Vortex Motion for the Landau-Lifshitz-Gilbert Equation with Applied Magnetic Field

Matthias Kurzke, Christof Melcher, and Roger Moser

1 Introduction

In micromagnetics, the fundamental evolution law for the magnetization $\mathbf{m}$ in a solid is given by the Landau-Lifshitz-Gilbert equation

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \left(\alpha \frac{\partial \mathbf{m}}{\partial t} - \gamma \, \boldsymbol{h}_{\text{eff}} \right), \tag{1}$$

which is used to describe the dynamics of a great variety of magnetic microstructures, in particularly the motion of domain walls and vortices in thin films, see e.g. [3]. Here $\boldsymbol{h}_{\text{eff}}$ is the *effective field*, essentially the L^2 gradient of the micromagnetic energy.

A collective coordinate ansatz $\mathbf{m} = \mathbf{m}(x - a(t))$, where $\mathbf{m}$ is the profile of the static problem and $a = a(t)$ describes its translation at time t, has been proposed by Thiele in [24] in order to drastically reduce the complexity of (1). Thiele's approach has been adapted by Huber [8] to the situation of a vortex system, giving rise to a system of ODEs typically called Thiele's equation of motion. More precisely, the

M. Kurzke (✉)
Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, D-53115 Bonn, Germany
e-mail: kurzke@iam.uni-bonn.de

C. Melcher
Department of Mathematics I & JARA – Fundamentals of Future Information Technology, RWTH Aachen University, Wüllnerstraße 5b, D-52056 Aachen, Germany
e-mail: melcher@rwth-aachen.de

R. Moser
Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK
e-mail: r.moser@bath.ac.uk

M. Griebel (ed.), *Singular Phenomena and Scaling in Mathematical Models*,
DOI 10.1007/978-3-319-00786-1_6,

resulting system for vortices with trajectories $t \mapsto a_j(t) \in \mathbb{R}^2 \times \{0\}$ $(j = 1, \dots d)$ takes the form

$$F_j(a) + G_j \times \dot{a}_j + D\dot{a}_j = 0.$$

Here $F_j = F_j(a_1, \dots, a_d)$ are interaction forces, $G_j = 4\pi q_j \hat{\mathbf{e}}_3$ is the gyro-vector of the jth vortex, which depends only on the topological index $q_j = \pm\frac{1}{2}$ of the vortex (which is half of the product of winding number and polarity), and D is an effective damping constant. In previous joint work with Spirn [14, 15] we have rigorously derived a Thiele equation from (1) in the limit of small vortex size, for an exchange-dominated model energy. In [13] we have generalized the result to an extended version of (1), modeling the influence of an in-plane spin-polarized current $v = v(t)$. More precisely, we have shown that the corresponding spin-torque terms give rise to an additive extension of Thiele's equation

$$F_j(a) + G_j \times (\dot{a}_j - v) + D(\dot{a}_j - \kappa v) = 0$$

where κ is a non-negative constant. The aim of the present work is to derive a Thiele equation from (1) under the influence of a (possibly time-dependent) applied field $\boldsymbol{h} \in \mathbb{R}^3$. Unlike the result for an external current, the effect of the magnetic field is visible only in the interaction force term. The precise result will be given in Theorem 3.

As our model energy we use

$$E_\epsilon(\boldsymbol{h}, \mathbf{m}) = \int_\Omega \left(\frac{1}{2}|\nabla \mathbf{m}|^2 + \frac{m_3^2}{\epsilon^2} - \boldsymbol{h} \cdot \mathbf{m} \right) dx, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded and simply connected domain, with a Dirichlet boundary condition $\mathbf{m} = \boldsymbol{g}$. The most physical choice of $\boldsymbol{g}$ is to use a unit tangent to $\partial\Omega$. We refer to [14] for a justification of this model.

2 Jacobian, Vorticity and Renormalized Energy

Suppose that we have a map $\mathbf{m} : \Omega \to \mathbb{S}^2$ in the Sobolev space H^1. It is convenient to consider the decomposition

$$\mathbf{m} = (m, m_3).$$

Recall that the Jacobian of $m : \Omega \to \mathbb{R}^2$ is defined as

$$J(m) = \det \nabla m.$$

Note that the Jacobian, considered as a differential 2-form, is exact. More precisely, $J(m) = \frac{1}{2}\operatorname{curl} j(m)$, where $j(m) = m \wedge \nabla m$ is the current, and we write $a \wedge b = a_1 b_2 - a_2 b_1$ for $a, b \in \mathbb{R}^2$. Observe that current and Jacobian are well-defined as distributions for maps $m \in L^\infty \cap W^{1,1}(\Omega; \mathbb{R}^2)$. Moreover, they carry topological information about the $\mathbb{S}^1$-degree of the map m. More precisely, if B is a ball, $m \in C^1(\overline{B}; \mathbb{R}^2)$ is such that $m|_{\partial B} \neq 0$ and $u = m/|m|$, then

$$\int_{\partial B} j(u) \cdot ds = 2\pi \deg(u, \partial B).$$

For $\mathbb{S}^2$-valued maps $\mathbf{m}$, the counterpart of the Jacobian is the vorticity

$$\omega(\mathbf{m}) = \left\langle \mathbf{m}, \frac{\partial \mathbf{m}}{\partial x_1} \times \frac{\partial \mathbf{m}}{\partial x_2} \right\rangle,$$

which is, considered as a differential 2-form, the pull-back of the standard volume form on $\mathbb{S}^2$ with respect to $\mathbf{m}$. Thus, if B is a ball, $\mathbf{m} \in C^1(\overline{B}; \mathbb{S}^2)$ is such that $\mathbf{m}|_{\partial B}$ is an equator map, then

$$\int_B \omega(\mathbf{m})\, dx = 4\pi q,$$

where q is the $\mathbb{S}^2$-degree of, i.e. the oriented number of covers of $\mathbb{S}^2$ by the map $\mathbf{m}$. Thus q is a half-integer if the winding number of $\deg(m, \partial B)$ is odd. In contrast to the Jacobian, however, $\omega(\mathbf{m})$ is not exact, i.e., $\omega(\mathbf{m})$ is not a null-Lagrangian.

2.1 Compactness

We have good compactness results for the Jacobian and, under assumptions on the energy excess, also on the maps themselves. The compactness properties of the vorticity are not as good as those for the Jacobians, and we will not discuss them here in general.

Proposition 1. *Assume that* $(\mathbf{m}_\epsilon)$ *is a sequence of maps* $m_\epsilon \in H^1(\Omega; \mathbb{S}^2)$ *with* $\mathbf{m}_\epsilon = (g, 0)$ *on* $\partial\Omega$ *and* $E_\epsilon(\boldsymbol{h}, \mathbf{m}_\epsilon) \leq C \log \frac{1}{\epsilon}$ *for some fixed* $\mathbf{h} \in \mathbb{R}^3$.

Then we can extract a subsequence (not relabeled) such that

$$J(m_\epsilon) \to \pi \sum_j^d \delta_{a_j} \tag{3}$$

in the dual of $C_0^{0,1}(\Omega)$.

Proof. As $\mathbf{h}$ is independent of ϵ, it follows that $E_\epsilon(0, \mathbf{m}_\epsilon) \le C \log \frac{1}{\epsilon}$. Hence the 2D Ginzburg-Landau energy of m_ϵ satisfies the same bound, and we can now apply standard compactness results [10]. □

Proposition 2. *Suppose that the sequence* $(\mathbf{m}_\epsilon)$ *satisfies the assumptions of Proposition 1 and suppose that* d *and* $a_1, \dots, a_d$ *are as in* (3)*. If additionally the sequence* $E_\epsilon(\boldsymbol{h}, \mathbf{m}_\epsilon) \le d\pi \log \frac{1}{\epsilon} + C$ *then* $\mathbf{m}_\epsilon$ *is bounded in* $W^{1,p}(\Omega; \mathbb{S}^2)$ *for* $1 \le p < 2$ *and in* $H^1_{\mathrm{loc}}(\Omega \backslash \{a_1, \dots, a_d\})$*. In particular, a subsequence converges strongly in* $L^q(\Omega; \mathbb{S}^2)$ *for every* $q < \infty$ *to a map* $\mathbf{m}_0 = (m_0, 0)$ *with* $|m_0| = 1$.

Proof. From the convergence of the Jacobians for a subsequence ϵ_n and lower bounds near the singularities [9, 19], we obtain for every $r > 0$

$$\limsup_{n\to\infty} \int_{\Omega_r(a)} |\nabla m_{\epsilon_n}|^2 dx \le 2\pi d \log \frac{1}{r} + C,$$

which shows the H^1_{loc} bound. Using an argument of Struwe [23] and appropriate diagonal subsequences, one can show by Hölder's inequality and summing a series that

$$\limsup_{\epsilon \searrow 0} \int_{\Omega} |\nabla m_\epsilon|^p dx \le C(p)$$

for all $p \in [1, 2)$. Alternatively, one can obtain the $W^{1,p}$ boundedness from the global bounds on ∇m_ϵ in the Lorentz space $L^{2,\infty}$ given in [21]. Rellich-Kondrachov embedding finally yields strong convergence. □

2.2 The Renormalized Energy

We introduce some notation. We fix a boundary condition $g \in C^\infty(\partial\Omega; \mathbb{S}^1)$ with $\deg(g) = d > 0$. For $a \in \Omega^d$, we set

$$\rho_a := \min(\min_i \operatorname{dist}(a_i, \partial\Omega), \frac{1}{2} \min_{i \ne j} |a_i - a_j|).$$

For $r \in [0, \rho_a)$ we define

$$\Omega_r(a) = \{x \in \Omega : |x - a_j| > r \text{ for } j = 1, \dots, d\}$$

and we write $\Omega^d_* = \{a \in \Omega_d : \rho_a > 0\}$. As in [2], for $a \in \Omega^d_*$, there exists a corresponding canonical harmonic map $M_* = M_*(\cdot, a)$ with vortex locations a and all local winding numbers equal to 1, i.e.

$$M_*(x;a) = \prod_{j=1}^{d} \frac{x-a_j}{|x-a_j|} e^{i\psi},$$

where ψ is a harmonic function chosen such that $M_*(x;a) = g$ on $\partial\Omega$. Recall that $M_*(\cdot,a) \in W_g^{1,p}(\Omega, \mathbb{S}^1)$ for all $p \in [1,2)$. We also verify by virtue of the explicit representation of $M_*(\cdot,a)$ that the mapping

$$\Omega_*^d \ni a \mapsto M_*(\cdot,a) \in L^p(\Omega;\mathbb{C}) \tag{4}$$

is continuously differentiable for $p \in [1,2)$. For $h \in \mathbb{R}^2$ sufficiently small, we consider

$$W(h,a) = W_0(a) + V(h,a)$$

where $W_0 = W_0(a)$ is the unperturbed renormalized energy as introduced by Bethuel, Brezis and Hélein [2]. The perturbation $V = V(h,a)$ is defined as the following energy minimum

$$V(h,a) = \min_{\theta \in H_0^1(\Omega)} \mathscr{G}(h,a;\theta),$$

where

$$\mathscr{G}(h,a;\theta) = \int_\Omega \frac{1}{2}|\nabla\theta|^2 - h \cdot \left(e^{i\theta} M_*(x;a)\right) dx. \tag{5}$$

Observe that, for h sufficiently small, $\mathscr{G}(h,a;\cdot)$ is a strictly convex functional on $H_0^1(\Omega)$, and hence there exists a unique minimizer $\theta = \theta(h,a) \in H_0^1(\Omega)$. Since $\mathscr{G}$ is a smooth function of h and $\theta = \theta(h,a)$ a critical point, it follows that

$$\frac{\partial W}{\partial h} = \frac{\partial V}{\partial h} = \frac{\partial \mathscr{G}}{\partial h}\Big|_{\theta=\theta(h,a)} = -\int_\Omega m_*(x;a)\, dx, \tag{6}$$

where

$$m_*(x;a) = e^{i\theta(x;h,a)} M_*(x;a).$$

Note that $m_*(\cdot,a) \in W_g^{1,p}(\Omega, \mathbb{S}^1)$ for all $p \in [1,2)$ with

$$J(m_*) = J(M_*) = \pi \sum_{j=1}^{d} \delta_{a_j}$$

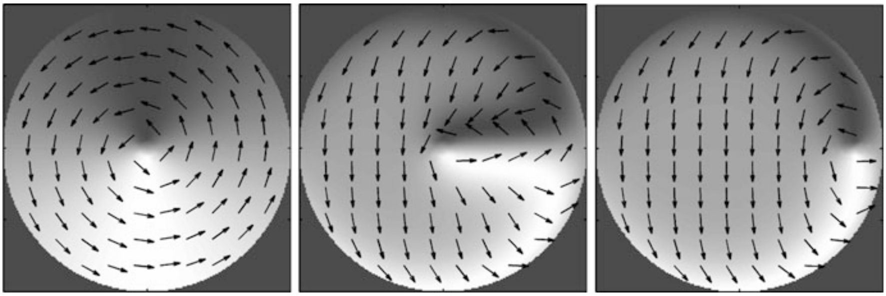

Fig. 1 Numerical plot of $M_*(\cdot;0)$ (*left*), $m_*(\cdot;0)$ (*center*) and $m_*(\cdot;a_{\min})$ (*right*) for $a_{\min}$ minimizing $W(h,a)$. The applied field is $h=(0,-40)$. Note that in the situation presented here, the external field exerts a force on the vortex that is perpendicular to the field. Numerical simulation by Jutta Steiner (using Matlab) based on Newton iteration for minimization of $\mathscr{G}(h,a;\theta)$ for fixed a and h

and that the Euler-Lagrange equation for (5) expressed in terms of m_* reads

$$\nabla\cdot j(m_*) = h\wedge m_*, \tag{7}$$

i.e., $m_* = m_*(\cdot,a)$ is the canonical h-harmonic map corresponding to g and $a\in\Omega_*^d$. We have the following characterization of the renormalized energy:

Lemma 1. *The renormalized energy can be calculated as*

$$W(h,a) = \lim_{r\to 0}\left(\int_{\Omega_r(a)} \frac{1}{2}|\nabla m_*|^2 - \mathbf{h}\cdot m_*\,dx - \pi d\log\frac{1}{r}\right). \tag{8}$$

Proof. As in [2], we can set $\Phi = 2\pi\sum_{j=1}^d \log|x-a_j|$. Then Φ is locally the conjugate harmonic map of the phase of $\prod_{j=1}^d \frac{x-a_j}{|x-a_j|}$. Using that $|m_*|=|M_*|=1$, we can now write $|\nabla M_*| = |\nabla^\perp\Phi+\nabla\psi|$ and $|\nabla m_*| = |\nabla^\perp\Phi+\nabla\psi+\nabla\theta|$, where ψ is the harmonic function and $\theta=\theta(\cdot;h,a)$ as above. It follows that

$$|\nabla m_*|^2 - |\nabla M_*|^2 = |\nabla\theta|^2 + 2(\nabla^\perp\Phi+\nabla\psi)\cdot\nabla\theta.$$

Integrating this expression over $\Omega_r(a)$ and using that ψ is harmonic, we obtain for $r\to 0$ the claimed result. □

We deduce from (4) a local Lipschitz condition for m_* as a mapping in a, which will be useful for identifying effective motion laws.

Lemma 2. *Suppose* $p\in[1,2)$, $a^0\in\Omega_*^d$. *Then there exists* $c>0$ *such that*

$$\|m_*(\cdot,a) - m_*(\cdot,\hat{a})\|_{L^p} \le c\,|a-\hat{a}|$$

for all $a,\hat{a}\in\Omega_*^d$ *such that* $\max\{|a-a_0|,|\hat{a}-a_0|\} < \rho(a_0)/2$.

Lemma 3. *Suppose* $\Phi \in C_0^\infty(\Omega;\mathbb{R}^2)$ *and* $\rho \in (0,\rho_a)$ *such that* $\Phi|B_\rho(a_\ell) =$ *const. and* $\Phi|B_\rho(a_k) = 0$ *for all* $k \neq \ell$*. Then, with* $m_* = m_*(\cdot, a)$*, we have*

$$\Phi(a_\ell)\cdot\frac{\partial W}{\partial a_\ell}(h,a) = \int_\Omega \nabla\Phi : \left(\left(\frac{1}{2}|\nabla m_*|^2 - h\cdot m_*\right)\mathbf{1} - \nabla m_*\otimes\nabla m_*\right)dx.$$

Proof. The claim of the lemma is in fact a singular version of Noether's formula for the Lagrangian $\frac{1}{2}|\nabla m_*|^2 - h\cdot m_*$ with respect to inner variations $s \mapsto m_*(x - s\,\Phi(x))$. Based on this observation, the argument in [12] for the case $h = 0$ carries over literally. □

We will need the following notion of energy excess for a map $\mathbf{m}$ and a configuration of points $a \in \Omega_*^d$:

$$D_\epsilon^{\mathbf{h}}(\mathbf{m};a) := E_\epsilon(\mathbf{h},\mathbf{m}) - \left(\pi d\log\frac{1}{\epsilon} + d\gamma + W(h,a)\right),$$

where γ is defined as $\lim_{\epsilon\searrow 0}(I_\epsilon - \pi\log\frac{1}{\epsilon})$, and

$$I_\epsilon = \inf\left\{\int_{B_1(0)} e_\epsilon(\mathbf{m})\,dx : \mathbf{m}(x) = (x,0) \text{ on } \partial B_1(0)\right\}.$$

To show that the name "energy excess" is justified, and to relate the micromagnetic energy to the renormalized energy, we have

Proposition 3. *If* $J(m_\epsilon) \to \pi\sum_{k=1}^d \delta_{a_k}$ *then* $\liminf_{\epsilon\searrow 0} D_\epsilon^{\mathbf{h}}(\mathbf{m}_\epsilon;a) \geq 0$.

Proof. Let $\epsilon_k \to 0$ be a sequence such that

$$A = \liminf_{\epsilon\searrow 0} D_\epsilon^{\mathbf{h}}(\mathbf{m}_\epsilon;a) = \lim_{k\to\infty} D_{\epsilon_k}^{\mathbf{h}}(\mathbf{m}_{\epsilon_k};a).$$

We can assume that $A < \infty$ (otherwise there is nothing to prove). By Proposition 2, we have (for a subsequence) that $\mathbf{m}_{\epsilon_k} \to \mathbf{m}_0 = (m_0, 0)$ weakly in $H^1_{\text{loc}}(\Omega_0(a);\mathbb{R}^3)$ and strongly in all $L^p(\Omega)$, $1 \leq p < \infty$. It follows that $|m_0| = 1$, i.e. $\mathbf{m}_0$ has values in $S^1\times\{0\}$.

Now $D_{\epsilon_k}^{\mathbf{h}}(\mathbf{m}_{\epsilon_k};a) = D_{\epsilon_k}^{0}(\mathbf{m}_{\epsilon_k};a) - \int_\Omega \mathbf{h}\cdot(\mathbf{m}_{\epsilon_k} - m_*)\,dx$. As in the proof of Theorem 5.3 of [14], for r sufficiently small we have

$$\begin{aligned} D_{\epsilon_k}^{\mathbf{h}}(\mathbf{m}_{\epsilon_k};a) \geq &\int_{\Omega_r(a)} (e_{\epsilon_k}(\mathbf{m}_{\epsilon_k}) - \frac{1}{2}|\nabla M_*|^2)\,dx \\ &+ \sum_{\ell=1}^d\left(\int_{B_r(a_\ell)} e_{\epsilon_k}(\mathbf{m}_{\epsilon_k})\,dx - I_{\epsilon_k/r}\right) - Cr^2 \\ &- \int_\Omega \mathbf{h}\cdot(\mathbf{m}_{\epsilon_k} - m_*). \end{aligned}$$

Using the convergence of $\mathbf{m}_{\epsilon_k}$ and Lemma 5.1 of [14] we obtain

$$\liminf_{k\to\infty} D^{\mathbf{h}}_{\epsilon_k}(\mathbf{m}_{\epsilon_k};a) \geq \int_{\Omega_r(a)} \left(\frac{1}{2}|\nabla m_0|^2 - \frac{1}{2}|\nabla m_*|^2\right) dx - \int_{\Omega} \mathbf{h}\cdot(m_0 - m_*) - Cr^2$$

We decompose $m_0 = e^{i\beta} M_*$. As in the derivation of (8), it is not difficult to see that

$$\int_{\Omega_r(a)} \left(\frac{1}{2}|\nabla m_0|^2 - \frac{1}{2}|\nabla M_*|^2\right) dx \to \int_{\Omega} \frac{1}{2}|\nabla\beta|^2\, dx$$

as $r \to 0$, and now we can use the minimality of θ to conclude the proof of the proposition. □

Now we show that the phase excess in $\Omega_r(a)$ (which measures the distance of $\mathbf{m}_\epsilon$ from an optimal map) can be bounded by the energy excess, up to errors that are small as $\epsilon \to 0$ and $r \to 0$. Unlike the quantitative theory of [11], our proof follows the idea of Lemma 3.7 in [20] and uses weak convergence.

We define

$$\tilde{e}_\epsilon(\mathbf{m}) = \frac{1}{2}\left(\big|\nabla|m|\big|^2 + |\nabla m_3|^2 + \frac{m_3^2}{\epsilon^2}\right)$$

and note the decomposition

$$e_\epsilon(\mathbf{m}) = \tilde{e}_\epsilon(\mathbf{m}) + \frac{1}{2}\left|\frac{j(m)}{|m|}\right|^2.$$

Proposition 4. *Assume $D_\epsilon = D^{\mathbf{h}}_\epsilon(\mathbf{m}_\epsilon;a) \leq C$. Then we have the following estimates for any $\rho < \rho_a$, $\ell = 1,\ldots,d$:*

$$\left|\int_{B_\rho(a_\ell)} |\nabla\mathbf{m}_\epsilon|^2\, dx - \pi|\log\epsilon|\right| \leq C, \tag{9}$$

$$\int_{\Omega_\rho(a)} \tilde{e}_\epsilon(\mathbf{m}_\epsilon)\, dx \leq D_\epsilon + o_\epsilon(1), \tag{10}$$

$$\int_{\Omega_\rho(a)} \frac{1}{2}\left|\frac{j(m_\epsilon)}{|m_\epsilon|} - j(m_*(\cdot;a))\right|^2 dx \leq \frac{1}{1-C|\mathbf{h}|} D_\epsilon + o_\epsilon(1). \tag{11}$$

Proof. As in the proof of Proposition 3, we have for a subsequence that $\mathbf{m}_\epsilon$ converges to $\mathbf{m}_0 = (m_0, 0)$ weakly in $H^1_{\mathrm{loc}}(\Omega_0(a))$ and strongly in $L^p(\Omega)$, with

$m_0 = e^{i\beta} M_*$, where $\beta \in H_0^1(\Omega)$. The proof of Proposition 3 also gives for any small $r > 0$

$$\liminf_{\epsilon\searrow 0} \left(\int_{\bigcup B_r(a_\ell)} e_\epsilon(\mathbf{m}_\epsilon)\, dx - \pi d \log \frac{r}{\epsilon} + d\gamma \right) \geq -Cr^2. \tag{12}$$

Furthermore,

$$\liminf_{\epsilon\searrow 0} \int_{\Omega_r(a)} \frac{1}{2} \left| \frac{j(m_\epsilon)}{|m_\epsilon|} \right|^2 dx \geq \int_{\Omega_r(a)} \frac{1}{2} |\nabla m_0|^2\, dx$$

and since $\mathbf{m}_\epsilon \to \mathbf{m}_0$ in $L^1(\Omega)$, we obtain

$$\liminf_{\epsilon\searrow 0} \left(\int_{\Omega_r(a)} \frac{1}{2} \frac{|j(m_\epsilon)|^2}{|m_\epsilon|^2}\, dx - \int_\Omega \mathbf{h} \cdot \mathbf{m}_\epsilon\, dx \right) \geq \int_{\Omega_r(a)} \frac{1}{2} |\nabla m_0|^2\, dx - \int_\Omega \mathbf{h} \cdot \mathbf{m}_0\, dx.$$

From (8) we obtain

$$\int_{\Omega_r(a)} \frac{1}{2} |\nabla m_*|^2\, dx - \int_\Omega \mathbf{h} \cdot m_*\, dx \geq W(h,a) + \pi d \log \frac{1}{r} - o_r(1)$$

so adding this to (12) we obtain

$$\liminf_{\epsilon\searrow 0} \left(D_\epsilon - \int_{\Omega_r(a)} \tilde{e}_\epsilon(\mathbf{m}_\epsilon)\, dx \right) \geq -o_r(1).$$

Since the right-hand side of the previous inequality tends to zero as $r \to 0$, we obtain by monotonicity of the left-hand side for any $\rho > 0$

$$\liminf_{\epsilon\searrow 0} \left(D_\epsilon - \int_{\Omega_\rho(a)} \tilde{e}_\epsilon(\mathbf{m}_\epsilon)\, dx \right) \geq 0.$$

This is (10). From $D_\epsilon \leq C$, we obtain that also (9) must hold.

From the definition of energy excess it follows that

$$\limsup_{\epsilon\searrow 0} \left(\int_{\Omega_r(a)} \frac{1}{2} |\nabla \mathbf{m}_\epsilon|^2 - \frac{1}{2} |\nabla m_*|^2\, dx - \int_\Omega \mathbf{h} \cdot (\mathbf{m}_\epsilon - m_*)\, dx - D_\epsilon \right) \leq -o_r(1)$$

so a fortiori

$$\limsup_{\epsilon\searrow 0} \left(\int_{\Omega_r(a)} \frac{1}{2} \frac{|j(m_\epsilon)|^2}{|m_\epsilon|^2} - \frac{1}{2} |j(m_*)|^2\, dx - \int_\Omega \mathbf{h} \cdot (\mathbf{m}_\epsilon - m_*)\, dx - D_\epsilon \right) \leq -o_r(1).$$

We calculate

$$\frac{1}{2}\left|\frac{j(m_\epsilon)}{|m_\epsilon|}-j(m_*)\right|^2=\frac{1}{2}\frac{|j(m_\epsilon)|^2}{|m_\epsilon|^2}-\frac{1}{2}|j(m_*)|^2-j(m_*)\cdot\left(\frac{j(m_\epsilon)}{|m_\epsilon|}-j(m_*)\right).$$

Using that $j(m_*)=\nabla^\perp\Phi+\nabla\psi+\nabla\theta$ and $j(m_0)=\nabla^\perp\Phi+\nabla\psi+\nabla\beta$, we have that

$$\lim_{\epsilon\searrow 0}\int\limits_{\Omega_r(a)} j(m_*)\cdot\left(\frac{j(m_\epsilon)}{|m_\epsilon|}-j(m_*)\right)dx=\int\limits_{\Omega_r(a)}\left(\nabla^\perp\Phi+\nabla\psi+\nabla\theta\right)\cdot(\nabla\beta-\nabla\theta)\,dx$$

For $r\to 0$, this expression converges using the harmonicity of ψ to

$$\int_\Omega \nabla\theta\cdot(\nabla\beta-\nabla\theta)\,dx.$$

We obtain

$$\limsup_{\epsilon\searrow 0}\left(\int_{\Omega_r(a)}\frac{1}{2}\left|\frac{j(m_\epsilon)}{|m_\epsilon|}-j(m_*)\right|^2-D_\epsilon\right)$$
$$\leq -o_r(1)+\int_\Omega \mathbf{h}\cdot M_*(e^{i\beta}-e^{i\theta})+\nabla\theta\cdot(\nabla\beta-\nabla\theta)\,dx.$$

The Euler-Lagrange for θ in weak form reads as

$$\int_\Omega \nabla\theta\cdot(\nabla\beta-\nabla\theta)\,dx=\int_\Omega \mathbf{h}\cdot(iM_*e^{i\theta})(\beta-\theta)\,dx.$$

We study the expression

$$\mathbf{h}\cdot\left(M_*e^{i\theta}(e^{i(\beta-\theta)}-1-i(\beta-\theta))\right)$$

and note that it can be written using an application of Taylor's theorem to the function $f(t)=\mathbf{h}\cdot(M_*e^{i(\theta+t(\beta-\theta))})$. In fact, we have

$$f(1)=f(0)+f'(0)+\int_0^1 f''(t)(1-t)\,dt,$$

where $f''(t)=\mathbf{h}\cdot M_*e^{i(\theta+t(\beta-\theta))}(\beta-\theta)^2$. Taking absolute values and integrating, it follows that

$$\left|\int_\Omega \mathbf{h}\cdot\left(M_*e^{i\theta}(e^{i(\beta-\theta)}-1-i(\beta-\theta))\right)dx\right|\leq|\mathbf{h}|\int_\Omega(\beta-\theta)^2\,dx.$$

By weak convergence,

$$\int_{\Omega_r(a)} \frac{1}{2}|\nabla(\beta-\theta)|^2\,dx \le \liminf_{\epsilon\searrow 0}\int_{\Omega_r(a)} \frac{1}{2}\left|\frac{j(m_\epsilon)}{|m_\epsilon|} - j(m_*)\right|^2\,dx.$$

Using Poincaré's inequality, we obtain that

$$\limsup_{\epsilon\searrow 0}\left((1-C\mathbf{h})\int_{\Omega_r(a)}\frac{1}{2}\left|\frac{j(m_\epsilon)}{|m_\epsilon|}-j(m_*)\right|^2 dx - D_\epsilon\right) \le -o_r(1),$$

and letting $r\to 0$ on the right as before we obtain (11). □

2.3 The Thiele Equation

For $h\in W^{1,1}(0,T;\mathbb{R}^2)$, which is small enough so that $W = W(h(t),\cdot)$ corresponds to a unique minimizer $\theta = \theta(h(t),\cdot)$ for all $t\in[0,T]$, we consider the equation

$$(4\pi q_\ell i + \alpha_0\pi)\dot{a}_\ell(t) + \frac{\partial W}{\partial a_\ell}(h(t),a(t)) = 0 \quad (\ell = 1,\dots,d). \tag{13}$$

Lemma 4. *For initial data $a(0) = a_0 \in \Omega_*^d$ the Cauchy problem for (13) has a unique solution $a\in C^1([0,T];\Omega^d)$, which satisfies the energy identity*

$$W(h(t_1),a(t_1)) - W(h(t_2),a(t_2)) = \alpha_0\,\pi\int_{t_1}^{t_2}|\dot{a}(s)|^2 ds + \int_{t_1}^{t_2}\int_\Omega \dot{h}(s)\cdot m_*\,dx\,ds$$

for all $0\le t_1 < t_2\le T$, where $m_*(x;a) = e^{i\theta(x)}M_*(x;a)$.

Proof. Using (6) and (13) we compute for the (unique) local solution $a = a(t)$

$$\begin{aligned}\frac{d}{dt}W(h(t),a(t)) &= \frac{\partial W}{\partial h}(h(t),a(t))\dot{h}(t) + \sum_{j=1}^d \frac{\partial W}{\partial a_j}(h(t),a(t))\cdot\dot{a}_j(t)\\ &= -\alpha_0\,\pi\,|\dot{a}(t)|^2 - \int_\Omega \dot{h}(t)\cdot m_*(x;a(t))\,dx.\end{aligned}$$

The energy identity follows, and the local solution $a = a(t)$ extends to $[0,T]$. □

3 LLG Equation with External Fields

Let us now consider the Landau-Lifshitz-Gilbert equation

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \left(\alpha_\epsilon \frac{\partial \mathbf{m}}{\partial t} - \boldsymbol{h}_{\text{eff}} \right), \tag{14}$$

where, for an external field $\boldsymbol{h} \in W^{1,1}(0, T; \mathbb{R}^3)$, the effective field is given by

$$\boldsymbol{h}_{\text{eff}} = \Delta \mathbf{m} - \frac{m_3}{\epsilon^2} \hat{\boldsymbol{e}}_3 + \boldsymbol{h}.$$

We consider a specific asymptotic behavior for α_ϵ such that $\alpha_\epsilon \log \frac{1}{\epsilon} \to \alpha_0 \in (0, \infty)$ as $\epsilon \to 0$. The effective field corresponds to minus the L^2 gradient of

$$E_\epsilon(\boldsymbol{h}, \mathbf{m}) = \int_\Omega e_\epsilon(\mathbf{m}) - \boldsymbol{h} \cdot \mathbf{m} \, dx.$$

where, as usual,

$$e_\epsilon(\mathbf{m}) = \frac{1}{2} |\nabla \mathbf{m}|^2 + \frac{m_3^2}{2\epsilon^2}$$

is the energy density of the Ginzburg-Landau type energy $E_\epsilon(\mathbf{m}) = E_\epsilon(0, \mathbf{m})$, which we have considered in [13–15]. In this section we study the equation for a fixed $\epsilon \in (0, 1)$. We impose Dirichlet boundary data given by a smooth map $\mathbf{g} = (g, 0)$ where $g : \partial\Omega \to \mathbb{S}^1$ with $\deg(g) = d$ and initial data $\mathbf{m}^0 \in H^1_{\mathbf{g}}(\Omega; \mathbb{S}^2)$ with

$$E_\epsilon(\mathbf{m}^0) \le d\pi \log \frac{1}{\epsilon} + C_0. \tag{15}$$

3.1 Conservation Laws

Let us assume $\mathbf{m}$ is a smooth solution of (1) in a space-time cylinder. The vorticity $\omega(\mathbf{m})$ makes contact to the LLG equation through the identity

$$\frac{\partial}{\partial t} \omega(\mathbf{m}) = \operatorname{curl} \left\langle \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t}, \nabla \mathbf{m} \right\rangle$$

leading to

$$\frac{\partial}{\partial t} \omega(\mathbf{m}) + \alpha_\epsilon \operatorname{curl} \left\langle \frac{\partial \mathbf{m}}{\partial t}, \nabla \mathbf{m} \right\rangle = \operatorname{curl} \operatorname{div} (\nabla \mathbf{m} \otimes \nabla \mathbf{m}) \,. \tag{16}$$

This conservation law for the vorticity will be crucial when identifying motion laws for vortices, which are the concentration points of $\omega(\mathbf{m})$ in the singular limit $\epsilon \searrow 0$.

Moreover, the energy identity for (14) reads

$$\frac{\partial}{\partial t}\Big(e_\epsilon(\mathbf{m}) - \boldsymbol{h}(t)\cdot\mathbf{m}\Big) + \alpha_\epsilon \left|\frac{\partial \mathbf{m}}{\partial t}\right|^2 = \operatorname{div}\left\langle \frac{\partial \mathbf{m}}{\partial t}, \nabla \mathbf{m}\right\rangle + \left\langle \dot{\boldsymbol{h}}(t), \frac{\partial \mathbf{m}}{\partial t}\right\rangle \tag{17}$$

Finally, we have conservation of spin

$$\frac{\partial m_3}{\partial t} + \operatorname{div} j(m) = \alpha_\epsilon\, m \wedge \frac{\partial m}{\partial t} + h \wedge m, \tag{18}$$

which is just the third component of (14), will imply that in the singular limit $\epsilon \searrow 0$, $\mathbf{m}$ will converge to an $h(t)$-harmonic map.

3.2 Weak Solutions and Bubbling

The LLG equation (14), for $\epsilon > 0$ fixed, is a lower order perturbation of the conformally invariant LLG equation $\mathbf{m}_t = \mathbf{m} \times (\alpha\, \mathbf{m}_t - \Delta \mathbf{m})$ which is traditionally studied in mathematical analysis. In dimension two, this equation is critical with respect to the natural energy estimate, and the formation of singularities in finite time must be expected, [1]. On the other hand, a well-known construction of what is called energy decreasing weak solutions, which has been introduced by Struwe [22] for the harmonic map heat flow, see also [4] and [6, 7] for LLG, can be carried out. In this framework, the possible blow-up scenario is precisely characterized through the formation of bubbles at the energy concentration points.

This is in fact the new fundamental difficulty compared with the corresponding problem for the complex Ginzburg-Landau theory, where at the finite ϵ level, evolution equations admit smooth solutions for all times, [12]. Since vortex trajectories are retraced in terms of concentration sets of the energy density $e_\epsilon(\mathbf{m})$ and the vorticity $\omega(\mathbf{m})$, precise information about their behavior near the singular points is a crucial ingredient to our analysis. This information can be obtained from the well-developed bubbling analysis for harmonic maps and flows, established e.g. in [5, 16–18, 25]. Applied to (14) we obtain the following result (cf. [14, Sect. 4] for more information):

Theorem 1. *For initial data* $\mathbf{m}^0 \in C^\infty(\overline{\Omega}; \mathbb{S}^2)$ *there exists a weak solution* $\mathbf{m}$ *of* (14) *which satisfies the energy inequality*

$$\alpha_\epsilon \int_0^{t_0}\int_\Omega \left|\frac{\partial \mathbf{m}}{\partial t}\right|^2 dx\,dt + E_\epsilon(\boldsymbol{h}(t_0), \mathbf{m}^0) \le E_\epsilon(\boldsymbol{h}(0), \mathbf{m}^0) - \int_0^{t_0}\int_\Omega \dot{\boldsymbol{h}}\cdot \mathbf{m}\, dx\,dt$$

for all $0 \le t_0 \le T$ *and is smooth away from a finite number of points* (x_i, t_i) *in space time. Moreover, there exists, for every* i, *an integer* q_i *such that for every sufficiently small* $r > 0$

$$\int_{B_r(x^i)\times\{t_i\}} e_\epsilon(\mathbf{m})\,dx + 4\pi|q_i| \le \liminf_{t\nearrow t^i} \int_{B_r(x^i)\times\{t_i\}} e_\epsilon(\mathbf{m})\,dx$$

and

$$\int_{B_r(x^i)\times\{t_i\}} \omega(\mathbf{m})\,dx + 4\pi q_i = \lim_{t\nearrow t^i} \int_{B_r(x^i)\times\{t_i\}} \omega(\mathbf{m})\,dx.$$

Finally, the (energy decreasing) solution $\mathbf{m}$ *is unique in its class.*

Form the energy inequality we deduce that for $E_\epsilon(\mathbf{m}^0) \le d\pi\log(1/\epsilon) + C_0$,

$$\alpha_\epsilon \int_0^{t_0}\int_\Omega \left|\frac{\partial \mathbf{m}}{\partial t}\right|^2 dx\,dt + E_\epsilon(\mathbf{m}(t_0)) \le d\pi\log(1/\epsilon) + C_1, \tag{19}$$

where $0 \le C_1 - C_0$ can be bounded above by a multiple of $\int_0^T |\dot{\boldsymbol{h}}(t)|dt + |\boldsymbol{h}(0)|$.

4 Convergence and Vortex Trajectories

Now we consider a sequence of initial data $\mathbf{m}_\epsilon^0 \in H_g^1(\Omega;\mathbb{S}^2)$ such that

$$\alpha_\epsilon e_\epsilon(\mathbf{m}_\epsilon^0) \to \alpha_0\pi\delta_{a^0}, \quad \omega_0(\mathbf{m}_\epsilon^0) \to 4\pi\sum_{\ell=1}^d q_\ell\delta_{a_\ell^0} \quad \text{and} \quad \lim_{\epsilon\searrow 0} D_\epsilon(\mathbf{m}_\epsilon^0; a^0) = 0$$

for a certain $a_0 \in \Omega_*^d$ and $q_1,\dots,q_d = \pm\frac{1}{2}$ and the corresponding weak solution $\mathbf{m}_\epsilon$ from Theorem 1. As in [14, Theorem 4.1] (see [13] for more details) and in view of Proposition 2 we obtain the following convergence result.

Theorem 2. *There exist a time* $T_0 \in (0, T]$, *a sequence* $\epsilon_k \searrow 0$, *and a curve*

$$a \in H^1(0, T_0; \Omega^d) \quad \text{with} \quad a(0) = a^0 \quad \text{and} \quad \inf_{t\in(0,T_0)} \rho(a(t)) > 0$$

such that for every $t \in [0, T_0]$ *and* $1 \le p < 2$

$$m_{\epsilon_k}(\cdot, t) \rightharpoonup m_*(\cdot, a(t)) \quad \text{weakly in} \quad W^{1,p}(\Omega;\mathbb{R}^2),$$

$$\alpha_{\epsilon_k} e_{\epsilon_k}(\mathbf{m}_{\epsilon_k}(\cdot,t)) \overset{*}{\rightharpoonup} \alpha_0\pi\delta_{a(t)} \quad \text{weakly* in} \quad (C_0^0(\Omega))^*,$$

$$J(m_{\epsilon_k}(\cdot,t)) \to \pi\delta_{a(t)}, \quad \omega(\mathbf{m}_{\epsilon_k}(\cdot,t)) \to 4\pi\sum_{\ell=1}^d q_\ell\delta_{a_\ell(t)} \quad \text{in} \quad (C_0^{0,1}(\Omega))^*.$$

Moreover, for all $t_1, t_2 \in [0, T_0]$ *with* $t_1 \le t_2$ *and* $\eta \in C^1(\overline{\Omega})$

$$\alpha_0 \pi \sum_{\ell=1}^{d} \eta(a_\ell(t))\Big|_{t=t_1}^{t_2} = \lim_{k\to\infty} \left(\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \nabla\eta \cdot \left\langle \frac{\partial \mathbf{m}_{\epsilon_k}}{\partial t}, \nabla \mathbf{m}_{\epsilon_k} \right\rangle dx\, dt \right)$$

and

$$\alpha_0 \pi \int_{t_1}^{t_2} |\dot{a}|^2\, dt \le \liminf_{k\to\infty} \left(\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \left| \frac{\partial \mathbf{m}_{\epsilon_k}}{\partial t} \right|^2 dx\, dt \right).$$

From the energy inequality in Theorem 1, the convergence of m_{ϵ_k} in Theorem 2 and conservation of spin identity (18) we deduce in particular that

$$j(m_{\epsilon_k}(t,\cdot)) \rightharpoonup j(m_*(t,\cdot)) \quad \text{weakly in } L^p(\Omega;\mathbb{R}^2) \tag{20}$$

for every $t \in [0, T_0)$, where

$$\operatorname{div} j(m_*(t,\cdot) = h(t) \wedge m_* \quad \text{and} \quad \operatorname{curl} j(m_*(t,\cdot)) = 2\pi\delta_a(t).$$

5 Motion Law

Theorem 3. *There exist positive numbers* h_0 *and* ϵ_0 *with the following property: For every* $\epsilon \in (0, \epsilon_0)$ *and every smooth* $\boldsymbol{h} : [0, T] \to \mathbb{R}^3$ *with*

$$\int_0^T |\dot{\boldsymbol{h}}(t)| dt + |\boldsymbol{h}(0)| < h_0,$$

there exists a smooth solution $\mathbf{m}_\epsilon \in C^\infty(\overline{\Omega} \times [0, T]; \mathbb{S}^2)$ *of the Landau-Lifshitz-Gilbert equation* (14) *with* $\mathbf{m}_\epsilon(\cdot, 0) = \mathbf{m}_\epsilon^0$ *and* $\mathbf{m}_\epsilon(\cdot, t)|_{\partial\Omega} = \mathbf{g}$ *for every* $t \ge 0$. *Moreover, for every* $t \in [0, T]$,

$$\alpha_\epsilon e_\epsilon(\mathbf{m}_\epsilon(\cdot, t)) \to \pi\alpha_0 \sum_{\ell=1}^{d} \delta_{a_\ell(t)} \quad \textit{and} \quad \omega(\mathbf{m}_\epsilon(\cdot, t)) \to 4\pi \sum_{\ell=1}^{d} q_\ell \delta_{a_\ell(t)}$$

as $\epsilon \searrow 0$, *in the sense of distributions, where* $a \in C^\infty([0, T]; \Omega^d)$ *is the solution of Thiele's equation*

$$G_\ell \times \dot{a}_\ell + D\, \dot{a}_\ell + \frac{\partial W(h, a)}{\partial a_\ell} = 0 \quad (\ell = 1, \ldots, d) \tag{21}$$

with $a(0) = a^0$ *and where* $G_\ell = 4\pi q_\ell \hat{\boldsymbol{e}}_3$ *and* $D = \pi\, \alpha_0$ *with* $\alpha_0 = \lim_{\epsilon\searrow 0} \alpha_\epsilon \log \frac{1}{\epsilon}$.

The rest of this section is devoted to the proof of the Theorem. Let $\hat{a} \in C^\infty([0,\infty);\Omega^d)$ be the unique solution of the initial value problem for (21) with initial values $\hat{a}(0) = a^0 \in \Omega^d$. We choose $T_0 > 0$ and a sequence $\epsilon_k \searrow 0$ that satisfy the conclusions of Theorem 2, and let a be the corresponding curve in Ω^d. We recall that solutions remain smooth in $(0, T_0)$ for small ϵ as shown in [14, Theorem 3], so we can concentrate on the verification of the motion law.

We fix a radius $r \in (0, \rho(a^0)/2]$ and adapt the terminal time T_0 such that the trajectories of a_ℓ and $\hat{a}_\ell$ do not exit $B_{r/2}(a_\ell^0)$ before time T_0 for all $\ell = 1, \dots, d$. As in [14] we choose $\phi, \psi \in C_0^\infty(\Omega)$ such that for every ℓ, both ϕ and ψ are affine with $\nabla\psi = \nabla^\perp\phi$ in $B_r(a_\ell^0)$. We define

$$\xi_k(t) = \int_{\Omega\times\{t\}} \big(\alpha_{\epsilon_k}\psi\, e_{\epsilon_k}(\mathbf{m}_{\epsilon_k}) + \phi\,\omega_0(\mathbf{m}_{\epsilon_k})\big)\,dx - \pi\sum_{\ell=1}^d (\alpha_0\psi(\hat{a}_\ell(t)) + 4q_\ell\phi(\hat{a}_\ell(t)))\,,$$

converging, for every $t \in [0, T)$, to

$$\xi(t) = \pi\sum_{\ell=1}^d \Big(\alpha_0\Big(\psi(a_\ell(t)) - \psi(\hat{a}_\ell(t))\Big) + 4q_\ell\Big(\phi(a_\ell(t)) - \phi(\hat{a}_\ell(t))\Big)\Big)\,.$$

In order to apply Proposition 4 we fix h_0 sufficiently small.

Lemma 5. *There exists a constant C such that for all $t_1, t_2 \in [0, T_0]$ with $t_1 \le t_2$ and every $k \in \mathbb{N}$,*

$$\xi_k(t_2) - \xi_k(t_1) \le C\int_{t_1}^{t_2}\Big(D_{\epsilon_k}^{h(t)}(\mathbf{m}_{\epsilon_k};\hat{a}(t)) + |a(t) - \hat{a}(t)|\Big)\,dt + o_{\epsilon_k}(1).$$

Proof. From (13) we obtain

$$\pi\sum_{\ell=1}^d \frac{d}{dt}(\alpha_0\psi(\hat{a}_\ell(t)) + 4q_\ell\phi(\hat{a}_\ell(t))) = -\frac{\partial W(h,\hat{a})}{\partial a_\ell}\cdot\nabla\psi(\hat{a}_\ell(t))$$

while from Lemma 3 with $\hat{m}_* := m_*(\,\cdot\,;\hat{a})$ and $\Phi = \nabla^\perp\phi$

$$-\sum_{\ell=1}^d \nabla\psi(\hat{a}_\ell(t))\cdot\frac{\partial W(h,\hat{a})}{\partial a_\ell} = \int_{\Omega\times\{t\}}\nabla^\perp\nabla\phi : (\nabla\hat{m}_*\otimes\nabla\hat{m}_*)\,dx.$$

Using conservation of vorticity (16), we find after integration by parts in space and integration in time

$$\int_\Omega \phi\,\omega(\mathbf{m}_{\epsilon_k}(t_2)) - \phi\,\omega(\mathbf{m}_{\epsilon_k}(t_1))\,dx = \alpha_{\epsilon_k}\int_{t_1}^{t_2}\int_\Omega \nabla^\perp\phi\cdot\Big\langle\frac{\partial\mathbf{m}_{\epsilon_k}}{\partial t}, \nabla\mathbf{m}_{\epsilon_k}\Big\rangle\,dx\,dt$$
$$+\int_{t_1}^{t_2}\int_\Omega \nabla^\perp\nabla\phi : \big(\nabla\mathbf{m}_{\epsilon_k}\otimes\nabla\mathbf{m}_{\epsilon_k}\big)\,dx\,dt.$$

For the terms on the left we use convergence of the vorticity provided by Theorem 2. Concerning the first term on the right we deduce from the energy estimate in Theorem 1

$$\left(\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega (\nabla^\perp \phi - \nabla \psi) \cdot \left\langle \nabla \mathbf{m}_{\epsilon_k}, \frac{\partial \mathbf{m}_{\epsilon_k}}{\partial t} \right\rangle dx\, dt \right)^2$$
$$\le c \int_{t_1}^{t_2} \int_\Omega |\nabla^\perp \phi - \nabla \psi|^2 \, \alpha_{\epsilon_k} e_{\epsilon_k}(\mathbf{m}_{\epsilon_k})\, dx\, dt \to 0$$

while by convergence of the kinetic term in Theorem 2

$$\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \nabla \psi \cdot \left\langle \nabla \mathbf{m}_{\epsilon_k}, \frac{\partial \mathbf{m}_{\epsilon_k}}{\partial t} \right\rangle dx\, dt \to -\pi \alpha_0 \sum_{\ell=1}^{d} \Big(\psi(a_\ell(t_2)) - \psi(a_\ell(t_1)) \Big)$$

as $\epsilon_k \searrow 0$. Therefore, it suffices to estimate the integrals

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla \phi : (\nabla \mathbf{m}_{\epsilon_k} \otimes \nabla \mathbf{m}_{\epsilon_k} - \nabla \hat{m}_* \otimes \nabla \hat{m}_*)\, dx\, dt,$$

which, by virtue of the usual decomposition argument and Proposition 4 (see [14, Sect. 6]), reduces to the estimation of

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla \phi : ((j(m_{\epsilon_k}) - j(\hat{m}_*)) \otimes j(\hat{m}_*))\, dx\, dt$$

and

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla \phi : (j(\hat{m}_*) \otimes (j(m_{\epsilon_k}) - j(\hat{m}_*)))\, dx\, dt.$$

Taking into account that both integrands are products of the form

$$\sigma \cdot (j(m_{\epsilon_k}) - j(\hat{m}_*))$$

for smooth vector fields $\sigma \in C^\infty(\overline{\Omega} \times [0, T_0]; \mathbb{R}^2)$ independent of k, we obtain from (20) with $m_* = m_*(\cdot, a(t))$ and $\hat{m}_* = m_*(\cdot, \hat{a}(t))$

$$\int_{t_1}^{t_2} \int_\Omega \sigma \cdot (j(m_{\epsilon_k}) - j(\hat{m}_*))\, dx\, dt = \int_{t_1}^{t_2} \int_\Omega \sigma \cdot (j(m_*) - j(\hat{m}_*))\, dx\, dt + o_{\epsilon_k}(1).$$

Next we adopt the Hodge decomposition argument used in [14, Lemma 7]. Writing $-\sigma = \nabla u + \nabla^\perp v$, where $u, v \in C^\infty(\overline{\Omega} \times [0, T_0])$ with $u = 0$ on $\partial\Omega \times [0, T_0]$ we infer, also taking into account Lemma 2,

$$\int_{t_1}^{t_2}\int_{\Omega}\sigma\cdot(j(m_*)-j(\hat m_*))\,dx\,dt=$$
$$\int_{t_1}^{t_2}\int_{\Omega}h\wedge(m_*-\hat m_*)\,u\,dx\,dt+2\pi\sum_{\ell=1}^{d}\int_{t_1}^{t_2}\left(v(a_\ell)-v(\hat a_\ell)\right)\,dt$$
$$\leq c\int_{t_1}^{t_2}|a(t)-\hat a(t)|\,dt. \qquad \square$$

Proof (Theorem 3). The proof follows by the usual Gronwall argument. For $t\in[0,T_0]$, we consider the functions

$$\zeta_k(t)=D_{\epsilon_k}^{\boldsymbol{h}(t)}(\mathbf{m}_{\epsilon_k}(t);\hat a(t))\quad\text{and}\quad\chi(t)=|\hat a_\ell(t)-a_\ell(t)|.$$

First we show $\zeta_k\to\zeta$ in $L^1(0,T_0)$ for a function $\zeta\in BV(0,T_0)$ with

$$\dot\zeta\leq c\big(|\dot{\hat a}-\dot a|+\chi\big). \tag{22}$$

In fact, we obtain from Lemma 4

$$W(h(t_1),\hat a(t_1))-W(h(t_2),\hat a(t_2))=\pi\alpha_0\int_{t_1}^{t_2}|\dot{\hat a}|^2\,dt-\int_{t_1}^{t_2}\int_{\Omega}\dot h(t)\cdot m_*(\cdot,\hat a(t))\,dx\,dt$$

and from (19)

$$E_{\epsilon_k}(\boldsymbol{h}(t_2),\mathbf{m}_{\epsilon_k}(t_2))-E_{\epsilon_k}(\boldsymbol{h}(t_1),\mathbf{m}_{\epsilon_k}(t_1))=$$
$$-\int_{t_1}^{t_2}\int_{\Omega}\left(\alpha_{\epsilon_k}\left|\frac{\partial\mathbf{m}_{\epsilon_k}}{\partial t}\right|^2-\dot{\boldsymbol{h}}(t)\cdot\mathbf{m}_{\epsilon_k}\right)\,dx\,dt,$$

respectively, for $0\leq t_1\leq t_2\leq T_0$, while

$$\left|\int_{t_1}^{t_2}\int_{\Omega}\left(\dot{\boldsymbol{h}}(t)\cdot m_{\epsilon_k}-\dot h(t)\cdot m_*(\cdot,\hat a(t))\right)\,dx\,dt\right|\leq c\int_{t_1}^{t_2}\chi(t)\,dt+o_{\epsilon_k}(1).$$

In view of Theorem 2 we can select a subsequence such that $\zeta_k(t)\to\zeta(t)$ almost everywhere for a bounded function $\zeta:[0,T_0]\to\mathbb{R}$ with

$$\zeta(t_2)-\zeta(t_1)\leq\int_{t_1}^{t_2}\pi\alpha_0\left(|\dot{\hat a}|^2-|\dot a|^2\right)+c\,\chi(t)\,dt\leq c\int_{t_1}^{t_2}|\dot{\hat a}-\dot a|+\chi(t)\,dt$$

for almost all $t_1\leq t_2$, which implies (22). Now Lemma 5 implies, by virtue of (22), for $0\leq t_1\leq t_2\leq T_0$,

$$\xi(t_2) - \xi(t_1) \leq c \int_{t_1}^{t_2} (\zeta(t) + \chi(t))\, dt.$$

With an appropriate choice of ϕ and ψ we obtain the desired inequality

$$|\dot{\hat{a}}(t) - \dot{a}(t)| \leq c \int_0^t |\dot{\hat{a}}(\tau) - \dot{a}(\tau)|\, d\tau.$$

As $\hat{a}(0) = a(0)$, Gronwall's lemma implies $\hat{a} = a$ in $[0, T_0]$. Moreover,

$$\limsup_{k\to\infty} D_{\epsilon_k}^{h(T_0)}(\mathbf{m}_{\epsilon_k}(T_0); a(T_0)) \leq 0,$$

which enables us to iterate the argument for new initial times T_0, and we eventually obtain the motion law for all times before T_0. Note that by uniqueness of energy decreasing solutions, solutions $\mathbf{m}_\epsilon$ extend, for small ϵ, smoothly to $[0, T]$. Finally, thanks to the unique solvability of the limiting ODE, the convergence result for energy density and vorticity can be seen to hold without taking subsequences, as any subsequence of $\epsilon \searrow 0$ will have a further subsequence converging to the same limit. □

References

1. Bartels, S., Ko, J., Prohl, A.: Numerical analysis of an explicit approximation scheme for the Landau-Lifshitz-Gilbert equation. Math. Comput. **77**(262), 773–788 (2008)
2. Bethuel, F., Brezis, H., Hélein, F.: Ginzburg-Landau Vortices. Progress in Nonlinear Differential Equations and their Applications, vol. 13. Birkhäuser Boston Inc., Boston (1994)
3. Capella, A., Melcher, C., Otto, F.: Wave-type dynamics in ferromagnetic thin films and the motion of Néel walls. Nonlinearity **20**(11), 2519–2537 (2007)
4. Chang, K.-C.: Heat flow and boundary value problem for harmonic maps. Ann. Inst. H. Poincaré Anal. Non Linéaire **6**(5), 363–395 (1989)
5. Ding, W., Tian, G.: Energy identity for a class of approximate harmonic maps from surfaces. Commun. Anal. Geom. **3**(3–4), 543–554 (1995)
6. Guo, B.L., Hong, M.C.: The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps. Calc. Var. Partial Differ. Equ. **1**(3), 311–334 (1993)
7. Harpes, P.: Uniqueness and bubbling of the 2-dimensional Landau-Lifshitz flow. Calc. Var. Partial Differ. Equ. **20**(2), 213–229 (2004)
8. Huber, D.L.: Dynamics of spin vortices in two-dimensional planar magnets. Phys. Rev. B **26**(7), 3758–3765 (1982)
9. Jerrard, R.L.: Lower bounds for generalized Ginzburg-Landau functionals. SIAM J. Math. Anal. **30**(4), 721–746 (1999) (electronic)
10. Jerrard, R.L., Soner, H.M.: The Jacobian and the Ginzburg-Landau energy. Calc. Var. Partial Differ. Equ. **14**(2), 151–191 (2002)
11. Jerrard, R.L., Spirn, D.: Refined Jacobian estimates and Gross-Pitaevsky vortex dynamics. Arch. Ration. Mech. Anal. **190**(3), 425–475 (2008)
12. Kurzke, M., Melcher, C., Moser, R., Spirn, D.: Dynamics for Ginzburg-Landau vortices under a mixed flow. Indiana Univ. Math. J. **58**(6), 2597–2621 (2009)

13. Kurzke, M., Melcher, C., Moser, R.: Vortex motion for the Landau–Lifshitz–Gilbert equation with spin-transfer torque. SIAM J. Math. Anal. **43**(3), 1099–1121 (2011)
14. Kurzke, M., Melcher, C., Moser, R., Spirn, D.: Ginzburg-Landau vortices driven by the Landau-Lifshitz-Gilbert equation. Arch. Ration. Mech. Anal. **199**(3), 843–888 (2011)
15. Kurzke, M., Melcher, C., Moser, R., Spirn, D.: Vortex dynamics in the presence of excess energy for the Landau-Lifshitz-Gilbert equation. (Calc. Var. PDE SFB 611 Preprint 526) (2012, to appear)
16. Lin, F., Wang, C.: Energy identity of harmonic map flows from surfaces at finite singular time. Calc. Var. Partial Differ. Equ. **6**(4), 369–380 (1998)
17. Qing, J.: On singularities of the heat flow for harmonic maps from surfaces into spheres. Commun. Anal. Geom. **3**, 297–315 (1995)
18. Qing, J., Tian, G.: Bubbling of the heat flows for harmonic maps from surfaces. Commun. Pure Appl. Math. **50**(4), 295–310 (1997)
19. Sandier, E.: Lower bounds for the energy of unit vector fields and applications. J. Funct. Anal. **152**(2), 379–403 (1998)
20. Sandier, E., Serfaty, S.: Gamma-convergence of gradient flows with applications to Ginzburg-Landau. Commun. Pure Appl. Math. **57**(12), 1627–1672 (2004)
21. Serfaty, S., Tice, I.: Lorentz space estimates for the Ginzburg-Landau energy. J. Funct. Anal. **254**(3), 773–825 (2008)
22. Struwe, M.: On the evolution of harmonic mappings of Riemannian surfaces. Comment. Math. Helv. **60**(4), 558–581 (1985)
23. Struwe, M.: On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions. Differ. Integral Equ. **7**(5–6), 1613–1624 (1994)
24. Thiele, A.A.: Steady-state motion of magnetic domains. Phys. Rev. Lett. **30**(6), 230–233 (1973)
25. Wang, C.: Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets. Houston J. Math. **22**(3), 559–590 (1996)