Adapted Function Spaces for Dispersive Equations

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Abstract The study of the p variation of functions of one variable has a long history. It has been discussed by Wiener in [21]. Here we define the space of functions of finite p Variation, and the predual space U^q, and we use them to study dispersive equations.

1 Introduction

The study of the p variation of functions of one variable has a long history. It has been discussed by Wiener in [21]. The generalization of the Riemann-Stieltjes integral to functions of bounded p variation against the deriative of a function of bounded q variation $1/p + 1/q > 1$ is due to Young [22]. Much later Lyons developed his theory of rough path [13] and [14], buildung on Young's ideas, but going much further.

In parallel Tataru realized that the spaces of bounded p variation, and their close relatives, the U^p spaces, allow a powerful sharping of Bourgain's technique of function spaces adapted to the dispersive equation at hand. These ideas were applied for the first time in the work of the author and Tataru in [11]. Since then there has been a number of questions in dispersive equations where these function spaces have been used. For example they play a crucial role in [12], but there they could probably be replaced by Bourgain's Fourier restriction spaces $X^{s,b}$. On the other hand, for wellposedness for the Kadomtsev-Petviashvili II equation in a critical function space (see [3]) the $X^{s,b}$ spaces seem to be insufficient. The theory of the spaces U^p and V^p and some of their basic properties like duality and logarithmic interpolation have been worked out in [3], with a focus on what

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was needed there. Until very recently the developments in stochastic differential equations and dispersive equations were entirely independent. The present treatment considerably extends the theory of [3].

We will introduce the spaces U^p and V^p, study their properties and indicate their role for dispersive equations. After that we turn to wellposedness questions for several dispersive PDEs, where we select a number of relevant and representative problems.

In the sequel $p \in [1,\infty]$. Unless explicitly stated otherwise we consider $p \in (1,\infty)$.

2 The Bounded *p* Variation

Definition 1. Let I be an interval, $1 \le p < \infty$ and $f : I \to X$. We define

$$\omega_p(v,I) := \sup_{t_i \in I, t_1 < t_2 \dots t_{n+1}} \left(\sum \|v(t_{i+1}) - v(t_i)\|_X^p \right)^{1/p} \in [0,\infty].$$

There are obvious properties. The function $t \to \omega_p(v,[a,t))$ is monotonically increasing. The same is true if we consider closed or open intervals. Moreover

$$\omega_p(v,[a,b)) \le \omega_p(v,[a,c)) \le 2(\omega_p(v,[a,b]) + \omega_p(v,[b,c))).$$

Finiteness of the p variation implies existence of one sided limits. It is not hard to see that $v \to \omega_p(v,[a,b))$ defines a norm, up to constants. If v is continuous and the p variation is bounded then it is a continuous function of the endpoint. Moreover

$$\omega_p(v,(a,b)) \le |b-a|^{1/p} \|v\|_{\dot C^{1/p}}$$

where $\dot C^{1/p}$ denotes the homogeneous Hölder space.

2.1 *Step Functions and Ruled Functions*

We introduce and study functions from an interval $[a,b)$ to $\mathbb{R}$, $\mathbb{R}^n$, a Hilbert space or a Banach space X, and spaces of such functions which are invariant under continuous monotone reparametrizations of the interval. For the most part of this section there are no more than the obvious modifications when considering Banach space valued functions.

We call a function f a ruled function if at every point (including the endpoints, which may be $\pm\infty$) left and right limits exist. This set is closed with respect to uniform convergence. We denote the Banach space of ruled functions equipped with the supremum norm by $\mathscr{R}$.

A partition τ of $[a,b)$ is a strictly increasing finite sequence

$$a < t_1 < t_2 < \cdots < t_{n+1} < b$$

where we allow $b = \infty$ and $a = -\infty$. A step function is a function f for which there exists a partition so that f is constant on every interval (a,t_1), (t_i,t_{i+1}) and (t_n,b). We do not require that the value at a point coincides with the limit from either side. Step functions are dense in $\mathscr{R}$. We denote the set of step functions by $\mathscr{S}$. Let $\mathscr{R}_{rc}$ be the closed subset of $\mathscr{R}$ of right continuous functions f with $\lim_{t\to a} f(t) = 0$. Similarly, if $X \subset \mathscr{R}$ we denote by X_{rc} the intersection with $\mathscr{R}_{rc}$.

The step functions

$$f_t = \chi_{[t,b)}$$

satisfy

$$\|f_t - f_s\|_{\sup} = 1 \tag{1}$$

for $s \neq t$. We will study Banach spaces Z most of which contain the right continuous step functions $\mathscr{S}_{rc}$, and which embed into $\mathscr{R}$. Moreover we will always have

$$1 \le \|f_t - f_s\|_Z \le 2 \tag{2}$$

and hence none of those spaces is separable.

It will be convenient to extend every function on $[a,b)$ by zero to $[a,b]$, i.e. we will always set $f(b) = 0$, even if $a = -\infty$ or $b = \infty$.

Definition 2. For $f \in \mathscr{R}$ and a partition

$$\tau = (t_1, t_2 \ldots t_n), \quad a < t_1 < t_2 < t_3 \cdots < t_n < b$$

we define (denoting the limit from the right by $f(t+)$)

$$f_\tau(t) = \begin{cases} f(t) & \text{if } t = t_j \\ f(a+) & \text{if } a < t < t_1 \\ f(t_i+) & \text{if } t_i < t < t_{i+1} \\ f(t_n) & \text{if } t_n < t \end{cases}$$

We observe that f_τ is a step function, and it is right continuous if f is right continuous.

2.2 *The Spaces V^p and U^p*

In this subsection we consider functions on (a,b) where we allow the cases $a = -\infty$ and $b = \infty$.

Definition 3. Let X be a Banach space, $1 \leq p < \infty$ and $v : (a,b) \to X$. We define

$$\|v\|_{V^p((a,b),X)} = \max\{\|v\|_{sup}, \omega_p(v,(a,b))\}.$$

Let $V^p = V^p((a,b)) = V^p(X) = V^p((a,b);X)$ be the set of all functions for which this expression is finite. We often suppress the interval and/or the Banach space in the notation when this seems appropriate.

The interval will usually be of minor importance. We omit it often in the sequel. The following properties are immediate:

1. $V^p(I)$ is closed with respect to this norm and hence $V^p(I)$ is a Banach subspace of $\mathscr{R}$. Moreover $V^p_{rc}(I)$ is a closed subspace.
2. We set $V^\infty = \mathscr{R}$.
3. If $1 \leq p \leq q \leq \infty$ then

$$\|v\|_{V^q} \leq \|v\|_{V^p}.$$

4. Let X_i be Banach spaces, $T : X_1 \times X_2 \to X_3$ a bounded bilinear operator, $v \in V^p(X_1)$ and $w \in V^p(X_2)$. Then $T(v,w) \in V^p(X_3)$ and

$$\|T(v,w)\|_{V^p(X_3)} \leq 2\|T\|\|v\|_{V^p(X_1)}\|w\|_{V^p(X_2)}.$$

5. We embed $V^p((a,b))$ into $V^p(\mathbb{R})$ by extending v by 0.
6. The space V^1 has some additional structure: Every bounded monotone function is in V^1, and functions in V^1 can be written as the difference of two bounded monotone functions.

The space of bounded p variation is build on the sequence space l^p. We may also replace it by the weak space l^p_w with

$$\|(x_j)\|_{l^p_w} = \sup_\lambda \lambda \#\{|x_j| > \lambda\}^{1/p}.$$

Definition 4. Let $1 \leq p < \infty$. The weak V^p_w space consists of all functions such that

$$\|v\|_{V^p_w} = \max\{\sup_{t_1 < \cdots < t_n} \|(v(t_{i+1}) - v(t_i))_{1 \leq i \leq n-1}\|_{l^p_w}, \|v\|_{sup}\}$$

is finite.

The spaces of bounded p variation are of considerable importance in probability and harmonic analysis. We shall see that V^p is the dual space of a space U^q, $1/p + 1/q = 1$, $1 < p < \infty$, with a duality pairing closely related to the Stieltjes integral, and its variant, the Young integral [22].

Definition 5. A p-atom a is a step function in $\mathscr{S}_{rc}$,

$$a(t) = \sum_{i=1}^{n} \phi_i \chi_{[t_i, t_{i+1})}(t)$$

where $\tau = (t_1 \dots t_n)$ is a partition, $t_{n+1} = b$, with $\sum |\phi_i|^p \le 1$. A p-atom a is called a strict p atom if

$$\max \|\phi_i\|_X (\#\tau)^{1/p} \le 1.$$

Let a_j be a sequence of atoms and λ_j a summable sequence. Then

$$u = \sum \lambda_j a_j$$

is a U^p function. The right hand side converges in $\mathscr{R}$. We define

$$\|u\|_{U^p} = \inf\{\sum |\lambda_j| : u = \sum \lambda_j a_j\}.$$

The strict space U^p_{strict} is defined in the same fashion using strict p atoms.

We collect a number of elementary properties.

1. If a is a p-atom then $\|a\|_{U^p} \le 1$. In general the norm is less than 1.
2. Functions in U^p are continuous from the right. The limit as $t \to a$ vanishes.
3. The expression $\|.\|_{U^p}$ defines a norm on U^p, and U^p is closed with respect to this norm. Hence $U^P \subset \mathscr{R}_{rc}$ is a Banach subspace.
4. If $p < q$ then $U^p \subset U^q$ and
$$\|u\|_{U^q} \le \|u\|_{U^p}.$$
5. If $1 \le p < \infty$ then for all $u \in U^p$
$$\|u\|_{V^p} \le 2^{1/p} \|u\|_{U^p}.$$
6. Let Y be a Banach space, and let the linear operator $T : \mathscr{S}_{rc} \to Y$ satisfy
$$\|Ta\|_Y \le C \tag{3}$$
for every p atom. Then T has a unique extension to a bounded linear operator from U^p to Y which satisfies
$$\|Tf\|_Y \le C \|f\|_{U^p}. \tag{4}$$

7. Let X_i be Banach spaces, $T : X_1 \times X_2 \to X_3$ a bounded bilinear operator, $v \in U^p(X_1)$ and $w \in U^p(X_2)$. Then $T(v,w) \in U^p(X_3)$ and

$$\|T(v,w)\|_{U^p(X_3)} \le 2\|T\|\|v\|_{U^p(X_1)}\|w\|_{U^p(X_2)}.$$

8. We consider $U^p([a,b))$ in the same way as subspace of $U^p(\mathbb{R})$ as for V^p.

The following decomposition is crucial for most of the following. It is related to Young's generalization of the Stieltjes integral, and it deals with a crucial point in the theory. A proof is contained in [11].

Lemma 1. *There exists $\delta > 0$ such that for v right continuous with $\|v\|_{V^p_w} = \delta$ there are strict p atoms a_i with*

$$\|a_j(t)\|_{sup} \le 2^{1-j} \qquad \text{and} \qquad \#\tau_j \le 2^{jp}$$

such that

$$v = \sum a_j.$$

There are a number of simple interesting and useful consequences.

Lemma 2. *Let $1 < p < q < \infty$. There exists $\kappa > 0$, depending only on p and q, such that for all $v \in V^p_{w,rc}$ and $M \ge 1$ there exist $u \in U^p_{strict}$ and $w \in U^q_{strict}$ with*

$$v = u + w$$

and

$$\frac{\kappa}{M}\|u\|_{V^p_{strict}} + e^M\|w\|_{U^q_{strict}} \le \|v\|_{V^p_w}.$$

Observe that we may replace U^p_{strict} by U^p (since $U^p_{strict} \subset U^p$) and V^p_w by V^p (since $V^p \subset V^p_w$). The proof is simple: We apply Lemma 1 and define u as the sum of the first m a_j. We obtain the following embedding

Lemma 3. *Let $1 < p < q < \infty$. Then*

$$V^p_{rc} \subset V^p_{w,rc} \subset U^q_{strict} \subset U^q.$$

Proof. Apply Lemma 2 with $M = 1$. □

The Riemann-Stieltjes integral defines

$$\int f\,dg = \int f g_t\,dt$$

for $f \in \mathscr{R}$ and $g \in V^1$. If $f \in \mathscr{S}_{rc}$ then

$$\int f g_t dt = \sum f(t_i)(g(t_i) - g(t_{i-1})). \tag{5}$$

We take this formula as our starting point for a similar integral for $f \in V^p$ and $g \in U^q$, for $1/p + 1/q = 1$, $q \geq 1$. Results become much cleaner when we use an equivalent norm in V^p,

$$\|v\|_{V^p} = \sup_{a<t_1 \dots t_n<b} \left(\sum_j |v(t_{j+1}) - v(t_j)|^p + |v(t_n)|^p \right)^{1/p}$$

which we do in the sequel. We also set $v(b) = 0$ and, for any partition, $t_{n+1} = b$.

Theorem 1. *There is a unique continuous bilinear map*

$$B : U^q(X) \times V^p(X^*) \to \mathbb{R}$$

which satisfies (with $t_0 = a$ and $u(t_0) = 0$, and a somewhat sloppy notation for the duality map $X^* \times X \to \mathbb{R}$)

$$B(u, v) = \sum_{i=1}^{n} (u(t_i) - u(t_{i-1}))v(t_i)$$

for $u \in \mathscr{S}_{rc}$ with associated partition $(t_1, \dots t_n)$ and

$$|B(u, v)| \leq \|u\|_{U^q(X)} \|v\|_{V^p(X^*)}. \tag{6}$$

The map

$$V^p(X^*) \ni v \to (u \to B(u, v)) \in (U^q(X))^*$$

is a surjective isometry. Moreover

$$\|v\|_{V^p(X^*)} = \sup_{u \in U^q(X), \|u\|_{U^q(X)}=1} B(u, v) = \sup_{a \text{ is a } q-\text{atom}} B(a, v). \tag{7}$$

The same statements are true if we replace U^p by U^p_{strict} and V^q by V^q_w.

See [3] for a proof. The previous results show that $U^p \subset V^p_{rc}$, and both spaces are very close. They are, however, not equal. The following example goes back to Young [22] with the same intention, but in a slightly different context. Let with a smooth function ϕ

$$v_p^N = \phi \sum_{j=1}^{N} 2^{-j/p} \sin(2^j x).$$

It is not hard to see that $\sup_N \|v_p^N\|_{C^{1/p}} < \infty$, and hence in $v_p^\infty \in C^{1/p} \subset V^p_{rc}$. Let $u_q = \phi \sum_{j=1}^\infty 2^{-j/q} \cos(2^j x)$. Now, assuming that $u_q \in U^q$,

$$\|u_q\|_{U^p} \|v_p^N\|_{V^q} \geq \left| \int (u_q^\infty)' v_p^N dx \right| = N/2 \int \phi^2 dx + O(1)$$

which is unbounded, hence a contradiction and $V^p_{rc} \ni u_p^\infty \notin U^p$.

Lemma 4. *For all $v \in V^p$ we have (recall Definition 2)*

$$\|v_\tau\|_{V^p(I)} \leq \|v\|_{V^p(I)} \tag{8}$$

and for all $u \in U^p$

$$\|u_\tau\|_{U^p(I)} \leq \|u\|_{U^p(I)}. \tag{9}$$

For $v \in V^p$ and $\varepsilon > 0$ there is a partition τ so that

$$\|v - v_\tau\|_{V^p} < \varepsilon. \tag{10}$$

Given $u \in U^p$ and $\varepsilon > 0$ there exists τ with

$$\|u - u_\tau\|_{U^p} < \varepsilon. \tag{11}$$

In particular $\mathscr{S}$ is dense in V^p and $\mathscr{S}_{rc}$ is dense in U^p.

Proof. When we take the supremum over partitions for v_τ we may restrict to subsets of τ and the first statement becomes obvious. For U^p it suffices to check p atoms a,

$$\|a_\tau\|_{U^p} \leq 1.$$

Density of step functions in U^p follows from the atomic definition of the space: Let $u \in U^p$ and $\varepsilon > 0$. By definition there exists a finite sum of atoms (which is a right continuous step function u_{step}) such that

$$\|u - u_{step}\|_{U^p} < \varepsilon/2.$$

Let τ be the step function associated to u_{step}. Then

$$\begin{aligned} \|u - u_\tau\|_{U^p} &\leq \|u_{step} - u_\tau\|_{U^p} + \|u - u_{step}\|_{U^p} \\ &< \|(u_{step} - u)_\tau\|_{U^p} + \varepsilon/2 < \varepsilon \end{aligned}$$

which is the claim for U^p. Let $\tilde{V}^p$ be the closure of the step functions in V^p. Suppose there exists $v \in V^p$ with distance 1 to $\tilde{V}^p$, and $\|v\|_{V^p} < 1 + \varepsilon$. Such a function exists when $\tilde{V}^p$ is not V^p. Let $D \subset U^q$ be the subset such $B(u,v) = 0$ whenever $u \in D$ and $v \in \tilde{V}^p$. There exists $u \in D$ with $B(u,v) = 1$, and a partition τ so that $\|u - u_\tau\|_{U^p} < \varepsilon$. However

$$0 = B(u, v_\tau) = B(u_\tau, v) = B(u,v) + B(u_\tau - u, v) \geq 1 - \varepsilon(1+\varepsilon)$$

which is a contradiction. Hence the step functions are dense in V^p. We complete the proof as for U^p. □

2.3 Embeddings

The first part of the next result it due to Hardy and Littlewood [4], and the second one follows by duality.

Lemma 5. *If* $1 < p < \infty$,

$$c_p^{-1} \|v\|_{\dot{B}^{1/p,p}_\infty} \leq \|v\|_{\tilde{V}^p} \leq 2^{1/p} \|u\|_{U^p} \leq c_p \|u\|_{\dot{B}^{1/p,p}_1}.$$

Let $\tilde{V}^p \subset V^p$ be the closed subspace of functions with

$$f(t) = \frac{1}{2} (\lim_{h \to 0} (f(t+h) + f(t-h))).$$

Choose a symmetric function $\phi \in L^1$ with $\int \phi = 1$ and $\phi_h(x) = h^{-1}\phi(x/h)$. The following claims can be easily verified for step functions, which suffices since they are dense.

Lemma 6. *Let* $a = -\infty$, $b = \infty$, $\phi \in L^1$ *symmetric with* $\int \phi dx = 1$. *We denote* $\phi_h(x) = h^{-1}\phi(x/h)$. *Then*

$$\phi_h * v \to v$$

*in the weak * topology for* $v \in \tilde{V}^p(\mathbb{R})$. *Moreover test functions are weak* dense in* V^p.

There is a second duality statement.

Lemma 7. *The bilinear map* B *defines a surjective isometry*

$$\tilde{V}^p(X^*) \to (U^q \cap C(X))^*, \frac{1}{p} + \frac{1}{q} = 1, 1 < p, q < \infty.$$

Proof. The kernel of the duality map restricted to $U^p \cap C(X)$ consists exactly of those elements of V^p which are nonzero at most at countably many points. Let $v \in \tilde V^p$. Then, by the previous lemma,

$$\phi_h * v \to v$$

in the weak $*$ topology of V^p. Moreover, for atoms

$$B(a, \phi_h * v) = B(\phi_h * a, v)$$

and hence this is true for functions in U^p. Now

$$\|v\|_{V^p} = \sup_{a\ q\text{-atom}} B(a,v) = \sup_a \lim_{h\to 0} B(a, \phi_h * v) = \sup \lim_{h\to 0} B(\phi_h * a, v) = \sup_a B(v,a).$$

It remains to prove surjectivity. Let $L : U^p \cap C(X) \to \mathbb{R}$ be linear. By the theorem of Hahn-Banach there is a extension with the same norm to U^p, and by duality there is $v \in V^q$ with $\|v\|_{V^q} = \|L\|$. Changing v at a countable set does not change the image in $(U^p \cap C(X))^*$, hence we may choose $v \in \tilde V^p$. □

We define

$$V^q_C = \{v \in V^q \cap C : \lim_{t\to a} v(t) = \lim_{t \to b} v(t) = 0\}. \tag{12}$$

Lemma 8. *The map*

$$U^p(X^*) \to (V^q_C(X))^*,$$

$$u \to (v \to B(u,v))$$

is a surjective isometry.

Proof. By the duality estimates the duality map is defined, and it is an isometry since the space V^q_C is weak star dense in V^q.

Let $L : V^q_C \to \mathbb{R}$. By Hahn-Banach there is an extension $\tilde L$ to V^q. We define (with obvious modifications for Banach space valued maps)

$$\tilde u(t) = -\tilde L(\chi_{[t,\infty)}).$$

As above we see that $(v \to B(\tilde u, v))$ coincides with $\tilde L$ on step functions. We define u as the unique right continuous function obtained by modifying $\tilde u$ at points of discontinuity. This does not change $B(u,.)$ on V^q_C. Moreover, by the definition of the quadratic form we may assume

$$\tilde u(t) \to 0 \qquad \text{as } t \to a.$$

Now $u \in U^{\tilde p}$ for all $p \geq 0$. The duality estimate allows to conclude that

$$\|u\|_{U^p} \leq \|L\|.$$

There is an immediate consequence. □

Lemma 9. *Test functions C_0^∞ are weak* dense in U^p.*

3 Dispersive Equations

3.1 Adapted Function Spaces

Here we briefly survey constructions going back to Bourgain, which have become standard. Details can be found in [11] and [3].

The following situation will be of particular interest. Let $t \to S(t)$ be a continuous unitary group on a Hilbert space H. We define U_S^p and V_S^p by

$$\|u\|_{U_S^p} = \|S(-t)u(t)\|_{U^p(H)}.$$

Now atoms are piecewise solutions. By Stone's theorem unitary groups are in one-one correspondence with selfadjoint operators, in the sense that

$$i\partial_t u = AU$$

with a self adjoint operator defines a unitary group $S(t)$ and vice versa. At least formally

$$i\partial_t(S(-t)u(t)) = S(-t)(i\partial_t u - Au)$$

and hence the duality assertion is

$$\|u\|_{U_S^q} = \sup_{\|v\|_{V_S^p}\leq 1} B(S(-t)u(t), S(-t)v(t)).$$

Now suppose that – again formally –

$$i\partial_t u - Au = f$$

then, by Duhamels formula, we obtain the solution

$$u(t) = \int_{-\infty}^t S(t-s)f(s)ds.$$

Thus,

$$\|u\|_{U^q_S} = \sup_{\|v\|_{V^p_S}\le 1} |B(S(-t)u(t), S(-t)v(t))| \tag{13}$$

$$= \sup_{\|v\|_{V^p_S}\le 1} \left| \int_{\mathbb{R}} \langle \partial_t S(-t)u(t), S(-t)v(t)\rangle dt \right| \tag{14}$$

$$= \sup_{\|v\|_{V^p_S}\le 1} |-i\,\langle S(-t)(i\,\partial_t u - Au), S(-t)v\rangle dt| \tag{15}$$

$$= \sup_{\|v\|_{V^p_S}\le 1} \int_{\mathbb{R}} \langle f, v\rangle dt \tag{16}$$

with a similar statement for V^p_S. This observation will be crucial for nonlinear dispersive equations. It is not hard to justify using our knowledge about weak* dense subspaces.

We want to use this construction for dispersive equations. There A is often defined by a Fourier multiplier, most often even by a partial differential operator with constant coefficients.

In order to be specific we consider the Airy equation – the situation would be similar for many other dispersive equations –

$$v_t + v_{xxx} = 0 \qquad \text{in } [0,\infty)$$

$$v(0) = u_0 \qquad \text{on } \mathbb{R}.$$

Let $v(t) = 0$ for $t < 0$ and the solution otherwise. Then

$$\|v\|_{V^1_{Airy}} = \|u_0\|_{L^2(\mathbb{R}^d)}.$$

There are three types of basic estimates: The *Strichartz estimate*

$$\|v\|_{L^p_t L^q_x} \le \||D|^{-1/p} u_0\|_{L^2} \tag{17}$$

whenever

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \qquad 2 \le p, q. \tag{18}$$

Here $|D|^s$ is defined by the Fourier multiplier $|\xi|^s$. The Strichartz estimate quantifies the effect of dispersion.

The Strichartz estimate immediately transfers to estimates with respect to U^p_{Airy}:

$$\|v\|_{L^p_t L^q_x} \le c \| |D|^{-1/p} u\|_{U^p_{Airy}}. \tag{19}$$

It suffices to verify this if $S_{Airy}(-t)v$ is an atom with partition $(t_1, t_2 \dots t_n)$. Then, with $t_{n+1} = \infty$, by the Strichartz estimate we can estimate the mixed norm

$$\|v\|_{L^p_t((t_j,t_{j+1});L^q_x(\mathbb{R}))} \le c \| |D|^{-1/p} v(t_j)\|_{L^2(\mathbb{R})}.$$

We raise this to the pth power, and add over j. Then

$$\|v\|_{L^pL^q} \le c \left(\sum \| |D|^{-1/p} v(t_j)\|^p_{L^2} \right)^{1/p} \le c$$

since $S_{Airy}(-t)u$ is a p atom.

Consider now $v(t) = \int_{-\infty}^t S_{Airy}(t-s) f(s) ds$. By duality (13), and with p and q satisfying (18)

$$\begin{aligned} \|v\|_{V^{p'}_{S_{Airy}}} &= \sup_{\|u\|_{U^p_{Airy}} \le 1} \left| B(S_{Airy}(-t)u, S_{Airy}(-t)v) \right| \\ &= \sup_{\|u\|_{U^p_{Airy}} \le 1} \left| \int u \bar{f} \, dx \, dt \right| \\ &\le \sup_{\|u\|_{U^p_{Airy}} \le 1} \|u\|_{L^pL^q} \|f\|_{L^{p'}L^{q'}} \\ &\le c \|f\|_{L^{p'}L^{q'}}. \end{aligned}$$

This implies the dual estimate of (17). If $p > 2$ we may combine the estimates with an embedding to obtain the full Strichartz estimate.

Waves with different velocity interact at most in a time interval which is the inverse of the differences of the velocities. Bilinear estimates quantify this fact. For the Airy equation the group velocity is $-3\xi^2$. We define the Fourier projection u_λ by

$$\hat{u}_\lambda = \chi_{1 \le |\xi|/\lambda \le 2}(\xi) \hat{u}$$

where $\hat{u}$ denotes the Fourier transform with respect to x. For solutions to the Airy equation we obtain the estimate

$$\|u_\lambda u_\mu\|_{L^2(\mathbb{R}^2)} \le c \mu^{-1} \|u_\lambda(0)\|_{L^2(R)} \|u_\mu(0)\|_{L^2(\mathbb{R})} \tag{20}$$

provided $\lambda \le \mu/4$. Again this implies for functions in U^2

$$\|u_\lambda v_\mu\|_{L^2} \le c\mu^{-1} \|u_\lambda\|_{U^2_{Airy}} \|u_\mu\|_{U^2_{Airy}}. \tag{21}$$

The imbedding estimate (5) immediately implies the high modulation estimate

$$\|u^\Lambda\|_{L^2(\mathbb{R}^2)} \le c\Lambda^{-1/2} \|u\|_{V^2_{Airy}} \tag{22}$$

where u^Λ is defined by the space-time Fourier multiplier $\chi_{|\tau-\xi^3|>\Lambda}(\tau,\xi)$.
This set of tools is complemented by the interpolation estimate (2).

3.2 The Generalized KdV Equation

For integers $p \ge 1$ we consider the initial value problems

$$u_t + u_{xxx} + (u^p u)_x = 0 \tag{23}$$

$$u(0) = u_0 \tag{24}$$

– the case $p = 1$ is the Korteweg-de-Vries equation, and $p = 2$ the modified Korteweg-de-Vries equation, and

$$u_t + u_{xxx} + (|u|^p u)_x = 0 \tag{25}$$

$$u(0) = u_0 \tag{26}$$

for positive real p.

Both equations have soliton solutions. They are invariant with respect to scaling: $\lambda^{2/p} u(\lambda x, \lambda^3 t)$ is a solution if u satisfies the equation. The mass $\int u^2 dx$ and energy $\int \frac{1}{2}u_x^2 - \frac{1}{p+2}u^{p+2}$ are conserved. The energy however is not bounded from below. The space $\dot H^{\frac{1}{2}-\frac{2}{p}}$ (with norm $\|v_0\|_{\dot H^s} = \||\xi|^s \hat v_0\|_{L^2}$) is invariant with respect to this scaling and it is not hard to see that the generalized *KdV* equation is globally wellposed in H^1 if $p < 4$. For $p \ge 4$ one expects blow-up. This has been proven in series of seminal papers by Martel, Merle and Martel, Merle and Raphael for $p = 4$, see [15–17] and the references therein.

The most prominent equation here is the KdV equation

$$u_t + u_{xxx} + (u^2)_x = 0.$$

The tools described here allow an alternative argument to prove local wellposedness in $H^{-3/4}(\mathbb{R})$. The order of derivatives cannot be improved by contraction arguments. There are however apriori estimates in H^{-1} by different techniques, [1]. For the modified KdV equation

$$u_t + u_{xxx} + (u^3)_x = 0$$

one obtains local wellposedness in $H^{1/4}$, which again is optimal in terms of the number of derivatives.

For the quartic KdV equation

$$u_t + u_{xxx} + (u^4)_x = 0$$

one obtains global existence by a contraction argument in the space

$$\|u\|_X = \sup_\lambda \lambda^{-1/6} \|u_\lambda\|_{U^2_{Airy}}$$

for initial data in a Besov space $\dot{B}^{-1/6}_{2,\infty}(\mathbb{R})$. Statement and proof are contained in [10], where it was one step to prove stability of the soliton in $\dot{B}_\infty^{-1/6,2}$, and scattering. This is probably the first stability result of solitons for gKdV which is not based on Weinstein's convexity argument in the energy space.

Wellposedness in a slightly smaller spaces has been proven by Grünrock [2] and Tao [20] based on a modification of the Fourier restriction spaces of Bourgain at the critical level.

The quintic KdV equation

$$u_t + u_{xxx} + u^5_x = 0$$

is of particular interest since it is L^2 critical. Since the work by Kenig, Ponce and Vega it is known to be locally wellposed in L^2. The local existence result has been extended to all equations

$$u_t + u_{xxx} + |u|^p u_x = 0$$

with $p \geq 4$ in [19] in critical function spaces using the techniques above. The case of polynomial (analytic) nonlinearities had been dealt with by Molinet and Ribaud [18] using different techniques.

3.3 The Kadomtsev-Petviashvili II Equation

The Kadomtsev-Petviashvili-II (KP-II) equation

$$\partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) + \partial_y^2 u = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^2 \tag{27}$$

$$u(0,x,y) = u_0(x,y) \qquad (x,y)\in\mathbb{R}^2 \tag{28}$$

has been introduced by Kadomtsev and Petviashvili [9] to describe weakly transverse water waves in the long wave regime with small surface tension. It generalizes the Korteweg – de Vries equation, which is spatially one dimensional and thus neglects transversal effects. The KP-II equation has a remarkably rich structure.

Here we describe a setup leading to global wellposedness and scattering for small data. The Hilbert space will be denoted by $\dot{H}^{-1/2,0}$ which is defined by through the norm

$$\|u_0\|_{\dot{H}^{-1/2}} = \||\xi|^{-1/2}\hat{u}_0\|_{L^2}$$

where ξ is the Fourier multiplier with respect to x. The Fourier multiplier $|\xi|^{-1/2}$ defines an isomorphism from L^2 to $\dot{H}^{-1/2}$.

For $\lambda > 0$ we define the Fourier projection to the $1 \leq |\xi|/\lambda < 2$ by

$$\hat{u}_\lambda = \chi_{\lambda \leq |\xi| \leq 2\lambda}\hat{u}$$

where $\mathscr{F}$ denotes the Fourier transform and ξ the Fourier variable of x. Usually we choose $\lambda \in 2^{\mathbb{Z}}$, the set of integer powers of 2. We define X by

$$\|u\|_X = \left(\sum_{\lambda \in 2^{\mathbf{Z}}} (\lambda^{-1/6}\|u_\lambda\|_{V^2_{KP}})^2\right)^{1/2}.$$

The following theorem has been proven in [3] with a proof relying on the space U^2 and V^2.

Theorem 2. *There exists $\delta > 0$ such that for all u_0 with $\|u_0\|_{\dot{H}^{-1/2}}$ there exists a unique solution*

$$u \in X \subset C([0,T]; \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2))$$

of the KP-II equation (27) on $(0,\infty)$. Moreover, the flow map

$$B_{\delta,R}(0) \ni u_0 \mapsto u \in X$$

is analytic.

A duality argument reduces the proof to an estimate of a trilinear integral. The functions there are expanded according to the Fourier projection, and the key estimate is a bound for

$$\left|\int u_{\lambda_1} v_{\lambda_2} w_{\lambda_3} dx\, dy\, dt\right|$$

Due to symmetry we may assume that $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Since there is only a contribution to the integral if there a point in the support of the Fourier transforms adding up to zero we can only get a contribution if $\lambda_2 \sim \lambda_3$, and only λ_1 may be smaller. Since

$$\begin{aligned}\tau_1 - \xi_1^3 + \eta_1^2/\xi_1 + \tau_2 - \xi_2^3 + \eta_2^2/\xi_2 \\ -[(\tau_1+\tau_2) - (\xi_1+\xi_2)^3 + (\eta_1+\eta_2)^2/(\xi_1+\xi_2)] \\ = -3\xi_1\xi_2(\xi_1+\xi_2) - \frac{(\xi_1\eta_2 - \xi_2\eta_1)^2}{\xi_1\xi_2(\xi_1+\xi_2)}\end{aligned}$$

there is only a contribution if at least for one $j \in \{1,2,3\}$

$$|\tau_j - \xi_j^3 + \eta_j^2/\xi_j| \geq |\xi_1\xi_2(\xi_1+\xi_2)|,$$

$j = 1,2,3$ and $\tau_3 = -\tau_1 - \tau_2$, $\xi_3 = -\xi_1 - \xi_2$, $\eta_3 = -\eta_1 - \eta_2$ in the support of the Fourier transforms. We set $\Lambda = \lambda_1\lambda_2\lambda_3/10$ and expand $u_{\lambda_1} = u_{\lambda_1}^{\Lambda} + u_{\lambda_1}^{low}$, where u^{Λ} is defined by the space-time Fourier multiplier

$$\chi_{|\tau-\xi^3+\eta^2/\xi|\geq\Lambda},$$

and similarly we decompose the other factors. We expand the trilinear integral. There is only a nontrivial contribution if at least one of the terms $u_{\lambda_1}^{\Lambda}$, $v_{\lambda_2}^{\Lambda}$ or $w_{\lambda_3}^{\Lambda}$ occurs. We apply Cauchy-Schwartz and estimate the corresponding term in L^2. For the other product we apply an L^4 space-time Strichartz estimate, or a bilinear estimate.

We obtain

$$\left|\int u_{\lambda_1} v_{\lambda_2} w_{\lambda_3}^{\Lambda}\, dx\, dy\, dt\right| \leq c(\lambda_1\lambda_2\lambda_3)^{-1/2}(\lambda_{min}/\lambda_{max})^{\frac{1}{4}} \|u_{\lambda_1}\|_{V^2_{KP}} \|v_{\lambda_2}\|_{V^2_{KP}} \|w_{\lambda_3}\|_{V^2_{KP}}$$

and

$$\left|\sum_{\lambda_1\leq\lambda_2} \int u_{\lambda_1} v_{\lambda_2} w_{\lambda_3}^{\Lambda}\, dx\, dy\, dt\right| \leq c\lambda_{max}^{-1}\|u\|_X \|v_{\lambda_2}\|_{V^2} \|w_{\lambda_2}\|_{V^2}$$

which suffices to conclude the proof.

3.4 The Energy Critical Nonlinear Schrrödinger Equation on Compact Manifolds

We consider the quintic nonlinear Schrödinger equation on the three dimensional torus $\mathbb{T}^3$, either focusing or defocusing

$$i\partial_t u + \Delta u = \pm|u|^4 u. \qquad (29)$$

On $\mathbb{R}^3$ the space $\dot{H}^1$ is critical. We consider solutions on a unit time interval with small initial data in H^1. We define the function space X by

$$\|u\|_X = \left\| (1 + k_1^2 + k_2^2 + k_3^3)^{1/2} \| e^{-it\pi^2(k_1^2+k_2^2+k_3^3)} \hat{u} \|_{V^2(0,1)} \right\|_{l^2_{\mathbf{k}}(\mathbb{Z}^3)}.$$

The following depends on a mix of previous arguments, and estimates for Gaussian sums.

Theorem 3 ([6]). *There exists $\delta > 0$ such that given $u_0 \in H^1$ with $\|u_0\|_{H^1} < \delta$ there exists a unique solution $u \in X$. This solution can be extended to a global solution in time. The map from initial data to solution is real analytic. If $u_0 \in H^1$ there is a local in time solution.*

This result has been extended to global wellposedness on T^3 for large data in H^1 by Ionescu and Pausader [8].

A similar mix of adapted function spaces, eigenfunction estimates and bounds on Gaussian sums has been applied by the same authors to energy critical partial periodic domains in $\mathbb{R}^4$ [7], and by Herr to the quintic Schrödinger equation on the three dimensional sphere [5].

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