On Prandtl-Reuss Mixtures

Jens Frehse and Josef Málek

Abstract We study mathematical properties of the model that has been proposed to explain the phenomenon of hardening due to cyclic loading. The model considers two elastic plastic materials, soft and hard, that co-exist while the soft material can be transformed into the hard material. Regarding elastic responses we remain in a simplified framework of linearized elasticity. Incorporating tools such as variational inequalities, penalty approximations and Sobolev spaces, we prove the existence of weak solution to the corresponding boundary-value problem and investigate its uniqueness and regularity.

1 Introduction

In the article [10], Kratochvíl, Rajagopal, Srinivasa and the second author of the present contribution developed a thermodynamically consistent model within the framework of finite elastic plasticity that is capable of 'explaining' the phenomenon of hardening of the material due to cyclic loadings. They consider the mixture of two elastic plastic materials, soft and hard, that coexist. The material that can be thought to be originally almost consisting of soft region builds the hard regions by a process of 'recruitment' of the soft material and its conversion into a hard material. The study in [10] carries on some ideas from Kratochvíl [9]. The authors then also consider a simplified model that is obtained by assuming that the gradient of the

J. Frehse (✉)
Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany
e-mail: erdbeere@iam.uni-bonn.de

J. Málek
Faculty of Mathematics and Physics, Mathematical Institute, Charles University in Prague, Prague, Czech Republic
e-mail: malek@karlin.mff.cuni.cz

M. Griebel (ed.), *Singular Phenomena and Scaling in Mathematical Models*, DOI 10.1007/978-3-319-00786-1_7,

displacement is small. This results in a model that can be viewed as the mixture of two (soft and hard) Prandtl-Reuss models of the linearized elastic perfect plasticity where the conversion of soft regions to hard regions is modeled through the variation of the volume fraction α of the soft material within the mixture. Note that $(1-\alpha)$ is the volume fraction of the hard material. The present paper intends to elaborate a rigorous mathematical treatment for this simplified model using the framework of variational inequalities, penalty approximations and Sobolev spaces. We call the materials that are based on the coexistence of the two (or more) Prandtl-Reuss elastic plastic materials 'Prandtl-Reuss mixtures'.

Although a mathematical treatment is very similar to the one of the classical Prandtl-Reuss-problems, several complications arise. A first study of this model has been performed in the thesis of Khasina, see [8].

The paper is organized in the following way. Section 2 starts with a basic mathematical setting, and contains the formulation of the mixture problem via a variational inequality for the convex combination of the soft and hard material, with the side condition that the convex combination $\alpha\sigma_s + (1-\alpha)\sigma_h$ satisfies the balance of linear momentum, and σ_s and σ_h satisfy the relevant yield conditions. The main theorem states that, under appropriate conditions on the data, in particular, under a safe load condition for the mixture, the considered problem for the Prandtl-Reuss mixture has unique solution σ_s, σ_h in the spaces $L^\infty(L^2)$ and $H^{1,2}(L^2)$.

Furthermore, from the formulation via a variational inequality one concludes, see Sect. 3, the existence of partial velocity gradients (or more precisely their symmetric parts) $\frac{1}{2}(\nabla\dot{u}_s + \nabla\dot{u}_s^T)$ and $\frac{1}{2}(\nabla\dot{u}_h + \nabla\dot{u}_h^T)$ so that

$$\frac{1}{2}(\nabla\dot{u}_s + \nabla\dot{u}_s^T) = A_s\frac{\partial}{\partial t}(\alpha\sigma_s) + \dot{e}_{ps},\ \frac{1}{2}(\nabla\dot{u}_h + \nabla\dot{u}_h^T) = A_h\frac{\partial}{\partial t}(\alpha\sigma_h) + \dot{e}_{ph}, \tag{1}$$

where $\dot{e}_{ps}$ and $\dot{e}_{ph}$ are the rates of plastic strains for the soft and hard material, A_s and A_h are the inverse fourth order elastic tensors, so that $A_s\sigma_s$ and $A_h\sigma_h$ correspond to the elastic strains for the soft and hard materials. Then we conclude that

$$\frac{1}{2}(\nabla\dot{u}_s + \nabla\dot{u}_s{}^T) = \frac{1}{2}(\nabla\dot{u}_h + \nabla\dot{u}_h{}^T) =: \frac{1}{2}(\nabla\dot{u} + \nabla\dot{u}^T). \tag{2}$$

Here and below, for any quantity w

$$\dot{w} = \frac{\partial w}{\partial t}, \tag{3}$$

we shall use both notation in what follows. We confine ourselves to the von Mises yield conditions

$$|\sigma_{sD}| \le \kappa_s, |\sigma_{hD}| \le \kappa_h, \tag{4}$$

where κ_s and κ_h may depend on t and x and $B_D = B - (\operatorname{tr} B/3)\, I$ for any $B \in \mathbb{R}^{n\times n}$. The plastic strains are nontrivial only if $|\sigma_{sD}| = \kappa_s$ and $|\sigma_{hD}| = \kappa_h$, and then they are proportional to the outer normal 'vectors' associated with the surfaces $|\sigma_{sD}| = \kappa_s$ and $|\sigma_{hD}| = \kappa_h$, it means that

$$\dot{e}_{ps} = \lambda_s \frac{\sigma_{sD}}{|\sigma_{sD}|}, \dot{e}_{ph} = \lambda_h \frac{\sigma_{hD}}{|\sigma_{hD}|} \text{ with } \lambda_s \text{ and } \lambda_h > 0. \tag{5}$$

These conditions can be rewritten in the compact Kuhn-Tucker forms

$$\lambda_s, \lambda_h \geq 0, \lambda_s(|\sigma_{sD}| - \kappa_s) = 0, \lambda_h(|\sigma_{hD}| - \kappa_h) = 0. \tag{6}$$

Unfortunately, in the rigorous mathematical treatment, similar as for the analysis of the classical Prandtl-Reuss model, the quantities $\frac{1}{2}(\nabla \dot{u}_s + \nabla \dot{u}_s^T)$, $\frac{1}{2}(\nabla \dot{u}_h + \nabla \dot{u}_h^T)$ are only elements of C^*, i.e. they are not functions. This holds also for $\dot{e}_{ps}$ and $\dot{e}_{ph}$, so the above Kuhn-Tucker rule has to be interpreted correctly, see Sect. 8. Fortunately, due to Temam's imbedding theorem, the quantity $\dot{u}$ is an element of $L^\infty(L^{\frac{n}{n-1}})$, i.e. it is a 'function'.

The proof of the main theorem starts in Sect. 3 by introducing a penalty approximation where the yield conditions (4) are penalized. The penalty approximation, in turn, is discretized by a Rothe approximation, and in Sect. 4 up to 6 we establish uniform estimates for Rothe approximations and take the limit in the Rothe approximation in order to obtain the solvability of the penalty approximation. In Sect. 7 we establish uniform $L^\infty(L^2)$-estimates for the stress velocities $\dot{\sigma}_{s\mu}, \dot{\sigma}_{h\mu}$, where $\mu \to 0$ is the penalty parameter.

Finally, in Sect. 8 we pass to the limit with respect to the penalty parameter and complete the proof of the main theorem. As mentioned above we also discuss how the Kuhn-Tucker forms have to be formulated rigorously. In Sect. 9, we consider a generalized model (derived in [10]) in which α, κ_s and κ_h may depend on history of the rate of the plastic strain of the soft material. We discuss how to treat this in the framework of the present paper. In a continuation of this study we intend to focus on the regularity properties of the solution.

We refer to [3] for a detailed survey of the results concerning the mathematical analysis of relevant results concerning initial and boundary value problems for classical Prandtl-Reuss model of the linearized elastic perfect plasticity.

2 Mathematical Formulation of the Problem

2.1 Basic Setting

Let Ω be a bounded domain of $\mathbb{R}^n$ occupied by a body which is supposed to be a mixture of a soft and a hard linearized elastic-perfect-plastic materials in the sense defined below.

We imagine the following deformation process with (slow) cyclic loading in which the mixture with a large portion of soft material is gradually deformed and transforms into a mixture with a large portion of hard material.

If $t \in [0, T]$ is the loading parameter, the interior stresses of the soft or hard material are denoted by $\sigma_s(t, x)$ and $\sigma_h(t, x)$, respectively. Let M^n_{sym} be a set of symmetric $n \times n$ matrices. We require the σ's to be symmetric, i.e.

$$\sigma_s, \sigma_h : [0,T] \times \Omega \to M^n_{sym}. \tag{7}$$

Let $\alpha : [0,T] \times \Omega \to [0,1]$ describe the fraction of the soft material in the mixture such that the stress of the mixture σ is given by

$$\sigma(t,x) = \alpha(t,x)\sigma_s(t,x) + (1-\alpha(t,x))\sigma_h(t,x). \tag{8}$$

Assumption 1. *(a) α is Lipschitz continuous and decreasing.*
(b) $0 < \alpha_0 < \alpha(t,x) < 1-\alpha_0 < 1$ with a constant α_0 (for all $t \in [0,T]$ and $x \in \Omega$).

Remark 1. In general, α depends on the history of σ_s, but readability is better if we start the theory with the above assumption, see also Sect. 9.

Remark 2. We are not able to treat the case $\alpha_0 = 0$, i.e. an analysis starting with a 'pure' soft material is not possible, up to now. This corresponds also to the results of numerical experiments performed and presented in [8].

The notion 'hard' or 'soft' material is given by the yield condition. We confine the presentation to the von Mises-yield condition:

Condition 1. *Let κ_s, κ_h :$[0,T] \times \Omega \to \mathbb{R}$ be Lipschitz continuous functions such that $0 < \kappa_0 < \kappa_s \leq \kappa$, $0 < \kappa_0 < \kappa_h \leq \kappa$ with some $\kappa_0, \kappa \in \mathbb{R}$. We say that σ_s, σ_h satisfy the von Mises-yield condition if $|\sigma_{sD}| \leq \kappa_s$, $|\sigma_{hD}| \leq \kappa_h$. If $\kappa_s < \kappa_h$ then σ_s is said to be the 'soft' material and σ_h the hard material.*

This means that the modulus of the deviator $\sigma_{sD} = \sigma_s - (\mathrm{tr}\,\sigma_s/n)I$ and $\sigma_{hD} = \sigma_h - (\mathrm{tr}\,\sigma_h/n)I$ may not exceed the yield boundary.

Remark 3. The theory presented here works quite similar for other yield functions of the type $F(\sigma_D) \leq \kappa$; the function F has to be Lipschitz continuous, convex and coercive. We believe that the readability improves if we confine ourselves to the above case given in Condition 1.

Remark 4. For $n = 2$, in applications, σ_D might be defined in a different way, namely $\sigma_D = \sigma - (\mathrm{tr}\,\sigma/3)\,\boldsymbol{I}$, see [2].

2.2 Balance of Linear Momentum

The mixture is supposed to satisfy the balance of linear momentum, it means that we have

$$-\,div(\alpha(t,x)\sigma_s(t,x) + (1-\alpha(t,x))\sigma_h(t,x)) = f(t,x), \tag{9}$$

where $f : [0,1] \times \Omega \to \mathbb{R}^n$ is a given volume force (density).

The mixture underlies a mixed boundary condition

$$\begin{aligned} \nu(x)[\alpha(t,x)\sigma_s(t,x) + (1-\alpha(t,x))\sigma_h(t,x)] = p_0(t,x) \quad &\text{on } (0,T) \times \partial\Omega \setminus \Gamma, \\ u = 0 \quad &\text{on } (0,T) \times \Gamma. \end{aligned} \tag{10}$$

Here Γ is a portion of the boundary of Ω, possibly empty,[1] $p_0(t,x) : [0,T]\times (\partial\Omega \setminus \Gamma) \to \mathbb{R}^n$ is a boundary force and $\nu(x)$ is the outer unit vector at $x \in \partial\Omega$, normal to $\partial\Omega$. We extend the definition of p_0 to the whole boundary by setting $p_0 = 0$ on $(0,T)\times\Gamma$.

The precise version of the weak formulation to (9) and (10) reads as:

$$(\alpha(t,\cdot)\sigma_s(t,\cdot) + (1-\alpha(t,\cdot))\sigma_h(t,\cdot), \nabla\phi) = \int_{\partial\Omega} p_0(t,.)\phi \,\mathrm{d}o + (f(t,.),\phi),$$
$$t \in [0,T], \quad \forall\phi \in H^1_\Gamma(\Omega,\mathbb{R}^n). \tag{11}$$

The brackets (.,.) denote the usual $L^2(\Omega)$ scalar product, for scalar, vector or tensor valued functions as well. H^1_Γ denotes the subspace of the Sobolev space $H^{1,2}$ whose elements vanish on Γ in the sense of traces.

The weak form of the balance equation (11) is well defined if we assume

$$\begin{aligned} f &\in L^\infty((0,T);L^2(\Omega;\mathbb{R}^n)) =: L^\infty(L^2),\\ p_0 &\in L^\infty((0,T);L^2(\partial\Omega;\mathbb{R}^n)). \end{aligned} \tag{12}$$

Furthermore, $\partial\Omega$ and Γ are $(n-1)$-dimensional Lipschitz-manifolds. As follows from above, we use the shortened notation for the Sobolev and Bochener spaces.

2.3 Elasticity

Let

$$\tau = (\tau_s,\tau_h), \quad \hat\tau = (\hat\tau_s,\hat\tau_h) : [0,T]\times\Omega \to M^n_{sym}\times M^n_{sym}, \tag{13}$$

and

$$Q\begin{pmatrix}\hat\tau_s\,\tau_s\\ \hat\tau_h\,\tau_h\end{pmatrix} = \int_\Omega [A_s\hat\tau_s : \tau_s + A_h\hat\tau_h : \tau_h]\,\mathrm{d}x. \tag{14}$$

Here A_s and A_h are inverse elasticity tensors, say of the same structure as in the Lame-Navier linearized elasticity (with possibly different material coefficients). They model the elastic interaction within the soft and hard material. (It is possible to treat additional interaction terms $A_{sh}\hat\tau_h : \tau_s$.)

Assumption 2. *For simplicity we assume that A_s, A_h do not depend on $x \in \Omega$, $t \in [0,T]$, we assume that A_s and A_h are positively definite.*

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[1]Here for simplicity (in order to avoid the compatibility condition on the data), we assume that $\Gamma \neq \emptyset$.

Note that the matrix Q represents the total elastic energy of the mixture corresponding to the stresses τ_s and τ_h of a hard and soft materials at the loading 'time' $t \in [0, T]$. It is possible to carry out the theory presented here also for nonlinear convex potential with quadratic growth, but we restrict ourselves to the simplified case described above in order not to be overburdened.

2.4 The Variational Inequality for Elastic-Perfect-Plastic Mixtures

With these preparations, we are able to rigorously formulate the loading process of an elastic-perfect-plastic mixture by a variational inequality. The approach is similar to the standard 'Prandtl-Reuss model', see [5,7,11,12]. For the formulation, we need the following convex set $\mathbb{K}$ of pairs (τ_s, τ_h) of functions $\tau_s, \tau_h \colon [0, T] \times \Omega \to M^n_{sym}$ such that the following holds:

Integrability:

$$\tau_s, \tau_h \in L^\infty(L^2), \quad \dot{\tau}_s, \dot{\tau}_h \in L^\infty(L^2); \tag{15}$$

Initial condition:

$$\tau_s(0) = \sigma_{s0}, \quad \tau_h(0) = \sigma_{h0}; \tag{16}$$

Balance of linear momentum:

$$(\alpha\tau_s + (1-\alpha)\tau_h, \nabla\phi) = (f, \phi) + \int\limits_{\partial\Omega} p_0\phi \mathrm{d}o, \ \ \forall\phi \in H^1_\Gamma(\Omega, \mathbb{R}^3), t \in (0, T); \tag{17}$$

Yield conditions:

$$|\tau_{sD}| \le \kappa_s, |\tau_{hD}| \le \kappa_h. \tag{18}$$

Then the variational inequality, i.e. the problem for the Prandtl-Reuss mixture, reads as follows: *Find a pair* $(\sigma_s, \sigma_h) \in \mathbb{K}$ *such that*

$$\int_0^T Q \begin{pmatrix} \frac{\partial}{\partial t}(\alpha\sigma_s) & \alpha(\sigma_s - \tau_s) \\ \frac{\partial}{\partial t}((1-\alpha)\sigma_h) & (1-\alpha)(\sigma_h - \tau_h) \end{pmatrix} \mathrm{d}t \le 0 \quad \text{for all } (\tau_s, \tau_h) \in \mathbb{K}. \tag{19}$$

Remark 5. The function α is Lipschitz in x and t. In the model investigated in [10], α depends also on σ_s. Then we have a quasi variational inequality. For the mathematical treatment of this more complicated case we first have to analyze α's that are σ-independent in order to apply a fixed point theorem for more general case. This is why we restrict ourselves to the simpler case in this study.

For the proof of the main theorem, performed via several levels of approximation and estimates, the following condition seems to be crucial.

Condition 2 (Safe load condition for mixtures). *There exists a pair* $(\tilde{\sigma}_s, \tilde{\sigma}_h) \in \mathbb{K}$ *and a number* $s_0 > 0$ *such that*

$$|\tilde{\sigma}_{sD}| \leq \kappa_s - s_0, \quad |\tilde{\sigma}_{hD}| \leq \kappa_h - s_0. \tag{20}$$

In addition we will deal with differentiability assumptions with respect to the loading parameter t.

Assumption 3. *We assume the following differentiability properties of the data:*

$$\dot{\alpha},\ \dot{\tilde{\sigma}}_s,\ \dot{\tilde{\sigma}}_h,\ \dot{\kappa}_s,\ \dot{\kappa}_h \in L^\infty(L^\infty) \tag{21}$$

and, for refined regularity estimates,

$$\ddot{\alpha},\ \ddot{\tilde{\sigma}}_s,\ \ddot{\tilde{\sigma}}_h,\ \ddot{\kappa}_s,\ \ddot{\kappa}_h \in L^\infty(L^\infty). \tag{22}$$

Now we may state our main result.

Theorem 1 (Main theorem). *Let* α *satisfy Assumption 1, let* f *and* p_0 *satisfy* (12) *and let* κ_s, κ_h *satisfy Condition 1. Assume the safe load Condition 2 with the regularity* (21). *Furthermore let* A_s *and* A_h *be positively definite. Then there exists a unique solution of the variational inequality* (19).

Proof. (i) The uniqueness in the case $\alpha = \alpha(x,t)$ is a simple consequence of Assumption 2. Indeed, if σ_s, σ_h and $\hat{\sigma}_s, \hat{\sigma}_h$ are solutions to (19), choose $\tau_s = \hat{\sigma}_s$, $\tau_h = \hat{\sigma}_h$ in the equation for σ_s, σ_h, and use a similar argument with σ_s, σ_h and $\hat{\sigma}_s, \hat{\sigma}_h$ interchanged. Then one concludes that for $w_s = \sigma_s - \hat{\sigma}_s$, $w_h = \sigma_h - \hat{\sigma}_h$

$$\frac{1}{2}\int_0^T \int_\Omega A_s \frac{\partial}{\partial t}(\alpha w_s) : (\alpha w_s) + A_h \frac{\partial}{\partial t}((1-\alpha) w_h) : ((1-\alpha) w_h)\,\mathrm{d}x\,\mathrm{d}t \leq 0, \tag{23}$$

which implies that $w_s = w_h = 0$.

(ii) The existence result is established in the following Sects. 3–6 and 8. In Sect. 3 the approximation of the variational inequality by a penalty method is presented. The variational inequality is approximated by an equation with the balance of linear momentum free functions as test functions and a penalty term where the yield condition is penalized. This is a familiar approach in the framework of classical models, like those of Hencky or Prandtl-Reuss, or hardening models. In Sect. 4 the penalty approximation is discretized via a Rothe method. There we also derive uniform discrete $L^\infty(L^2)$-estimates for the approximate stresses for the soft and hard material, as well as discrete uniform $L^1(L^1)$-estimates for the approximate strain velocities. In Sect. 5 we prove uniform discrete $L^2(L^2)$-estimates for first difference quotients of the

Rothe Approximation. This allows us, in Sect. 6, to pass to the limit in the Rothe Approximation and we obtain, via weak compactness and monotonicity arguments, a solution $\sigma_{s\mu}$, $\sigma_{h\mu}$ of the penalty equation.

Finally, in Sect. 8 we pass to the limit $\mu \to +0$ and obtain a solution of the variational inequality. Again, the proof runs via weak convergence and monotonicity, since the $L^2(L^2)$-estimates for the stress velocities in Sects. 4–6 turned out to be uniform also with respect to μ. The estimates of the stresses and their velocities depend on the $L^\infty(L^\infty)$ norms of $\dot{\alpha}, \dot{\kappa}_s, \dot{\kappa}_h, \dot{\tilde{\sigma}}_s, \dot{\tilde{\sigma}}_h$. For $L^\infty(L^2)$-estimates for $\dot{\sigma}_s, \dot{\sigma}_h$ (rather than $L^2(L^2)$-estimates) we need to assume that $\ddot{\alpha}, \ddot{\kappa}_s, \ddot{\kappa}_h, \ddot{\tilde{\sigma}}_s, \ddot{\tilde{\sigma}}_h \in L^\infty(L^\infty)$. But this additional derivative in the assumption would be very restrictive to the considered class of nonlinear models, like the one in [10], where α, κ_s, κ_h depend on the history of the stress of the soft material, see the discussion in Sect. 9. For this reason, we arranged the existence theory in the $L^2(L^2)$ setting for $\dot{\sigma}_s$, $\dot{\sigma}_h$. □

Remark 6. The uniqueness needs not hold if α is σ-dependent.

In the classical theory of the Prandtl-Reuss model the inclusion $\dot{\sigma} \in L^\infty(L^2)$ follows in a natural way. A similar theorem is also possible in the present setting.

Theorem 2. *Under the assumptions of the main theorem and* (22)*, the solution couple σ_s, σ_h for the Prandtl-Reuss mixture satisfies*

$$\dot{\sigma}_s, \dot{\sigma}_h \in L^\infty(L^\infty((0,T) \times \Omega; M^n_{sym})) \tag{24}$$

with corresponding $L^\infty(L^\infty)$ bounds depending additionally on the $L^\infty(L^\infty)$ norms of $\ddot{\alpha}, \ddot{\kappa}_s, \ddot{\kappa}_h, \ddot{\tilde{\sigma}}_s, \ddot{\tilde{\sigma}}_h$.

The proof is done in Sect. 7, where a corresponding bound for the solution of the penalty approximation is established.

The variational inequality (19) is a complete dual formulation of the mechanical problem, i. e. the strains and strain velocities do not appear a priori. However, due to uniqueness and the construction of a solution via the penalty method we conclude the following theorem.

Theorem 3. *Under the assumptions of the main theorem, there exist a Riesz measure $\dot{u} \in C^*((0,T) \times \Omega; \mathbb{R}^n)$ such that $\frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}^T)$ is also a Riesz measure and*

$$\int_0^T \left(A_s \frac{\partial}{\partial t}(\alpha\sigma_s) - \frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}^T), \sigma_s - \tau_s \right) \mathrm{d}t \le 0$$

$$\int_0^T \left(A_s \frac{\partial}{\partial t}((1-\alpha)\sigma_h) - \frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}^T), \sigma_h - \tau_h \right) \mathrm{d}t \le 0 \tag{25}$$

for all τ_s, $\tau_h \in C((0,T) \times \bar{\Omega}; M^n_{sym})$ such that $|\tau_{sD}| \le \kappa_s$, $|\tau_{hD}| \le \kappa_s$. If the assumptions of Theorem 2 are satisfied, the

$$\dot{u} \in L^\infty(L^{\frac{n}{n-1}}). \tag{26}$$

In Sect. 3, from the penalty equation, we derive partial approximate strains and interpret the penalty terms as approximate plastic strain velocities. It turns out that the rates of partial plastic strain for hard and soft materials are equal. In Sects. 4–7, the corresponding $L^1(L^1)$ and $L^\infty(L^1)$-estimates for the strain velocities are proved. This works analogously as in the classical Prandtl-Reuss case via the safe load condition. For the $L^\infty(L^{\frac{n}{n-1}})$ inclusion the tools from Sect. 7 are needed.

There is a lot of further analogy between the problem for the Prandtl-Reuss mixture model considered here and the classical Prandtl-Reuss model. For example, if $n = 2$ one can prove an $L^\infty(L^{2+\delta})$ estimate for the strains, based on the reverse Hölder-inequality and Gehring's lemma. Furthermore the H^1_{loc}-differentiability of the stresses can be done similarly as in the classical Prandtl-Reuss case, see [1, 4]. However, in our case, we need extra differentiability properties and corresponding estimates for the volume fraction α and the yield quantities. This decreases the possibilities to establish the same result if α depends nonlinearly on $\dot{e}_{ps}$.

In the next sections we establish the existence theory needed to complete the proof of Theorem 1.

3 The Penalty Equation

Analogously to the classical Prandtl-Reuss problem we approximate the variational inequality by penalizing the yield conditions. The approximation we use reads:

Find a pair $(\sigma_s, \sigma_h) = (\sigma_{s\mu}, \sigma_{s\mu})$ *such that the properties* (15)–(18) *are satisfied and the following penalty equation holds a.e. with respect to* $t \in [0, T]$

$$Q\begin{pmatrix} \frac{\partial}{\partial t}(\alpha\sigma_s) & \alpha\tau_s \\ \frac{\partial}{\partial t}((1-\alpha)\sigma_h) & (1-\alpha)\tau_h \end{pmatrix}$$
$$+ (\mu^{-1}[|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|}, \alpha\tau_s) + (\mu^{-1}[|\sigma_{hD}| - \kappa_h]_+ \frac{\sigma_{hD}}{|\sigma_{hD}|}, (1-\alpha)\tau_h) = 0 \tag{27}$$

for all $\tau_s, \tau_h : \Omega \to M^n_{sym}$, $\tau_s, \tau_h \in L^2$ *satisfying the balance of linear momentum with force zero*

$$(\alpha\tau_s + (1-\alpha)\tau_h, \nabla\phi) = 0, \qquad \forall\phi \in H^1_\Gamma(\Omega, \mathbb{R}^n). \tag{28}$$

μ *is here the penalty-parameter,* $\mu > 0$.

The penalty equation (27) has a solution:

Theorem 4. *Under the assumptions of Theorem 1, Eq.* (27) *has a unique solution* $\sigma_{s\mu}, \sigma_{h\mu} \in L^\infty(L^2)$ *such that* $\dot{\sigma}_{s\mu}, \dot{\sigma}_{h\mu} \in L^2(L^2)$, *with corresponding uniform bounds as* $\mu \to +0$.

As $\mu \to +0$ *the solutions* $\sigma_{s\mu}, \sigma_{h\mu}$ *converge strongly in* $L^2(L^2)$ *to the solution* σ_s, σ_h *of the variational inequality* (19)*. Furthermore we have the uniform*

$L^1(L^1)$-bound for the penalty part

$$\mu^{-1}\int_0^T\int_\Omega [|\sigma_{s\mu D}|-\kappa_s]_+(|\sigma_{s\mu D}|+1)+[|\sigma_{h\mu D}|-\kappa_h]_+(\sigma_{h\mu D}+1)\,\mathrm{d}x\,\mathrm{d}t \\ \leq K(\dot\alpha,\dot\kappa_s,\dot\kappa_h) \quad (29)$$

and the bound

$$\operatorname*{ess\,sup}_t \quad \mu^{-1}\int_\Omega [|\sigma_{s\mu D}|-\kappa_s]_+^2+[|\sigma_{h\mu D}|-\kappa_h]_+^2\,\mathrm{d}x \leq K(\dot\alpha,\dot\kappa_s,\dot\kappa_h) \quad (30)$$

uniformly as $\mu \to +0$.

The proof of Theorem 4 is established in Sects. 4–6.

3.1 Reconstruction of Partial Strains

In Eq. (27), we choose

$$\tau_s = \alpha^{-1}\tau_0, \quad \tau_h = 0,$$

or vice versa

$$\tau_s = 0, \quad \tau_h = (1-\alpha)^{-1}\tau_0,$$

where $(\tau_0, \nabla\phi) = 0$ for all $\phi \in H^1_\Gamma$. These pairs (τ_s, τ_h) of test functions are admissible since they satisfy (28). Thus we obtain two equations

$$\left(A_s\frac{\partial}{\partial t}\left(\alpha\sigma_{s\mu}\right), \tau_0\right) + \left(\mu^{-1}[|\sigma_{s\mu D}|-\kappa_s]_+\frac{\sigma_{s\mu D}}{|\sigma_{s\mu D}|}, \tau_0\right) = 0 \quad (31)$$

$$\left(A_h\frac{\partial}{\partial t}\left((1-\alpha)\sigma_{h\mu}\right), \tau_0\right) + \left(\mu^{-1}[|\sigma_{h\mu D}|-\kappa_h]_+\frac{\sigma_{h\mu D}}{|\sigma_{h\mu D}|}, \tau_0\right) = 0 \quad (32)$$

a.e. in $[0,T]$, for all $\tau_0 \in L^2(\Omega, M^n_{sym})$ fulfilling (11). Conversely, from (31), (32) we reach (27).

Now, we use the symmetric Helmholtz decomposition in L^2 to conclude that there exists $v_{s\mu}, v_{h\mu} \in L^2(0,T; H^1_\Gamma(\Omega,\mathbb{R}^n))$ such that $v_{s\mu} = \dot u_{s\mu}$, $v_{h\mu} = \dot u_{h\mu}$ and

$$A_s\frac{\partial}{\partial t}\left(\alpha\sigma_{s\mu}\right) + \mu^{-1}[|\sigma_{s\mu D}|-\kappa_s]_+\frac{\sigma_{s\mu D}}{|\sigma_{s\mu D}|} = \frac{1}{2}(\nabla\dot u_{s\mu} + \nabla\dot u_{s\mu}{}^T), \quad (33)$$

$$A_h \frac{\partial}{\partial t}\left((1-\alpha)\sigma_{h\mu}\right) + \mu^{-1}[|\sigma_{h\mu D}| - \kappa_h]_+ \frac{\sigma_{h\mu D}}{|\sigma_{h\mu D}|} = \frac{1}{2}(\nabla \dot{u}_{h\mu} + \nabla \dot{u}_{h\mu}{}^T). \tag{34}$$

Interestingly, there is a relation between $\dot{u}_{s\mu}$ and $\dot{u}_{h\mu}$ which follows from the penalty equation, namely

$$\left(\dot{u}_{s\mu}, \operatorname{div}(\alpha\tau_s)\right) + \left(\dot{u}_{h\mu}, \operatorname{div}((1-\alpha)\tau_h)\right) = 0, \tag{35}$$

that is valid for all τ_s, $\tau_h \in L^2(\Omega; M^n_{sym})$ such that $\alpha\tau_s + (1-\alpha)\tau_h$ satisfies the balance of linear momentum with zero force.

Theorem 5. *Let $\dot{u}_{s\mu}$ and $\dot{u}_{h\mu}$ be the partial strain velocities arising in* (33) *and* (34)*. Then we have*

$$\frac{1}{2}(\nabla \dot{u}_{s\mu} + \nabla \dot{u}_{s\mu}{}^T) = \frac{1}{2}(\nabla \dot{u}_{h\mu} + \nabla \dot{u}_{h\mu}{}^T). \tag{36}$$

Proof. From (28) and (35) we conclude that

$$\left(\frac{1}{2}(\nabla \dot{u}_{s\mu} + \nabla \dot{u}_{s\mu}{}^T), \alpha\tau_s\right) - \left(\frac{1}{2}(\nabla \dot{u}_{h\mu} + \nabla \dot{u}_{h\mu}{}^T), \alpha\tau_s\right) = 0 \tag{37}$$

for all τ_s, τ_h such that $(\alpha\tau_h + (1-\alpha)\tau_s, \nabla\phi) = 0$, $\phi \in H^1_\Gamma$. For arbitrary $\tau_s^0 \in L^2$ we define

$$\tau_h^0 = -\frac{\alpha}{1-\alpha}\tau_s^0. \tag{38}$$

Then obviously (τ_s^0, τ_h^0) satisfies the balance of linear momentum with zero force and we conclude that

$$\frac{1}{2}(\nabla \dot{u}_{s\mu} + \nabla \dot{u}_{s\mu}{}^T) = \frac{1}{2}(\nabla \dot{u}_{h\mu} + \nabla \dot{u}_{h\mu}{}^T) \qquad \text{in } L^2. \tag{39}$$

Remark 7. Clearly, (36) extends to the limit $\mu \to +0$ in the space $C^*([0,T] \times \Omega; \mathbb{R}^n)$ of Riesz measures.

From the mathematical point of view, we believe that (33) and (34) are the 'best' equations to understand the analysis of Prandtl-Reuss mixtures. The functions $v_{s\mu}$ and $v_{h\mu}$ can be interpreted as the approximate total strain velocities for the soft and the hard material. We have written $v_{s\mu} = \dot{u}_{s\mu}$, $v_{h\mu} = \dot{u}_{h\mu}$, assuming some initial condition for $u_{s\mu}, u_{h\mu}$. The penalty terms correspond to the (approximate) velocities of plastic deformation and the terms $A_s \frac{\partial}{\partial t}\left(\alpha\sigma_{s\mu}\right)$, $A_h \frac{\partial}{\partial t}\left(\alpha\sigma_{h\mu}\right)$ model the elastic deformation of the hard and soft material. Note that (33) and (34) are equivalent to (27).

From the estimates of the penalty term, proved in the following sections, we state the following theorem.

Theorem 6. *Under the assumptions of Theorem 4 we have the uniform estimate*

$$\sup_{\mu}\left\{||\nabla\dot{u}_{s\mu}+\nabla\dot{u}_{s\mu}{}^{T}||_{L^1(L^1)}+||\nabla\dot{u}_{h\mu}+\nabla\dot{u}_{h\mu}{}^{T}||_{L^1(L^1)}\right\}\leq K(\dot{\alpha},\dot{\kappa}_s,\dot{\kappa}_h). \quad (40)$$

Due to Temam's imbedding theorem we derive

Corollary 1.

$$\sup_{\mu}\left\{||\dot{u}_{s\mu}||_{L^1(L^{\frac{n}{n-1}})}+||\dot{u}_{h\mu}||_{L^1(L^{\frac{n}{n-1}})}\right\}\leq K(\dot{\alpha},\dot{\kappa}_s,\dot{\kappa}_h) \quad \textit{as} \quad \mu\to+0. \quad (41)$$

From the $L^1(L^1)$-estimates for the strain velocities and the penalty terms we have, for a subsequence

$$\frac{1}{2}(\nabla\dot{u}_{s\mu}+\nabla\dot{u}_{s\mu}{}^{T})\rightharpoonup\frac{1}{2}(\nabla\dot{u}_s+\nabla\dot{u}_s{}^{T}) \quad \text{weakly in } C^*([0,T]\times\bar{\Omega};\mathbb{R}^n), \quad (42)$$

$$\frac{1}{2}(\nabla\dot{u}_{h\mu}+\nabla\dot{u}_{h\mu}{}^{T})\rightharpoonup\frac{1}{2}(\nabla\dot{u}_h+\nabla\dot{u}_h{}^{T}) \quad \text{weakly in } C^*([0,T]\times\bar{\Omega};\mathbb{R}^n), \quad (43)$$

i.e. the limiting strains are only Riesz-measures. If more regularity is assumed in the safe load condition (see further theorems) we have that $\dot{u}_s,\dot{u}_h\in L^\infty(L^{\frac{n}{n-1}})$, i.e. the velocities and the displacements, are at least functions.

For the penalty terms we have

$$\mu^{-1}[|\sigma_{s\mu D}|-\kappa_s]_+\frac{\sigma_{s\mu D}}{|\sigma_{s\mu D}|}\rightharpoonup\dot{e}_{ps}, \qquad \mu^{-1}[|\sigma_{h\mu D}|-\kappa_h]_+\frac{\sigma_{h\mu D}}{|\sigma_{h\mu D}|}\rightharpoonup\dot{e}_{ph} \quad (44)$$

both weakly in $C^*([0,T]\times\bar{\Omega})$, as $\mu\to+0$.

If we would know that $\sigma_{s\mu}\to\sigma_s$, $\sigma_{h\mu}\to\sigma_h$ in C (= space of continuous functions), we could prove the representation

$$\dot{e}_{ps}=\lambda_s\frac{\sigma_{sD}}{|\sigma_{sD}|}, \quad \dot{e}_{ph}=\lambda_h\frac{\sigma_{hD}}{|\sigma_{hD}|}, \quad (45)$$

where λ_s, λ_h is the weak C^*-limit of $\mu^{-1}[|\sigma_{s\mu D}|-\kappa_s]_+$, $\mu^{-1}[|\sigma_{h\mu D}|-\kappa_h]_+$. The support of λ_s and λ_h is on the set $|\sigma_{sD}|\geq\kappa_s$ and $|\sigma_{hD}|\geq\kappa_h$, respectively. In the case of two dimensions there is a substitute of the argument, taking into account that $\sigma_s,\sigma_h\in C$ is not known, see the discussion in Sect. 9. With the above convergences in C^* the solution of the variational inequality satisfies the equations

$$\frac{1}{2}(\nabla\dot{u}_s+\nabla\dot{u}_s{}^{T})=A_s\frac{\partial}{\partial t}(\alpha\sigma)+\dot{e}_{ps}, \quad (46)$$

$$\frac{1}{2}(\nabla\dot{u}_h+\nabla\dot{u}_h{}^{T})=A_h\frac{\partial}{\partial t}((1-\alpha)\sigma)+\dot{e}_{ph}. \quad (47)$$

With a more restrictive assumption we gain $L^\infty(L^2)$-bounds for $\dot\sigma_{s\mu}$, $\dot\sigma_{h\mu}$ and $L^\infty(L^1)$-bounds for the partial strains. This is proved in Sect. 7.

Theorem 7. *Under the assumption of Theorem 4 and the additional requirement that the safe loads and the data* α, κ_s, κ_h *satisfy*

$$\ddot{\tilde\sigma}_s, \ddot{\tilde\sigma}_h, \ddot\alpha, \ddot\kappa_s, \ddot\kappa_h \in L^\infty(L^\infty) \tag{48}$$

there holds the uniform bound

$$\sup_\mu \left\{ \|\dot\sigma_{s\mu}\|_{L^\infty(L^2)} + \|\dot\sigma_{h\mu}\|_{L^\infty(L^2)} \right\} \le K(\ddot\alpha, \ddot\kappa_s, \ddot\kappa_h) \tag{49}$$

and

$$\left\| \nabla\dot u_{s\mu} + \nabla\dot u_{s\mu}{}^T \right\|_{L^\infty(L^1)} + \left\| \nabla\dot u_{h\mu} + \nabla\dot u_{h\mu}{}^T \right\|_{L^\infty(L^1)} \le K(\ddot\alpha, \ddot\kappa_s, \ddot\kappa_h). \tag{50}$$

Remark 8. In the above estimates we indicate how the bounds depend on the derivative of α, κ_s, κ_h. This is relevant, later, for the treatment of the quasivariational inequality, where α, κ_s, κ_h depend on σ_s (and also σ_h). A dependence of $\dot\alpha$, $\dot\kappa_s$, $\dot\kappa_h$ does not give problems modelling α, κ_s, κ_h, but a dependence of $\ddot\alpha$, $\ddot\kappa_s$, $\ddot\kappa_h$ leads to restrictions.

It is useful to observe that the solutions of the penalty problems satisfy the variational inequalities

$$\left(A_s \frac{\partial}{\partial t}\left(\alpha\sigma_{s\mu}\right), \sigma_{s\mu} - \omega_s \right) \le \left(\frac{1}{2}(\nabla\dot u_{s\mu} + \nabla\dot u_{s\mu}{}^T), \sigma_{s\mu} - \omega_s \right) \tag{51}$$

$$\left(A_h \frac{\partial}{\partial t}\left((1-\alpha)\sigma_{h\mu}\right), \sigma_{h\mu} - \omega_h \right) \le \left(\frac{1}{2}(\nabla\dot u_{h\mu} + \nabla\dot u_{h\mu}{}^T), \sigma_{h\mu} - \omega_h \right) \tag{52}$$

a.e. with respect to t, for all $\omega_s, \omega_h \in L^2(\Omega, M^n_{sym})$ such that $|\omega_{sD}| \le \kappa_s$, $|\omega_{hD}| \le \kappa_h$.

This follows from the

$$\left([|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|} : (\sigma_{sD} - \omega_{sD}) \right) =$$
$$\left([|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|} - \underbrace{[|\omega_{sD}| - \kappa_s]_+}_{=0} \frac{\omega_{sD}}{|\omega_{sD}|}, \sigma_{sD} - \omega_{sD} \right) \ge 0 \tag{53}$$

and, correspondingly, for the hard material.

4 The Rothe Approximation

4.1 Definition and Solvability of the Rothe Approximation

We discretize the loading interval [0,T] by a discrete set $I_\delta = \{k\delta | k = 0, \ldots, N\}$ with mesh size $\delta = T/N$ and approximate $\frac{\partial}{\partial t} w(t,.)$ by the backward difference quotient

$$D^{-\delta} w(t,.) = \delta^{-1}(w(t,.) - w(t-\delta,.)). \tag{54}$$

Then the **Rothe approximation** of the penalty approximation (27) reads:

Find a pair $(\sigma_s, \sigma_h) = (\sigma_{s\mu\delta}, \sigma_{h\mu\delta}) : I_\delta \times \Omega \to M^n_{sym} \times M^n_{sym}$ *such that* (σ_s, σ_h) *satisfies* (11) *for* $t \in I_\delta$ *and that*

$$Q \begin{pmatrix} D^{-\delta}(\alpha\sigma_s) & \alpha\tau_s \\ D^{-\delta}((1-\alpha)\sigma_h) & (1-\alpha)\tau_h \end{pmatrix} \tag{55}$$

$$+ (\mu^{-1}[|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|}, \alpha\tau_s) + (\mu^{-1}[|\sigma_{hD}| - \kappa_h]_+ \frac{\sigma_{hD}}{|\sigma_{hD}|}, (1-\alpha)\tau_h) = 0$$

for all $(\tau_s, \tau_h) : \Omega \to M^n_{sym} \times M^n_{sym}$, $\tau_s, \tau_h \in L^2$ *such that* (28) *is satisfied.*

Lemma 1. *Let* A_s, A_h *be positively definite and let the set of all* (σ_s, σ_h) *satisfying the balance of linear momentum not be empty. Let the Assumption 1 on* α *be satisfied. Then* (55) *has a unique solution* (σ_s, σ_h).

Proof. We assume that $(\sigma_s, \sigma_h)(t)$ has been constructed for $t = 0, \delta, \ldots, (k-1)\delta$ and we want to construct $(\sigma_s, \sigma_h)(t^*)$, $t^* = k\delta$. This is done by minimizing the functional

$$J\ (\sigma_s^*, \sigma_h^*) = \frac{1}{2\delta}(\alpha^2(t^*)A_s\sigma_s^*, \sigma_s^*) + \frac{1}{2\delta}((1-\alpha(t^*))^2 A_h\sigma_h^*, \sigma_h^*) \tag{56}$$

$$-\frac{1}{\delta}(\alpha(t^*-\delta)A_s\sigma_s(t^*-\delta), \alpha(t^*)\sigma_s^*) - \frac{1}{\delta}((1-\alpha(t^*-\delta))A_h\sigma_h(t^*-\delta), \sigma_h^*)$$

$$+\frac{1}{2\mu}\int_\Omega \alpha(t^*)[|\sigma_{sD}^*| - \kappa_s(t^*)]_+^2\,dx + \frac{1}{2\mu}\int_\Omega (1-\alpha(t^*))[|\sigma_{hD}^*| - \kappa_h(t^*)]_+^2\,dx$$

on the set of pairs $(\sigma_s^*, \sigma_h^*) : \Omega \to M^n_{sym} \times M^n_{sym}$, $\sigma_s^*, \sigma_h^* \in L^2$ which satisfies the balance of linear momentum

$$(\alpha(t^*)\sigma_s^* + (1-\alpha(t^*))\sigma_h^*, \nabla\phi) = (f(t^*), \phi) + \int_{\partial\Omega} p_0(t^*)\phi,\ \phi \in H^{1,2}_\Gamma. \tag{57}$$

Since the functional J is strictly convex, coercive, and continuous in the strong topology of L^2, a unique minimizer (σ_s^*, σ_h^*) exists and we define $\sigma_s(t^*) = \sigma_s^*$ and $\sigma_h(t^*) = \sigma_h^*$.

It is easy to see that the Lagrange-Euler equation to the above minimization problem is just the Rothe approximation (55). The uniqueness follows with a monotonicity argument. □

4.2 First Estimates for the Rothe Approximation

In this section, we derive discrete versions of $L^\infty(L^2)$-estimates for the solutions (σ_s, σ_h) of the Rothe equation and also discrete versions of $L^1(L^1)$-estimates for the penalty term. These estimates are uniform as $\delta \to +0$. Since it is convenient to have the uniformity of these estimates also with respect to $\mu \to +0$, we assume a compatibility condition for the yield conditions and the balance of linear momentum.

Condition 3 (Weak safe load condition). *There exists $(\tilde{\sigma}_s, \tilde{\sigma}_h) \in \mathbb{K}$ (cf. Sect. 2.4) such that $\tilde{\sigma}_s, \tilde{\sigma}_h, \dot{\tilde{\sigma}}_s, \dot{\tilde{\sigma}}_h \in L^\infty(L^2)$.*

Theorem 8. *Let $(\sigma_s, \sigma_h) = (\sigma_{s\mu\delta}, \sigma_{h\mu\delta})$ be a solution of the Rothe problem, and let Assumptions 1 and 2 and Condition 3 hold. Then*

$$\max_{t=0,\ldots,N\delta} \int_\Omega |\sigma_s|^2 + |\sigma_h|^2 \, dx$$
$$+ \delta \sum_{t=\delta,\ldots,N\delta} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+ ||\sigma_{sD}| - |\tilde{\sigma}_{sD}|| + [|\sigma_{hD}| - \kappa_h]_+ ||\sigma_{hD}| - |\tilde{\sigma}_{hD}|| \, dx$$
$$\leq K + K \int_0^t \int_\Omega |\frac{\partial}{\partial t}(\alpha \tilde{\sigma}_s)|^2 + |\frac{\partial}{\partial \alpha}((1-\alpha)\tilde{\sigma}_h)|^2 \, dx \, dt, \quad (58)$$

where the constant K does not depend on $\delta \to +0$ and $\mu \to +0$.

Proof. We use the pair $(\sigma_s - \tilde{\sigma}_s, \sigma_h - \tilde{\sigma}_h)$ as a test function in (55) and obtain

$$\delta \sum_{t=\delta,\ldots,N\delta} Q \begin{pmatrix} D^{-\delta}(\alpha\sigma_s) & \alpha(\sigma_s - \tilde{\sigma}_s) \\ D^{-\delta}((1-\alpha)\sigma_h) & (1-\alpha)(\sigma_h - \tilde{\sigma}_h) \end{pmatrix}$$
$$+ \mu^{-1}\delta \sum_{t=\delta,\ldots,N\delta} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_s}{|\sigma_s|} : (\sigma_s - \tilde{\sigma}_s) \quad (59)$$
$$+ [|\sigma_{hD}| - \kappa_h]_+ \frac{\sigma_h}{|\sigma_h|} : (\sigma_h - \tilde{\sigma}_h) \, dx = 0.$$

We abbreviate

$$E_t = Q\begin{pmatrix} \alpha\sigma_s & \alpha\sigma_s \\ (1-\alpha)\sigma_h & (1-\alpha)\sigma_h \end{pmatrix}\Big|_t \tag{60}$$

and we use Hölder's inequality

$$Q\begin{pmatrix} D^{-\delta}(\alpha\sigma_s) & \alpha(\sigma_s-\tilde{\sigma}_s) \\ D^{-\delta}((1-\alpha)\sigma_h) & (1-\alpha)(\sigma_h-\tilde{\sigma}_h) \end{pmatrix}\Big|_t \geq \frac{1}{2\delta}(E_t - E_{t-\delta}) - Q\begin{pmatrix} D^{-\delta}(\alpha\sigma_s) & \alpha\tilde{\sigma}_s \\ D^{-\delta}((1-\alpha)\sigma_h) & (1-\alpha)\tilde{\sigma}_h \end{pmatrix}\Big|_t. \tag{61}$$

Using the arguments similar to (53) both for the soft and hard material we observe that the third term in (59), which comes from the penalty, is nonnegative and that

$$\text{penalty terms in (59)} \geq \mu^{-1}\delta \sum_{t=\delta,\dots,N\delta}\int_\Omega [|\sigma_{sD}|-\kappa_s]_+(|\sigma_{sD}|-|\tilde{\sigma}_{sD}|)+ + [|\sigma_{hD}|-\kappa_h]_+(|\sigma_{hD}|-|\tilde{\sigma}_{hD}|)\,\mathrm{d}x \geq 0. \tag{62}$$

Note that $|\sigma_{sD}| - |\tilde{\sigma}_{sD}| \geq 0$ on $[|\sigma_{sD}| - \kappa_s]_+$, similar for $(|\sigma_{hD}| - |\tilde{\sigma}_{hD}|)$, and it also holds that

$$\frac{\sigma_{sD}}{|\sigma_{sD}|} : (\sigma_s - \tilde{\sigma}_s) \geq |\sigma_{sD}| - |\tilde{\sigma}_s|. \tag{63}$$

From (59), (61), and (62) we obtain

$$\begin{aligned} T_1 + T_2 + T_3 := \\ \delta \sum_{t=\delta,\dots,N\delta} \Bigg\{ & \frac{1}{2\delta}(E_t - E_{t-\delta}) - Q\begin{pmatrix} D^{-\delta}(\alpha\sigma_s) & \alpha\tilde{\sigma}_s \\ D^{-\delta}((1-\alpha)\sigma_h) & (1-\alpha)\tilde{\sigma}_h \end{pmatrix} \\ & + \frac{1}{\mu}\int_\Omega \alpha[|\sigma_{sD}|-\kappa_s]_+(|\sigma_{sD}|-|\hat{\sigma}_{sD}|) \\ & + (1-\alpha)[|\sigma_{hD}|-\kappa_h]_+(|\sigma_{hD}|-|\hat{\sigma}_{hD}|)\,\mathrm{d}x \Bigg\} \leq 0. \end{aligned} \tag{64}$$

Finally, we obtain via partial summation and Hölder's inequality

$$\delta \sum_{t=\delta,\dots,N\delta} Q\begin{pmatrix} D^{-\delta}(\alpha\sigma_s) & \alpha\tilde{\sigma}_s \\ D^{-\delta}((1-\alpha)\sigma_h) & (1-\alpha)\tilde{\sigma}_h \end{pmatrix}$$

$$= -\delta \sum_{t=\delta,\dots,N\delta} Q\begin{pmatrix} \alpha\sigma_s & D^{\delta}(\alpha\tilde{\sigma}_s) \\ (1-\alpha)\sigma_h & D^{\delta}((1-\alpha)\tilde{\sigma}_h) \end{pmatrix}$$

$$+ \int_\Omega Q\begin{pmatrix} \alpha\sigma_s & \alpha\tilde{\sigma}_s \\ (1-\alpha)\sigma_h & (1-\alpha)\tilde{\sigma}_h \end{pmatrix} \mathrm{d}x \Big|_{t=0}^{t=N\delta}$$

$$\leq K\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega |\sigma_s|^2 + |\sigma_h|^2 \,\mathrm{d}x \tag{65}$$

$$+ K\int_0^t \int_\Omega |\frac{\partial}{\partial t}(\alpha\tilde{\sigma}_s)|^2 + |\frac{\partial}{\partial t}((1-\alpha)\tilde{\sigma}_h)|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \epsilon_0 \int_\Omega |\sigma_s|^2 + |\sigma_h|^2 \,\mathrm{d}x\Big|_{t=N\delta} + K_{\epsilon_0}\int_\Omega |\tilde{\sigma}_s|^2 + |\tilde{\sigma}_h|^2\,\mathrm{d}x\Big|_{t=N\delta}$$

$$+ K + \int_\Omega |\sigma_{os}|^2 + |\sigma_{oh}|^2 \,\mathrm{d}x.$$

Recall that σ_{os}, σ_{oh} are the initial values of σ_s, σ_h. We use this inequality in (64) for estimating from below. Observe also that

$$\sum_{t=\delta,\dots,N\delta} E_t - E_{t-\delta} = E_{N\delta} - E_0. \tag{66}$$

Since $T_3 \geq 0$ the statement of Theorem 8 then follows by using a discrete version of Gronwall's inequality. □

Corollary 2. *Under the additional assumption of the safe load Condition 2 for $\tilde{\sigma}_s$, $\tilde{\sigma}_h$ we have, for the solutions of the Rothe approximation, the discrete $L^1(L^1)$ estimate*

$$\mu^{-1}\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+ + [|\sigma_{hD}| - \kappa_h]_+ \,\mathrm{d}x$$

$$+ \mu^{-1}\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+|\sigma_{sD}| + [|\sigma_{hD}| - \kappa_h]_+|\sigma_{hD}| \,\mathrm{d}x$$

$$\leq K + K\int_0^t\int_\Omega |\frac{\partial}{\partial t}(\alpha\tilde{\sigma}_s)| + |\frac{\partial}{\partial \alpha}((1-\alpha)\tilde{\sigma}_h)|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{67}$$

holds uniformly as $\delta \to +0$, $\mu \to +0$.

Proof. If we have the safe load condition, the penalty part in (64) can be estimated from below by

$$P_{\delta\mu 1} = \mu^{-1}\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega \alpha[|\sigma_{sD}| - \kappa_s]_+ s_0 + (1-\alpha)[|\sigma_{hD}| - \kappa_h]_+ s_0 \,\mathrm{d}x, \quad (68)$$

where $s_0 > 0$ comes from the safe load condition. Thus, this term contributes to the estimates and we obtain

$$|P_{\delta\mu 1}| \le K \text{ uniformly.} \quad (69)$$

Once knowing this,by inspection of (64), we observe that also

$$P_{\delta\mu 2} = \mu^{-1}\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega \alpha[|\sigma_{sD}| - \kappa_s]_+ |\sigma_{sD}| + (1-\alpha)[|\sigma_{hD}| - \kappa_h]_+ |\sigma_{hD}| \,\mathrm{d}x \quad (70)$$

remains bounded as $\mu \to 0$ and $\delta \to +0$. □

5 Estimates for the Rothe Approximation

We finally present a discrete analogue of an $H^{1,2}(L^2)$-estimate for the solutions of the Rothe-equation.

Theorem 9. *Assume the safe load condition with an admissible pair* $(\tilde\sigma_s, \tilde\sigma_h) \in \mathbb{K}$ *such that* $\dot{\tilde\sigma}_s, \dot{\tilde\sigma}_h \in L^\infty(L^\infty)$. *Let* $\dot\alpha, \dot\kappa_s, \dot\kappa_h \in L^\infty(L^\infty)$, *and* A_s, A_h *be positively definite and symmetric. Then there is a constant* $C(\dot\alpha, \dot\kappa_s, \dot\kappa_h, \dot{\tilde\sigma}_s, \dot{\tilde\sigma}_h)$ *such that for the solution* $(\sigma_s, \sigma_h) = (\sigma_{s\mu\delta}, \sigma_{h\mu\delta})$

$$\begin{aligned}&\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega |D^{-\delta}\sigma_s|^2 + |D^{-\delta}\sigma_h|^2 \,\mathrm{d}x \\ &+ \mu^{-1} \sup_{t=0,\dots,N\delta} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+^2 + [|\sigma_{hD}| - \kappa_h]_+^2 \,\mathrm{d}x \le C(\dot\alpha, \dot\kappa_s, \dot\kappa_h, \dot{\tilde\sigma}_s, \dot{\tilde\sigma}_h)\end{aligned} \quad (71)$$

uniformly as $\delta \to 0$, $\mu \to 0$.

Proof. We use the shift operator $S^{-\delta}$ defined by

$$S^{-\delta}(t) = w(t-\delta). \quad (72)$$

It is easy to see that the following pair (τ_s^*, τ_h^*) satisfies the balance of linear momentum with zero force

$$\tau_s^* = \sigma_s - \tilde{\sigma}_s - \alpha^{-1} S^{-\delta}\left(\alpha(\sigma_s - \tilde{\sigma}_s)\right), \tag{73}$$

$$\tau_h^* = \sigma_h - \tilde{\sigma}_h - (1-\alpha)^{-1} S^{-\delta}\left((1-\alpha)(\sigma_h - \tilde{\sigma}_h)\right). \tag{74}$$

Hence we use this pair as test function in the Rothe approximation and we obtain, multiplying with δ^{-1},

$$\int_\Omega A_s D^{-\delta}(\alpha\sigma_s) : D^{-\delta}(\alpha(\sigma_s - \tilde{\sigma}_s)) \\ + A_h D^{-\delta}((1-\alpha)(\sigma_h - \tilde{\sigma}_h)) : D^{-\delta}((1-\alpha)(\sigma_h - \tilde{\sigma}_h))\,\mathrm{d}x + \text{penalty part} = 0. \tag{75}$$

The penalty part consists of a contribution of the soft material, namely

$$P_s = \int_\Omega \mu^{-1}[|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|} D^{-\delta}(\alpha(\sigma_s - \tilde{\sigma}_s))\,\mathrm{d}x, \tag{76}$$

and an analogous term P_h. We rewrite and estimate P_s in view of the discrete Leibniz rule. Thus,

$$P_s = \mu^{-1}\int_\Omega [|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|}(S^{-\delta}\alpha D^{-\delta}\sigma_{sD} + D^{-\delta}\alpha\sigma_{sD})\,\mathrm{d}x \\ - \mu^{-1}\int_\Omega [|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|} D^{-\delta}(\alpha\tilde{\sigma}_{sD})\,\mathrm{d}x = P_{1s} + P_{2s} + P_{3s}. \tag{77}$$

Since

$$\frac{\sigma_{sD}}{|\sigma_{sD}|} D^{-\delta}\sigma_{sD} \geq D^{-\delta}|\sigma_{sD}|, \tag{78}$$

we estimate and rewrite

$$P_{1s} \geq \mu^{-1}\int_\Omega [|\sigma_{sD}| - \kappa_s]_+ S^{-\delta}\alpha D^{-\delta}|\sigma_{sD}|\,\mathrm{d}x \\ = \mu^{-1}\int_\Omega [|\sigma_{sD}| - \kappa_s]_+ S^{-\delta}\alpha D^{-\delta}(|\sigma_{sD}| - \kappa_s)\,\mathrm{d}x \\ + \mu^{-1}\int_\Omega [|\sigma_{sD}| - \kappa_s]_+ S^{-\delta}\alpha D^{-\delta}\kappa_s\,\mathrm{d}x = P_{11s} + P_{12s}. \tag{79}$$

Since

$$[|\sigma_{sD}| - \kappa_s]_+ D^{-\delta} \left(|\sigma_{sD}| - \kappa_s\right) \geq \frac{1}{2} D^{-\delta} \left([|\sigma_{sD}| - \kappa_s]_+^2\right), \tag{80}$$

we obtain

$$P_{11s} \geq \frac{1}{2}\mu^{-1}\alpha_0 \int_\Omega D^{-\delta} \left([|\sigma_{sD}| - \kappa_s]_+^2\right) \mathrm{d}x. \tag{81}$$

Furthermore, due to the Lipschitz continuity of κ_s and the $L^1(L^1)$ property stated in Corollary 2, we have

$$\delta \sum_{t=\delta,\dots,N\delta} P_{12s} \geq -\mu^{-1}\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+ |D^{-\delta}\kappa_s| \mathrm{d}x \geq -C_{1s}(\dot{\kappa}_s). \tag{82}$$

With a similar argument, we obtain

$$\delta \sum_{t=\delta,\dots,N\delta} P_{2s} \geq -\mu^{-1}\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+ |D^{-\delta}\alpha||\sigma_{sD}| \,\mathrm{d}x \geq -C_{2s}(\dot{\alpha}) \tag{83}$$

and analogously

$$P_{3s} \geq -C_{3s}(\alpha), \tag{84}$$

where we used the Lipschitz continuity of α and the assumption that $\dot{\tilde{\sigma}}_s \in L^\infty$. From (81) we obtain

$$\begin{aligned} \delta \sum_{t=\delta,\dots,N\delta} P_{11s} &\geq \mu^{-1} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+^2 \alpha \,\mathrm{d}x|_0^T \\ &\quad -\mu^{-1}\delta \sum_{t=\delta,\dots,N\delta} \int_\Omega [|\sigma_{sD}| - \kappa_s]_+^2 |D^{\delta}\alpha| \,\mathrm{d}x \\ &= P_{111s} + P_{112s}. \end{aligned} \tag{85}$$

Again

$$P_{112s} \geq -C_{112}(\dot{\alpha}). \tag{86}$$

The constants $C_{1s}, \dots, C_{112}$ depend on the $L^1(L^1)$ estimate for the penalty term, so, they depend on $\dot{\alpha}, \dot{\kappa}_s, \dot{\kappa}_h$. The penalty parts for the hard material are treated in a similar manner.

We now sum (75) from $t = \delta$ up to $t = N\delta = T$ and obtain, using the estimates for the penalty parts,

$$\frac{1}{2}\sum_{t=\delta}^{N\delta}\int_{\Omega} A_s D^{-\delta}(\alpha\sigma_s) : D^{-\delta}(\alpha\sigma_s) + A_h D^{-\delta}((1-\alpha)\sigma_h) : D^{-\delta}((1-\alpha)\sigma_h)\,\mathrm{d}x$$

$$\le K(\dot{\alpha}, \dot{\kappa}_s, \dot{\kappa}_h) + \frac{1}{2}\delta \sum_{t=\delta,\dots,N\delta}\int_{\Omega} A_s D^{-\delta}(\alpha\tilde{\sigma}_s) : D^{-\delta}(\alpha\tilde{\sigma}_s) \tag{87}$$

$$+ A_h D^{-\delta}((1-\alpha)\sigma_h) : D^{-\delta}((1-\alpha)\tilde{\sigma}_h)\,\mathrm{d}x.$$

The theorem then follows, taking into account that $0 < \alpha_0 < \alpha < 1 - \alpha_0$. □

6 Convergence of the Rothe Method

In this section, we extend the solutions $\sigma_{s\mu\delta}, \sigma_{h\mu\delta}$ of the Rothe problem to the step functions $J_\delta\sigma_{s\mu\delta}$ and $J_\delta\sigma_{h\mu\delta}$ and show that they strongly converge in $L^2(L^2)$ to a solution $\sigma_{s\mu}, \sigma_{h\mu}$ of the penalty approximation as $\delta \to 0$. Throughout this section, except in the formulation of the theorem, we drop the index μ. For any function w defined on $I_\delta = \{k\delta | k = 0, \dots, N\}$ we define the extension as a step function by

$$J_\epsilon w(k\delta + \eta) = w(k\delta), \quad 0 \le \eta \le \delta. \tag{88}$$

If w is not defined in $[0, \delta]$, we define $J_\delta w = 0$ on $[0, \delta]$. In our setting, this acts on the argument of the loading parameter t.

Theorem 10. *Let $\mu > 0$ be fixed, α, κ_s, κ_h be Lipschitz, $0 < \alpha_0 < \alpha < 1-\alpha_0 < 1$, $\kappa_s, \kappa_h \ge \kappa_0 > 0$. Let A_h, A_s be symmetric and positive definite. Assume the safe load condition with an admissible pair $(\tilde{\sigma}_s, \tilde{\sigma}_h) \in \mathbb{K}$ such that $\dot{\tilde{\sigma}}_s, \dot{\tilde{\sigma}}_h \in L^\infty(L^\infty)$. Then the solutions $\sigma_{s\delta\mu}, \sigma_{h\delta\mu}$ converge to a pair $(\sigma_{s\mu}, \sigma_{h\mu})$ which is a solution of the penalty problem in the sense*

$$J_\delta\sigma_{s\mu\delta} \to \sigma_{s\mu}, \quad J_\delta\sigma_{h\mu\delta} \to \sigma_{h\mu} \quad (\delta \to +0) \tag{89}$$

strongly in $L^2(L^2)$, and

$$J_\delta D^{-\delta}\left(\sigma_{s\mu\delta}\right) \to \dot{\sigma}_{s\mu}, \quad J_\delta D^{-\delta}\left(\sigma_{h\mu\delta}\right) \to \dot{\sigma}_{h\mu} \quad (\delta \to +0) \tag{90}$$

weakly in $L^2(L^2)$.

Proof. By the uniform estimates for $\sigma_{s\delta}, \sigma_{h\delta}$ from Theorem 8 we have uniform $L^2(L^2)$-estimates for the functions $J_\delta\sigma_{s\delta}$, $J_\delta\sigma_{h\delta}$, $J_\delta D^{-\delta}\sigma_{s\delta}$, $J_\delta D^{-\delta}\sigma_{h\delta}$ and, by weak compactness in $L^2(L^2)$, any sequence $(\delta_i \to +0)$ has a subsequence such that (89) holds with weak limits σ_s, σ_h $(\delta_i \to 0)$. Furthermore the functions $J_\delta D^{-\delta}\sigma_{s\delta}$, $J_\delta D^{-\delta}\sigma_{h\delta}$ have weak limits which turn out to have the form $\dot{\sigma}_s, \dot{\sigma}_h$, i.e. they are the derivatives of σ_s, σ_h. This is classical and easy to prove. As a consequence, $\sigma_s(t, .)$ and $\sigma_h(t, .)$ are defined for $t \in [0, T]$ as $L^2(\Omega)$ functions. By weak convergence, one sees immediately that σ_s, σ_h satisfy the balance of linear momentum. Due to the representation

$$\sigma_{s0} = -\delta \sum_{t=\delta,\dots,N\delta} D^{-\delta}\sigma_{s\delta} + \sigma_s(T,.) \tag{91}$$

and by averaging with respect to T, we see via weak convergence that $\sigma_s(0,.) = \sigma_{s0}, \sigma_h(0,.) = \sigma_{h0}$, i.e. the weak limit satisfies the initial condition. The main task is to establish strong convergence in order to pass to the limit in the nonlinear penalty term. For this purpose, we define the restriction operator, which assigns to functions $w \in L^2(L^2)$ with $\dot{w} \in L^2(L^2)$ a function $R_\delta w$ on $I_\delta = \{\delta, \dots, N\delta\}$ defined by

$$R_\delta w(k\delta) = w(k\delta). \tag{92}$$

One has

$$J_\delta R_\delta w \to w, \qquad J_\delta D^{-\delta} R_\delta w \to \dot{w} \quad \text{strongly in } L^2(L^2) \tag{93}$$

as $\delta \to +0$, provided that $\dot{w} \in L^2(L^2)$. We now turn to the Rothe equation and use the pair $(\sigma_{s\delta} - R_\delta\sigma_s, \sigma_{h\delta} - R_\delta\sigma_h)$ as test function. Note that this test function satisfies the balance of linear momentum with zero force for $t \in I_\delta$. Rewriting the resulting equation and employing the extension operator J_δ we conclude

$$\begin{aligned}
\int_0^T &(A_s J_\delta D^{-\delta}(\alpha\sigma_{s\delta}), J_\delta(\alpha(\sigma_{s\delta} - R_\delta\sigma_s))) \\
&+ (A_h J_\delta D^{-\delta}((1-\alpha)\sigma_{h\delta}), J_\delta((1-\alpha)(\sigma_{s\delta} - R_\delta\sigma_s))) \\
&+ \mu^{-1}([|J_\delta\sigma_{s\delta D}| - J_\delta R_\delta\kappa_s]_+ J_\delta \frac{\sigma_{s\delta D}}{|\sigma_{s\delta D}|}, J_\delta(\alpha(\sigma_{s\delta D} - R_\delta\sigma_{sD}))) \\
&+ \mu^{-1}([|J_\delta\sigma_{h\delta D}| - J_\delta R_\delta\kappa_h]_+ J_\delta \frac{\sigma_{h\delta D}}{|\sigma_{h\delta D}|}, J_\delta(\alpha(\sigma_{h\delta D} - R_\delta\sigma_{hD}))) = 0.
\end{aligned} \tag{94}$$

In (55) we may add the terms

$$\int_0^T \mu^{-1}\left([J_\delta\sigma_{sD} - J_\delta R_\delta\kappa_s]_+ J_\delta\frac{\sigma_{sD}}{|\sigma_{sD}|}, J_\delta(\alpha\sigma_{s\delta D} - \alpha R_\delta\sigma_{sD})\right) \mathrm{d}t = o(1) \text{ as } \delta \to 0 \tag{95}$$

since the left hand factor in the scalar product is compact in L^2 for μ fixed, and there is a similar term for the hard material. The resulting penalty terms (i.e. summands with factor μ^{-1}) are ≥ 0 due to monotonicity and will be dropped, replacing $=$ by $\leq$. Furthermore, we may add the term

$$-\int_0^T \left(A_s J_\delta D^{-\delta} R_\delta(\alpha\sigma_s), J_\delta(\alpha\sigma_{s\delta} - \alpha R_\delta\sigma_s)\right) \mathrm{d}t = o(1) \tag{96}$$

and a similar term for the hard material due to weak convergence

$$z_{s\delta} := J_\delta(\alpha\sigma_{s\delta} - \alpha R_\delta \sigma_s) \rightharpoonup 0 \quad \text{in } L^2(L^2) \tag{97}$$

and due to strong $L^2(L^2)$ convergence of

$$J_\delta D^{-\delta} R_\delta(\alpha\sigma_s) \to \frac{\partial}{\partial t}(\alpha\sigma_s)\,. \tag{98}$$

Similarly $z_{h\delta} = J_\delta(\alpha\sigma_{h\delta} - \alpha R_\delta \sigma_h) \rightharpoonup 0$. Thus we are left with

$$\int_0^T \left(A_s D^{-\delta} z_{s\delta}, z_{s\delta}\right) + \left(A_h D^{-\delta} z_{h\delta}, z_{h\delta}\right) \mathrm{d}t \le o(1) \tag{99}$$

from which we conclude (see the analogous reasoning in Sect. 5) that

$$\delta^{-1} \int_{T-\delta}^T |z_{s\delta}|^2 + |z_{h\delta}|^2 \,\mathrm{d}t \le o(1). \tag{100}$$

This holds for all $T \in \{k\delta | k = 1, \ldots, N\}$ and we conclude $z_{s\delta} \to 0$, $z_{h\delta} \to 0$ strongly in $L^2(L^2)$ which implies

$$J_\delta \sigma_{s\delta} \to \sigma_s, \quad J_\delta \sigma_{h\delta} \to \sigma_h \text{ strongly in } L^2(L^2). \tag{101}$$

This allows us to pass to the limit in the Rothe equation (employing the extension operator J_δ) and we arrive at the equation

$$\begin{aligned} 0 = \int_{t_1}^{t_2} &\left(A_s \frac{\partial}{\partial t}(\alpha\sigma_s), \tau_s\right) + \left(A_h \frac{\partial}{\partial t}((1-\alpha)\sigma_h), \tau_h\right) \\ &+ \left(\mu^{-1}[|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|}, \alpha\tau_s\right) + \left(\mu^{-1}[|\sigma_{hD}| - \kappa_h]_+ \frac{\sigma_{hD}}{|\sigma_{hD}|}, (1-\alpha)\tau_h\right) \mathrm{d}t \end{aligned} \tag{102}$$

valid for all τ_s, τ_h satisfying the balance of linear momentum with zero force.

This proves the convergence of the Rothe method and the existence of solutions $\sigma_s, \sigma_h \in L^2(L^2), \dot{\sigma}_s, \dot{\sigma}_h \in L^2(L^2)$ of the penalty equation. □

Since the discrete $L^2(L^2)$ norms of $\sigma_{s\mu\delta}$, $\sigma_{h\mu\delta}$, $D^{-\delta}\left(\sigma_{s\mu\delta}\right)$, $D^{-\delta}\left(\sigma_{h\mu\delta}\right)$ are uniformly bounded with respect to $\mu \to +0$ (with error terms converging to zero as $\delta \to 0$, μ fixed) we obtain the corresponding bounds for $\sigma_{s\mu}$, $\sigma_{h\mu}$ as $\mu \to +0$.

With a similar reasoning we have a uniform $L^\infty(L^2)$ bound for the penalty potentials $\mu^{-1}[|\sigma_{s\mu D}| - \kappa_s]_+^2$, $\mu^{-1}[|\sigma_{h\mu D}| - \kappa_h]_+^2$ as well as a uniform $L^1(L^1)$ bound for the terms $\mu^{-1}[|\sigma_{s\mu D}| - \kappa_s]_+(|\sigma_{s\mu D}| + 1)$, $\mu^{-1}[|\sigma_{h\mu D}| - \kappa_h]_+(|\sigma_{h\mu D}| + 1)$. This proves the Theorems 4 and 6 of Sect. 3.

We finish the section with a conditional $L^\infty(L^1)$ estimate for the penalty term which is needed for the $L^\infty(L^2)$ estimate of $\dot{\sigma}_{s\mu}, \dot{\sigma}_{h\mu}$ in the next section.

Lemma 2 **($L^\infty(L^1)$).** *Under the assumption of the main theorem there is a constant $K(\dot{\alpha}, \dot{\tilde{\sigma}}_s, \dot{\tilde{\sigma}}_h)$ such that, for a.e. $t \in [0, T]$,*

$$\mu^{-1} \int_\Omega [|\sigma_{s\mu D}| - \kappa_s]_+ (|\sigma_{s\mu D} + 1|) + [|\sigma_{h\mu D}| - \kappa_h]_+ (|\sigma_{h\mu D} + 1|) \, \mathrm{d}x \Big|_t \leq K(\dot{\alpha}, \dot{\tilde{\sigma}}_s, \dot{\tilde{\sigma}}_h) \left(\|\dot{\sigma}_{s\mu}\|_{L^2(\Omega)} + \|\dot{\sigma}_{h\mu}\|_{L^2(\Omega)} \right)\Big|_t . \quad (103)$$

Proof. Use $\sigma_{s\mu} - \tilde{\sigma}_{s\mu}, \sigma_{h\mu} - \tilde{\sigma}_{h\mu}$ as test functions and use the safe load condition similar as in the $L^1(L^1)$ estimate for the penalty term before. This implies an estimate for the left hand side of (103) by

$$\left| \left(A_s \frac{\partial}{\partial t} (\alpha \sigma_s) , \sigma_{s\mu} - \tilde{\sigma}_{s\mu} \right) \right| \quad (104)$$

and a corresponding term for the hard material. Since an $L^\infty(L^2)$ bound is available for $\sigma_{s\mu}, \tilde{\sigma}_{s\mu}, \sigma_{h\mu}, \tilde{\sigma}_{h\mu}$ we obtain (103). □

7 $L^\infty(L^2)$-Estimate for the Time Derivatives of the Stresses

In the theory of the classical Prandtl-Reuss-problem there is the well known inclusion $\dot{\sigma} \in L^\infty(L^2)$ for the stress σ. A similar theorem holds also for Prandtl-Reuss-mixtures, but the proof is a bit involved, although it is obviously motivated by the classical theory.

Theorem 11. *Let $\sigma_{\mu s}, \sigma_{\mu h} \in L^\infty(L^2)$ be the solution to the penalty approximation of the Prandtl-Reuss-mixture problem. Besides the hypotheses of the main theorem let*

$$\ddot{\alpha}, \ddot{\kappa}_s, \ddot{\kappa}_h, \ddot{\tilde{\sigma}}_s, \ddot{\tilde{\sigma}}_h \in L^\infty(L^\infty) \quad (105)$$

where $\tilde{\sigma}_s, \tilde{\sigma}_h$ are safe loads. Then

$$\|\dot{\sigma}_{s\mu}\|_{L^\infty(L^2)} + \|\dot{\sigma}_{h\mu}\|_{L^\infty(L^2)} \leq K(\ddot{\alpha}, \ddot{\kappa}_s, \ddot{\kappa}_h, \ddot{\tilde{\sigma}}_s, \ddot{\tilde{\sigma}}_h) \quad (106)$$

uniformly as $\mu \to +0$.

Proof. Let $D^\eta = \eta^{-1}(S^\eta - I)$, $D^{-\eta} = \eta^{-1}(I - S^{-\eta})$ be the forward and backward difference operators with stepsize η, with respect to the loading variable t; $S^\eta w(t) = w(t+\eta)$. We write σ_s, σ_h rather than $\sigma_{s\mu}, \sigma_{h\mu}$. In the penalty equation we use the test functions

$$\begin{aligned} &-\eta^{-2} \ \left(\alpha^{-1} S^\eta\left(\alpha(\sigma_s - \tilde{\sigma}_s)\right) - 2(\sigma_s - \tilde{\sigma}_s) + \alpha^{-1} S^{-\eta}\left(\alpha(\sigma_s - \tilde{\sigma}_s)\right)\right), \\ &-\eta^{-2} \ \left((1-\alpha)^{-1} S^\eta\left((1-\alpha)(\sigma_h - \tilde{\sigma}_h)\right)\right. \\ &\qquad \left. -2(\sigma_h - \tilde{\sigma}_h) + (1-\alpha)^{-1} S^{-\eta}\left((1-\alpha)(\sigma_h - \tilde{\sigma}_h)\right)\right). \end{aligned} \tag{107}$$

They obey the balance of linear momentum with zero force. This yields

$$\begin{aligned} &-\left(A_s \frac{\partial}{\partial t}(\alpha\sigma_s), D^\eta D^{-\eta}\left(\alpha(\sigma_s - \tilde{\sigma}_s)\right)\right) \\ &-\left(A_h \frac{\partial}{\partial t}((1-\alpha)\sigma_h), D^\eta D^{-\eta}\left((1-\alpha)(\sigma_h - \tilde{\sigma}_h)\right)\right) \\ &-\left(\mu^{-1}[|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|}, D^\eta D^{-\eta}\left(\alpha(\sigma_s - \tilde{\sigma}_s)\right)\right) \\ &-\left(\mu^{-1}[|\sigma_{hD}| - \kappa_h]_+ \frac{\sigma_{hD}}{|\sigma_{hD}|}, D^\eta D^{-\eta}\left((1-\alpha)(\sigma_h - \tilde{\sigma}_h)\right)\right) = 0. \end{aligned} \tag{108}$$

We first get rid of the terms where $\tilde{\sigma}_s, \tilde{\sigma}_h$ occurs. We simply estimate (after integration $\int_{t_1}^{t_2} \mathrm{d}t$)

$$\begin{aligned} \left\| \left(A_s \frac{\partial}{\partial t}(\alpha\sigma_s), D^\eta D^{-\eta}(\alpha\tilde{\sigma}_s)\right) \right\| &\le K \left\| \frac{\partial}{\partial t}(\alpha\sigma_s) \right\|_{L^2(L^2)} \left\| \frac{\partial^2}{\partial t^2}(\alpha\tilde{\sigma}_s) \right\|_{L^2(L^2)} \\ &\le K_2(\ddot{\alpha}, \ddot{\tilde{\sigma}}_s) \end{aligned} \tag{109}$$

for $\mu \to \infty$, since a uniform $L^2(L^2)$ estimate for

$$\frac{\partial}{\partial t}(\alpha\sigma_s) = \frac{\partial}{\partial t}\left(\alpha\sigma_{s\mu}\right) \tag{110}$$

has been established in Sect. 4 and an appropriate estimate for $\tilde{\sigma}$ has been assumed. A similar reasoning holds for the hard material.

The penalty part where the factor $\tilde{\sigma}$ occurs is simply estimated by a constant $K(\ddot{\alpha}, \ddot{\tilde{\sigma}}_s)$ using the uniform $L^1(L^1)$ estimate for $\mu^{-1}[|\sigma_{sD}| - \kappa_s]_+$ and the $L^\infty(L^\infty)$ estimate for $\frac{\partial^2}{\partial t^2}(a\tilde{\sigma}_s)$ from the assumption. Again, a similar reasoning is done for the hard material. Thus we arrive at

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\left(A_s\frac{\partial}{\partial t}(\alpha\sigma_s), D^\eta D^{-\eta}(\alpha\sigma_s)\right)\,\mathrm{d}t\\
&-\int_{t_1}^{t_2}\left(A_h\frac{\partial}{\partial t}((1-\alpha)\sigma_h), D^\eta D^{-\eta}((1-\alpha)\sigma_h)\right)\,\mathrm{d}t\\
&-\int_{t_1}^{t_2}\mu^{-1}\left([|\sigma_{sD}|-\kappa_s]_+\frac{\sigma_{sD}}{|\sigma_{sD}|}, D^\eta D^{-\eta}(\alpha\sigma_s)\right)\,\mathrm{d}t\\
&-\int_{t_1}^{t_2}\mu^{-1}\left([|\sigma_{hD}|-\kappa_h]_+\frac{\sigma_{hD}}{|\sigma_{hD}|}, D^\eta D^{-\eta}((1-\alpha)\sigma_h)\right)\,\mathrm{d}t\\
&\le K(\ddot{\alpha},\ddot{\tilde{\sigma}}_s,\ddot{\tilde{\sigma}}_h).
\end{aligned}
\tag{111}
$$

We now move the operator D^η to the first function in the scalar products ('partial summation'). It changes into $D^{-\eta}$, we obtain boundary terms $S_{t_1t_2}$ and see that

$$
\begin{aligned}
&\int_{t_1}^{t_2}\left(A_sD^{-\eta}\left(\frac{\partial}{\partial t}(\alpha\sigma_s)\right), D^{-\eta}(\alpha\sigma_s)\right)\\
&\quad+\left(A_hD^{-\eta}\left(\frac{\partial}{\partial t}((1-\alpha)\sigma_h)\right), D^{-\eta}((1-\alpha)\sigma_h)\right)\,\mathrm{d}t\\
&\quad+\int_{t_1}^{t_2}\mu^{-1}\left(D^{-\eta}\left([|\sigma_{sD}|-\kappa_s]_+\frac{\sigma_{sD}}{|\sigma_{sD}|}\right), D^{-\eta}(\alpha\sigma_{sD})\right)\,\mathrm{d}t\\
&\quad+\int_{t_1}^{t_2}\mu^{-1}\left(D^{-\eta}\left([|\sigma_{hD}|-\kappa_h]_+\frac{\sigma_{hD}}{|\sigma_{hD}|}\right), D^{-\eta}((1-\alpha)\sigma_{hD})\right)\,\mathrm{d}t+S_{t_1t_2}\\
&\quad= S_A+S_{pen}+S_{t_1t_2}\le K(\ddot{\alpha},\ddot{\tilde{\sigma}}_s,\ddot{\tilde{\sigma}}_h).
\end{aligned}
\tag{112}
$$

Then

$$
\begin{aligned}
S_A &= \frac{1}{2}\left(A_sD^{-\eta}(\alpha\sigma_s), D^{-\eta}(\alpha\sigma_s)\right)\Big|_{t_1}^{t_2}\\
&\quad+\frac{1}{2}\left(A_hD^{-\eta}((1-\alpha)\sigma_h), D^{-\eta}((1-\alpha)\sigma_h)\right)\Big|_{t_1}^{t_2}
\end{aligned}
\tag{113}
$$

and we take the limit $\eta\to 0$.

Since $\dot{\sigma}_s,\dot{\sigma}_h\in L^2(L^2)$, this limit $\eta\to +0$ exists a.e. with respect to t_1, t_2 and we obtain

$$
\begin{aligned}
&\frac{1}{2}\left(A_s \frac{\partial}{\partial t}(\alpha\sigma_s), \frac{\partial}{\partial t}(\alpha\sigma_s)\right)\Big|_{t_1}^{t_2} \\
&+\frac{1}{2}\left(A_h \frac{\partial}{\partial t}((1-\alpha)\sigma_h), \frac{\partial}{\partial t}((1-\alpha)\sigma_h)\right)\Big|_{t_1}^{t_2} \\
&+\int_{t_1}^{t_2} \mu^{-1}\left(\frac{\partial}{\partial t}\left([|\sigma_{sD}|-\kappa_s]_+\frac{\sigma_{sD}}{|\sigma_{sD}|}\right), \frac{\partial}{\partial t}(\alpha\sigma_{sD})\right) \mathrm{d}t \\
&+\int_{t_1}^{t_2} \mu^{-1}\left(\frac{\partial}{\partial t}\left([|\sigma_{hD}|-\kappa_h]_+\frac{\sigma_{hD}}{|\sigma_{hD}|}\right), \frac{\partial}{\partial t}((1-\alpha)\sigma_{hD})\right) \mathrm{d}t \\
&+\lim_{\eta\to 0} S_{t_1t_2} = A_1 + Pen_{soft} + Pen_{hard} + \lim_{\eta\to 0} S_{t_1t_2} \le K(\ddot{\alpha}, \ddot{\tilde{\sigma}}_s, \ddot{\tilde{\sigma}}_h).
\end{aligned}
\tag{114}
$$

We write

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left([|\sigma_{sD}|-\kappa_s]_+\frac{\sigma_{sD}}{|\sigma_{sD}|}\right) : \frac{\partial}{\partial t}(\alpha\sigma_{sD}) \\
&= [|\sigma_{sD}|-\kappa_s]_+\alpha|\sigma_{sD}|^{-1}\left(\left|\frac{\partial}{\partial t}(\sigma_{sD})\right|^2 - \left|\frac{\partial}{\partial t}|\sigma_{sD}|\right|^2\right) + \left|\frac{\partial}{\partial t}[|\sigma_{sD}|-\kappa_s]_+\right|^2 \alpha \\
&\qquad + \frac{\partial}{\partial t}[|\sigma_{sD}|-\kappa_s]_+\alpha\frac{\partial}{\partial t}\kappa_s + \frac{\partial}{\partial t}[|\sigma_{sD}|-\kappa_s]_+|\sigma_{sD}|\dot{\alpha} \\
&= P_1 + P_2 + P_3 + P_4.
\end{aligned}
\tag{115}
$$

We have $P_1 \ge 0$, $P_2 \ge 0$ and these terms contribute to the final estimate. For P_3 we find via partial integration

$$
\begin{aligned}
\mu^{-1}\int_{t_1}^{t_2}\int_{\Omega} P_3\,\mathrm{d}x\,\mathrm{d}t =& -\mu^{-1}\int_{t_1}^{t_2}\int_{\Omega}[|\sigma_{sD}|-\kappa_s]_+\frac{\partial}{\partial t}\left(\alpha\frac{\partial}{\partial t}\kappa_s\right)\mathrm{d}x\,\mathrm{d}t \\
&+\mu^{-1}\int_{\Omega}[|\sigma_{sD}|-\kappa_s]_+\alpha\frac{\partial}{\partial t}\kappa_s\,\mathrm{d}x\Big|_{t_1}^{t_2} = \tilde{P}_{31} + \tilde{P}_{32}
\end{aligned}
\tag{116}
$$

The term $\tilde{P}_{31}$ is uniformly bounded due to the $L^1(L^1)$ bound for the penalty term and the hypotheses on α and κ_s. The term $\tilde{P}_{32}$ is estimated via the $L^\infty(L^2)$ lemma for the penalty term. This yields

$$
\left|\mu^{-1}\int_{\Omega}\tilde{P}_{32}\mathrm{d}x\Big|_{t_1}^{t_2}\right| \le K(\dot{\kappa}_s)\left(\|\dot{\sigma}_s(t_2,.)\|_{L^2(\Omega)} + \|\dot{\sigma}_h(t_2,.)\|_{L^2(\Omega)}\right) + o(t_1) \tag{117}
$$

where $o(t_1)$ needs not be uniform in μ.

The term P_4 is treated in a similar manner like P_3. Thus we obtain

$$\begin{aligned} Pen_{soft} &\geq \mu^{-1} \int_{t_1}^{t_2} \int_{\Omega} \left| \frac{\partial}{\partial t} \left| [|\sigma_{sD}| - \kappa_s]_+ \right| \right|^2 \alpha \, \mathrm{d}x \, \mathrm{d}t \\ &= K - \|\dot{\sigma}_s(t_2, .)\|_{L^2(\Omega)} - \|\dot{\sigma}_h(t_2, .)\|_{L^2(\Omega)} - o(t_1) \end{aligned} \tag{118}$$

and a similar inequality for the hard material.

It remains to analyze the boundary terms coming from the partial summation

$$\lim_{\eta \to 0} S_{t_1 t_2} = S^1_{t_1 t_2} + S^2_{t_1 t_2}. \tag{119}$$

From the result of the previous partial summation we obtain

$$\begin{aligned} \lim_{\eta \to 0} S_{t_1 t_2} = &- \frac{1}{2} \left(A_s \frac{\partial}{\partial t} (\alpha \sigma_s), \frac{\partial}{\partial t} (\alpha \sigma_s) \right) \Bigg|_{t_1}^{t_2} \\ &- \frac{1}{2} \left(A_h \frac{\partial}{\partial t} ((1-\alpha)\sigma_h), \frac{\partial}{\partial t} ((1-\alpha)\sigma_h) \right) \Bigg|_{t_1}^{t_2} \\ &- \mu^{-1} \left(\frac{\partial}{\partial t} \left([|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|} \right), \frac{\partial}{\partial t} (\alpha \sigma_{sD}) \right) \Bigg|_{t_1}^{t_2} \\ &- \mu^{-1} \left(\frac{\partial}{\partial t} \left([|\sigma_{hD}| - \kappa_h]_+ \frac{\sigma_{hD}}{|\sigma_{hD}|} \right), \frac{\partial}{\partial t} ((1-\alpha)\sigma_{hD}) \right) \Bigg|_{t_1}^{t_2}. \end{aligned} \tag{120}$$

We have

$$\begin{aligned} S^1_{t_1 t_2} \quad &= \left(A_s \tfrac{\partial}{\partial t} (\alpha \sigma_s), \tfrac{\partial}{\partial t} (\alpha \sigma_s) \right) \Big|_{t_1}^{t_2} \\ &\quad + \textit{corresponding term for hard material.} \end{aligned} \tag{121}$$

$$\begin{aligned} S^2_{t_1 t_2} &= \mu^{-1} \left([|\sigma_{sD}| - \kappa_s]_+ \tfrac{\sigma_{sD}}{|\sigma_{sD}|}, \tfrac{\partial}{\partial t} (\alpha \sigma_s) \right) \Big|_{t_1}^{t_2} \\ &\quad + \textit{corresponding term for hard material.} \end{aligned} \tag{122}$$

Let

$$\begin{aligned} S_t = \left(A_s \frac{\partial}{\partial t} (\alpha \sigma_s), \frac{\partial}{\partial t} (\alpha \sigma_s) \right) \Bigg|_t + \mu^{-1} \left([|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|}, \frac{\partial}{\partial t} (\alpha \sigma_s) \right) \Bigg|_t \\ + \textit{ corresponding term for hard material.} \end{aligned} \tag{123}$$

With this notation $\lim_{\eta \to 0} S_{t_1 t_2} = S_{t_1} - S_{t_2}$. Now we present an argument which shows that $\lim_{\eta \to 0} S_t$ remains unchanged if we replace the right hand factors

$\frac{\partial}{\partial t}(\alpha\sigma_s)$ and $\frac{\partial}{\partial t}((1-\alpha)\sigma_h)$ in the scalar products by $\frac{\partial}{\partial t}(\alpha\tilde{\sigma}_s)$ and $\frac{\partial}{\partial t}((1-\alpha)\tilde{\sigma}_h)$. In fact, this follows if we use the test functions

$$\delta_1^{-1}\left(\sigma_s-\tilde{\sigma}_s-\alpha^{-1}S^{-\delta_1}\left(\alpha\sigma_s-\alpha\tilde{\sigma}_s\right)\right),$$
$$\delta_1^{-1}\left(\sigma_h-\tilde{\sigma}_h-(1-\alpha)^{-1}S^{-\delta_1}\left((1-\alpha)(\sigma_h-\tilde{\sigma}_h)\right)\right) \tag{124}$$

and pass to the limit in the penalty equation $\delta_1 \to 0$, performing this procedure at t_1 and t_2.

This gives us the equation

$$\left(A_s\frac{\partial}{\partial t}(\alpha\sigma_s),\frac{\partial}{\partial t}(\alpha\sigma_s-\alpha\tilde{\sigma}_s)\right)+\mu^{-1}\left([|\sigma_{sD}|-\kappa_s]_+\frac{\sigma_{sD}}{|\sigma_{sD}|},\frac{\partial}{\partial t}(\alpha\sigma_s)-\alpha\tilde{\sigma}_s\right)$$
$$+\textit{ a similar term for the hard material} = 0 \tag{125}$$

for all $t = t_1$ and a.e. t_2.

We conclude that

$$\left|S^1_{t_1t_2}+S^2_{t_1t_2}\right| \le K\left(\left\|\frac{\partial}{\partial t}(\alpha\sigma_s)\right\|_{L^2(\Omega)}\left\|\frac{\partial}{\partial t}(\alpha\tilde{\sigma}_s)\right\|_{L^2(\Omega)}\right)\Bigg|_{t_2} \tag{126}$$
$$+K\left(\left\|\mu^{-1}[|\sigma_{sD}|-\kappa_s]_+\right\|_{L^1(L^1)}\left\|\frac{\partial}{\partial t}(\alpha\tilde{\sigma}_s)\right\|_{L^\infty(L^\infty)}\right)\Bigg|_{t_2}$$
$$+\textit{ a similar term}\big|_{t_1}$$
$$+\textit{ the related summand for hard material at } t_1 \textit{ and } t_2 \textit{ a.e.}$$

Thus, we see that

$$\left|S^1_{t_1t_2}+S^2_{t_1t_2}\right| \le K(\dot{\alpha},\dot{\tilde{\sigma}}_s,\dot{\tilde{\sigma}}_h)(1+\|\sigma_s\|_{L^2(\Omega)}|_{t_2}+\|\sigma_s\|_{L^2(\Omega)}|_{t_1}$$
$$+\|\sigma_h\|_{L^2(\Omega)}|_{t_2}+\|\sigma_h\|_{L^2(\Omega)}|_{t_1}). \tag{127}$$

Collecting our results we obtain from (114)

$$\left|\frac{1}{2}\left(A_s\frac{\partial}{\partial t}(\alpha\sigma_s),\frac{\partial}{\partial t}(\alpha\sigma_s)\right)\Bigg|_{t_1}^{t_2}+\frac{1}{2}\left(A_h\frac{\partial}{\partial t}((1-\alpha)\sigma_h),\frac{\partial}{\partial t}((1-\alpha)\sigma_h)\right)\Bigg|_{t_1}^{t_2}\right|$$
$$\le K\left(\int_\Omega|\dot{\sigma}_s|^2\,\mathrm{d}x\Bigg|_{t_2}+\int_\Omega|\dot{\sigma}_h|^2\,\mathrm{d}x\Bigg|_{t_2}\right)^{\frac{1}{2}} \tag{128}$$

$$+K\left(\int_{\Omega}|\dot{\sigma}_s|^2\,\mathrm{d}x\bigg|_{t_1}+\int_{\Omega}|\dot{\sigma}_h|^2\,\mathrm{d}x\bigg|_{t_1}\right)^{\frac{1}{2}}$$

$$+K(\ddot{\alpha},\ddot{\kappa}_s,\ddot{\kappa}_h,\ddot{\tilde{\sigma}}_s,\ddot{\tilde{\sigma}}_h).$$

We finally get rid of the terms

$$\left(A_s\frac{\partial}{\partial t}(\alpha\sigma_s),\frac{\partial}{\partial t}(\alpha\sigma_s)\right)\bigg|_{t_1},\qquad \int_{\Omega}|\dot{\sigma}_s|^2\,\mathrm{d}x\bigg|_{t_1}. \tag{129}$$

In fact, from the penalty equation, with the above reasoning, we obtain

$$\begin{aligned}&\left(A_s\frac{\partial}{\partial t}(\alpha\sigma_s),\frac{\partial}{\partial t}(\alpha\sigma_s-\alpha\tilde{\sigma}_s)\right)\bigg|_{t_1}\\&+\mu^{-1}\left([|\sigma_{sD}|-\kappa_s]_+\frac{\sigma_{sD}}{|\sigma_{sD}|},\frac{\partial}{\partial t}(\alpha\sigma_s-\alpha\tilde{\sigma}_s)\right)\bigg|_{t_1}\\&+\textit{corresponding term with hard material}=0.\end{aligned} \tag{130}$$

Now, since, for fixed μ,

$$[|\sigma_{sD}|-\kappa_s]_+\frac{\sigma_{sD}}{|\sigma_{sD}|}\to 0\quad\text{in}\quad L^2(L^2)\qquad\text{as } t_1\to 0, \tag{131}$$

(similarly for the hard material) we conclude that

$$\begin{aligned}&\lim_{t_1\to 0}\left(A_s\frac{\partial}{\partial t}(\alpha\sigma_s),\frac{\partial}{\partial t}(\alpha\sigma_s)\right)+\textit{corresponding term for hard material}\\&\le \operatorname*{ess\,sup}_{0\le t\le\delta}\left\{\left(A_s\frac{\partial}{\partial t}(\alpha\tilde{\sigma}_s),\frac{\partial}{\partial t}(\alpha\tilde{\sigma}_s)\right)+\textit{corresponding term for hard material}\right\}.\end{aligned} \tag{132}$$

The theorem now follows from (128) and (132). □

Corollary 3.

$$\begin{aligned}&\left\|\mu^{-1}[|\sigma_{s\mu D}|-\kappa_s]_+(|\sigma_{s\mu D}|+1)\right\|_{L^\infty(L^1)}+\\&\left\|\mu^{-1}[|\sigma_{h\mu D}|-\kappa_h]_+(|\sigma_{h\mu D}|+1)\right\|_{L^\infty(L^1)}\le C_0\end{aligned} \tag{133}$$

uniformly as $\mu\to +0$.

8 Passage to the Limit as the Penalty Parameter μ Tends to Zero

Theorem 12. *Under the assumption of the main theorem the solutions* $(\sigma_{s\mu}, \sigma_{h\mu})$ *of the penalty problem converge to the solution* (σ_s, σ_h) *of the variational inequality* (19) *for the Prandtl-Reuss mixture. The convergence is strong in* $L^\infty(L^2)$ *and weak in* $H^1(L^2)$.

Proof. Since $\sigma_{s\mu}, \sigma_{h\mu}, \dot{\sigma}_{s\mu}, \dot{\sigma}_{h\mu}$ are uniformly bounded in $L^2(L^2)$ as $\mu \to +0$ we may subtract a subsequence $\Lambda = \{\mu_m | \mu_m \to +0\}$ such that $\sigma_{s\mu} \rightharpoonup \sigma_s$, $\sigma_{h\mu} \rightharpoonup \sigma_h$, $\dot{\sigma}_{s\mu} \rightharpoonup \dot{\sigma}_s$, $\dot{\sigma}_{h\mu} \rightharpoonup \dot{\sigma}_h$ weakly in $L^2(L^2)$.

We may pass to the limit in the equation of balance of linear momentum and obtain (11) for σ_s, σ_h. Furthermore, the symmetry of σ_s and σ_h is preserved. From the $L^1(L^1)$-estimate (see the corollary to Theorem 9 in Sect. 6) we have that the penalty term is bounded in $L^1(L^1)$ as $\mu \to +0$. This implies $[|\sigma_{s\mu D}| - \kappa_s]_+^2 \le K\mu$, $[|\sigma_{h\mu D}| - \kappa_h]_+^2 \le K\mu$ and, since $[|\xi| - \kappa_s]_+^2$ is convex and continuous, we obtain $[|\sigma_{sD}| - \kappa_s]_+^2 \le 0$, $[|\sigma_{hD}| - \kappa_h]_+^2 \le 0$ i.e. $|\sigma_{sD}| \le \kappa_s$, $|\sigma_{hD}| \le \kappa_h$. □

The variational inequality (19) follows from the penalty equations

$$\begin{aligned}
&\int_{t_1}^{t_2} \left(A_s \frac{\partial}{\partial t} \left(\alpha \sigma_{s\mu}\right), \alpha(\sigma_{s\mu} - \hat{\sigma}_s) \right) \\
&\qquad + \left(A_h \frac{\partial}{\partial t} \left((1-\alpha)\sigma_{h\mu}\right), (1-\alpha)(\sigma_{h\mu} - \hat{\sigma}_h) \right) \mathrm{d}t \\
&\le -\mu^{-1} \int_{t_1}^{t_2} \left([|\sigma_{s\mu D}| - \kappa_s]_+ \frac{\sigma_{s\mu D}}{|\sigma_{s\mu D}|}, \alpha(\sigma_{s\mu} - \hat{\sigma}_s) \right) \\
&\qquad + \left([|\sigma_{h\mu D}| - \kappa_h]_+ \frac{\sigma_{h\mu D}}{|\sigma_{h\mu D}|}, (1-\alpha)(\sigma_{h\mu} - \hat{\sigma}_h) \right) \mathrm{d}t \\
&\le 0 \qquad \text{for all } (\hat{\sigma}_s, \hat{\sigma}_h) \in \mathbb{K}.
\end{aligned} \tag{134}$$

The last step concerning that the left hand side is ≤ 0 follows from the monotonicity property of

$$[|\tau| - \kappa]_+ \frac{\tau_D}{|\tau_D|} \tag{135}$$

and the fact that $[|\hat{\sigma}_s| - \kappa_s]_+ = 0$, $[|\hat{\sigma}_h| - \kappa_h]_+ = 0$ by definition of $\mathbb{K}$. We may pass to the weak limit $\mu \to 0$ in (134), keeping the inequality ≤ 0 due to lower semicontinuity. This yields (19). The strong convergence

$$\sigma_{s\mu} \to \sigma_s \qquad \sigma_{h\mu} \to \sigma_h \qquad \text{in } L^2(L^2) \tag{136}$$

follows by setting $\hat{\sigma}_s = \sigma_s$, $\hat{\sigma}_h = \sigma_h$ in (134) and adding the terms

$$-\left(A_s \frac{\partial}{\partial t}(\alpha\sigma_s), \alpha(\sigma_{s\mu} - \sigma_s)\right) \\ -\left(A_h \frac{\partial}{\partial t}((1-\alpha)\sigma_h), (1-\alpha)(\sigma_{h\mu} - \sigma_h)\right) \to 0 \tag{137}$$

In fact, this yields (for $t_1 = 0$)

$$\limsup_{\mu\to+0} \big(A_s\alpha(\sigma_{s\mu} - \sigma_s), \alpha(\sigma_{s\mu} - \sigma_s)\big) \\ + \big(A_h(1-\alpha)(\sigma_{h\mu} - \sigma_h), (1-\alpha)(\sigma_{h\mu} - \sigma_h)\big) \le 0 \tag{138}$$

which even implies

$$\sigma_{s\mu} \to \sigma_s, \qquad \sigma_{h\mu} \to \sigma_h, \qquad \text{in } L^\infty(L^2). \tag{139}$$

Since the solution σ_s, σ_h is unique, (136) holds for the full sequence (via the usual contradiction argument). The convergence (139) for the full sequence can be derived with an additional simple $C(L^2)$ argument.

We now want to incorporate the partial strain velocities and the plastic strain velocities into the discussion. Similar, as to the classical Prandtl-Reuss problem, the situation is not quite satisfactory due to the fact that only L^1-estimates are available. Under the assumptions of Theorem (9) (in particular, no assumptions on $\ddot{\alpha}, \ddot{\kappa}_s, \ddot{\kappa}_h$) we have uniform $L^1(L^1)$-bounds for

$$\frac{1}{2}(\nabla\dot{u}_{s\mu} + \nabla\dot{u}_{s\mu}{}^T), \qquad \frac{1}{2}(\nabla\dot{u}_{h\mu} + \nabla\dot{u}_{h\mu}{}^T), \tag{140}$$

and the corresponding penalty terms.

With Temam's imbedding theorem this implies a uniform $L^1(L^{\frac{n}{n-1}})$-bound for $\dot{u}_{s\mu}$, $\dot{u}_{h\mu}$ and we obtain that, for a subsequence,

$$\dot{u}_{s\mu} \rightharpoonup \dot{u}_s, \qquad \dot{u}_{h\mu} \rightharpoonup \dot{u}_h \tag{141}$$

and

$$\frac{1}{2}(\nabla\dot{u}_{s\mu} + \nabla\dot{u}_{s\mu}{}^T) \to \frac{1}{2}(\nabla\dot{u}_s + \nabla\dot{u}_s{}^T) \tag{142}$$

$$\frac{1}{2}(\nabla\dot{u}_{h\mu} + \nabla\dot{u}_{h\mu}{}^T) \to \frac{1}{2}(\nabla\dot{u}_h + \nabla\dot{u}_h{}^T) \tag{143}$$

weakly in $C^*([0,T]\times\bar{\Omega})$, $\mu\to+0$. This means that the strain velocities need not be functions, they are only Riesz measures.

In case that an $L^\infty(L^1)$-bound is available for the penalty terms, see Theorem 11 , $\dot{u}_{s\mu}$ and $\dot{u}_{h\mu}$ are bounded in $L^\infty(L^{\frac{n}{n-1}})$, the convergence in (141) takes place in $L^{\frac{n}{n-1}}(L^{\frac{n}{n-1}})$, and the limiting deformation velocities are (at least) $L^2(L^{\frac{n}{n-1}})$-functions. We want to derive a variational inequality which takes the strain velocity into account.

From (33) and (34) we conclude

$$\left(A_s\frac{\partial}{\partial t}\left(\alpha\sigma_{s\mu}\right),\sigma_{s\mu}-\tau\right)=\left(\frac{1}{2}(\nabla\dot{u}_{s\mu}+\nabla\dot{u}_{s\mu}{}^T),\sigma_{s\mu}-\tau\right)\le 0 \tag{144}$$

for all $\tau\in C(\bar{\Omega})$ such that $\tau=\tau^T$ and $|\tau_D|\le\kappa_s$, a.e. (no balance of linear momentum for τ is assumed).

The $\le$ inequality in (144) follows from the monotonicity property of the penalty term and the fact that $[|\tau_D|-\kappa_s]_+=0$. An inequality similar to (144) holds for the hard material.

We want to pass to the limit $\mu\to+0$ in (144). For the left hand side this is possible due to weak and strong $L^2(L^2)$ convergence of the functions in the scalar product. For the left hand side, obviously

$$\frac{1}{2}\left(\nabla\dot{u}_{s\mu}+\nabla\dot{u}_{s\mu}{}^T,\tau\right)\to\frac{1}{2}(\nabla\dot{u}_s+\nabla\dot{u}_\mu{}^T,\tau) \tag{145}$$

but for $\frac{1}{2}\left(\nabla\dot{u}_{s\mu}+\nabla\dot{u}_{s\mu}{}^T,\sigma_{s\mu}\right)$ this convergence is not clear, since we do not know that $\sigma_{s\mu}\to\sigma_s$ in $C(\bar{\Omega})$. Thus we assume the hypothesis of Theorem 10 and we use the $L^\infty(L^{\frac{n}{n-1}})$ for $\dot{u}_{s\mu}$, $\dot{u}_{h\mu}$ and write

$$\frac{1}{2}\left(\nabla\dot{u}_{s\mu}+\nabla\dot{u}_{s\mu}{}^T,\sigma_{s\mu}\right)=(\dot{u}_{s\mu},f_s)+\int_{\partial\Omega}p\sigma_{s\mu}\,\mathrm{d}o \tag{146}$$

due to the balance of linear momentum.

In the right hand side we may pass to the limit and obtain as limit

$$(u_s,f)+\int_{\partial\Omega}p\sigma_s\,\mathrm{d}o. \tag{147}$$

So we arrive at the variational inequality

$$\left(A_s\frac{\partial}{\partial t}\left(\alpha\sigma_{s\mu}\right),\sigma_s-\tau\right)\le(f,u_s)+\int_{\partial\Omega}p\sigma_s\,\mathrm{d}o-\frac{1}{2}(\nabla u_s+\nabla u_s{}^T,\tau) \tag{148}$$

for all $\tau\in C(\bar{\Omega},M^n_{sym})$, $|\tau_D|\le\kappa_s$ and a similar inequality for the hard material. Of course, this is not satisfactory.

In the case of the Hencky problem, in two dimension there is an interesting way to overcome the formulation (148). An analogue approach might work also for the Prandtl Reuss problem. For Hencky's problem, and similar for Prandtl-Reuss's problem, via a technique using a reverse Hölder inequality, there are $L^{\frac{n}{n-1}+\delta}$ and $L^\infty(L^{\frac{n}{n-1}+\delta})$-estimates available for the displacements u (see [6]) or the displacement velocities $\dot{u}$, respectively. Thus we have a $(\frac{n}{n-1}-\delta')$ capacity potential ϕ of the set where the $(\frac{n}{n-1}-\delta')$ capacity is small and, testing the penalty equation with $\sigma_s\phi$, we obtain uniform smallness of

$$\mu^{-1}\int_\Omega [|\sigma_{\mu D}|-\kappa_s]_+|\sigma_{\mu D}|\phi\,\mathrm{d}x \tag{149}$$

on sets of small capacity. Hence the limiting Riesz measure shares this property.

Since interior H^1-estimates for σ are available and thus σ_μ is uniformly continuous except on a set of small $2-\delta'$ capacity we may give a meaning to $\left(\frac{1}{2}(\nabla\dot{u}+\nabla\dot{u}^T),\zeta\right)$, $\zeta\in C_0^\infty(\Omega)$ by extending the measure to functions which are $2-\delta'$ quasicontinuous in the sense of capacities.

We do not state the above discussion concerning capacity methods since, for a rigorous discussion, this would take more space than available here. We confine to fix the statement concerning convergence in C^* of the strain velocities.

Theorem 13. *Assume the hypotheses of the main theorem. Then the partial strain velocities constructed in Sect. 3 converge weakly in C^**

$$\begin{aligned}\frac{1}{2}(\nabla\dot{u}_{s\mu}+\nabla\dot{u}_{s\mu}{}^T) &\rightharpoonup \frac{1}{2}(\nabla\dot{u}_s+\nabla\dot{u}_s{}^T)\\ \frac{1}{2}(\nabla\dot{u}_{h\mu}+\nabla\dot{u}_{h\mu}{}^T) &\rightharpoonup \frac{1}{2}(\nabla\dot{u}_h+\nabla\dot{u}_h{}^T)\\ \textit{and}\;\dot{u}_{s\mu}\rightharpoonup\dot{u}_s\quad,&\quad\dot{u}_{h\mu}\rightharpoonup\dot{u}_h\;\textit{weakly in}\;C^*.\end{aligned} \tag{150}$$

If, in addition, the assumption of Theorem 10 are satisfied (150) *($\mu\to+0$) holds in $L^{\frac{n}{n-1}}(L^{\frac{n}{n-1}})$.*

Remark 9. (150) holds strongly in $L^{\frac{n}{n-1}-\delta'}(L^{\frac{n}{n-1}-\delta'})$ due to Temam's imbedding theorem, and in fact in $L^{\frac{n}{n-1}+\delta}(L^{\frac{n}{n-1}+\delta})$, δ small, if the reverse Hölder inequality technique is used (which we did not do here).

Corollary 4. *If the assumptions of Theorem 10 are satisfied, the variational inequality* (148) *holds.*

Concerning the Kuhn Tucker rule there are similar problems like the interpretation of the 'pointwise' inequality (144). The penalty terms converge weakly in C^* as $\mu\to+0$. We have

$$\begin{aligned}\mu^{-1}[|\sigma_{s\mu D}|-\kappa_s]_+\frac{\sigma_{s\mu D}}{|\sigma_{s\mu D}|} &\rightharpoonup \dot{e}_{ps},\\ \mu^{-1}[|\sigma_{h\mu D}|-\kappa_h]_+\frac{\sigma_{h\mu D}}{|\sigma_{h\mu D}|} &\rightharpoonup \dot{e}_{hs}\end{aligned} \tag{151}$$

and one would like to conclude from

$$\left.\begin{aligned} \mu^{-1}[|\sigma_{s\mu D}| - \kappa_s]_+ &\to \lambda_s \\ \mu^{-1}[|\sigma_{h\mu D}| - \kappa_h]_+ &\to \lambda_h \end{aligned}\right\} \text{*-weakly in } C \tag{152}$$

that

$$\dot{e}_{ps} = \lambda_s \frac{\sigma_{sD}}{|\sigma_{sD}|}, \qquad \dot{e}_{ph} = \lambda_h \frac{\sigma_{hD}}{|\sigma_{hD}|}. \tag{153}$$

In the case of the two dimensional Hencky-Model this can be proved by the tools indicated above, using the reverse Hölder inequality for the displacements, smallness of the support of λ_s and λ_h on sets of small $(2 + \delta')$ capacity and the fact that $\sigma_{s\mu}, \sigma_{h\mu}$ converges $(2 + \delta')$ uniformly for a subsequence.

9 A Model for the Volume Fraction α and the Yield Parameter Depending on the History of the Rate of the Plastic Strain of the Soft Material

In [10], the following model for α, κ_h, κ_s is suggested. Let

$$l(t, x) = \int_0^t |\dot{e}_{ps}(\xi, x)| \,\mathrm{d}\xi \tag{154}$$

where $\dot{e}_{ps}$ is the plastic deformation velocity of the soft material. Then the volume fraction α is defined by

$$\alpha(t, .) = \alpha_0 + (1 - \alpha_0)e^{-c_0 l(t,.)} \tag{155}$$

and the yield parameters by

$$\kappa_s = const > 0, \qquad \kappa_h = \kappa_0 + r_0 l \tag{156}$$

with constants $\alpha_0 > 0$, $c_0 > 0$, $\kappa_0 > 0$, $r_0 > 0$. In the rigorous setting $\dot{e}_{ps}(s, x)$ may be a Riesz measure and we have to approach it via the penalty approximations, see below. Since l could be unbounded we apply a simple (from our point of view acceptable) modification by setting

$$l(t, x) = \int_0^t g(|\dot{e}_{ps}(s, x)|) \,\mathrm{d}s + \delta_0, \qquad \delta_0 > 0 \tag{157}$$

with a bounded, non negative function $g \in C^1$.

This also guarantees our condition that $\alpha \geq \delta_1 > 0$ and $(1 - \alpha) \leq 1 - \delta_1$, and the condition

$$\|\dot{\alpha}\|_{L^\infty(L^\infty)} + \|\dot{\kappa}_s\|_{L^\infty(L^\infty)} + \|\dot{\kappa}_h\|_{L^\infty(L^\infty)} \leq K \tag{158}$$

whatever function $\dot{e}_{ps}$ is chosen.

On the level of the penalty approximation, we have

$$\dot{e}_{ps} = \mu^{-1}[|\sigma_{sD}| - \kappa_s]_+ \frac{\sigma_{sD}}{|\sigma_{sD}|} \text{ and} \tag{159}$$

$$|\dot{e}_{ps}| = \mu^{-1}[|\sigma_{sD}| - \kappa_s]_+ \tag{160}$$

$$\alpha = \alpha_0 + (1-\alpha_0)\exp\left(-c_0 \int_0^t g(\mu^{-1}[|\sigma_{sD}| - \kappa_s]_+)\,\mathrm{d}\xi\right) \tag{161}$$

$$\kappa_h = \kappa_0 + r_0 \int_0^t g(\mu^{-1}[|\sigma_{sD}| - \kappa_s]_+)\,\mathrm{d}\xi. \tag{162}$$

With this definition of α, κ_h, κ_s (which is constant) we assign to every pair (σ_s, σ_h) satisfying the symmetry condition a solution $\bar{\sigma}_{\mu s}$, $\bar{\sigma}_{\mu h}$ to the penalty equation (27), and all the a priori estimates of this paper requiring the L^∞ property on $\dot{\alpha}$, $\dot{\kappa}_h$ (not $\ddot{\alpha}$, $\ddot{\kappa}_h$) are true.

For applying Schauder's fix point theorem to obtain a solution $\bar{\sigma}_{\mu s} = \sigma_{\mu s}$, $\bar{\sigma}_{\mu h} = \sigma_{\mu h}$ one needs an additional compactness condition in space direction which would be achieved by a non local dependence of α and κ_h in terms of σ_s, say

$$l(t,x) = \int_0^t g(\mu^{-1}[|\int_\Omega K(x,y)\sigma_{sD}(t,y)\,\mathrm{d}y| - \kappa_s]_+)\,\mathrm{d}\xi \tag{163}$$

with a compact singular integral operator $K : L^2 \to L^2$. With this compact dependence it is possible to solve the penalty equation, due to the a priori estimates given in Sects. 4 and 5, and also to prove the convergence of the penalty equation as $\mu \to 0$, which leads to a solution of a quasivariational inequality with the above interpretation.

Acknowledgements Jens Frehse work was supported by SFB611 at University of Bonn, Josef Málek work was supported by the project GAČR 107/12/0121.

References

1. Bensoussan, A., Frehse, J.: Asymptotic behaviour of the time dependent Norton-Hoff law in plasticity theory and H^1 regularity. Commentationes Mathematicae Universitatis Carolinae **37**(2), 285–304 (1996)
2. Blum, H., Frehse, J.: Boundary differentiability for the solution to Hencky's law of elastic plastic plane stress. Preprint 435, SFB 611 University of Bonn (2008)
3. Bulíček, M., Frehse, J., Málek, J.: On boundary regularity for the stress in problems of linearized elasto-plasticity. Int. J. Adv. Eng. Sci. Appl. Math. **1**(4), 141–156 (2009)
4. Demyanov, A.: Regularity of stresses in Prandtl-Reuss perfect plasticity. Calc. Var. Partial Differ. Equ. **34**(1), 23–72 (2009)
5. Duvaut, G., Lions, J.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)

6. Hardt, R., Kinderlehrer, D.: Elastic plastic deformation. Appl. Math. Optim. **10**(1), 203–246, (1983)
7. Hlaváček, I., Haslinger, J., Nečas, J., Lovíšek, J.: Solution of Variational Inequalities in Mechanics. Springer, New York (1988)
8. Khasina, L.: Mathematische Behandlung von Mischungen Elastoplastischer Substanzen. Ph.D. thesis, University of Bonn (2007)
9. Kratochvíl, J.: A theory of non-proportional cyclic plasticity based on micromechanical approach. In: Tokuda, M., Xu, B., Senoo, M. (eds.) Macro/Micro/Meso Mechanical Properties of Materials, Proceedings of IMMM'93, 3–5 Aug 1993, Mie University. Mie Academic Press, Japan (1993)
10. Kratochvíl, J., Málek, J., Rajagopal, K., Srinivasa, A.: Modeling of the response of elastic plastic materials treated as a mixture of hard and soft regions. Zeitschrift für Angewandte Mathematik und Physik (ZAMP) **55**(3), 500–518 (2004)
11. Temam, R.: Mathematical Problems in Plasticity, vol. 15. Gauthier-Villars, Paris (1985)
12. Temam, R.: A generalized Norton-Hoff model and the Prandtl-Reuss law of plasticity. Arch. Ration. Mech. Anal. **95**, 137–183 (1986)