Chapter 6
Exact and Almost Exact Simulation

The aim of this chapter is to discuss the simulation of tractable models, illustrated in the context of stochastic volatility models. For two popular stochastic volatility models, the Heston model, see Heston (1993), and the $3/2$ model, see Heston (1997) and Lewis (2000), we present exact simulation algorithms, where we use results from Sect. 5.4. These techniques are based on the *inverse transform method*, which we firstly recall. Moving to higher dimensions, it seems more difficult to generalize these techniques except for the trivial dependence structure, the independent case. Consequently, we recall almost exact simulation schemes from Platen and Bruti-Liberati (2010), which are applicable in the multidimensional case. Finally, we point out that in Chap. 11 we will discuss advanced multidimensional stochastic volatility models based on the Wishart process, which have been successfully applied to the modeling of stochastic volatility.

We introduce these simulation methods in an equity context, in particular, we concentrate on modeling stocks and stock indices. However, these methods are also applicable in other areas, for example in interest rate modeling: the *stochastic volatility Brace-Gątarek-Musiela model* introduces stochastic volatility processes in the context of the *LIBOR market model*. The techniques discussed in this chapter are also applicable in such a context, see e.g. Chap. 16 in Brace (2008), in particular, Sect. 16.4, which deals with simulation.

6.1 Sampling by Inverse Transform Methods

Conceptually, we simulate the given models, one- and multidimensional models, using the inverse transform method, which was discussed e.g. in Chap. 2 in Platen and Bruti-Liberati (2010). The forthcoming brief description of the inverse transform method follows this discussion closely.

The well-known inverse transform method can be applied for the generation of a continuous random variable Y with given probability distribution function F_Y. From

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a uniformly distributed random variable $0 < U < 1$, we obtain an F_Y distributed random variable $y(U)$ by realizing that

$$U = F_Y\big(y(U)\big), \tag{6.1.1}$$

so that

$$y(U) = F_Y^{-1}(U). \tag{6.1.2}$$

Here F_Y^{-1} denotes the inverse function of F_Y. More generally, one can still set

$$y(U) = \inf\big\{y\colon\ U \le F_Y(y)\big\} \tag{6.1.3}$$

in the case when F_Y is no longer continuous, where $\inf\{y\colon\ U \le F_Y(y)\}$ denotes the lower limit of the set $\{y\colon\ U \le F_Y(y)\}$. If U is a $U(0, 1)$ uniformly distributed random variable, then the random variable $y(U)$ in (6.1.2) will be F_Y-distributed. The above calculation in (6.1.2) may need to apply a root finding method, for instance, a Newton method, see Press et al. (2002). Obviously, given an explicit transition distribution function for the solution of a one-dimensional SDE we can sample a trajectory directly from this transition law at given time instants. One simply starts with the initial value, generates the first increment and sequentially the subsequent random increments of the simulated trajectory, using the inverse transform method for the respective transition distributions that emerge.

Also in the case of a two-dimensional SDE we can simulate by sampling from the bivariate transition distribution. We first identify the marginal transition distribution function F_{Y_1} of the first component. Then we use the inverse transform method, as above, for the exact simulation of an outcome of the first component of the two-dimensional random variable based on its marginal distribution function. Afterwards, we exploit the conditional transition distribution function $F_{Y_2|Y_1}$ of the second component Y_2, given the simulated first component Y_1, and use again the inverse transform method to simulate also the second component of the considered SDE. This simulation method is exact as long as the root finding procedure involved can be interpreted as being exact. It exploits a well-known basic result on multivariate distribution functions, see for instance Rao (1973).

It is obvious that this simulation technique can be generalized to the exact simulation of increments of solutions of some d-dimensional SDEs. Based on a given d-variate transition distribution function one needs to find the marginal distribution F_{Y_1} and the conditional distributions $F_{Y_2|Y_1}, F_{Y_3|Y_1,Y_2}, \ldots, F_{Y_d|Y_1,Y_2,\ldots,Y_{d-1}}$. Then the inverse transform method can be applied to each conditional transition distribution function one after the other. This also shows that it is sufficient to characterize explicitly in a model just the marginal and conditional transition distribution functions.

Note also that nonparametrically described transition distribution functions are sufficient for application of the inverse transform method. Of course, explicitly known parametric distributions are preferable for a number of practical reasons. They certainly reduce the complexity of the problem itself by splitting it into a sequence of problems. Finally, we recall that explicit transition densities have already been presented in Chaps. 2, 3, and 5.

Regarding the simulation of stochastic volatility models describing the evolution of a stock or index price, we proceed as follows, assuming a price process with SDE

$$dS_t = \mu S_t\, dt + \sqrt{V_t} S_t\, dW_t,$$

where $V = \{V_t,\ t \geq 0\}$ is a square-root process, see Sect. 3.1, if we deal with the Heston model; or a $3/2$ process, see Sect. 3.1, if we deal with the $3/2$ model. For both models, to obtain a realization of S_t, we firstly simulate V_t, subsequently we simulate $\int_0^t V_s\, ds$ conditional on V_t, and lastly S_t, which, conditional on V_t and $\int_0^t V_s\, ds$, follows a conditional Gaussian distribution. As discussed in Chap. 3, the distribution of V_t is known for the square-root and the $3/2$ process, see Sect. 3.1. Regarding the conditional distribution of $\int_0^t V_s\, ds$, we compute the Laplace transform of $\int_0^t V_s\, ds$, conditional on V_t. Subsequently, the probability distribution is easily recovered by an approach due to Feller, see Feller (1971). Having obtained the conditional probability distribution, the inversion method is applicable. To compute the Laplace transform of $\int_0^t V_s ds$ conditional on V_t, we rely on the results from Sect. 5.4, especially the fundamental solutions. We compute the relevant conditional Laplace transforms in Sect. 6.2, and also compute additional conditional Laplace transforms, such as the Hartman-Watson law for squared Bessel processes. Subsequently, in Sects. 6.3 and 6.4, we show how to apply the results from Sect. 6.2 to the Heston and the $3/2$ model.

6.2 Computing Conditional Laplace Transforms

In this section, we discuss how Laplace transforms of the form

$$E\left(\exp\left\{-\frac{b^2}{2}\int_0^t X_s\, ds - \nu \int_0^t \frac{ds}{X_s}\right\} \,\middle|\, X_t\right), \tag{6.2.4}$$

where $X = \{X_t,\ t \geq 0\}$ is a one-dimensional diffusion process to be specified below, can be computed using the results from Sect. 5.4. Such Laplace transforms turn out to play important roles in the design of exact simulation methods for stochastic volatility models, as we will show in Sects. 6.3 and 6.4. We point out when computing conditional Laplace transforms of the above form that Lie symmetry methods turn out to be crucial.

Formally, we consider the computation of the functional

$$u(x,t) = E\left(\exp\left\{-\lambda X_t - \frac{b^2}{2}\int_0^t X_s\, ds - \nu \int_0^t \frac{ds}{X_s}\right\}\right),$$

where $X = \{X_t,\ t \geq 0\}$ is such that its drift f satisfies one of the Ricatti equations (4.4.34), (4.4.35), or (4.4.36), and $X_0 = x$. We identify the corresponding PDE for u and denote the fundamental solution by $p(x, y, t)$.

However,

$$
\begin{aligned}
u(x,t) &= E\left(\exp\left\{-\lambda X_t - \frac{b^2}{2}\int_0^t X_s\,ds - \nu\int_0^t \frac{ds}{X_s}\right\}\right)\\
&= \int_0^\infty \exp\{-\lambda y\} E\left(\exp\left\{-\frac{b^2}{2}\int_0^t X_s\,ds - \nu\int_0^t \frac{ds}{X_s}\right\} \,\Big|\, X_t = y\right)\\
&\quad \times q(x,y,t)\,dy,
\end{aligned}
$$

where $q(x,y,t)$ denotes the transition density of $X = \{X_t,\ t \ge 0\}$. Since $p(x,y,t)$ is a fundamental solution of the associated PDE we immediately have

$$
E\left(\exp\left\{-\frac{b^2}{2}\int_0^t X_s\,ds - \nu\int_0^t \frac{ds}{X_s}\right\} \,\Big|\, X_t = y\right) = \frac{p(t,x,y)}{q(t,x,y)}.
$$

Assuming the fundamental solution $p(x,y,t)$ and the transition density $q(x,y,t)$ are available in closed-form, the simple steps presented above outline a systematic approach to computing conditional Laplace transforms. As an illustration, we compute the Hartman-Watson law for squared Bessel processes, see also Jeanblanc et al. (2009), Proposition 6.5.1.1.

Proposition 6.2.1 *Assume that* $\delta \ge 2$, *and that* $X = \{X_t,\ t \ge 0\}$ *is given by the SDE*

$$
dX_t = \delta\,dt + 2\sqrt{X_t}\,dW_t,
$$

where $X_0 = x > 0$. *Then*

$$
E\left(\exp\left\{-\frac{b^2}{2}\int_0^t \frac{ds}{X_s}\right\} \,\Big|\, X_t = y\right) = \frac{I_{\sqrt{b^2+\nu^2}}(\sqrt{xy}/t)}{I_\nu(\sqrt{xy}/t)},
$$

where $\nu = \delta/2 - 1$.

Proof The proof follows immediately from Proposition 5.4.1, where the fundamental solution of the PDE

$$
u_t = 2x u_{xx} + \delta u_x - \frac{b^2}{2}\frac{u}{x}
$$

is given by

$$
p(x,y,t) = \frac{1}{2t}\left(\frac{x}{y}\right)^{(1-\delta/2)/2} I_{2d+\frac{\delta}{2}-1}\left(\frac{\sqrt{xy}}{t}\right)\exp\left\{-\frac{(x+y)}{2t}\right\},
$$

where $d = \frac{1}{4}(2 - \delta + \sqrt{(\delta-2)^2 + 4b^2})$ and the transition density of the squared Bessel process is of the form

$$
q(x,y,t) = \frac{1}{2t}\left(\frac{x}{y}\right)^{(1-\delta/2)/2} I_{\delta/2-1}\left(\frac{\sqrt{xy}}{t}\right)\exp\left\{-\frac{(x+y)}{2t}\right\}.
$$

Lastly, note that

$$
2d + \frac{n}{2} - 1 = \sqrt{b^2+\nu^2},
$$

where $\nu = \frac{\delta}{2} - 1$ is the index of the squared Bessel process, which finishes the proof. □

The next result is due to Pitman and Yor (1982). However, we present an alternative proof which employs Lie symmetry methods.

Proposition 6.2.2 *Assume* $\delta \geq 2$, *and that* $X = \{X_t,\, t \geq 0\}$ *is given by*

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dW_t,$$

$X_0 = x > 0$. *Then*

$$\begin{aligned} &E\left(\exp\left\{-\frac{b^2}{2}\int_0^t X_s\, ds\right\} \Bigg|\, X_t = y\right) \\ &\quad = \frac{bt}{\sinh(bt)} \exp\left\{\frac{x+y}{2t}\big(1 - bt\coth(bt)\big)\right\} \frac{I_\nu\left(\frac{b\sqrt{xy}}{\sinh(bt)}\right)}{I_\nu\left(\frac{\sqrt{xy}}{t}\right)}. \end{aligned}$$

Proof The proof follows along the lines of the proof of Proposition 6.2.1. From Proposition 5.4.2, we have that the fundamental solution of the PDE

$$u_t = 2xu_{xx} + \delta u_x - \frac{b^2}{2}\frac{u}{x},$$

is given by

$$p(x, y, t) = \frac{b}{2\sinh(bt)}\left(\frac{y}{x}\right)^{\frac{\delta/2-1}{2}} \exp\left\{-\frac{b(x+y)}{2\tanh(bt)}\right\} I_{(\delta-2)/2}\left(\frac{b\sqrt{xy}}{\sinh(bt)}\right).$$

Recalling the transition density of the squared Bessel process, the result follows. □

Proposition 6.2.2 plays a crucial role in the Broadie-Kaya exact simulation scheme for the Heston model, see Broadie and Kaya (2006). Consequently, the fundamental solutions presented in Chap. 5 can be used for this stochastic volatility model.

Finally, the following result can be used in the design of an exact simulation scheme for the 3/2 model, which is another stochastic volatility model.

Proposition 6.2.3 *Let* $X = \{X_t,\, t \geq 0\}$ *be a squared Bessel process of dimension* δ, *where* $\delta \geq 2$. *Then*

$$\begin{aligned} &E\left(\exp\left\{-\frac{b^2}{2}\int_0^t X_s\, ds - \mu\int_0^t \frac{ds}{X_s}\right\} \Bigg|\, X_t = y\right) \\ &\quad = \frac{bt}{\sinh(bt)} \exp\left\{\frac{(x+y)}{2t}\big(1 - tb\coth(bt)\big)\right\} \frac{I_{\sqrt{\nu^2+2\mu}}\left(\frac{b\sqrt{xy}}{\sinh(bt)}\right)}{I_\nu\left(\frac{\sqrt{xy}}{t}\right)}. \end{aligned}$$

Proof From Proposition 5.4.4, we have that the fundamental solution of

$$u_t = 2xu_{xx} + \delta u_x - \left(\frac{b^2 x}{2} + \frac{\mu}{x}\right)u$$

is

$$p(x, y, t) = \frac{b}{2\sinh(bt)} \exp\left\{-\frac{b(x+y)}{2\tanh(bt)}\right\}\left(\frac{y}{x}\right)^{\frac{\delta-2}{4}} I_{\sqrt{\nu^2+2\mu}}\left(\frac{b\sqrt{xy}}{\sinh(bt)}\right).$$

Recalling the transition density of the squared Bessel process, the result follows. □

We remind the reader that the fundamental solutions obtained via Lie symmetry methods sit at the heart of the computations of the results, not the Laplace transform of the solutions.

These conditional Laplace transforms are now applied to two stochastic volatility models, the Heston and the 3/2 model.

6.3 Exact Simulation of the Heston Model

In this section, we present the approach proposed by Broadie and Kaya (2006) to simulate the stock price under the Heston model exactly. We recall that the dynamics of the stock price and squared volatility under the Heston model satisfy the SDE,

$$dS_t = \mu S_t\,dt + \rho\sqrt{V_t}S_t\,dB_t + \sqrt{1-\rho^2}\sqrt{V_t}S_t\,dW_t, \tag{6.3.5}$$

$$dV_t = \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dB_t, \tag{6.3.6}$$

respectively, where $W = \{W_t,\ t \geq 0\}$ and $B = \{B_t,\ t \geq 0\}$ are independent Brownian motions. Integrating the stock price, we have

$$S_t = S_0 \exp\left\{\mu t - \frac{1}{2}\int_0^t V_s\,ds + \rho\int_0^t \sqrt{V_s}\,dB_s + \sqrt{1-\rho^2}\int_0^t \sqrt{V_s}\,dW_s\right\}.$$

We now integrate the squared volatility or the variance process

$$V_t = V_0 + \kappa\theta t - \kappa\int_0^t V_s\,ds + \sigma\int_0^t \sqrt{V_s}\,dB_s.$$

Hence one obtains

$$\int_0^t \sqrt{V_s}\,dB_s = \frac{V_t - V_0 - \kappa\theta t + \kappa\int_0^t V_s\,ds}{\sigma}. \tag{6.3.7}$$

Consequently, it follows

$$S_t = S_0 \exp\left\{\mu t - \frac{1}{2}\int_0^t V_s\,ds + \frac{\rho}{\sigma}\left(V_t - V_0 - \kappa\theta t + \kappa\int_0^t V_s\,ds\right)\right.$$
$$\left. + \sqrt{1-\rho^2}\int_0^t \sqrt{V_s}\,dW_s\right\}.$$

We now present the exact simulation algorithm, and subsequently explain the individual steps in detail.

Algorithm 6.1 Exact simulation for the Heston model

1: Generate a sample of V_t given V_0
2: Generate a sample of $\int_0^t V_s\,ds$ given V_t
3: Compute $\int_0^t \sqrt{V_s}\,dB_s$ from (6.3.7) given V_t and $\int_0^t V_s\,ds$
4: Generate a sample from S_t, given $\int_0^t \sqrt{V_s}dB_s$ and $\int_0^t V_s\,ds$

6.3.1 Simulating V_t

In Sect. 5.3, the transition density of the square-root process of dimension δ was derived, see also Sect. 3.1, from which we can obtain the following equality in distribution

$$V_t \stackrel{d}{=} \frac{\sigma^2(1-\exp\{-\kappa t\})}{4\kappa}\chi_\delta^2\left(\frac{4\kappa\exp\{-\kappa t\}}{\sigma^2(1-\exp\{-\kappa t\})}\right),$$

where $\delta = \frac{4\theta\kappa}{\sigma^2}$ and $\chi_\delta^2(\lambda)$ denotes a non-central χ^2 random variable with δ degrees of freedom and non-centrality parameter λ. One way of sampling non-central χ^2 random variables, which was also used in Broadie and Kaya (2006), proceeds as follows: from Johnson et al. (1995), it is known that for $\delta > 1$,

$$\chi_\delta^2(\lambda) = \chi_1^2(\lambda) + \chi_{\delta-1}^2,$$

and hence

$$\chi_\delta^2(\lambda) = (Z+\sqrt{\lambda})^2 + \chi_{\delta-1}^2,$$

where Z is a standard normal random variable independent of $\chi_{\delta-1}^2$. Furthermore, for $\delta > 0$, we have the following equality in distribution:

$$\chi_\delta^2(\lambda) \stackrel{d}{=} \chi_{\delta+2N}^2,$$

where N is a Poisson random variable with mean $\frac{\lambda}{2}$. Since a χ^2-distributed random variable is a special case of a gamma random variable, we can use algorithms to sample from the gamma distribution. Lastly, we remark that in Sect. 13.2, we will present an algorithm to compute the cumulative distribution function of a non-central χ^2 random variable with $\delta \geq 0$ degrees of freedom, and hence we can also sample by inverting the cumulative distribution function as discussed in Sect. 6.1.

6.3.2 Simulating $\int_0^t V_s\,ds$ Given V_t

We point out that the challenging step in Algorithm 6.1 is the simulation of the integrated variance, $\int_0^t V_s\,ds$, conditional on the end point of the integral, V_t. This problem is solved by computing the Laplace transform of $\int_0^t V_s\,ds$, conditional on V_t, by combining a probabilistic result with a result from Sect. 5.4.

The method illustrates that results obtained via Lie symmetry analysis can be powerfully combined with results from probability theory, see also Sect. 5.5 for additional examples. Having obtained the Laplace transform, we use it to compute the characteristic function, which in turn can be used to compute the probability distribution function.

In fact, the approach is similar to the approach presented in Sect. 5.5. We introduce a time-change, i.e. we set $\rho_t = V_{\frac{4t}{\sigma^2}}$ to obtain the following SDE for $\rho = \{\rho_t,\ t \geq 0\}$,

$$d\rho_t = (2j\rho_t + \delta)\,dt + 2\sqrt{\rho_t}\,d\tilde{B}_t, \tag{6.3.8}$$

where $\tilde{B} = \{\tilde{B}_t,\ t \geq 0\}$ is a standard Brownian motion, $j = -\frac{2\kappa}{\sigma^2}$, and $\delta = \frac{4\kappa\theta}{\sigma^2}$. We now recall formula (6.d) from Pitman and Yor (1982), which reads

$${}^{j}P^{\delta,t}_{\rho_0\to y} = \frac{\exp\{-\frac{j^2}{2}\int_0^t \rho_s\,ds\}}{P^{\delta,t}_{\rho_0\to y}}\,P^{\delta,t}_{\rho_0\to y}, \tag{6.3.9}$$

using ${}^{j}P^{\delta,t}_{\rho_0\to y}$ to denote the bridge for $\{\rho_s,\ 0 \leq s \leq t\}$ obtained by conditioning ${}^{j}P^{\delta}_{\rho_0}$ on $\rho_t = y$, where ${}^{j}P^{\delta}_{\rho_0}$ denotes the law of $\rho = \{\rho_t,\ t \geq 0\}$ started at ρ_0. Equation (6.3.9) is the analogue of Proposition 3.1.6, but for bridge constructions. We are now in a position to prove the following theorem.

Theorem 6.3.1 *Let $V = \{V_t,\ t \geq 0\}$ be given by Eq. (6.3.6). Then*

$$\begin{aligned}
&E\left(\exp\left\{-a\int_0^t V_s\,ds\right\}\,\bigg|\,V_t\right)\\
&\quad= \frac{\gamma(a)\exp\{-\frac{(\gamma(a)-\kappa)t}{2}\}(1-\exp\{-\kappa t\})}{\kappa(1-\exp\{-\gamma(a)t\})}\\
&\qquad\times \exp\left\{\frac{V_0+V_t}{\sigma^2}\left(\frac{\kappa(1+\exp\{-\kappa t\})}{1-\exp\{-\kappa t\}} - \frac{\gamma(a)(1+\exp\{-\gamma(a)t\})}{1-\exp\{-\gamma(a)t\}}\right)\right\}\\
&\qquad\times \frac{I_{\frac{\delta}{2}-1}\left(\frac{4\gamma(a)\sqrt{V_0V_t}}{\sigma^2}\frac{\exp\{-\frac{\gamma(a)t}{2}\}}{(1-\exp\{-\gamma(a)t\})}\right)}{I_{\frac{\delta}{2}-1}\left(\frac{4\kappa\sqrt{V_0V_t}}{\sigma^2}\frac{\exp\{-\frac{\kappa t}{2}\}}{(1-\exp\{-\kappa t\})}\right)},
\end{aligned}$$

where $\gamma(a) = \sqrt{\kappa^2 + 2\sigma^2 a}$.

Proof The steps of the proof are as follows: in fact, they are similar to the proof of Theorem 5.5.1. We firstly change the volatility coefficient of V_t from σ to 2, using the well-known time-change discussed above. Subsequently, we apply the bridge construction from Eq. (6.3.9), which is analogous to Eq. (5.5.20), to reduce the problem to the computation of conditional Laplace transforms involving a squared Bessel process, which we derived in Sect. 6.2.

As in Sect. 5.5, we set $\rho_t = V_{\frac{4t}{\sigma^2}}$ to obtain the SDE (6.3.8) for $\rho = \{\rho_t,\ t \geq 0\}$.

Recalling Eq. (6.3.9), we compute

$$\begin{aligned}
&E\left(\exp\left\{-a\int_0^t V_s\,ds\right\}\,\middle|\,V_t\right)\\
&\quad= E\left(\exp\left\{-a\int_0^t \rho_{\frac{\sigma^2 s}{4}}\,ds\right\}\,\middle|\,\rho_{\frac{\sigma^2 t}{4}}\right)\\
&\quad= E\left(\exp\left\{-\frac{4a}{\sigma^2}\int_0^{\frac{\sigma^2 t}{4}} \rho_s\,ds\right\}\,\middle|\,\rho_{\frac{\sigma^2 t}{4}}\right)\\
&\quad= \frac{\tilde{E}\big(\exp\{-(\frac{j^2}{2}+\frac{4a}{\sigma^2})\int_0^{\frac{\sigma^2 t}{4}}\rho_s\,ds\}\mid \rho_{\frac{\sigma^2 t}{4}}\big)}{\tilde{E}(\exp\{-\frac{j^2}{2}\int_0^{\frac{\sigma^2 t}{4}}\rho_s\,ds\}\mid \rho_{\frac{\sigma^2 t}{4}})},
\end{aligned}$$

where we use E to denote the expectation with respect to ${}^{j}P^{\delta,t}_{\rho_0\to y}$, and $\tilde{E}$ the expectation with respect to $P^{\delta,t}_{\rho_0\to y}$. Applying Proposition 6.2.2 to both, the numerator and the denominator, and recalling that $\rho_{\frac{\sigma^2 t}{4}} = V_t$ and $j = -\frac{2\kappa}{\sigma^2}$, the result follows. □

As described in Broadie and Kaya (2006), we now obtain the characteristic function $\Phi(b)$ by setting $a = -\imath b$,

$$\Phi(b) = E\left(\exp\left\{\imath b\int_0^t V_s\,ds\right\}\,\middle|\,V_t\right).$$

The probability distribution function can be obtained by Fourier inversion methods, see Feller (1971):

$$\begin{aligned}
P\left(\int_0^t V_s\,ds \le x\,\middle|\,V_t\right) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(ux)}{u}\Phi(u)\,du\\
&= \frac{2}{\pi}\int_0^{\infty}\frac{\sin(ux)}{u}\Re\big(\Phi(u)\big)\,du, \qquad (6.3.10)
\end{aligned}$$

where $\Re(\Phi(u))$ denotes the real part of $\Phi(u)$.

The final integral in Eq. (6.3.10) can be computed numerically and one can then sample by inversion.

6.3.3 Generating S_t

We recall that in Step 3 of Algorithm 6.1, we computed $\int_0^t \sqrt{V_s}\,dB_s$ in terms of V_t and $\int_0^t V_s\,ds$. Due to the independence of $V = \{V_t,\ t \ge 0\}$ and $W = \{W_t,\ t \ge 0\}$, it is clear that $\int_0^t \sqrt{V_s}\,dW_s$ given $\int_0^t V_s\,ds$ follows a normal distribution with mean 0 and variance $\int_0^t V_s\,ds$. Hence $\log(S_t)$ follows a conditionally normal distribution with mean

$$\log(S_0) + \mu t - \frac{1}{2}\int_0^t V_s\,ds + \frac{\rho}{\sigma}\left(V_t - V_0 - \kappa\theta t + \kappa\int_0^t V_s\,ds\right)$$

and random variance

$$\left(1-\rho^2\right)\int_0^t V_s\,ds.$$

In this way, we can obtain samples of S_t satisfying the dynamics (6.3.5).

6.4 Exact Simulation of the 3/2 Model

It is very useful to have exact simulation algorithms for important models. In this section, we closely follow the approach from Baldeaux (2012a) to simulate exactly the stock price or index under the 3/2 model, see e.g. Carr and Sun (2007), Heston (1997), Itkin and Carr (2010), and Lewis (2000). We remark that this approach is similar to the approach from Broadie and Kaya (2006), which we discussed in the previous section.

The dynamics of the stock price under the $3/2$ model are described by the system of SDEs,

$$dS_t = \mu S_t\,dt + \rho\sqrt{V_t}S_t\,dB_t + \sqrt{1-\rho^2}\sqrt{V_t}S_t\,dW_t, \tag{6.4.11}$$

$$dV_t = \kappa V_t(\theta - V_t)\,dt + \sigma(V_t)^{3/2}\,dB_t, \tag{6.4.12}$$

where $B = \{B_t,\ t\ge 0\}$ and $W = \{W_t,\ t\ge 0\}$ are independent Brownian motions. The key observation, as already discussed in Sects. 3.1 and 5.5, is that V_t is the inverse of a square-root process. Defining $X_t = \frac{1}{V_t}$, we obtain

$$dX_t = \left(\kappa + \sigma^2 - \kappa\theta X_t\right)dt - \sigma\sqrt{X_t}\,dB_t. \tag{6.4.13}$$

Expressing the stock price in terms of the process $X = \{X_t,\ t\ge 0\}$, we obtain

$$S_t = S_0\exp\left\{\mu t - \frac{1}{2}\int_0^t (X_s)^{-1}\,ds + \rho\int_0^t(\sqrt{X_s})^{-1}\,dB_s + \sqrt{1-\rho^2}\int_0^t(\sqrt{X_s})^{-1}\,dW_s\right\}. \tag{6.4.14}$$

It is useful to study $\log(X_t)$, for which we obtain the following SDE

$$d\log(X_t) = \left(\frac{\kappa+\frac{\sigma^2}{2}}{X_t} - \kappa\theta\right)dt - \sigma(\sqrt{X_t})^{-1}\,dB_t.$$

Hence

$$\log(X_t) = \log(X_0) + \left(\kappa + \frac{\sigma^2}{2}\right)\int_0^t\frac{ds}{X_s} - \kappa\theta t - \sigma\int_0^t(\sqrt{X_s})^{-1}\,dB_s,$$

or equivalently

$$\int_0^t(\sqrt{X_s})^{-1}\,dB_s = \frac{1}{\sigma}\left(\log\left(\frac{X_0}{X_t}\right) + \left(\kappa+\frac{\sigma^2}{2}\right)\int_0^t\frac{ds}{X_s} - \kappa\theta t\right). \tag{6.4.15}$$

Algorithm 6.2 describes how to simulate the stock price given by (6.4.11) exactly.

We now discuss the individual steps of the algorithm. Clearly, Steps (1), (3), and (4) are very similar to steps (1), (3), and (4) of Algorithm 6.1.

Algorithm 6.2 Exact simulation for the 3/2 model

1: Generate a sample of X_t given X_0
2: Generate a sample of $\int_0^t \frac{ds}{X_s}$ given X_t
3: Compute $\int_0^t (\sqrt{X_s})^{-1}\, dB_s$ from (6.4.15) given X_t and $\int_0^t \frac{ds}{X_s}$
4: Generate a sample from S_t, given $\int_0^t (\sqrt{X_s})^{-1}\, dB_s$ and $\int_0^t (X_s)^{-1}\, ds$

6.4.1 Simulating X_t

Since $X = \{X_t,\ t \geq 0\}$ is a square-root process, see Eq. (6.4.13), we can immediately apply the methodology from Sect. 6.3.

6.4.2 Simulating $\int_0^t \frac{ds}{X_s}$ Given X_t

We approach Step (2) of Algorithm 6.2 in the same manner as Step (2) of Algorithm 6.1. However, we end up having to compute a different conditional Laplace transform. Fortunately, the relevant Laplace transform can be computed as shown in Sect. 6.2 using Lie symmetry methods. As before, we change the volatility coefficient of X from σ to 2, using the standard time change, which was also used in Sect. 6.3: we define $\rho_t = X_{\frac{4t}{\sigma^2}}$ to obtain the SDE

$$d\rho_t = (2j\rho_t + \delta)\, dt + 2\sqrt{\rho_t}\, d\tilde{B}_t,$$

where

$$\delta = \frac{4(\kappa + \sigma^2)}{\sigma^2}$$

and $j = -\frac{2\kappa\theta}{\sigma^2}$ and $\tilde{B} = \{\tilde{B}_t,\ t \geq 0\}$ is a standard Brownian motion. Now we use formula (6.3.9) again, but this time to obtain a different conditional Laplace transform in the numerator.

$$\begin{aligned}
&E\left(\exp\left\{-a\int_0^t \frac{ds}{X_s}\right\} \,\middle|\, X_t\right) \\
&\quad = E\left(\exp\left\{-a\int_0^t \frac{ds}{\rho_{\frac{\sigma^2 s}{4}}}\right\} \,\middle|\, \rho_{\frac{\sigma^2 t}{4}}\right) \\
&\quad = E\left(\exp\left\{-\frac{4a}{\sigma^2}\int_0^{\frac{\sigma^2 t}{4}} \frac{ds}{\rho_s}\right\} \,\middle|\, \rho_{\frac{\sigma^2 t}{4}}\right) \\
&\quad = \tilde{E}\left(\exp\left\{-\frac{4a}{\sigma^2}\int_0^{\frac{\sigma^2 t}{4}} \frac{ds}{\rho_s} - \frac{j^2}{2}\int_0^{\frac{\sigma^2 t}{4}} \rho_s\, ds\right\} \,\middle|\, \rho_{\frac{\sigma^2 t}{4}}\right) \Big/ \\
&\qquad \tilde{E}\left(\exp\left\{-\frac{j^2}{2}\int_0^{\frac{\sigma^2 t}{4}} \rho_s\, ds\right\} \,\middle|\, \rho_{\frac{\sigma^2 t}{4}}\right), \qquad (6.4.16)
\end{aligned}$$

where we use E to denote the expectation with respect to ${}^{j}P^{\delta,t}_{\rho_0 \to y}$, and $\tilde{E}$ denotes the expectation with respect to $P^{\delta,t}_{\rho_0 \to y}$, see Eq. (6.3.9).

Computing the numerator in (6.4.16) using Proposition 6.2.3 and the denominator using Proposition 6.2.2 yields the following result.

Theorem 6.4.1 *Let X be given by* (6.4.13). *Then*

$$E\left(\exp\left\{-a\int_0^t \frac{ds}{X_s}\right\} \middle| X_t\right) = \frac{I_{\sqrt{\nu^2+8a/\sigma^2}}\left(-\frac{2\kappa\theta\sqrt{X_t X_0}}{\sigma^2 \sinh(-\frac{\kappa\theta t}{2})}\right)}{I_\nu\left(-\frac{2\kappa\theta\sqrt{X_t X_0}}{\sigma^2 \sinh(-\frac{\kappa\theta t}{2})}\right)},$$

where $\delta = \frac{4(\kappa+\sigma^2)}{\sigma^2}$ *and* $\nu = \frac{\delta}{2} - 1$.

Consequently, we can proceed as in Sect. 6.3: we compute the Laplace transform using Theorem 6.4.1, compute the probability distribution of $\int_0^t \frac{ds}{X_s}$ conditional on X_t, and sample by inversion.

6.4.3 Simulating S_t

As in Sect. 6.3, in Step 3) of Algorithm 6.2, we compute $\int_0^t (\sqrt{X_s})^{-1}\, dB_s$ in terms of X_t and $\int_0^t \frac{ds}{X_s}$. Due to the independence of $X = \{X_t,\ t \geq 0\}$ and $W = \{W_t,\ t \geq 0\}$, it follows that $\int_0^t (\sqrt{X_s})^{-1}\, dW_s$ given $\int_0^t (X_s)^{-1}\, ds$ follows a normal distribution with mean 0 and variance $\int_0^t (X_s)^{-1}\, ds$. Hence $\log(S_t)$ follows a normal distribution with mean

$$\log(S_0) + \mu t - \frac{1}{2}\int_0^t (X_s)^{-1}\, ds + \rho \int_0^t (\sqrt{X_s})^{-1}\, dB_s$$

and variance

$$\left(1 - \rho^2\right)\int_0^t (X_s)^{-1}\, ds.$$

6.5 Stochastic Volatility Models with Jumps in the Stock Price

In this section, we extend the model to the case where the stock price process is also subjected to jumps. We follow the presentation in Broadie and Kaya (2006), see also Korn et al. (2010), Sect. 7.2.3, and deal with the Heston model. However, the argument does not rely on the specification of the volatility process, but tells us how to modify the approach from Sects. 6.3 and 6.4 to allow for jumps. Hence the discussion presented in this section also applies to the 3/2 model. The following model was presented in Bates (1996), we also refer the reader to Chap. 5 in Gatheral (2006), where it is referred to as the *SVJ model*,

$$dS_t = S_{t-}\big((r-\lambda\bar{\mu})\,dt + \sqrt{V_t}\big(\rho\,dB_t + \sqrt{1-\rho^2}\,dW_t\big) + (Y_t - 1)\,dN_t\big), \tag{6.5.17}$$
$$dV_t = \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dB_t,$$

where $N = \{N_t,\ t \geq 0\}$ is a Poisson process with constant intensity λ. The processes $B = \{B_t,\ t \geq 0\}$ and $W = \{W_t,\ t \geq 0\}$ are independent Brownian motions and independent of the Poisson process, and the jump variables $Y = \{Y_t,\ t \geq 0\}$ are a family of independent random variables all having the same lognormal distribution with mean μ_s and variance σ_s^2. Furthermore,

$$E(Y_t - 1) = \bar{\mu},$$

and hence

$$\mu_s = \log(1+\bar{\mu}) - \frac{1}{2}\sigma_s^2.$$

Integrating the SDE for the stock price (6.5.17), we obtain

$$S_t = \tilde{S}_t \prod_{j=1}^{N_t} \tilde{Y}_j, \tag{6.5.18}$$

where

$$\tilde{S}_t = S_0 \exp\bigg\{(r-\lambda\bar{\mu})t - \frac{1}{2}\int_0^t V_s\,ds + \rho\int_0^t \sqrt{V_s}\,dB_s + \sqrt{1-\rho^2}\int_0^t \sqrt{V_s}\,dW_s\bigg\},$$

and $\tilde{Y}_j$, $j = 1, \ldots, N_t$, denotes the size of the j-th jump. As discussed in Broadie and Kaya (2006), Korn et al. (2010), Eq. (6.5.18) motivates the simulation algorithm for the SVJ model: we firstly simulate the diffusion part as in Sect. 6.3 and consequently take care of the jump part, $\prod_{j=1}^{N_t} \tilde{Y}_j$. Algorithm 6.3 is the analogue of Algorithm 6.1 and also appeared in Broadie and Kaya (2006) and in similar form in Korn et al. (2010).

Algorithm 6.3 Exact Simulation Algorithm for the SVJ model

1: Generate a sample of V_t given V_0
2: Generate a sample from the distribution of $\int_0^t V_s\,ds$ given V_t and V_0
3: Recover $\int_0^t \sqrt{V_s}\,dB_s$ from (6.3.6) given V_t, V_0 and $\int_0^t V_s\,ds$
4: Generate $\tilde{S}_t$
5: Generate N_t
6: Generate $\prod_{j=1}^{N_t} \tilde{Y}_j$, given N_t

Since the $\tilde{Y}_j$, $j = 1, \ldots, N_t$, are mutually independent and each follows a lognormal distribution with mean μ_s and variance σ_s^2, it is clear that

$$\sum_{j=1}^{N_t} \log(\tilde{Y}_j) | N_t \sim N\big(N_t \mu_s, N_t \sigma_s^2\big).$$

There are alternative approaches to simulating $\prod_{j=1}^{N_t} \tilde{Y}_j$: in Sect. 3.5 in Glasserman (2004), it was shown how to simulate N_t by simulating the jump times of the Poisson process. Furthermore, as discussed in Broadie and Kaya (2006), given N_t, one can simulate the jump sizes $\tilde{Y}_j$, $j = 1, \ldots, N_t$, individually. However, Algorithm 6.3 results in a problem that is of fixed dimension. More precisely, the dimension of the problem in Algorithm 6.3 is five, i.e. five random numbers are used to obtain a realization of S_t. Having a problem of fixed dimensionality is important when applying quasi-Monte Carlo methods, permitting an effective way of tackling multidimensional problems, see Chap. 12, hence we choose the formulation presented in Algorithm 6.3.

6.6 Stochastic Volatility Models with Simultaneous Jumps in the Volatility Process and the Stock Price

In this section, we briefly extend the SVJ model from Sect. 6.5 to allow for simultaneous jumps in the stock price and the volatility process, the *SVCJ model*. As argued in Gatheral (2006), it is unrealistic to assume that the instantaneous volatility would not jump if the stock price did. Hence the following model, introduced in Duffie et al. (2000), allows for simultaneous jumps in the stock price and the volatility,

$$\begin{aligned} dS_t &= S_{t-}\Big((r - \lambda\bar{\mu})\,dt + \sqrt{V_t}\big(\rho\, dB_t + \sqrt{1-\rho^2}\, dW_t\big) + \big(Y_t^s - 1\big)\, dN_t\Big), \\ dV_t &= \kappa(\theta - V_t)\, dt + \sigma\sqrt{V_t}\, dB_t + Y^v dN_t, \end{aligned}$$

where $N = \{N_t,\, t \geq 0\}$ is again a Poisson process with constant intensity λ, $Y^s = \{Y_t^s,\, t \geq 0\}$ is the relative jump size of the stock price, and $Y^v = \{Y_t^v,\, t \geq 0\}$ is the jump size of the variance. The magnitudes of the jumps in the stock price and variance processes are dependent, via the parameter ρ_J, in the following way: the distribution of Y_t^v is exponential with mean μ_v and given Y^v, Y^s is lognormally distributed with mean $\mu_s + \rho_J Y^v$ and variance σ_s^2. The parameters μ_s and $\bar{\mu}$ are related via

$$\mu_s = \log\big((1+\bar{\mu})(1-\rho_J\mu_v)\big) - \frac{1}{2}\sigma_s^2,$$

hence only one needs to be specified. Due to the occurrence of jumps in the volatility, we have to modify the previous procedure. Essentially, we simulate the variance and the stock price process at each jump time. Algorithm 6.4 is the analogue of Algorithms 6.1 and 6.3 and we point out that this algorithm also appeared in Broadie and Kaya (2006), see Sect. 6.2.

Algorithm 6.4 Exact Simulation Algorithm for the SVCJ model

1: Simulate the arrival time of the next jump, τ_j.
2: **if** $\tau_j > T$ **then**
3: Set $\tau_j \to T$
4: **end if**
5: Simulate $V_{\tau_j^-}$ and $S_{\tau_j^-}$, using the time step $\Delta t \to \tau_j - t^*$
6: **if** $\tau_j = T$ **then**
7: Go to Step 13
8: **else**
9: Generate Y^v from an exponential distribution with mean μ_v and set

$$V_{\tau_j} \to V_{\tau_j^-} + Y^v.$$

10: **end if**
11: Generate Y^s by sampling from a lognormal distribution with mean $(\mu_s + \rho_J Y^s)$ and variance σ_s^2. Set $S_{\tau_j} \to S_{\tau_j^-} Y^s$.
12: Set $S_{t^*} \to S_{\tau_j}$, $V_{t^*} \to V_{\tau_j}$, $t^* \to \tau_j$ and go to Step 1
13: Set $S_T \to S_{\tau_j^-}$

6.7 Multidimensional Stochastic Volatility Models

In this section, we discuss the extension of the methodology presented in Sects. 6.3 and 6.4 to the multidimensional case. We firstly explain why a generalization of this methodology is not straightforward, which motivates us to consider almost exact simulation schemes, see Platen and Bruti-Liberati (2010), Chap. 2, for more information on this topic. Furthermore, in Chap. 11 we study advanced stochastic volatility models based on Wishart processes.

Consider the following simple case, with SDEs

$$dS_t^1 = \mu^1 S_t^1\,dt + \sqrt{V_t^1}S_t^1\,dW_t^1, \tag{6.7.19}$$

$$dS_t^2 = \mu^2 S_t^2\,dt + \sqrt{V_t^2}S_t^2\,dW_t^2, \tag{6.7.20}$$

where the two Brownian motions $W^1 = \{W_t^1,\, t \geq 0\}$ and $W^2 = \{W_t^2,\, t \geq 0\}$ co-vary, say $d[W^1, W^2]_t = \rho\,dt$. The volatility processes, V^1 and V^2, which can be square-root or 3/2 processes, see Sect. 3.1, are here driven by Brownian motions independent of W^1 and W^2. Of course, S^1 and S^2 can be simulated as discussed in Sects. 6.3 and 6.4, however, S^1 and S^2 are not independent. In particular, given V_t^j, $\int_0^t V_s^j\,ds$, $j = 1, 2$, we have that, for $j = 1, 2$,

$$\log\bigl(S_t^j\bigr) \sim N\bigl(\mu_j, \sigma_j^2\bigr),$$

with

$$\mu_j = \log\bigl(S_0^j\bigr) - \frac{1}{2}\int_0^t V_s^j\,ds$$

and

$$\sigma_j^2 = \int_0^t V_s^j \, ds.$$

Here the conditional covariance is given by

$$\rho \int_0^t \sqrt{V_s^1}\sqrt{V_s^2}\, ds, \tag{6.7.21}$$

where we recall that ρ denotes the correlation between W^1 and W^2. The computation of the integral in Eq. (6.7.21) does not follow immediately from the methods discussed in Sect. 5.4. We hence recall the almost exact simulation methodology from Platen and Bruti-Liberati (2010).

6.7.1 Matrix Square-Root Processes via Time-Changed Wishart Processes

In this subsection, we briefly recall from Platen and Bruti-Liberati (2010) how to obtain a matrix square-root process from a time-changed Wishart process. The diagonal elements of this process will play the role of V^1 and V^2 in Eqs. (6.7.19) and (6.7.20). We point out that this discussion is based on the simple Wishart process from Sect. 3.2. Once we fully develop the theory of Wishart processes in Chap. 11, we can employ more advanced stochastic volatility models, as in Da Fonseca et al. (2008c).

Recall from Sect. 3.2 that square-root processes can be obtained by time-changing a squared Bessel process. As in Platen and Bruti-Liberati (2010), we consider the function

$$s_t = s_0 \exp\{ct\},$$

where $s_0 > 0$ and consider the transformed time

$$\varphi(t) = \varphi(0) + \frac{1}{4}\int_0^t \frac{b^2}{s_u}\, du,$$

and compute

$$\varphi(t) = \varphi(0) + \frac{b^2}{4cs_0}\big(1 - \exp\{-ct\}\big).$$

Let $X = \{X_t,\, t \geq 0\}$ denote a squared Bessel process of dimension $\delta > 0$, then we obtain a square-root process $Y = \{Y_t,\, t \geq 0\}$ of the same dimension $\delta > 0$ as follows: setting

$$Y_t = s_t X_{\varphi(t)},$$

we obtain the following dynamics for Y,

$$dY_t = \left(\frac{\delta}{2}b^2 + cY_t\right) dt + b\sqrt{Y_t}\, dU_t,$$

where

$$dU_t = \sqrt{\frac{4s_t}{b^2}}\, dW_{\varphi(t)},$$

and since

$$[U]_t = \int_0^t \frac{4s_z}{b^2}\, d\varphi(z) = t,$$

$U = \{U_t,\, t \geq 0\}$ is a Brownian motion, by Levy's characterization theorem, see Sect. 15.3. This procedure is easily generalized. Recall the Wishart process from Sect. 3.2, so $\boldsymbol{W}_t$ is an $n \times p$ matrix, whose elements are independent scalar Brownian motions and $\boldsymbol{W}_0 = \boldsymbol{C}$ is the initial state matrix. We set

$$\boldsymbol{X}_t = \boldsymbol{W}_t^\top \boldsymbol{W}_t, \qquad \boldsymbol{X}_0 = \boldsymbol{C}^\top \boldsymbol{C},$$

so $\boldsymbol{X} = \{\boldsymbol{X}_t,\, t \geq 0\}$ is a Wishart process $WIS_p(\boldsymbol{X}_0, n, \boldsymbol{0}, \boldsymbol{I}_p)$. Following Platen and Bruti-Liberati (2010), we generalize the idea of time-changing a squared Bessel process to time-changing a Wishart process and set

$$\boldsymbol{\Sigma}_t = s_t \boldsymbol{X}_{\varphi(t)},$$

to obtain the SDE

$$d\boldsymbol{\Sigma}_t = \left(\frac{\delta}{4} b^2 \boldsymbol{I} + c\boldsymbol{\Sigma}_t\right) dt + \frac{b}{2}\left(\sqrt{\boldsymbol{\Sigma}_t}\, d\boldsymbol{U}_t + d\boldsymbol{U}_t^\top \sqrt{\boldsymbol{\Sigma}_t}\right), \qquad (6.7.22)$$

for $t \geq 0$, $\boldsymbol{\Sigma}_0 = s_0 \boldsymbol{X}_{\varphi(0)}$, and $d\boldsymbol{U}_t = \sqrt{\frac{4s_t}{b_t^2}}\, d\boldsymbol{W}_{\varphi(t)}$ is the differential of a matrix Wiener process.

6.7.2 Multidimensional Heston Model with Independent Prices

We firstly focus on the case where the volatility process and the Brownian motion driving the stock price are independent. We study the following model

$$d\boldsymbol{S}_t = \boldsymbol{A}_t(\boldsymbol{r}\, dt + \sqrt{\boldsymbol{B}_t}\, d\boldsymbol{W}_t),$$

where $\boldsymbol{S} = \{\boldsymbol{S}_t = (S_t^1, S_t^2, \ldots, S_t^d)^\top,\, t \geq 0\}$ is a vector process and $\boldsymbol{A} = \{\boldsymbol{A}_t = [A_t^{i,j}]_{i,j=1}^d,\, t \geq 0\}$ is a diagonal matrix process with elements

$$A_t^{i,j} = \begin{cases} S_t^i & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases} \qquad (6.7.23)$$

Additionally, $\boldsymbol{r} = (r_1, r_2, \ldots, r_d)^\top$ is a d-dimensional vector and $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, W_t^2, \ldots, W_t^d)^\top,\, t \geq 0\}$ is a d-dimensional vector of correlated Wiener processes. Moreover, $\boldsymbol{B} = \{\boldsymbol{B}_t = [B_t^{i,j}]_{i,j=1}^d,\, t \geq 0\}$ is a matrix process with elements

$$B_t^{i,j} = \begin{cases} \Sigma_t^{i,i} & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases} \qquad (6.7.24)$$

Note that $\boldsymbol{B}$ is the generalization of V in the one-dimensional case. Here, the matrix process $\boldsymbol{\Sigma} = \{\boldsymbol{\Sigma}_t = [\Sigma_t^{i,j}]_{i,j=1}^d,\ t \geq 0\}$ is a matrix square-root process given by the SDE (6.7.22). Therefore, $\boldsymbol{B}_t$ can be constructed from the diagonal elements of $\boldsymbol{\Sigma}_t$. Recall that these elements $\Sigma_t^{1,1}, \Sigma_t^{2,2}, \ldots, \Sigma_t^{d,d}$ form square-root processes and that, for simplicity, we assumed that $\boldsymbol{B}$ is independent of $\boldsymbol{W}$.

We illustrate the simulation in a two-dimensional example. The corresponding two-dimensional SDE for the two prices can be represented as

$$\begin{aligned}
dS_t^1 &= S_t^1 r_1\,dt + S_t^1\sqrt{\Sigma_t^{1,1}}\,d\tilde{W}_t^1,\\
dS_t^2 &= S_t^2 r_2\,dt + S_t^2\sqrt{\Sigma_t^{2,2}}\big[\varrho\,d\tilde{W}_t^1 + \sqrt{1-\varrho^2}\,d\tilde{W}_t^2\big],
\end{aligned}$$

where $t \geq 0$. Here, $\Sigma^{1,1}$ and $\Sigma^{2,2}$ are diagonal elements of the 2×2 matrix given by (6.7.22) and $\tilde{W}^1$ and $\tilde{W}^2$ are independent Wiener processes. The logarithmic transformation $\boldsymbol{X}_t = \log(\boldsymbol{S}_t)$ yields the following SDE

$$\begin{aligned}
dX_t^1 &= \left(r_1 - \frac{1}{2}\Sigma_t^{1,1}\right)dt + \sqrt{\Sigma_t^{1,1}}\,dW_t^1,\\
dX_t^2 &= \left(r_2 - \frac{1}{2}\Sigma_t^{2,2}\right)dt + \sqrt{\Sigma_t^{2,2}}\big[\varrho\,d\tilde{W}_t^1 + \sqrt{1-\varrho^2}\,d\tilde{W}_t^2\big],
\end{aligned}$$

for $t \geq 0$. This results in the following representations:

$$\begin{aligned}
X_{t_{i+1}}^1 &= X_{t_i}^1 + r_1(t_{i+1}-t_i) - \frac{1}{2}\int_{t_i}^{t_{i+1}}\Sigma_u^{1,1}\,du + \int_{t_i}^{t_{i+1}}\sqrt{\Sigma_u^{1,1}}\,d\tilde{W}_u^1,\\
X_{t_{i+1}}^2 &= X_{t_i}^2 + r_2(t_{i+1}-t_i) - \frac{1}{2}\int_{t_i}^{t_{i+1}}\Sigma_u^{2,2}\,du + \varrho\int_{t_i}^{t_{i+1}}\sqrt{\Sigma_u^{2,2}}\,d\tilde{W}_u^1\\
&\quad + \sqrt{1-\varrho^2}\int_{t_i}^{t_{i+1}}\sqrt{\Sigma_u^{2,2}}\,d\tilde{W}_u^2.
\end{aligned}$$

We approximate the integral $\int_{t_i}^{t_{i+1}}\Sigma_u^{j,j}\,du$, $j=1,2$, using e.g. the trapezoidal rule. Consequently, we can simulate the model, noting that conditional on $\int_{t_i}^{t_{i+1}}\Sigma_u^{j,j}\,du$ and $X_{t_i}^j$, $j=1,2$, we obtain that $X_{t_{i+1}}^j$ follows a normal distribution with mean

$$X_{t_i}^j + r_j(t_{i+1}-t_i) - \frac{1}{2}\int_{t_i}^{t_{i+1}}\Sigma_u^{j,j}\,du, \quad j=1,2,$$

and variance

$$\int_{t_i}^{t_{i+1}}\Sigma_u^{j,j}\,du.$$

Furthermore, $X_{t_{i+1}}^1$ and $X_{t_{i+1}}^2$ have the conditional covariance

$$\varrho\int_{t_i}^{t_{i+1}}\sqrt{\Sigma_u^{1,1}}\sqrt{\Sigma_u^{2,2}}\,du,$$

which we approximate, for example, using the trapezoidal rule and the trajectories of $\Sigma^{1,1}$ and $\Sigma^{2,2}$.

6.7.3 Multidimensional Heston Model with Correlated Prices

We now consider a multidimensional version of the Heston model, which allows for correlation of the volatility vector $\boldsymbol{\Sigma}$ with the vector asset price process $\boldsymbol{S}$. We define the generalization by the system of SDEs

$$d\boldsymbol{S}_t = \boldsymbol{A}_t\big(\boldsymbol{r}\,dt + \sqrt{\boldsymbol{B}_t}\big(\boldsymbol{C}\,d\boldsymbol{W}_t^1 + \boldsymbol{D}\,d\boldsymbol{W}_t^2\big)\big),$$
$$d\boldsymbol{\Sigma}_t = (\boldsymbol{a} - \boldsymbol{E}\boldsymbol{\Sigma}_t)\,dt + \boldsymbol{F}\sqrt{\boldsymbol{B}_t}\,d\boldsymbol{W}_t^1,$$

for $t \geq 0$. Here, $\boldsymbol{S} = \{\boldsymbol{S}_t = (S_t^1, S_t^2, \ldots, S_t^d)^\top, t \geq 0\}$ and $\boldsymbol{r} = (r_1, r_2, \ldots, r_d)^\top$. The matrix $\boldsymbol{A}_t = [A_t^{i,j}]_{i,j=1}^d$ is given by (6.7.23) and $\boldsymbol{B}_t = [B_t^{i,j}]_{i,j=1}^d$ is a matrix with elements as in (6.7.24). Additionally, $\boldsymbol{C} = [C^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements

$$C^{i,j} = \begin{cases} \varrho_i & \text{for } i = j \\ 0 & \text{otherwise,} \end{cases}$$

and $\boldsymbol{D} = [D^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements

$$D^{i,j} = \begin{cases} \sqrt{1 - \varrho_i^2} & \text{for } i = j \\ 0 & \text{otherwise,} \end{cases}$$

where $\varrho_i \in [-1, 1]$, $i \in \{1, 2, \ldots, d\}$. Moreover, $\boldsymbol{\Sigma} = \{\boldsymbol{\Sigma}_t = (\Sigma_t^{1,1}, \Sigma_t^{2,2}, \ldots, \Sigma_t^{d,d})^\top, t \in [0, \infty)\}$ and $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)^\top$. The matrix $\boldsymbol{E} = [E^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements

$$E^{i,j} = \begin{cases} b_i & \text{for } i = j \\ 0 & \text{otherwise,} \end{cases}$$

and $\boldsymbol{F} = [F^{i,j}]_{i,j=1}^d$ is a diagonal matrix with elements

$$F^{i,j} = \begin{cases} \sigma_i & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $\boldsymbol{W}^1 = \{\boldsymbol{W}_t^1 = (W_t^{1,1}, W_t^{1,2}, \ldots, W_t^{1,d})^\top, t \geq 0\}$ is a vector of independent Wiener processes and $\boldsymbol{W}^2 = \{\boldsymbol{W}_t^2 = (W_t^{2,1}, W_t^{2,2}, \ldots, W_t^{2,d})^\top, t \geq 0\}$ is a vector of correlated Wiener processes which are independent of $\boldsymbol{W}^1$. In two dimensions, the model looks as follows:

$$d\Sigma_t^{1,1} = \big(a_1 - b_1\Sigma_t^{1,1}\big)\,dt + \sigma_1\sqrt{\Sigma_t^{1,1}}\,dW_t^{1,1},$$
$$d\Sigma_t^{2,2} = \big(a_2 - b_2\Sigma_t^{2,2}\big)\,dt + \sigma_2\sqrt{\Sigma_t^{2,2}}\,dW_t^{1,2},$$

for $t \geq 0$. The two-dimensional asset price process is given by

$$dS_t^1 = r_1 S_t^1\,dt + S_t^1\sqrt{\Sigma_t^{1,1}}\Big(\varrho_1\,dW_t^{1,1} + \sqrt{1 - \varrho_1^2}\,dW_t^{2,1}\Big),$$
$$dS_t^2 = r_2 S_t^2\,dt + S_t^2\sqrt{\Sigma_t^{2,2}}\Big(\varrho_2\,dW_t^{1,2} + \sqrt{1 - \varrho_2^2}\,dW_t^{2,2}\Big),$$

for $t \geq 0$. Hence we can simulate $\Sigma^{1,1}$ and $\Sigma^{2,2}$ via the non-central χ^2-distribution, see Sect. 3.1, or the elements of a matrix square-root process. We can now generate samples of the logarithm of the stock price, $\boldsymbol{X}_t = \log(\boldsymbol{S}_t)$, using the representation

$$
\begin{aligned}
X^1_{t_{i+1}} &= X^1_{t_i} + r_1(t_{i+1} - t_i) + \frac{\varrho_1}{\sigma_1}\left(\Sigma^{1,1}_{t_{i+1}} - \Sigma^{1,1}_{t_i} - a_1(t_{i+1} - t_i)\right) \\
&\quad + \left(\frac{\varrho_1 b_1}{\sigma_1} - \frac{1}{2}\right)\int_{t_i}^{t_{i+1}} \Sigma^{1,1}_u\, du + \sqrt{1-\varrho_1^2}\int_{t_i}^{t_{i+1}} \sqrt{\Sigma^{1,1}_u}\, dW^{2,1}_u, \\
X^2_{t_{i+1}} &= X^2_{t_i} + r_2(t_{i+1} - t_i) + \frac{\varrho_2}{\sigma_2}\left(\Sigma^{2,2}_{t_{i+1}} - \Sigma^{2,2}_{t_i} - a_2(t_{i+1} - t_i)\right) \\
&\quad + \left(\frac{\varrho_2 b_2}{\sigma_2} - \frac{1}{2}\right)\int_{t_i}^{t_{i+1}} \Sigma^{2,2}_u\, du + \sqrt{1-\varrho_2^2}\int_{t_i}^{t_{i+1}} \sqrt{\Sigma^{2,2}_u}\, dW^{2,2}_u.
\end{aligned}
$$

Hence we approximate $\int_{t_i}^{t_{i+1}} \Sigma^{j,j}_u\, du,\ j = 1, 2$, using e.g. the trapezoidal rule. We recall that given

$$
\Sigma^{j,j}_{t_{i+1}},\, \Sigma^{j,j}_{t_i},\, \int_{t_i}^{t_{i+1}} \Sigma^{j,j}_u\, du,\, X^j_{t_i}, \quad j = 1, 2,
$$

the random variables $X^j_{t_{i+1}}$, $j = 1, 2$, are conditionally Gaussian with mean

$$
\begin{aligned}
&X^j_{t_i} + r_j(t_{i+1} - t_i) + \frac{\varrho_j}{\sigma_j}\left(\Sigma^{j,j}_{t_{i+1}} - \Sigma^{j,j}_{t_i} - a_j(t_{i+1} - t_i)\right) \\
&\quad + \left(\frac{\varrho_j b_j}{\sigma_j} - \frac{1}{2}\right)\int_{t_i}^{t_{i+1}} \Sigma^{j,j}_u\, du
\end{aligned}
$$

and variance

$$
\left(1 - \varrho_j^2\right)\int_{t_i}^{t_{i+1}} \Sigma^{j,j}_u\, du.
$$

Lastly, if $d[W^{2,1}, W^{2,2}]_t = \rho\, dt$, then the covariance between $X^1_{t_{i+1}}$ and $X^2_{t_{i+1}}$, conditional on $\Sigma^{j,j}_{t_{i+1}}, \Sigma^{j,j}_{t_i}, \int_{t_i}^{t_{i+1}} \Sigma^{j,j}_u\, du,\ X^j_{t_i},\ j = 1, 2$, is

$$
\rho\sqrt{1-\varrho_1^2}\sqrt{1-\varrho_2^2}\int_{t_i}^{t_{i+1}} \sqrt{\Sigma^{1,1}_u}\sqrt{\Sigma^{2,2}_u}\, du.
$$

Concluding the chapter we mention that in Chap. 11 we will introduce another Heston model based on the Wishart process.