Chapter 8
Pricing Using Affine Diffusions

The aim of this chapter is to illustrate how to price derivatives using affine diffusions in the classical risk-neutral setting and under the benchmark approach. In the classical risk-neutral setting, the affine transform plays a crucial role in the pricing of derivatives. In particular, there are essentially two ways in which this transform has been employed:

- the affine transform can be used to determine the law of the vector of random variables under consideration, if necessary numerically;
- the affine transform can be employed together with the Fourier transform.

In this chapter, we first show how to use the affine transform to determine the law of a vector of random variables. Later, we combine this with the Fourier transform. We present the theory, mainly relying on Filipović and Mayerhofer (2009). Subsequently, we illustrate the theory by using two one-dimensional examples.

Under the benchmark approach, we can work under the real world probability measure, using the Craddock-Lennox-Platen approach from Sect. 7.3.1, or benchmarked Laplace transforms, or we can employ the forward measure from Sect. 7.3.3. In Sect. 8.5, we illustrate the usage of benchmarked Laplace transforms, and in Sect. 8.6, we work under the forward measure.

8.1 Theoretical Background

As in Chap. 7, we work on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ and use $\boldsymbol{X}$ to denote an affine process that assumes values in the canonical state space $\mathcal{X} = (\Re^+)^m \times \Re^n$. The dynamics of $\boldsymbol{X}$ are given by

$$d\boldsymbol{X}_t = \boldsymbol{b}(\boldsymbol{X}_t)\,dt + \boldsymbol{\rho}(\boldsymbol{X}_t)\,d\boldsymbol{W}_t, \tag{8.1.1}$$

where $\boldsymbol{X}_0 = \boldsymbol{x}$ and

$$\boldsymbol{\rho}(\boldsymbol{x})\boldsymbol{\rho}(\boldsymbol{x})^\top = \boldsymbol{a}(\boldsymbol{x}).$$

Affine processes are frequently used in the context of short rate models, and we restate Assumption 7.2.1, also to recall the notation used therein.

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Assumption 8.1.1 *The process* $r = \{r_t, t \geq 0\}$ *is an affine transform of* $\boldsymbol{X} = \{\boldsymbol{X}_t, t \geq 0\}$,

$$r_t = c + \boldsymbol{\gamma}^\top \boldsymbol{X}_t,$$

where $\boldsymbol{X}$ *is an affine process on the canonical state space* $(\Re^+)^m \times \Re^n$ *given by Eq.* (8.1.1) *with admissible parameters* $\boldsymbol{a}, \boldsymbol{\alpha}_i, \boldsymbol{b}$, *and* $\boldsymbol{\beta}_i$, *where* $i \in \{1, \ldots, d\}$, *given in Eq.* (7.1.7), *and* $c \in \Re$, $\boldsymbol{\gamma} \in \Re^d$.

We are interested in computing conditional expectations of the form

$$\pi(t) = E\left(\exp\left\{-\int_t^T r_s\, ds\right\} f(\boldsymbol{X}_T) \,\middle|\, \mathcal{A}_t\right) \tag{8.1.2}$$

and hence impose the integrability condition

$$E\left(\exp\left\{-\int_0^T r_s\, ds\right\} \left|f(\boldsymbol{X}_T)\right|\right) < \infty$$

for the remainder of this chapter. In Eq. (8.1.2), the expectation is taken with respect to the measure P. This refers either to the case when P denotes some assumed equivalent risk-neutral probability measure or the case when P denotes the real world probability measure. In the remainder of the section, we discuss how to compute such discounted Laplace transforms. We recall Theorem 7.2.2, where we assume that the expectation is taken with respect to the measure P, irrespective of whether this refers to an assumed risk-neutral measure or the real world probability measure. We point out that if P corresponds to an assumed risk neutral probability measure and if f is simply the constant one, then the computation of (8.1.2) yields the price at time t of a zero coupon bond maturing at time T. For the remainder of the section, we assume that the conditions of Theorem 7.2.2 are satisfied. We have the following result, see Corollary 4.2 in Filipović and Mayerhofer (2009).

Theorem 8.1.2 *Let* $\tau > 0$ *and assume that the conditions of Theorem* 7.2.2 *are satisfied. Then for any maturity* $T \leq \tau$, *the* T*-zero coupon bond price at* $t \leq T$ *is given as*

$$E\left(\exp\left\{-\int_t^T r_s\, ds\right\} \,\middle|\, \mathcal{A}_t\right) = \exp\left\{-A(T-t) - \boldsymbol{B}(T-t)^\top \boldsymbol{X}_t\right\} \tag{8.1.3}$$

where we denote

$$A(t) = -\Phi(t, \mathbf{0}), \qquad \boldsymbol{B}(t) = -\boldsymbol{\Psi}(t, \mathbf{0}).$$

Moreover, for $t \leq T \leq S \leq \tau$, *the* $\mathcal{A}_t$*-conditional characteristic function of* $\boldsymbol{X}_T$ *is given by*

$$E\left(\exp\left\{-\int_t^S r_s\, ds + \boldsymbol{u}^\top \boldsymbol{X}_T\right\} \,\middle|\, \mathcal{A}_t\right)$$
$$= e^{-A(S-T)+\Phi(T-t,\boldsymbol{u}-\boldsymbol{B}(S-T))+\boldsymbol{\Psi}(T-t,\boldsymbol{u}-\boldsymbol{B}(S-T))^\top \boldsymbol{X}_t} \tag{8.1.4}$$

for all $\boldsymbol{u} \in \mathcal{S}(U + B(S-T))$, *where* U *is the neighborhood of* $\mathbf{0}$ *in* $\Re^d$ *from Theorem* 7.2.2.

We remark that if P corresponds to the risk-neutral probability measure, then equality (8.1.4) gives the law of $\boldsymbol{X}_T$ under a forward measure P^S, defined via the Radon-Nikodym derivative

$$\Lambda_F = \frac{dP^S}{dP} = \frac{1}{E((S_S^0)^{-1})} \frac{1}{S_S^0},$$

where $S_t^0 = \exp\{\int_0^t r_s\, ds\}$. From Bayes' Theorem, see Sect. 15.8,

$$E_{P^S}\big(\exp\{\boldsymbol{u}^\top \boldsymbol{X}_T\} \mid \mathcal{A}_t\big) = \frac{E(\exp\{-\int_t^S r_s\, ds + \boldsymbol{u}^\top \boldsymbol{X}_T\} \mid \mathcal{A}_t)}{E(\exp\{-\int_t^S r_s\, ds\} \mid \mathcal{A}_t)}. \tag{8.1.5}$$

The expression $E(\exp\{-\int_t^S r_s\, ds\} \mid \mathcal{A}_t)$ was computed in (8.1.3) and

$$E\left(\exp\left\{-\int_t^S r_s\, ds + \boldsymbol{u}^\top \boldsymbol{X}_T\right\} \middle| \mathcal{A}_t\right)$$

in Eq. (8.1.4). One can now recognize the law of $\boldsymbol{X}_T$ under P_S, or compute it numerically. Finally, we point out that computations using forward measures under the benchmark approach will be performed in Sect. 8.6.

We now illustrate how to apply Theorem 8.1.2. Clearly, this requires the solution of the system of Riccati equations (7.2.1). In some cases, such as the Vasiček and the CIR model, explicit solutions can be found, and we now show how to obtain them.

8.2 One-Dimensional Examples

In this section, we discuss two one-dimensional examples, which feature prominently in the finance literature.

8.2.1 Vasiček Model

The state space of the Vasiček model, see Vasiček (1977), is $\Re$, and we set $r_t = X_t$, so that we consider the one-dimensional affine process

$$dr_t = (b + \beta r_t)\, dt + \sigma\, dW_t, \tag{8.2.6}$$

where $\sigma \geq 0$, $b, \beta \in \Re$. Given this parametrization, the system of Riccati equations (7.2.1) now reads

$$\begin{aligned}
\partial_t \Phi(t,u) &= \frac{1}{2}\Psi^2(t,u)\sigma^2 + b\Psi(t,u), \\
\Phi(0,u) &= 0, \\
\partial_t \Psi(t,u) &= \beta\Psi(t,u) - 1, \\
\Psi(0,u) &= u.
\end{aligned} \tag{8.2.7}$$

This system is easily solved, in particular, we obtain

$$\Psi(t,u) = \exp\{\beta t\}u - \frac{\exp\{\beta t\} - 1}{\beta}$$

and

$$\begin{aligned}\Phi(t,u) = \frac{1}{2}\sigma^2\bigg[&\frac{u^2}{2\beta}\big(\exp\{2\beta t\} - 1\big) + \frac{1}{2\beta^3}\big(\exp\{2\beta t\} - 4\exp\{\beta t\} + 3 + 2\beta t\big)\\ &- \frac{u}{\beta^2}\big(\exp\{2\beta t\} - 2\exp\{\beta t\} + 1\big)\bigg]\\ &+ b\bigg[\frac{u}{\beta}\big(\exp\{\beta t\} - 1\big) - \frac{\exp\{\beta t\} - 1 - t\beta}{\beta^2}\bigg],\end{aligned}$$

which holds for all $u \in \mathbf{C}$, and hence (8.1.4) holds for all $u \in \mathbf{C}$. This allows us, via Theorem 8.1.2, to compute

$$E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\Big|\,\mathcal{A}_t\right) = \exp\big\{-A(T-t) - B(T-t)r_t\big\},$$

where

$$\begin{aligned}A(t) &= -\Phi(t,0)\\ &= -\frac{b}{\beta^2}\big(1 - \exp\{\beta t\} + \beta t\big) - \frac{\sigma^2}{4\beta^3}\big(3 - 4\exp\{\beta t\} + \exp\{2\beta t\} + 2\beta t\big),\end{aligned}$$

and

$$B(t) = -\Psi(t,0) = \frac{\exp\{\beta t\} - 1}{\beta}.$$

Furthermore, we have, by invoking Eq. (8.1.5),

$$\begin{aligned}&E_{P^S}\big(\exp\{ur_T\}\,\big|\,\mathcal{A}_t\big)\\ &= \exp\bigg\{u\bigg(\exp\big\{\beta(T-t)\big\}r_t - \frac{\sigma^2}{2\beta^2}\big(2 - \exp\big\{\beta(S-T)\big\} + \exp\big\{\beta(S+T-2t)\big\}\\ &\quad - 2\exp\big\{\beta(T-t)\big\}\big) - \frac{b}{\beta}\big(1 - \exp\big\{\beta(T-t)\big\}\big)\\ &\quad + \frac{\sigma^2 u}{4\beta}\big(\exp\big\{2\beta(T-t)\big\} - 1\big)\bigg)\bigg\}.\end{aligned}$$

This means, we identify the distribution of r_T under P^S conditional on $\mathcal{A}_t$ as Gaussian with mean

$$\begin{aligned}\exp\big\{\beta(T-t)\big\}r_t &- \frac{\sigma^2}{2\beta^2}\big(2 - \exp\big\{\beta(S-T)\big\}\\ &+ \exp\big\{\beta(S+T-2t)\big\} - 2\exp\big\{\beta(T-t)\big\}\big)\\ &- \frac{b}{\beta}\big(1 - \exp\big\{\beta(T-t)\big\}\big)\end{aligned}$$

and variance

$$\frac{\sigma^2\beta}{2}\big(\exp\{2\beta(T-t)\}-1\big). \tag{8.2.8}$$

For the special case $S = T$, i.e. $P^T = P^S$, this distribution reduces to the well-known law of a Gaussian random variable with mean

$$\exp\{\beta(T-t)\}r_t - \frac{\sigma^2}{2\beta^2}\big(\exp\{2\beta(T-t)\} - \exp\{\beta(T-t)\}\big)$$
$$- \left(\frac{b}{\beta} + \frac{\sigma^2}{2\beta^2}\right)\big(1 - \exp\{\beta(T-t)\}\big)$$

and variance (8.2.8). These results are in line with well-known results on pricing under the Vasiček model, see e.g. Mamon (2004).

8.2.2 CIR Model

We now discuss the CIR model, see Cox et al. (1985), following the presentation in Filipović and Mayerhofer (2009). In this case, the state space is $\Re^+$. We set $r_t = X_t$, and deal with the following model for the short rate

$$dr_t = (b + \beta r_t)\,dt + \sigma\sqrt{r_t}\,dW_t, \tag{8.2.9}$$

where $b, \sigma > 0$ and $\beta < 0$. The system of Ricatti equations (7.2.1) now reads

$$\begin{aligned}
\partial_t \Phi(t,u) &= b\Psi(t,u),\\
\Phi(0,u) &= 0,\\
\partial_t \Psi(t,u) &= \frac{1}{2}\sigma^2\Psi^2(t,u) + \beta\Psi(t,u) - 1,\\
\Psi(0,u) &= u.
\end{aligned} \tag{8.2.10}$$

To solve system (8.2.10), we use the following lemma, which appeared as Lemma 5.2 in Filipović and Mayerhofer (2009).

Lemma 8.2.1 *Consider the Riccati differential equation*

$$\partial_t G = AG^2 + BG - C, \quad G(0,u) = u, \tag{8.2.11}$$

where $A, B, C \in \mathbf{C}$ *and* $u \in \mathbf{C}$, *with* $A \neq 0$ *and* $B^2 + 4AC \in \mathbf{C}\setminus\Re^-$. *Let* $\sqrt{\cdot}$ *denote the analytic extension of the real square root to* $\mathbf{C}\setminus\Re^-$, *and define* $\lambda = \sqrt{B^2 + 4AC}$.

- *The function*

$$G(t,u) = -\frac{2C(\exp\{\lambda t\}-1) - (\lambda(\exp\{\lambda t\}+1) + B(\exp\{\lambda t\}-1))u}{\lambda(\exp\{\lambda t\}+1) - B(\exp\{\lambda t\}-1) - 2A(\exp\{\lambda t\}-1)u}$$

  is the unique solution of (8.2.11) *on its maximal interval of existence* $[0, t_+(u))$.

Moreover,

$$\int_0^t G(s,u)\,ds$$
$$= \frac{1}{A}\log\left(\frac{2\lambda\exp\{\frac{\lambda-B}{2}t\}}{\lambda(\exp\{\lambda t\}+1) - B(\exp\{\lambda t\}-1) - 2A(\exp\{\lambda t\}-1)u}\right). \tag{8.2.12}$$

- *If, in addition,* $A > 0$, $B \in \Re$, $\Re(C) \geq 0$ *and* $u \in \mathbf{C}^-$, *then* $t_+(u) = \infty$ *and* $G(t,u)$ *is* $\mathbf{C}^-$*-valued.*

Invoking Lemma 8.2.1, we conclude that $A = \frac{1}{2}\sigma^2$, $B = \beta$, $C = 1$, $\lambda = \sqrt{\beta^2 + 2\sigma^2}$ and

$$\Psi(t,u) = -\frac{2(\exp\{\lambda t\}-1) - (\lambda(\exp\{\lambda t\}+1) + \beta(\exp\{\lambda t\}-1))u}{\lambda(\exp\{\lambda t\}+1) - \beta(\exp\{\lambda t\}-1) - \sigma^2(\exp\{\lambda t\}-1)u}$$
$$= -\frac{L_1(t) - L_2(t)u}{L_3(t) - L_4(t)u},$$

where

$$L_1(t) = 2\big(\exp\{\lambda t\}-1\big)$$
$$L_2(t) = \lambda\big(\exp\{\lambda t\}+1\big) + \beta\big(\exp\{\lambda t\}-1\big)$$
$$L_3(t) = \lambda\big(\exp\{\lambda t\}+1\big) - \beta\big(\exp\{\lambda t\}-1\big)$$
$$L_4(t) = \sigma^2\big(\exp\{\lambda t\}-1\big)$$

and

$$\Phi(t,u) = \frac{2b}{\sigma^2}\log\left(\frac{2\lambda\exp\{\frac{\lambda-\beta}{2}t\}}{\lambda(\exp\{\lambda t\}+1) - \beta(\exp\{\lambda t\}-1) - \sigma^2(\exp\{\lambda t\}-1)u}\right)$$
$$= \frac{2b}{\sigma^2}\log\left(\frac{L_5(t)}{L_3(t) - L_4(t)u}\right),$$

i.e. we set

$$L_5(t) = 2\lambda\exp\left\{\frac{\lambda-\beta}{2}t\right\},$$

where $(\Phi(\cdot,u), \Psi(\cdot,u)) : \Re^+ \to \mathbf{C}^- \times \mathbf{C}^-$ and (8.1.4) holds for all $u \in \mathbf{C}^-$ and $t \leq T$. As an application of the above result, we can obtain from Theorem 8.1.2

$$E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right) = \exp\big\{-A(T-t) - B(T-t)r_t\big\},$$

where

$$A(t) = -\Phi(t,0) = \frac{2b}{\sigma^2}\log\left(\frac{L_3(t)}{L_5(t)}\right)$$

and

$$B(t) = -\Psi(t,0) = \frac{L_1(t)}{L_3(t)},$$

which is the same result as the one derived in Sect. 5.5 using Lie symmetry methods.

We can also compute the law of r_T under P^S, conditional on $\mathcal{A}_t$. Applying Eq. (8.1.5), this gives

$$\begin{aligned}
&E_{P^S}\big(\exp\{ur_T\}\,\big|\,\mathcal{A}_t\big)\\
&\quad=\frac{\exp\{-A(S-T)+\Phi(T-t,u-B(S-T))+\Psi(T-t,u-B(S-T))r_t\}}{P_S(t)}\\
&\quad=\exp\big\{-A(S-T)+\Phi\big(T-t,u-B(S-T)\big)+\Psi\big(T-t,u-B(S-T)\big)r_t\\
&\qquad+A(S-t)+B(S-t)r_t\big\}\\
&\quad=\left(\frac{L_5(S-T)L_5(T-t)L_3(S-t)}{L_3(S-T)(L_3(T-t)-L_4(T-t)(u-B(S-T)))L_5(S-t)}\right)^{\frac{2b}{\sigma^2}}\\
&\qquad\times\exp\left\{r_t\left(\frac{L_1(S-t)}{L_3(S-t)}-\frac{L_1(T-t)-L_2(T-t)(u-B(S-T))}{L_3(T-t)-L_4(T-t)(u-B(S-T))}\right)\right\}.
\end{aligned}$$

It can be confirmed that

$$\frac{L_5(S-T)L_5(T-t)L_3(S-t)}{L_3(S-T)(L_3(T-t)-L_4(T-t)(u-B(S-T)))L_5(S-t)}=\frac{1}{1-C_1(t,T,S)u}$$

and also that

$$\begin{aligned}
&\frac{L_1(S-t)}{L_3(S-t)}-\frac{L_1(T-t)-L_2(T-t)(u-B(S-T))}{L_3(T-t)-L_4(T-t)(u-B(S-T))}\\
&\quad=-C_2(t,T,S)+\frac{C_2(t,T,S)}{1-C_1(t,T,S)u},
\end{aligned}$$

where

$$\begin{aligned}
C_1(t,T,S)&=\frac{L_3(S-T)L_4(T-t)}{2\lambda L_3(S-t)}\quad\text{and}\\
C_2(t,T,S)&=\frac{L_2(T-t)}{L_4(T-t)}-\frac{L_1(S-t)}{L_3(S-t)}.
\end{aligned}$$

To identify the distribution of r_T under P^S conditional on $\mathcal{A}_t$, we recall the following well-known result, which in this form appeared as Lemma 5.1 in Filipović and Mayerhofer (2009), see also Sects. 3.1 and 13.1.

Lemma 8.2.2 *The non-central χ^2-distribution with $\delta>0$ degrees of freedom and non-centrality parameter $\lambda>0$ has the density function*

$$p(x,\delta,\lambda)=\frac{1}{2}\exp\left\{-\frac{x+\lambda}{2}\right\}\left(\frac{x}{\lambda}\right)^{\frac{\delta}{4}-\frac{1}{2}}I_{\frac{\delta}{2}-\frac{1}{2}}(\sqrt{\lambda x}),\quad x\geq 0$$

and characteristic function

$$\int_{\Re^+}\exp\{ux\}p(x,\delta,\lambda)\,dx=\frac{\exp\{\frac{\lambda u}{1-2u}\}}{(1-2u)^{\frac{\delta}{2}}},\quad u\in\mathbf{C}^-.$$

Here $I_\nu(x)=\sum_{j\geq 0}\frac{1}{j!\Gamma(j+\nu+1)}(\frac{x}{2})^{2j+\nu}$ denotes the modified Bessel function of the first kind of order $\nu>-1$, see e.g. Abramowitz and Stegun (1972), *Sect.* 9.6.

Using Lemma 8.2.2, we conclude that under P^S, the random variable $\frac{2r_T}{C_1(t,T,S)}$, conditional on $\mathcal{A}_t$, follows a non-central χ^2-distribution with $\frac{4b}{\sigma^2}$ degrees of freedom and non-centrality parameter $2C_2(t, T, S)r_t$. These results are consistent with well-known pricing formulas under the CIR model.

8.3 Fourier Transform Approach

We recall that the methodology in the previous section relied on using the characteristic function to identify the law of $\boldsymbol{X}_T$, either by inspection or numerical inversion. The approach presented in the current section also uses the characteristic function, but in a different manner. We follow the approach presented in Filipović (2009), where the following economic interpretation was presented.

We start with the economic interpretation and later present the approach in a rigorous fashion. Its applications to some examples will conclude the section. Essentially, we express the payoff function $f(\boldsymbol{x})$ as follows

$$f(\boldsymbol{x}) = \int_{\Re^q} \exp\{(\boldsymbol{v} + \imath \boldsymbol{L}\boldsymbol{\lambda})^\top \boldsymbol{x}\} \tilde{f}(\boldsymbol{\lambda})\, d\boldsymbol{\lambda}, \quad d\boldsymbol{x}\text{-a.s.},$$

where $\tilde{f}(\boldsymbol{\lambda})$ denotes an integrable function. Economically, this means that we set up a static hedge using claims with complex payoffs $\exp\{(\boldsymbol{v} + \imath \boldsymbol{L}\boldsymbol{\lambda})^\top \boldsymbol{x}\}$, each weighted by $\tilde{f}(\boldsymbol{\lambda})$. The linearity of pricing rules ensures that the price of the claim with payoff $f(\boldsymbol{x})$ is given by the weighted average of the prices of the claims with payoffs $\exp\{(\boldsymbol{v} + \imath \boldsymbol{L}\boldsymbol{\lambda})^\top \boldsymbol{x}\}$, each weighted by $\tilde{f}(\boldsymbol{\lambda})$. The following theorem, which appeared as Theorem 10.5 in Filipović (2009), makes this argument rigorous.

Theorem 8.3.1 *Suppose either condition* (i) *or* (ii) *of Theorem* 7.2.2 *is met for some* $\tau \geq T$, *and let* $\mathcal{D}_{\Re}(T)$ *denote the maximal domain for the system of Riccati equations* (7.2.1). *Assume that* f *satisfies*

$$f(\boldsymbol{x}) = \int_{\Re^q} \exp\{(\boldsymbol{v} + \imath \boldsymbol{L}\boldsymbol{\lambda})^\top \boldsymbol{x}\} \tilde{f}(\boldsymbol{\lambda})\, d\boldsymbol{\lambda}, \quad d\boldsymbol{x}\text{-a.s.}, \tag{8.3.13}$$

for some $\boldsymbol{v} \in \mathcal{D}_{\Re}(T)$ *and* $d \times q$ *matrix* $\boldsymbol{L}$, *and some integrable function* $\tilde{f} : \Re^q \to \mathbf{C}$, *for some positive integer* $q \leq d$. *Then the price* (8.1.2) *is well defined and given by the formula*

$$\pi(t) = \int_{\Re^q} \exp\{\Phi(T - t, \boldsymbol{v} + \imath \boldsymbol{L}\boldsymbol{\lambda}) + \boldsymbol{\Psi}(T - t, \boldsymbol{v} + \imath \boldsymbol{L}\boldsymbol{\lambda})^\top \boldsymbol{X}_t\} \tilde{f}(\boldsymbol{\lambda})\, d\boldsymbol{\lambda}. \tag{8.3.14}$$

If f is continuous in $\boldsymbol{x}$, then (8.3.13) holds for all $\boldsymbol{x}$, which follows since the right-hand side of (8.3.14) is continuous in $\boldsymbol{x}$, by the Riemann-Lebesgue theorem.

Of course, the applicability of Theorem 8.3.1 depends on how easy it is to come up with a representation of the form (8.3.13). Following Filipović (2009), we can find some examples useful for finance. We refer also to Sect. 8.4 for a more constructive approach.

8.3.1 Examples of Fourier Decompositions

Following Filipović (2009) and Hurst and Zhou (2010), we discuss European call and put options, exchange options, and spread options. For the proofs of the following results, we refer the reader to Filipović (2009).

Lemma 8.3.2 *Let $K > 0$. For any $y \in \Re$ the following identities hold*:

$$\frac{1}{2\pi}\int_{\Re} \exp\{(w+\imath\lambda)y\}\frac{K^{-(w-1+\imath\lambda)}}{(w+\imath\lambda)(w-1+\imath\lambda)}\,d\lambda = \begin{cases} (K-e^y)^+ & \textit{if } w < 0 \\ (e^y-K)^+ - e^y & \textit{if } 0 < w < 1, \\ (e^y-K)^+ & \textit{if } w > 1. \end{cases}$$

Clearly, the case $0 < w < 1$ also equals $(K-e^y)^+ - K$.

By setting $K = e^z$ in Lemma 8.3.2, we obtain the payoff of an exchange option.

Corollary 8.3.3 *For any $y, z \in \Re$ the following identities hold*:

$$\frac{1}{2\pi}\int_{\Re}\frac{\exp\{(w+\imath\lambda)y-(w-1+\imath\lambda)z\}}{(w+\imath\lambda)(w-1+\imath\lambda)}\,d\lambda = \begin{cases} (e^y-e^z)^+ & \textit{if } w > 1, \\ (e^y-e^z)^+ - e^y & \textit{if } 0 < w < 1. \end{cases}$$

Lastly, we discuss the payoff of a spread-option.

Lemma 8.3.4 *Let $\boldsymbol{w} = (w_1, w_2)^\top \in \Re^2$ be such that $w_2 < 0$ and $w_1 + w_2 > 1$. Then for any $\boldsymbol{y} = (y_1, y_2)^\top \in \Re^2$ the following identity holds*:

$$\big(e^{y_1} - e^{y_2} - 1\big)^+(2\pi)^2 = \int_{\Re^2}\exp\big\{(\boldsymbol{w}+\imath\boldsymbol{\lambda})^\top\boldsymbol{y}\big\} \times \frac{\Gamma(w_1+w_2-1+\imath(\lambda_1+\lambda_2))\Gamma(-w_2-\imath\lambda_2)}{\Gamma(w_1+1+\imath\lambda_1)}\,d\lambda_1\,d\lambda_2,$$

where the gamma function $\Gamma(z) = \int_0^\infty t^{-1+z}e^{-t}dt$ is defined for all complex z with $\Re(z) > 0$.

8.4 A Special Class of Payoff Functions

Following Filipović (2009), we point out that for a special class of payoff functions, we can apply both approaches, the one from Sect. 8.3 and the one from Sect. 8.1. For particular payoff functions, we can compute $\tilde{f}$, as needed for the Fourier transform approach from Sect. 8.3, but one can also compute the relevant densities. The following theorem is Theorem 10.6 in Filipović (2009).

Theorem 8.4.1 *Suppose either condition* (i) *or* (ii) *of Theorem* 7.2.2 *is met for some* $\tau \geq T$, *and let* $\mathcal{D}_{\Re}$ *denote the maximal domain for the system of Riccati equations* (7.2.1). *Assume that* f *is of the form*

$$f(\boldsymbol{x}) = e^{\boldsymbol{v}^\top \boldsymbol{x}} h\big(\boldsymbol{L}^\top \boldsymbol{x}\big)$$

for some $\boldsymbol{v} \in \mathcal{D}_{\Re}(T)$ *and* $d \times q$*-matrix* $\boldsymbol{L}$, *and some integrable function* $h : \Re^q \to \Re$, *for a positive integer* $q \leq d$. *Define the bounded function*

$$\tilde{f}(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^q} \int_{\Re^q} e^{-\imath \boldsymbol{\lambda}^\top \boldsymbol{y}} h(\boldsymbol{y})\, d\boldsymbol{y}, \quad \boldsymbol{\lambda} \in \Re^q.$$

- *If* $\tilde{f}$ *is an integrable function in* $\boldsymbol{\lambda} \in \Re^q$, *then the assumptions of Theorem* 8.3.1 *are met.*
- *If* $\boldsymbol{v} = \boldsymbol{L}\boldsymbol{w}$, *for some* $\boldsymbol{w} \in \Re^q$, *and* $e^{\Phi(T-t,\boldsymbol{v}+\imath \boldsymbol{L}\boldsymbol{\lambda})+\boldsymbol{\Psi}(T-t,\boldsymbol{v}+\imath \boldsymbol{L}\boldsymbol{\lambda})^\top \boldsymbol{X}_t}$ *is an integrable function in* $\boldsymbol{\lambda} \in \Re^q$, *then the* $\mathcal{A}_t$*-conditional distribution of the* $\Re^q$*-valued random variable* $\boldsymbol{Y} = \boldsymbol{L}^\top \boldsymbol{X}_T$ *under the* T*-forward measure* P^T *admits the continuous density function*

$$q(t,T,\boldsymbol{y}) = \frac{1}{(2\pi)^q} \int_{\Re^q} e^{-(\boldsymbol{w}+\imath\boldsymbol{\lambda})^\top \boldsymbol{y}} \frac{e^{\Phi(T-t,\boldsymbol{v}+\imath \boldsymbol{L}\boldsymbol{\lambda})+\boldsymbol{\Psi}(T-t,\boldsymbol{v}+\imath \boldsymbol{L}\boldsymbol{\lambda})^\top \boldsymbol{X}_t}}{P_T(t)}\, d\boldsymbol{\lambda}.$$

In either case, the integral in (8.3.14) *is well-defined and the pricing formula* (8.3.14) *holds.*

8.5 Pricing Using Benchmarked Laplace Transforms

In this section, we discuss pricing under the benchmark approach using benchmarked Laplace transforms. We have two applications:

- a standard European put option;
- realized variance derivatives.

8.5.1 Put Options Under the Stylized MMM

In this subsection, we motivate how benchmarked Laplace transforms naturally arise when pricing options. For simplicity, we place ourselves in the stylized MMM, see Sect. 3.3, which we now briefly recall, as it is used in this and the next subsection, and Sect. 8.6. The filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, where the filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\geq 0}$ is assumed to satisfy the usual conditions, carries one source of uncertainty, a standard Brownian motion $W = \{W_t, t \geq 0\}$. As in Sect. 3.3, we assume a constant short rate and model the savings account using the differential equation

$$dS_t^0 = r S_t^0\, dt,$$

for $t \geq 0$ with $S_0^0 = 1$. We recall that the GOP is modeled using the SDE

$$S_t^{\delta_*} = S_t^0 \, \bar{S}_t^{\delta_*} = S_t^0 \, Y_t \, \alpha_t^{\delta_*}, \tag{8.5.15}$$

where $Y_t = \frac{\bar{S}_t^{\delta_*}}{\alpha_t^{\delta_*}}$ is a square-root process of dimension four, satisfying the SDE

$$dY_t = (1 - \eta \, Y_t) \, dt + \sqrt{Y_t} \, dW_t, \tag{8.5.16}$$

for $t \geq 0$ with initial value $Y_0 > 0$ and net growth rate $\eta > 0$. As before, $\alpha_t^{\delta_*}$ is a deterministic function of time, given by

$$\alpha_t^{\delta_*} = \alpha_0 \exp\{\eta t\},$$

with scaling parameter $\alpha_0 > 0$. The following lemma shows how benchmarked Laplace transforms arise when pricing options.

Lemma 8.5.1 *Let g denote a positive $\mathcal{A}_T$-measurable random variable, and define*

$$h(K) := E\left(\frac{(K-g)^+}{Y_T}\right).$$

We have for $\lambda > 0$,

$$\int_0^\infty \exp\{-\lambda K\} h(K) \, dK = \frac{1}{\lambda^2} E\left(\frac{\exp\{-\lambda g\}}{Y_T}\right).$$

Proof By the Fubini theorem it follows

$$\begin{aligned}\int_0^\infty \exp\{-\lambda K\} h(K) \, dK &= \int_0^\infty \exp\{-\lambda K\} E\left(\frac{(K-g)^+}{Y_T}\right) dK \\ &= E\left(\int_0^\infty \exp\{-\lambda K\} \frac{(K-g)^+}{Y_T} dK\right).\end{aligned}$$

We obtain

$$\begin{aligned}&\int_0^\infty \exp\{-\lambda K\} \frac{(K-g)^+}{Y_T} \, dK \\ &= \int_g^\infty \exp\{-\lambda K\} \frac{(K-g)}{Y_T} \, dK \\ &= \frac{1}{Y_T} \int_g^\infty \exp\{-\lambda K\} K \, dK - \frac{g}{Y_T} \int_g^\infty \exp\{-\lambda K\} \, dK \\ &= \frac{1}{Y_T} \left(\frac{g \exp\{-\lambda g\}}{\lambda} + \frac{\exp\{-\lambda g\}}{\lambda^2}\right) - \frac{g}{Y_T} \frac{\exp\{-\lambda g\}}{\lambda} \\ &= \frac{1}{\lambda^2} \frac{\exp\{-\lambda g\}}{Y_T},\end{aligned}$$

and the result follows. □

Now, for a put option with strike K and maturity date T, we compute

$$p_{T,K}(0) = S_0^{\delta_*} E\left(\frac{(K - S_T^{\delta_*})^+}{S_T^{\delta_*}}\right) = S_0^{\delta_*} E\left(\frac{(\tilde{K} - Y_T)^+}{Y_T}\right),$$

where $\tilde{K} = \frac{K}{S_T^0 \alpha_T^{\delta_*}}$. We are interested in the Laplace transform with respect to the modified strike $\tilde{K}$, and obtain, for

$$h(\tilde{K}) = E\left(\frac{(\tilde{K} - Y_T)^+}{Y_T}\right)$$

the following equality

$$\int_0^\infty \exp\{-\lambda \tilde{K}\} h(\tilde{K})\, d\tilde{K} = \frac{1}{\lambda^2} E\left(\frac{\exp\{-\lambda Y_T\}}{Y_T}\right).$$

We recall from Sect. 3.1, that $Y_t \exp\{\eta t\}/c(t) \sim \chi_4^2(\alpha)$, where $\alpha = \frac{Y_0}{c(t)}$, $c(t) = \frac{\exp\{\eta t\}-1}{4\eta}$, and $\chi_\nu^2(\lambda)$ denotes a non-central χ^2-distributed random variable with ν degrees of freedom and non-centrality parameter λ. Consequently,

$$\begin{aligned} E\left(\frac{\exp\{-\mu Y_T\}}{Y_T}\right) &= E\left(\frac{\exp\{-\tilde{\mu}\chi_4^2(\alpha)\}}{\chi_4^2(\alpha)}\right)\frac{\exp\{\eta T\}}{c(T)} \\ &= \frac{\exp\{-\alpha/2\}(\exp\{\frac{\alpha}{4\tilde{\mu}+2}\} - 1)}{\alpha}\frac{\exp\{\eta T\}}{c(T)}, \end{aligned} \tag{8.5.17}$$

where $\tilde{\mu} = \mu \frac{c(T)}{\exp\{\eta T\}}$. Equality (8.5.17) is easily verified using the probability density function of $\chi_4^2(\alpha)$. This illustrates how benchmarked Laplace transforms arise naturally in the context of option pricing. Finally, we remark that using the techniques from Sect. 13.5, options can now be priced.

8.5.2 Derivatives on Realized Variance Under the Stylized MMM

We remind the reader that in Sect. 3.3, we had already derived the price of a put option under the stylized MMM, without using Laplace transforms. However, we now discuss an example, where the availability of benchmarked Laplace transforms is crucial. In particular, we discuss the pricing of *derivatives on realized variance* of an index. We point out that derivatives on the realized variance of an index, such as the VIX, and options on the VIX, as traded on the Chicago Board Options Exchange, have become important risk management tools.

In this subsection, we show how to price call and put options on realized variance, variance swaps, and volatility swaps. The formulas derived in this subsection are in the spirit of the pricing formulas presented in Sect. 3.3. However, the results needed to price derivatives on realized variance, rely on the benchmarked Laplace transform, see Proposition 7.3.8. Hence we discuss realized variance derivatives in this subsection. Furthermore, we remark that the results presented here have appeared in Baldeaux et al. (2011a), Chan and Platen (2011), and Lennox (2011).

We place ourselves in the stylized MMM and model realized variance as the quadratic variation of the logarithm of the index,

$$\big[\ln\big(S^{\delta_*}\big)\big]_T,$$

which admits the following representation.

Lemma 8.5.2 *The realized variance of the index is given by the integral*

$$\big[\ln\big(S^{\delta_*}\big)\big]_T = \int_0^T \frac{dt}{Y_t}. \tag{8.5.18}$$

Proof Clearly,

$$\big[\ln\big(S^{\delta_*}\big)\big]_T = \big[\ln(Y)\big]_T,$$

as $S^0_\cdot$ and $\alpha^{\delta_*}_\cdot$ are deterministic functions of time. Now one has by the Itô formula

$$\begin{aligned} d\ln(Y_t) &= \frac{dY_t}{Y_t} - \frac{1}{2}\frac{d[Y]_t}{Y_t^2} \\ &= \frac{1}{Y_t}(1-\eta Y_t)\,dt + \frac{dW_t}{\sqrt{Y_t}} - \frac{1}{2}\frac{dt}{Y_t} \\ &= \frac{1}{Y_t}\left(\frac{1}{2} - \eta Y_t\right)dt + \frac{dW_t}{\sqrt{Y_t}}, \end{aligned}$$

which completes the proof. □

We now study call and put options on realized variance. In particular, we present Laplace transforms of prices of options on realized variance and show how benchmarked Laplace transforms naturally arise in this context. We will focus on put options, as prices of call options can be recovered from the following put-call parity.

Lemma 8.5.3 *The following put-call parity relation holds for payoffs of options on realized variance*

$$\begin{aligned} &E\left(\frac{(\frac{1}{T}\int_0^T \frac{dt}{Y_t} - K)^+}{S_T^{\delta_*}}\right) \\ &\quad = E\left(\frac{\frac{1}{T}\int_0^T \frac{dt}{Y_t} - K}{S_T^{\delta_*}}\right) + E\left(\frac{(K - \frac{1}{T}\int_0^T \frac{dt}{Y_t})^+}{S_T^{\delta_*}}\right). \end{aligned}$$

Note that the put-call parity involves the fair zero coupon bond and not the savings bond even when the short rate is constant.

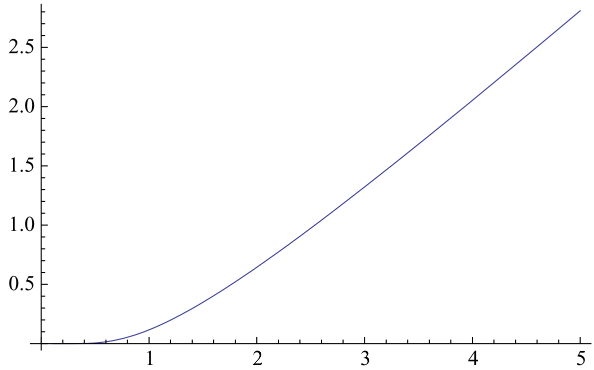

Fig. 8.5.1 Prices of put options on realized variance versus strike prices

We address the problem of pricing put options on realized variance, which by Lemma 8.5.3 also covers the case of call options. For notational convenience, we focus on the case $t = 0$, and are, therefore, interested in computing the expectation

$$h(K) := E\left(\frac{(K - \frac{1}{T}\int_0^T \frac{dt}{Y_t})^+}{Y_T}\right). \tag{8.5.19}$$

Inspired by Carr et al. (2005), we first compute the Laplace transform of $h(K)$ with respect to the strike K, which we obtain from Lemma 8.5.1. Setting $g = \frac{1}{T}\int_0^T \frac{dt}{Y_t}$, we compute

$$\int_0^\infty \exp\{-\lambda K\} h(K)\, dK = \frac{1}{\lambda^2} E\left(\frac{\exp\{-\frac{\lambda}{T}\int_0^T \frac{dt}{Y_t}\}}{Y_T}\right). \tag{8.5.20}$$

The quantity

$$E\left(\frac{\exp\{-\frac{\lambda}{T}\int_0^T \frac{dt}{Y_t}\}}{Y_T}\right)$$

is easily computed using Proposition 7.3.8. We can hence price put options on realized variance by inverting the Laplace transform given in Eq. (8.5.20) and invoking Proposition 7.3.8. To demonstrate that this methodology works reliably, Fig. 8.5.1 displays put option prices for different strikes, that have been confirmed to the shown accuracy via numerical methods to be introduced in Sects. 12.2 and 13.5, where we choose

$$Y_0 = 1, \qquad T = 1, \qquad \eta = 0.052, \qquad r = 0.05.$$

In Sect. 13.5, we will discuss how to invert Laplace transforms, and also present examples relevant to the pricing of realized variance derivatives.

We remark that the approach presented in this subsection cannot immediately be extended to the pricing of call and put options on volatility. This is due to the fact that the approach presented in this subsection requires the computation of the expectation

$$E\left(\frac{\exp\{-\lambda\sqrt{\int_0^T \frac{dt}{Y_t}}\}}{Y_T}\right). \tag{8.5.21}$$

However, there seems to exist no explicit formula for (8.5.21). This motivates us to apply numerical methods to the problem, which we will develop in Sects. 12.2 and 13.5. In particular, we demonstrate how to recover the joint distribution of $(\int_0^t \frac{ds}{Y_s}, Y_t)$ by inverting the one-dimensional Laplace transform given in Eq. (5.4.16). Subsequently, we can apply quadrature methods to compute prices, see Sect. 12.2.

We now discuss variance and volatility swaps. Again, the benchmarked Laplace transforms are useful in this context. The payoff of a *variance swap* maturing at $T > 0$ is given by

$$\left[\ln\left(S^{\delta_*}\right)\right]_T - K,$$

where K is a fixed swap rate, chosen in such a way that the time $t = 0$ value of the variance swap is zero. Hence from the real world pricing formula (1.3.19), we need to solve the following equation for K,

$$S_0^{\delta_*} E\left(\frac{[\ln(S^{\delta_*})]_T - K}{S_T^{\delta_*}}\right) = 0,$$

which by Eq. (8.5.15) and Lemma 8.5.2 is equivalent to

$$\frac{S_0^{\delta_*}}{\alpha_T^{\delta_*} S_T^0} E\left(\frac{\int_0^T \frac{ds}{Y_s}}{Y_T}\right) - K\, P_T(0) = 0,$$

where $P_T(t)$ denotes the time t price of a zero coupon bond maturing at T. Regarding the computation of

$$E\left(\frac{\int_0^T \frac{ds}{Y_s}}{Y_T}\right),$$

we use the following proposition, see Lennox (2011), Proposition 2.0.41, and also Chan and Platen (2011), Proposition 8.1. We present the result in generality. We consider the square-root process

$$dX_t = (a - bX_t)\, dt + \sqrt{2\sigma X_t}\, dW_t, \tag{8.5.22}$$

where $X_0 = x > 0$, and remark that this proposition follows immediately from the benchmarked Laplace transform given in Proposition 7.3.8.

Proposition 8.5.4 *Let* $X = \{X_t,\ t \geq 0\}$ *be given by* (8.5.22), *let* $\beta(\mu) = 1 + m - \alpha + \frac{\nu(\mu)}{2}$, $m = \frac{1}{2}(\frac{a}{\sigma} - 1)$, *and* $\nu(\mu) = \frac{1}{\sigma}\sqrt{(a-\sigma)^2 + 4\mu\sigma}$, *and assume that* $\frac{2a}{\sigma} \geq 2$.

Then if $m > \alpha - 1$,

$$E\left(\frac{\int_0^t \frac{ds}{X_s}}{X_t^\alpha}\right)$$
$$= -x^{-m}\exp\left\{-\frac{bx}{\sigma(e^{bt}-1)} + bmt\right\}\frac{d}{d\mu}\left(\left(\frac{be^{bt}}{(e^{bt}-1)\sigma}\right)^{-m+\alpha-\frac{\nu(\mu)}{2}}\right.$$
$$\times\left(\frac{b^2 x}{4\sigma^2\sinh^2(\frac{bt}{2})}\right)^{\nu(\mu)/2}\frac{\Gamma(1+m-\alpha+\frac{\nu(\mu)}{2})}{(1+\nu(\mu))}$$
$$\left.\times\, {}_1F_1\left(\beta(\mu), 1+\nu(\mu), \frac{bx}{\sigma(e^{bt}-1)}\right)\right)\bigg|_{\mu=0},$$

where ${}_1F_1$ *denotes the confluent hypergeometric function, see e.g. Chap.* 13 *in Abramowitz and Stegun* (1972).

To price variance swaps, we simply set $a = 1$, $b = \eta$, and $\sigma = \frac{1}{2}$ in (8.5.22) and note that

$$m = \frac{1}{2} > 0 = \alpha - 1,$$

hence the result applies to the stylized MMM.

We now study *volatility swaps*. A volatility swap pays

$$\sqrt{\left[\ln\left(S^{\delta_*}\right)\right]_T} - K,$$

at maturity $T > 0$, where again K is chosen so that the initial value of the volatility swap is zero. Hence we solve the following equation for K:

$$S_0^{\delta_*} E\left(\frac{\sqrt{[\ln(S^{\delta_*})]_T}}{S_T^{\delta_*}}\right) - E\left(\frac{S_0^{\delta_*}}{S_T^{\delta_*}}\right)K = 0,$$

where again $E(\frac{S_0^{\delta_*}}{S_T^{\delta_*}})$ is the time 0 price of a fair zero coupon bond maturing at T. The following representation is useful, and is, for example, also used in Gatheral (2006), Eq. (11.6):

$$\sqrt{x} = \frac{1}{2\pi}\int_0^\infty \frac{1-\exp\{-ux\}}{u^{3/2}}\,du, \quad x \geq 0. \tag{8.5.23}$$

Hence by Eq. (8.5.15) and Lemma 8.5.2,

$$E\left(\frac{\sqrt{[\ln(S^{\delta_*})]_T}}{S_T^{\delta_*}}\right) = \frac{1}{\alpha_T^{\delta_*} S_T^0} E\left(\frac{\sqrt{\int_0^T \frac{ds}{Y_s}}}{Y_T}\right),$$

and

$$E\left(\frac{\sqrt{\int_0^T \frac{ds}{Y_s}}}{Y_T}\right) = \frac{1}{2\pi}\int_0^\infty \frac{E(\frac{1}{Y_T}) - E(\frac{\exp\{-u\int_0^T \frac{ds}{Y_s}\}}{Y_T})}{u^{3/2}}\,du.$$

We recall that $E(\frac{1}{Y_T})$ is easily computed using the transition density of Y, see Eq. (3.1). Finally, we observe that

$$E\left(\frac{\exp\{-u\int_0^T \frac{ds}{Y_s}\}}{Y_T}\right)$$

is again the benchmarked Laplace transform, which was computed in Proposition 7.3.8.

We conclude that when pricing derivatives under the benchmark approach, benchmarked Laplace transforms feature prominently, but are easily computed via Lie symmetry methods, for those tractable models we consider in this book under the benchmark approach.

8.6 Pricing Under the Forward Measure Using the Benchmark Approach

In this section, we illustrate how to combine the results from Sect. 8.3 with the benchmark approach. For simplicity, we begin with the one-dimensional case, but we consequently also discuss a two-dimensional example. Assume that the payoff function f admits the representation

$$f(x) = \int_{\Re} \exp\big\{(w + \imath\lambda)x\big\} \tilde{f}(\lambda)\, d\lambda, \quad dx\text{-a.s.}$$

Also recall from Sect. 8.3.1, that $f(\cdot)$ is typically a function of the log-price $\ln(S_T^{\delta_*})$. Consequently, Proposition 7.3.10 yields the formula

$$\begin{aligned}
&P_T(t) E_{P^T}\big(f\big(S_T^{\delta_*}\big) \mid \mathcal{A}_t\big) \\
&\quad = P_T(t) \int_{\Re} E_{P^T}\big(\exp\big\{(w + \imath\lambda)\ln\big(S_T^{\delta_*}\big)\big\} \mid \mathcal{A}_t\big) \tilde{f}(\lambda)\, d\lambda \\
&\quad = P_T(t) \int_{\Re} E_{P^T}\big(\big(S_T^{\delta_*}\big)^{w+\imath\lambda} \mid \mathcal{A}_t\big) \tilde{f}(\lambda)\, d\lambda.
\end{aligned}$$

We have

$$E_{P^T}\big(\big(S_T^{\delta_*}\big)^{w+\imath\lambda} \mid \mathcal{A}_t\big) = \frac{S_t^{\delta_*}}{P_T(t)} E\big(\big(S_T^{\delta_*}\big)^{w-1+\imath\lambda} \mid \mathcal{A}_t\big).$$

For the stylized MMM, we use Eq. (8.5.15) to compute, for $u \in \mathbf{C}$,

$$\begin{aligned}
E_{P^T}\big(\exp\big\{u\ln\big(S_T^{\delta_*}\big)\big\} \mid \mathcal{A}_t\big) &= \frac{S_t^{\delta_*}}{P_T(t)} E\left(\frac{\exp\{u\ln(S_T^{\delta_*})\}}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) \\
&= \frac{E((S_T^{\delta_*})^{u-1} \mid \mathcal{A}_t)}{E((S_T^{\delta_*})^{-1} \mid \mathcal{A}_t)} \\
&= \big(\alpha_T^{\delta_*} S_T^0\big)^u \frac{E(Y_T^{u-1} \mid \mathcal{A}_t)}{E(Y_T^{-1} \mid \mathcal{A}_t)}.
\end{aligned}$$

Recall that we use $\chi^2_\nu(\lambda)$ to denote a non-central χ^2-distributed random variable with ν degrees of freedom and non-centrality parameter λ. In Sect. 3.1 we established that conditional on $\mathcal{A}_t$, $\frac{Y_T \exp\{\eta(T-t)\}}{c(T-t)}$ follows a non-central χ^2-distribution with 4 degrees of freedom and non-centrality parameter $\beta = \frac{Y_t}{c(T-t)}$, where $c(t) = (\exp\{\eta t\} - 1)/(4\eta)$. We hence obtain

$$E_{P^T}\left(\exp\left\{u \ln\left(S_T^{\delta_*}\right)\right\} \mid \mathcal{A}_t\right)$$
$$= \left(S_T^0\left(\varphi(T) - \varphi(t)\right)\right)^u \frac{E((\chi_4^2(\beta))^{u-1} \mid \mathcal{A}_t)}{E((\chi_4^2(\beta))^{-1} \mid \mathcal{A}_t)}, \tag{8.6.24}$$

where

$$\varphi(t) = \frac{1}{4} \int_0^t \alpha_s^{\delta_*} \, ds.$$

Due to the tractability of the stylized MMM, we can compute explicitly

$$E\left(\left(\chi_4^2(\beta)\right)^{u-1}\right) = 2^{u-1} \Gamma(1+u) \; {}_1F_1\left(-u+1, 2, -\frac{\beta}{2}\right), \tag{8.6.25}$$

for $\Re(u) > -1$, where ${}_1F_1$ denotes the confluent hypergeometric function, see Chap. 13 in Abramowitz and Stegun (1972). For $u = 0$, this evaluates to

$$E\left(\left(\chi_4^2(\beta)\right)^{-1}\right) = \frac{(1 - \exp\{-\beta/2\})}{\beta}. \tag{8.6.26}$$

The forward measure can also be employed in a bivariate context. We consider the GOP denominated in two currencies: S^a denotes the GOP denominated in the domestic currency, and S^b denotes the GOP denominated in the foreign currency. As in Sect. 3.3, we model both discounted GOPs as independent squared Bessel processes of dimension four, i.e. we assume that

$$S_t^k = S_t^{0,k} \alpha_t^k Y_t^k, \quad k \in \{a, b\},$$

where $S_t^{0,k} = \exp\{r_k t\}$ denotes the savings account denominated in currency k,

$$\alpha_t^k = \alpha_0^k \exp\left\{\eta^k t\right\},$$

and

$$dY_t^k = \left(1 - \eta^k Y_t^k\right) dt + \sqrt{Y_t^k} \, dW_t^k,$$

where we assume that $d\langle W^a, W^b \rangle_t = 0$.

We consider an exchange option, i.e. the payoff is given by

$$\left(S_T^a - S_T^b\right)^+.$$

Using the forward measure which employs the zero coupon bond in the domestic currency, the real world pricing formula yields

$$S_t^a E\left(\frac{(S_T^a - S_T^b)^+}{S_T^a} \,\middle|\, \mathcal{A}_t\right) = P_T^a(t) E_{P^T}\left(\left(S_T^a - S_T^b\right)^+ \mid \mathcal{A}_t\right),$$

where $P_T^k(t)$ denotes the time t price of a zero coupon bond in currency $k \in \{a, b\}$, maturing at T. From Corollary 8.3.3, we get

$$\begin{aligned} &P_T^a(t) E_{P^T}\big(\big(S_T^a - S_T^b\big)^+ \mid \mathcal{A}_t\big) \\ &\quad = \frac{P_T^a(t)}{2\pi} \int_{\Re} \frac{E_{P^T}((S_T^a)^{w+\imath\lambda}(S_T^b)^{-(w-1+\imath\lambda)} \mid \mathcal{A}_t)}{(w+\imath\lambda)(w-1+\imath\lambda)}\, d\lambda, \end{aligned}$$

where $w > 1$. From the assumed independence of S^a and S^b,

$$\begin{aligned} &E_{P^T}\big(\big(S_T^a\big)^{w+\imath\lambda}\big(S_T^b\big)^{-(w-1+\imath\lambda)} \mid \mathcal{A}_t\big) \\ &\quad = E_{P^T}\big(\big(S_T^a\big)^{w+\imath\lambda} \mid \mathcal{A}_t\big) E_{P^T}\big(\big(S_T^b\big)^{-(w-1+\imath\lambda)} \mid \mathcal{A}_t\big), \end{aligned}$$

which can be computed as demonstrated above. This leads to the formula

$$\begin{aligned} &S_t^a E\bigg(\frac{(S_T^a - S_T^b)^+}{S_T^a} \bigg| \mathcal{A}_t\bigg) \\ &\quad = \frac{P_T^a(t)}{2\pi} \int_{\Re} \frac{E_{P^T}((S_T^a)^{w+\imath\lambda} \mid \mathcal{A}_t) E_{P^T}((S_T^b)^{-(w-1+\imath\lambda)} \mid \mathcal{A}_t)}{(w+\imath\lambda)(w-1+\imath\lambda)}\, d\lambda, \end{aligned}$$

where $w > 1$. Furthermore, we compute using Eqs. (8.6.24), (8.6.25), and (8.6.26),

$$\begin{aligned} &E_{P^T}\big(\big(S_T^a\big)^{u_a} \mid \mathcal{A}_t\big) \\ &\quad = \frac{\beta_a (S_T^{0,a}(\varphi^a(T) - \varphi^a(t)))^{u_a} 2^{u_a - 1} \Gamma(1+u_a)\ {}_1F_1(-u_a+1, 2, -\frac{\beta_a}{2})}{(1 - \exp\{-\beta_a/2\})}, \end{aligned}$$

where $u_a = w - 1 + \imath\lambda$,

$$\begin{aligned} \varphi^k(t) &= \frac{1}{4}\int_0^t \alpha_s^k\, ds, \quad k \in \{a, b\}, \\ \beta^k &= \frac{Y_t^k}{c^k(T-t)}, \quad k \in \{a, b\}, \\ c^k(t) &= \frac{(\exp\{\eta^k t\} - 1)}{4\eta^k}, \quad k \in \{a, b\}. \end{aligned}$$

We now turn to the computation of $E_{P^T}((S_T^b)^{u_b} \mid \mathcal{A}_t)$. Recall that we used the zero coupon bond in the domestic currency to define the forward measure. It follows that

$$\begin{aligned} E_{P^T}\big(\big(S_T^b\big)^{u_b} \mid \mathcal{A}_t\big) &= \frac{S_t^a}{P_T^a(t)} E\bigg(\frac{(S_T^b)^{u_b}}{(S_T^a)} \bigg| \mathcal{A}_t\bigg) \\ &= \frac{S_t^a}{P_T^a(t)} E\bigg(\frac{1}{S_T^a} \bigg| \mathcal{A}_t\bigg) E\big(\big(S_T^b\big)^{u_b} \mid \mathcal{A}_t\big) \\ &= E\big(\big(S_T^b\big)^{u_b} \mid \mathcal{A}_t\big). \end{aligned}$$

As above, we compute

$$\begin{aligned} &E\big(\big(S_T^b\big)^{u_b} \mid \mathcal{A}_t\big) \\ &\quad = \big(S_T^{0,b}\big(\varphi^b(T) - \varphi^b(t)\big)\big)^{u_b} 2^{u_b} \Gamma(u_b + 2)\ {}_1F_1\bigg(-u_b, 2, -\frac{\beta_b}{2}\bigg). \end{aligned}$$