Chapter 5
Transition Densities via Lie Symmetry Methods

In this chapter, we discuss how to obtain explicit transition densities and Laplace transforms of joint transition densities for various diffusions using Lie symmetry methods. We begin with a motivating example, and subsequently present two cautionary examples. The chapter continues with transition densities, which could have useful applications in finance or other areas of application, but are new and have therefore not received so far much attention in the literature. It is hoped that this chapter encourages readers to construct their own examples and apply them to problems they encounter. Subsequently, we present Laplace transforms of joint transition densities in Sect. 5.4. Section 5.5 illustrates how Lie symmetry methods can be powerfully combined with probability theory to enlarge the scope of results that can be obtained.

5.1 A Motivating Example

In this section, we firstly present an example, which exemplifies how explicit transition densities can be found via Lie symmetry methods. The squared Bessel process sits at the heart of the developments in Chap. 3, and our motivating example is also based on this process. Consequently, we consider a squared Bessel process of dimension δ, $\delta \geq 2$,

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dW_t,$$

where $X_0 = x > 0$, whose transition density satisfies the Kolmogorov backward equation

$$u_t = 2x u_{xx} + \delta u_x.$$

Hence in Eq. (4.4.1), we set $\sigma = 2$, $f = \delta$, $g = 0$, and $\gamma = 1$, and in Eq. (4.4.34), we set $h = \delta$, $A = 0$, $B = -2\delta + \frac{\delta^2}{2}$. Now, we employ Theorem 4.4.3 with $u(x,t) = 1$ and $F(x) = \delta \ln x$ to obtain

$$\overline{U}_\epsilon(x,t) = \exp\left\{-\frac{4\epsilon x}{\sigma(1+4\epsilon t)}\right\}(1+4\epsilon t)^{-\frac{\delta}{\sigma}}, \tag{5.1.1}$$

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where $\sigma = 2$. Setting $\epsilon = \frac{\sigma\lambda}{4}$ in Eq. (5.1.1), we obtain the Laplace transform

$$\begin{aligned} U_\lambda(x,t) &= \int_0^\infty \exp\{-\lambda y\} p(t,x,y)\,dy \\ &= \exp\left\{-\frac{x\lambda}{1+2\lambda t}\right\}(1+2\lambda t)^{-\frac{\delta}{2}}, \end{aligned}$$

which is easily inverted to yield

$$p(t,x,y) = \frac{1}{2t}\left(\frac{x}{y}\right)^{\frac{\nu}{2}} I_\nu\left(\frac{\sqrt{xy}}{t}\right)\exp\left\{-\frac{(x+y)}{2t}\right\}, \tag{5.1.2}$$

where $\nu = \frac{\delta}{2} - 1$ denotes the index of the squared Bessel process. Of course, Eq. (5.1.2) shows the transition density of a squared Bessel process started at time 0 in x for being at time t in y. Recall that I_ν denotes the modified Bessel function of the first kind, and that we plotted this transition density in Fig. 3.1.1. We also show it in Fig. 5.3.1.

5.2 Two Cautionary Examples

The previous example begs the question whether a fundamental solution is necessarily a transition density. Fundamental solutions are known not to be unique, and the following example, which is again based on a squared Bessel process and taken from Craddock and Lennox (2009), shows that a fundamental solution is not necessarily a transition density.

Example 5.2.1 Consider a squared Bessel process of dimension three, $\delta = 3$, the transition density of which satisfies the Kolmogorov backward equation

$$u_t = 2xu_{xx} + 3u_x, \tag{5.2.3}$$

a stationary solution of which is $u_1(x) = 1/\sqrt{x}$. Again, we employ Theorem 4.4.3, to obtain

$$\begin{aligned} &\int_0^\infty \frac{1}{\sqrt{y}} \exp\{-\lambda y\} p(t,x,y)\,dy \\ &\quad = \exp\left\{-\frac{x\lambda}{1+2\lambda t}\right\}(1+2\lambda t)^{-\frac{3}{2}} u_1\left(\frac{x}{(1+2\lambda t)^2}\right) \\ &\quad = \exp\left\{-\frac{x\lambda}{1+2\lambda t}\right\}(1+2\lambda t)^{-\frac{3}{2}} \frac{(1+2\lambda t)}{\sqrt{x}} \\ &\quad = \exp\left\{-\frac{x\lambda}{1+2\lambda t}\right\}(1+2\lambda t)^{-\frac{1}{2}} \frac{1}{\sqrt{x}}, \end{aligned}$$

so that we have

$$p(t,x,y) = \frac{\exp\{-\frac{y+x}{2t}\}\cosh(\frac{\sqrt{xy}}{t})}{\sqrt{2t\pi x}}.$$

We note that

$$\int_0^\infty p(t,x,y)\,dy = \frac{\sqrt{2t}}{\sqrt{\pi x}}\exp\left\{-\frac{x}{2t}\right\} + \operatorname{erf}\left(\frac{\sqrt{x}}{\sqrt{2t}}\right).$$

This fundamental solution does not integrate to 1, and hence is not a transition probability density.

We conclude that not all fundamental solutions are transition probability densities. From Example 5.2.1, it is tempting to deduce that fundamental solutions integrating to 1 are transition probability densities. The next example, which stems from Craddock (2009), see Proposition 2.10, shows that also this conjecture is false. As the preceding two examples, it is again based on a squared Bessel process. The example makes use of the following proposition, Proposition 2.4 in Craddock (2009), which shows how to invert a Laplace transform when studying squared Bessel processes.

Proposition 5.2.2 *For a nonnegative integer n, the following equality holds,*

$$L_y^{-1}\left(\lambda^n \exp\left\{\frac{k}{\lambda}\right\}\right) = \sum_{l=0}^{n} \frac{k^l}{l!}\delta^{n-l}(y) + \left(\frac{k}{y}\right)^{\frac{n+1}{2}} I_{n+1}(2\sqrt{ky}),$$

where L_λ is the Laplace transform, $\delta(y)$ is the Dirac delta function and I_n is a modified Bessel function of the first kind with index n.

We now present the example.

Example 5.2.3 Consider a squared Bessel process of dimension 2δ. The transition density satisfies the Kolmogorov backward equation

$$u_t = 2xu_{xx} + 2\delta u_x. \tag{5.2.4}$$

It is easily verified that the stationary solutions $u_0(x) = 1$ and $u_1(x) = x^{1-\delta}$ satisfy (5.2.4). In Sect. 5.1, it was shown that the stationary solution $u_0(x) = 1$ produces the correct transition density. We will now investigate the fundamental solution produced by $u_1(x) = x^{1-\delta}$. Applying Theorem 4.4.3 with $A = 0$, we obtain

$$U_\lambda(x,t) = x^{1-\delta}\exp\left\{-\frac{\lambda x}{(1+2\lambda t)}\right\}(1+2\lambda t)^{\delta-2},$$

i.e.

$$\begin{aligned} L_\lambda &= \int_0^\infty \exp\{-\lambda y\} u_1(y) q(t,x,y)\,dy \\ &= x^{1-\delta}\exp\left\{-\frac{\lambda x}{1+2\lambda t}\right\}(1+2\lambda t)^{\delta-2}. \end{aligned}$$

We now invert the Laplace transform, which yields

$$u_1(y)q(t,x,y) = x^{1-\delta}\exp\left\{-\frac{x+y}{2t}\right\}(2t)^{\delta-2} L_y^{-1}\left(\lambda^{\delta-2}\exp\left\{\frac{k}{\lambda}\right\}\right),$$

where $k = \frac{x}{(2t)^2}$. We now apply Proposition 5.2.2 to yield

$$q(t,x,y) = (2t)^{-1}\left(\frac{y}{x}\right)^{\frac{\delta-1}{2}} \exp\left\{-\frac{(x+y)}{2t}\right\} I_{\delta-1}\left(\frac{\sqrt{xy}}{t}\right) + (2t)^{\delta-2}\left(\frac{y}{x}\right)^{\delta-1} \exp\left\{-\frac{(x+y)}{2t}\right\} \sum_{l=0}^{\delta-2} \frac{k^l}{l!}\delta^{\delta-2-l}(y).$$

We have

$$\int_0^\infty (2t)^{\delta-2}\left(\frac{y}{x}\right)^{\delta-1} \exp\left\{-\frac{x+y}{2t}\right\} \sum_{l=0}^{\delta-2} \frac{x^l}{(2t)^{2l} l!}\delta^{(\delta-2-l)}(y)\, dy = 0,$$

since the Dirac delta function and their derivatives select the value of the test function $y^{\delta-1}$ and its derivatives at zero. Also, we recognize that

$$(2t)^{-1}\left(\frac{y}{x}\right)^{\frac{\delta-1}{2}} \exp\left\{-\frac{(x+y)}{2t}\right\} I_{\delta-1}\left(\frac{\sqrt{xy}}{t}\right)$$

is the transition density of a squared Bessel process of dimension 2δ, cf. (3.1.4), and hence

$$\int_0^\infty (2t)^{-1}\left(\frac{y}{x}\right)^{\frac{\delta-1}{2}} \exp\left\{-\frac{(x+y)}{2t}\right\} I_{\delta-1}\left(\frac{\sqrt{xy}}{t}\right) dy = 1.$$

Finally, we observe that $U_0(x,t) = u_1(x)$ and

$$U_\lambda(x,t) = \int_0^\infty \exp\{-\lambda y\} u_1(y) q(t,x,y)\, dy,$$

which yields

$$\int_0^\infty u_1(y) q(t,x,y)\, dy = u_1(x),$$

and hence $q(t,x,y)$ is not the transition density.

However, in Craddock (2009), the following useful check for processes satisfying

$$X_t = X_0 + \int_0^t f(X_s)\, ds + \int_0^t \sqrt{2\sigma X_s}\, dW_s$$

was presented, see Proposition 2.11 in Craddock (2009), which we now recall.

Proposition 5.2.4 *Let $X = \{X_t, t \geq 0\}$ be an Itô diffusion which is the unique strong solution of*

$$X_t = X_0 + \int_0^t f(X_s)\, ds + \int_0^T \sqrt{2\sigma X_s}\, dW_s,$$

where $W = \{W_t, t \geq 0\}$ is a standard Wiener process and $X_0 = x > 0$. Suppose further that f is measurable and there exist constants $K > 0$, $a > 0$ such that

$\|f(x)\| \leq K \exp\{ax\}$ *for all* x. *Then there exists a* $T > 0$ *such that* $u(x,t,\lambda) = E(\exp\{-\lambda X_t\})$ *is the unique strong solution of the first order PDE*

$$\frac{\partial u}{\partial t} + \lambda^2 \sigma \frac{\partial u}{\partial \lambda} + \lambda E\big(f(X_t)\exp\{-\lambda X_t\}\big) = 0, \tag{5.2.5}$$

subject to $u(x,0,\lambda) = \exp\{-\lambda x\}$, *for* $0 \leq t < T$, $\lambda > a$.

Finally, we show that Proposition 5.2.4 can be used to confirm that the fundamental solution $u_0(x) = 1$ produces the correct fundamental solution, see Example 2.3 in Craddock (2009).

Example 5.2.5 For the squared Bessel process $X = \{X_t, t \geq 0\}$ of dimension δ given by the SDE

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dW_t,$$

where $X_0 = x > 0$, Eq. (5.2.5) yields that $E(\exp\{-\lambda X_t\})$ is the unique solution of the PDE

$$u_t + 2\lambda^2 u_\lambda + \lambda\delta u = 0,$$

where $u(x,0,\lambda) = \exp\{-\lambda x\}$. It can be confirmed that

$$u(x,t,\lambda) = E\big(\exp\{-\lambda X_t\}\big) = \frac{1}{(1+2\lambda t)^{\frac{\delta}{2}}} \exp\left\{-\frac{\lambda x}{1+2\lambda t}\right\},$$

satisfies the PDE and boundary conditions, and coincides with the result produced by the fundamental solution corresponding to $u_0(x) = 1$ in Example 5.2.3.

The above examples indicate that one has to be careful when deciding which fundamental solution yields the desired transition probability density.

5.3 One-Dimensional Examples

In this section, we aim to illustrate how to derive one-dimensional transition densities using the results from Chap. 4. We emphasize that the process of deriving transition densities is mechanical and easily applied to the study of novel stochastic processes. In this regard, we recall examples of transition densities studied in Craddock and Platen (2004) and provide the reader with additional references. It is intended that this section encourages readers to study stochastic processes that are tractable and potentially more suitable to their applications than those processes that have been employed in the past mainly because they were considered to be tractable from a conventional perspective.

We illustrate the derivation of the transition density of the square-root process, where we follow the presentation in Craddock (2009). In particular, we assume that

$$dX_t = (a - bX_t)\, dt + \sqrt{2\sigma X_t}\, dW_t, \tag{5.3.6}$$

where $X_0 = x > 0$ and a, b, and σ are assumed to be positive and $\frac{a}{\sigma} \geq 1$.

Proposition 5.3.1 *The transition density of the process X as specified in the SDE (5.3.6) started in x at time 0 being in y at time t is given by the explicit formula*

$$p(x,t,y) = \frac{b\exp\{bt(\frac{a}{\sigma}+1)\}}{\sigma(\exp\{bt\}-1)}\left(\frac{y}{x}\right)^{\frac{\nu}{2}}\exp\left\{\frac{-b(x+\exp\{bt\}y)}{\sigma(\exp\{bt\}-1)}\right\} \times I_\nu\left(\frac{b\sqrt{xy}}{\sigma\sinh(\frac{bt}{2})}\right), \tag{5.3.7}$$

where $\nu = \frac{a}{\sigma} - 1 \geq 0$.

Proof We note that the transition density of X satisfies the Kolmogorov backward equation

$$u_t = \sigma x u_{xx} + (a - bx)$$

and that Eq. (4.4.67) is satisfied with

$$h(x) = (a - bx), \qquad g = 0, \qquad \gamma = 1.$$

Hence we employ Theorem 4.4.5 with $\gamma = 1$, $u_0 = 1$, $A = b^2$, $B = -ab$, $C = \frac{1}{2}a^2 - a\sigma$, and $F(x) = a\ln(x) - bx$ to obtain

$$\begin{aligned}
\overline{U}_\epsilon(x,t) &= \exp\left\{\frac{-b^2\epsilon x}{\sigma}\left(\frac{\cosh(bt)+b\epsilon\sinh(bt)}{1+2b\epsilon\sinh(bt)+2b^2\epsilon^2(\cosh(bt)-1)}\right)\right\} \\
&\quad \times \exp\left\{\frac{tab}{2\sigma}\right\}\left|\frac{\cosh(\frac{bt}{2})+(1+2b\epsilon)\sinh(\frac{bt}{2})}{\cosh(\frac{bt}{2})-(1-2b\epsilon)\sinh(\frac{bt}{2})}\right|^{\frac{-ab}{2\sigma b}} \\
&\quad \times \exp\left\{\frac{1}{2\sigma}F\left(\frac{x}{(1+2b^2\epsilon^2(\cosh(bt)-1)+2b\epsilon\sinh(bt))}\right) - \frac{F(x)}{2\sigma}\right\} \\
&= \exp\left\{b(at-2b\epsilon x)\cosh\left(\frac{bt}{2}\right) + at\sinh\left(\frac{bt}{2}\right) + 2ab\epsilon t\sinh\left(\frac{bt}{2}\right)\right. \\
&\quad \left. + 2b\epsilon x\sinh\left(\frac{bt}{2}\right)\right\}\left(\cosh\left(\frac{bt}{2}\right) + (1+2b\epsilon)\sinh\left(\frac{bt}{2}\right)\right)^{-\frac{a}{\sigma}},
\end{aligned}$$

where the last equality can be shown using MATHEMATICA. We have

$$\begin{aligned}
&\left(\cosh\left(\frac{bt}{2}\right) + (1+2b\epsilon)\sinh\left(\frac{bt}{2}\right)\right)^{-\frac{a}{\sigma}} \\
&\quad = \exp\left\{\frac{bta}{2\sigma}\right\}\big(\exp\{bt\} + b\epsilon\big(\exp\{bt\}-1\big)\big)^{-\frac{a}{\sigma}}.
\end{aligned}$$

Also, it follows that

$$\begin{aligned}
&\exp\left\{b(at-2b\epsilon x)\cosh\left(\frac{bt}{2}\right) + at\sinh\left(\frac{bt}{2}\right) + 2ab\epsilon t\sinh\left(\frac{bt}{2}\right)\right. \\
&\qquad \left. + 2b\epsilon x\sinh\left(\frac{bt}{2}\right)\right\} \\
&\quad = \exp\left\{\frac{abt}{2\sigma}\right\}\exp\left\{-\frac{b^2\epsilon x}{\sigma(\exp\{bt\}+b\epsilon(\exp\{bt\}-1))}\right\}.
\end{aligned}$$

Now, one obtains

$$
\begin{aligned}
\overline{U}_\epsilon(x,t) &= \exp\left\{\frac{abt}{2\sigma}\right\}\big(\exp\{bt\} + b\epsilon\big(\exp\{bt\} - 1\big)\big)^{-\frac{a}{\sigma}} \\
&\quad \times \exp\left\{\frac{abt}{2\sigma}\right\}\exp\left\{-\frac{b^2\epsilon x}{\sigma(\exp\{bt\} + b\epsilon(\exp\{bt\} - 1))}\right\} \\
&= \exp\left\{\frac{abt}{\sigma}\right\}\big(\exp\{bt\} + b\epsilon\big(\exp\{bt\} - 1\big)\big)^{-\frac{a}{\sigma}} \\
&\quad \times \exp\left\{-\frac{b^2\epsilon x}{\sigma(\exp\{bt\} + b\epsilon(\exp\{bt\} - 1))}\right\}.
\end{aligned}
$$

Substituting $\epsilon = \frac{\lambda\sigma}{b^2}$, we get

$$
\begin{aligned}
\overline{U}_\epsilon(x,t) &= U_\lambda(x,t) \\
&= \int_0^\infty \exp\{-\lambda y\}p(t,x,y)\,dy \\
&= \exp\left\{\frac{abt}{\sigma}\right\}\big(b\exp\{bt\} + \lambda\sigma\big(\exp\{bt\} - 1\big)\big)^{-\frac{a}{\sigma}} b^{\frac{a}{\sigma}} \\
&\quad \times \exp\left\{-\frac{b\lambda x}{(b\exp\{bt\} + \lambda\sigma(\exp\{bt\} - 1))}\right\}.
\end{aligned}
$$

This Laplace transform can be easily inverted to yield (5.3.7). It can be confirmed via Proposition 5.2.4 that the density in (5.3.7) is the correct transition probability density. □

In Fig. 3.1.2, a plot of the transition density of a square-root process was shown. We now recall some results from Craddock and Platen (2004). In particular, we study generalizations of the squared Bessel process. We focus on the process $X = \{X_t,\ t \geq 0\}$, given by the SDE

$$dX_t = a(X_t)\,dt + \sqrt{2X_t}\,dW_t, \tag{5.3.8}$$

for $t \geq 0$ with $X_0 > 0$. Then, following Craddock and Platen (2004), Platen and Heath (2010), and Platen and Bruti-Liberati (2010), by applying the results of Chap. 4, we distinguish ten cases:

(i) for the constant drift function

$$a(x) = \alpha > 0,$$

we recover the squared Bessel process of dimension $\delta = 2\alpha$ with transition density

$$p(0,x;t,y) = \frac{1}{t}\left(\frac{x}{y}\right)^{\frac{1-\alpha}{2}} I_{\alpha-1}\left(\frac{2\sqrt{x\,y}}{t}\right)\exp\left\{-\frac{(x+y)}{t}\right\}.$$

Here $I_{\alpha-1}$ is again the modified Bessel function of the first kind with index $\alpha - 1$, see also Eq. (3.1.4) and Fig. 3.1.1

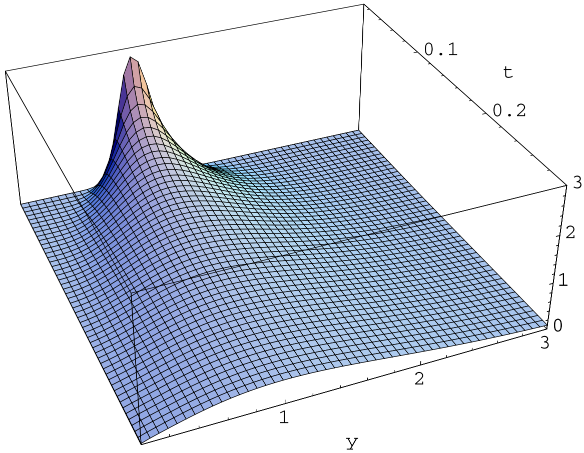


Fig. 5.3.1 Transition density for a squared Bessel process, case (i)

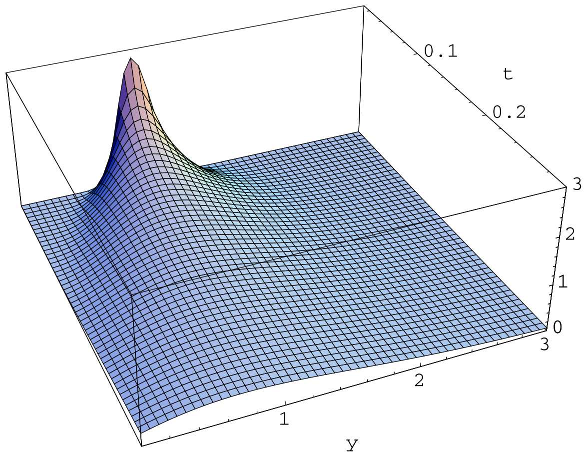


Fig. 5.3.2 Transition density for case (ii)

(ii) setting the drift function to

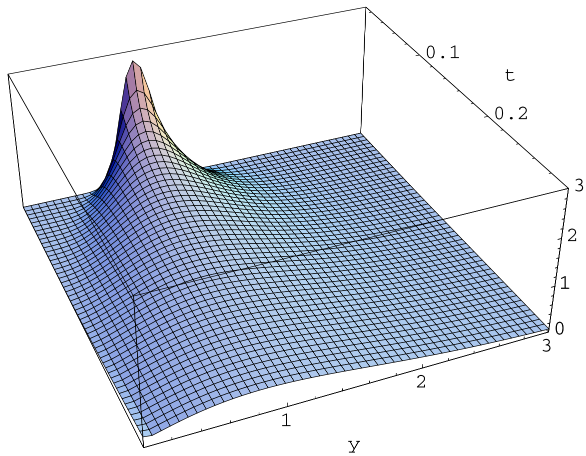
$$a(x) = \frac{\mu\, x}{1 + \frac{\mu}{2}\, x}$$

for $\mu > 0$, we obtain the transition density

$$p(0, x; t, y) = \frac{\exp\{-\frac{(x+y)}{t}\}}{(1 + \frac{\mu}{2}\, x)t} \left[\left(\sqrt{\frac{x}{y}} + \frac{\mu \sqrt{x\, y}}{2} \right) I_1\left(\frac{2\sqrt{x\, y}}{t} \right) + t\, \delta(y) \right]$$

with $\delta(\cdot)$ denoting the Dirac delta function. For $y = 0$ one can interpret $\frac{\exp\{-\frac{x}{t}\}}{(1+\frac{\mu}{2}\, x)}$ as the probability of absorption at zero. In Fig. 5.3.2 we show the above transition density for $x = 1$ and $\mu = 1$

Fig. 5.3.3 Transition density for case (iii)



(iii) the drift function

$$a(x) = \frac{1 + 3\sqrt{x}}{2\,(1 + \sqrt{x})},$$

results in the transition density

$$p(0, x; t, y) = \frac{\cosh(\frac{2\sqrt{x\,y}}{t})}{\sqrt{\pi y t}(1 + \sqrt{x})}\left(1 + \sqrt{y}\tanh\left(\frac{2\sqrt{x\,y}}{t}\right)\right)$$
$$\times \exp\left\{-\frac{(x + y)}{t}\right\}.$$

In Fig. 5.3.3 we display the corresponding transition density for $x = 1$

(iv) studying the drift function

$$a(x) = 1 + \mu \tanh\left(\mu + \frac{1}{2}\mu \ln(x)\right)$$

for $\mu = \frac{1}{2}\sqrt{\frac{5}{2}}$, we obtain the transition density

$$p(0, x; t, y) = \left(\frac{x}{y}\right)^{\frac{\mu}{2}}\left[I_{-\mu}\left(\frac{2\sqrt{x\,y}}{t}\right) + e^{2\mu} y^{\mu} I_{\mu}\left(\frac{2\sqrt{x\,y}}{t}\right)\right]$$
$$\times \frac{\exp\{-\frac{x+y}{t}\}}{(1 + \exp\{2\mu\}x^{\mu})\,t}. \tag{5.3.9}$$

The shape of the density (5.3.9) for $x = 1$ looks quite similar to that in Fig. 5.3.3

(v) given the drift function

$$a(x) = \frac{1}{2} + \sqrt{x},$$

we obtain the transition density

$$p(0, x; t, y) = \cosh\left(\frac{(t + 2\sqrt{x})\sqrt{y}}{t}\right) \frac{\exp\{-\sqrt{x}\}}{\sqrt{\pi\, y\, t}} \times \exp\left\{-\frac{(x + y)}{t} - \frac{t}{4}\right\}. \tag{5.3.10}$$

Also the transition density (5.3.10) for $x = 1$ shows a lot of similarity with that in Fig. 5.3.3

(vi) the drift function

$$a(x) = \frac{1}{2} + \sqrt{x}\tanh(\sqrt{x}),$$

results in the transition density

$$p(0, x; t, y) = \frac{\cosh(\frac{2\sqrt{xy}}{t})}{\sqrt{\pi yt}} \frac{\cosh(\sqrt{y})}{\cosh(\sqrt{x})} \exp\left\{-\frac{(x + y)}{t} - \frac{t}{4}\right\}. \tag{5.3.11}$$

The above transition density (5.3.11) for $x = 1$ has also a similar shape as that in Fig. 5.3.3

(vii) when the drift function satisfies

$$a(x) = \frac{1}{2} + \sqrt{x}\coth(\sqrt{x})$$

the process has the transition density

$$p(0, x; t, y) = \frac{\sinh(\frac{2\sqrt{x\, y}}{t})}{\sqrt{\pi\, y\, t}} \frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})} \exp\left\{-\frac{(x + y)}{t} - \frac{t}{4}\right\}.$$

This transition density has for $x = 1$ some similarity with that shown in Fig. 5.3.1

(viii) using the drift function

$$a(x) = 1 + \cot\big(\ln(\sqrt{x})\big)$$

for $x \in (\exp\{-2\pi\}, 1)$, then we obtain the real valued transition density

$$p(0, x; t, y) = \frac{\exp\{-\frac{(x+y)}{t}\}}{2\imath t \sin(\ln(\sqrt{x}))} \left(y^{\frac{\imath}{2}} I_\imath\left(\frac{2\sqrt{x\, y}}{t}\right) - y^{-\frac{\imath}{2}} I_{-\imath}\left(\frac{2\sqrt{x\, y}}{t}\right)\right), \tag{5.3.12}$$

where $\imath$ denotes the imaginary unit. We plot in Fig. 5.3.4 the transition density (5.3.12) for $x = \frac{1}{2}$. Note that the process X lives on the bounded interval $(\exp\{-2\pi\}, 1)$

(ix) choosing the drift function

$$a(x) = x \coth\left(\frac{x}{2}\right),$$

Fig. 5.3.4 Transition density for case (viii)

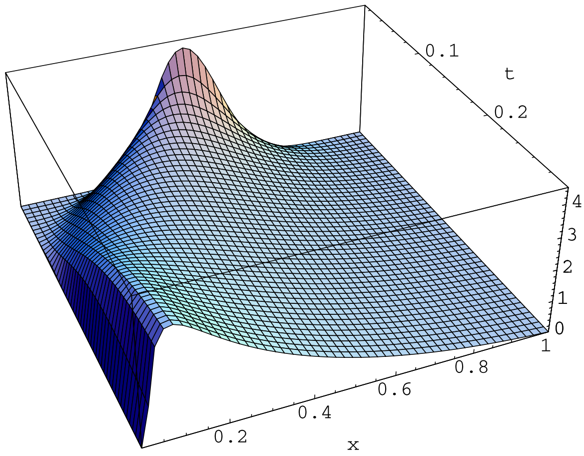
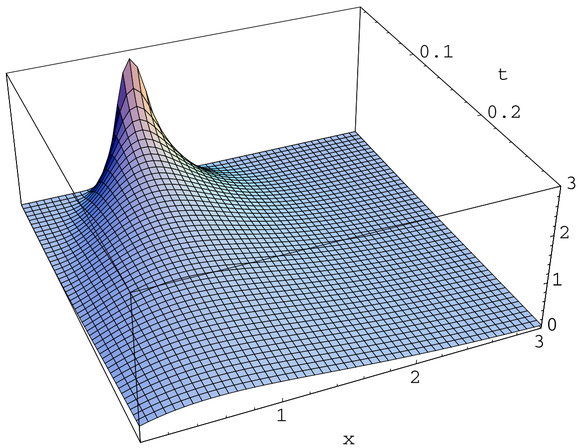


Fig. 5.3.5 Transition density for case (x)



then we obtain the transition density

$$p(0,x;t,y) = \frac{\sinh(\frac{y}{2})}{\sinh(\frac{x}{2})} \exp\left\{-\frac{(x+y)}{2\,\tanh(\frac{t}{2})}\right\} \times \left[\frac{\exp\{\frac{t}{2}\}}{\exp\{t\}-1}\sqrt{\frac{x}{y}}\, I_1\left(\frac{\sqrt{x\,y}}{\sinh(\frac{t}{2})}\right) + \delta(y)\right],$$

where $\delta(\cdot)$ is again the Dirac delta function. Figure 5.3.1 displayed a transition density of similar shape

(x) lastly, setting the drift function to

$$a(x) = x\,\tanh\left(\frac{x}{2}\right)$$

we obtain the transition density

$$p(0,x;t,y) = \frac{\cosh(\frac{y}{2})}{\cosh(\frac{x}{2})} \exp\left\{-\frac{(x+y)}{2\tanh(\frac{t}{2})}\right\} \times \left[\frac{\exp\{\frac{t}{2}\}}{\exp\{t\}-1}\sqrt{\frac{x}{y}} I_1\left(\frac{\sqrt{x\,y}}{\sinh(\frac{t}{2})}\right) + \delta(y)\right].$$

We plot in Fig. 5.3.5 the transition density for $x = 1$.

Many of the above diffusion processes are very recent in the literature and essentially discovered in Craddock and Platen (2004). They offer new dynamics ready to be employed in modeling, for instance, in finance.

5.4 Laplace Transforms of Joint Transition Densities

In this section, we present Laplace transforms of the type

$$E\left(\exp\left\{-\lambda X_t - \mu \int_0^t X_s\,ds - \gamma \int_0^t \frac{ds}{X_s}\right\}\right), \tag{5.4.13}$$

for suitable stochastic processes $X = \{X_t,\ t \ge 0\}$. These Laplace transforms have important applications. For example, if X is the independent short rate process and $\lambda = \gamma = 0$ and $\mu = 1$, then Eq. (5.4.13) contributes to the price of a zero coupon bond, see also Sect. 5.5. However, there are many applications beyond interest rate modeling. For instance, in Chap. 6 we will design exact Monte Carlo schemes for stochastic volatility models based on results from this section. In Sect. 8.5.2, we will focus on exact and quasi-Monte Carlo methods for realized variance derivatives, to illustrate further possible applications of the results presented in this section. At the heart of such applications sits the observation that for some tasks, the fundamental solution is sometimes more interesting than its Laplace transform, see Sect. 8.5.2 and Chap. 6. Hence even though we might not always be able to integrate the fundamental solution to calculate the Laplace transform, we may be nevertheless able to calculate and subsequently use the fundamental solution.

We illustrate this type of technique in the following result, see Craddock and Lennox (2009).

Proposition 5.4.1 *Let $X = \{X_t,\ t \ge 0\}$ be a squared Bessel process,*

$$dX_t = \delta\,dt + 2\sqrt{X_t}\,dW_t,$$

where $\delta \ge 2$ and $X_0 = x > 0$. Then the function $u(x,t)$ given by

$$u(x,t) = E\left(\exp\left\{-\lambda X_t - \mu \int_0^t \frac{ds}{X_s}\right\}\right) = \exp\{-x/2t\}\left(\frac{x}{2t}\right)^d \frac{\Gamma(\alpha)\,{}_1F_1(\alpha,\beta,x/(2t+4t^2\lambda))}{\Gamma(\beta)(1+2\lambda t)^\alpha},$$

where ${}_1F_1(a,b,z)$ *is Kummer's confluent hypergeometric function, satisfies the PDE*

$$u_t = 2xu_{xx} + \delta u_x - \frac{\mu}{x}u,$$

whose fundamental solution is given by

$$p(t,x,y) = \frac{1}{2t}\left(\frac{x}{y}\right)^{(1-\delta/2)/2} I_{2d+\delta/2-1}\left(\frac{\sqrt{xy}}{t}\right)\exp\left\{-\frac{(x+y)}{2t}\right\}, \tag{5.4.14}$$

where $d = \frac{1}{4}(2-\delta+\sqrt{(\delta-2)^2+8\mu})$, $\alpha = d+\frac{\delta}{2}$, *and* $\beta = 2d+\frac{\delta}{2}$.

Proof The drift function $f(x)=\delta$ satisfies the first Ricatti equation (4.4.34), where $\sigma=2$, $\gamma=1$, $g(x)=\frac{\mu}{x}$, and $A=0$. Choosing the stationary solution $u_0(x)=x^d$, where $d=\frac{1}{4}(2-\delta+\sqrt{(\delta-2)^2+8\mu})$, we obtain from Theorem 4.4.3

$$\overline{U}_\epsilon(x,t) = \exp\left\{-\frac{4\epsilon x}{2(1+4\epsilon t)}\right\}\frac{x^d}{(1+4\epsilon t)^{2d+\frac{\delta}{2}}}.$$

Next, we set $\epsilon = \frac{\sigma\lambda}{4} = \frac{\lambda}{2}$ to obtain

$$\begin{aligned}\overline{U}_\epsilon(x,t) = U_\lambda(x,t) &= \int_0^\infty y^d p(t,x,y)\exp\{-\lambda y\}\,dy\\ &= \frac{x^d}{(1+2\lambda t)^{\frac{\delta}{2}+2d}}\exp\left\{-\frac{\lambda x}{(1+2\lambda t)}\right\}.\end{aligned}$$

Inverting this Laplace transform, we obtain the fundamental solution

$$p(t,x,y) = \frac{1}{2t}\exp\left\{-\frac{x+y}{2t}\right\}\left(\frac{x}{y}\right)^{\frac{1-\delta/2}{2}} I_{\frac{\delta}{2}+2d-1}\left(\frac{\sqrt{xy}}{t}\right).$$

We obtain

$$\int_0^\infty e^{-\lambda y}p(t,x,y)\,dy = \frac{\Gamma(\alpha)}{\Gamma(\beta)}\left(\frac{x}{2t}\right)^d e^{-\frac{x}{2t}}\,{}_1F_1\left(\alpha,\beta,\frac{x}{2t+4t^2\lambda}\right)(1+2\lambda t)^{-\alpha},$$

by integrating the modified Bessel function of the first kind term-by-term. □

We now recall results from Craddock and Lennox (2009), where Eq. (4.4.35) was handled via group invariant solutions. In particular, this approach produced Whittaker transforms of fundamental solutions. Although such integral transforms have known inversion integrals, explicit inversion is usually not possible, as few of these transforms have been computed and tabulated. However, in Craddock (2009), Eqs. (4.4.35) and (4.4.36) were handled via symmetry methods, namely by using the full group of symmetries, see also the proof of Theorem 4.4.5. This approach produces generalized Laplace transforms of the fundamental solutions.

As fundamental solutions will play an important role in Chap. 6, we present both, Laplace transforms and fundamental solutions themselves.

Theorem 5.4.2 *Let $X = \{X_t,\ t \geq 0\}$ be a squared Bessel process where*

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dW_t,$$

for $\delta \geq 2$ and $X_0 = x > 0$. Then the function $u(x,t)$ given by

$$u(x,t) = E\left(\exp\left\{-\lambda X_t - \frac{b^2}{2}\int_0^t X_s\, ds\right\}\right)$$
$$= \frac{\exp\{-(xb/2)(1 + 2\lambda b^{-1}\coth(bt))/(\coth(bt) + 2\lambda b^{-1})\}}{(\cosh(bt) + 2\lambda b^{-1}\sinh(bt))^{\delta/2}}.$$

satisfies the PDE

$$u_t = 2xu_{xx} + \delta u_x - \frac{b^2}{2}xu,$$

whose fundamental solution is given by

$$p(t,x,y) = \frac{b}{2\sinh(bt)}\left(\frac{y}{x}\right)^{\delta/4-1/2}\exp\left\{-\frac{b(x+y)}{2\tanh(bt)}\right\}I_{(\delta-2)/2}\left(\frac{b\sqrt{xy}}{\sinh(bt)}\right).$$

We have the following result pertaining to square-root processes satisfying the SDE,

$$dX_t = (a - bX_t)\, dt + \sqrt{2\sigma X_t}\, dW_t, \tag{5.4.15}$$

where $X_0 = x > 0$.

Proposition 5.4.3 *Let $X = \{X_t,\ t \geq 0\}$ be a square-root process of dimension $\delta = \frac{4a}{2\sigma} \geq 2$, whose dynamics satisfy the SDE (5.4.15). Then the function $u(x,t)$ is given by*

$$u(x,t) = E\left(\exp\left\{-\lambda X_t - \mu\int_0^t \frac{ds}{X_s}\right\}\right)$$
$$= \frac{\Gamma(k + \nu/2 + 1/2)}{\Gamma(\nu+1)}\beta x^{-k}\exp\left\{\frac{b}{2\sigma}\left(at + x - \frac{x}{\tanh(bt/2)}\right)\right\}$$
$$\times \frac{e^{\beta^2/(2\alpha)}}{\beta\alpha^k}M_{-k,\nu/2}\left(\frac{\beta^2}{\alpha}\right),$$

where $\nu = \frac{1}{\sigma}\sqrt{(a-\sigma)^2 + 4\mu\sigma}$, $k = \frac{a}{2\sigma}$, $\alpha = \frac{b}{2\sigma}(1 + \coth(\frac{bt}{2})) + \lambda$, $\beta = \frac{b\sqrt{x}}{2\sigma\sinh(\frac{bt}{2})}$, and $M_{s,r}(z)$ denotes the Whittaker function of the first kind. Furthermore, $u(x,t)$ satisfies the PDE

$$u_t = \sigma x u_{xx} + (a - bx)u_x - \frac{\mu}{x}u,$$

whose fundamental solution is given by

$$p(t,x,y) = \frac{b}{2\sigma\sinh(bt/2)}\left(\frac{y}{x}\right)^{a/(2\sigma)-1/2}\exp\left\{\frac{b}{2\sigma}\left(at + (x-y) - \frac{x+y}{\tanh(bt/2)}\right)\right\}$$
$$\times I_\nu\left(\frac{b\sqrt{xy}}{\sigma\sinh(bt/2)}\right). \tag{5.4.16}$$

Finally, we present the Laplace transform of the joint density for

$$\left(X_t, \int_0^t X_s\,ds, \int_0^t \frac{ds}{X_s}\right).$$

In particular, we consider the function

$$u(x,t) = E\left(\exp\left\{-\lambda X_t - \left(b^2/2\right)\int_0^t X_s\,ds - \nu\int_0^t \frac{ds}{X_s}\right\}\right),$$

where $X_0 > 0$. We have the following result.

Proposition 5.4.4 *Let $X = \{X_t,\ t \geq 0\}$ be a squared Bessel process of dimension $\delta \geq 2$. Then*

$$\begin{aligned} u(x,t) &= E\left(\exp\left\{-\lambda X_t - \left(b^2/2\right)\int_0^t X_s\,ds - \nu\int_0^t \frac{ds}{X_s}\right\}\right) \\ &= \exp\left\{-bx/\left(2\tanh(bt)\right)\right\}\frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{b^{a/2}(x\exp\{bt\})^{\gamma}(\exp\{2bt\}-1)^{-\gamma}}{(\cosh(bt)+(2\lambda/b)\sinh(bt))^{\delta}} \\ &\quad \times {}_1F_1\left(\alpha,\beta,\frac{b^2x\,\mathrm{csch}(bt)}{2b\cosh(bt)+4\lambda\sinh(bt)}\right), \end{aligned}$$

where $a = \sqrt{(\delta-2)^2+8\nu}$, $\delta = \frac{1}{4}(2+a+\delta)$, $\gamma = \frac{1}{4}(2+a-\delta)$, $\alpha = \frac{1}{4}(a+\delta+2)$, $\beta = \frac{a+2}{2}$, and ${}_1F_1(a,b,z)$ is Kummer's confluent hypergeometric function and csch denotes the hyperbolic cosecant, $\mathrm{csch}(x) = \frac{2\exp\{x\}}{\exp\{2x\}-1}$. *Furthermore*, $u(x,t)$ *satisfies the PDE*

$$u_t = 2xu_{xx} + \delta u_x - \frac{b^2}{2}xu - \nu\frac{u}{x},$$

whose fundamental solution is given by

$$\begin{aligned} p(t,x,y) &= \frac{b}{2\sinh(bt)}\exp\left(-b(x+y)/\left(2\tanh(bt)\right)\right)\left(\frac{y}{x}\right)^{(\delta-2)/4} \\ &\quad \times I_{\sqrt{(\delta-2)^2+8\nu}/2}\left(\frac{b\sqrt{xy}}{\sinh(bt)}\right). \end{aligned}$$

This result provides important access to functionals of squared Bessel processes that have explicit formulas. We point out that Proposition 5.4.4 will be applied in Sects. 5.5, 6.3, and 6.4.

5.5 Bond Pricing in Quadratic Models

So far in this chapter, we have illustrated how Lie symmetry methods can be used to obtain transition densities and Laplace transforms of joint transition densities. This section illustrates that by combining results obtained via Lie symmetry methods with probability theory, the scope of results that can be obtained is increased. We illustrate this using two examples. Firstly, we use Proposition 5.4.4, which provides the Laplace transform of joint transition densities of the squared Bessel process with the change of law result from Pitman and Yor (1982), see Proposition 3.1.6, which connects squared Bessel and square-root processes, to price zero coupon bonds in the Cox, Ingersoll, Ross (CIR) model introduced in Cox et al. (1985). Secondly, we recall from Sect. 3.1, that a $3/2$ process is simply the inverse of a squared Bessel process, and use this observation and Proposition 5.4.3, which deals with square-root processes, to price zero coupon bonds under a $3/2$ process for the short-rate.

We begin with the pricing of a zero coupon bond in the CIR model. Recall that in the CIR model, the short rate is modeled using a square-root process,

$$dr_t = k(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t, \tag{5.5.17}$$

where $r_0 \geq 0$ and $\frac{4k\theta}{\sigma^2} \geq 2$. Consequently, we are interested in computing

$$E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right), \tag{5.5.18}$$

where we use $\mathcal{A}_t$ to denote $\mathcal{A}_t = \sigma\{r_s,\ s \leq t\}$. We find it convenient to reduce the pricing problem to the study of Laplace transforms of squared Bessel processes. As discussed in Sect. 3.1, we recall that there are at least two methods for reducing the study of square-root processes to the study of squared Bessel processes. These are transformation of space-time and the change of law, see Propositions 3.1.5 and 3.1.6. As discussed in Sect. 3.1, using the standard change of time technique, we transform (5.5.17) into a square-root process with volatility coefficient 2: we introduce the process $\rho = \{\rho_t, t \geq 0\}$ via $\rho_t = r_{\frac{4t}{\sigma^2}}$, and obtain the following SDE for ρ_t:

$$d\rho_t = (2j\rho_t + \delta)\,dt + 2\sqrt{\rho_t}\,d\tilde{W}_t, \tag{5.5.19}$$

where $\tilde{W} = \{\tilde{W}_t,\ t \geq 0\}$ is a standard Brownian motion, $j = -\frac{2k}{\sigma^2}$, and $\delta = \frac{4k\theta}{\sigma^2}$. We use ${}^jP^n_{\rho_0}$ to denote the law of ρ, and set $\mathcal{F}_t = \sigma\{\rho_s, s \leq t\}$. Due to the functional dependence of r and ρ, we have $\mathcal{A}_{\frac{4t}{\sigma^2}} = \mathcal{F}_t$, $t \geq 0$. By Proposition 3.1.6, the following absolute continuity relationship between square-root and squared Bessel processes holds:

$${}^jP^{\delta}_{\rho_0}\big|_{\mathcal{F}_t} = \exp\left\{\frac{j}{2}(\rho_t - \rho_0 - \delta t) - \frac{j^2}{2}\int_0^t \rho_s\,ds\right\} P^{\delta}_{\rho_0}\bigg|_{\mathcal{F}_t}. \tag{5.5.20}$$

We now use Eq. (5.5.20) to change the pricing problem (5.5.18) into one that can be solved using the Laplace transforms of densities of squared Bessel processes from Sect. 5.4. We point out that this technique will also be used in Chap. 6. The next theorem shows how to derive the well-known bond pricing formula in the CIR model by combining the results from Sect. 5.4, in particular Proposition 5.4.4, with the change of law formula from Pitman and Yor (1982).

Theorem 5.5.1 *Assume that the dynamics of* r_t *are given by* (5.5.17) *and* $\frac{4k\theta}{\sigma^2} \geq 2$. *Then we have the following formula for a zero coupon price at time t with maturity date* $T > t$:

$$E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right) = A(t,T)\exp\{-B(t,T)r_t\},$$

where

$$A(t,T) = \left(\frac{2h\exp((k+h)(T-t)/2)}{2h+(k+h)(\exp(h(T-t))-1)}\right)^{\frac{2k\theta}{\sigma^2}}$$
$$B(t,T) = \frac{2(\exp((T-t)h)-1)}{2h+(k+h)(\exp((T-t)h)-1)}$$
$$h = \sqrt{k^2+2\sigma^2}.$$

Proof Setting $\tilde{t} := \frac{t\sigma^2}{4}$ and $\tilde{T} := \frac{T\sigma^2}{4}$, we employ Eq. (5.5.20) to obtain

$$\begin{aligned}
&E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right)\\
&= E\left(\exp\left\{-\int_t^T \rho_{\frac{s\sigma^2}{4}}\,ds\right\}\,\middle|\,\mathcal{F}_{\frac{t\sigma^2}{4}}\right)\\
&= E\left(\exp\left\{-\frac{4}{\sigma^2}\int_{\tilde{t}}^{\tilde{T}} \rho_{\tilde{s}}\,d\tilde{s}\right\}\,\middle|\,\mathcal{F}_{\tilde{t}}\right)\\
&= \tilde{E}\left(\exp\left\{\frac{j}{2}\rho_{\tilde{T}} - \frac{j}{2}\rho_{\tilde{t}} - \frac{j\delta(\tilde{T}-\tilde{t})}{2} - \left(\frac{j^2}{2}+\frac{4}{\sigma^2}\right)\int_{\tilde{t}}^{\tilde{T}} \rho_{\tilde{s}}\,d\tilde{s}\right\}\,\middle|\,\mathcal{F}_{\tilde{t}}\right),
\end{aligned}$$

where we use E to denote the expectation with respect to ${}^{j}P^{\delta}_{\rho_0}$ and $\tilde{E}$ to denote the expectation with respect to $P^{\delta}_{\rho_0}$. Also, we recall that $\delta = \frac{4k\theta}{\sigma^2}$ and $j = -\frac{2k}{\sigma^2}$. Now we define

$$\frac{b^2}{2} = \frac{j^2}{2} + \frac{4}{\sigma^2}$$

to obtain

$$b = \frac{2}{\sigma^2}\sqrt{k^2+2\sigma^2} = \frac{2}{\sigma^2}h$$

and we also set $\lambda = -\frac{j}{2}$. It now follows from Theorem 5.4.2 that

$$\begin{aligned}
&\tilde{E}\left(\exp\left\{\frac{j}{2}\rho_{\tilde{T}} - \frac{j}{2}\rho_{\tilde{t}} - \frac{j\delta(\tilde{T}-\tilde{t})}{2} - \left(\frac{j^2}{2}+\frac{4}{\sigma^2}\right)\int_{\tilde{t}}^{\tilde{T}} \rho_{\tilde{s}}\,d\tilde{s}\right\}\,\middle|\,\mathcal{F}_{\tilde{t}}\right)\\
&\quad = \exp\left\{\frac{k^2\theta(T-t)}{\sigma^2} + r_t\frac{k}{\sigma^2}\right\}
\end{aligned}$$

$$\times \frac{\exp\left\{\dfrac{-r_t \frac{h(1+2\lambda \frac{\coth(b(\tilde{T}-\tilde{t}))\sigma^2}{2\sqrt{k^2+2\sigma^2}})}{\sigma^2(\coth(b(\tilde{T}-\tilde{t}))+2\frac{\lambda\sigma^2}{2\sqrt{k^2+2\sigma^2}})}}{\coth(b(\tilde{T}-\tilde{t}))+2\lambda\frac{\sigma^2}{2\sqrt{k^2+2\sigma^2}}}\right\}}{(\cosh(b(\tilde{T}-\tilde{t}))+2\lambda b^{-1}\sinh(b(\tilde{T}-\tilde{t})))^{\delta/2}}.$$

It can be checked that

$$\frac{\exp\{\frac{k^2\theta(T-t)}{\sigma^2}\}}{(\cosh(b(\tilde{T}-\tilde{t}))+2\lambda b^{-1}\sinh(b(\tilde{T}-\tilde{t})))^{\delta/2}} = A(t,T).$$

Finally,

$$\exp\left\{-r_t\left(\frac{h}{\sigma^2}\left(\frac{1+\frac{k}{h}\coth(b(\tilde{T}-\tilde{t}))}{\coth(b(\tilde{T}-\tilde{t}))+\frac{k}{h}}\right)-\frac{k}{\sigma^2}\right)\right\} = \exp\{-r_t B(t,T)\},$$

is completing the proof. □

Next, we discuss zero coupon bond pricing in the 3/2 model. However, we point out that these techniques are also useful when studying volatility derivatives, see e.g. Carr and Sun (2007). We recall the 3/2 process from Sect. 3.1, which is given by

$$dr_t = \kappa r_t(\theta - r_t)\,dt + \sigma r_t^{3/2}\,dW_t,$$

where $r_0 > 0$. Consequently, we are interested in computing

$$E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right),$$

where $\mathcal{A}_t = \sigma\{r_s,\, s \le t\}$. Now, we define $v_t = \frac{1}{r_t}$, and obtain by Itô's formula

$$dv_t = \left(\kappa + \sigma^2 - \kappa\theta v_t\right)dt - \sigma\sqrt{v_t}\,dW_t.$$

Since $v_t = \frac{1}{r_t}$, we have

$$E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right) = E\left(\exp\left\{-\int_t^T \frac{ds}{v_s}\right\}\,\middle|\,\mathcal{A}_t\right).$$

We now simply use Proposition 5.4.3 to yield

$$E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right)$$
$$= \frac{\Gamma(k+\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\nu+1)}\beta r_t^k \exp\left\{\frac{b}{\sigma^2}\left(a\tau + r_t^{-1} - \frac{r_t^{-1}}{\tanh(b\tau/2)}\right)\right\}$$
$$\times \frac{\exp\{\beta^2/(2\alpha)\}}{\beta\alpha^k} M_{-k,\nu/2}\left(\frac{\beta^2}{\alpha}\right),$$

where $\nu = \frac{2}{\sigma^2}\sqrt{(\kappa + \sigma^2 - \frac{\sigma^2}{2})^2 + 2\sigma^2}$, $k = \frac{\kappa+\sigma^2}{\sigma^2}$, $\alpha = \frac{\kappa\theta}{\sigma^2}(1 + \coth(\frac{\kappa\theta\tau}{2}))$, $\beta = \frac{\kappa\theta v_t^{-\frac{1}{2}}}{\sigma^2 \sinh(\frac{\kappa\theta\tau}{2})}$, $a = \kappa + \sigma^2$, and $b = \kappa\theta$. This result can be shown to match Theorem 3 in Carr and Sun (2007).

Note that similar calculations yield corresponding results for other diffusion processes captured in Chap. 4.