Chapter 2
Functionals of Wiener Processes

In this chapter, we discuss scalar- and multidimensional processes, which are based on the Wiener process, and consequently apply them in the context of the benchmark approach.

2.1 One-Dimensional Functionals of Wiener Processes

We summarize well-known SDEs and transition densities for models and processes closely related to the Wiener process or Brownian motion, including:

- the Bachelier model;
- the Black-Scholes model;
- the Ornstein-Uhlenbeck-process;
- the geometric Ornstein-Uhlenbeck-process.

Also we collect results from the literature on *functionals* of *Wiener processes* and add new results and presentations. We remark that parts of this section are based on Borodin and Salminen (2002), Jeanblanc et al. (2009), Chap. 3, and Platen and Heath (2010), Chap. 4.

2.1.1 Wiener Process

The Wiener process is a continuous Markov process and has the following transition density:

$$p(s,x;t,y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(y-x)^2}{2(t-s)}\right\}, \tag{2.1.1}$$

for $t \in [0,\infty)$, $s \in [0,t]$ and $x, y \in \Re$. For the purpose of illustration, we display some transition densities in Fig. 2.1.1 as functions of time t and final value y, where we set the initial time to $s = 0$ and the initial value to $x = 0$.

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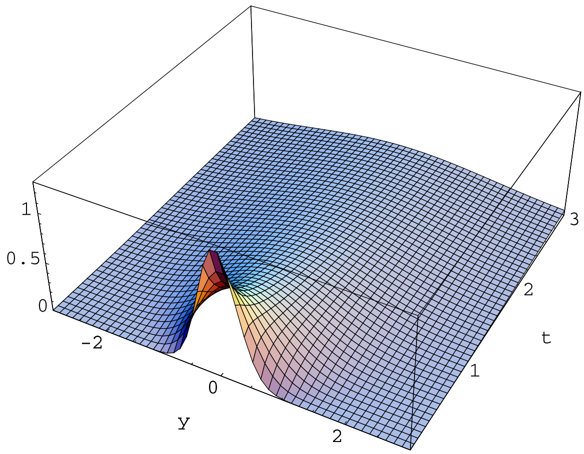


Fig. 2.1.1 Probability densities for the standard Wiener process

The Wiener process enjoys the *strong Markov property*, which allows us to formulate the following lemma:

Lemma 2.1.1 *For a finite stopping time* τ, *the process* $\tilde{W} = \{\tilde{W}_t,\ t \geq 0\}$, *where*

$$\tilde{W}_t = W_{\tau+t} - W_\tau, \tag{2.1.2}$$

is a Wiener process with respect to its natural filtration.

We now introduce the following notation

$$T_a = \inf\{t \geq 0:\ W_t = a\}$$
$$M_t = \sup_{0 \leq s \leq t} W_s$$
$$m_t = \inf_{0 \leq s \leq t} W_s.$$

The following proposition, commonly referred to as reflection principle, employs Lemma 2.1.1 and the symmetry of the Wiener process, see Lemma 15.1.3.

Proposition 2.1.2 *Let* $y \geq 0$, $x \leq y$, *then one has*

$$P(W_t \leq x,\ M_t \geq y) = P(W_t \geq 2y - x). \tag{2.1.3}$$

For a proof, see e.g. Jeanblanc et al. (2009), Proposition 3.1.1.1. Next, we discuss the joint distribution of (M_t, W_t), see Theorem 3.1.1.2 in Jeanblanc et al. (2009).

Proposition 2.1.3 *For a Brownian motion* W_t *and its running maximum* M_t, *the following formulas hold*:

$$P(W_t \leq x, M_t \leq y) = N\left(\frac{x}{\sqrt{t}}\right) - N\left(\frac{x-2y}{\sqrt{t}}\right), \quad y \geq 0,\ x \leq y,$$

$$P(W_t \le x, M_t \le y) = P(M_t \le y) = N\left(\frac{y}{\sqrt{t}}\right) - N\left(\frac{-y}{\sqrt{t}}\right), \quad y \ge 0,\ x \ge y,$$
$$P(W_t \le x, M_t \le y) = 0, \quad y \le 0.$$

The distribution of (W_t, M_t) *is given by*

$$P(W_t \in dx,\ M_t \in dy) = \mathbf{1}_{y\ge 0}\mathbf{1}_{x\le y}\frac{2(2y-x)}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(2y-x)^2}{2t}\right\}dx\,dy.$$

The law of the maximum satisfies the following equality, see Proposition 3.1.3.1 in Jeanblanc et al. (2009),

$$P(M_t \le y) = N\left(\frac{y}{\sqrt{t}}\right) - N\left(\frac{-y}{\sqrt{t}}\right), \quad y \ge 0.$$

We remark that the law of the maximum of a process finds important applications in derivative pricing, see Sect. 2.3.

Proposition 2.1.4 *For a Brownian motion* W_t *and its running minimum* m_t, *the following formulas hold*:

$$P(W_t \ge x,\ m_t \ge y) = N\left(\frac{-x}{\sqrt{t}}\right) - N\left(\frac{2y-x}{\sqrt{t}}\right), \quad y \le 0,\ x \ge y$$
$$P(W_t \ge x,\ m_t \ge y) = N\left(\frac{-y}{\sqrt{t}}\right) - N\left(\frac{y}{\sqrt{t}}\right), \quad y \le 0,\ x \le y$$
$$P(W_t \ge x,\ m_t \ge y) = 0, \quad y \ge 0.$$

The law of the minimum satisfies, for $y \le 0$,

$$P(m_t \ge y) = N\left(\frac{-y}{\sqrt{t}}\right) - N\left(\frac{y}{\sqrt{t}}\right).$$

Finally, we turn to hitting times, which are also used in derivative pricing, for example when studying rebates, see Sect. 2.3.

Proposition 2.1.5 *Let* T_y *be the first hitting time of* $y \in \Re$ *for a standard Brownian motion. Then for* $\lambda > 0$,

$$E\left(\exp\left\{-\frac{\lambda^2}{2}T_y\right\}\right) = \exp\{-|y|\lambda\}.$$

We can also compute the density

$$P(T_y \in dt) = \frac{x}{\sqrt{2\pi t^3}}\exp\left\{-\frac{x^2}{2t}\right\}\mathbf{1}_{t\ge 0}\,dt.$$

This section concludes with results on integrals of Brownian motion, taken from Borodin and Salminen (2002). Such formulas are useful when studying Asian options and related contracts, such as Australian options, see Sect. 2.3:

$$P\left(\int_0^t W_s\,ds \in dy\right) = \frac{\sqrt{3}}{\sqrt{2\pi t^3}}\exp\left\{-\frac{3y^2}{2t^3}\right\}dy$$

$$P\left(\int_0^t W_s\,ds \in dy,\ W_t \in dz\right) = \frac{\sqrt{3}}{\pi t^2}\exp\left\{-\frac{z^2}{2t} - \frac{3(2y-zt)^2}{2t^3}\right\}dy\,dz.$$

2.1.2 Bachelier Model

The results in the previous section can be extended to the case

$$X_t = \nu t + W_t, \quad t \geq 0,$$

a Brownian motion with drift. This process corresponds to the *Bachelier model*, which models the stock price S_t via

$$S_t = S_0 + \mu t + \sigma W_t, \quad t \geq 0,$$

see Bachelier (1900). Again, we employ the notation

$$T_a = \inf\{t \geq 0\colon\ X_t = a\}$$
$$M_t = \sup_{0\leq s\leq t} X_s$$
$$m_t = \inf_{0\leq s\leq t} X_s.$$

We start our discussion with the transition density of the process X,

$$p(s,x;t,y) = \frac{1}{\sqrt{2\pi(t-s)}}\exp\left\{-\frac{(y-x-\nu(t-s))^2}{2(t-s)}\right\}, \tag{2.1.4}$$

for $t \in [0,\infty)$, $s \in [0,t]$ and $x, y \in \Re$. The following result corresponds to Proposition 2.1.3 and uses Proposition 3.2.1.1 and Corollary 3.2.1.2 from Jeanblanc et al. (2009).

Proposition 2.1.6 *For a Brownian motion with drift X_t and its running maximum M_t, the following formulas hold*:

$$P(X_t \leq x, M_t \leq y) = N\left(\frac{x-\nu t}{\sqrt{t}}\right) - \exp\{2\nu y\}N\left(\frac{x-2y-\nu t}{\sqrt{t}}\right), \quad y \geq 0,\ x \leq y.$$

The density of (W_t, M_t) is given by

$$P(X_t \in dx,\ M_t \in dy)$$
$$= \mathbf{1}_{x<y}\mathbf{1}_{0<y}\frac{2(2y-x)}{\sqrt{2\pi t^3}}\exp\left\{\nu x - \frac{1}{2}\nu^2 t - \frac{(2y-x)^2}{2t}\right\}dx\,dy.$$

Furthermore, the law of the maximum satisfies

$$P(M_t \leq y) = N\left(\frac{y-\nu t}{\sqrt{t}}\right) - \exp\{2\nu y\}N\left(\frac{-y-\nu t}{\sqrt{t}}\right), \quad y \geq 0.$$

Next, we present results corresponding to Proposition 2.1.4.

Proposition 2.1.7 *For a Brownian motion with drift X_t and its running minimum m_t, the following formulas hold*:

$$P(X_t \geq x,\ m_t \geq y) = N\left(\frac{-x+\nu t}{\sqrt{t}}\right) - \exp\{2\nu y\} N\left(\frac{-x+2y+\nu t}{\sqrt{t}}\right).$$

Furthermore, the law of the minimum is given by

$$P(m_t \geq y) = N\left(\frac{-y+\nu t}{\sqrt{t}}\right) - \exp\{2\nu y\} N\left(\frac{y+\nu t}{\sqrt{t}}\right), \quad y \leq 0.$$

We now turn to hitting times, see Eq. (3.2.3) in Jeanblanc et al. (2009).

Proposition 2.1.8 *Let T_y be the first hitting time of the level y for a Brownian motion with drift. Then*

$$P(T_y \in dt) = \frac{|y|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{1}{2t}(y-\nu t)^2\right\} \mathbf{1}_{t \geq 0}\, dt.$$

Furthermore,

$$E\left(\exp\left\{-\frac{\lambda^2}{2} T_y\right\}\right) = \exp\{\nu y\} \exp\left\{-|y|\sqrt{\nu^2+\lambda^2}\right\}.$$

Results on integrals of Brownian motion with drift can be found in Borodin and Salminen (2002), see Eqs. (1.8.4) and (1.8.8) in their Appendix 1,

$$P\left(\int_0^t X_s\, ds \in dy\right) = \frac{\sqrt{3}}{\sqrt{2\pi t^3}} \exp\left\{-\frac{3(y-\nu t^2/2)^2}{2t^3}\right\} dy$$

$$P\left(\int_0^t X_s\, ds \in dy, X_t \in dz\right) = \frac{\sqrt{3}}{\pi t^2} \exp\left\{-\frac{(z-\nu t)^2}{2t} - \frac{3(2y-zt)^2}{2t^3}\right\} dy\, dz.$$

We now derive the transition density of a Brownian motion with drift killed at $z \in \Re$. To do so, we firstly recall Lemma 2.1 from Hulley and Platen (2008), which requires us to introduce the following notation: let $Y = \{Y_t,\ t \geq 0\}$ be a regular one-dimensional time-homogeneous diffusion process, whose state space is an interval $I \subseteq \Re$, which is typically $\Re$, $[0, \infty)$ or $(0, \infty)$ and which starts at $x \in I$. We shall denote the transition density of Y with respect to its speed measure by $q(.,.,.)$, where we omit the dependence on the initial time $s = 0$, so that

$$P(Y_t \in A) = \int_A q(t, x, y) m(y)\, dy,$$

for all $t \geq 0$ and $x \in I$ and for every Borel set $A \in \mathcal{B}(I)$. Furthermore, for any $z \in I$, let

$$T_z^Y := \inf\{t > 0\colon\ Y_t = z\}$$

be the first-passage time of Y to z. We shall denote its density with respect to the Lebesgue measure by $p_z(.,.)$, so that

$$P\left(T_z^Y \leq t\right) = \int_0^t p_z(x, s)\, ds.$$

Furthermore, let $\tilde{q}_z(.,.,.)$ denote its transition density, with respect to the speed measure of Y killed at z, so that

$$P\big(Y_t \in A,\, T_z^Y > t\big) = \int_A \tilde{q}_z(t,x,y)m(y)\,dy,$$

for all $A \in \mathcal{B}(I)$. We are now in a position to state Lemma 2.1 from Hulley and Platen (2008):

Lemma 2.1.9 *Let $x, y, z \in I$ and suppose that $t > 0$. Then*

$$q(t,x,y) = \tilde{q}_z(t,x,y) + \int_0^t p_z(x,s)q(t-s,z,y)\,ds. \tag{2.1.5}$$

Intuitively speaking, the first term in (2.1.5) corresponds to those trajectories which travel from x to y without visiting z, whereas the second includes those trajectories which do visit z between 0 and t. We now use Lemma 2.1.9 to derive the density of a Brownian motion with drift started at x killed at z. We remark that this density will be employed in the pricing of Barrier options under the Black-Scholes model in Sect. 2.3. From Borodin and Salminen (2002), we obtain for a Brownian motion with drift

$$X_t = \nu t + W_t$$

started at x and

$$T_a = \inf\{t > 0\colon\, X_t = a\}$$

that

$$q(t,x,y) = \frac{1}{2\sqrt{2\pi t}} \exp\left\{-\mu(y+x) - \frac{\mu^2 t}{2} - \frac{(x-y)^2}{2t}\right\} \tag{2.1.6}$$

and

$$p_z(x,t) = \frac{|z-x|}{\sqrt{2\pi} t^{3/2}} \exp\left\{-\frac{(z-x-\mu t)^2}{2t}\right\} \tag{2.1.7}$$

and hence the following corollary:

Corollary 2.1.10 *For a Brownian motion with drift $X = \{X_t, t \geq 0\}$ started at x, we have*

$$\tilde{q}_z(t,x,y) = \begin{cases} \frac{1}{2\sqrt{2\pi t}} \exp\{-\mu(x+y) - \frac{\mu^2 t}{2}\} & \\ \quad \times \big(\exp\{-\frac{(x-y)^2}{2t}\} - \exp\{-\frac{(x+y-2z)^2}{2t}\}\big) & y, x > z \\ 0 & y < z \leq x \end{cases} \tag{2.1.8}$$

and

$$\tilde{q}_z(t,x,y) = \begin{cases} 0 & y > z > x \\ \frac{1}{2\sqrt{2\pi t}} \exp\{-\mu(x+y) - \frac{\mu^2 t}{2}\} \\ \quad \times \left(\exp\{-\frac{(x-y)^2}{2t}\} - \exp\{-\frac{(x+y-2z)^2}{2t}\}\right) & x, y < z. \end{cases} \tag{2.1.9}$$

Proof Assume that $x, y > z$ or $x, y < z$, then from Lemma 2.1.9, we need to compute

$$\begin{aligned}
&\int_0^t p_z(x,s) q(t-s,z,y)\, ds \\
&\quad = \int_0^t \frac{|z-x|}{\sqrt{2\pi} t^{3/2}} \exp\left\{-\frac{(z-x-\mu s)^2}{2s}\right\} \frac{1}{2\sqrt{2\pi(t-s)}} \\
&\qquad \times \exp\left\{-\mu(y+x) - \frac{\mu^2(t-s)}{2} - \frac{(z-y)^2}{2(t-s)}\right\} ds \\
&\quad = \frac{|z-x|}{2(2\pi)} \exp\left\{-\mu(x+y) - \frac{\mu^2 t}{2}\right\} \int_0^t \frac{\exp\{-\frac{(z-x)^2}{2s} - \frac{(z-y)^2}{2(t-s)}\}}{s^{3/2}\sqrt{t-s}}\, ds.
\end{aligned}$$

Noting that for $y, x > z$ and $x, y < z$ we have $\frac{z-x}{z-y} > 0$, we employ the following change of variables

$$\sqrt{t/s - 1}\sqrt{\frac{z-x}{z-y}} = \xi,$$

to obtain

$$\begin{aligned}
&\int_0^t p_z(x,s) q(t-s,z,y)\, ds \\
&\quad = \frac{|z-x|}{2\pi} \exp\left\{-\mu(x+y) - \frac{\mu^2 t}{2} - \frac{((z-y)^2 + (z-x)^2)^2}{2t}\right\} \frac{1}{t}\sqrt{\frac{z-y}{z-x}} \\
&\qquad \times \int_0^\infty \exp -\frac{1}{2}\left(\frac{1}{\xi^2} + \xi^2\right) \frac{(z-x)(z-y)}{t}\, d\xi \\
&\quad = \frac{|z-x|}{2\pi} \exp\left\{-\mu(x+y) - \frac{((z-y)^2 + (z-x)^2)}{2t}\right\} \frac{1}{t}\sqrt{\frac{z-y}{z-x}} \\
&\qquad \times \exp\left\{-\frac{(z-x)(z-y)}{t}\right\} \sqrt{\frac{\pi}{2}} \sqrt{\frac{t}{(z-x)(z-y)}} \\
&\quad = \frac{1}{2\sqrt{2\pi t}} \exp\left\{-\mu(x+y) - \frac{\mu^2 t}{2} - \frac{((z-y) + (z-x))^2}{2t}\right\},
\end{aligned}$$

where we used *MATHEMATICA* to arrive at the second last equation. Consequently,

$$\tilde{q}_z(t,x,y) = \frac{1}{2\sqrt{2\pi t}} \exp\left\{-\mu(x+y) - \frac{\mu^2 t}{2}\right\}$$

$$\times \left(\exp\left\{ -\frac{(x-y)^2}{2t} \right\} - \exp\left\{ -\frac{(2z-x-y)^2}{2t} \right\} \right)$$

for $x, y > z$ and $x, y < z$. □

Now, we focus on occupation times, firstly deriving the result for standard Brownian motion and subsequently for Brownian motion with drift. Occupation times measure the amount of time a stochastic process spends above or below a particular level. They have important applications in finance, as there are products whose pay-offs depend on the amount of time the asset price spends above or below a particular barrier. We are particularly interested in obtaining the distribution of occupation times explicitly. The approach to obtain such distributions we present here is based on Jeanblanc et al. (2009) and is motivated by the following result, see Theorem 2.5.1.1 in Jeanblanc et al. (2009): for convenience, we use E_x to denote the expectation with respect to the probability distribution of a Brownian motion started at x.

Theorem 2.1.11 *Let $\alpha \in \Re^+$ and let $k : \Re \to \Re^+$ and $g : \Re \to \Re$ be continuous functions and let g be bounded. Then the function*

$$f(x) = E_x\left(\int_0^\infty g(W_t) \exp\left\{ -\alpha t - \int_0^t k(W_s)\, ds \right\} dt \right) \tag{2.1.10}$$

is piecewise twice differentiable and satisfies the differential equation

$$(\alpha + k) f = \frac{1}{2} f'' + g. \tag{2.1.11}$$

We firstly consider $A_t^+ := \int_0^t \mathbf{1}_{[0,\infty)}(W_s)\, ds$, which measures the amount of time the standard Brownian motion $W = \{W_t, t \in [0,\infty)\}$ spends above 0 during the time interval $[0, t]$. Consider an exponentially distributed random variable τ, $\tau \sim \text{Exp}(\lambda)$, which is independent of W. Clearly,

$$E_x\left(\exp\left\{ -\beta A_\tau^+ \right\} \right) = \lambda f(x),$$

where

$$f(x) := E_x\left(\int_0^\infty \exp\left\{ -\alpha t - \beta \int_0^t \mathbf{1}_{[0,\infty)}(W_s)\, ds \right\} dt \right).$$

However, $f(x)$ can be interpreted as a double Laplace transform of the density of A_τ^+, with respect to occupation time and the upper limit of the time interval. Inversion of the double Laplace transform will provide us with the desired density. Theorem 2.1.11 provides us with a useful expression for f, which can be inverted, if necessary numerically. To illustrate the technique used to obtain the distribution of occupation times, we present below the proof of the next result, see also Proposition 2.5.2.1 in Jeanblanc et al. (2009).

Proposition 2.1.12 *The law of $A_t^+ := \int_0^t \mathbf{1}_{[0,\infty)}(W_s)\, ds$ is given by*

$$P\left(A_t^+ \in ds \right) = \frac{ds}{\pi \sqrt{s(t-s)}} \mathbf{1}_{0 \le s < t}.$$

Proof We set $k(x) = \beta \mathbf{1}_{x\geq 0}$ and $g(x) = 1$ in Theorem 2.1.11. Then we obtain

$$f(x) = E_x\left(\int_0^\infty \exp\left\{-\alpha t - \beta \int_0^t \mathbf{1}_{[0,\infty)}(W_s)\,ds\right\}dt\right),$$

which solves

$$\begin{cases} \alpha f(x) = \dfrac{1}{2} f''(x) - \beta f(x) + 1, & x \geq 0 \\ \alpha f(x) = \dfrac{1}{2} f''(x) + 1, & x \geq 0. \end{cases}$$

In Jeanblanc et al. (2009), an explicit solution for $f(x)$ is obtained. We are particularly interested in the special case

$$f(0) = \int_0^\infty \exp\{-\alpha t\} E_0\left(e^{-\beta A_t^+}\right) dt = \frac{1}{\sqrt{\alpha(\alpha+\beta)}}. \tag{2.1.12}$$

However, we recall

$$\int_0^\infty e^{-\alpha t}\left(\int_0^\infty du\, \mathbf{1}_{s<t} \frac{\exp\{-\beta u\}}{\pi\sqrt{u(t-u)}}\right) dt = \frac{1}{\sqrt{\alpha(\alpha+\beta)}},$$

so we can explicitly invert the double Laplace transform (2.1.12) to complete the proof. □

We remark that the same technique can be used to compute the corresponding result for the occupation time of a Brownian motion with drift. Let $X_t = \nu t + W_t$, and consider the occupation time of this Brownian motion above the level $L > 0$

$$A_t^{+,L,\nu} = \int_0^t \mathbf{1}_{X_s > L}\, ds,$$

and we define $A_t^{-,L,\nu}$ analogously. Using the same idea as before, together with the relevant Feynman-Kac result, we get

$$\begin{aligned} &P\left(A_t^{-,0,\nu} \in du\right) \\ &\quad = \left(\sqrt{\frac{2}{\pi u}} \exp\left\{-\frac{\nu^2}{2} u\right\} - 2\nu \Theta(\nu\sqrt{u})\right) \\ &\qquad \times \left(\nu + \frac{1}{\sqrt{2\pi(t-u)}} \exp\left\{-\frac{\nu^2}{2}(t-u)\right\} - \nu\Theta(\nu\sqrt{t-u})\right), \end{aligned} \tag{2.1.13}$$

where $\Theta(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\{-\frac{y^2}{2}\}\,dy$. Finally,

$$P\left(A_t^{-,L,\nu} \leq u\right) = \int_0^u \varphi(s, L; \nu) P\left(A_{t-s}^{-,0,\nu} < u - s\right) ds,$$

where $\varphi(s, L; \nu)$ is the density $P(T_L \in ds)/ds$, where T_L denotes the first time the Brownian motion with drift hits the level L, $T_L = \inf\{t\colon\ X_t = L\}$, and

$$\varphi(s, L; \nu) = \frac{L}{\sqrt{2\pi s^3}} \exp\left\{-\frac{1}{2s}(y - \nu s)^2\right\} \mathbf{1}_{s\geq 0}.$$

Finally, the law of $A_t^{+,L,\nu}$ follows from $A_t^{+,L,\nu} + A_t^{-,L,\nu} = t$.

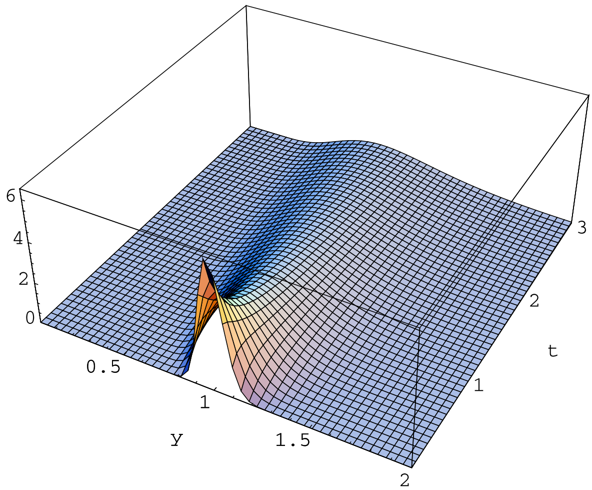


Fig. 2.1.2 Transition density for geometric Brownian motion

2.1.3 Geometric Brownian Motion

Geometric Brownian motion is a process of significant importance in finance, as the Black-Scholes model (BSM), Black and Scholes (1973), is based on it, see also Sect. 2.3. We can describe geometric Brownian motion via the SDE

$$dX_t = X_t\left(\left(g + \frac{1}{2}b^2\right)dt + b\,dW_t\right), \tag{2.1.14}$$

subject to $X_0 > 0$. Equation (2.1.14) can be explicitly solved to yield

$$X_t = X_0 \exp(gt + bW_t).$$

Its transition density function satisfies

$$p(s, x; t, y) = \frac{1}{\sqrt{2\pi(t-s)}by} \exp\left\{-\frac{(\ln(y) - \ln(x) - g(t-s))^2}{2b^2(t-s)}\right\}, \tag{2.1.15}$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $x, y \in (0, \infty)$. Figure 2.1.2 shows the transition density for a geometric Brownian motion with growth rate $g = 0.05$, volatility $b = 0.2$ and initial value $x = 1$ at time $s = 0$ for the period from 0.1 to 3 years.

The corresponding laws of first hitting times, maximum, and minimum follow easily from the corresponding results for a Brownian motion with drift. Regarding the integrals, we have the following result, see Yor (2001) and Pintoux and Privault (2011):

$$\begin{aligned} P&\left(\int_0^t \exp\{\sigma W_s - p\sigma^2 s/2\}\,ds \in du,\, W_t \in dy\right) \\ &= \frac{\sigma}{2}\exp\{-p\sigma y/2 - p^2\sigma^2 t/8\}\exp\left\{-2\frac{1+\exp\{\sigma y\}}{\sigma^2 u}\right\} \\ &\quad \times \theta\left(\frac{4\exp\{\sigma y/2\}}{\sigma^2 u}, \frac{\sigma^2 t}{4}\right)\frac{du}{u}\,dy, \end{aligned}$$

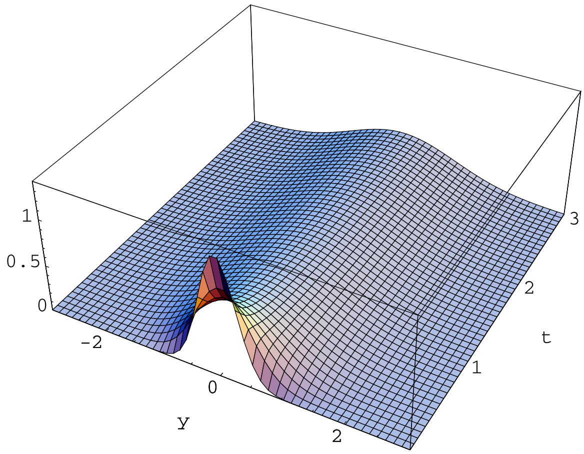


Fig. 2.1.3 Transition density of standard OU process starting at $(s, x) = (0, 0)$

where $p = -\frac{2g}{b^2}$, $u > 0$, $y \in \Re$ and

$$\theta(v, t) = \frac{v \exp\{\frac{\pi^2}{2t}\}}{\sqrt{2\pi^3 t}} \int_0^\infty \exp\left\{-\frac{\xi^2}{2t} - v\cosh(\xi)\right\} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi, \quad v > 0.$$

2.1.4 Ornstein-Uhlenbeck Process

The *Ornstein-Uhlenbeck process* is also a process of importance in finance and forms the basis of the *Vasiček model*, see Vasiček (1977). We consider the standard Ornstein-Uhlenbeck process,

$$dX_t = -X_t\, dt + \sqrt{2}\, dW_t,$$

where $X_0 = x \in \Re$. Its transition density is Gaussian,

$$p(s, x; t, y) = \frac{1}{\sqrt{2\pi(1 - e^{-2(t-s)})}} \exp\left\{-\frac{(y - x e^{-(t-s)})^2}{2(1 - e^{-2(t-s)})}\right\}, \tag{2.1.16}$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $x, y \in \Re$, with mean $x\, e^{-(t-s)}$ and variance $1 - e^{-2(t-s)}$.

To illustrate the stochastic dynamics of this process we show in Fig. 2.1.3 the transition density of a standard OU process for the period from 0.1 to 3 years with initial value $x = 0$ at time $s = 0$. As can be seen from Fig. 2.1.3 the transition densities for the standard OU process seem to stabilize after a period of about one year. In fact, as can be seen from (2.1.16) these transition densities asymptotically approach, as $t \to \infty$, a standard Gaussian density. This is in contrast, for example, to transition densities for the Wiener process, which do not converge to a *stationary density*, see (2.1.1) and Fig. 2.1.1. For illustration, we plot in Fig. 2.1.4 the transition

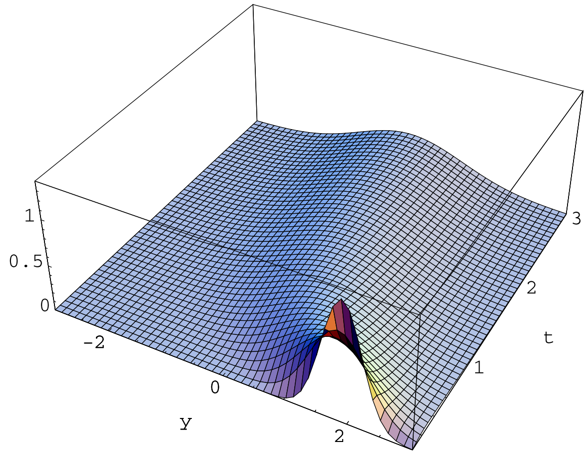


Fig. 2.1.4 Transition density of standard OU process starting at $(s, x) = (0, 2)$

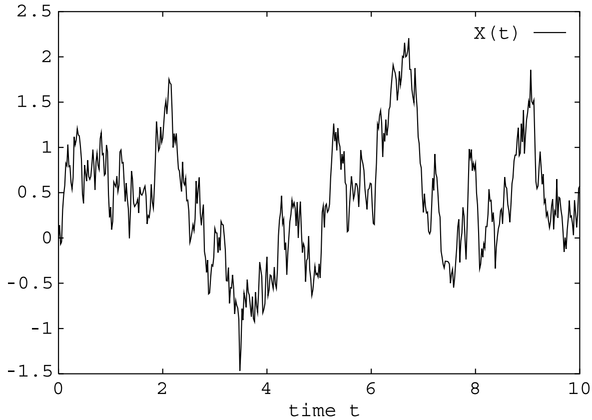


Fig. 2.1.5 Path of a standard Ornstein-Uhlenbeck process

density for a standard OU process that starts at the initial value $x = 2$ at time $t = 0$. Note how the transition density evolves towards a median that is close to 0.

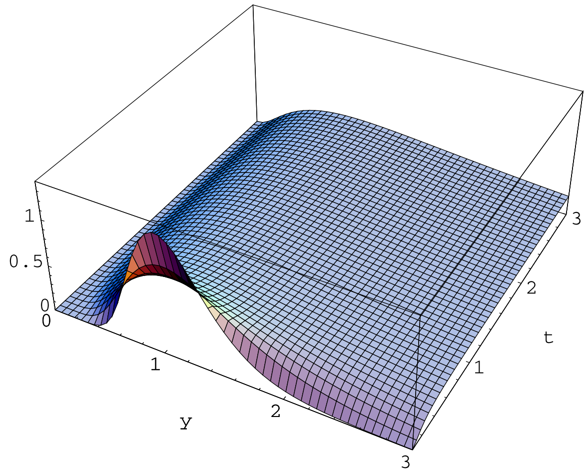
In Fig. 2.1.5 a path of a standard OU process is shown. It can be observed that this trajectory fluctuates around some reference level. Indeed, as already indicated, the standard OU process has a stationary density. This can be seen from (2.1.16) when $t \to \infty$. Note also that the Gaussian property of the standard OU process means that the process itself and even a scaled and shifted OU process may become negative. We now recall Proposition 3.4.1.1 from Jeanblanc et al. (2009), which characterizes the first hitting time of the level 0,

$$T_0 = \inf\{t \geq 0\colon\ X_t = 0\}.$$

Proposition 2.1.13 *The density function of T_0 is given by*

$$f(t) = \frac{x}{2\sqrt{\pi}} \exp\left\{\frac{x^2}{4}\right\} \exp\left\{\frac{1}{2}\left(t - \frac{x^2}{2}\coth(t)\right)\right\} \left(\frac{1}{\sinh(t)}\right)^{\frac{3}{2}}.$$

Fig. 2.1.6 Transition density of a geometric Ornstein-Uhlenbeck process



Furthermore, integrals of the Ornstein-Uhlenbeck process are of importance in finance, as they impact bond prices for example. Defining

$$n(t,T) = \big(1 - \exp\{-(T-t)\}\big),$$

we have that

$$\int_0^T X_s\, ds \sim N\left(n(0,T)X_0, 2\int_0^T n^2(u,T)\, du\right).$$

2.1.5 Geometric Ornstein-Uhlenbeck Process

Exponentiating an Ornstein-Uhlenbeck process, as discussed in the previous subsection, we obtain a geometric Ornstein-Uhlenbeck process. Its transition density is lognormal satisfying

$$p(s,x;t,y) = \frac{1}{y\sqrt{2\pi(1-e^{-2(t-s)})}} \exp\left\{-\frac{(\ln(y) - \ln(x)e^{-(t-s)})^2}{2(1-e^{-2(t-s)})}\right\}, \quad (2.1.17)$$

for $t \in [0,\infty)$, $s \in [0,t]$ and $x, y \in (0,\infty)$. In Fig. 2.1.6 we display the corresponding probability densities for the time period from 0.1 to 3 years with initial value $x = 1$ at the initial time $s = 0$. In this case the transition density converges over time to a limiting lognormal density as stationary density, as can be seen from (2.1.17). Figure 2.1.7 shows a trajectory for the geometric OU process. We note that it stays positive and shows large fluctuations for large values. Since it is the exponential of an Ornstein-Uhlenbeck process, one can use the result on the hitting time from the previous subsection.

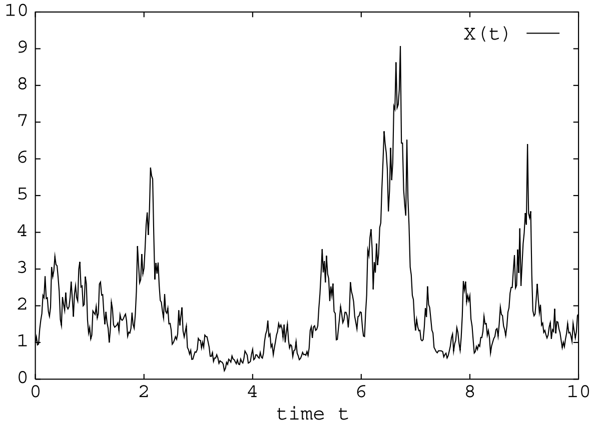


Fig. 2.1.7 Path of a geometric Ornstein-Uhlenbeck process

2.2 Functionals of Multidimensional Wiener Processes

In this section, we discuss functionals of multidimensional Wiener processes or Brownian motions, in particular their SDEs and transition densities. When modeling complex systems, such as a financial market, it is often necessary to employ a multidimensional stochastic process to model the uncertainty. It is crucial to understand the dependence structure between the individual stochastic processes, hence we briefly discuss copulas before discussing stochastic processes.

2.2.1 Copulas

Each multivariate distribution function has its, so called *copula*, which characterizes the dependence structure between the components. Roughly speaking, the copula is the joint density of the components when they are each transformed into uniformly $U(0,1)$ distributed random variables. Essentially, every multivariate distribution has a corresponding copula. Conversely, each copula can be used together with some given marginal distributions to obtain a corresponding multivariate distribution function. This is a consequence of Sklar's theorem, see for instance McNeil et al. (2005).

If, for instance, $\boldsymbol{Y} \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Omega})$ is a Gaussian random vector, then the copula of $\boldsymbol{Y}$ is the same as the copula of $\boldsymbol{X} \sim N_d(\boldsymbol{0}, \boldsymbol{\Omega})$, where $\boldsymbol{0}$ is the zero vector and $\boldsymbol{\Omega}$ is the correlation matrix of $\boldsymbol{Y}$. By the definition of the d-dimensional Gaussian copula we obtain

$$\begin{aligned} C^{Ga}_{\boldsymbol{\Omega}} &= P\big(N(X_1) \le u_1, \ldots, N(X_d) \le u_d\big) \\ &= N_{\Omega}\big(N^{-1}(u_1), \ldots, N^{-1}(u_d)\big), \end{aligned} \tag{2.2.18}$$

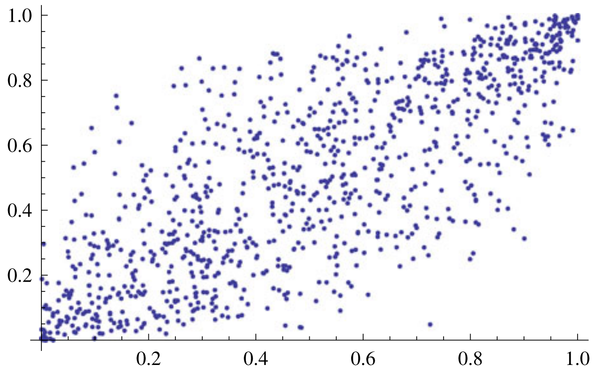


Fig. 2.2.8 Gaussian copula with parameter $\varrho = 0.5$

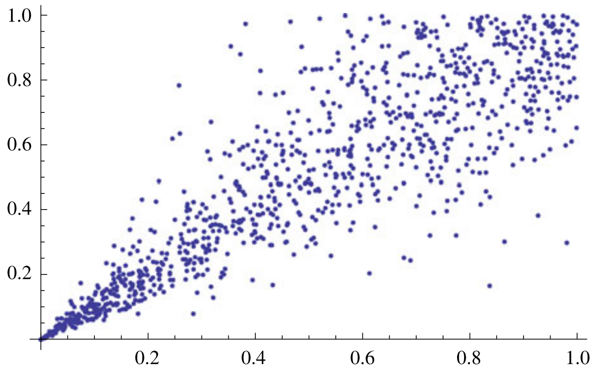


Fig. 2.2.9 Clayton copula with parameter $\theta = 0.5$

where N denotes the standard univariate normal distribution function and N_{Ω} denotes the joint distribution function of $\boldsymbol{X}$. Hence, in two dimensions we obtain

$$C_{\boldsymbol{\Omega}}^{Ga}(u_1, u_2) = \int_{-\infty}^{N^{-1}(u_1)} \int_{-\infty}^{N^{-1}(u_2)} \frac{1}{2\pi(1-\varrho^2)^{1/2}} \times \exp\left\{ \frac{-(s_1^2 - 2\varrho s_1 s_2 + s_2^2)}{2(1-\varrho^2)} \right\} ds_1\, ds_2, \qquad (2.2.19)$$

where $\varrho \in [-1, 1]$ is the correlation parameter in $\boldsymbol{\Omega}$. In Fig. 2.2.8, we simulate from a Gaussian copula with parameter $\varrho = 0.5$.

Another example of a copula is the Clayton copula. This copula can be expressed in the d-dimensional case as

$$C_{\theta}^{Cl} = \left(u_1^{-\theta} + \cdots + u_d^{-\theta} - d + 1\right)^{-1/\theta}, \quad \theta \geq 0, \qquad (2.2.20)$$

where the limiting case $\theta = 0$ is the d-dimensional independence copula. For purposes of comparison, in Fig. 2.2.9, we simulate from a Clayton copula with $\theta = 0.5$. It is evident from Figs. 2.2.8 and 2.2.9, that the Gaussian copula does not allow for tail dependence, whereas the Clayton copula does.

Moreover, d-dimensional Archimedian copulas can be expressed in terms of Laplace-Stieltjes transforms of distribution functions on $\Re^+$. If F is a distribution

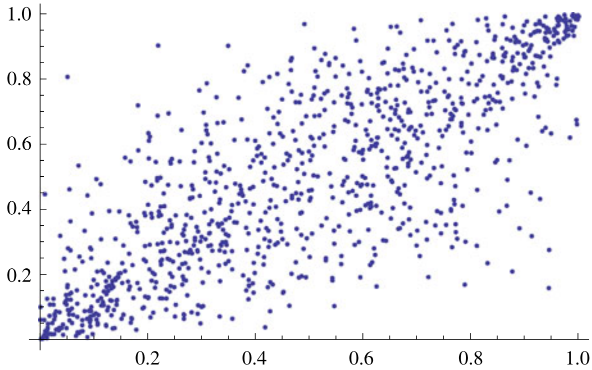


Fig. 2.2.10 Student t copula with four degrees of freedom and $\rho = 0.8$

function on $\Re^+$ satisfying $F(0) = 0$, then the Laplace-Stieltjes transform can be expressed by

$$\hat{F}(t) = \int_0^\infty e^{-tx} \, dF(x), \quad t \geq 0. \tag{2.2.21}$$

Using the Laplace-Stieltjes transform the d-dimensional Archimedian copula has the form

$$C^{Ar}(u_1, \ldots, u_d) = E\left(\exp\left\{-V \sum_{i=1}^{d} \hat{F}^{-1}(u_i)\right\}\right) \tag{2.2.22}$$

for strictly positive random variables V with Laplace-Stieltjes transform $\hat{F}$. We show in Fig. 2.2.10 the Student t copula for four degrees of freedom, which has been shown in Ignatieva et al. (2011) to reflect extremely well the dependence of log-returns of well-diversified indices in different currencies. Compared to Fig. 2.2.8, we notice a marked difference in the tails of the distribution, the Student t copula allows for higher dependence in the extreme values.

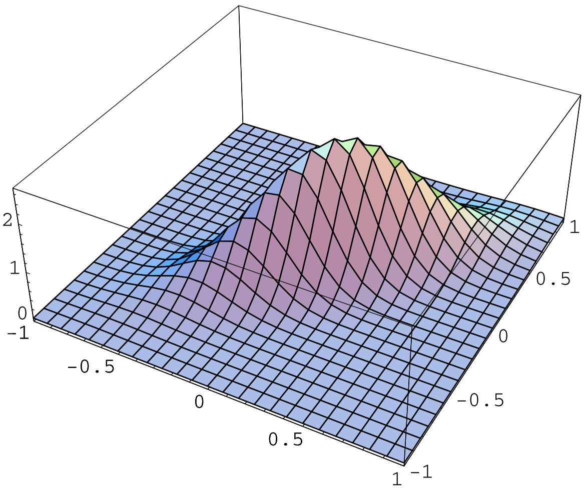
A simulation method follows directly from this representation, see Marshall and Olkin (1988). More examples of multidimensional copulas can be found in McNeil et al. (2005).

Note that each of the following transition densities relate to their own copulas. We will list the transition densities for:

- Multidimensional Wiener processes;
- Multidimensional geometric Brownian motions;
- Multidimensional OU-processes;
- Multidimensional geometric OU-processes.

It is well-known that more analytical results are available for one-dimensional than for multidimensional processes. Hence it is important to have access to the transition densities, so that important functionals can be computed numerically, using e.g. the techniques to be presented in Chap. 12.

Fig. 2.2.11 Bivariate transition density of the two-dimensional Wiener process for fixed time step $\Delta = 0.1$, $x_1 = x_2 = 0.1$ and $\varrho = 0.8$



2.2.2 Multidimensional Wiener Process

As a first example of a continuous multidimensional stochastic process, whose transition density can be expressed explicitly, we focus on the d-dimensional Wiener process. This fundamental stochastic process has a multivariate Gaussian transition density of the form

$$p(s, \boldsymbol{x}; t, \boldsymbol{y}) = \frac{1}{(2\pi(t-s))^{d/2}\sqrt{\det \boldsymbol{\Sigma}}} \exp\left\{\frac{(\boldsymbol{y}-\boldsymbol{x})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{x})}{2(t-s)}\right\}, \qquad (2.2.23)$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $\boldsymbol{x}, \boldsymbol{y} \in \Re^d$. Here $\boldsymbol{\Sigma}$ is a normalized covariance matrix. Its copula is the Gaussian copula (2.2.18), which is simply derived from the corresponding multivariate Gaussian distribution function. In the bivariate case with correlated Wiener processes this transition probability density simplifies to

$$\begin{aligned} p(s, x_1, x_2; t, y_1, y_2) &= \frac{1}{2\pi(t-s)\sqrt{1-\varrho^2}} \\ &\quad \times \exp\left\{-\frac{(y_1-x_1)^2 - 2(y_1-x_1)(y_2-x_2)\varrho + (y_2-x_2)^2}{2(t-s)(1-\varrho^2)}\right\}, \end{aligned} \qquad (2.2.24)$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $x_1, x_2, y_1, y_2 \in \Re$. Here the correlation parameter ϱ varies in the interval $[-1, 1]$. In the case of correlated Wiener processes one can first simulate independent Wiener processes and then form from these, by linear transforms, correlated ones.

In Fig. 2.2.11 we illustrate the bivariate transition density of the two-dimensional Wiener process for the time increment $\Delta = t - s = 0.1$, initial values $x_1 = x_2 = 0.1$ and correlation $\varrho = 0.8$. One can also generate dependent Wiener processes that have a joint distribution with a given copula.

2.2.3 Transition Density of a Multidimensional Geometric Brownian Motion

The multidimensional geometric Brownian motion is a componentwise exponential of linearly transformed Wiener processes. Given a vector of correlated Wiener processes $\boldsymbol{W}$ with the transition density (2.2.23) we consider the following transformation

$$\boldsymbol{S}_t = \boldsymbol{S}_0 \exp\{\boldsymbol{a}t + \boldsymbol{B}\boldsymbol{W}_t\}, \tag{2.2.25}$$

for $t \in [0, \infty)$, where the exponential is taken componentwise. Here $\boldsymbol{a}$ is a vector of length d, while the elements of the matrix $\boldsymbol{B}$ are as follows

$$B^{i,j} = \begin{cases} b^j & \text{for } i = j \\ 0 & \text{otherwise,} \end{cases} \tag{2.2.26}$$

where $i, j \in \{1, 2, \ldots, d\}$. Then the transition density of the above defined geometric Brownian motion has the following form

$$\begin{aligned} p(s, \boldsymbol{x}; t, \boldsymbol{y}) &= \frac{1}{(2\pi(t-s))^{d/2}\sqrt{\det \boldsymbol{\Sigma}}\prod_{i=1}^{d} b^i y_i} \\ &\quad \times \exp\left\{-\frac{(\ln(\boldsymbol{y}) - \ln(\boldsymbol{x}) - \boldsymbol{a}(t-s))^\top \boldsymbol{B}^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{-1}}{2}\right. \\ &\quad \left. \times \frac{(\ln(\boldsymbol{y}) - \ln(\boldsymbol{x}) - \boldsymbol{a}(t-s))}{t-s}\right\} \end{aligned} \tag{2.2.27}$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $\boldsymbol{x}, \boldsymbol{y} \in \Re_+^d$. Here the logarithm is understood componentwise. In the bivariate case this transition density takes the particular form

$$\begin{aligned} &p(s, x_1, x_2; t, y_1, y_2) \\ &\quad = \frac{1}{2\pi(t-s)\sqrt{1-\varrho^2}\, b^1 b^2 y_1 y_2} \\ &\qquad \times \exp\left\{-\frac{(\ln(y_1) - \ln(x_1) - a^1(t-s))^2}{2(b^1)^2(t-s)(1-\varrho^2)}\right\} \\ &\qquad \times \exp\left\{-\frac{(\ln(y_2) - \ln(x_2) - a^2(t-s))^2}{2(b^2)^2(t-s)(1-\varrho^2)}\right\} \\ &\qquad \times \exp\left\{\frac{(\ln(y_1) - \ln(x_1) - a^1(t-s))(\ln(y_2) - \ln(x_2) - a^2(t-s))\varrho}{b^1 b^2 (t-s)(1-\varrho^2)}\right\}, \end{aligned}$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $x_1, x_2, y_1, y_2 \in \Re^+$, where $\varrho \in [-1, 1]$.

In Fig. 2.2.12 we illustrate the bivariate transition density of the two-dimensional geometric Brownian motion for the time increment $\Delta = t - s = 0.1$, initial values $x_1 = x_2 = 0.1$, correlation $\varrho = 0.8$, volatilities $b^1 = b^2 = 2$ and growth parameters $a^1 = a^2 = 0.1$.

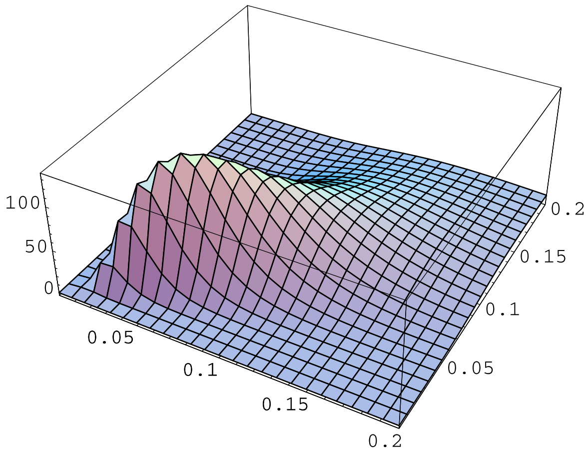


Fig. 2.2.12 Bivariate transition density of the two-dimensional geometric Brownian motion for $\Delta = 0.1$, $x_1 = x_2 = 0.1$, $\varrho = 0.8$, $b^1 = b^2 = 2$ and $a_1 = a_2 = 0.1$

2.2.4 Transition Density of a Multidimensional OU-Process

Another example is the standard d-dimensional *Ornstein-Uhlenbeck* (OU)*-process*. This process has a Gaussian transition density of the form

$$p(s, \boldsymbol{x}; t, \boldsymbol{y}) = \frac{1}{(2\pi(1 - e^{-2(t-s)}))^{d/2}\sqrt{\det \boldsymbol{\Sigma}}} \times \exp\left\{-\frac{(\boldsymbol{y} - \boldsymbol{x}e^{-(t-s)})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{y} - \boldsymbol{x}e^{-(t-s)})}{2(1 - e^{-2(t-s)})}\right\}, \quad (2.2.28)$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $\boldsymbol{x}, \boldsymbol{y} \in \Re^d$, with mean $\boldsymbol{x}e^{-(t-s)}$ and covariance matrix $\boldsymbol{\Sigma}(1 - e^{-2(t-s)})$, $d \in \{1, 2, \ldots\}$, see e.g. Platen and Bruti-Liberati (2010). In the bivariate case the transition density of the standard OU-process is expressed by

$$p(s, x_1, x_2; t, y_1, y_2) = \frac{1}{2\pi(1 - e^{-2(t-s)})\sqrt{1 - \varrho^2}} \times \exp\left\{-\frac{(y_1 - x_1e^{-(t-s)})^2 + (y_2 - x_2e^{-(t-s)})^2}{2(1 - e^{-2(t-s)})(1 - \varrho^2)}\right\} \times \exp\left\{\frac{(y_1 - x_1e^{-(t-s)})(y_2 - x_2e^{-(t-s)})\varrho}{(1 - e^{-2(t-s)})(1 - \varrho^2)}\right\}, \quad (2.2.29)$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $x_1, x_2, y_1, y_2 \in \Re$, where $\varrho \in [-1, 1]$.

2.2.5 Transition Density of a Multidimensional Geometric OU-Process

The transition density of a d-dimensional *geometric OU-process* can be obtained from the transition density of the multidimensional OU-process by applying the exponential transformation. Therefore, it can be expressed as

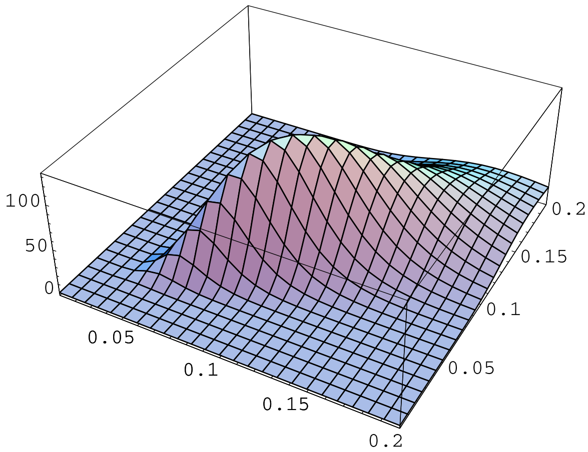


Fig. 2.2.13 Bivariate transition density of the two-dimensional geometric OU-process for $\Delta = 0.1$, $x_1 = x_2 = 0.1$ and $\varrho = 0.8$

$$p(s, \boldsymbol{x}; t, \boldsymbol{y}) = \frac{1}{(2\pi(1 - e^{-2(t-s)}))^{d/2}\sqrt{\det \boldsymbol{\Sigma}}\prod_{i=1}^{d} y_i} \times \exp\left\{-\frac{(\ln(\boldsymbol{y}) - \ln(\boldsymbol{x})e^{-(t-s)})^\top \boldsymbol{\Sigma}^{-1}(\ln(\boldsymbol{y}) - \ln(\boldsymbol{x})e^{-(t-s)})}{2(1 - e^{-2(t-s)})}\right\}, \tag{2.2.30}$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $\boldsymbol{x}, \boldsymbol{y} \in \Re_+^d$, $d \in \{1, 2, \ldots\}$. In the bivariate case the transition density of the multidimensional geometric OU-process is of the form

$$\begin{aligned} p(s, x_1, x_2; t, y_1, y_2) &= \frac{1}{2\pi(1 - e^{-2(t-s)})\sqrt{1 - \varrho^2}\, y_1 y_2} \\ &\quad \times \exp\left\{-\frac{(\ln(y_1) - \ln(x_1)e^{-(t-s)})^2 + (\ln(y_2) - \ln(x_2)e^{-(t-s)})^2}{2(1 - e^{-2(t-s)})(1 - \varrho^2)}\right\} \\ &\quad \times \exp\left\{\frac{(\ln(y_1) - \ln(x_1)e^{-(t-s)})(\ln(y_2) - \ln(x_2)e^{-(t-s)})\varrho}{(1 - e^{-2(t-s)})(1 - \varrho^2)}\right\}, \end{aligned} \tag{2.2.31}$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $x_1, x_2, y_1, y_2 \in \Re_+$, where $\varrho \in [-1, 1]$.

In Fig. 2.2.13 we illustrate the bivariate transition density of the two-dimensional geometric OU-process for the time increment $\Delta = t - s = 0.1$, initial values $x_1 = x_2 = 0.1$ and correlation $\varrho = 0.8$. It is now obvious how to obtain the transition density of the componentwise exponential of other Gaussian vector processes.

2.3 Real World Pricing Under the Black-Scholes Model

In this section, we continue to discuss a continuous financial market as introduced in Chap. 1. We illustrate real world pricing under the benchmark approach using the Black-Scholes model (BSM), see Black and Scholes (1973) and Sect. 1.1. The resulting explicit formulas are of importance not only for the BSM but also for more general models when used in variance reduction techniques, see Platen and Bruti-Liberati (2010). In addition, we illustrate that real world pricing does, in fact, recover the well-known risk neutral pricing as special case and is hence consistent with the classical approach. Finally, we remark that we could, of course, in the case of the BSM perform the relevant change of measure to directly obtain the risk neutral prices. However, this section aims to illustrate real world pricing, and hence we proceed by computing the expected value in (1.3.19) directly in the case of the BSM.

To alleviate notation we define the *benchmarked volatility* $\sigma_t^{j,k}$ by setting

$$\sigma_t^{0,k} = \theta_t^k \tag{2.3.32}$$

for $j = 0$ and $k \in \{1, 2, \dots, d\}$, and

$$\sigma_t^{j,k} = \theta_t^k - b_t^{j,k} \tag{2.3.33}$$

for $k \in \{1, 2, \dots, d\}$ and $j \in \{1, 2, \dots, d\}$, $t \geq 0$. Consequently, it follows from (1.2.12) that the SDE governing the dynamics of the GOP becomes

$$dS_t^{\delta_*} = S_t^{\delta_*}\left(r_t\,dt + \sum_{k=1}^d \sigma_t^{0,k}\left(\sigma_t^{0,k}\,dt + dW_t^k\right)\right), \tag{2.3.34}$$

which can be solved explicitly to yield

$$S_t^{\delta_*} = S_0^{\delta_*} \exp\left\{\int_0^t \left(r_s + \frac{1}{2}\sum_{k=1}^d \left(\sigma_s^{0,k}\right)^2\right) ds + \sum_{k=1}^d \int_0^t \sigma_s^{0,k}\,dW_s^k\right\} \tag{2.3.35}$$

for all $t \geq 0$. Furthermore, the jth benchmarked primary security account $\hat{S}_t^j = \frac{S_t^j}{S_t^{\delta_*}}$ can be shown to satisfy

$$d\hat{S}_t^j = -\hat{S}_t^j \sum_{k=1}^d \sigma_t^{j,k}\,dW_t^k, \tag{2.3.36}$$

for all $j \in \{0, 1, \dots, d\}$ and $t \geq 0$, with $\hat{S}_0^j = S_0^j$, which follows from (1.3.16) by setting $\pi_{\delta,t}^i = 1$ for $i = j$ and $\pi_{\delta,t}^i = 0$ otherwise. Consequently, we obtain the following explicit expression for the jth benchmarked primary security account

$$\hat{S}_t^j = \hat{S}_0^j \exp\left\{-\frac{1}{2}\int_0^t \sum_{k=1}^d \left(\sigma_s^{j,k}\right)^2 ds - \sum_{k=1}^d \int_0^t \sigma_s^{j,k}\,dW_s^k\right\} \tag{2.3.37}$$

for $j \in \{0, 1, \dots, d\}$ and $t \geq 0$.

We now illustrate that under the benchmark approach, the benchmarked primary security accounts $\hat{S}_t^j$, $j \in \{0, 1, \dots, d\}$ are the pivotal objects of study: in particular, specifying the savings account S_t^0 and the benchmarked primary security accounts suffices to determine the entire investment universe. The ratio $S_t^{\delta_*} = \frac{S_t^0}{\hat{S}_t^0}$, for all $t \geq 0$, see (1.3.15), expresses the GOP and NP in terms of the savings account and the benchmarked savings account. The product $S_t^j = \hat{S}_t^j S_t^{\delta_*}$ recovers each primary security account from the corresponding benchmarked primary security account and the GOP for each $j \in \{1, 2, \ldots, d\}$ and $t \geq 0$.

Next, we introduce the processes $|\sigma^j| = \{|\sigma_t^j|, t \geq 0\}$ for $j \in \{0, 1, \ldots, d\}$, by setting

$$|\sigma_t^j| = \sqrt{\sum_{k=1}^{d} (\sigma_t^{j,k})^2}. \tag{2.3.38}$$

These processes enable us to introduce the *aggregate continuous noise processes* $\hat{W}^j = \{\hat{W}_t^j, t \in [0, \infty)\}$ for $j \in \{0, 1, \ldots, d\}$, defined by

$$\hat{W}_t^j = \sum_{k=1}^{d} \int_0^t \frac{\sigma_s^{j,k}}{|\sigma_s^j|} \, dW_s^k. \tag{2.3.39}$$

An application of Lévy's Theorem for the characterization of the Wiener process, see Chap. 15, Theorem 15.3.3, allows us to conclude that $\hat{W}^j$ is a Wiener process for each $j \in \{0, 1, \ldots, d\}$. We point out that the Wiener processes $\hat{W}^0, \hat{W}^1, \ldots, \hat{W}^d$ can be correlated. Furthermore, we enforce Assumption 1.1.1, so that the volatility matrix $\boldsymbol{b}_t = [b_t^{j,k}]_{j,k=1}^d$ becomes invertible for all $t \geq 0$. Note that so far in this section the short rate and volatility processes are not specified and remain still general.

2.3.1 The Black-Scholes Model

The stylized *Black-Scholes model* (BSM) arises if we assume that all parameter processes, that is, the short rate and the volatilities, are constant, i.e. if we set $r_t = r$ and $\sigma_t^{j,k} = \sigma^{j,k}$ for each $j \in \{0, 1, \ldots, d\}$, $k \in \{1, 2, \ldots, d\}$ and $t \geq 0$. Consequently, (2.3.34) and (2.3.36) become in this case

$$S_t^{\delta_*} = S_0^{\delta_*} \exp\left\{ r\,t + \frac{t}{2} |\sigma^0|^2 + |\sigma^0|\, \hat{W}_t^0 \right\} \tag{2.3.40}$$

and

$$\hat{S}_t^j = S_0^j \exp\left\{ -\frac{t}{2} |\sigma^j|^2 - |\sigma^j|\, \hat{W}_t^j \right\} \tag{2.3.41}$$

for each $j \in \{0, 1, \ldots, d\}$ and all $t \geq 0$. From (2.3.41) it is clear that the benchmarked primary security accounts $\hat{S}_t^j$, $j \in \{0, 1, \ldots, d\}$, are continuous martingales, as they are driftless geometric Brownian motions. As this holds, in particular, for the benchmarked savings account, the Radon-Nikodym derivative process $\Lambda_\theta(t) = \frac{\hat{S}_t^0}{\hat{S}_0^0}$ in (1.3.20) is an $(\underline{\mathcal{A}}, P)$-martingale. We conclude that the standard risk neutral pricing approach could, therefore, be used for derivative pricing under the BSM making use of the risk neutral pricing formula (1.3.21). Finally, we emphasize that we do not advocate the BSM as a reasonably realistic description of observed market dynamics. However, given its familiarity, it is useful for illustrating real world pricing under the benchmark approach for classical models, which produces the same answers as risk neutral pricing. Furthermore, the fact that explicit formulas can be obtained for many derivatives is extremely useful in practice. We will derive below explicit formulas and descriptions of a range of derivative prices under the BSM by using real world pricing.

2.3.2 *Zero Coupon Bonds*

We firstly demonstrate how to price a standard default-free *zero coupon bond* that pays one unit of the domestic currency at its maturity date $T \in [0, \infty)$. It follows from the real world pricing formula (1.3.19) that the value of the zero coupon bond at time t is given by

$$P_T(t) = S_t^{\delta_*} E\left(\frac{1}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t \right) = \frac{1}{\hat{S}_t^0} E\left(\exp\left\{ -\int_t^T r_s \, ds \right\} \hat{S}_T^0 \,\Big|\, \mathcal{A}_t \right) \tag{2.3.42}$$

for all $t \in [0, T]$. Since $\hat{S}_t^0 = \frac{S_t^0}{S_t^{\delta_*}}$ is an $(\underline{\mathcal{A}}, P)$-martingale and $r_t = r$ is constant we obtain

$$P_T(t) = \exp\{-r(T-t)\} \frac{1}{\hat{S}_t^0} E\left(\hat{S}_T^0 \mid \mathcal{A}_t\right) = \exp\{-r(T-t)\} \tag{2.3.43}$$

for all $t \in [0, T]$. As expected, this is the usual bond pricing formula that is determined by the deterministic short rate r, which one can also obtain via risk neutral pricing, see (1.3.20) and Harrison and Kreps (1979). As long as the benchmarked savings account, and with this the Radon-Nikodym derivative of the risk neutral measure, is a martingale one obtains this classical zero coupon bond price.

2.3.3 *Forward Contracts*

We now aim to price a *forward contract* with the delivery of one unit of the jth primary security account at the maturity date T, which is written or initiated at time $t \in [0, T]$ for $j \in \{0, 1, \dots, d\}$. The value of the forward contract written at initiation time t is zero by definition. The real world pricing formula (1.3.19) yields then the following relation, which determines the forward price $F_T^j(t)$ at time $t \in [0, T]$ via the relation

$$S_t^{\delta_*} E\left(\frac{F_T^j(t) - S_T^j}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t \right) = 0. \tag{2.3.44}$$

By (2.3.42) and $\hat{S}_T^j = \frac{S_T^j}{S_T^{\delta_*}}$, we obtain

$$F_T^j(t) = \frac{S_t^{\delta_*} E(\hat{S}_T^j \mid \mathcal{A}_t)}{S_t^{\delta_*} E(\frac{1}{S_T^{\delta_*}} \mid \mathcal{A}_t)} = \frac{S_t^j}{P_T(t)} \frac{1}{\hat{S}_t^j} E\big(\hat{S}_T^j \mid \mathcal{A}_t\big) \tag{2.3.45}$$

for a given $t \in [0, T]$. Again, as the benchmarked primary security accounts are $(\underline{\mathcal{A}}, P)$-martingales under classical models as the BSM, it follows using (2.3.43) that

$$F_T^j(t) = S_t^j \exp\big\{r(T-t)\big\} \tag{2.3.46}$$

for all $t \in [0, T]$. This is then also the standard risk neutral formula for the forward price, see for instance Musiela and Rutkowski (2005).

2.3.4 Asset-or-Nothing Binaries

Binary options can be considered to be building blocks for several more complex derivatives. This is useful to know when it comes to the valuation and hedging of various exotic options, see e.g. Ingersoll (2000), Buchen (2004), and Baldeaux and Rutkowski (2010).

The derivative contract we consider in this subsection is an *asset-or-nothing binary* on a market index, which we interpret here as the GOP. At its maturity date T, this derivative pays its holder one unit of the market index if its value is greater than the strike K, and nothing otherwise. Using the real world pricing formula (1.3.19) and (2.3.40), we obtain under the BSM

$$\begin{aligned}
A_{T,K}(t) &= S_t^{\delta_*} E\left(\mathbf{1}_{\{S_T^{\delta_*} \geq K\}} \frac{S_T^{\delta_*}}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) \\
&= S_t^{\delta_*} P\big(S_T^{\delta_*} \geq K \mid \mathcal{A}_t\big) \\
&= S_t^{\delta_*} P\left(S_t^{\delta_*} \exp\left\{\left(r + \frac{1}{2}|\sigma^0|^2\right)(T-t) + |\sigma^0|\big(\hat{W}_T^0 - \hat{W}_t^0\big)\right\} \geq K \,\middle|\, \mathcal{A}_t\right) \\
&= S_t^{\delta_*} N(d_1)
\end{aligned} \tag{2.3.47}$$

for all $t \in [0, T]$, where

$$d_1 = \frac{\ln(\frac{S_t^{\delta_*}}{K}) + (r + \frac{1}{2}|\sigma^0|^2)(T-t)}{|\sigma^0|\sqrt{T-t}} \tag{2.3.48}$$

and $N(\cdot)$ denotes the Gaussian distribution function.

2.3.5 Bond-or-Nothing Binaries

In this subsection, we consider pricing a *bond-or-nothing binary*, which pays the strike $K \in \Re^+$ at maturity T in the event that the market index at time T is not less than K. As before, the market index is interpreted as the GOP.

$$\begin{aligned} B_{T,K}(t) &= S_t^{\delta_*} E\left(\mathbf{1}_{\{S_T^{\delta_*} \ge K\}} \frac{K}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) \\ &= S_t^{\delta_*} E\left(\mathbf{1}_{\{S_T^{\delta_*} \ge K\}} K \frac{\hat{S}_T^0}{\hat{S}_0^0} \frac{\hat{S}_0^0}{S_T^0} \,\middle|\, \mathcal{A}_t\right) \\ &= S_t^{\delta_*} \frac{\hat{S}_0^0}{S_T^0} E\big(\mathbf{1}_{\{S_T^{\delta_*} \ge K\}} K \Lambda_{|\sigma^0|}(T) \,\big|\, \mathcal{A}_t\big). \end{aligned} \tag{2.3.49}$$

Under the BSM, making use of Girsanov's theorem and the Bayes rule facilitates pricing, in particular, we recall that the benchmarked savings account is an $(\underline{\mathcal{A}}, P)$-martingale and one has the Radon-Nikodym derivative process $\Lambda_{|\sigma^0|} = \{\Lambda_{|\sigma^0|}(t),\ t \in [0,T]\}$, where

$$\Lambda_{|\sigma^0|}(t) = \frac{\hat{S}_t^0}{\hat{S}_0^0} = \exp\left\{-\frac{t}{2}\left|\sigma^0\right|^2 - \left|\sigma^0\right| \hat{W}_t^0\right\}. \tag{2.3.50}$$

This process is used to define a measure $P_{|\sigma^0|}$ via

$$\frac{dP_{|\sigma^0|}}{dP} = \Lambda_{|\sigma^0|}(T), \tag{2.3.51}$$

by setting

$$P_{|\sigma^0|}(A) = E\big(\Lambda_{|\sigma^0|}(T)\mathbf{1}_A\big) = E_{|\sigma^0|}(\mathbf{1}_A) \tag{2.3.52}$$

for $A \in \mathcal{A}_T$. We use $E_{|\sigma^0|}$ to denote the expectation with respect to $P_{|\sigma^0|}$. By Girsanov's theorem, $W^{|\sigma^0|} = \{W_t^{|\sigma^0|},\ t \in [0,T]\}$, where

$$W_t^{|\sigma^0|} = \hat{W}_t^0 + \left|\sigma^0\right| t \tag{2.3.53}$$

is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P_{|\sigma^0|})$. This yields for (2.3.49) the relations

$$\begin{aligned} B_{T,K}(t) &= S_t^{\delta_*} \frac{\hat{S}_0^0}{S_T^0} K P_{|\sigma^0|}\big(S_T^{\delta_*} \ge K \,\big|\, \mathcal{A}_t\big) E\big(\Lambda_{|\sigma^0|}(T) \,\big|\, \mathcal{A}_t\big) \\ &= S_t^{\delta_*} \frac{\hat{S}_0^0}{S_T^0} K P_{|\sigma^0|}\big(S_T^{\delta_*} \ge K \,\big|\, \mathcal{A}_t\big) \frac{\hat{S}_t^0}{\hat{S}_0^0} \\ &= K \exp\big\{-r(T-t)\big\} \\ &\quad \times P_{|\sigma^0|}\left(W_{T-t}^{|\sigma^0|} \ge \frac{\ln(\frac{K}{S_t^{\delta_*}}) + (r - \frac{1}{2}|\sigma^0|^2)(T-t)}{|\sigma^0|} \,\middle|\, \mathcal{A}_t\right) \\ &= K \exp\big\{-r(T-t)\big\} N(d_2) \end{aligned}$$

for $t \in [0, T]$, where

$$d_2 = \frac{\ln(\frac{S_t^{\delta_*}}{K}) + (r - \frac{1}{2}|\sigma^0|^2)(T-t)}{|\sigma^0|\sqrt{T-t}} \tag{2.3.54}$$

and $N(\cdot)$ is again the Gaussian distribution function.

2.3.6 European Options

We now focus on pricing a European call option with maturity $T \in [0, \infty)$ and strike $K \in \Re^+$ on a market index, which is again interpreted as the GOP. Invoking the real world pricing formula (1.3.19), and recalling the previously obtained binaries, we obtain the price of the European call option

$$\begin{aligned} c_{T,K}(t) &= S_t^{\delta_*} E\left(\frac{(S_T^{\delta_*} - K)^+}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t\right) \\ &= S_t^{\delta_*} E\left(\mathbf{1}_{\{S_T^{\delta_*} \geq K\}} \frac{S_T^{\delta_*} - K}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t\right) \\ &= A_{T,K}(t) - B_{T,K}(t) \end{aligned} \tag{2.3.55}$$

for all $t \in [0, T]$. Combining (2.3.47) and (2.3.54) gives

$$c_{T,K}(t) = S_t^{\delta_*} N(d_1) - K \exp\{-r(T-t)\} N(d_2) \tag{2.3.56}$$

for all $t \in [0, T]$, where d_1 and d_2 are given by (2.3.48) and (2.3.54), respectively.

The above explicit formula corresponds to the original pricing formula for a European call on a stock under the BSM, as given in Black and Scholes (1973). Similarly, the price of a European put option is given by

$$p_{T,K}(t) = K \exp\{-r(T-t)\} N(-d_2) - S_t^{\delta_*} N(-d_1),$$

for all $t \in [0, T]$.

2.3.7 Rebates

In this subsection, we consider the valuation of a *rebate* written on a market index, which is again interpreted as the GOP. This claim pays one unit of the domestic currency as soon as the index hits a certain level, assuming this occurs before a contracted expiry date $T > 0$. Following Hulley and Platen (2008), the trigger level for the rebate is a deterministic barrier $Z_t := z \exp\{rt\}$, for some $z > 0$. We mention the fact that the deterministic barrier grows at the risk-free rate, which is economically attractive, as it makes the price of the rebate dependent on the performance of the index relative to that of the savings account. We make use of the following stopping times

$$\sigma_{z,t} := \inf\big\{u > 0\colon\ S^{\delta_*}_{t+u} = Z_{t+u}\big\} \tag{2.3.57}$$

and

$$\tau_z := \inf\{t > 0\colon\ Y_t = z\}, \tag{2.3.58}$$

where $Y = \{Y_t, t \ge 0\}$ satisfies

$$Y_t = x \exp\left\{\frac{1}{2}\left|\sigma^0\right|^2 t + \left|\sigma^0\right| \hat{W}^0_t\right\} \tag{2.3.59}$$

and $x := \exp\{-rt\} S^{\delta_*}_t$. Furthermore, we introduce the auxiliary process $X = \{X_t,\ t \ge 0\}$, where

$$X_t = \nu t + \hat{W}^0_t, \tag{2.3.60}$$

and $\nu = \frac{1}{2}|\sigma^0|$. This means X is a Brownian motion with drift. Additionally, we define

$$T_a := \inf\{t > 0\colon\ X_t = a\}. \tag{2.3.61}$$

It is easily seen that we have the following equality in distribution

$$\sigma_{z,t} \stackrel{d}{=} \tau_z \stackrel{d}{=} T_{\tilde{z}} \tag{2.3.62}$$

under P, where $\tilde{z} := \ln(\frac{z}{x}) \frac{1}{|\sigma^0|}$.

First, we consider the valuation of a *perpetual rebate*, for which $T = \infty$. It follows by applying real world pricing that

$$\begin{aligned} R_{\infty,z}(t) &= S^{\delta_*}_t E\left(\frac{1}{S^{\delta_*}_{t+\sigma_{z,t}}} \,\Big|\, \mathcal{A}_t\right) \\ &= \frac{S^{\delta_*}_t}{Z_t} E\big(\exp\{-r\sigma_{z,t}\} \,\big|\, \mathcal{A}_t\big) \\ &= \frac{S^{\delta_*}_t}{Z_t} E\big(\exp\{-r\tau_z\} \,\big|\, \mathcal{A}_t\big). \end{aligned}$$

Making use of the known moment generating function of $T_{\tilde{z}}$, see Proposition 2.1.8, we get

$$E\big(\exp\{-rT_{\tilde{z}}\} \,\big|\, \mathcal{A}_t\big) = \left(\frac{z}{x}\right)^{1/2} \exp\left\{-\left|\ln\left(\frac{z}{x}\right)\right| \frac{\sqrt{2r + (\frac{|\sigma^0|}{2})^2}}{|\sigma^0|}\right\} \tag{2.3.63}$$

and hence

$$R_{\infty,z}(t) = \left(\frac{S^{\delta_*}_t}{Z_t}\right)^{\frac{1}{2}} \exp\left\{-\left|\ln(Z_t) - \ln\big(S^{\delta_*}_t\big)\right| \frac{\sqrt{2r + (\frac{|\sigma^0|}{2})^2}}{|\sigma^0|}\right\}. \tag{2.3.64}$$

Now we turn our attention to the rebate with finite maturity $T < \infty$. Using the real world pricing formula (1.3.19) we obtain

$$
\begin{aligned}
R_{T,z}(t) &= S_t^{\delta_*} E\left(\frac{\mathbf{1}_{t+\sigma_{z,t} \le T}}{S_{t+\sigma_{z,t}}^{\delta_*}} \,\Big|\, \mathcal{A}_t \right) \\
&= \frac{S_t^{\delta_*}}{Z_t} E\left(\mathbf{1}_{\sigma_{z,t} \le T-t} \exp\{-r\sigma_{z,t}\} \,\big|\, \mathcal{A}_t\right) \\
&= \frac{S_t^{\delta_*}}{Z_t} E\left(\mathbf{1}_{T_{\tilde{z}} \le T-t} \exp\{-rT_{\tilde{z}}\} \,\big|\, \mathcal{A}_t\right) \\
&= \frac{S_t^{\delta_*}}{Z_t} \int_0^{T-t} \exp\{-ru\} \frac{|\tilde{z}|}{\sqrt{2\pi} u^{3/2}} \exp\left\{ -\frac{(\tilde{z} - \nu u)^2}{2u} \right\} du,
\end{aligned}
$$

where the last equality employs the distribution of $T_{\tilde{z}}$. Using the change of variables $l := u^{-1/2}$, we obtain

$$
\begin{aligned}
&\int_0^{T-t} \frac{\exp\{-ru - \frac{(\tilde{z}-\nu u)^2}{2u}\}}{u^{3/2}} du \\
&= 2 \int_{(T-t)^{-1/2}}^{\infty} \exp\left\{ -\frac{(\tilde{z})^2}{2} l^2 - \left(r + \frac{\nu^2}{2} \right) l^{-2} \right\} dl \\
&= 2 \frac{\exp\{-2\sqrt{bc}\}\sqrt{\pi}(erfc(a\sqrt{b} - \frac{\sqrt{c}}{a}) + \exp\{4\sqrt{bc}\} erfc(a\sqrt{b} + \frac{\sqrt{c}}{a}))}{4\sqrt{b}},
\end{aligned}
\tag{2.3.65}
$$

where $a := (T-t)^{-1/2}$, $b := \frac{(\tilde{z})^2}{2}$ and $c := (r + \frac{\nu^2}{2})$ and c is assumed to be positive. Furthermore, *erfc* denotes the complement of the error function *erf*, i.e. $erfc(z) = 1 - erf(z)$, where $erf(z) = \frac{2}{\pi} \int_0^z \exp\{-t^2\} dt$. Finally, we remark that (2.3.65) can also be easily confirmed using *Mathematica*.

2.3.8 Barrier Options

In this subsection we consider a *barrier option* on a market index, the GOP or NP. As in the previous subsection, the payoff of this contingent claim is determined by whether or not the index hits a certain level prior to its maturity $T > 0$. In particular, we consider a European call with strike price $K > 0$, that is knocked out if the index breaches the same deterministic barrier Z as in the previous subsection, sometime before expiry.

Using the real world pricing formula and the notation introduced in the previous subsection, we obtain the following price for this claim

$$
\begin{aligned}
C_{T,K,z}^{uo}(t) &= S_t^{\delta_*} E\left(\mathbf{1}_{t+\sigma_{z,t} > T} \frac{(S_T^{\delta_*} - K)^+}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t \right) \\
&= S_t^{\delta_*} E\left(\mathbf{1}_{\sigma_{z,t} > T-t} \left(1 - \frac{K}{S_T^{\delta_*}} \right)^+ \,\Big|\, \mathcal{A}_t \right)
\end{aligned}
$$

$$= S_t^{\delta_*} E\left(\mathbf{1}_{\tau_z > T-t}\left(1 - \frac{K\exp\{-rT\}}{Y_{T-t}}\right)^+ \Bigg| \mathcal{A}_t\right)$$

$$= S_t^{\delta_*} E\left(\mathbf{1}_{T_{\tilde{z}} > T-t}\left(1 - \frac{k}{x\exp\{\sigma X_{T-t}\}}\right)^+ \Bigg| \mathcal{A}_t\right) \qquad (2.3.66)$$

where, as in the previous subsection, $\nu := \frac{1}{2}|\sigma^0|$, $x := S_t^{\delta_*}\exp\{-rt\}$, $k := K \times \exp\{-rT\}$, $\tilde{z} := \ln(\frac{z}{x})\frac{1}{|\sigma^0|}$, $\sigma := |\sigma^0|$ and $X = \{X_t,\ t \geq 0\}$, where X_t is given by (2.3.60) and T_a denotes the first time the process X hits the level a. As X is a Brownian motion with drift and T_a denotes the associated first hitting time of the level a, we can apply Corollary 2.1.10 to obtain the price of the above Barrier option. We remark that an alternative derivation of this formula, based on Girsanov's theorem and the Bayes' rule, is presented in Musiela and Rutkowski (2005).

Following Hulley and Platen (2008), we find it convenient to distinguish the following two cases: firstly $S_t^{\delta_*} \leq Z_t \Leftrightarrow x \leq z$, in which case we deal with an *up-and-out call* and $S_t^{\delta_*} \geq Z_t \Leftrightarrow x \geq z$, in which case we deal with a *down-and-out call*. Regarding the up-and-out call, we remark that the Brownian motion with drift X started at 0 killed at $\tilde{z}$ lives on the domain $(-\infty, \tilde{z})$. Finally, setting $a := \ln(\frac{k}{x})\frac{1}{|\sigma^0|}$, we obtain from (1.3.19) the following pricing formula for an up-and-out call option:

$$C^{uo}_{T,K,z}(t) = S_t^{\delta_*}\int_a^{\tilde{z}}\left(1 - \frac{k}{x\exp\{\sigma y\}}\right)\tilde{q}_{\tilde{z}}(T-t, 0, y)m(y)\,dy$$

$$= S_t^{\delta_*}\int_a^{\tilde{z}}\left(1 - \frac{k}{x\exp\{\sigma y\}}\right)\frac{1}{\sqrt{2\pi(T-t)}}\exp\left\{\nu y - \frac{\nu^2(T-t)}{2}\right\}$$

$$\times\left(\exp\left\{-\frac{y^2}{2(T-t)}\right\} - \exp\left\{-\frac{(y-2\tilde{z})^2}{2(T-t)}\right\}\right)dy \qquad (2.3.67)$$

using the fact that the speed measure of a Brownian motion with drift is given by $m(y) = 2\exp\{2\nu y\}$, see Borodin and Salminen (2002). Consequently, to compute the price of a barrier option, we need to compute four integrals: firstly, we calculate

$$I_1 = S_t^{\delta_*}\int_a^{\tilde{z}}\frac{1}{\sqrt{2\pi(T-t)}}\exp\left\{\nu y - \frac{\nu^2(T-t)}{2} - \frac{y^2}{2(T-t)}\right\}dy$$

$$= S_t^{\delta_*}\left(N\left(\frac{\tilde{z} - \nu(T-t)}{\sqrt{T-t}}\right) - N\left(\frac{a - \nu(T-t)}{\sqrt{T-t}}\right)\right). \qquad (2.3.68)$$

The second integral is given by

$$I_2 = -S_t^{\delta_*}\int_a^{\tilde{z}}\frac{1}{\sqrt{2\pi(T-t)}}\exp\left\{\nu y - \frac{\nu^2(T-t)}{2} - \frac{(y-2\tilde{z})^2}{2(T-t)}\right\}dy$$

$$= -S_t^{\delta_*}\exp\{2\tilde{z}\nu\}\left(N\left(\frac{-\tilde{z} - \nu(T-t)}{\sqrt{T-t}}\right) - N\left(\frac{a - 2\tilde{z} - \nu(T-t)}{\sqrt{T-t}}\right)\right)$$

$$= -Z_t\left(N\left(\frac{-\tilde{z} - \nu(T-t)}{\sqrt{T-t}}\right) - N\left(\frac{a - 2\tilde{z} - \nu(T-t)}{\sqrt{T-t}}\right)\right), \qquad (2.3.69)$$

which is obtained by completing the square. Next the third integral can be computed as follows:

$$
\begin{aligned}
I_3 &= -S_t^{\delta_*}\frac{k}{x}\int_a^{\tilde{z}} \frac{\exp\{y(\nu-\sigma)-\frac{\nu^2(T-t)}{2}-\frac{y^2}{2(T-t)}\}}{\sqrt{2\pi(T-t)}}\,dy \\
&= -S_t^{\delta_*}\frac{k}{x}\exp\left\{\frac{\sigma^2(T-t)}{2}-\nu\sigma(T-t)\right\}\left(N\left(\frac{\tilde{z}-(\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)\right. \\
&\quad \left. - N\left(\frac{a-(\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)\right) \\
&= -K\exp\{-r(T-t)\}\left(N\left(\frac{\tilde{z}-(\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)\right. && (2.3.70) \\
&\quad \left. - N\left(\frac{a-(\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)\right), && (2.3.71)
\end{aligned}
$$

where we also completed the square. Finally, the fourth integral is given by

$$
\begin{aligned}
I_4 &= S_t^{\delta_*}\frac{k}{x}\int_a^{\tilde{z}} \frac{1}{\sqrt{2\pi(T-t)}}\exp\left(y(\nu-\sigma)-\frac{\nu^2(T-t)}{2}-\frac{(y-2\tilde{z})^2}{2(T-t)}\right)dy \\
&= S_t^{\delta_*}\frac{k}{x}\exp\left(2\tilde{z}(\nu-\sigma)-\nu\sigma(T-t)+\frac{\sigma^2(T-t)}{2}\right) \\
&\quad \times\left(N\left(\frac{-\tilde{z}-(\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)-N\left(\frac{a-2\tilde{z}-(\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)\right) \\
&= S_t^{\delta_*}\frac{k}{z}\left(N\left(\frac{-\tilde{z}-(\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)-N\left(\frac{a-2\tilde{z}-(\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)\right).
\end{aligned}
\tag{2.3.72}
$$

Obviously, the price of the up-and-out call is given by the sum of the four terms in (2.3.68), (2.3.69), (2.3.71), and (2.3.72). In summary, this yields the explicit formula

$$
C^{uo}_{T,K,z}\left(t, S_t^{\delta_*}\right) = I_1 + I_2 + I_3 + I_4. \tag{2.3.73}
$$

We now turn our attention to the down-and-out call, i.e. the case $S_t \geq Z_t \Leftrightarrow x \geq z$. As for the up-and-out call, we remark that the Brownian motion with drift started at 0 killed at $\tilde{z}$ lives on the domain $(\tilde{z}, \infty)$. Recalling that $a = \ln(\frac{k}{x})\frac{1}{|\sigma^0|}$, we obtain the following pricing formula for a down-and-out call option from (1.3.19),

$$
\begin{aligned}
C^{do}_{T,K,z}(t) &= S_t^{\delta_*}\int_{\tilde{z}\vee a}^{\infty}\left(1-\frac{k}{x\exp\{\sigma y\}}\right)\tilde{q}_{\tilde{z}}(T-t,0,y)m(y)\,dy \\
&= S_t^{\delta_*}\int_{\tilde{z}\vee a}^{\infty}\left(1-\frac{k}{x\exp\{\sigma y\}}\right)\frac{1}{\sqrt{2\pi(T-t)}}\exp\left\{\nu y-\frac{\nu^2(T-t)}{2}\right\} \\
&\quad \times\left(\exp\left\{-\frac{y^2}{2(T-t)}\right\}-\exp\left\{-\frac{(y-2\tilde{z})^2}{2(T-t)}\right\}\right)dy.
\end{aligned}
\tag{2.3.74}
$$

As for the up-and-out call, the pricing of the down-and-out call entails the computation of the following four integrals

$$\bar{I}_1 = S_t^{\delta_*} \int_{\tilde{z}\vee a}^{\infty} \frac{\exp\{\nu y - \frac{\nu^2(T-t)}{2} - \frac{y^2}{2(T-t)}\}}{\sqrt{2\pi(T-t)}}\, dy$$
$$= S_t^{\delta_*} N\left(-\frac{(\tilde{z}\vee a) + \nu(T-t)}{\sqrt{T-t}}\right). \tag{2.3.75}$$

The second integral is given by

$$\bar{I}_2 = -S_t^{\delta_*} \int_{\tilde{z}\vee a}^{\infty} \exp\left\{\nu y - \frac{\nu^2(T-t)}{2} - \frac{(y-2\tilde{z})^2}{2(T-t)}\right\} dy$$
$$= -S_t^{\delta_*} \exp\{2\tilde{z}\nu\} N\left(-\frac{(\tilde{z}\vee a) + 2\tilde{z} + \nu(T-t)}{\sqrt{T-t}}\right)$$
$$= -Z_t N\left(-\frac{(\tilde{z}\vee a) + 2\tilde{z} + \nu(T-t)}{\sqrt{T-t}}\right). \tag{2.3.76}$$

The third integral is given by

$$\bar{I}_3 = -S_t^{\delta_*} \frac{k}{x} \int_{\tilde{z}\vee a}^{\infty} \exp\left\{\nu y - \sigma y - \frac{\nu^2(T-t)}{2} - \frac{y^2}{2(T-t)}\right\} dy$$
$$= -S_t^{\delta_*} \frac{k}{x} \exp\left\{\frac{\sigma^2(T-t)}{2} - \nu\sigma(T-t)\right\} N\left(-\frac{(\tilde{z}\vee a) + (\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)$$
$$= -K \exp\bigl\{-r(T-t)\bigr\} N\left(\frac{-(\tilde{z}\vee a) + (\nu-\sigma)(T-t)}{\sqrt{T-t}}\right), \tag{2.3.77}$$

and the last integral is given by

$$\bar{I}_4 = S_t^{\delta_*} \frac{k}{x} \exp\left\{2\tilde{z}(\nu-\sigma) - \nu\sigma(T-t) + \frac{\sigma^2(T-t)}{2}\right\}$$
$$\times \int_{\tilde{z}\vee a}^{\infty} \frac{\exp\{-\frac{(y-(2\tilde{z}+(\nu-\sigma)(T-t)))^2}{2(T-t)}\}}{\sqrt{2\pi(T-t)}}\, dy$$
$$= S_t^{\delta_*} \frac{k}{x} \exp\left\{2\tilde{z}(\nu-\sigma) - \nu\sigma(T-t) + \frac{\sigma^2(T-t)}{2}\right\}$$
$$\times N\left(\frac{-(\tilde{z}\vee a) + 2\tilde{z} + (\nu-\sigma)(T-t)}{\sqrt{T-t}}\right)$$
$$= S_t^{\delta_*} \frac{k}{z} N\left(\frac{-(\tilde{z}\vee a) + 2\tilde{z} + (\nu-\sigma)(T-t)}{\sqrt{T-t}}\right). \tag{2.3.78}$$

It follows for the down-and-out call option the formula

$$C^{do}_{T,K,z}\bigl(t, S_t^{\delta_*}\bigr) = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4.$$

By the same methodology one obtains also other barrier options.

2.3.9 Lookback Options

In this subsection, we consider the valuation of a *lookback option* written on a market index, which is again interpreted as the GOP. A standard lookback call option pays

$$\left(S_T^{\delta_*} - m_T^{S^{\delta_*}}\right)^+ = S_T^{\delta_*} - m_T^{S^{\delta_*}},$$

where $m_T^{S^{\delta_*}} = \min_{t\in[0,T]} S_t^{\delta_*}$. We remark that lookback options are always exercised. In Musiela and Rutkowski (2005), the price of a lookback option is derived via a measure change. In this subsection, we proceed by directly integrating the relevant probability density function derived in Sect. 2.1. For ease of presentation, we consider the pricing of a call option at time $t = 0$, but consequently present the formulas for the general case. The real world pricing formula (1.3.19) gives the following price $LC(0)$ for a lookback call option

$$LC(0) = S_0^{\delta_*} E\left(\frac{(S_T^{\delta_*} - m_T^{S^{\delta_*}})^+}{S_T^{\delta_*}}\right) = S_0^{\delta_*} - S_0^{\delta_*} E\left(\frac{m_T^{S^{\delta_*}}}{S_T^{\delta_*}}\right).$$

From (2.3.40),

$$m_T^{S^{\delta_*}} = S_0^{\delta_*} \min_{t\in[0,T]} \exp\left\{\left(r + \frac{\sigma^2}{2}\right)t + \sigma \hat{W}_t^0\right\},$$

where $\sigma := |\sigma^0|$ and hence

$$\frac{m_T^{S^{\delta_*}}}{S_T^{\delta_*}} = \exp\left\{\min_{t\in[0,T]}\left(\left(r + \frac{\sigma^2}{2}\right)(t - T) + \sigma\left(\hat{W}_t^0 - \hat{W}_T^0\right)\right)\right\}.$$

From the time reversibility of Brownian motion, see Chap. 15, we have the following equality in distribution,

$$\min_{t\in[0,T]}\left(\left(r + \frac{\sigma^2}{2}\right)(t - T) + \sigma\left(\hat{W}_t^0 - \hat{W}_T^0\right)\right) \stackrel{d}{=} \min_{\tau\in[0,T]}\left(-\left(r + \frac{\sigma^2}{2}\right)\tau + \sigma \hat{W}_\tau^0\right).$$

We use the notation

$$X_t = \nu t + \hat{W}_t^0,$$

where $\nu = -\frac{(r+\frac{\sigma^2}{2})}{\sigma}$, and recall from Sect. 2.1, that the probability density of $\min_{t\in[0,T]} X_t$ satisfies

$$P\left(m_T^X \in dy\right) = \left(\phi\left(\frac{-y + \nu T}{\sqrt{T}}\right)\frac{1}{\sqrt{T}} + 2\nu \exp\{2\nu y\} N\left(\frac{y + \nu T}{\sqrt{T}}\right) + \exp\{2\nu y\}\frac{1}{\sqrt{T}}\phi\left(\frac{y + \nu T}{\sqrt{T}}\right)\right) dy.$$

Hence

$$
\begin{aligned}
E\left(\frac{m_T^{S^{\delta_*}}}{S_T^{\delta_*}}\right) &= E\big(\exp\{\sigma m_T^X\}\big) \\
&= \int_{-\infty}^{0} \exp\{\sigma y\}\phi\left(\frac{-y+\nu T}{\sqrt{T}}\right)\frac{1}{\sqrt{T}}\,dy \\
&\quad + \int_{-\infty}^{0} \exp\{\sigma y\}2\nu\exp\{2\nu y\}N\left(\frac{y+\nu T}{\sqrt{T}}\right)dy \\
&\quad + \int_{-\infty}^{0} \exp\{\sigma y\}\exp\{2\nu y\}\frac{1}{\sqrt{T}}\phi\left(\frac{y+\nu T}{\sqrt{T}}\right)dy.
\end{aligned}
$$

We now compute these three integrals

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{0} \exp\{\sigma y\}\phi\left(\frac{-y+\nu T}{\sqrt{T}}\right)\frac{1}{\sqrt{T}}\,dy \\
&= \int_{-\infty}^{0} \exp\{\sigma y\}\frac{\exp\{-\frac{1}{2}(\frac{y-\nu T}{\sqrt{T}})^2\}}{\sqrt{2\pi T}}\,dy \\
&= \exp\left\{\frac{T}{2}\big(\sigma^2+2\sigma\nu\big)\right\}\int_{-\infty}^{-(\nu+\sigma)\sqrt{T}} \frac{\exp\{-\frac{z^2}{2}\}}{\sqrt{2\pi}}\,dz \\
&= \exp\{-rT\}N\big(-(\nu+\sigma)\sqrt{T}\big) \\
&= \exp\{-rT\}N(d-\sigma\sqrt{T}\,),
\end{aligned}
$$

where

$$
d = \frac{(r+\frac{1}{2}\sigma^2)\sqrt{T}}{\sigma}. \tag{2.3.79}
$$

Regarding the second integral, we introduce

$$
I_2 = \int_{-\infty}^{0} \exp\{\sigma y\}2\nu\exp\{2\nu y\}N\left(\frac{y+\nu T}{\sqrt{T}}\right)dy.
$$

Using integration by parts, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{0} \exp\{\sigma y\}2\nu\exp\{2\nu y\}N\left(\frac{y+\nu T}{\sqrt{T}}\right)dy \\
&= 2\nu\left(\frac{N(\nu\sqrt{T})}{2\nu+\sigma} - \int_{-\infty}^{0}\frac{\exp\{y(2\nu+\sigma)\}}{(2\nu+\sigma)}\frac{\exp\{-\frac{1}{2}\frac{(y+\nu T)^2}{T}\}}{\sqrt{2\pi T}}\,dy\right) \\
&= \frac{2\nu}{2\nu+\sigma}\big(N(\nu\sqrt{T}\,) - \exp\{-rT\}N\big(-(\nu+\sigma)\sqrt{T}\big)\big) \\
&= N(-d) + \frac{\sigma^2}{2r}N(-d) - \exp\{-rT\}N(d-\sigma\sqrt{T}\,) \\
&\quad - \frac{\exp\{-rT\}\sigma^2}{2r}N(d-\sigma\sqrt{T}\,),
\end{aligned}
$$

where the quantity d is defined in (2.3.79). Finally, regarding the third integral, one has

$$\begin{aligned}
I_3 &= \int_{-\infty}^{0} \exp\{\sigma y\}\exp\{2\nu y\}\frac{1}{\sqrt{T}}\phi\left(\frac{y+\nu T}{\sqrt{T}}\right)dy \\
&= \int_{-\infty}^{0} \exp\big\{y(2\nu+\sigma)\big\}\frac{\exp\{-\frac{1}{2T}(y+\nu T)^2\}}{\sqrt{2\pi T}}\,dy \\
&= \int_{-\infty}^{0} \frac{\exp\{-\frac{(y-(\nu+\sigma)T)^2}{2T}\}}{\sqrt{2\pi T}}\exp\left\{\frac{T}{2}\big(2\nu\sigma+\sigma^2\big)\right\}dy \\
&= \exp\{-rT\}N(d-\sigma\sqrt{T}\,).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&E\big(\exp\{\sigma m_T^X\}\big) \\
&= \exp\{-rT\}N(d-\sigma\sqrt{T}\,)+N(-d)+\frac{\sigma^2}{2r}N(-d)-\exp\{-rT\}N(d-\sigma\sqrt{T}\,) \\
&\quad -\exp\{-rT\}\frac{\sigma^2}{2r}N(d-\sigma\sqrt{T}\,)+\exp\{-rT\}N(d-\sigma\sqrt{T}\,) \\
&= N(-d)+\frac{\sigma^2}{2r}N(-d)-\exp\{-rT\}\frac{\sigma^2}{2r}N(d-\sigma\sqrt{T}\,) \\
&\quad +\exp\{-rT\}N(d-\sigma\sqrt{T}\,) \\
&= 1-N(d)+\frac{\sigma^2}{2r}N(-d)-\exp\{-rT\}\frac{\sigma^2}{2r}N(d-\sigma\sqrt{T}\,) \\
&\quad +\exp\{-rT\}N(d-\sigma\sqrt{T}\,).
\end{aligned}$$

The time 0 price of a lookback call option is then given by

$$\begin{aligned}
LC(0) = S_0^{\delta_*}\bigg(&N(d)-\frac{\sigma^2}{2r}N(-d)-\exp\{-rT\}N(d-\sigma\sqrt{T}\,) \\
&+\exp\{-rT\}\frac{\sigma^2}{2r}N(d-\sigma\sqrt{T}\,)\bigg).
\end{aligned}$$

We now recall for a general $t<T$ the following result from Musiela and Rutkowski (2005), see their Proposition 6.7.1.

Proposition 2.3.1 *Assume that $r>0$. Then the price at time $t<T$ of a European lookback call option equals*

$$\begin{aligned}
LC(t) = S_t^{\delta_*}N\left(\frac{\ln(S_t^{\delta_*}/m_t^{S^{\delta_*}})+r_1(T-t)}{\sigma\sqrt{T-t}}\right) \\
-m_t^{S^{\delta_*}}N\left(\frac{\ln\big(\frac{S_t^{\delta_*}}{m_t^{S^{\delta_*}}}\big)+r_2(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

$$- \frac{S_t^{\delta_*}\sigma^2}{2r} N\left(\frac{\ln\left(\frac{m_t^{S^{\delta_*}}}{S_t^{\delta_*}}\right) - r_1(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$+ \exp\{-r(T-t)\} \frac{S_t^{\delta_*}\sigma^2}{2r} \left(\frac{m_t^{S^{\delta_*}}}{S_t^{\delta_*}}\right)^{2r\sigma^{-2}} N\left(\frac{\ln\left(\frac{m_t^{S^{\delta_*}}}{S_t^{\delta_*}}\right) + r_2(T-t)}{\sigma\sqrt{T}}\right),$$

where $r_{1,2} = r \pm \frac{1}{2}\sigma^2$.

The payoff of a lookback put option is given by

$$\left(M_T^{S^{\delta_*}} - S_T^{\delta_*}\right)^+ = M_T^{S^{\delta_*}} - S_T^{\delta_*},$$

where $M_T^{\delta_*} = \max_{t\in[0,T]} S_t^{\delta_*}$.

Proposition 2.3.2 *Assume that* $r > 0$. *The price of a European lookback put option at time* $t < T$ *equals*

$$LP(t) = -S_t^{\delta_*} N\left(-\frac{\ln\left(\frac{S_t^{\delta_*}}{M_t^{S^{\delta_*}}}\right) + r_1(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$+ M_t^{S^{\delta_*}} \exp\{-r(T-t)\} N\left(-\frac{\ln\left(\frac{S_t^{\delta_*}}{M_t^{S^{\delta_*}}}\right) + r_2(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$+ \frac{S_t^{\delta_*}\sigma^2}{2r} N\left(\frac{\ln\left(\frac{S_t^{\delta_*}}{M_t^{S^{\delta_*}}}\right) + r_1(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$- \exp\{-r(T-t)\} \frac{S_t^{\delta_*}\sigma^2}{2r} \left(\frac{M_t^{S^{\delta_*}}}{S_t^{\delta_*}}\right)^{2r\sigma^{-2}} N\left(\frac{\ln\left(\frac{S_t^{\delta_*}}{M_t^{S^{\delta_*}}}\right) - r_2(T-t)}{\sigma\sqrt{T-t}}\right),$$

where again $r_{1,2} = r \pm \frac{1}{2}\sigma^2$.

2.3.10 Asian Options

In this subsection, we consider *Asian options* on a market index, the GOP. Unlike the derivatives presented in the previous subsections, the pay-off of Asian options is based on average values of the market index. In particular, the pay-off of an Asian call option is given by

$$\left(\frac{1}{T}\int_0^T S_u^{\delta_*}\, du - K\right)^+$$

and

$$\left(K - \frac{1}{T}\int_0^T S_u^{\delta_*}\, du\right)^+.$$

We point out that closed-form solutions, as presented in the preceding subsections, are not available for Asian options. However, using the explicitly derived joint density of $(\int_0^T S_u^{\delta_*}\,du, S_T^{\delta_*})$ from Sect. 2.1, we can obtain an integral representation for the price. In particular, using the notation

$$\begin{aligned} P\bigg(\int_0^t \exp\big\{\sigma \hat{W}_s^0 - p\sigma^2 s/2\big\}\,ds \in du, \hat{W}_t^0 \in dy\bigg) &= \frac{\sigma}{2}\exp\big\{-p\sigma y/2 - p^2\sigma^2 t/8\big\}\exp\bigg\{-2\frac{1+\exp\{\sigma y\}}{\sigma^2 u}\bigg\} \\ &\quad \times \theta\bigg(\frac{4\exp\{\sigma y/2\}}{\sigma^2 u}, \frac{\sigma^2 t}{4}\bigg)\frac{du}{u}\,dy \\ &= f(y,u)\,dy\,du, \end{aligned}$$

where $p = -(1+\frac{2r}{|\sigma^0|^2})$ and $\sigma := |\sigma^0|$, we obtain from the real-world pricing formula (1.3.19) the following representation for the price of a call option at time 0, struck at K with maturity T,

$$\begin{aligned} C_{T,K}^A(0) &= S_0^{\delta_*} E\Bigg(\frac{\big(\frac{\int_0^T S_u^{\delta_*}\,du}{T} - K\big)^+}{S_T^{\delta_*}}\Bigg) \\ &= \frac{S_0^{\delta_*}}{T}\int_0^\infty\int_0^\infty \frac{(u - \frac{TK}{S_0^{\delta_*}})^+}{\exp\{-p\sigma^2 T/2 + \sigma y\}} f(y,u)\,dy\,du. \end{aligned} \qquad (2.3.80)$$

The above expression needs to be computed numerically, using e.g. the techniques to be presented in Chap. 12. Finally, we alert the reader to a quasi-analytical result shown in Geman and Yor (1993). They computed the Laplace transform with respect to time to maturity. We point out that the proof uses a connection between geometric Brownian motion and time-changed Bessel processes, also referred to as Lampert's Theorem, see Theorem 6.2.4.1 in Jeanblanc et al. (2009). The following result appeared as Proposition 6.8.1 in Musiela and Rutkowski (2005) and is based on Eq. (3.10) in Geman and Yor (1993).

Proposition 2.3.3 *The price of an Asian call option admits the representation*

$$C_{T,K}^A(t) = \frac{4\exp\{-r(T-t)\}S_t^{\delta_*}}{\sigma^2 T} C^w(h,q)$$

where

$$w = \frac{2r}{\sigma^2} - 1, \qquad h = \frac{\sigma^2}{4}(T-t), \qquad q = \frac{\sigma^2}{4S_t^{\delta_*}}\bigg(KT - \int_0^t S_u^{\delta_*}\,du\bigg).$$

Moreover, the Laplace transform of $C^w(h,q)$ with respect to h is given by the formula

$$\int_0^\infty \exp\{-\lambda h\}C^w(h,q)\,dh = \int_0^{\frac{1}{2q}} \big(d\exp\{-x\}x^{\gamma-2}(1-2qx)^{\gamma+1}\big)\,dx,$$

where $\mu = \sqrt{2\lambda + w^2}$, $\gamma = \frac{1}{2}(\mu - w)$, and $d = (\lambda(\lambda - 2 - 2\nu)\Gamma(\gamma - 1))^{-1}$.

We remark that the techniques to be presented in Sect. 13.5 can be used to invert the above Laplace transform.

2.3.11 Australian Options

Australian options are closely related to Asian options. In this case the pay-off depends on the quotient of the average of the market index over a specific time interval and the market index at maturity, i.e. the quotient $\frac{\int_0^T S_u^{\delta_*}\,du}{S_T^{\delta_*}}$, see Handley (2000), Handley (2003), Moreno and Navas (2008), and Ewald et al. (2011). In the BSM framework, a connection between Australian and Asian options is known to exist, see Ewald et al. (2011). The real-world pricing formula (1.3.19) yields the following expression for an Australian call option on the market index:

$$C_{T,K}^{AU}(t) = S_t^{\delta_*} E\left(\left(\frac{\int_0^T S_u^{\delta_*}\,du}{T S_T^{\delta_*}} - K\right)^+ \frac{1}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right).$$

We now follow Ewald et al. (2011),

$$C_{T,K}^{AU}(t) = S_t^{\delta_*} E\left(\frac{(\frac{\int_0^T S_u^{\delta_*}\,du}{T} - K S_T^{\delta_*})^+}{(S_T^{\delta_*})^2} \,\middle|\, \mathcal{A}_t\right).$$

Next we introduce the same auxiliary measure as for the bond-or-nothing binaries, i.e. we recall the Radon-Nikodym derivative process from (2.3.50),

$$\Lambda_{|\sigma^0|}(t) = \frac{\hat{S}_t^0}{\hat{S}_0^0} = \exp\left\{-\frac{t}{2}\left|\sigma^0\right|^2 - \left|\sigma^0\right|\hat{W}_t^0\right\},$$

and define the measure $P_{|\sigma^0|}$ via

$$\frac{dP_{|\sigma^0|}}{dP} = \Lambda_{|\sigma^0|}(T),$$

by setting

$$P_{|\sigma^0|}(A) = E\left(\Lambda_{|\sigma^0|}(T)\mathbf{1}_A\right) = E_{|\sigma^0|}(\mathbf{1}_A)$$

for $A \in \mathcal{A}_T$. We use $E_{|\sigma^0|}$ to denote the expectation with respect to $P_{|\sigma^0|}$. By Girsanov's theorem, $W^{|\sigma^0|} = \{W_t^{|\sigma^0|},\ t \in [0, T]\}$, where

$$W_t^{|\sigma^0|} = \hat{W}_t^0 + \left|\sigma^0\right|t$$

is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P_{|\sigma^0|})$. Hence

$$C_{T,K}^{AU}(t) = \exp\{-r(T-t)\} E_{|\sigma^0|}\left(\frac{\left(\frac{\int_0^T S_u^{\delta_*}\,du}{T} - K S_T^{\delta_*}\right)^+}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right). \qquad (2.3.81)$$

We remark that under $P_{|\sigma^0|}$, the dynamics of S^{δ_*} are given by

$$S_t^{\delta_*} = S_0^{\delta_*} \exp\left\{ \left(r - \left|\sigma^0\right|^2\right)t + \frac{t}{2}\left|\sigma^0\right|^2 + \left|\sigma^0\right| W_t^{|\sigma^0|} \right\}.$$

Hence, comparing (2.3.81) to (2.3.80), we point out that computing (2.3.81) amounts to pricing an Asian option with variable strike, but at a different interest rate, namely $r - |\sigma^0|^2$. Clearly, this relation required the candidate measure $P_{|\sigma^0|}$ to be equivalent to P. This does not hold for all models considered in this book, see e.g. Chap. 3. However, assuming suitable integrability conditions are satisfied, we can express the price of an Australian option as an integral over the relevant probability density function and use the techniques from Chap. 12.

2.3.12 Exchange Options

In this subsection, we price *exchange options* on the market index, i.e. the option to exchange the market index denominated in one currency for the market index denominated in another currency. This is our first example of a derivative whose payoff is a functional of two assets. We point out that in the classical literature, see e.g. Margrabe (1978), such contracts are often priced by computing prices under an appropriately chosen probability measure. This is not the case under the benchmark approach, where we only need to compute prices under one measure, the real world probability measure. In particular, we define the time t exchange price as

$$X_t^{i,j} = \frac{S_t^{\delta_*,i}}{S_t^{\delta_*,j}},$$

where $S_t^{\delta_*,i}$ denotes the GOP denominated in currency i, and $S_t^{\delta_*,j}$ denotes the GOP denominated in currency j. We assume that the dynamics of the GOP in currency k are given by

$$dS_t^{\delta_*,k} = S_t^{\delta_*,k}\left(\left(r^k + \left|\sigma^k\right|^2\right)dt + \left|\sigma^k\right| d\hat{W}_t^k\right), \tag{2.3.82}$$

where $k \in \{i, j\}$ and $d[\hat{W}^i, \hat{W}^j]_t = \rho\, dt$. The joint transition density of $S^{\delta_*,i}$ and $S^{\delta_*,j}$ was derived in Sect. 2.2. For deriving the following result, we employ a change of variables. This reduces the computation to one which involves the standard Gaussian bivariate density. As with European call options on the GOP, we find it convenient to firstly price asset binary options on an exchange price, and subsequently bond binary options on an exchange price. We use the notation $A_{T,K}^i(t)$ for an asset binary option on an exchange price in the ith currency, which, based on the real world pricing formula, satisfies

$$A_{T,K}^i(t) = S_t^{\delta_*,i} E\left(\frac{S_T^{\delta_*,i}}{S_T^{\delta_*,j}} \frac{1}{S_T^{\delta_*,i}} \mathbf{1}_{\frac{S_T^{\delta_*,i}}{S_T^{\delta_*,j}} \geq K} \,\middle|\, \mathcal{A}_t \right).$$

Using $q(t, z_t^j, z_t^i, T, z_T^i, z_T^j)$ to denote the joint density of $(S_t^{\delta_*,j}, S_t^{\delta_*,i})$ to $(S_T^{\delta_*,j}, S_T^{\delta_*,i})$, we have

$$\begin{aligned} A^i_{T,K}(t) &= E\left(\frac{S_t^{\delta_*,i}}{S_T^{\delta_*,i}} X_T^{i,j} \mathbf{1}_{X_T^{i,j} \geq K} \,\middle|\, \mathcal{A}_t\right) \\ &= X_t^{i,j} E\left(\frac{S_t^{\delta_*,j}}{S_T^{\delta_*,j}} \mathbf{1}_{S_T^{\delta_*,j} \leq S_T^{\delta_*,i}/K} \,\middle|\, \mathcal{A}_t\right) \\ &= X_t^{i,j} \int_0^\infty \int_0^{\frac{z_T^i}{K}} \frac{z_t^i}{z_T^i} q\big(t, z_t^j, z_t^i, T, z_T^j, z_T^i\big)\, dz_T^j\, dz_T^i. \end{aligned}$$

We now use a change of variables to perform computations in terms of the bivariate Gaussian density

$$u_T^k = \frac{\ln(\frac{z_T^k}{z_t^k}) - (r_k + \frac{1}{2}|\sigma^k|^2)(T-t)}{|\sigma^k|\sqrt{T-t}}, \quad k \in \{i, j\}.$$

Hence we obtain

$$A^i_{T,K}(t) = X_t^{i,j} \int_{-\infty}^\infty \int_{-\infty}^{\bar{d}_1(X_t^{i,j})} \frac{z_t^j}{z_T^j} p\big(u_T^j, u_T^i, \rho\big)\, du_T^j\, du_T^i,$$

where

$$\bar{d}_1(x) = \frac{\ln(\frac{x}{K}) - (r_j - r_i + \frac{1}{2}(|\sigma^j|^2 - |\sigma^i|^2))(T-t)}{|\sigma^j|\sqrt{T-t}} + u_T^i \frac{|\sigma^i|}{|\sigma^j|},$$

and

$$p(z_1, z_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{(z_1^2 - 2\rho z_1 z_2 + z_2^2)}{2(1-\rho^2)}\right\}$$

denotes the density of two correlated standard Gaussian random variables. We now set

$$\begin{aligned} \bar{u}_T^j &= u_T^j + \big|\sigma^j\big|\sqrt{T-t}, \\ \bar{u}_T^i &= u_T^i + \rho\big|\sigma^j\big|\sqrt{T-t}, \end{aligned}$$

which allows us to write

$$A^i_{T,K}(t) = X_t^{i,j} \exp\big\{-r_j(T-t)\big\} \int_{-\infty}^\infty \int_{-\infty}^{\tilde{d}_1(X_t^{i,j})} p\big(\bar{u}_T^j, \bar{u}_T^i, \rho\big)\, d\bar{u}_T^j\, d\bar{u}_T^i, \quad (2.3.83)$$

where

$$\begin{aligned} \tilde{d}_1(x) &= \frac{\ln(\frac{x}{K}) + (r_i - r_j + \frac{1}{2}(|\sigma^j|^2 - 2\rho|\sigma^j||\sigma^i| + |\sigma^i|^2))(T-t)}{|\sigma^j|\sqrt{T-t}} + \bar{u}_T^i \frac{|\sigma^i|}{|\sigma^j|} \\ &= \hat{d}\big(X_t^{i,j}\big) + \bar{u}_T^i \frac{|\sigma^i|}{|\sigma^j|}. \end{aligned}$$

The expression in equation (2.3.83) can be interpreted as the probability that a standard normal random variable Z_1 is less than a constant $\hat{d}_1(X_t^{i,j})$ plus another standard normal random variable Z_2 multiplied by $\frac{|\sigma^i|}{|\sigma^j|}$, i.e.

$$P\left(Z^1 < \hat{d}_1\left(X_t^{i,j}\right) + \frac{|\sigma^i|}{|\sigma^j|} Z^2\right).$$

But since $Z^1 - \frac{|\sigma^i|}{|\sigma^j|} Z^2$ is normal with mean zero and variance $\frac{\sigma_{i,j}^2}{|\sigma^j|^2}$, we obtain the following result:

$$A_{T,K}^i(t) = X_t^{i,j} \exp\left\{-r_j(T-t)\right\} N\left(d_1\left(X_t^{i,j}\right)\right),$$

where

$$d_1\left(X_t^{i,j}\right) = \frac{\ln(\frac{X_t^{i,j}}{K}) + (r_i - r_j + \frac{1}{2}\sigma_{i,j}^2)(T-t)}{\sigma_{i,j}\sqrt{T-t}},$$

$$\sigma_{i,j}^2 = \left|\sigma^i\right|^2 - 2\rho\left|\sigma^i\right|\left|\sigma^j\right| + \left|\sigma^j\right|^2.$$

Using similar calculations, we obtain the following result for a binary bond option on the exchange price,

$$B_{T,K}^i(t) = E\left(\frac{S_t^{\delta_*,i}}{S_T^{\delta_*,i}} \mathbf{1}_{X_T^{i,j} > K} \,\bigg|\, \mathcal{A}_t\right) = \exp\left\{-r_i(T-t)\right\} N\left(d_2\left(X_t^{i,j}\right)\right),$$

where

$$d_2\left(X_t^{i,j}\right) = \frac{\ln(\frac{X_t^{i,j}}{K}) + (r_i - r_j - \frac{1}{2}\sigma_{i,j}^2)(T-t)}{\sigma_{i,j}\sqrt{T-t}}.$$

Finally, we arrive at prices for call and put options in the ith currency on an exchange price at time t with expiry T and strike price K,

$$\begin{aligned} c_{T,K}^i(t) &= X_t^{i,j} \exp\left\{-r_j(T-t)\right\} N\left(d_1\left(X_t^{i,j}\right)\right) \\ &\quad - K \exp\left\{-r_i(T-t)\right\} N\left(d_2\left(X_t^{i,j}\right)\right), \\ p_{T,K}^i(t) &= -X_t^{i,j} \exp\left\{-r_j(T-t)\right\} N\left(-d_1\left(X_t^{i,j}\right)\right) \\ &\quad + K \exp\left\{-r_i(T-t)\right\} N\left(-d_2\left(X_t^{i,j}\right)\right). \end{aligned}$$

2.3.13 American Options

The derivative contracts discussed until now were all *European style options*, i.e. could only be exercised at maturity. We now briefly discuss *American style options*, which allow the holder to exercise the option at any time before maturity. This additional feature makes the pricing of American options more difficult than the pricing of European options. For more information on the mathematics of American option pricing, we refer the reader to McKean (1965), van Moerbeke (1976), Bensoussan (1984), Karatzas (1988), Karatzas (1989), and to Myneni (1992) for a survey. We point out that closed-form solutions similar to the ones derived in the preceding subsections are not available for American options, except for the perpetual case, see for example the discussion in Musiela and Rutkowski (2005). However, we also refer the reader to Zhu (2006).

Consequently, for results that provide almost closed-form solutions numerical methods have to be employed to price American options. A popular method involves restricting the dates at which the option can be exercised to a finite set, i.e. turning the American option into a *Bermudan option*. Using dynamic programming, one can compute prices via backward induction. This in turn can be done via Monte Carlo simulation, see e.g. Broadie and Glasserman (1997). In this context, the transition densities collected in this book are of importance, as they are used to perform the simulation step. Furthermore, the Monte Carlo technique is of course general, one only needs to have access to the relevant transition densities.

There exist more explicit formulas for derivatives under the BSM. It is mainly its explicitly known transition density and the well researched area of functionals of Brownian motions that give access to such a rich set of pricing formulas for the standard market model. It is unfortunate that the BSM provides only a poor reflection of the real market dynamics, in particular, over longer periods of time and for extreme market movements. Therefore, it is essential to find more realistic tractable market models with a similar set of explicit formulas.