices to show that its cumulants converge to the cumulants of Z_1^{2H-1} (recall that the law of a multiple integral of order 2 is given by the cumulants). The k-cumulant of a random variable $I_2(f)$ in the second Wiener chaos can be computed by (3.16) and thus

$$
\begin{aligned}
& c_k\big(N_a^{2-2H} I_2\big(f_N^{(a)}\big)\big) \\
&= N_a^{(2H-2)k} N_a^{-k} \sum_{i_1,\ldots,i_k=1}^{N_a} \int_{[0,1]^k} dy_1 \cdots dy_k \\
&\quad \times \int_{[0,1]^{2k}} dx_1 dz_1 \cdots dx_k dz_k \psi(x_1)\psi(z_1)\psi(x_2)\psi(z_2)\cdots\psi(x_k)\psi(z_k) \\
&\quad \times K^H\big(a(x_1+i_1), y_1\big) K^H\big(a(z_1+i_1), y_2\big) \\
&\quad \times K^H\big(a(x_2+i_2), y_2\big) K^H\big(a(z_2+i_2), y_3\big) \\
&\quad \times \cdots \\
&\quad \times K^H\big(a(x_{k-1}+i_{k-1}), y_{k-1}\big) K^H\big(a(z_k+i_k), y_k\big) \\
&\quad \times K^H\big(a(x_k+i_k), y_k\big) K^H\big(a(z_k+i_k), y_1\big).
\end{aligned}
$$

Using Fubini's theorem and the fact that

$$
\int_0^{a(x+i)\wedge a(x'+j)} K^H\big(a(x+i), y_1\big) K^H\big(a\big(x'+j\big), y_1\big) dy_1 = Z^H\big(a(x+i), a\big(x'+j\big)\big)
$$

we get

$$
\begin{aligned}
& c_k\big(N_a^{2-2H} I_2\big(f_N^{(a)}\big)\big) \\
&= N_a^{(2H-2)k} a^{2Hk} N_a^{-k} \\
&\quad \times \sum_{i_1,\ldots,i_k=1}^{N_a} \int_{[0,1]^{2k}} dx_1 dz_1 \cdots dx_k dz_k \psi(x_1)\psi(z_1)\psi(x_2)\psi(z_2)\cdots\psi(x_k)\psi(z_k)
\end{aligned}
$$

$$
\begin{aligned}
&\times Z^H(z_1+i_1, x_2+i_2)Z^H(z_2+i_2, x_3+i_3)\\
&\times \cdots\\
&\times Z^H(z_{k-1}+i_{k-1}, x_k+i_k)Z^H(z_k+i_k, x_1+i_1)\\
&= N_a^{(2H-2)k} a^{2Hk} N_a^{-k}\\
&\times \sum_{i_1,\ldots,i_k=1}^{N_a} \int_{[0,1]^{2k}} dx_1 dz_1 \cdots dx_k dz_k \psi(x_1)\psi(z_1)\psi(x_2)\psi(z_2)\cdots\psi(x_k)\psi(z_k)\\
&\times \big[|z_1-x_2+i_1-i_2|\cdot|z_2-x_3+i_2-i_3|\\
&\times \cdots\\
&\times |z_{k-1}-x_k+i_{k-1}-i_k|\cdot|z_k-x_1+i_k-i_1|\big]^{2H}\\
&= N_a^{(2H-2)k} a^{2Hk} N_a^{-k} \sum_{i_1,\ldots,i_k=1}^{N_a} \big(|i_1-i_2|\cdot\cdots\cdot|i_{k-1}-i_k|\cdot|i_k-i_1|\big)^{2H}\\
&\times \int_{[0,1]^{2k}} dx_1 dz_1 \cdots dx_k dz_k \psi(x_1)\psi(z_1)\psi(x_2)\psi(z_2)\cdots\psi(x_k)\psi(z_k)\\
&\times \left|\left(1+\frac{z_1-x_2}{i_1-i_2}\right)^{2H}\cdots\left(1+\frac{z_k-x_1}{i_k-i_1}\right)^{2H}\right|\\
&\sim N_a^{(2H-2)k} a^{2Hk} H^{2k}(2H-1)^{2k} N_a^{-k}\\
&\times \sum_{i_1,\ldots,i_k=1}^{N_a} \big(|i_1-i_2|\cdot\cdots\cdot|i_{k-1}-i_k|\cdot|i_k-i_1|\big)^{2H-2}\\
&\times \int_{[0,1]^{2k}} dx_1 dz_1 \cdots dx_k dz_k \psi(x_1)\psi(z_1)\psi(x_2)\psi(z_2)\\
&\times \cdots\\
&\times \psi(x_k)\psi(z_k)x_1 z_1\cdots x_k z_k
\end{aligned}
$$

where we used the fact that the integral of the mother wavelet vanishes and a Taylor expansion of second order of the function $(1+x)^{2H}$. As a consequence, by a Riemann sum argument it is clear that the cumulant of $\ell_2^{-1}(H)N_a^{2-2H}I_2(f_N^{(a)})$ converges to

$$
\int_{[0,1]^{2k}} \big[|x_1-x_2|\cdot\cdots\cdot|x_{k-1}-x_k|\cdot|x_k-x_1|\big]^{2H-2} dx_1\cdots dx_k
$$

which represents the k cumulant of the Rosenblatt random variable Z_1^{2H-1} (see [167, 174]). □

In the case of the statistics based on the variations of fBm, in the case $H \in (3/4, 1)$ the statistic $\frac{1}{N}\sum_{i=0}^{N-1} \frac{(B^H_{\frac{i+1}{N}} - B^H_{\frac{i}{N}})^2}{N^{-2H}} - 1$, renormalized by a constant times N^{2-2H}, converges in $L^2(\Omega)$ to a Rosenblatt random variable at time 1 (see Sect. 5.1). In the wavelet world, the above result gives only convergence in law. The following question is then natural: can we get L^2 convergence for the renormalized statistics $V_N(a)$? The answer is negative (see Exercise 5.17).

5.7.2 Wavelet Variations in the Rosenblatt Case

Chaotic Expansion of the Wavelet Variation We study in this section the wavelet-based statistics V_N given by (5.91) in the situation when the observed process is the Rosenblatt process. Throughout this section, we assume that Z^H is a Rosenblatt process with self-similarity order H given by the right-hand side of (3.17). In this case, the wavelet coefficient can be written as

$$\begin{aligned} d(a,i) &= \sqrt{a}\int_0^1 \psi(x) Z^H_{a(x+i)}dx \\ &= \sqrt{a}\int_0^1 \psi(x)dx\left(\int_0^{a(x+i)}\int_0^{a(x+i)} L^H_{a(x+i)}(y_1,y_2)dW_{y_1}dW_{y_2}\right) \\ &= I_2\big(g_{a,i}(\cdot,\cdot)\big) \end{aligned}$$

with

$$\begin{aligned} g_{a,i}(y_1,y_2) &:= d_H\sqrt{a}1_{[0,a(i+1)]}(y_1)1_{[0,a(i+1)]}(y_2) \\ &\quad \times \int_{\frac{y_1\vee y_2}{a}-i}^1 dx\,\psi(x)\left(\int_{y_1\vee y_2}^{a(x+i)} \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(u,y_2)du\right). \end{aligned} \tag{5.98}$$

The product formula for multiple stochastic integrals (C.4) gives

$$I_2(f)I_2(g) = I_4(f\otimes g) + 4I_2(f\otimes_1 g) + 2\langle f,g\rangle_{L^2[0,N]^2}$$

if $f, g \in L^2([0,N]^2)$ are two symmetric functions and the contraction $f \otimes_1 g$ is defined by

$$(f\otimes_1 g)(y_1,y_2) = \int_0^N f(y_1,x)g(y_2,x)dx.$$

Thus, we obtain

$$d^2(a,i) = I_4\big(g_{a,i}^{\otimes 2}\big) + 4I_2(g_{a,i}\otimes_1 g_{a,i}) + 2\|g_{a,i}\|^2_{L^2[0,N]^2}$$

and noting that, since the covariance of the Rosenblatt process is the same as the covariance of the fractional Brownian motion, we will also have

$$\mathbf{E}\big(d^2(a,i)\big) = \mathbf{E}\big(I_2(g_{a,i})\big)^2 = 2\|g_{a,i}\|^2_{L^2[0,N]^2} = a^{2H+1}C_\psi(H).$$

Therefore, we obtain the following decomposition for the statistic $V_N(a)$:

$$V_N(a) = a^{-2H-1}C_\psi(H)^{-1}\frac{1}{N_a}\left[\sum_{i=1}^{N_a} I_4\big(g_{a,i}^{\otimes 2}\big) + 4\sum_{i=1}^{N_a} I_2(g_{a,i}\otimes_1 g_{a,i})\right] = T_2 + T_4$$

$$\text{with } \begin{cases} T_2 := a^{-2H-1}C_\psi(H)^{-1}\frac{4}{N_a}\sum_{i=1}^{N_a} I_2(g_{a,i}\otimes_1 g_{a,i}), \\ T_4 := a^{-2H-1}C_\psi(H)^{-1}\frac{1}{N_a}\sum_{i=1}^{N_a} I_4(g_{a,i}^{\otimes 2}). \end{cases} \tag{5.99}$$

To understand the limit of the sequence V_N we need to regard the two terms above (note that similar terms appear in the decomposition of the variation statistics of the Rosenblatt process, see [181]). In essence, the following will happen: the term T_4 which lives in the fourth Wiener chaos retains some characteristics of the fBm case (since it has to be renormalized by $\sqrt{N_a}$ except in the case $Q=1$ where the normalization is N_a^{2-2H} for $H>\frac{3}{4}$) and its limit will be Gaussian (except for $Q=1$ and $H>\frac{3}{4}$). Unfortunately, this apparent good behavior does not affect the limit of V_N which is non-normal. The same phenomenon occurs for the limit behavior of the quadratic variations of the Rosenblatt process, see Sect. 5.2.

Now, let us study the asymptotic behavior of the term $\mathbf{E}T_4^2$. From (5.99), we have

$$T_4 = I_4\big(g_N^{(a)}\big)$$

where

$$g_N^{(a)} := a^{-2H-1}C_\psi(H)^{-1}\frac{1}{N_a}\sum_{i=1}^{N_a} g_{a,i}^{\otimes 2}, \tag{5.100}$$

and thus, by the isometry of multiple stochastic integrals,

$$\begin{aligned}\mathbf{E}T_4^2 &= 4!C_\psi(H)^{-2}a^{-4H-2}\frac{1}{N_a^2}\sum_{i,j=1}^{N_a}\big\langle g_{a,i}^{\otimes 2}, g_{a,j}^{\otimes 2}\big\rangle_{L^2[0,N]^4} \\ &= 4!C_\psi(H)^{-2}a^{-4H-2}\frac{1}{N_a^2}\sum_{i,j=1}^{N_a}\langle g_{a,i}, g_{a,j}\rangle^2_{L^2[0,N]^2}.\end{aligned}$$

But,

$$\langle g_{a,i}, g_{a,j}\rangle_{L^2[0,N]^2} = \frac{1}{2}\mathbf{E}\big(d(a,i)d(a,j)\big)$$

and we obtain the same behavior (up to a multiplicative constant) as in the case of fractional Brownian motion. That is, using (5.95) and the proof of Theorem 5.22, we will have

Proposition 5.15 *If $Q > 1$ and $H \in (\frac{1}{2}, 1)$ or if $Q = 1$ and $H \in (\frac{1}{2}, \frac{3}{4})$, then*

$$\frac{N}{a}\mathbf{E}\big(T_4^2\big) \to_N 3\ell_1(1,1,H) \tag{5.101}$$

and, if $Q = 1$ and $H \in (\frac{3}{4}, 1)$, then

$$\left(\frac{N}{a}\right)^{4-4H}\mathbf{E}\big(T_4^2\big) \to_N 3\ell_2(H). \tag{5.102}$$

The constants $\ell(1,1,H)$ and $\ell_2(H)$ are given by (5.95), (5.97) respectively.

Asymptotic Behavior of the Term T_2 Recall that we have

$$T_2 = I_2\big(h_N^{(a)}\big)$$

with

$$h_N^{(a)} := 4\frac{1}{a^{2H+1}C_\psi(H)}\frac{1}{N_a}\sum_{i=1}^{N_a} g_{a,i} \otimes_1 g_{a,i}. \tag{5.103}$$

We compute the contraction $g_{a,i} \otimes_1 g_{a,i}$. We have

$$\begin{aligned}
&(g_{a,i} \otimes_1 g_{a,i})(y_1, y_2)\\
&= \int_0^{N_a} g_{a,i}(y_1,z)g_{a,i}(y_2,z)dz\\
&= ad_H^2 1_{[0,a(i+1)]}(y_1)1_{[0,a(i+1)]}(y_2)\\
&\quad\times \int_0^{a(i+1)} dz\bigg[\int_{\frac{y_1\vee z}{a}-i}^1 dx\psi(x)\bigg(\int_{y_1\vee z}^{a(x+i)} \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(u,z)du\bigg)\bigg]\\
&\quad\times \bigg[\int_{\frac{y_2\vee z}{a}-i}^1 dx'\psi\big(x'\big)\bigg(\int_{y_2\vee z}^{a(x'+i)} \partial_1 K^{H'}\big(u',y_2\big)\partial_1 K^{H'}\big(u',z\big)du'\bigg)\bigg]\\
&= ad_H^2 1_{[0,a(i+1)]}(y_1)1_{[0,a(i+1)]}(y_2)\bigg[\int_{\frac{y_1}{a}-i}^1 dx\psi(x)\int_{\frac{y_2}{a}-i}^1 dx'\psi\big(x'\big)\\
&\quad\times \int_{y_1}^{a(x+i)}\int_{y_2}^{a(x'+i)} M\big(u,y_1,u',y_2\big)dudu'\int_0^{u\wedge u'} M\big(u,z,u',z\big)dz\bigg]
\end{aligned}$$

where $M(u,y_1,u',y_2) := \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(u',y_2)$ and $H' = (H+1)/2$. Now, we have already seen that $\int_0^{t\wedge s} K^H(t,z)K^H(s,z)dz = Z^H(t,s)$ with $Z^H(t,s)$ given in (1.1) and therefore

$$\int_0^{u\wedge u'} M\big(u,z,u',z\big)dz = H'\big(2H'-1\big)\big|u-u'\big|^{2H'-2}. \tag{5.104}$$

(In fact, this relation can easily be derived from $\int_0^{u\wedge v} K^{H'}(u,y_1)K^{H'}(v,y_1)dy_1 = R^{H'}(u,v)$, and will be used repeatedly in the sequel.) Thus putting $\alpha_H := H'(2H'-1) = H(H+1)/2$ and since ψ is $[0,1]$-supported, we obtain

$$(g_{a,i} \otimes_1 g_{a,i})(y_1,y_2) = ad_H^2\alpha_H 1_{[0,a(i+1)]}(y_1)1_{[0,a(i+1)]}(y_2) \times \int_{(\frac{y_1}{a}-i)\vee 0}^1 \int_{(\frac{y_2}{a}-i)\vee 0}^1 dxdx'\psi(x)\psi(x') \times \int_{y_1}^{a(x+i)} \int_{y_2}^{a(x'+i)} |u-u'|^{2H'-2} M(u,y_1,u',y_2)dudu'.$$

By direct computation, it is possible to evaluate the expectation of T_2^2. It is as follows

$$N_a^{2-2H}\mathbf{E}T_2^2 \to_N 32\frac{\alpha_H^4 d_H^4}{H(2H-1)C_\psi^2(H)} \times \left(\int_{[0,1]^4} \psi(x)\psi(x')xx'\left|ux-vx'\right|^{2H'-2}dxdx'dudv\right)^2 := C_{T_2}^2(H). \tag{5.105}$$

We shall not prove this estimate here because it is a consequence of the following proposition which shows that the sequence $C_{T_2}^{-1}(H)N_a^{1-H}T_2$ (and therefore the sequence $V_N(a)$) converges in $L^2(\Omega)$ to a Rosenblatt random variable with self-similarity index H.

Proposition 5.16 *Let $(Z_t^H)_{t\geq 0}$ be a Rosenblatt process with self-similarity index $H \in (\frac{1}{2},1)$ and let T_2 be the sequence given by (5.99) and computed from $(Z_t^H)_{t\geq 0}$. Then, for any $Q \geq 1$ and $H \in (\frac{1}{2},1)$, there exists a Rosenblatt random variable Z_1^H with self-similarity order H such that*

$$C_{T_2}^{-1}(H)N_a^{1-H}T_2 \underset{N}{\to} Z_1^H$$

where C_{T_2} is given by (5.105).

Proof This proof follows the lines of the proof of Theorem 5.22. With $T_2 = I_2(h_N^{(a)})$ in mind, as in the proof of Theorem 5.22, a direct proof that the cumulants of the sequence $N_a^{1-H}I_2(h_N^{(a)})$ converge to those of the Rosenblatt process can be given. Indeed, by combining formula (3.16), the proof of Theorem 5.22 and the estimation of the square mean of T_2 we will obtain

$$c_k\left(N_a^{1-H}I_2\left(h_N^{(a)}\right)\right) = c_{a,H}N_a^{k(1-H)}N_a^{-k}\sum_{i_1,\dots,i_k=1}^{N_a}\int_{[0,1]^{4k}}\prod_{j=1}^k \psi(x_j)\psi(x_j')\psi(z_j)\psi(z_j')dx_jdx_j'dz_jdz_j'$$

$$\times \int_{[0,1]^{4k}} du_j du'_j dv_j dv'_j \prod_{j=1}^{k} \left(|u_j x_j - v_j x'_j| \cdot |u'_j z_j - v'_j z'_j| \right)^{2H'-2}$$

$$\times \prod_{j=1}^{k} \left(|u_j x_j - u'_j z_j + i_j - i_{j+1}| \cdot |v_j x'_j - v'_j z'_j + i_k - i_{j+1}| \right)^{2H'-2}$$

with the convention $i_{k+1} := i_1$. The key fact is that the sequence

$$S^k_{N_a} = N_a^{-k} \times \sum_{i_1,\ldots,i_k=1}^{N_a} \prod_{j=1}^{k} \left(\frac{|u_j x_j - u'_j z_j + i_j - i_{j+1}| \cdot |v_j x'_j - v'_j z'_j + i_k - i_{j+1}|}{N_a} \right)^{2H'-2}$$

converges as a Riemann sum (for fixed $x_j, x'_j, z_j, z'_j, u_j, v_j, u'_j, v'_j$) to, modulo a constant, the integral

$$\int_{[0,1]^k} dx_1 \cdots dx_k \left(|x_1 - x_2| \cdot |x_2 - x_3| \cdot \cdots \cdot |x_k - x_1| \right)^{2H'-2}$$

which is the cumulant of Z_1^H. □

We will finally state our main result on the convergence of the wavelet statistic constructed from a Rosenblatt process. Its proof is a consequence of (5.101) and Proposition 5.16.

Theorem 5.23 *Let $(Z_t^H)_{t\geq 0}$ be a Rosenblatt process. Then, for any $Q \geq 1$ and $H \in (\frac{1}{2}, 1)$, there exists a Rosenblatt random variable Z_1^H with self-similarity order H such that*

$$C_{T_2}^{-1}(H) N_a^{1-H} V_N(a) \underset{N}{\rightarrow} Z_1^H,$$

where C_{T_2} is given by (5.105).

It is also possible to give a multidimensional counterpart of Theorem 5.23 in the case when the scale a is the vector $(ai)_{1\leq i\leq m}$.

Theorem 5.24 *Let $V_N(a)$ be the wavelet variation statistic of the Rosenblatt process. Then for every $Q > 1$ and $H \in (\frac{1}{2}, 1)$*

$$\left(\left(\frac{N}{a}\right)^{1-H} V_N(ai) \right)_{1\leq i\leq m} \underset{N}{\rightarrow} \left(Z^H_{1,1}, \ldots, Z^H_{1,m} \right)$$

where $Z^H_{1,j}$ is a Rosenblatt random variable for every $j = 1, \ldots, m$ and

$$\mathbf{E} Z^H_{1,p} Z^H_{1,q} = \frac{c_{p,q,H}}{\sqrt{c_{p,p,H} c_{q,q,H}}}$$

with

$$c_{p,q,H} = 32(pq)^{-1} \frac{\alpha_H^4 d_H^4}{C_\psi^2(H)} \left(\int_{[0,1]^4} \psi(x)\psi(x') x x' \left| ux - vx' \right|^{2H'-2} dx dx' du dv \right)^2 \times \left(\int_0^1 \int_0^1 |px - qy|^{2H-2} dy dx \right).$$

Proof Since for every $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ the linear combination $\sum_{j=1}^m \lambda_j (\frac{N}{a_j})^{1-H} \times V_N(a_j)$ behaves as a multiple integral of order two and it is possible to compute its cumulants by using the formula (3.16) and to show that they converge to the corresponding cumulants of the Rosenblatt vector. □

It is possible and instructive to study the behavior of the term T_4 in the cases $Q > 1$ and $H \in (\frac{1}{2}, 1)$ or $Q = 1$ and $H \in (\frac{1}{2}, \frac{3}{4})$. It can be already seen from its asymptotic variance that it is very close to the Gaussian case. Actually this term converges in law to a Gaussian random variable (see Exercise 5.19). This fact does not influence the limit of the statistic V_N but we find that it is interesting from a theoretical point of view.

5.8 Bibliographical Notes

In martingale theory, the quadratic variations play a crucial role. The construction of the Itô integral is based in a significant measure on this object. For self-similar processes, the original motivation to study the quadratic variations is rather closely related to statistics and to the construction of consistent and asymptotically normal estimators for the self-similarity parameter. This research direction has an old history. We refer to the monographs [28, 68, 75, 160]. Some important papers that first derived the limit of the long range dependent time series are [41, 67, 82, 167, 168].

The motivation to study the quadratic variations based on higher order increments comes from Theorem 5.4 which shows that in the case of fractional Brownian motion the asymptotic normality of the quadratic variation statistic depends on whether the index H is smaller or greater than $\frac{3}{4}$ and this fact is not very convenient for practical purposes (confidence intervals, simulation etc.). The asymptotic normality of the second order quadratic variations for every $H \in (0, 1)$ was first noticed in [85] and [97]. The method was then applied to other Gaussian processes and sequences. Other references related to the study of higher order variations are [27, 57, 58] in the Gaussian case, and [49] in the non-Gaussian case. See also [22–24] for limit theorems for the power variations of fBm and related processes. The wavelet estimator is an alternative approach to the estimator based on quadratic variations. To study self-similar and long range phenomena in data (that is, to put in light their presence and to estimate the relevant parameters) wavelet transforms have proved to be tools of particular interest. There also exists a slightly different approach, based on discrete wavelet transforms and applied to stationary sequences instead of continuous time stochastic processes. We refer, among other references, to [2–4, 54–56, 123, 124, 156, 187]. This research has had a strong impulse in recent years since the publication of the papers [138] and [138] (see also [137]) which brought new tools, based on the Malliavin calculus and multiple stochastic integrals, and found important applications to limit theorems and statistics. Later, the paper [127] by Nourdin and Peccati on Stein's method combined with the Malliavin calculus provided an elegant method to find the rate of convergence in the Central Limit Theorem. A series of papers, by several authors, then followed, with various extensions of the Stein method. We refer, among a long list, to [132] for the multidimensional Stein method, to [130] or [105] for the approximation of other probability distributions, to [135] for a density formula in terms of the Malliavin derivatives, to [38] and [175] for applications to Cramer's theorem, [93] for applications to local times etc. See also the monographs [118, 125, 126] and [141].

5.9 Exercises

Exercise 5.1 Consider the sequences given by (5.43) and (5.44). Prove that these sequences converge in law, as $N \to \infty$, to the standard normal law.

Hint Use Theorem 5.1 in order to prove that the distance between these sequences and the standard normal law tends to zero as $n \to \infty$.

Exercise 5.2 ([48]) Prove Theorem 5.13.

Exercise 5.3 ([181]) Let

$$f_{1,H} := 32 d(H)^4 a(H)^2 \sum_{k=1}^{\infty} k^{2H-2} F\left(\frac{1}{k}\right) \tag{5.106}$$

where the function F is defined by

$$\begin{aligned} F(x) = \int_{[0,1]^4} & du dv du' dv' \left|(u-u')x+1\right|^{2H'-2} \\ & \times \left[a(H)^2 \left(|u-v||u'-v'|\left|(v-v')x+1\right|\right)^{2H'-2}\right. \\ & \left. - 2a(H)\left(|u-v|\left|(v-u')x+1\right|\right)^{2H'-2} + \left|(u-u')x+1\right|^{2H'-2}\right]. \end{aligned} \tag{5.107}$$

Let $(Z(t), t \in [0,1])$ be a Rosenblatt process with self-similarity parameter $H \in (1/2, 2/3)$ and let previous notations for constants prevail. Prove that the following convergence occurs in distribution:

$$\lim_{N\to\infty} \frac{\sqrt{N}}{\sqrt{e_{1,H} + f_{1,H}}} \left[V_N - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1)\right] = N(0,1)$$

where V_N is given by (5.24) and the constants $a(H), d(H), e_{1,H}, f_{1,H}, c_{3,H}$ are those in Sect. 5.2.

Exercise 5.4 Consider the notation from Sect. 5.1.5. Define, for $p = 1, \dots, M$, the sequence $\bar{F}_N(a^p) = b_{p,H} N^{2-2H} V_N(2, a^p)$ where $b_{p,H}$ is a suitable normalizing constant such that $\mathbf{E}(\bar{F}_N(a^p))^2$ converges to 1 as $N \to \infty$. Then show that for $H > \frac{3}{4}$ the vector $(\bar{F}_N(a^1), \dots, F_N(a^M))$ converges, as $N \to \infty$, in $L^2(\Omega)$ to the vector $(Z_1^{2H-1}(1), \dots, Z_M^{2H-1}(1))$ with $Z_p^{2H-1}(1)$ $(p = 1, \dots, M)$ Rosenblatt random variables with self-similarity index $2H - 1$. Give the covariance matrix of the limit.

Exercise 5.5 Suppose that $H > \frac{1}{2}$, $X = Z$ (the Rosenblatt process with self-similarity parameter H) in (5.73) and consider the estimator (5.74). Then, strong consistency holds for $\hat{H}_N$, i.e. almost surely,

$$\lim_{N\to\infty} \hat{H}_N = H. \tag{5.108}$$

In addition, prove that we have the following convergence in $L^2(\Omega)$:

$$\lim_{N\to\infty} \frac{N^{1-H}}{2d(H)} \log(N)\, (\hat{H}_N - H) = Z(1), \tag{5.109}$$

where $Z(1)$ is the Rosenblatt process at time 1.

Exercise 5.6 (see [48]) Study the asymptotic behavior of the estimator (5.74) when X is the Hermite process.

Exercise 5.7 (see [1]) Let $\frac{1}{N}$, $i = 0, \dots, N$ be a partition of the unit interval [0, 1] and let

$$V_N = \sum_{i=0}^{N-1} \left(N^{2HK} \mathbf{E}\big(B^{H,K}_{\frac{i+1}{N}} - B^{H,K}_{\frac{i}{N}}\big)^2 - \theta(i,i) \right)$$

where $B^{H,K}$ is a bi-fBm with $K \in (0, 1]$, $H \in (0, 1)$ and

$$\begin{aligned}\theta(i,j) = 2^{-K}\Big[&\big((i+1)^{2H} + (j+1)^{2H}\big)^K - \big((i+1)^{2H} + j^{2H}\big)^K \\ &- \big(i^{2H} + (j+1)^{2H}\big)^K + \big(i^{2H} + j^{2H}\big)^K + 2^{-K+1}\rho(i-j)\Big]\end{aligned}$$

with ρ given by $\rho(r) = \frac{1}{2}(|r+1|^{2HK} + |r-1|^{2HK} - 2|r|^{2HK})$.

1. Assume $0 < HK < \frac{3}{4}$. Show that

$$\frac{\mathrm{Var}\, V_N}{N} \underset{N}{\to} c_1$$

where c_1 is a positive constant.

2. Assume $HK = \frac{3}{4}$. Prove that

$$\frac{\operatorname{Var} V_N}{N \log N} \underset{N}{\to} c_2$$

where c_2 is a positive constant.
3. Define

$$\tilde{V}_N = \frac{V_N}{\operatorname{Var} V_N}.$$

Show that for any $HK \leq \frac{3}{4}$, the sequence $\tilde{V}_N$ converges in distribution as $N \to \infty$ to the standard normal law.
4. Give a bound for the Kolmogorov distance between the law of $\tilde{V}_N$ and the standard normal law.

Exercise 5.8 ([184]) Let S^H be a sub-fBm and define

$$V_N = \sum_{i=0}^{N-1} \left[N^{2H} \mathbf{E}\left(S^H_{\frac{i+1}{N}} - S^H_{\frac{i}{N}}\right)^2 - \operatorname{Var}\left(S^H_{\frac{i+1}{N}} - S^H_{\frac{i}{N}}\right) \right].$$

1. Define

$$\tilde{V}_N = \frac{V_N}{\operatorname{Var} V_N}.$$

Show that for any $H \leq \frac{3}{4}$, the sequence $\tilde{V}_N$ converges in distribution as $N \to \infty$ to the standard normal law.
2. Give a bound for the Kolmogorov distance between the law of $\tilde{V}_N$ and the standard normal law.

Exercise 5.9 ([114]) Define, for every $N \geq 2, t \geq 0$, the sequence

$$V_N(t) = \sum_{i=0}^{[Nt]-1} \left[\frac{(B^{H_1}_{\frac{i+1}{N}} - B^{H_1}_{\frac{i}{N}})(B^{H_2}_{\frac{i+1}{N}} - B^{H_2}_{\frac{i}{N}})}{\mathbf{E}(B^{H_1}_{\frac{i+1}{N}} - B^{H_1}_{\frac{i}{N}})(B^{H_2}_{\frac{i+1}{N}} - B^{H_2}_{\frac{i}{N}})} - 1 \right]. \tag{5.110}$$

1. Show that, in the case $H_1 = H_2 = H \in (\frac{3}{4}, 1)$, the (renormalized) sequence $(V_N(t))_{t \geq 0}$ converges, as $N \to \infty$, in the sense of finite dimensional distributions, to a symmetric Rosenblatt process with self-similarity parameter $2H - 1$.
2. Show that, after suitable normalization, the sequence (5.110) converges in the sense of finite dimensional distributions to the non-symmetric Rosenblatt process Y^{H_1,H_2} given by (3.25).

Exercise 5.10 ([114]) Let us define, for every $t \geq 0$

$$W_N(t) = N^{1-2H} \sum_{i=0}^{[Nt]-1} \left[\left(B_{i+1}^{H_1} - B_i^{H_1} \right) g \left(B_{i+1}^{H_2} - B_i^{H_2} \right) - c_0 \right] \tag{5.111}$$

where $c_0 = \mathbf{E}[(B_{i+1}^{H_1} - B_i^{H_1}) g(B_{i+1}^{H_2} - B_i^{H_2})]$ and where g is a deterministic function with Hermite rank equal to one which has a finite expansion into Hermite polynomials of the form

$$g(x) = \sum_{q=1}^{M} c_q H_q(x) \tag{5.112}$$

where $M \geq 1$ and H_n denotes the nth Hermite polynomial

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left(\exp\left(-\frac{x^2}{2}\right) \right), \quad x \in \mathbb{R}. \tag{5.113}$$

Consider two fractional Brownian motions B^{H_1} and B^{H_2} given by (1.22) with $H_1 + H_2 = 2H > \frac{3}{2}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a deterministic function given by (5.112) such that for every $q \geq 2$

$$(2H_2 - 2)(q - 1) < -1. \tag{5.114}$$

Prove that the sequence of stochastic processes $(W_N(t))_{t \geq 0}$ converges in the sense of finite dimensional distributions as $N \to \infty$ to the process $c_1 c(H_1, H_2)^{-1}$ $c(H_1) c(H_2) b(H_1, H_2)^{-1} Y^{H_1, H_2}$ with Y^{H_1, H_2} defined in (3.25).

Notice that assumption (5.114) excludes the existence of terms with $q = 2$ in the expansion of g.

Exercise 5.11 ([172]) Consider the Gaussian process $(u(t, x), t \in [0, T], x \in \mathbb{R}^d)$ given by the solution (2.31) to the linear stochastic heat equation driven by a fractional-white noise.

1. Express $u(t, x)$ as a Wiener integral with respect to the Brownian sheet.
2. Express the cumulants of the random variable (5.67) using the formula for the cumulants of a random variable in the second Wiener chaos (see (3.16)).
3. Prove Theorem 5.16.

Exercise 5.12 ([49]) Consider the quadratic variation statistic for a filter α of order p based on the observations of our Rosenblatt process Z:

$$S_N := \frac{1}{N} \sum_{i=\ell}^{N} \left(\sum_{q=0}^{\ell} \alpha_q Z\left(\frac{i-q}{N}\right) \right)^2. \tag{5.115}$$

1. Prove that $\mathbf{E}[S_N] = -\frac{N^{-2H}}{2} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |q - r|^{2H} = c(H)$ where $c(H)$ is as defined in (5.80).

2. Consider the following non-linear equation for fixed N:

$$-\frac{N^{-2x}}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|q-r|^{2x}-S_N(2,\alpha)=0. \tag{5.116}$$

   Show that there exists a non-random value N_0 depending only on α such that if $N\ge N_0$, (5.116) has exactly one solution in $(1/2,1)$.

   Define the estimator $\hat{H}_N$ of H to be the unique solution of (5.116).
3. Prove that for any $H\in(1/2,1)$, almost surely, $\lim_{N\to\infty}N^{2H}S_N=c(H)/2$.
4. Prove that strong consistency holds for $\hat{H}_N$, i.e.

$$\lim_{N\to\infty}\hat{H}_N=H, \quad \text{a.s.}$$

5. Prove that for any $H\in(\frac{1}{2},1)$, we have

$$\lim_{N\to\infty}2c_{2,H}^{-1/2}N^{1-H}(\hat{H}_N-H)\log N=Z(1)$$

   in $L^2(\Omega)$, where $Z(1)$ is a Rosenblatt random variable ($c_{2,H}$ is given by (5.86)).

Exercise 5.13 ([49]) Compute

$$d\big(G_N,N(0,1)\big) \tag{5.117}$$

where G_N is defined by (5.83).

Exercise 5.14 Prove Proposition 5.14.

Exercise 5.15 Let $c_{1,H}$ be the constant (5.88) and T_4 from (5.84). Show that the sequence

$$G_N:=\frac{\sqrt{N}}{c_{1,H}}T_4$$

converges in distribution, as $N\to\infty$, to a standard normal random variable.

Exercise 5.16 Prove Theorem 5.21 using the Malliavin calculus.

Hint Use Theorem 5.1 in order to show that every component of the vector converges in distribution to the normal law. Then use Theorem 5.3 to obtain the convergence of the vector.

Exercise 5.17 Suppose $Q=1$ and $H>\frac{3}{4}$. Consider the sequence $V_N(a)$ (5.91) and let $f_N(a)$ be the kernel of $V_N(a)$.

1. Show that the term $f_N^{(a)}$ can be written as

$$\begin{aligned}
f_N^{(a)}(y_1,y_2) &= \frac{1}{a^{2H}C_\psi(H)}\frac{1}{N_a} \\
&\quad\times\sum_{i=1}^{N_a} 1_{[0,a(i+1)]}(y_1)1_{[0,a(i+1)]}(y_2)\Big(1_{[0,ai]}(y_1)1_{[0,ai]}(y_2) \\
&\quad\times\int_0^1\int_0^1 dxdz\psi(x)\psi(z)K^H\big(a(x+i),y_1\big)K^H\big(a(z+i),y_2\big) \\
&\quad+1_{[0,ai]}(y_1)1_{[ai,a(1+i)]}(y_2) \\
&\quad\times\int_0^1\int_{\frac{y_2}{a}-i}^1 dxdz\psi(x)\psi(z)K^H\big(a(x+i),y_1\big)K^H\big(a(z+i),y_2\big) \\
&\quad+1_{[0,ai]}(y_2)1_{[ai,a(1+i)]}(y_1) \\
&\quad\times\int_{\frac{y_1}{a}-i}^1\int_0^1 dxdz\psi(x)\psi(z)K^H\big(a(x+i),y_1\big)K^H\big(a(z+i),y_2\big) \\
&\quad+1_{[ai,a(1+i)]}(y_1)1_{[ai,a(1+i)]}(y_2)\int_{\frac{y_1}{a}-i}^1\int_{\frac{y_2}{a}-i}^1 dxdz\psi(x)\psi(z)K^H \\
&\quad\times\big(a(x+i),y_1\big)K^H\big(a(z+i),y_2\big)\Big) \\
&:= f_N^{(a,1)}(y_1,y_2)+f_N^{(a,2)}(y_1,y_2)+f_N^{(a,3)}(y_1,y_2)+f_N^{(a,4)}(y_1,y_2).
\end{aligned}$$

2. Show that the terms $N_a^{2-2H}f_N^{(a,2)}$, $N_a^{2-2H}f_N^{(a,3)}$ and $N_a^{2-2H}f_N^{(a,4)}$ converge to zero in $L^2([0,\infty)^2)$ as $N_a\to\infty$.
3. Show that $\ell_2^{-1}(H)N_a^{2-2H}f_N^{(a,1)}$ is equivalent (in the sense that it has the same pointwise limit) to $N^{1-2H}L_N^{2H-1}$ where L_N^{2H-1} is the kernel of the Rosenblatt process with self-similarity index $2H-1$,

$$L_t^H(y_1,y_2):=d_H 1_{[0,t]}(y_1)1_{[0,t]}(y_2)\int_{y_1\vee y_2}^t \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(u,y_2)du, \tag{5.118}$$

with K^H the standard kernel defined in (1.4) and $H'=\frac{H+1}{2}$.
4. Deduce that $\ell_2^{-1}(H)N_a^{2-2H}V_N(a)$ is equivalent to $N^{1-2H}I_2(L_N^{2H-1}) = N^{1-2H}Z_N^{2H-1}=Z_1^{2H-1}$ where the equivalence is asymptotically in law.
5. Prove that the sequence $N^{1-2H}Z_N^{2H-1}$ is not Cauchy in L^2.
6. Deduce that $V_N(a)$ (renormalized) does not converge in L^2 to the Rosenblatt random variable with index $2H-1$ given by (3.17).

Exercise 5.18 Prove the limit (5.105).

Exercise 5.19 Use the notation in Sect. 5.7. Denote by $C_{T_4}(H)$ the positive constant such that

$$C_{T_4}^2(H) := 3\ell(1, 1, H),$$

where $\ell(p, q, H)$ is defined in (5.95).

Suppose that Z^H is a Rosenblatt process with self-similarity order H. Suppose that $Q > 1$ or $Q = 1$ and $H \in (\frac{1}{2}, \frac{3}{4})$. Then prove that

$$\sqrt{N_a} T_4 \to_N \mathcal{N}\big(0, C_{T_4}^2(H)\big).$$

Exercise 5.20 Use the notation in Sect. 5.7. Use the argument of Exercise 5.17 to show that in the Rosenblatt case $V_N(a)$ (5.91) (renormalized) does not converge in L^2 to the Rosenblatt random variable with index H.

Chapter 6
Hermite Variations for Self-similar Processes

The quadratic variation of a stochastic process $(X_t)_{t\in[0,T]}$ defined by the expression (5.1) involves the square of the increments $(X_{t_{i+1}} - X_{t_i})^2$ along a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the time interval $[0, T]$. It can be generalized to p-variations, with $p \geq 2$, meaning that the power two in (5.1) is replaced by a power $p \geq 2$. A variant of the p-variation is the so called *Hermite variation* of order p. This is usually defined as $v_n(X) = \sum_{i=0}^{n-1} H_p(\frac{X_{t_{i+1}} - X_{t_i}}{\sqrt{\mathbf{E}(X_{t_{i+1}} - X_{t_i})^2}})$ where H_p denotes the Hermite polynomial of order p. To employ the method based on multiple stochastic integrals and the Malliavin calculus, it is often more convenient to study the Hermite variations of order p instead of the p-variations.

6.1 Hermite Variations of Fractional Brownian Motion

Consider a fBm $(B_t^H)_{t\in[0,1]}$ with $H \in (0, 1)$ and define its Hermite variations of order q by

$$V_n = \sum_{k=0}^{n-1} H_q\big(n^H (B_{\frac{k+1}{n}} - B_{\frac{k}{n}})\big) \tag{6.1}$$

(in the sequel we will omit the superscript H for B) where H_q is the Hermite polynomial of degree $q \geq 1$ given by (5.113).

The behavior of the sequence V_n (6.1) is as follows.

Theorem 6.1 *Let $q \geq 2$ be an integer and let $(B_t)_{t\geq 0}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Then, with some explicit positive constants $c_{1,q,H}, c_{2,q,H}, c_{3,q,H}$ depending only on q and H, we have:*

(i) *If $0 < H < 1 - \frac{1}{2q}$ then*

$$\frac{V_n}{c_{1,q,H}\sqrt{n}} \xrightarrow[n\to\infty]{\text{Law}} N(0, 1). \tag{6.2}$$

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Moreover, if $Z_n^{(1)} := \frac{V_n}{c_{1,q,H}\sqrt{n}}$

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)} > x\right) - P\left(Z^{(1)} > x\right)\right| \le c \begin{cases} \frac{1}{\sqrt{n}}, & H \in (0, \frac{1}{2}] \\ n^{H-1}, & H \in [\frac{1}{2}, \frac{2q-3}{2q-2}) \\ n^{qH-q+\frac{1}{2}}, & H \in [\frac{2q-3}{2q-2}, 1-\frac{1}{2q}). \end{cases} \tag{6.3}$$

(ii) *If* $1 - \frac{1}{2q} < H < 1$ *then*

$$\frac{V_n}{c_{2,q,H}n^{1-q(1-H)}} \xrightarrow[n\to\infty]{\mathrm{L}^2} Z \tag{6.4}$$

where Z *is a Hermite random variable given by* (3.2). *Moreover, if* $Z_n^{(2)} := \frac{V_n}{c_{2,q,H}n^{1-q(1-H)}}$ *then*

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(2)} > x\right) - P\left(Z^{(2)} > x\right)\right| \le cn^{1-\frac{1}{2q}-H}. \tag{6.5}$$

(iii) *If* $H < 1 - \frac{1}{2q}$ *then*

$$\frac{V_n}{c_{3,q,H}\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathrm{Law}} N(0,1). \tag{6.6}$$

Moreover, if $Z_n^{(3)} := \frac{V_n}{c_{3,q,H}\sqrt{n\log n}}$, *then*

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(3)} > x\right) - P\left(Z^{(3)} > x\right)\right| \le c\frac{1}{\sqrt{\log n}}.$$

The proof of point (i) and (iii) is based on Stein's method combined with the Malliavin calculus. Notice that, since

$$H_q\left(n^H(B_{\frac{k+1}{n}} - B_{\frac{k}{n}})\right) = H_q\left(I_1\left(n^H 1_{(\frac{k}{n},\frac{k+1}{n})}\right)\right) = \frac{1}{q!}n^{qH} I_q\left(1^{\otimes q}_{(\frac{k}{n},\frac{k+1}{n})}\right)$$

the Hermite variation (6.1) can be expressed as a multiple integral of order q and then one can apply Theorem 5.1 in order to obtain a bound for the distance between the law of the renormalized sequence V_n and the standard normal law. To prove point (ii), one can apply Theorem 5.6. We will not give the details (the reader may consult [126] for a detailed study of the Hermite variations of the fBm) but the proof of Theorem 6.1 can be obtained by following the lines of Theorems 6.2 and 6.3 for moving average processes in the next section. Notice that Theorem 6.1 covers the results in Sect. 5.1. for the quadratic variations of fractional Brownian motion (by letting $q = 2$).

6.2 Hermite Variations of the Moving Average Process

We will consider a long memory moving average sequence defined by

$$X_n = \sum_{i \geq 1} a_i \varepsilon_{n-i}, \quad n \in \mathbb{Z}$$

where the innovations ε_i are centered i.i.d. random variables having at least finite second moments and the moving averages a_i are of the form $a_i = i^{-\beta} L(i)$ with $\beta \in (\frac{1}{2}, 1)$ and L slowly varying towards infinity. The covariance function $\rho(m) = \mathbf{E}(X_0 X_m)$ behaves as $c_\beta m^{-2\beta+1}$ when $m \to \infty$ and consequently is not summable since $\beta < 1$. Therefore X_n is usually called a long memory or "long-range dependence" moving average. The long memory moving average processes considered in this part cover the model known as the fractional ARIMA process (cf. [83, 89]), which has motivated considerable interest in applied areas such as econometrics and hydrology (see, e.g., [87, 111]).

Note that the autocorrelation function of the sequence X_m behaves when m goes to infinity as the autocorrelation of the fractional Brownian motion with Hurst parameter $H = \frac{3}{2} - \beta$.

Let K be a deterministic function which has Hermite rank q and satisfies $\mathbf{E}(K^2(X_n)) < \infty$ and define

$$S_N = \sum_{n=1}^{N} \big[K(X_n) - \mathbf{E}\big(K(X_n)\big) \big]. \tag{6.7}$$

Then it has been proven in [88] (see also [192]) that, with $c_1(\beta, q), c_2(\beta, q)$ being positive constants depending only on q and β: (a) if $q > \frac{1}{2\beta-1}$, then the sequence $c_1(\beta, q) \frac{1}{\sqrt{N}} S_N$ converges in law to a standard normal random variable; and (b) if $q < \frac{1}{2\beta-1}$, then the sequence $c_2(\beta, q) N^{\beta q - \frac{q}{2} - 1} S_N$ converges in law to a Hermite random variable of order q. We will prove this result by using the Stein's method and the Malliavin calculus and we will compute the rate of the convergence of the sequence S_N toward its limit.

In order to apply the techniques based on the Malliavin calculus and multiple Wiener-Itô integrals, we will restrict our focus to the following situation: the innovations ε_i are chosen to be the increments of a Brownian motion W on the real line while the function K is a Hermite polynomial of order q. In this case the random variable X_n is a Wiener integral with respect to W, and $H_q(X_n)$ can be expressed as a multiple Wiener-Itô stochastic integral of order q with respect to W.

Here, we will focus on the sequence S_N (6.7) where

$$X_n = \sum_{i=1}^{\infty} \alpha_i (W_{n-i} - W_{n-i-1}), \tag{6.8}$$

with $\alpha_i \in \mathbb{R}, \alpha_i = ci^{-\beta}, \beta \in (\frac{1}{2}, 1)$ and $\sum_{i=1}^{\infty} \alpha_i^2 = 1$. The autocorrelation function of X_n is given by (see Exercise 6.3)

$$\rho(m) := \sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|}. \tag{6.9}$$

Note that X_n can also be written as (I_n is the multiple integral with respect to W)

$$\begin{aligned} X_n &= \sum_{i=1}^{\infty} \alpha_i (W_{n-i-1} - W_{n-i}) = \sum_{i=1}^{\infty} \alpha_i I_1(\mathbf{1}_{[n-i-1,ni]}) \\ &= I_1\underbrace{\left(\sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{[n-i-1,n-i]}\right)}_{f_n} = I_1(f_n). \end{aligned} \tag{6.10}$$

As $K = H_q$, we have

$$S_N = \sum_{n=1}^{N} \big[H_q(X_n) - \mathbf{E}\big(H_q(X_n)\big)\big] = \sum_{n=1}^{N} \big[H_q\big(I_1(f_n)\big) - \mathbf{E}\big(H_q\big(I_1(f_n)\big)\big)\big].$$

We know that, if $\|f\|_{\mathcal{H}} = 1$, we have $H_q(I_1(f)) = \frac{1}{q!} I_q(f^{\otimes q})$. In this part $\mathcal{H}$ will be $L^2(\mathbb{R})$. Furthermore, we have

$$\begin{aligned} \|f_n\|_{\mathcal{H}}^2 = \langle f_n, f_n \rangle_{\mathcal{H}} &= \left\langle \sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{[n-i-1,ni]}, \sum_{r=1}^{\infty} \alpha_r \mathbf{1}_{[n-r-1,nr]} \right\rangle_{\mathcal{H}} \\ &= \sum_{i,r=1}^{\infty} \alpha_i \alpha_r \langle \mathbf{1}_{[n-i-1,n-i]}, \mathbf{1}_{[n-r-1,nr]} \rangle_{\mathcal{H}}. \end{aligned}$$

It is easily verified that if $i > r \Leftrightarrow n - i \le n - r - 1$ or $i < r \Leftrightarrow n - r \le n - i - 1$, we have $[n - i - 1, n - i] \cap [n - r - 1, n - r] = \emptyset$ and thus $\langle \mathbf{1}_{[n-i-1,n-i]}, \mathbf{1}_{[n-r-1,n-r]} \rangle_{\mathcal{H}} = 0$. It follows that

$$\|f_n\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \alpha_i^2 \|\mathbf{1}_{[n-i-1,n-i]}\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \alpha_i^2 = 1.$$

Thanks to this result, S_N can be represented as

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \big[H_q\big(I_1(f_n)\big) - \mathbf{E}\big(H_q\big(I_1(f_n)\big)\big)\big] = \frac{1}{q!} \sum_{n=1}^{N} \big[I_q\big(f_n^{\otimes q}\big) - \mathbf{E}\big(I_q\big(f_n^{\otimes q}\big)\big)\big] \\ &= \frac{1}{q!} \sum_{n=1}^{N} I_q\big(f_n^{\otimes q}\big) = \frac{1}{q!} I_q\left(\sum_{n=1}^{N} f_n^{\otimes q}\right). \end{aligned}$$

6.2.1 Berry-Esséen Bounds for the Central Limit Theorem

We will first focus on the case where $q > (2\beta - 1)^{-1}$, i.e. the Central Limit Theorem. Let $Z_N = \frac{1}{\sigma\sqrt{N}} S_N$ where $\sigma_{q,\beta}$ is given by

$$\sigma := \sigma_{q,\beta}^2 = q! \sum_{m=-\infty}^{+\infty} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q = q! \sum_{m=-\infty}^{+\infty} \rho^q(m). \tag{6.11}$$

The following result gives the Berry-Esséen bounds in the Central Limit Theorem.

Theorem 6.2 *Under the condition $q > (2\beta - 1)^{-1}$, Z_N converges in law towards $Z \sim \mathcal{N}(0,1)$. Moreover, there exists a constant C_β, depending uniquely on β, such that, for any $N \geq 1$,*

$$\sup_{z\in\mathbb{R}} \left| \mathbf{P}(Z_N \leq z) - \mathbf{P}(Z \leq z) \right| \leq C_\beta \begin{cases} N^{\frac{q}{2}+\frac{1}{2}-q\beta} & \text{if } \beta \in (\frac{1}{2}, \frac{q}{2q-2}] \\ N^{\frac{1}{2}-\beta} & \text{if } \beta \in [\frac{q}{2q-2}, 1). \end{cases}$$

Remark 6.1

(a) The same result, modulo a change of the constant, holds for other distances between the laws of random variables (e.g. total variations distance, Wasserstein etc. See Theorem 5.1).
(b) Actually, the condition $\beta \in (\frac{1}{2}, \frac{q}{2q-2}]$ reads $\beta \in (\frac{1}{2q} + \frac{1}{2}, \frac{q}{2q-2}]$ since $q > (2\beta - 1)^{-1}$.

Proof To apply Theorem 5.1, we need to evaluate the quantity

$$\mathbf{E}\left(\left(1 - q^{-1}\|DZ_N\|_{\mathcal{H}}^2\right)^2\right).$$

We will start by computing $\|DZ_N\|_{\mathcal{H}}^2$. We have the following lemma.

Lemma 6.1 *The following result on $\|DZ_N\|_{\mathcal{H}}$ holds.*

$$\frac{1}{q}\|DZ_N\|_{\mathcal{H}}^2 - 1 = \sum_{r=0}^{q-1} A_r(N) - 1$$

where

$$A_r(N) = \frac{qr!}{\sigma^2 N} \binom{q-1}{r}^2 \sum_{k,l=1}^{N} I_{2q-2-2r}\left(f_k^{\otimes q-1-r} \tilde{\otimes} f_l^{\otimes q-1-r}\right) \langle f_k, f_l \rangle_{\mathcal{H}}^{r+1}. \tag{6.12}$$

Proof We have

$$DZ_N = D\left(\frac{1}{\sigma\sqrt{N}}\sum_{n=1}^{N} I_q\big(f_n^{\otimes q}\big)\right) = \frac{q}{\sigma\sqrt{N}}\sum_{n=1}^{N} I_{q-1}\big(f_n^{\otimes q-1}\big) f_n$$

and

$$\|DZ_N\|_{\mathcal{H}}^2 = \frac{q^2}{\sigma^2 N}\sum_{k,l=1}^{N} I_{q-1}\big(f_k^{\otimes q-1}\big) I_{q-1}\big(f_l^{\otimes q-1}\big)\langle f_k, f_l\rangle_{\mathcal{H}}. \tag{6.13}$$

The multiplication formula between multiple stochastic integrals gives us that

$$\begin{aligned} &I_{q-1}\big(f_k^{\otimes q-1}\big) I_{q-1}\big(f_l^{\otimes q-1}\big) \\ &= \sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I_{2q-2-2r}\big(f_k^{\otimes q-1-r}\tilde{\otimes} f_l^{\otimes q-1-r}\big)\langle f_k, f_l\rangle_{\mathcal{H}}^r. \end{aligned}$$

By substituting this into (6.13), we obtain

$$\|DZ_N\|_{\mathcal{H}}^2 = \frac{q^2}{\sigma^2 N}\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 \sum_{k,l=1}^{N} I_{2q-2-2r}\big(f_k^{\otimes q-1-r}\tilde{\otimes} f_l^{\otimes q-1-r}\big)\langle f_k, f_l\rangle_{\mathcal{H}}^{r+1}$$

and the conclusion follows easily. □

By using Lemma 6.1 and the fact that $\mathbf{E}(I_m I_n) = 0$ if $m \neq n$, we can now evaluate $\mathbf{E}((1 - q^{-1}\|DZ_N\|_{\mathcal{H}}^2)^2)$. We have

$$\mathbf{E}\big(\big(1 - q^{-1}\|DZ_N\|_{\mathcal{H}}^2\big)^2\big) = \sum_{r=0}^{q-2}\mathbf{E}\big(A_r^2(N)\big) + \mathbf{E}\big(A_{q-1}(N) - 1\big)^2. \tag{6.14}$$

We need to evaluate the behavior of those two terms as $N \to \infty$, but first, recall that the α_i are of the form $\alpha_i = i^{-\beta}$ with $\beta \in (1/2, 1)$. We will use the notation $a_n \sim b_n$ meaning that a_n and b_n have the same limit as $n \to \infty$ and $a_n \preceq b_n$ meaning that $\sup_{n\geq 1}|a_n|/|b_n| < \infty$. Below is a useful lemma we will use throughout.

Lemma 6.2

1. *We have*

$$\rho(n) \sim c c_\beta n^{-2\beta+1}$$

with $c_\beta = \int_0^\infty y^{-\beta}(y+1)^{-\beta}dy = \boldsymbol{\beta}(2\beta - 1, 1-\beta)$.

2. *For any* $\alpha \in \mathbb{R}$, *we have*

$$\sum_{k=1}^{n-1} k^\alpha \preceq 1 + n^{\alpha+1}.$$

3. *If* $\alpha \in (-\infty, -1)$, *we have*

$$\sum_{k=n}^{\infty} k^{\alpha} \preceq n^{\alpha+1}.$$

Proof Points 2. and 3. follow from [127], Lemma 4.3. We will only prove the first point of the lemma. We know that $\rho(n) = \sum_{i=1}^{\infty} i^{-\beta}(i+|n|)^{-\beta}$ behaves as $\int_0^{\infty} x^{-\beta}(x+|n|)^{-\beta}dx$ and the following holds

$$\begin{aligned}\int_0^{\infty} x^{-\beta}\big(x+|n|\big)^{-\beta}dx &= \int_0^{\infty} x^{-\beta}|n|^{-\beta}\left(\frac{x}{|n|}+1\right)^{-\beta}dx \\ &= |n|^{-2\beta+1}\underbrace{\int_0^{\infty} y^{-\beta}(y+1)^{-\beta}dy}_{c_{\beta}}.\end{aligned}$$

Thus,

$$\rho(n) \sim \sum_{i=1}^{\infty} i^{-\beta}\big(i+|n|\big)^{-\beta} \sim c_{\beta} n^{-2\beta+1}.$$ □

We will start the evaluation of (6.14) with the term $\mathbf{E}(A_{q-1}(N)-1)^2$. Note that we have $\mathbf{E}(A_{q-1}(N)-1)^2 = (A_{q-1}(N)-1)^2$ because $A_{q-1}(N)-1$ is deterministic. We can write

$$A_{q-1}(N)-1 = \frac{q!}{\sigma^2 N}\sum_{k,l=1}^{N}\langle f_k, f_l\rangle_{\mathcal{H}}^q - 1.$$

Note that we have

$$\langle f_k, f_l\rangle_{\mathcal{H}} = \sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|} = \rho(l-k).$$

Hence

$$\begin{aligned}A_{q-1}(N)-1 &= \frac{q!}{\sigma^2 N}\sum_{k,l=1}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q - 1 \\ &= \frac{1}{\sigma^2 N}\left(q!\sum_{k,l=1}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q - N\sigma^2\right) \\ &= \frac{1}{\sigma^2 N}\left(q!\sum_{k,l=1}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q\right. \\ &\qquad \left. - Nq!\sum_{m=-\infty}^{+\infty}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right). \qquad (6.15)\end{aligned}$$

Observe that

$$
\begin{aligned}
&\sum_{k,l=1}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q \\
&=\sum_{k\le l}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q+\sum_{k>l}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q \\
&=\sum_{k=1}^{N}\sum_{l=k}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q+\sum_{l=1}^{N}\sum_{k=l+1}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q.
\end{aligned}
$$

Let $m=l-k$. We obtain

$$
\begin{aligned}
&\sum_{k,l=1}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|l-k|}\right)^q \\
&=\sum_{k=1}^{N}\sum_{m=0}^{N-k}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q+\sum_{l=1}^{N}\sum_{m=-N+l}^{-1}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q \\
&=\sum_{m=0}^{N-1}\sum_{k=1}^{N-m}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q+\sum_{m=-(N-1)}^{-1}\sum_{l=1}^{N+m}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q \\
&=\sum_{m=0}^{N-1}(N-m)\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q+\sum_{m=-(N-1)}^{-1}(N+m)\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q \\
&=N\sum_{m=-(N-1)}^{N-1}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q-2\sum_{m=0}^{N-1}m\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q.
\end{aligned}
$$

Substituting this into (6.15), we get

$$
\begin{aligned}
A_{q-1}(N)-1&=\frac{q!}{\sigma^2N}\left(N\sum_{m=-(N-1)}^{N-1}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q-N\sum_{m=-\infty}^{+\infty}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right. \\
&\qquad\left.-2\sum_{m=0}^{N-1}m\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right) \\
&=\frac{q!}{\sigma^2N}\left(-N\sum_{m=-\infty}^{-N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q-N\sum_{m=N}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right. \\
&\qquad\left.-2\sum_{m=0}^{N-1}m\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right)
\end{aligned}
$$

$$= \frac{q!}{\sigma^2 N}\left(-2N\sum_{m=N}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q - 2\sum_{m=0}^{N-1} m\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right).$$

By noticing that the condition $q > (2\beta-1)^{-1}$ is equivalent to $-q(2\beta-1) < -1$, we can apply Lemma 6.2 to get

$$\begin{aligned} A_{q-1}(N) - 1 &\preceq \sum_{m=N}^{\infty} m^{-q(2\beta-1)} + N^{-1}\sum_{m=0}^{N-1} m^{-q(2\beta-1)+1} \\ &\preceq N^{-q(2\beta-1)+1} + N^{-1}\big(1 + N^{-q(2\beta-1)+2}\big) \end{aligned}$$

and finally

$$A_{q-1}(N) - 1 \preceq N^{-1} + N^{q-2q\beta+1}.$$

Thus, we obtain a bound on $(A_{q-1}(N)-1)^2 = \mathbf{E}(A_{q-1}(N)-1)^2$,

$$\mathbf{E}\big(A_{q-1}(N) - 1\big)^2 \preceq N^{-2} + N^{q-2q\beta} + N^{2q-4q\beta+2}. \tag{6.16}$$

Let us now treat the second term of (6.14), i.e. $\sum_{r=0}^{q-2}\mathbf{E}(A_r^2(N))$. Here we can assume that $r \le q-2$ is fixed. We have

$$\begin{aligned} \mathbf{E}\big(A_r^2(N)\big) &= \mathbf{E}\bigg(\frac{q^2 r!^2}{\sigma^4 N^2}\binom{q-1}{r}^4 \sum_{i,j,k,l=1}^{N} \langle f_k, f_l\rangle_{\mathcal{H}}^{r+1}\langle f_i, f_j\rangle_{\mathcal{H}}^{r+1} \\ &\quad \times I_{2q-2-2r}\big(f_k^{\otimes q-1-r}\widetilde{\otimes} f_l^{\otimes q-1-r}\big) I_{2q-2-2r}\big(f_i^{\otimes q-1-r}\widetilde{\otimes} f_j^{\otimes q-1-r}\big)\bigg) \\ &= c(r,q)N^{-2}\sum_{i,j,k,l=1}^{N}\langle f_k, f_l\rangle_{\mathcal{H}}^{r+1}\langle f_i, f_j\rangle_{\mathcal{H}}^{r+1} \\ &\quad \times \big\langle f_k^{\otimes q-1-r}\widetilde{\otimes} f_l^{\otimes q-1-r}, f_i^{\otimes q-1-r}\widetilde{\otimes} f_j^{\otimes q-1-r}\big\rangle_{\mathcal{H}^{\otimes 2q-2r-2}} \\ &= \sum_{\substack{\alpha,\nu\ge 0\\ \alpha+\nu=q-r-1}}\ \sum_{\substack{\gamma,\delta\ge 0\\ \gamma+\delta=q-r-1}} c(r,q,\alpha,\nu,\gamma,\delta) B_{r,\alpha,\nu,\gamma,\delta}(N) \end{aligned}$$

where

$$\begin{aligned} &B_{r,\alpha,\nu,\gamma,\delta}(N) \\ &\quad = N^{-2}\sum_{i,j,k,l=1}^{N}\langle f_k, f_l\rangle_{\mathcal{H}}^{r+1}\langle f_i, f_j\rangle_{\mathcal{H}}^{r+1}\langle f_k, f_i\rangle_{\mathcal{H}}^{\alpha}\langle f_k, f_j\rangle_{\mathcal{H}}^{\nu}\langle f_l, f_i\rangle_{\mathcal{H}}^{\gamma}\langle f_l, f_j\rangle_{\mathcal{H}}^{\delta} \\ &\quad = N^{-2}\sum_{i,j,k,l=1}^{N}\rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}. \end{aligned}$$

When α,ν,γ and δ are fixed, we can decompose the sum $\sum_{i,j,k,l=1}^{N}$ which appears in $B_{r,\alpha,\nu,\gamma,\delta}(N)$ just above, as follows:

$$\sum_{i=j=k=l} + \Big(\sum_{\substack{i=j=k \\ l\neq i}} + \sum_{\substack{i=j=l \\ k\neq i}} + \sum_{\substack{i=l=k \\ j\neq i}} + \sum_{\substack{j=k=l \\ i\neq j}} \Big) + \Big(\sum_{\substack{i=j,k=l \\ k\neq i}} + \sum_{\substack{i=k,j=l \\ j\neq i}} + \sum_{\substack{i=l,j=k \\ j\neq i}} \Big)$$

$$+ \Big(\sum_{\substack{i=j,k\neq i \\ k\neq l,l\neq i}} + \sum_{\substack{i=k,j\neq i \\ j\neq l,k\neq l}} + \sum_{\substack{i=l,k\neq i \\ k\neq j,j\neq i}} + \sum_{\substack{j=k,k\neq i \\ k\neq l,l\neq i}} + \sum_{\substack{j=l,k\neq i \\ k\neq l,l\neq i}} + \sum_{\substack{k=l,k\neq i \\ k\neq j,j\neq i}} \Big) + \sum_{\substack{i,j,k,l \\ i\neq j\neq k\neq l}} .$$

We will have to evaluate each of these fifteen sums separately. Before that, we will give a useful lemma that we will be using regularly throughout.

Lemma 6.3 *For any $\alpha \in \mathbb{R}$, we have*

$$\sum_{i\neq j=1}^{n} |i-j|^{\alpha} = \sum_{i,j=0}^{n-1} |i-j|^{\alpha} \preceq n \sum_{j=0}^{n-1} j^{\alpha}.$$

Proof The following upper bounds prove this lemma

$$\left| \frac{\sum_{i,j=0}^{n-1} |i-j|^{\alpha}}{n\sum_{j=0}^{n-1} j^{\alpha}} \right| = \left| \frac{\sum_{m=0}^{n-1} (n-m) m^{\alpha}}{n\sum_{j=0}^{n-1} j^{\alpha}} \right| \leq \left| \frac{n\sum_{m=0}^{n-1} m^{\alpha}}{n\sum_{j=0}^{n-1} j^{\alpha}} \right| + \left| \frac{\sum_{m=0}^{n-1} m^{\alpha+1}}{n\sum_{j=0}^{n-1} j^{\alpha}} \right|$$

$$\leq 1 + \left| \frac{\sum_{m=0}^{n-1} m^{\alpha+1}}{\sum_{j=0}^{n-1} j^{\alpha+1}} \right| \leq 2.$$

□

Let us return to our sums and begin by treating the first one. The first sum can be rewritten as

$$N^{-2} \sum_{i=j=k=l} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$

$$= N^{-2} \sum_{i=1}^{N} \rho(0)^{2r+2+\alpha+\nu+\gamma+\delta} = N^{-2} N \preceq N^{-1}.$$

For the second sum, we have

$$N^{-2} \sum_{\substack{i=j=k \\ l\neq i}} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$

$$= N^{-2} \sum_{\substack{i=j=k \\ l\neq i}} \rho(l-i)^{r+1+\gamma+\delta} = N^{-2} \sum_{i\neq l} \rho(l-i)^{q}.$$

At this point, we will use Lemma 6.2 and then Lemma 6.3 to write

$$N^{-2} \sum_{\substack{i=j=k \\ l \neq i}} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$

$$\preceq N^{-2} \sum_{i \neq l=1}^{N} |l-i|^{q(-2\beta+1)} \preceq N^{-1} \sum_{l=1}^{N-1} l^{q(-2\beta+1)} \preceq N^{-1}\left(1+N^{-2\beta q+q+1}\right)$$

$$\preceq N^{-1}+N^{-2\beta q+q}.$$

For the third sum, we are in exactly the same case, therefore we obtain the same bound $N^{-1}+N^{-2\beta q+q}$. The fourth sum can be handled as follows

$$N^{-2} \sum_{\substack{i=k=l \\ j \neq i}} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$

$$= N^{-2} \sum_{\substack{i=k=l \\ j \neq i}} \rho(i-j)^{r+1+\nu+\delta} \preceq N^{-2} \sum_{j \neq i} |i-j|^{(r+1+\nu+\delta)(-2\beta+1)}.$$

Note that $r+1+\nu+\delta \geq 1$, so we get

$$N^{-2} \sum_{\substack{i=k=l \\ j \neq i}} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$

$$\preceq N^{-2} \sum_{j \neq i} |i-j|^{-2\beta+1} \preceq N^{-1} \sum_{j=1}^{N-1} j^{-2\beta+1} \preceq N^{-1}\left(1+N^{-2\beta+2}\right)$$

$$\preceq N^{-1}+N^{-2\beta+1}.$$

For the fifth sum, we are in exactly the same case and we obtain the same bound $N^{-1}+N^{-2\beta+1}$. For the sixth sum, we can proceed as follows

$$N^{-2} \sum_{\substack{i=j,k=l \\ k \neq i}} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$

$$= N^{-2} \sum_{k \neq i} \rho(k-i)^{\alpha+\nu+\gamma+\delta} = N^{-2} \sum_{k \neq i} \rho(k-i)^{2q-2r-2}.$$

Recalling that $r \leq q-2 \Leftrightarrow 2(q-r-1) \geq 2$, we obtain

$$N^{-2} \sum_{\substack{i=j,k=l \\ k \neq i}} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$

$$\preceq N^{-2} \sum_{k \neq i} |k-i|^{(2q-2r-2)(-2\beta+1)} \preceq N^{-2} \sum_{k \neq i} |k-i|^{-4\beta+2} \preceq N^{-1} \sum_{k=1}^{N-1} k^{-4\beta+2}$$

$$\preceq N^{-1}+N^{-4\beta+2}.$$

We obtain the same bound, $N^{-1} + N^{-4\beta+2}$, for the seventh and eighth sums. For the ninth sum, we have to deal with the following quantity:

$$N^{-2} \sum_{\substack{i=j,k\neq i\\ k\neq l, l\neq i}} \rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}$$
$$= N^{-2} \sum_{\substack{k\neq i\\ k\neq l, l\neq i}} \rho(k-l)^{r+1}\rho(k-i)^{q-r-1}\rho(l-i)^{q-r-1}.$$

For $\sum_{\substack{k\neq i\\ k\neq l, l\neq i}}$, observe that it can be decomposed into

$$\sum_{k>l>i} + \sum_{k>i>l} + \sum_{l>i>k} + \sum_{i>l>k} + \sum_{i>k>l}. \tag{6.17}$$

For the first of the above sums, we can write

$$N^{-2} \sum_{k>l>i} \rho(k-l)^{r+1}\rho(k-i)^{q-r-1}\rho(l-i)^{q-r-1}$$
$$\preceq N^{-2} \sum_{k>l>i} (k-l)^{(r+1)(-2\beta+1)}(k-i)^{(q-r-1)(-2\beta+1)}(l-i)^{(q-r-1)(-2\beta+1)}$$
$$\preceq N^{-2} \sum_{k>l>i} (k-l)^{q(-2\beta+1)}(l-i)^{(q-r-1)(-2\beta+1)} \quad \text{since } k-i > k-l$$
$$= N^{-2} \sum_{k}\sum_{l<k} (k-l)^{q(-2\beta+1)} \sum_{i<l} (l-i)^{(q-r-1)(-2\beta+1)}$$
$$\preceq N^{-2} \sum_{k}\sum_{l<k} (k-l)^{q(-2\beta+1)} \sum_{i<l} (l-i)^{-2\beta+1} \quad \text{since } q-r-1 \geq 1$$
$$\preceq N^{-2} \sum_{k=1}^{N}\sum_{l=1}^{k-1} (k-l)^{q(-2\beta+1)} \sum_{i=1}^{l-1} (l-i)^{-2\beta+1}.$$

Note that $\sum_{l=1}^{k-1}(k-l)^{q(-2\beta+1)} = \sum_{l=1}^{k-1} l^{q(-2\beta+1)}$ and that $\sum_{i=1}^{l-1}(l-i)^{-2\beta+1} = \sum_{i=1}^{l-1} i^{-2\beta+1}$. We can also bound the terms $\sum_{l=1}^{k-1} l^{q(-2\beta+1)}$ (resp. $\sum_{i=1}^{l-1} i^{-2\beta+1}$) from above by $\sum_{l=1}^{N-1} l^{q(-2\beta+1)}$ (resp. $\sum_{i=1}^{N-1} i^{-2\beta+1}$). It follows that

$$N^{-2} \sum_{k>l>i} \rho(k-l)^{r+1}\rho(k-i)^{q-r-1}\rho(l-i)^{q-r-1}$$
$$\preceq N^{-2} \sum_{k=1}^{N}\sum_{l=1}^{N-1} l^{q(-2\beta+1)} \sum_{i=1}^{N-1} i^{-2\beta+1}$$

$$\preceq N^{-1}\sum_{l=1}^{N-1} l^{q(-2\beta+1)}\sum_{i=1}^{N-1} i^{-2\beta+1}$$
$$\preceq N^{-1}\big(1+N^{-2\beta q+q+1}\big)\big(1+N^{-2\beta+2}\big)$$
$$\preceq N^{-1}+N^{-2\beta+1}+N^{-2\beta q+q}+N^{-2\beta q-2\beta+2}.$$

Since $-2\beta+1<0$, $-2\beta q+q<0$ and $-2\beta q-2\beta+2<0$, it is easy to check that

$$-2\beta q-2\beta+2<-2\beta q+q<-2\beta+1.$$

Consequently,

$$N^{-2}\sum_{k>l>i}\rho(k-l)^{r+1}\rho(k-i)^{q-r-1}\rho(l-i)^{q-r-1}\preceq N^{-1}+N^{-2\beta+1}.$$

We obtain exactly the same bound $N^{-1}+N^{-2\beta+1}$ for the other terms of the decomposition (6.17) as well as for the tenth, eleventh, twelfth, thirteenth and fourteenth sums by applying the same method.

This leaves us with the last (fifteenth) sum. We can decompose $\sum_{\substack{i,j,k,l\\ i\neq j\neq k\neq l}}$ as follows

$$\sum_{k>l>i>j}+\sum_{k>l>j>i}+\cdots. \tag{6.18}$$

For the first term, we have

$$N^{-2}\sum_{k>l>i>j}\rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}$$
$$\preceq N^{-2}\sum_{k>l>i>j}(k-l)^{q(-2\beta+1)}(i-j)^{(r+1)(-2\beta+1)}(l-i)^{(q-r-1)(-2\beta+1)}$$
$$=N^{-2}\sum_{k}\sum_{l<k}(k-l)^{q(-2\beta+1)}\sum_{i<l}(l-i)^{(q-r-1)(-2\beta+1)}\sum_{j<i}(i-j)^{(r+1)(-2\beta+1)}$$
$$\preceq N^{-1}\sum_{l=1}^{N-1}l^{q(-2\beta+1)}\sum_{i=1}^{N-1}i^{(q-r-1)(-2\beta+1)}\sum_{j=1}^{N-1}j^{(r+1)(-2\beta+1)}$$
$$\preceq N^{-1}\big(1+N^{-2\beta q+q+1}\big)\big(1+N^{(q-r-1)(-2\beta+1)+1}\big)\big(1+N^{(r+1)(-2\beta+1)+1}\big)$$
$$\preceq N^{-1}\big(1+N^{-2\beta q+q+1}\big)$$
$$\times\big(1+N^{(r+1)(-2\beta+1)+1}+N^{q(-2\beta+1)-(r+1)(-2\beta+1)+1}+N^{q(-2\beta+1)+2}\big)$$
$$\preceq N^{-1}\big(1+N^{-2\beta q+q+1}\big)$$
$$\times\big(1+N^{-2\beta+2}+N^{-2\beta+2}+N^{q(-2\beta+1)+2}\big)\quad\text{since } r+1,\, q-r-1\geq 1$$

$$\preceq N^{-1}\big(1 + N^{-2\beta+2} + N^{q(-2\beta+1)+2}\big)$$
$$\preceq N^{-1} + N^{-2\beta+1} + N^{q(-2\beta+1)+1}.$$

We find the same bound $N^{-1} + N^{-2\beta+1} + N^{q(-2\beta+1)+1}$ for the other terms of the decomposition (6.18).

Finally, by combining all these bounds, we find that

$$\max_{r=1,\ldots,q-1} \mathbf{E}\big(A_r^2\big) \preceq N^{-2\beta+1} + N^{q(-2\beta+1)+1},$$

and we obtain

$$\mathbf{E}\left(\left(\frac{1}{q}\|DZ_N\|_{\mathcal{H}}^2 - 1\right)^2\right) \preceq N^{-2\beta+1} + N^{q(-2\beta+1)+1},$$

which allow us to complete the proof. □

Remark 6.2

1. When $q = 2$, $\frac{q}{2q-2} = 1$, so the second line of Theorem 6.2 vanishes. If $q > 2$, both lines exist and $\frac{q}{2q-2} \underset{q\to+\infty}{\longrightarrow} \frac{1}{2}$.
2. When $q < (2\beta - 1)^{-1}$, the sequence Z_N does not converge in law towards $\mathcal{N}(0, 1)$. It converges (with another normalization) to a Hermite random variable.
3. The results in the above theorem are consistent with those found in [127], Theorem 4.1. Indeed, in [127] one works with $Y_n = B_{n+1}^H - B_n^H$ instead of X_n, where B^H is a fractional Brownian motion. Note that the covariance function $\rho'(m) = \mathbf{E}(Y_0 Y_m)$ of Y behaves as m^{2H-2} while, as follows from Lemma 6.2, the covariance of X behaves as $m^{-2\beta+1}$. Thus β corresponds to $\frac{3}{2} - H$. It can be seen that Theorem 6.2 is in concordance with Theorem 6.1 (or Theorem 4.1 in [127]).

6.2.2 Error Bounds in the Non-Central Limit Theorem

We will now turn our attention to the case where $q < (2\beta - 1)^{-1}$, where we will use the total variation distance instead of the Kolmogorov distance because that is the distance which appears in Theorem 5.6.

We will use the scaling property of Brownian motion to introduce a new sequence U_N that has the same law as S_N. Recall that S_N is defined by

$$S_N = \sum_{n=1}^{N} H_q\left(\sum_{i=1}^{\infty} \alpha_i (W_{n-i} - W_{n-i-1})\right).$$

Let U_N be defined by

$$U_N = \sum_{n=1}^{N} H_q \left(\sum_{i=1}^{\infty} \alpha_i N^{\frac{1}{2}} (W_{\frac{n-i}{N}} - W_{\frac{n-i-1}{N}}) \right).$$

Based on the scaling property, U_N has the same law as S_N for every fixed N. We will show that

$$h_{q,\beta}^{-1} N^{\beta q - \frac{q}{2} - 1} S_N \underset{N \to +\infty}{\longrightarrow} Z^{(q)}$$

where $Z^{(q)}$ is a Hermite random variable of order q (it is actually the value at time 1 of the Hermite process of order q with self-similarity index

$$\frac{q}{2} - q\beta + 1$$

defined in (3.2)). Let us first prove the following renormalization result.

Lemma 6.4 *Let*

$$h_{q,\beta}^2 = \frac{2c_\beta^q}{q!(-2\beta q + q + 1)(-2\beta + q + 2)}. \tag{6.19}$$

Then

$$\mathbf{E}\big(h_{q,\beta}^{-1} N^{\beta q - \frac{q}{2} - 1} S_N\big)^2 \underset{N \to +\infty}{\longrightarrow} 1.$$

Proof Define $f_N = \sum_{n=1}^N f_n^{\otimes n}$. Since $S_N = \frac{1}{q!} I_q(f_N)$ we have

$$\begin{aligned}
\mathbf{E}\big(h_{q,\beta}^{-1} N^{\beta q - \frac{q}{2} - 1} S_N\big)^2 &= h_{q,\beta}^{-2} \frac{1}{(q!)} N^{2\beta q - q - 2} \sum_{n,m=1}^{N} \rho\big(|n-m|\big)^q \\
&= h_{q,\beta}^{-2} \frac{1}{(q!)} N^{2\beta q - q - 2} N \rho(0)^q \\
&\quad + 2 h_{q,\beta}^{-2} \frac{1}{(q!)} N^{2\beta q - q - 2} \sum_{n,m=1; n>m}^{N} \rho(n-m)^q \\
&\sim 2 h_{q,\beta}^{-2} \frac{1}{(q!)^2} N^{2\beta q - q - 2} \sum_{n,m=1; n>m}^{N} \rho(n-m)^q
\end{aligned}$$

where for the last equivalence we notice that the diagonal term $h_{q,\beta}^{-2} \frac{1}{(q!)} N^{2\beta q - q - 2} N \rho(0)^q$ converges to zero since $q < \frac{1}{2\beta - 1}$. Therefore, by using the change of indices

$n-m=k$ we can write

$$\begin{aligned}\mathbf{E}\big(h_{q,\beta}^{-1}N^{\beta q-\frac{q}{2}-1}S_N\big)^2 &= h_{q,\beta}^{-2}\frac{1}{(q!)}N^{2\beta q-q-2}\sum_{n,m=1}^{N}\rho\big(|n-m|\big)^q\\ &\sim 2h_{q,\beta}^{-2}\frac{1}{(q!)}N^{2\beta q-q-2}\sum_{k=1}^{N}(N-k)\rho(k)^q\\ &\sim 2h_{q,\beta}^{-2}\frac{c_\beta^q}{(q!)}N^{2\beta q-q-2}\sum_{k=1}^{N}(N-k)k^{-2\beta q+q}\end{aligned}$$

because, according to Lemma 6.2, $\rho(k)$ behaves as $c_\beta k^{-2\beta+1}$ when k goes to ∞. Consequently,

$$\mathbf{E}\big(h_{q,\beta}^{-1}N^{\beta q-\frac{q}{2}-1}S_N\big)^2 \sim 2h_{q,\beta}^{-2}\frac{c_\beta^q}{q!}\frac{1}{N}\sum_{k=1}^{N}\left(1-\frac{k}{N}\right)\left(\frac{k}{N}\right)^{-2\beta q+q}$$

and this converges to 1 as $N\to\infty$ because $\frac{1}{N}\sum_{k=1}^{N}(1-\frac{k}{N})(\frac{k}{N})^{-2\beta q+q}$ converges to

$$\int_0^1(1-x)x^{-2\beta q+q}dx=\frac{1}{(-2\beta q+q+1)(-2\beta q+q+2)}. \qquad \square$$

Let Z_N be defined here by

$$Z_N=N^{\beta q-\frac{q}{2}-1}U_N=N^{\beta q-\frac{q}{2}-1}\sum_{n=1}^{N}H_q\left(\sum_{i=1}^{\infty}\alpha_i N^{\frac{1}{2}}(W_{\frac{n-i}{N}}-W_{\frac{n-i-1}{N}})\right).$$

We also know that $h_{q,\beta}^{-1}Z_N \underset{N\to+\infty}{\longrightarrow} Z^{(q)}$ in law (because U_N has the same law as S_N), with $Z^{(q)}$ given by (3.2). Let us give a proper representation of Z_N as an element of the qth-chaos. We have

$$\begin{aligned}Z_N &= N^{\beta q-\frac{q}{2}-1}\sum_{n=1}^{N}H_q\left(\sum_{i=1}^{\infty}\alpha_i N^{\frac{1}{2}}(W_{\frac{n-i}{N}}-W_{\frac{n-i-1}{N}})\right)\\ &= N^{\beta q-\frac{q}{2}-1}\sum_{n=1}^{N}H_q\left(I_1\left(N^{\frac{1}{2}}\sum_{i=1}^{\infty}\alpha_i\mathbf{1}_{[\frac{n-i-1}{N},\frac{n-i}{N}]}\right)\right)\\ &= N^{\beta q-\frac{q}{2}-1}\sum_{n=1}^{N}\frac{1}{q!}I_q\left(\left(N^{\frac{1}{2}}\sum_{i=1}^{\infty}\alpha_i\mathbf{1}_{[\frac{n-i-1}{N},\frac{n-i}{N}]}\right)^{\otimes q}\right)\\ &= \frac{1}{q!}I_q\left(N^{\beta q-1}\sum_{n=1}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\mathbf{1}_{[\frac{n-i-1}{N},\frac{n-i}{N}]}\right)^{\otimes q}\right)\end{aligned}$$

$$:= \frac{1}{q!} I_q \Bigg(\underbrace{N^{\beta q-1} \sum_{n=1}^{N} g_n^{\otimes q}}_{g_N} \Bigg)$$

with $g_n = \sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{[\frac{n-i-1}{N}, \frac{n-i}{N}]}$ and $g_N = N^{\beta q-1} \sum_{n=1}^{N} g_n^{\otimes q} \in \mathcal{H}^{\odot q}$. We will see that $h_{q,\beta}^{-1} Z_N$ converges towards $Z^{(q)}$ in $L^2(\Omega)$, or equivalently that $\{\frac{1}{q!} h_{q,\beta}^{-1} g_N\}_{N \geq 1}$ converges in $L^2(\mathbb{R}^{\otimes q}) = \mathcal{H}^{\otimes q}$ to the kernel

$$g(y_1, \dots, y_q) = h_{q,\beta}^{-1} \int_{y_1 \vee \dots \vee y_q}^{1} du (u - y_1)_+^{-\beta} \cdots (u - y_q)_+^{-\beta} \tag{6.20}$$

(which is the kernel of the Hermite random variable, see (3.2)) by computing the following L^2-norm

$$\mathbf{E}\big(\big|h_{q,\beta}^{-1} Z_N - Z^{(q)}\big|^2\big) = \mathbf{E}\left(\left|I_q\left(\frac{1}{q!} h_{q,\beta}^{-1} g_N\right) - I_q(g)\right|^2\right) = q! \left\| \frac{1}{q!} h_{q,\beta}^{-1} g_N - g \right\|_{\mathcal{H}^{\otimes q}}^2.$$

We will now study $\|g_N - g\|_{\mathcal{H}^{\otimes q}}^2$ and establish the rate of convergence of this quantity.

Proposition 6.1 *We have*

$$\left\| h_{q,\beta}^{-1} \frac{1}{q!} g_N - g \right\|_{\mathcal{H}^{\otimes q}}^2 = \mathcal{O}\big(N^{2\beta q - q - 1}\big).$$

In particular the sequence $h_{q,\beta}^{-1} \frac{1}{q!} g_N$ converges in $L^2(\mathbb{R}^{\otimes q})$ as $N \to \infty$ to the kernel of the Hermite process g (6.20).

Proof We have

$$\begin{aligned}
\|g_N\|_{\mathcal{H}^{\otimes q}}^2 &= N^{2\beta q-2} \sum_{n,k=1}^{N} \langle g_n, g_k \rangle_{\mathcal{H}}^q \\
&= N^{2\beta q-2} \sum_{n,k=1}^{N} \left(\int_{\mathbb{R}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_i \alpha_j \mathbf{1}_{[\frac{n-i-1}{N}, \frac{n-i}{N}]}(u) \mathbf{1}_{[\frac{k-j-1}{N}, \frac{k-j}{N}]}(u) du \right)^q \\
&= N^{2\beta q-2} \sum_{n,k=1}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|n-k|} \int_{\frac{n-i-1}{N}}^{\frac{n-i}{N}} du \right)^q \\
&= N^{2\beta q-q-2} \sum_{n,k=1}^{N} \left(\sum_{i=1}^{\infty} i^{-\beta} \big(i + |n-k|\big)^{-\beta} \right)^q .
\end{aligned} \tag{6.21}$$

In addition, based on the definition of the Hermite process (3.2) (with self-similarity order $\frac{q}{2}-\beta+1$ and with the notation $d(q,\beta)=c(\frac{q}{2}-\beta+1,q)$ in (3.4)), we have

$$d(q,\beta)^2 q!\|g\|^2_{\mathcal{H}^{\otimes q}}=1.$$

Let us now compute the scalar product $\langle g_N, g\rangle^{\otimes q}_{\mathcal{H}}$ where g is given by (6.20). We have

$$\begin{aligned}
&\langle g_N, g\rangle_{\mathcal{H}^{\otimes q}}\\
&=d(q,\beta)N^{\beta q-1}\sum_{n=1}^{N}\langle g_n^{\otimes q}, g\rangle_{\mathcal{H}^{\otimes q}}\\
&=d(q,\beta)N^{\beta q-1}\sum_{n=1}^{N}\int_0^1\left(\sum_{i\geq 1}\alpha_i\int_{\mathbb{R}}(u-y)_+^{-\beta}1_{(\frac{n-i-1}{N},\frac{n-i}{N}]}(y)dy\right)^q du\\
&=d(q,\beta)N^{\beta q-1}\sum_{n=1}^{N}\sum_{k=1}^{N}\int_{\frac{k-1}{N}}^{\frac{k}{N}}\left(\sum_{i\geq 1}\alpha_i\int_{\mathbb{R}}(u-y)_+^{-\beta}1_{(\frac{n-i-1}{N},\frac{n-i}{N}]}(y)dy\right)^q du.
\end{aligned}$$

We will now perform the change of variables $u'=(u-\frac{k-1}{N})N$ and $y'=(y-\frac{n-i-1}{N})N$ (renaming the variables u and y), obtaining

$$\begin{aligned}
\langle g_N, g\rangle_{\mathcal{H}^{\otimes q}}&=d(q,\beta)N^{\beta q-1}N^{-q-1}\\
&\quad\times\sum_{n=1}^{N}\sum_{k=1}^{N}\int_0^1\left(\sum_{i\geq 1}\alpha_i\int_0^1\left(\frac{u-y+k-n+i}{N}\right)_+^{-\beta}dy\right)^q du\\
&\sim d(q,\beta)N^{\beta q-q-2}\sum_{n=1}^{N}\sum_{k=1}^{N-1}\left(\sum_{i\geq 1}\alpha_i\left(\frac{k-n+i}{N}\right)_+^{-\beta}\right)^q
\end{aligned}$$

where we used the fact that, when $N\to\infty$, the quantity $\frac{u-y}{N}$ is negligible. Hence, by eliminating the diagonal term as above,

$$\begin{aligned}
\langle g_N, g\rangle_{\mathcal{H}^{\otimes q}}&\sim d(q,\beta)N^{2\beta q-q-2}\sum_{k,n=1;k>n}\left(\sum_{i\geq 1}\alpha_i(i+k-n)^{-\beta}\right)^q\\
&\quad+d(q,\beta)N^{2\beta q-q-2}\sum_{k,n=1;k<n}\left(\sum_{i\geq n-k}\alpha_i(i+k-n)^{-\beta}\right)^q
\end{aligned}$$

and by using the change of indices $k-n=l$ in the first summand above and $n-k=l$ in the second summand we observe that

$$\langle g_N, g\rangle_{\mathcal{H}^{\otimes q}} \sim d(q,\beta)N^{2\beta q-q-2}\sum_{l=1}^{N}(N-l)\left(\sum_{i\geq 1} i^{-\beta}(i+l)^{-\beta}\right)^q$$
$$+\, d(q,\beta)N^{2\beta q-q-2}\sum_{l=1}^{N}(N-l)\left(\sum_{i\geq l} i^{-\beta}(i-l)^{-\beta}\right)^q. \qquad (6.22)$$

By summarizing the above estimates (6.21) and (6.22), we establish that

$$\begin{aligned}
&\left\| h_{q,\beta}^{-1}\frac{1}{q!}g_N - g\right\|^2_{\mathcal{H}^{\otimes q}}\\
&\sim N^{2\beta q-q-1}\Bigg[2h_{q,\beta}^{-2}\frac{1}{(q!)^2}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq 1} i^{-\beta}(i+k)^{-\beta}\right)^q\\
&\quad - 2d(q,\beta)h_{q,\beta}^{-1}\frac{1}{q!}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq 1} i^{-\beta}(i+k)^{-\beta}\right)^q\\
&\quad - 2d(q,\beta)h_{q,\beta}^{-1}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq k} i^{-\beta}(i-k)^{-\beta}\right)^q\\
&\quad + \frac{1}{d(q,\beta)^2 q!}N^{-2\beta q+q+1}\Bigg].
\end{aligned}$$

To obtain the conclusion, it suffices to check that the sequence

$$\begin{aligned}
a_N := {}& 2h_{q,\beta}^{-2}\frac{1}{(q!)^2}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq 1} i^{-\beta}(i+k)^{-\beta}\right)^q\\
&-2d(q,\beta)h_{q,\beta}^{-1}\frac{1}{q!}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq 1} i^{-\beta}(i+k)^{-\beta}\right)^q\\
&-2d(q,\beta)h_{q,\beta}^{-1}\frac{1}{q!}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq k} i^{-\beta}(i-k)^{-\beta}\right)^q + \frac{1}{q!}N^{-2\beta q+q+1}
\end{aligned}$$

is uniformly bounded by a constant with respect to N. Since $d(q,\beta)h_{q,\beta}^{-1} = \frac{1}{q!}h_{q,\beta}^{-2}$, $\sum_{i\geq 1} i^{-\beta}(i+k)^{-\beta} \sim c_\beta k^{-2\beta q+q}$ and

$$\sum_{i\geq k} i^{-\beta}(i-k)^{-\beta} = \sum_{i\geq 1} i^{-\beta}(i+k)^{-\beta}$$

(by the change of notation $i-k=j$), the sequence a_N can be written as

$$a_N \sim \frac{1}{q!}\Bigg(-(-2\beta q+q+1)(-2\beta q+q+2)\frac{1}{N} \times \sum_{k=1}^{N}(N-k)k^{-2\beta q+q} + N^{-2\beta q+q+1}\Bigg).$$

It is easy to check that

$$\begin{aligned} N^{-2\beta q+q+1} &= N^{-2\beta q+q+1}(-2\beta q+q+1)(-2\beta q+q+2)\int_0^1 (1-x)x^{-2\beta q+q}dx \\ &= (-2\beta q+q+1)(-2\beta q+q+2)\frac{1}{N}\int_0^N (N-y)y^{-2\beta q+q}dy \end{aligned}$$

(by the change of variables $xN=y$). Thus,

$$\begin{aligned} q!a_N &\sim c\frac{1}{N}\sum_{k=1}^{N}\int_{k-1}^{k} dy\big((N-y)y^{-2\beta q+q}-(N-k)k^{-2\beta q+q}\big) \\ &\le \sum_{k=1}^{N}\int_{k-1}^{k} dy\big|y^{-2\beta q+q}-k^{-2\beta q+q}\big| \\ &\quad + \frac{1}{N}\sum_{k=1}^{N}\int_{k-1}^{k} dy\big|y^{-2\beta q+q+1}-k^{-2\beta q+q+1}\big| \\ &\le \sum_{k=1}^{N}\big((k-1)^{-2\beta q+q}-k^{-2\beta q+q}\big) + \frac{1}{N}\sum_{k=1}^{N}\big(k^{-2\beta q+q+1}-(k-1)^{-2\beta q+q+1}\big) \end{aligned}$$

and elementary computations show that the terms in the last line above are of order $N^{-2\beta q+q+1}$. □

As a consequence of Proposition 6.1 and of Theorem 5.6, we obtain

Theorem 6.3 *Let $q < \frac{1}{2\beta-1}$ and let S_N be given by (6.7).*

$$d_{TV}\big(h_{q,\beta}^{-1}N^{\beta q-\frac{q}{2}-1}S_N, Z^{(q)}\big) \le C_0(q,\beta)N^{2\beta q-q-1}$$

where $Z^{(q)}$ is a Hermite random variable with self-similarity index $H=\frac{q}{2}-q\beta+1$ given by (3.2), $h_{q,\beta}$ is given by (6.19) and $C_0(q,\beta)$ is a positive constant.

6.3 Hsu-Robbins and Spitzer's Theorems for Fractional Brownian Motion

A famous result by Hsu and Robbins [91] says that if $X_1, X_2, \dots$ is a sequence of independent identically distributed random variables with zero mean and finite variance and $S_n := X_1 + \cdots + X_n$, then

$$\sum_{n \geq 1} P\big(|S_n| > \varepsilon n\big) < \infty$$

for every $\varepsilon > 0$. Later, Erdös ([73, 74]) showed that the converse implication also holds, namely if the above series is finite for every $\varepsilon > 0$ and $X_1, X_2, \dots$ are independent and identically distributed, then $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. Since then, many authors have extended this result in several directions.

Spitzer showed in [165] that

$$\sum_{n \geq 1} \frac{1}{n} P\big(|S_n| > \varepsilon n\big) < \infty$$

for every $\varepsilon > 0$ if and only if $\mathbf{E}X_1 = 0$ and $\mathbf{E}|X_1| < \infty$. Spitzer's theorem has also been the object of various generalizations and variants. One of the problems related to Hsu-Robbins' and Spitzer's theorems is to find the precise asymptotic as $\varepsilon \to 0$ of the quantities $\sum_{n \geq 1} P(|S_n| > \varepsilon n)$ and $\sum_{n \geq 1} \frac{1}{n} P(|S_n| > \varepsilon n)$. Heyde [86] showed that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n \geq 1} P\big(|S_n| > \varepsilon n\big) = \mathbf{E}X_1^2 \tag{6.23}$$

whenever $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$.

Our purpose is to prove Hsu-Robbins' and Spitzer's theorems for sequences of correlated random variables, related to the increments of fractional Brownian motion, in the spirit of [86]. This gives a better picture of the convergences in Theorem 6.1. Concretely, we will study the behavior of the tail probabilities of the sequence (6.1). Recall that the sequence V_n behaves as follows (see Theorem 6.1): if $0 < H < 1 - \frac{1}{2q}$, a central limit theorem holds for the renormalized sequence $Z_n^{(1)} = \frac{V_n}{c_{1,q,H}\sqrt{n}}$ while if $1 - \frac{1}{2q} < H < 1$, the sequence $Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}}$ converges in $L^2(\Omega)$ to a Hermite random variable of order q.

We note that the techniques generally used in the literature to prove the Hsu-Robbins and Spitzer's results are strongly related to the independence of the random variables $X_1, X_2, \dots$. In our case the variables are correlated. Indeed, for any $k, l \geq 1$ we have $\mathbf{E}(H_q(B_{k+1} - B_k) H_q(B_{l+1} - B_l)) = \frac{1}{(q!)^2} \rho_H(k-l)$ where the correlation function is $\rho_H(k) = \frac{1}{2}((k+1)^{2H} + (k-1)^{2H} - 2k^{2H})$ which is not equal to zero unless $H = \frac{1}{2}$ (which is the case for standard Brownian motion). We use new techniques based on the estimates for the multiple Wiener-Itô integrals obtained in

Theorem 5.1 and Theorem 5.6. Concretely, we study the behavior as $\varepsilon \to 0$ of the quantities

$$\sum_{n\geq 1} \frac{1}{n} P(V_n > \varepsilon n) = \sum_{n\geq 1} \frac{1}{n} P\big(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\big) \tag{6.24}$$

and

$$\sum_{n\geq 1} P(V_n > \varepsilon n) = \sum_{n\geq 1} P\big(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\big) \tag{6.25}$$

if $0 < H < 1 - \frac{1}{2q}$ and of

$$\sum_{n\geq 1} \frac{1}{n} P\big(V_n > \varepsilon n^{2-2q(1-H)}\big) = \sum_{n\geq 1} \frac{1}{n} P\big(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\big) \tag{6.26}$$

and

$$\sum_{n\geq 1} P\big(V_n > \varepsilon n^{2-2q(1-H)}\big) = \sum_{n\geq 1} P\big(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\big) \tag{6.27}$$

if $1 - \frac{1}{2q} < H < 1$. The basic idea in the proofs is that, if we replace $Z_n^{(1)}$ and $Z_n^{(2)}$ by their limits (standard normal random variable or Hermite random variable) in the above expressions, the behavior as $\varepsilon \to 0$ can be obtained by standard calculations. Then we need to estimate the difference between the tail probabilities of $Z_n^{(1)}$, $Z_n^{(2)}$ and the tail probabilities of their limits. To this end, we will use the estimates obtained in Theorems 5.1 and 5.6 via the Malliavin calculus and we will be able to prove that this difference converges to zero in all cases.

6.3.1 Spitzer's Theorem

We set

$$Z_n^{(1)} = \frac{V_n}{c_{1,q,H}\sqrt{n}}, \qquad Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \tag{6.28}$$

with the constants $c_{1,q,H}, c_{2,q,H}$ from Theorem 6.1.

For every $\varepsilon > 0$ let

$$f_1(\varepsilon) = \sum_{n\geq 1} \frac{1}{n} P(V_n > \varepsilon n) = \sum_{n\geq 1} \frac{1}{n} P\big(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\big) \tag{6.29}$$

and

$$f_2(\varepsilon) = \sum_{n\geq 1} \frac{1}{n} P\big(V_n > \varepsilon n^{2-2q(1-H)}\big) = \sum_{n\geq 1} \frac{1}{n} P\big(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\big). \tag{6.30}$$

Remark 6.3 It is natural to consider the tail probability of order $n^{2-2q(1-H)}$ in (6.30) because the L^2-norm of the sequence V_n is in this case of order $n^{1-q(1-H)}$.

We are interested in studying the behavior of $f_i(\varepsilon)$ $(i = 1, 2)$ as $\varepsilon \to 0$. For a given random variable X, we set $\Phi_X(z) = 1 - P(X < z) + P(X < -z)$.

The first lemma gives the asymptotics of the functions $f_i(\varepsilon)$ as $\varepsilon \to 0$ when $Z_n^{(i)}$ are replaced by their limits.

Lemma 6.5 *Let* $c > 0$.

(i) *Let* $Z^{(1)}$ *be a standard normal random variable. Then*

$$\frac{1}{-\log c\varepsilon} \sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) \underset{\varepsilon\to 0}{\longrightarrow} 2.$$

(ii) *Let* $Z^{(2)}$ *be a Hermite random variable of order* q *given by* (3.2). *Then, for any integer* $q \geq 1$

$$\frac{1}{-\log c\varepsilon} \sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(2)}}\left(c\varepsilon n^{1-q(1-H)}\right) \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{1-q(1-H)}.$$

Proof We can write (see [164])

$$\sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) = \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx - \frac{1}{2}\Phi_{Z^{(1)}}(c\varepsilon)$$
$$- \int_1^\infty P_1(x) d\left[\frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right]$$

with $P_1(x) = [x] - x + \frac{1}{2}$. Clearly as $\varepsilon \to 0$, $\frac{1}{\log \varepsilon}\Phi_{Z^{(1)}}(c\varepsilon) \to 0$ because $\Phi_{Z^{(1)}}$ is a bounded function and regarding the last term it is also trivial to observe that

$$\frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) d\left[\frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right]$$
$$= \frac{1}{-\log c\varepsilon}\left(-\int_1^\infty P_1(x)\left(\frac{1}{x^2}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx\right.\right.$$
$$\left.\left. + c\varepsilon \frac{1}{2} x^{-\frac{1}{2}} \frac{1}{x} \Phi'_{Z^{(1)}}(\varepsilon\sqrt{x})\right)dx\right) \underset{\varepsilon\to 0}{\longrightarrow} 0$$

since $\Phi_{Z^{(1)}}$ and $\Phi'_{Z^{(1)}}$ are bounded. Therefore the asymptotics of the function $f_1(\varepsilon)$ as $\varepsilon \to 0$ will be given by $\int_1^\infty \frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx$. By making the change of variables

$c\varepsilon\sqrt{x} = y$, we get

$$\lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx = \lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} 2 \int_{c\varepsilon}^\infty \frac{1}{y} \Phi_{Z^{(1)}}(y)dy$$
$$= \lim_{\varepsilon\to 0} 2\Phi_{Z^{(1)}}(c\varepsilon) = 2.$$

Let us consider now the case of the Hermite random variable. We will have as above

$$\lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} \sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(2)}}\left(c\varepsilon n^{1-q(1-H)}\right)$$
$$= \lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} \left(\int_1^\infty \frac{1}{x} \Phi_{Z^{(2)}}\left(c\varepsilon x^{1-q(1-H)}\right) dx - \int_1^\infty P_1(x) d\left[\frac{1}{x} \Phi_{Z^{(2)}}\left(c\varepsilon x^{1-q(1-H)}\right)\right]\right).$$

By making the change of variables $c\varepsilon x^{1-q(1-H)} = y$ we will obtain

$$\lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} \int_1^\infty \frac{1}{x} \Phi_{Z^{(2)}}\left(c\varepsilon x^{1-q(1-H)}\right) dx$$
$$= \lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} \frac{1}{1-q(1-H)} \int_{c\varepsilon}^\infty \frac{1}{y} \Phi_{Z^{(2)}}(y)dy$$
$$= \lim_{\varepsilon\to 0} \frac{1}{1-q(1-H)} \Phi_{Z^{(2)}}(c\varepsilon) = \frac{1}{1-q(1-H)}$$

where we used the fact that $\Phi_{Z^{(2)}}(y) \leq y^{-2}\mathbf{E}|Z^{(2)}|^2$ and so $\lim_{y\to\infty} \log y \Phi_{Z^{(2)}}(y) = 0$.

It remains to show that $\frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) d[\frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})]$ converges to zero as ε tends to 0. This is equal to

$$\lim_{\varepsilon} \frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) c\varepsilon \big(1-q(1-H)\big) x^{-q(1-H)-1} \Phi'_{Z^{(2)}}\left(c\varepsilon x^{1-q(1-H)}\right) dx$$
$$= c\frac{\varepsilon}{-\log\varepsilon} (c\varepsilon)^{\frac{q(1-H)}{1-q(1-H)}} \int_{c\varepsilon}^\infty P_1\left(\left(\frac{y}{c\varepsilon}\right)^{\frac{1}{1-q(1-H)}}\right) \Phi'_{Z^{(2)}}(y) y^{-\frac{1}{1-q(1-H)}} dy$$
$$\leq c\frac{1}{-\log\varepsilon} \int_{c\varepsilon}^\infty P_1\left(\left(\frac{1}{c\varepsilon}\right)^{\frac{1}{1-q(1-H)}}\right) \Phi'_{Z^{(2)}}(y)dy$$

which clearly tends to zero since P_1 is bounded and $\int_0^\infty \Phi'_{Z^{(2)}}(y)dy = 1$. □

The next result estimates the limit of the difference between the functions $f_i(\varepsilon)$ given by (6.29), (6.30) and the sequence in Lemma 6.5.

Proposition 6.2 *Let $q \geq 2$ and $c > 0$.*

(i) *If $H < 1 - \frac{1}{2q}$, let $Z_n^{(1)}$ be given by (6.28) and let $Z^{(1)}$ be a standard normal random variable. Then*

$$\frac{1}{-\log c\varepsilon}\left[\sum_{n\geq 1}\frac{1}{n}P\left(\left|Z_n^{(1)}\right| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(\left|Z^{(1)}\right| > c\varepsilon\sqrt{n}\right)\right] \xrightarrow[\varepsilon\to 0]{} 0.$$

(ii) *Let $Z^{(2)}$ be a Hermite random variable of order $q \geq 2$ and $H > 1 - \frac{1}{2q}$. Then*

$$\begin{aligned}\frac{1}{-\log c\varepsilon}\Bigg[&\sum_{n\geq 1}\frac{1}{n}P\left(\left|Z_n^{(2)}\right| > c\varepsilon n^{1-q(1-H)}\right)\\ &-\sum_{n\geq 1}\frac{1}{n}P\left(\left|Z^{(2)}\right| > c\varepsilon n^{1-q(1-H)}\right)\Bigg] \xrightarrow[\varepsilon\to 0]{} 0.\end{aligned}$$

Proof Let us start with point (i). Assume $H < 1 - \frac{1}{2q}$. We can write

$$\begin{aligned}&\sum_{n\geq 1}\frac{1}{n}P\left(\left|Z_n^{(1)}\right| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(\left|Z^{(1)}\right| > c\varepsilon\sqrt{n}\right)\\ &\quad= \sum_{n\geq 1}\frac{1}{n}\left[P\left(Z_n^{(1)} > c\varepsilon\sqrt{n}\right) - P\left(Z^{(1)} > c\varepsilon\sqrt{n}\right)\right]\\ &\qquad+\sum_{n\geq 1}\left[\frac{1}{n}P\left(Z_n^{(1)} < -c\varepsilon\sqrt{n}\right) - P\left(Z^{(1)} < -c\varepsilon\sqrt{n}\right)\right]\\ &\quad\leq 2\sum_{n\geq 1}\frac{1}{n}\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)} > x\right) - P\left(Z^{(1)} > x\right)\right|.\end{aligned}$$

Using point (i) of Theorem 6.1 we obtain

$$\begin{aligned}&\sum_{n\geq 1}\frac{1}{n}\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)} > x\right) - P\left(Z^{(1)} > x\right)\right|\\ &\quad\leq c\begin{cases}\sum_{n\geq 1}\frac{1}{n\sqrt{n}}, & H\in(0,\frac{1}{2}]\\ \sum_{n\geq 1}n^{H-2}, & H\in[\frac{1}{2},\frac{2q-3}{2q-2})\\ \sum_{n\geq 1}n^{qH-q-\frac{1}{2}}, & H\in[\frac{2q-3}{2q-2}, 1-\frac{1}{2q})\end{cases}\end{aligned}\tag{6.31}$$

and the last sums are finite (for the last one we use $H < 1 - \frac{1}{2q}$). The conclusion follows.

Concerning point (ii) (the case $H > 1 - \frac{1}{2q}$), by point (ii) in Theorem 6.1 (relation (6.5)) we have

$$\sum_{n\geq 1} \frac{1}{n} P\left(\left|Z_n^{(2)}\right| > c\varepsilon n^{1-q(1-H)}\right) - \sum_{n\geq 1} \frac{1}{n} P\left(\left|Z^{(2)}\right| > c\varepsilon n^{1-q(1-H)}\right) \leq c \sum_{n\geq 1} n^{-\frac{1}{2q}-H}$$

and the above series is convergent because $H > 1 - \frac{1}{2q}$. □

We now state Spitzer's theorem for variations of fractional Brownian motion.

Theorem 6.4 *Let* f_1, f_2 *be given by* (6.29), (6.30) *and the constants* $c_{1,q,H}, c_{2,q,H}$ *be those from Theorem* 6.1.

(i) *If* $0 < H < 1 - \frac{1}{2q}$ *then*

$$\lim_{\varepsilon \to 0} \frac{1}{\log(c_{1,H,q}^{-1}\varepsilon)} f_1(\varepsilon) = 2.$$

(ii) *If* $1 > H > 1 - \frac{1}{2q}$ *then*

$$\lim_{\varepsilon \to 0} \frac{1}{\log(c_{2,H,q}^{-1}\varepsilon)} f_2(\varepsilon) = \frac{1}{1 - q(1-H)}.$$

Proof This is a consequence of Lemma 6.5 and Proposition 6.2. □

Remark 6.4 Concerning the case $H = 1 - \frac{1}{2q}$, note that the correct normalization of V_n (6.1) is $\frac{1}{(\log n)\sqrt{n}}$. Because of the appearance of the term $\log n$ our approach is not directly applicable to this case.

6.3.2 The Hsu-Robbins Theorem

In this section we prove a version of the Hsu-Robbins theorem for variations of fractional Brownian motion. Concretely, for every $\varepsilon > 0$ we let

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n\right) \tag{6.32}$$

if $H < 1 - \frac{1}{2q}$ and

$$g_2(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2-2q(1-H)}\right) \tag{6.33}$$

if $H > 1 - \frac{1}{2q}$ and we estimate the behavior of the functions $g_i(\varepsilon)$ as $\varepsilon \to 0$. Note that we can write

$$g_1(\varepsilon) = \sum_{n\geq 1} P\big(\big|Z_n^{(1)}\big| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\big),$$
$$g_2(\varepsilon) = \sum_{n\geq 1} P\big(\big|Z_n^{(2)}\big| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\big)$$

with $Z_n^{(1)}, Z_n^{(2)}$ given by (6.28).

We decompose it as: for $H < 1 - \frac{1}{2q}$

$$\begin{aligned} g_1(\varepsilon) = {} & \sum_{n\geq 1} P\big(\big|Z^{(1)}\big| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\big) \\ & + \sum_{n\geq 1}\big[P\big(\big|Z_n^{(1)}\big| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\big) - P\big(\big|Z^{(1)}\big| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\big)\big] \end{aligned}$$

and for $H > 1 - \frac{1}{2q}$

$$\begin{aligned} g_2(\varepsilon) = {} & \sum_{n\geq 1} P\big(\big|Z^{(2)}\big| > \varepsilon c_{2,q,H}^{-1} n^{1-q(1-H)}\big) \\ & + \sum_{n\geq 1}\big[P\big(\big|Z_n^{(2)}\big| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\big) - P\big(\big|Z^{(2)}\big| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\big)\big]. \end{aligned}$$

We start again by consider the situation when the $Z_n^{(i)}$ are replaced by their limits.

Lemma 6.6

(i) *Let $Z^{(1)}$ be a standard normal random variable. Then*

$$\lim_{\varepsilon\to 0}(c\varepsilon)^2 \sum_{n\geq 1} P\big(\big|Z^{(1)}\big| > c\varepsilon\sqrt{n}\big) = 1.$$

(ii) *Let $Z^{(2)}$ be a Hermite random variable with $H > 1 - \frac{1}{2q}$. Then*

$$\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n\geq 1} P\big(\big|Z^{(2)}\big| > c\varepsilon n^{1-q(1-H)}\big) = \mathbf{E}\big|Z^{(2)}\big|^{\frac{1}{1-q(1-H)}}.$$

Proof Part (i) is a consequence of the result of Heyde [86]. Indeed take $X_i \sim N(0,1)$ in (6.23). Concerning part (ii) we can write

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}}\sum_{n\geq 1}\Phi_{Z^{(2)}}\big(c\varepsilon n^{1-q(1-H)}\big)\\
&=\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}}\bigg[\int_1^\infty \Phi_{Z^{(2)}}\big(c\varepsilon x^{1-q(1-H)}\big)dx\\
&\quad-\int_1^\infty P_1(x)d\big[\Phi_{Z^{(2)}}\big(c\varepsilon x^{1-q(1-H)}\big)\big]\bigg]\\
&:=\lim_{\varepsilon\to 0}\big(A(\varepsilon)+B(\varepsilon)\big)
\end{aligned}$$

with $P_1(x)=[x]-x+\frac{1}{2}$. Moreover

$$\begin{aligned}
A(\varepsilon)&=(c\varepsilon)^{\frac{1}{1-q(1-H)}}\int_1^\infty \Phi_{Z^{(2)}}\big(c\varepsilon x^{1-q(1-H)}\big)dx\\
&=\frac{1}{1-q(1-H)}\int_{c\varepsilon}^\infty \Phi_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}-1}dy.
\end{aligned}$$

Since $\Phi_{Z^{(2)}}(y)\leq y^{-2}$ we have $\Phi_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}}\to_{y\to\infty}0$ and therefore

$$A(\varepsilon)=-\Phi_{Z^{(2)}}(c\varepsilon)(c\varepsilon)^{\frac{1}{1-q(1-H)}}-\int_{c\varepsilon}^\infty \Phi'_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}}dy$$

where the first terms tends to zero and the second to $\mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}$. The proof that the term $B(\varepsilon)$ converges to zero is similar to the proof of Lemma 6.6, point (ii). □

Remark 6.5 The Hermite random variable has moments of all orders (in particular the moment of order $\frac{1}{1-q(1-H)}$ exists) since it is the value at time 1 of a self-similar process with self-similarity.

Proposition 6.3

(i) *Let $H<1-\frac{1}{2q}$ and let $Z_n^{(1)}$ be given by (6.28). Let also $Z^{(1)}$ be a standard normal random variable. Then*

$$(c\varepsilon)^2\sum_{n\geq 1}\big[P\big(\big|Z_n^{(1)}\big|>c\varepsilon\sqrt{n}\big)-P\big(\big|Z^{(1)}\big|>c\varepsilon\sqrt{n}\big)\big]\xrightarrow[\varepsilon\to 0]{}0.$$

(ii) *Let $H>1-\frac{1}{2q}$ and let $Z_n^{(2)}$ be given by (6.28). Let $Z^{(2)}$ be a Hermite random variable. Then*

$$\begin{aligned}
&(c\varepsilon)^{\frac{1}{1-q(1-H)}}\sum_{n\geq 1}\big[P\big(\big|Z_n^{(2)}\big|>c\varepsilon n^{1-q(1-H)}\big)\\
&\quad-P\big(\big|Z^{(2)}\big|>c\varepsilon n^{1-q(1-H)}\big)\big]\xrightarrow[\varepsilon\to 0]{}0.
\end{aligned}$$

Remark 6.6 Note that the bounds (6.3) and (6.5) do not help here because the series that appear after their use are not convergent.

Proof *Case* $H < 1 - \frac{1}{2q}$. We have, for some $\beta > 0$ to be chosen later,

$$\begin{aligned}
&\varepsilon^2 \sum_{n\geq 1} \big[P\big(\big|Z_n^{(1)}\big| > c\varepsilon\sqrt{n}\big) - P\big(\big|Z^{(1)}\big| > c\varepsilon\sqrt{n}\big)\big] \\
&= \varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} \big[P\big(\big|Z_n^{(1)}\big| > c\varepsilon\sqrt{n}\big) - P\big(\big|Z^{(1)}\big| > c\varepsilon\sqrt{n}\big)\big] \\
&\quad + \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} \big[P\big(\big|Z_n^{(1)}\big| > c\varepsilon\sqrt{n}\big) - P\big(\big|Z^{(1)}\big| > c\varepsilon\sqrt{n}\big)\big] \\
&:= I_1(\varepsilon) + J_1(\varepsilon).
\end{aligned}$$

Consider first the situation when $H \in (0, \frac{1}{2}]$. Let us choose a real number β such that $2 < \beta < 4$. By using (6.3),

$$I_1(\varepsilon) \leq c\varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} n^{-\frac{1}{2}} \leq c\varepsilon^2 \varepsilon^{-\frac{\beta}{2}} \xrightarrow[\varepsilon\to 0]{} 0$$

since $\beta < 4$. Next, by using the bound for the tail probabilities of multiple integrals and since $\mathbf{E}|Z_n^{(1)}|^2$ converges to 1 as $n \to \infty$

$$\begin{aligned}
J_1(\varepsilon) &= \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} P\big(Z_n^{(1)} > c\varepsilon\sqrt{n}\big) \leq c\varepsilon^{-2} \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\frac{-c\varepsilon\sqrt{n}}{(\mathbf{E}|Z_n^{(1)}|^2)^{\frac{1}{2}}}\right)^{\frac{2}{q}} \\
&\leq \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} \exp\big(\big(-cn^{-\frac{1}{\beta}}\sqrt{n}\big)^{\frac{2}{q}}\big)
\end{aligned}$$

which converges to zero for $\beta > 2$. The same argument shows that $\varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} P(Z^{(1)} > c\varepsilon\sqrt{n})$ converges to zero.

The case when $H \in (\frac{1}{2}, \frac{2q-3}{2q-2})$ can be obtained by taking $2 < \beta < \frac{2}{H}$ (which is possible since $H < 1$) while in the case $H \in (\frac{2q-3}{2q-2}, 1 - \frac{1}{2q})$ we have to choose $2 < \beta < \frac{2}{qH-q+\frac{3}{2}}$ (which is possible because $H < 1 - \frac{1}{2q}$!).

Case $H > 1 - \frac{1}{2q}$. We have, for some suitable $\beta > 0$

$$\begin{aligned}
&\varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n\geq 1} \big[P\big(\big|Z_n^{(2)}\big| > c\varepsilon n^{1-q(1-H)}\big) - P\big(\big|Z^{(2)}\big| > c\varepsilon n^{1-q(1-H)}\big)\big] \\
&= \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n=1}^{[\varepsilon^{-\beta}]} \big[P\big(\big|Z_n^{(2)}\big| > c\varepsilon n^{1-q(1-H)}\big) - P\big(\big|Z^{(2)}\big| > c\varepsilon n^{1-q(1-H)}\big)\big]
\end{aligned}$$

$$+\,\varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n\geq[\varepsilon^{-\beta}]} \left[P\left(\left|Z_n^{(2)}\right| > c\varepsilon n^{1-q(1-H)}\right) - P\left(\left|Z^{(2)}\right| > c\varepsilon n^{1-q(1-H)}\right)\right]$$

$$:= I_2(\varepsilon) + J_2(\varepsilon).$$

Choose $\frac{1}{1-q(1-H)} < \beta < \frac{1}{(1-q(1-H))(2-H-\frac{1}{2q})}$ (again, this is always possible when $H > 1 - \frac{1}{2q}$!). Then

$$I_2(\varepsilon) \leq c e^{\frac{1}{1-q(1-H)}} \varepsilon^{(-\beta)(2-H-\frac{1}{2q})} \xrightarrow[\varepsilon\to 0]{} 0$$

and by (C.6)

$$J_2(\varepsilon) \leq c \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\left(\frac{-c\varepsilon n^{1-q(1-H)}}{(\mathbf{E}|Z_n^{(2)}|^2)^{\frac{1}{2}}}\right)^{\frac{2}{q}}\right)$$

$$\leq c \sum_{n>[\varepsilon^{-\beta}]} \exp\left(cn^{-\frac{1}{\beta}} n^{1-q(1-H)}\right)^{\frac{2}{q}} \xrightarrow[\varepsilon\to 0]{} 0.$$

□

We state the main result of this section which is a consequence of Lemma 6.6 and Proposition 6.3.

Theorem 6.5 *Let $q \geq 2$ and let $c_{1,q,H}, c_{2,q,H}$ be the constants from Theorem 6.1. Let $Z^{(1)}$ be a standard normal random variable, $Z^{(2)}$ a Hermite random variable of order $q \geq 2$ and let g_1, g_2 be given by (6.32) and (6.33). Then*

(i) *If $0 < H < 1 - \frac{1}{2q}$, we have $(c_{1,q,H}^{-1}\varepsilon)^2 g_1(\varepsilon) \to_{\varepsilon\to 0} 1 = \mathbf{E}(Z^{(1)})^2$.*

(ii) *If $1 - \frac{1}{2q} < H < 1$, we have $(c_{2,q,H}^{-1}\varepsilon)^{\frac{1}{1-q(1-H)}} g_2(\varepsilon) \to_{\varepsilon\to 0} \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}$.*

Remark 6.7 In the case $H = \frac{1}{2}$ we retrieve the result (6.23) of [86]. The case $q = 1$ is trivial, because in this case, since $V_n = B_n$ and $\mathbf{E}V_n^2 = n^{2H}$, we obtain the following (by applying Lemma 6.5 and 6.6 with $q = 1$)

$$\frac{1}{\log \varepsilon} \sum_{n\geq 1} \frac{1}{n} P\left(|V_n| > \varepsilon n^{2H}\right) \xrightarrow[\varepsilon\to 0]{} \frac{1}{H}$$

and

$$\varepsilon^2 \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2H}\right) \to_{\varepsilon\to 0} \mathbf{E}\left|Z^{(1)}\right|^{\frac{1}{H}}.$$

6.4 Hermite Variations of the Fractional Brownian Sheet

Let $(W_{s,t}^{\alpha,\beta})_{s,t\geq 0}$ be an anisotropic fractional Brownian sheet with Hurst parameter $(\alpha, \beta) \in (0, 1)^2$ given in Definition 4.1. The reader should refer to Sect. 4.1 for its basic properties. We introduce a two-parameter counterpart of the sequence (6.1). Actually, we will define the Hermite variations of order $q \geq 1$ of the fractional Brownian sheet by

$$V_{N,M} := \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} H_q\big(N^\alpha M^\beta \big(W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}}\big)\big), \quad (6.34)$$

where H_q is the Hermite polynomial of order q. Note that the self-similarity and the stationarity of the increments of the fractional Brownian sheet (Propositions 4.1 and 4.2) imply that

$$\mathbf{E}\big(W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}}\big)^2 = N^{-2\alpha} M^{-2\beta},$$

which explains the appearance of the factor $N^\alpha M^\beta$ in (6.34): with this factor the random variable $N^\alpha M^\beta (W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}})$ has L^2-norm equal to 1.

We will use the notation

$$\Delta i = \left[\frac{i}{N}, \frac{i+1}{N}\right] \quad \text{and} \quad \Delta i, j = \left[\frac{i}{N}, \frac{i+1}{N}\right] \times \left[\frac{j}{M}, \frac{j+1}{M}\right] = \Delta i \times \Delta j,$$

for $i \in \{0, \dots, N-1\}$, $j \in \{0, \dots, M-1\}$. In principle $\Delta i = \Delta i^{(N)}$ depends on N but we will omit the superscript N to simplify the notation. With this notation we can write

$$\begin{aligned} W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}} &= I_1(\mathbf{1}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}) \\ &= I_1(\mathbf{1}_{\Delta i,j}) = I_1(\mathbf{1}_{\Delta i \times \Delta j}). \end{aligned}$$

Here, and in the sequel, I_n indicates the multiple integral of order $n > 1$ with respect to the fractional Brownian sheet $W^{\alpha,\beta}$. Since for any deterministic function $h \in \mathcal{H}^{\alpha,\beta}$ ($\mathcal{H}^{\alpha,\beta}$ represents the canonical Hilbert space associated with the fractional Brownian sheet, see Sect. 4.1) with norm one we have

$$H_q\big(I_1(h)\big) = \frac{1}{q!} I_q\big(h^{\otimes q}\big),$$

we derive

$$V_{N,M} = \frac{1}{q!} \sum_{i=1}^{N} \sum_{j=1}^{M} N^{\alpha q} M^{\beta q} I_q\big(\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\big).$$

We want to study the limit of the (suitably normalized) sequence $V_{N,M}$ as $N, M \to \infty$. Since this normalization depends on the choice of α and β, we will normalize it with a function $\varphi(\alpha, \beta, N, M)$.

Let us define

$$\tilde{V}_{N,M} := \frac{1}{q!}\varphi(\alpha,\beta,N,M)\sum_{i=1}^{N}\sum_{j=1}^{M} I_q\big(\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\big). \tag{6.35}$$

By a renormalization of the sequence $V_{N,M}$ we understand a function $\varphi(\alpha,\beta,N,M)$ fulfilling the property $\mathbf{E}\tilde{V}^2_{N,M} \overset{N,M\to\infty}{\to} 1$.

It turns out that the limit of the sequence $\tilde{V}_{N,M}$ is either Gaussian, or a Hermite random variable, which is the value at time $(1,1)$ of a two-parameter Hermite process.

In the case when $\tilde{V}_{N,M}$ converges to a Gaussian random variable, our proof will be based on Stein's bound given in Theorem 5.1. Throughout d will denote one of the distances mentioned in Theorem 5.1. We will also assume that $q \geq 2$ because for $q=1$ we have $H_1 = x$ and then $V_{N,M}$ is Gaussian; this case is trivial. As in the previous chapters, our argumentation has the following structure. We first compute the Malliavin derivative (with respect to the fractional Brownian sheet $W^{\alpha,\beta}$) $D\tilde{V}_{N,M}$ and we compute its norm in the space $\mathcal{H}^{\alpha,\beta}$. We will get

$$D\tilde{V}_{N,M} = \frac{1}{(q-1)!}\varphi(\alpha,\beta,N,M)\sum_{i=1}^{N}\sum_{j=1}^{M} I_{q-1}\big(\mathbf{1}^{\otimes q-1}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\big)\mathbf{1}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]},$$

and

$$\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}} = \frac{1}{(q-1)!^2}\big(\varphi(\alpha,\beta,N,M)\big)^2 \times \sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle_{\mathcal{H}^{\alpha,\beta}}$$
$$\times I_{q-1}\big(\mathbf{1}^{\otimes q-1}_{\Delta i,j}\big) I_{q-1}\big(\mathbf{1}^{\otimes q-1}_{\Delta i',j'}\big).$$

The product formula for multiple integrals (C.4) reads

$$I_{q-1}\big(\mathbf{1}^{\otimes q-1}_{\Delta i,j}\big) I_{q-1}\big(\mathbf{1}^{\otimes q-1}_{\Delta i',j'}\big) = \sum_{p=0}^{q-1} p!\big(C^p_{q-1}\big)^2 \langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^p_{\mathcal{H}^{\alpha,\beta}}$$
$$\times I_{2q-2-2p}\big(\mathbf{1}^{\otimes q-1-p}_{\Delta i,j}\tilde{\otimes}\mathbf{1}^{\otimes q-1-p}_{\Delta i',j'}\big),$$

where $C^p_{q-1} := \binom{q-1}{p}$ for $q \geq 2$, $p \leq q-1$ and $f\tilde{\otimes}g$ denotes the symmetrization of the function $f \otimes g$. Hence, we have

$$\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}} = \frac{1}{(q-1)!^2}\big(\varphi(\alpha,\beta,N,M)\big)^2$$
$$\times \sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\sum_{p=0}^{q-1}\langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}} p!$$
$$\times \big(C^p_{q-1}\big)^2 I_{2q-2-2p}\big(\mathbf{1}^{\otimes q-1-p}_{\Delta i,j}\tilde{\otimes}\mathbf{1}^{\otimes q-1-p}_{\Delta i',j'}\big).$$

Let us isolate the term $p = q - 1$ in the above expression. In this case $2q - 2 - 2p = 0$ and this term gives the expectation of $\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}}$.

$$\begin{aligned}\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}} &= \frac{1}{(q-1)!^2}\big(\varphi(\alpha,\beta,N,M)\big)^2 \qquad (6.36)\\ &\quad\times \sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\sum_{p=0}^{q-2}\langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}} p!\big(C^p_{q-1}\big)^2 I_{2q-2-2p}\\ &\quad\times \big(\mathbf{1}^{\otimes q-1-p}_{\Delta i,j}\tilde{\otimes}\mathbf{1}^{\otimes q-1-p}_{\Delta i',j'}\big)\\ &\quad+ \frac{1}{(q-1)!}\big(\varphi(\alpha,\beta,N,M)\big)^2\sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{q}_{\mathcal{H}^{\alpha,\beta}}\\ &=: T_1 + T_2. \qquad (6.37)\end{aligned}$$

The term T_2 is a deterministic term which is equal to $\mathbf{E}\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}}$.

With the correct choice of normalization we will show that T_2 converges to q as N, M goes to infinity and T_1 converges to zero in the L^2 sense. Using Theorem 5.1 we will prove the convergence to a standard normal random variable of $\tilde{V}_{N,M}$ and we give bounds for the speed of convergence. The distinction between the two cases (when the limit is normal and when the limit is non-Gaussian) will be made by the term T_1: it converges to zero if $\alpha \le 1 - \frac{1}{2q}$ or $\beta \le 1 - \frac{1}{2q}$, while for $\alpha, \beta > 1 - \frac{1}{2q}$ this term converges to a constant.

Let us first discuss the normalization $\varphi(\alpha,\beta,N,M)$ and the convergence of T_2. Given two sequences of real numbers $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$, recall that we write $a_n \trianglelefteq b_n$ for $\sup_{n\ge1}\frac{|a_n|}{|b_n|} < \infty$.

Lemma 6.7 *Let γ in $(0,1)$ and q be an integer with $q \ge 2$. We set*

$$r_\gamma(z) := \frac{1}{2}\big(|z+1|^{2\gamma} + |z-1|^{2\gamma} - 2|z|^{2\gamma}\big), \quad z \in \mathbb{Z}.$$

We have:

(i) *If $0 < \gamma < 1 - \frac{1}{2q}$, then*

$$\lim_{N\to\infty} N^{2\gamma q-1}\sum_{i,i'=0}^{N-1}\langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'}\rangle^q_{\mathcal{H}^\gamma} = \sum_{r\in\mathbb{Z}} r_\gamma(z)^q =: s_\gamma,$$

and $|N^{2\gamma q-1}\sum_{i,i'=0}^{N-1}\langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'}\rangle^q_{\mathcal{H}^\gamma} - s_\gamma| \trianglelefteq N^{-1} + N^{2q\gamma-2q+1}$.

(ii) *If $\gamma = 1 - \frac{1}{2q}$, then*

$$\lim_{N\to\infty}\log(N)^{-1}N^{2q-2}\sum_{i,i'=0}^{N-1}\langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'}\rangle^q_{\mathcal{H}^\gamma} = 2\left(\frac{(2q-1)(q-1)}{2q^2}\right)^q =: \iota_\gamma,$$

and $|\log(N)^{-1}N^{2q-2}\sum_{i,i'=0}^{N-1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}^q-\iota_\gamma|\trianglelefteq\log(N)^{-1}$.

(iii) *If* $\gamma>1-\frac{1}{2q}$, *then*

$$\lim_{N\to\infty}N^{2q-2}\sum_{i,i'=0}^{N-1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}^q=\frac{\gamma^q(2\gamma-1)^q}{(\gamma q-q+1)(2\gamma q-2q+1)}=:\kappa_\gamma,$$

and $|N^{2q-2}\sum_{i,i'=0}^{N-1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}^q-\kappa_\gamma|\trianglelefteq N^{2q-1-2\gamma q}$.

Proof The first two claims can be found respectively in [127, p. 102] and [39, pp. 491–492]. For the third part we define $f_N:=N^{q-1}\sum_{k=0}^{N-1}\mathbf{1}_{[\frac{k}{N},\frac{k+1}{N}]}^{\otimes q}$. Then f_N is a Cauchy sequence in $(\mathcal{H}^\gamma)^{\otimes q}$ with limit f and $\|f\|^2_{(H^\gamma)^{\otimes q}}=\kappa_\gamma$. For the rate of convergence we have

$$\begin{aligned}\|f_N\|^2_{(H^\gamma)^{\otimes q}}-\|f\|^2_{(H^\gamma)^{\otimes q}}&=\|f_N-f\|^2_{(H^\gamma)^{\otimes q}}+2\langle f_N-f,f\rangle_{(H^\gamma)^{\otimes q}}\\&\le\|f_N-f\|^2_{(H^\gamma)^{\otimes q}}+2\|f_N-f\|^2_{(H^\gamma)^{\otimes q}}\|f\|^2_{(H^\gamma)^{\otimes q}}.\end{aligned}$$

The reader should refer to [39, Proposition 3.1] for the details and to get that the order is $O(N^{2q-1-2\gamma q})$ (a direct argument as in the proof of the next lemma can be also employed). □

We also state the following estimates which have been obtained respectively in [127, pp. 102, 104] and in [39, pp. 491–492].

Lemma 6.8 *Let* γ *in* $(0,1)$. *We let* q,p,a,b *be integers such that*: $q\ge 2$, $p\in\{0,\dots,q-2\}$ *and* $a+b=q-1-p$. *We have*:

(i) *If* $0<\gamma<1-\frac{1}{2q}$, *then*

$$\begin{aligned}&N^{4q\gamma-2}\sum_{i,i',k,k'=0}^{N-1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta k},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{a}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{b}\\&\quad\times\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{b}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{a}\\&\quad\trianglelefteq N^{-1}+N^{2\gamma-2}+N^{2\gamma q-2q+1}.\end{aligned}$$

(ii) *If* $\gamma=1-\frac{1}{2q}$, *then*

$$\begin{aligned}&\frac{N^{4q-4}}{\log(N)^2}\sum_{i,i',k,k'=0}^{N-1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta k},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{a}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{b}\\&\quad\times\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{b}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{a}\\&\quad\trianglelefteq\log(N)^{-1}.\end{aligned}$$

(iii) *If* $\gamma > 1 - \frac{1}{2q}$, *then*

$$N^{4q-4} \sum_{i,i',k,k'=0}^{N-1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^\gamma}^{p+1} \langle \mathbf{1}_{\Delta k}, \mathbf{1}_{\Delta k'} \rangle_{\mathcal{H}^\gamma}^{p+1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k} \rangle_{\mathcal{H}^\gamma}^{a} \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k'} \rangle_{\mathcal{H}^\gamma}^{b}$$
$$\times \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k} \rangle_{\mathcal{H}^\gamma}^{b} \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k'} \rangle_{\mathcal{H}^\gamma}^{a}$$
$$\trianglelefteq 1.$$

Proof The first point is proved in [127, pp. 102, 105]. Point (ii) is proved in [39, pp. 491–492]. The proofs follows the lines of Sect. 6.2. The last case can be treated in the following way. The quantity $\langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^\gamma}$ is equivalent to a constant times $N^{-2\gamma}|i-i'|^{2\gamma-2}$ and the sum appearing in (iii) is then equivalent to

$$N^{4q-4} N^{-4\gamma q} \sum_{i,i',k,k'=0}^{N-1} \left|i-i'\right|^{(2\gamma-2)(p+1)} \left|k-k'\right|^{(2\gamma-2)(p+1)}$$
$$\times |i-k|^{(2\gamma-2)a} \left|i'-k'\right|^{(2\gamma-2)b} \left|i'-k\right|^{(2\gamma-2)c} \left|i-k'\right|^{(2\gamma-2)d}$$
$$= N^{-4} \sum_{i,i',k,k'=0}^{N-1} N^{-2q(2\gamma-2)} \left|i-i'\right|^{(2\gamma-2)(p+1)} \left|k-k'\right|^{(2\gamma-2)(p+1)}$$
$$\times |i-k|^{(2\gamma-2)a} \left|i'-k'\right|^{(2\gamma-2)b} \left|i'-k\right|^{(2\gamma-2)c} \left|i-k'\right|^{(2\gamma-2)d},$$

for N large enough and this is a Riemann sum which converges to a constant. □

Lemma 6.9 *Let* T_2 *be as in* (6.37). *Then* $q^{-1}T_2 \overset{N,M\to\infty}{\to} 1$ *for the following choices of* φ:

(1) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{s_\alpha s_\beta}} N^{\alpha q-1/2} M^{\alpha q-1/2}$, *if* $0 < \alpha, \beta < 1 - \frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq N^{-1} + N^{2q\alpha-2q+1} + M^{-1} + M^{2q\beta-2q+1}$;
(2) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{s_\alpha \iota_\beta}} N^{\alpha q-1} M^{q-1} (\log M)^{-1/2}$, *if* $0 < \alpha < 1 - \frac{1}{2q}$, $\beta = 1 - \frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq N^{-1} + N^{2q\alpha-2q+1} + (\log M)^{-1}$;
(3) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{\iota_\alpha \iota_\beta}} N^{q-1} (\log N)^{-1/2} M^{q-1} (\log M)^{-1/2}$, *if* $\alpha = \beta = 1 - \frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq (\log N)^{-1} + (\log M)^{-1}$;
(4) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{s_\alpha \kappa_\beta}} N^{\alpha q-1/2} M^{q-1}$, *if* $0 < \alpha < 1 - \frac{1}{2q}$, $\beta > 1 - \frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq N^{-1} + N^{-2q\alpha+2q-1} + M^{-2q\beta+2q-1}$;
(5) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{\iota_\alpha \kappa_\beta}} N^{q-1} (\log N)^{-1/2} M^{q-1}$, *if* $0 < \alpha = 1 - \frac{1}{2q}$, $\beta > 1 - \frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq (\log N)^{-1} + M^{-2q\beta+2q-1}$;

(6) $\varphi(\alpha,\beta,N,M)=\sqrt{\frac{q!}{\kappa_\alpha\kappa_\beta}}N^{q-1}M^{q-1}$, *if* $\alpha>1-\frac{1}{2q}$, $\beta>1-\frac{1}{2q}$ *and* $q^{-1}T_2-1\trianglelefteq N^{-2q\alpha+2q-1}+M^{-2q\beta+2q-1}$,

where $s_\cdot$, $\iota_\cdot$ *and* $\kappa_\cdot$ *are as defined in Lemma* 6.7.

Proof Using the properties of the scalar product in Hilbert spaces we have

$$\begin{aligned}T_2&=\frac{1}{(q-1)!}\big(\varphi(\alpha,\beta,N,M)\big)^2\times\sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\big\langle\mathbf{1}_{\Delta i,j}(\cdot),\mathbf{1}_{\Delta i',j'}(\cdot)\big\rangle^q_{\mathcal{H}^{\alpha,\beta}}\\&=\frac{1}{(q-1)!}\big(\varphi(\alpha,\beta,N,M)\big)^2\times\left(\sum_{i,i'=0}^{N-1}\big\langle\mathbf{1}_{\Delta i}(\cdot),\mathbf{1}_{\Delta i'}(\cdot)\big\rangle^q_{\mathcal{H}^{\alpha}}\right)\\&\quad\times\left(\sum_{j,j'=0}^{M-1}\big\langle\mathbf{1}_{\Delta j}(\cdot),\mathbf{1}_{\Delta j'}(\cdot)\big\rangle^q_{\mathcal{H}^{\beta}}\right).\end{aligned}$$

The result then follows from Lemma 6.7. □

Remark 6.8 As mentioned above, $T_2=\mathbf{E}\|D\tilde{V}_{N,M}\|_{\mathcal{H}^{\alpha,\beta}}$. On the other hand, we also have

$$qT_2=\mathbf{E}\tilde{V}^2_{N,M}.$$

Indeed, this is true because for every multiple integral $F=I_q(f)$, it holds that $\mathbf{E}F^2=q\mathbf{E}\|DF\|^2_{\mathcal{H}^{\alpha,\beta}}$.

6.4.1 The Central Limit Case

We will prove that for every $\alpha,\beta\in(0,1)^2\setminus(1-\frac{1}{2q},1-\frac{1}{2q})^2$ a Central Limit Theorem holds, where $\tilde{V}_{N,M}$ was defined in (6.35). Using Stein's method we also give the Berry-Esséen bounds for this convergence. The reader may compare the below result with Theorem 6.1.

Theorem 6.6 (Central Limits) *Let* $\tilde{V}_{N,M}$ *be defined by* (6.35). *For every* $(\alpha,\beta)\in(0,1)^2$, *we denote by* $c_{\alpha,\beta}$ *a generic positive constant which depends on* α,β,q *and on the distance* d *and which is independent of* N *and* M. *We have*:

(1) *If* $0<\alpha,\beta<1-\frac{1}{2q}$, *then* $\tilde{V}_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha,\beta,N,M)=\sqrt{\frac{q!}{s_\alpha s_\beta}}N^{\alpha q-1/2}M^{\alpha q-1/2}$. *In addition*

$$d(\tilde{V}_{N,M},\mathcal{N})\le c_{\alpha,\beta}\sqrt{N^{-1}+N^{2\alpha-2}+N^{2\alpha q-2q+1}+M^{-1}+M^{2\beta-2}+M^{2\beta q-2q+1}}.$$

(2) *If* $0<\alpha<1-\frac{1}{2q}$ *and* $\beta=1-\frac{1}{2q}$, *then* $\tilde{V}_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha,\beta,N,M)=\sqrt{\frac{q!}{s_\alpha\iota_\beta}}N^{\alpha q-1}M^{q-1}$

$(\log M)^{-1/2}$. *In addition*

$$d(\tilde{V}_{N,M}, \mathcal{N}) \leq c_{\alpha,\beta}\sqrt{N^{-1} + N^{2\alpha-2} + N^{2\alpha q-2q+1} + (\log M)^{-1}}.$$

(3) *If both* $\alpha = \beta = 1 - \frac{1}{2q}$, *then* $\tilde{V}_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha, \beta, N, M) = \sqrt{\frac{q!}{\iota_\alpha \iota_\beta}} N^{q-1}(\log N)^{-1/2} M^{q-1}(\log M)^{-1/2}$. *In addition*

$$d(\tilde{V}_{N,M}, \mathcal{N}) \leq c_{\alpha,\beta}\sqrt{\log N^{-1} + \log M^{-1}}.$$

(4) *If* $\alpha < 1 - \frac{1}{2q}$ *and* $\beta > 1 - \frac{1}{2q}$, *then* $\tilde{V}_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha, \beta, N, M) = \sqrt{\frac{q!}{s_\alpha \kappa_\beta}} N^{\alpha q-1/2} M^{q-1}$. *In addition*

$$d(\tilde{V}_{N,M}, \mathcal{N}) \leq c_{\alpha,\beta}\sqrt{N^{-1} + N^{2\alpha-2} + N^{2\beta q-2q+1} + M^{2\beta q-2q+1}}.$$

(5) *If* $\alpha = 1 - \frac{1}{2q}$ *and* $\beta > 1 - \frac{1}{2q}$, *then* $\tilde{V}_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha, \beta, N, M) = \sqrt{\frac{q!}{\iota_\alpha \kappa_\beta}} N^{q-1}(\log N)^{-1/2} M^{q-1}$. *In addition*

$$d(\tilde{V}_{N,M}, \mathcal{N}) \leq c_{\alpha,\beta}\sqrt{\log(N)^{-1} + M^{2\beta q-2q+1}}.$$

Proof Recall that

$$\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}} =: T_1 + T_2,$$

where the summands T_1 and T_2 are given as in (6.37). We apply Lemma 6.9 to see that $1 - q^{-1}T_2$ converges to zero as N, M goes to infinity.

Let us show that T_1 converges to zero in $L^2(\Omega)$. We use the orthogonality of the iterated integrals to compute

$$\begin{aligned}
\mathbf{E}T_1^2 &= \frac{1}{(q-1)!^4}\big(\varphi(\alpha,\beta,N,M)\big)^4 \sum_{i,i',k,k'=0}^{N-1} \sum_{j,j',\ell,\ell'=0}^{M-1} \sum_{p=0}^{q-2} (p!)^2 \\
&\quad \times \big(C_{q-1}^p\big)^4 \langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}} \langle \mathbf{1}_{\Delta k,\ell}(\cdot), \mathbf{1}_{\Delta k',\ell'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}} \\
&\quad \times \mathbf{E}\big[I_{2q-2-2p}\big(\mathbf{1}_{\Delta i,j}^{\otimes q-1-p} \tilde{\otimes} \mathbf{1}_{\Delta i',j'}^{\otimes q-1-p}\big) I_{2q-2-2p}\big(\mathbf{1}_{\Delta k,\ell}^{\otimes q-1-p} \tilde{\otimes} \mathbf{1}_{\Delta k',\ell'}^{\otimes q-1-p}\big)\big] \\
&= \frac{1}{(q-1)!^4}\big(\varphi(\alpha,\beta,N,M)\big)^4 \sum_{i,i',k,k'=0}^{N-1} \sum_{j,j',\ell,\ell'=0}^{M-1} \sum_{p=0}^{q-2} (p!)^2 \\
&\quad \times \big(C_{q-1}^p\big)^4 \langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}} \langle \mathbf{1}_{\Delta k,\ell}(\cdot), \mathbf{1}_{\Delta k',\ell'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}} \\
&\quad \times (2q-2-2p)! \langle \mathbf{1}_{\Delta i,j}^{\otimes q-1-p} \tilde{\otimes} \mathbf{1}_{\Delta i',j'}^{\otimes q-1-p}, \mathbf{1}_{\Delta k,\ell}^{\otimes q-1-p} \tilde{\otimes} \mathbf{1}_{\Delta k',\ell'}^{\otimes q-1-p}\rangle_{\mathcal{H}^{\alpha,\beta}}.
\end{aligned}$$

Now, let us discuss the tensorized terms. We use the fact that

$$\begin{aligned}
&\left\langle \mathbf{1}_{\Delta i,j}^{\otimes q-1-p}\tilde{\otimes}\mathbf{1}_{\Delta i',j'}^{\otimes q-1-p}, \mathbf{1}_{\Delta k,\ell}^{\otimes q-1-p}\tilde{\otimes}\mathbf{1}_{\Delta k,\ell'}^{\otimes q-1-p}\right\rangle_{\mathcal{H}^{\alpha,\beta}} \\
&\quad = \sum_{a+b=q-1-p;c+d=q-1-p} \langle \mathbf{1}_{\Delta i,j}, \mathbf{1}_{\Delta k,\ell}\rangle_{\mathcal{H}^{\alpha,\beta}}^{a} \langle \mathbf{1}_{\Delta i,j}, \mathbf{1}_{\Delta k',\ell'}\rangle_{\mathcal{H}^{\alpha,\beta}}^{b} \\
&\qquad \times \langle \mathbf{1}_{\Delta i',j'}, \mathbf{1}_{\Delta k,\ell}\rangle_{\mathcal{H}^{\alpha,\beta}}^{c} \langle \mathbf{1}_{\Delta i',j'}, \mathbf{1}_{\Delta k',\ell'}\rangle_{\mathcal{H}^{\alpha,\beta}}^{d} \\
&\quad = \sum_{a+b=q-1-p;c+d=q-1-p} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^{\alpha}}^{a} \langle \mathbf{1}_{\Delta j}, \mathbf{1}_{\Delta \ell}\rangle_{\mathcal{H}^{\beta}}^{a} \\
&\qquad \times \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^{\alpha}}^{b} \langle \mathbf{1}_{\Delta j}, \mathbf{1}_{\Delta \ell'}\rangle_{\mathcal{H}^{\beta}}^{b} \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^{\alpha}}^{c} \langle \mathbf{1}_{\Delta j'}, \mathbf{1}_{\Delta \ell}\rangle_{\mathcal{H}^{\beta}}^{c} \\
&\qquad \times \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^{\alpha}}^{d} \langle \mathbf{1}_{\Delta j'}, \mathbf{1}_{\Delta \ell'}\rangle_{\mathcal{H}^{\beta}}^{d}
\end{aligned}$$

(we recall that $\mathbf{1}_{\Delta i} := \mathbf{1}_{[\frac{i}{N},\frac{i+1}{N}]}$). Therefore we finally have

$$\begin{aligned}
\mathbf{E}T_1^2 &= \frac{1}{(q-1)!^4}\big(\varphi(\alpha,\beta,N,M)\big)^4 \sum_{p=0}^{q-2}\big(C_{q-1}^{p}\big)^4 (p!)^2 \\
&\quad \times \sum_{a+b=q-1-p;c+d=q-1-p} a_N(p,\alpha,a,b,c,d)b_M(p,\beta,a,b,c,d),
\end{aligned}$$

with

$$\begin{aligned}
a_N(p,\alpha,a,b,c,d) = \sum_{i,i',k,k'=0}^{N-1} &\langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^{\alpha}}^{a} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^{\alpha}}^{b} \\
&\times \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^{\alpha}}^{c} \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^{\alpha}}^{d} \\
&\times \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^{\alpha}}^{p+1} \langle \mathbf{1}_{\Delta k}, \mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^{\alpha}}^{p+1}
\end{aligned}$$

and $b_M(p,\beta,a,b,c,d)$ similarly defined. We apply Lemma 6.8 to the terms a_N and b_M to conclude the convergence of T_1 to zero. Hence, $\mathbf{E}[(1-q^{-1}\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}})^2] = q^{-2}\mathbf{E}[|T_1|^2] + (1-q^{-1}T_2)^2$ which converges to zero for $\alpha \le 1-\frac{1}{2q}$ or $\beta \le 1-\frac{1}{2q}$. The bounds on the rate of convergence are given by Lemmas 6.8 and 6.9. Using Theorem 5.1, the conclusion of the theorem follows. □

The fact that the term T_1 converges to zero is the essential difference between the situations treated in the above theorem and the non-central limit case proved in the next section.

6.4.2 The Non-Central Limit Theorem

We will assume throughout this section that the Hurst parameters α, β satisfy

$$1 > \alpha, \beta > 1 - \frac{1}{2q}.$$

We will study the limit of the sequence $\tilde{V}_{N,M}$ given by the formula (6.35) with the renormalization factor φ from Lemma 6.9, point (6). Let us denote by $h_{N,M}$ the kernel of the random variable $\tilde{V}_{N,M}$ which is an element of the qth Wiener chaos, i.e.

$$h_{N,M} = \frac{1}{q!}\varphi(\alpha, \beta, N, M) \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \mathbf{1}^{\otimes q}_{[\frac{i}{N}, \frac{i+1}{N}] \times [\frac{j}{M}, \frac{j+1}{M}]}.$$

We will prove that $(h_{N,M})_{N,M \geq 1}$ is a Cauchy sequence in the Hilbert space $(\mathcal{H}^{\alpha,\beta})^{\otimes q}$. Using relation (4.2), we obtain

$$\begin{aligned}
&\langle h_{N,M}, h_{N',M'} \rangle_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}} \\
&= \frac{1}{q!^2}\varphi(\alpha, \beta, N, M)\varphi\big(\alpha, \beta, N', M'\big) \\
&\quad \times \big(\alpha(2\alpha - 1)\big)^q \sum_{i=0}^{N-1} \sum_{i'=0}^{N'-1} \left(\int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i'}{N}}^{\frac{i'+1}{N}} |u - v|^{2\alpha - 2} du dv \right)^q \\
&\quad \times \big(\beta(2\beta - 1)\big)^q \sum_{j=0}^{M-1} \sum_{j'=1}^{M'-1} \left(\int_{\frac{j}{M}}^{\frac{j+1}{M}} \int_{\frac{j'}{M'}}^{\frac{j'+1}{M'}} |u - v|^{2\beta - 2} du dv \right)^q
\end{aligned}$$

and this converges to (see also [39] or [181])

$$\begin{aligned}
&c_2(\alpha, \beta)\frac{1}{q!^2}\big(\alpha(2\alpha - 1)\big)^q \big(\beta(2\beta - 1)\big)^q \int_0^1 \int_0^1 |u - v|^{(2\alpha - 2)q} du dv \\
&\quad \times \int_0^1 \int_0^1 |u - v|^{(2\beta - 2)q} du dv,
\end{aligned}$$

where $c_2(\alpha, \beta) = \frac{q!}{\kappa_\alpha \kappa_\beta}$. The above constant is equal to

$$\begin{aligned}
&c_2(\alpha, \beta)\frac{1}{q!^2}\big(\alpha(2\alpha - 1)\big)^q \big(\beta(2\beta - 1)\big)^q \frac{1}{(\alpha q - q + 1)(2\alpha q - 2q + 1)} \\
&\quad \times \frac{1}{(\beta q - q + 1)(2\beta q - 2q + 1)} = \frac{1}{q!}.
\end{aligned}$$

It follows that the sequence $h_{N,M}$ is Cauchy in the Hilbert space $(\mathcal{H}^{\alpha,\beta})^{\otimes q}$ and as $N, M \to \infty$ it has a limit in $(\mathcal{H}^{\alpha,\beta})^{\otimes q}$ denoted by $\mu^{(q)}$. In the same way, the sequence

$$h_{N,M}(t,s) = \frac{1}{q!}\varphi(\alpha,\beta,N,M) \sum_{i=0}^{[(N-1)t]} \sum_{j=0}^{[(M-1)s]} \mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}$$

is Cauchy in $(\mathcal{H}^{\alpha,\beta})^{\otimes q}$ for every fixed s, t and it has a limit in this Hilbert space which will be denoted by $\mu^{(q)}_{s,t}$. Notice that $\mu^{(q)} = \mu^{(q)}(1,1)$ and that $\mu^{(q)}$ is a normalized uniform measure on the set $([0,t] \times [0,s])^q$.

Definition 6.1 We define the Hermite sheet process of order q and with Hurst parameters $\alpha, \beta \in (0,1)$, denoted by $(Z^{(q)}_{t,s})_{s,t\in[0,1]}$, by

$$Z^{(q),\alpha,\beta}_{t,s} := Z^{(q)}_{t,s} = I_q\big(\mu^{(q)}_{s,t}\big), \quad \forall s,t \in [0,1].$$

The previous computations lead to the following theorem.

Theorem 6.7 *Let $\tilde{V}_{N,M}$ be given by* (6.35) *with the function φ defined in Lemma* 6.9*, point* (6)*. Consider the Hermite sheet introduced in Definition* 6.1*. Then for $q \geq 2$*

$$\lim_{N,M\to\infty} \mathbf{E}\big[|\tilde{V}_{N,M} - Z|^2\big] = 0,$$

where $Z := Z^{(q)}_{1,1}$.

Proof Note that $\frac{1}{q!}\mathbf{E}[|\tilde{V}_{N,M} - Z|^2] = \|h_{N,M}\|^2_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}} + \|\mu^{(q)}\|^2_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}} - 2\langle h_{N,M}, \mu^{(q)}\rangle_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}}$. The computations at the beginning of this section complete the proof. □

Remark 6.9 As we will see below, the Hermite sheet from Definition 6.1 has the same properties as the process defined in Sect. 4.2. These two processes should coincide in the sense of finite dimensional distributions but a proof has not been provided. For the one-parameter case, such a result has been proven in [148].

Let us prove below some basic properties of the Hermite sheet.

Proposition 6.4 *Let us consider the Hermite sheet $(Z^{(q)}_{s,t})_{s,t\in[0,1]}$ from Definition* 6.1*. We have the following*:

(a) *The covariance of the Hermite sheet is given by*

$$\mathbf{E}Z^{(q)}_{s,t}Z^{(q)}_{u,v} = R_{q(\alpha-1)+1}(s,u)R_{q(\beta-1)+1}(t,v).$$

Consequently, it has the same covariance as the fractional Brownian sheet with Hurst parameters $q(\alpha-1)+1$ and $q(\beta-1)+1$.

(b) *The Hermite process is self-similarin the following sense: for every* $c, d > 0$, *the process*

$$\hat{Z}^{(q)}_{s,t} := \left(Z^{(q)}\right)_{cs,dt}$$

has the same law as $c^{q(\alpha-1)+1} d^{q(\beta-1)+1} Z^{(q)}_{s,t}$.

(c) *The Hermite process has stationary increments in the sense of Definition* A.5.

(d) *The paths are Hölder continuous of order* (α', β') *with* $0 < \alpha' < \alpha$ *and* $0 < \beta' < \beta$.

Proof Let f be an arbitrary function in $(\mathcal{H}^{\alpha,\beta})^{\otimes q}$. It follows that

$$\begin{aligned}
&\left\langle h_{N,M}(t,s), f\right\rangle_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}} \\
&\quad = c_2(\alpha,\beta)^{-1/2} \frac{1}{q!} N^{q-1} M^{q-1} \sum_{i=0}^{[(N-1)t]} \sum_{j=0}^{[(M-1)s]} \left\langle \mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}, f\right\rangle_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}} \\
&\quad = a(\alpha)^q a(\beta)^q c_2(\alpha,\beta)^{-1/2} \frac{1}{q!} N^{q-1} M^{q-1} \\
&\qquad \times \sum_{i=0}^{[(N-1)t]} \sum_{j=0}^{[(M-1)s]} \int_{[0,1]^{2q}} dx_1 \cdots dx_q dy_1 \cdots dy_q f\big((x_1,y_1),\ldots,(x_q,y_q)\big) \\
&\qquad \times \int_{[\frac{i}{N},\frac{i+1}{N}]^q} da_1 \cdots da_q \int_{[\frac{j}{M},\frac{j+1}{M}]^q} db_1 \cdots db_q \\
&\qquad \times \prod_{k=1}^{q} |a_k - x_k|^{2\alpha-2} \prod_{k=1}^{q} |b_k - y_k|^{2\beta-2} \\
&\quad \xrightarrow{N,M\to\infty} a(\alpha)^q a(\beta)^q c_2(\alpha,\beta)^{-1/2} \frac{1}{q!} \int_0^t da \int_0^s db \int_{[0,1]^{2q}} dx_1 \cdots dx_q dy_1 \cdots dy_q \\
&\qquad \times f\big((x_1,y_1),\ldots,(x_q,y_q)\big) \prod_{k=1}^{q} |a - x_k|^{2\alpha-2} \prod_{k=1}^{q} |b - y_k|^{2\beta-2}.
\end{aligned}$$

By applying the above formula for $f = \mu^{(q)}_{u,v}$ and using the fact that

$$\mathbf{E}\big(Z^{(q)}_{s,t} Z^{(q)}_{u,v}\big) = q! \langle \mu^{(q)}_{s,t}, \mu^{(q)}_{u,v}\rangle_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}},$$

we prove assertion (a).

Concerning (b), let us denote by

$$H_{N,M}(t,s) = \frac{1}{q!} \varphi(\alpha,\beta,N,M) \sum_{i=0}^{[(N-1)t]} \sum_{j=0}^{[(M-1)]s} I_q\big(\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\big).$$

We know that

$$H_{cN,dM}(t,s) \xrightarrow{N,M\to\infty} Z^{(q)}_{t,s} \tag{6.38}$$

in $L^2(\Omega)$ for every $s,t \in [0,1]$. But

$$\begin{aligned} H_{cN,dM}(t,s) &= \frac{c_2(\alpha,\beta)^{-1/2}}{(cN)^{1-(1-\alpha)q}(dM)^{1-(1-\beta)q}} \sum_{i=0}^{[(N-1)t]} \sum_{j=0}^{[(M-1)s]} I_q\big(\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\big) \\ &= \frac{1}{(c)^{1-(1-\alpha)q}(d)^{1-(1-\beta)q}} H_{N,M}(t,s) \xrightarrow{N,M\to\infty} Z^{(q)}_{t,s}. \end{aligned} \tag{6.39}$$

Point (b) then follows easily from (6.38) and (6.39).

Point (c) is a consequence of the fact that the fractional Brownian sheet has stationary increments in the sense of Definition A.5 while point (d) can easily be proved by using the Kolmogorov continuity criterion together with points (b) and (c) above (see also Sect. 4, pp. 35–36 in [12]).

□

6.5 Bibliographical Notes

The Hermite variations, although related to the historical limit theorems by [41, 67, 82, 167, 168], started to be systematically studied after the publications of the papers [137, 138, 142] and [127]. This is because the Hermite polynomials fit well into the Malliavin calculus context. Some important references on the Hermite variations for fractional Brownian motion are [127, 133, 134]. These variations have in turn been applied to other Gaussian processes (bifractional Brownian motion, subfractional Brownian motion, moving averages [1, 37, 182] etc.) or to multiparameter processes ([40, 152]). The Hsu-Robbins and Spitzer theorems for fBm are proven in [176].

6.6 Exercises

Exercise 6.1 Find the constants $c_{1,q,H}, c_{2,q,H}, c_{3,q,H}$ in Theorem 6.1.

Exercise 6.2 Prove point (iii) in Theorem 6.1.

Exercise 6.3 The process $(Z_t)_{t\in\mathbb{Z}}$ is said to be a white noise with zero mean and variance σ^2, written

$$(Z_t) \sim \mathcal{WN}\big(0,\sigma^2\big),$$

if and only if $\{Z_t\}$ has zero mean and covariance function $\gamma(h) = \mathbf{E}(Z_{t+h}Z_t)$, $h \in \mathbb{N}$, defined by

$$\gamma(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0. \end{cases}$$

If $\{Z_t\} \sim \mathcal{WN}(0, \sigma^2)$ then we say that $\{X_t\}$ is a moving average (MA(∞)) of $\{Z_t\}$ if there exists a sequence $\{\psi_j\}$ with $\sum_{j=0}^{\infty} |\psi_j| < \infty$ such that

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t = 0, \pm 1, \pm 2, \dots. \tag{6.40}$$

Prove that the (MA(∞)) process defined by (6.40) is stationary with mean zero and covariance function

$$\gamma(k) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|}. \tag{6.41}$$

Exercise 6.4 ([127]) Give the rate of the convergences in Theorem 6.1.

Hint Follow the proofs in Sect. 6.2.

Exercise 6.5 ([78]) Let $(Z_t^{(H)})_{t\in[0,1]}$ be a Rosenblatt process with self-similarity index $H \in (\frac{1}{2}, 1)$ and define *its cubic variation* by

$$V^{3,N} = \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}})^3}{\mathbf{E}(Z^{(H)}_{\frac{i+1}{N}} - Z^{(H)}_{\frac{i}{N}})^3} - 1 \right). \tag{6.42}$$

Let L^H be the kernel of the Rosenblatt process

$$\begin{aligned} L_t^H(y_1, y_2) &:= L_t(y_1, y_2) \\ &= d(H) 1_{[0,t]}(y_1) 1_{[0,t]}(y_2) \int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \end{aligned}$$

with

$$H' := \frac{H+1}{2} \quad \text{and} \quad d(H) := \frac{1}{H+1} \left(\frac{H}{2(2H-1)} \right)^{-\frac{1}{2}}$$

and with K^H the standard kernel defined in (1.4) appearing in the Wiener integral representation of the fBm. For $= 1, \dots, N$ set

$$f_{i,N} = L^{(H)}_{\frac{i+1}{N}} - L^{(H)}_{\frac{i}{N}}.$$

1. Prove that for any symmetric function $f \in L^2([0,1]^2)$,

$$\begin{aligned} I_2(f)^3 &= I_6\big((f \tilde{\otimes} f) \otimes f\big) + 8 I_4\big((f \tilde{\otimes} f) \otimes_1 f\big) + 4 I_4\big((f \otimes_1 f) \otimes f\big) \\ &+ 12 I_2\big((f \tilde{\otimes} f) \otimes_2 f\big) + 16 I_2\big((f \otimes_1 f) \otimes_1 f\big) \\ &+ 2 \langle f, f \rangle_{L^2([0,1]^2)} I_2(f) + 8 \big\langle (f \otimes_1 f), f \big\rangle_{L^2([0,1]^2)}. \end{aligned}$$

2. Deduce that

$$\big(I_2(f_{i,N})\big)^3 = 8(f_{i,N} \otimes_1 f_{i,N}) \otimes_2 f_{i,N} + I_2(g_{i,N}) + 4I_4(h_{i,N}) + I_6\big((f_{i,N} \tilde{\otimes} f_{i,N}) \otimes f_{i,N}\big) \tag{6.43}$$

with

$$g_{i,N} = 2\|f_{i,N}\|^2_{L^2} f_{i,N} + 12(f_{i,N} \tilde{\otimes} f_{i,N}) \otimes_2 f_{i,N} + 16(f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N} \tag{6.44}$$

and

$$h_{i,N} = 2(f_{i,N} \tilde{\otimes} f_{i,N}) \otimes_1 f_{i,N} + f_{i,N} \otimes (f_{i,N} \otimes_1 f_{i,N}) := h^{(1)}_{i,N} + h^{(2)}_{i,N}. \tag{6.45}$$

3. Prove that $g_{i,N} \in L^2([0,1]^2)$ and $h_{i,N} \in L^2([0,1]^4)$.
4. Deduce the Wiener chaos expansion of (6.42).
5. Prove that

$$\mathbf{E}\big(N^{1-H} V^{3,N}\big)^2 \xrightarrow[N\to\infty]{} \bar{C}(H) \tag{6.46}$$

where $\bar{C}(H) := C(H)^2 C_0(H)$ with

$$C_0(H) = \big(9 + 36C'(H)H(2H-1) + 144\big[C'(H)H(2H-1)\big]^2\big)$$

and

$$C'(H) = \int_{[0,1]^3} |v_1 - v_2|^{2H'-2} |v_2 - v_3|^{2H'-2} dv_1 dv_2 dv_3.$$

6. Prove that the renormalized cubic variation statistics based on the Rosenblatt process $N^{1-H} V^{3,N}$ with $V^{3,N}$ given by (6.42) converges in $L^2(\Omega)$ as $N \to \infty$ to the Rosenblatt random variable $D(H) Z_1^{(H)}$ where $D(H) = C(H)^{-1}(3 + 24 d(H)^2 a(H)^2 C'(H))$.

Exercise 6.6 Consider the sequences (6.7) where

$$X_n = \sum_{i=1}^{\infty} \alpha_i (W_{n-i} - W_{n-i-1}),$$

with W a Brownian motion on the whole real line, $\alpha_i \in \mathbb{R}$, $\alpha_i = c i^{-\beta}$, $\beta \in (\frac{1}{2}, 1)$ and $\sum_{i=1}^{\infty} \alpha_i^2 = 1$. Recall that under the condition $q > (2\beta - 1)^{-1}$, Z_N converges in law towards $Z \sim \mathcal{N}(0,1)$ and the rate of convergence is given in Theorem 6.2.

Define

$$f_1(\varepsilon) = \sum_{N \geq 1} \frac{1}{N} P\big(|S_N| > \varepsilon N\big)$$

and

$$f_2(\varepsilon) = \sum_{N\geq 1} \frac{1}{N} P\big(|S_N| > \varepsilon N^{-2\beta q+q+2}\big).$$

Prove that when $q > \frac{1}{2\beta-1}$,

$$\lim_{\varepsilon\to 0} \frac{1}{-\log(\varepsilon)} f_1(\varepsilon) = 2$$

and when $q < \frac{1}{2\beta-1}$ then

$$\lim_{\varepsilon\to 0} \frac{1}{-\log(\varepsilon)} f_2(\varepsilon) = \frac{1}{1+\frac{q}{2}-\beta q}.$$

Hint Follow the proof of Theorem 6.2.

Exercise 6.7 Let the notation in Exercise 6.6 prevail and define

$$g_1(\varepsilon) = \sum_{N\geq 1} P\big(|S_N| > \varepsilon N\big)$$

and

$$g_2(\varepsilon) = \sum_{N\geq 1} P\big(|S_N| > \varepsilon N^{-2\beta q+q+2}\big).$$

Prove that when $q > \frac{1}{2\beta-1}$,

$$\lim_{\varepsilon\to 0} \big(\sigma_{q,\beta}^{-1}\varepsilon\big)^2 g_1(\varepsilon) = 1 = \mathbf{E}\big(Z^2\big)$$

and when $q < \frac{1}{2\beta-1}$ then

$$\lim_{\varepsilon\to 0} \big(h_{q\beta}^{-1}\varepsilon\big)^{\frac{1}{1+\frac{q}{2}-\beta q}} g_2(\varepsilon) = \mathbf{E}\big|Z^{(q)}\big|^{\frac{1}{1+\frac{q}{2}-\beta q}}.$$

Exercise 6.8 ([40]) Let $\tilde{V}_{N,M}$ be given by (6.35) with the function φ defined in Lemma 6.9, point (6). Show that

$$d_{TV}(\tilde{V}_{N,M}, Z) \leq cN^{\frac{2q-1-2q\alpha}{2q}} M^{\frac{2q-1-2q\beta}{2q}}$$

where d_{TV} denotes the total variation distance.

Hint Use Theorem 5.6.

Exercise 6.9 Prove that for $q = 2$ the process defined in Definition 6.1 coincides with the Hermite process of order $q = 2$ and self-similarity index $(q(\alpha-1)+1, q(\beta-1)+1)$ (recall Definition A.4).

Hint Use formula (3.16).

Appendix A
Self-similar Processes with Self-similarity: Basic Properties

The first paper to give a rigorous mathematical analysis of self-similar processes is [106]. There are several monographs (see e.g. [75, 160]) that the reader can consult in order to build a more complete picture of self-similarity and related topics.

A.1 One-Parameter Self-similar Processes

Let us define the concept of self-similarity.

Definition A.1 A stochastic process $(X_t)_{t\geq 0}$ is called self-similar if there exists a real number $H > 0$ such that for any $c > 0$ the processes $(X_{ct})_{t\geq 0}$ and $(c^H X_t)_{t\geq 0}$ have the same finite dimensional distributions.

Remark A.1 A self-similar process satisfies $X_0 = 0$ almost surely.

Definition A.2 A stochastic process $(X_t)_{t\geq 0}$ is said to be with stationary increments if for any $h > 0$ the distribution of the process

$$(X_{t+h} - X_h)_{t\geq 0}$$

does not depend on h.

The self-similar processes with stationary increments all have the same covariance.

Theorem A.1 *Let $(X_t)_{t\geq 0}$ be a non-trivial H-self-similar process with stationary increments such that $EX_1^2 < \infty$. Then*

$$\mathbf{E}X_t X_s = \frac{1}{2}\mathbf{E}X_1^2\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

C.A. Tudor, *Analysis of Variations for Self-similar Processes*,
Probability and Its Applications, DOI 10.1007/978-3-319-00936-0,

Proof Let $s \le t$. Writing

$$X_t X_s = \frac{1}{2}\big(X_t^2 + X_s^2 - (X_t - X_s)^2\big)$$

we get

$$\begin{aligned}\mathbf{E}X_t X_s &= \frac{1}{2}\big(X_t^2 + X_s^2 - \mathbf{E}X_{t-s}^2\big)\\ &= \frac{1}{2}\mathbf{E}X_1^2\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).\end{aligned}$$

□

Functions of the moments of a self-similar process with stationary increments can yield information concerning the self-similarity index.

Proposition A.1 *Let $(X_t)_{t\ge 0}$ be a non-trivial H-self-similar process with stationary increments. Then*

(i) *If $\mathbf{E}|X_1| < \infty$, then $0 < H \le 1$.*
(ii) *If $\mathbf{E}|X_1| < \infty$ and $H = 1$ then*

$$X_t = tX_1 \quad a.s.$$

(iii) *If $\mathbf{E}|X_1|^\alpha < \infty$ for some $\alpha \le 1$ then $H < \frac{1}{\alpha}$.*

Proposition A.2 *Let $(X_t)_{t\ge 0}$ be a non-trivial H-self-similar process with stationary increments such that $\mathbf{E}X_1^2 < \infty$. Define, for any integer $n \ge 1$*

$$r(n) = \mathbf{E}\big(X_1(X_{n+1} - X_n)\big).$$

Then, if $H \ne \frac{1}{2}$, as $n \to \infty$

$$r(n) \sim H(2H-1)n^{2H-2}\mathbf{E}X_1^2.$$

Proof From Proposition A.1,

$$r(n) = \frac{1}{2}\mathbf{E}X_1^2\big((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\big)$$

and it suffices to study the asymptotic behavior of the sequence on the right-hand side above when $n \to \infty$. □

Remark A.2 If $H = \frac{1}{2}$ then $r(n) = 0$ for any $n \ge 1$.

Definition A.3 We say that a process X exhibits long-range dependence (or it is a long-memory process) if

$$\sum_{n\ge 0} r_n = \infty$$

where $r(n) = \mathbf{E}(X_1 - X_0)(X_{n+1} - X_n)$. Otherwise, if

$$\sum_{n\ge 0} r_n < \infty$$

we say that X is a short-memory process.

From Proposition A.2 and Definition A.3 we conclude that if $(X_t)_{t\geq 0}$ is a non-trivial H-self-similar process with stationary increments and with $\mathbf{E}X_1^2 < \infty$ then X is with long-range dependence for $H > \frac{1}{2}$ and with short-memory if $H \leq \frac{1}{2}$.

A.2 Multiparameter Self-similar Processes

We first introduce the concept of self-similarity and stationary increments for two-parameters processes.

Definition A.4 A two-parameter stochastic process $(X_{s,t})_{(s,t)\in T}$, $T \subset \mathbb{R}^2$, is self-similar with order (α, β) if for any $h, k > 0$ the process $(\widehat{X}_{s,t})_{(s,t)\in T}$ defined as

$$\widehat{X}_{s,t} := h^{\alpha} k^{\beta} X_{\frac{s}{h},\frac{t}{k}}, \quad (s,t) \in T$$

has the same finite dimensional distributions as the process X.

Definition A.5 A process $(X_{s,t})_{(s,t)\in I}$ with $I \subset \mathbb{R}^2$ is said to be stationary if for every integer $n \geq 1$ and $(s_i, t_j) \in I$, $i, j = 1, \ldots, n$, the distribution of the random vector

$$(X_{s+s_1,t+t_1}, X_{s+s_2,t+t_2}, \ldots, X_{s+s_n,t+t_n})$$

does not depends on (s,t), where $s, t \geq 0$, $(s+s_i, t+t_i) \in I$, $i = 1, \ldots, n$.

We will say that a two-parameter stochastic process $(X_{s,t})_{(s,t)\in\mathbb{R}^2}$ has stationary increments if for every $h, k > 0$ the process

$$(X_{t+h,s+k} - X_{t,s+k} - X_{t+h,s} + X_{t,s})_{(s,t)\in\mathbb{R}^2}$$

is stationary.

So, a two-parameter stochastic process has stationary increments if its *rectangular* increments are stationary. The concept can be extended to multiparameter stochastic processes.

Definition A.6 A stochastic process $(X_{\mathbf{t}})_{\mathbf{t}\in T}$, where $T \subset \mathbb{R}^d$, is called self-similar with self-similarity order $\alpha = (\alpha_1, \ldots, \alpha_d) > 0$ if for any $\mathbf{h} = (h_1, \ldots, h_d) > 0$ the stochastic process $(\hat{X}_{\mathbf{t}})_{\mathbf{t}\in T}$ given by

$$\hat{X}_{\mathbf{t}} = \mathbf{h}^{\alpha} X_{\frac{\mathbf{t}}{\mathbf{h}}} = h_1^{\alpha_1} \cdots h_d^{\alpha_d} X_{\frac{t_1}{h_1},\ldots,\frac{t_d}{h_d}}$$

has the same law as the process X.

Let us recall the notion of the increment of a d-parameter process X on a rectangle $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^d$, $\mathbf{s} = (s_1, \ldots, s_d), \mathbf{t} = (t_1, \ldots, t_d)$, with $\mathbf{s} \leq \mathbf{t}$. This increment is denoted by $\Delta X_{[\mathbf{s},\mathbf{t}]}$ and it is given by

$$\Delta X_{[\mathbf{s},\mathbf{t}]} = \sum_{r\in\{0,1\}^d} (-1)^{d-\sum_i r_i} X_{\mathbf{s}+\mathbf{r}\cdot(\mathbf{t}-\mathbf{s})}. \tag{A.1}$$

When $d = 1$ one obtains $\Delta X_{[\mathbf{s},\mathbf{t}]} = X_t - X_s$ while for $d = 2$ one gets $\Delta X_{[\mathbf{s},\mathbf{t}]} = X_{t_1,t_2} - X_{t_1,s_2} - X_{s_1,t_2} + X_{s_1,s_2}$.

Definition A.7 A process $(X_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}^d)$ has stationary increments if for every $\mathbf{h} > 0$, $\mathbf{h} \in \mathbb{R}^d$ the stochastic processes $(\Delta X_{[0,\mathbf{t}]}, \mathbf{t} \in \mathbb{R}^d)$ and $(\Delta X_{[\mathbf{h},\mathbf{h}+\mathbf{t}]}, \mathbf{t} \in \mathbb{R}^d)$ have the same finite dimensional distributions.

Appendix B
The Kolmogorov Continuity Theorem

This result is used to obtain the continuity of sample paths of stochastic processes.

Theorem B.1 *Consider a stochastic process* $(X_t)_{t\in T}$ *where* $T\subset\mathbb{R}$ *is a compact set. Suppose that there exist constants* $p, C>0$ *and* $\beta>1$ *such that for every* $s,t\in T$

$$\mathbf{E}|X_t-X_s|^p\le C|t-s|^\beta.$$

Then X *has a continuous modification* $\tilde{X}$. *Moreover for every* $0<\gamma<\frac{\beta-1}{p}$

$$\mathbf{E}\biggl(\sup_{s,t\in T; s\neq t}\frac{|\tilde{X}_t-\tilde{X}_s|}{|t-s|^\gamma}\biggr)^p<\infty.$$

In particular X *admits a modification which is Hölder continuous of any order* $\alpha\in(0,\frac{\beta-1}{p})$.

There exists a two-parameter version of the Kolmogorov continuity theorem (see e.g. [12]).

Theorem B.2 *Let* $(X_{s,t})_{s,t\in T}$ *be a two-parameter process, vanishing on the axis, with* T *a compact subset of* $\mathbb{R}$. *Suppose that there exist constants* $C,p>0$ *and* $x,y>1$ *such that*

$$\mathbf{E}|X_{s+h,t+k}-X_{s+h,t}-X_{s,t+k}+X_{s,t}|^p\le Ch^xk^y$$

for every $h,k>0$ *and for every* $s,t\in T$ *such that* $s+h, t+k\in T$. *Then* X *admits a continuous modification* $\tilde{X}$. *Moreover* $\tilde{X}$ *has Hölder continuous paths of any orders* $x'\in(0,\frac{x-1}{p})$, $y'\in(0,\frac{y-1}{p})$ *in the following sense: for every* $\omega\in\Omega$, *there exists a* $C_\omega>0$ *such that for every a* $s,t,s',t'\in T$

$$\bigl|X_{s,t}(\omega)-X_{s,t'}(\omega)-X_{s',t}(\omega)+X_{s',t'}(\omega)\bigr|\le C_\omega\bigl|t-t'\bigr|\bigl|s-s'\bigr|.$$

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Appendix C
Multiple Wiener Integrals and Malliavin Derivatives

Here we describe the elements from the Malliavin calculus that we need in the monograph. As mentioned, we give only a flavor of the Malliavin calculus, the basic tools necessary to follow the exposition in Part II of the monograph. Consider a real separable Hilbert space $\mathcal{H}$ and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$, which is a centered Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$. Denote by I_n the multiple stochastic integrals with respect to B (see [136]). This I_n is actually an isometry between the Hilbert space $\mathcal{H}^{\odot n}$(symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_n is the Hermite polynomial of degree $n \geq 1$

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as: for m, n positive integers,

$$\begin{aligned} \mathbf{E}\big(I_n(f)I_m(g)\big) &= n!\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}\big(I_n(f)I_m(g)\big) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{C.1}$$

We also have

$$I_n(f) = I_n(\tilde{f})$$

where $\tilde{f}$ denotes the symmetrization of f defined by $\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!}\sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by B can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n \geq 0} I_n(f_n) \tag{C.2}$$

where $f_n \in \mathcal{H}^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

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Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n\geq 0} n I_n(f_n)$$

where F is given by (C.2) such that $\sum_{n=1}^{\infty} n^2 n \|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty$.

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \left\| \big((I-L)F\big)^{\frac{\alpha}{2}} \right\|_{L^p(\Omega)}$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(B(\varphi_1), \dots, B(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in \mathcal{H}$, $i = 1, \dots, n$)

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\big(B(\varphi_1), \dots, B(\varphi_n)\big)\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(\mathcal{H})$. What is important for the reader of this monograph is how the Malliavin derivative acts on the Wiener chaos. Actually, if $F = I_n(f)$ with $f \in \mathcal{H}^{\odot n}$ then

$$D_\alpha F = n I_{n-1}\big(f(\cdot, \alpha)\big) \tag{C.3}$$

for every $\alpha > 0$, where $\cdot$ represents $n-1$ variables. Also, the pseudo-inverse of L satisfies $(-L)^{-1} I_n(f) = \frac{1}{n} I_n(f)$ if $n \geq 1$.

We will need the general formula for calculating products of Wiener chaos integrals of any orders p, q for any symmetric integrands $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$; it is

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \otimes_r g) \tag{C.4}$$

as given for instance in [136, Proposition 1.1.3]; for example, if $\mathcal{H}$ is the space $L^2([0,T]^n)$, then the contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$ defined by

$$\begin{aligned}(f \otimes_\ell g)&(s_1, \dots, s_{n-\ell}, t_1, \dots, t_{m-\ell}) \\ &= \int_{[0,T]^{m+n-2\ell}} f(s_1, \dots, s_{n-\ell}, u_1, \dots, u_\ell) \\ &\quad \times g(t_1, \dots, t_{m-\ell}, u_1, \dots, u_\ell) du_1 \cdots du_\ell. \end{aligned} \tag{C.5}$$

We will need the following bound for the tail probabilities of multiple Wiener-Itô integrals (see [116], Theorem 4.1)

$$P\big(|I_n(f)| > u\big) \leq c \exp\left(\left(\frac{-cu}{\sigma}\right)^{\frac{2}{n}}\right) \tag{C.6}$$

for all $u > 0$, $n \geq 1$, with $\sigma = \|f\|_{L^2([0,1]^n)}$.

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