Chapter 17
Detecting Strict Local Martingales

In Sect. 3.3, we presented two models for the GOP, namely the MMM and the TCEV model. For both models, we established the property that a risk-neutral measure does not exist, because the Radon-Nikodym derivative of the putative risk-neutral measure is a strict local martingale. Figure 3.3.6 seems to suggest that this is a plausible feature of our financial market, in particular, when analyzing its history over long periods of time and taking into account that investors request extra long term growth in risky securities. We established the local martingale property by making use of the explicitly available transition density of squared Bessel processes. This highlights the usefulness of squared Bessel processes in finance, they produce both tractability but also realistic models.

In this chapter, we propose another class of processes for modeling the GOP. For this class one can easily establish whether the processes allow for a risk-neutral measure or not. An argument due to Sin, see Sin (1998), allows us to determine if a particular model for the GOP admits a risk-neutral measure by studying the boundary behavior of a one-dimensional diffusion. The boundary behavior of one-dimensional diffusions is well understood, we refer the reader e.g. to our Chap. 16. As demonstrated in Chap. 16, in particular in Table 16.2.1, it simply amounts to confirming if certain integrals explode or not. Hence for the class of processes studied in this chapter, we present simple tools that allow us to answer the crucial question whether a particular local martingale is a martingale or a strict local martingale.

We remark that the question whether a local martingale is a martingale or a strict local martingale, has received much attention in the literature. In Kotani (2006), Hulley and Platen (2011), necessary and sufficient conditions have been presented for one-dimensional regular strong Markov continuous local martingales. Furthermore, in Kallsen and Muhle-Karbe (2010), Kallsen and Shiryaev (2002), and Mayerhofer et al. (2011a) an exponential semimartingale framework with focus on affine processes is considered. For further background, the reader is referred to Mijatovic and Urusov (2012). At the end of Sect. 17.1, we will present one of the main results from Mijatovic and Urusov (2012).

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17.1 Sin's Argument

In this section, we work on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ carrying two Brownian motions W^1 and W^2. We use the following dynamics for the GOP,

$$dS_t^{\delta_*} = S_t^{\delta_*}\big((r_t + V_t)\,dt + \rho\sqrt{V_t}\,dW_t^1 + \rho^{\perp}\sqrt{V_t}\,dW_t^2\big),$$

and

$$dV_t = \kappa(\theta - V_t)\,dt + \sigma V_t^p\,dW_t^1, \tag{17.1.1}$$

where $S_0^{\delta_*} > 0$ and $V_0 > 0$. Here $r = \{r_t,\, t \geq 0\}$ is an adapted short-rate process, $\rho \in [-1, 1]$ denotes the correlation between the GOP and the variance process and $\rho^{\perp} = \sqrt{1-\rho^2}$. The parameters κ, θ, σ, and p are positive. We remark that this model is based on Andersen and Piterbarg (2007). For $p = \frac{1}{2}$, we obtain the Heston model, see also Sect. 6.3, for $b = 1$ we recover a GARCH model, and for $b = \frac{3}{2}$ we recover a 3/2-model with linear drift. Next we define the savings account $B_t = \exp\{\int_0^t r_s\,ds\}$, for $t \geq 0$, and the benchmarked savings account $\hat{B}_t = \frac{B_t}{S_t^{\delta_*}}$, $t \geq 0$, which satisfies the SDE

$$d\hat{B}_t = -\hat{B}_t\big(\rho\sqrt{V_t}\,dW_t^1 + \rho^{\perp}\sqrt{V_t}\,dW_t^2\big), \tag{17.1.2}$$

for $t \geq 0$. We recall the following properties of the process V from Andersen and Piterbarg (2007), see Proposition 2.1.

Proposition 17.1.1 *For the process V given by Eq.* (17.1.1), *the following properties hold*:

- 0 *is always an attainable boundary for* $0 < p < \frac{1}{2}$;
- 0 *is an attainable boundary for* $p = \frac{1}{2}$, *if* $2\kappa\theta < \sigma^2$;
- 0 *is an unattainable boundary for* $p > \frac{1}{2}$;
- ∞ *is an unattainable boundary for* $p > 0$.

Proof The proof is easily completed using Table 16.2.1 in Chap. 16. Recall that the speed measure is given by

$$m(dx) = m(x)\,dx$$

and the scale function by

$$s(x) = \int_c^x s'(y)\,dy,$$

where $c \in [0, \infty)$. Now,

$$m(x) = \frac{2}{a^2(x)s'(x)} \quad \text{and} \quad s'(x) = \exp\left(-\int_c^x \frac{2b(y)}{a^2(y)}\,dy\right),$$

for $x \in [0, \infty)$, where

$$a(x) = \sigma x^p \quad \text{and} \quad b(x) = \kappa(\theta - x).$$

□

For the remainder of this section, we assume that $p \geq \frac{1}{2}$ and $2\kappa\theta \geq \sigma^2$, so that V cannot reach 0. From Eq. (17.1.2), it is clear that $\hat{B}$ is a local martingale. As discussed in Sect. 3.3, if $\hat{B}$ is a martingale, a risk-neutral probability measure exists. If $\hat{B}$ is a strict local martingale, a risk-neutral measure does not exist. The following proposition identifies when $\hat{B}$ is a martingale. The proof is based on Lemma 2.3 in Andersen and Piterbarg (2007), which uses techniques from the proof of Lemma 4.2 in Sin (1998).

Lemma 17.1.2 *Let $\hat{B}$ and V be given by Eqs.* (17.1.1) *and* (17.1.2). *Denote by $\tilde{\tau}_\infty$ the explosion time for $\tilde{V}$,*

$$\tilde{\tau}_\infty = \lim_{n\to\infty} \tilde{\tau}_n, \qquad \tilde{\tau}_n = \inf\{t\colon\ \tilde{V}_t \geq n\}, \tag{17.1.3}$$

P-almost surely. Here the dynamics of $\tilde{V}$ under P are given by

$$d\tilde{V}_t = \bigl(\kappa(\theta - V_t) - \rho\sigma \tilde{V}_t^{p+\frac{1}{2}}\bigr)\,dt + \sigma \tilde{V}_t^p\, dW_t. \tag{17.1.4}$$

Then

$$E(\hat{B}_T) = \hat{B}_0 P(\tilde{\tau}_\infty > T).$$

Furthermore, when $p = \frac{1}{2}$ or $p > \frac{3}{2}$, then $\hat{B}$ is a martingale. When $\frac{1}{2} < p < \frac{3}{2}$, $\hat{B}$ is a martingale for $\rho \geq 0$ and a strict local martingale for $\rho < 0$. For $p = \frac{3}{2}$, $\hat{B}$ is a martingale for $\rho \geq -\frac{\sigma}{2}$ and a strict local martingale for $\rho < -\frac{\sigma}{2}$.

Proof We follow the technique of the proof of Lemma 2.3 in Andersen and Piterbarg (2007) and compute

$$\begin{aligned}
E(\hat{B}_T) &= \hat{B}_0 E\biggl(\exp\biggl\{-\frac{1}{2}\int_0^T V_s\,ds - \rho\int_0^T \sqrt{V_s}\,dW_s^1 - \rho^\perp \int_0^T \sqrt{V_s}\,dW_s^2\biggr\}\biggr) \\
&= \hat{B}_0 E\biggl(\exp\biggl\{-\frac{1}{2}\int_0^T V_s\,ds - \rho\int_0^T \sqrt{V_s}\,dW_s^1\biggr\} \\
&\quad \times E\biggl(\exp\biggl\{-\rho^\perp \int_0^T \sqrt{V_s}\,dW_s^2\biggr\}\,\bigg|\,\sigma\bigl\{W_t^1, t \leq T\bigr\}\biggr)\biggr) \\
&= \hat{B}_0 E\biggl(\exp\biggl\{-\rho\int_0^T \sqrt{V_s}\,dW_s^1 - \frac{\rho^2}{2}\int_0^T V_s\,ds\biggr\}\biggr).
\end{aligned}$$

Next, introduce a sequence of stopping times

$$\tau_n := \inf\biggl\{t\colon \int_0^t V_s\,ds \geq n\biggr\}$$

and define the stochastic Doléan exponential

$$\xi_t = \exp\biggl\{-\rho\int_0^t \sqrt{V_s}\,dW_s^1 - \frac{\rho^2}{2}\int_0^t V_s\,ds\biggr\}.$$

Clearly, $\xi = \{\xi_t,\ t \geq 0\}$ is a local martingale. We now define

$$\xi_t^{(n)} = \xi_{t\wedge\tau_n},$$

which is a martingale and use it to define the auxiliary measure,

$$\tilde{P}^n(A) = E\big(\mathbf{1}_A \xi_T^{(n)}\big),$$

for $A \in \mathcal{A}_T$. We now use the argument in Lemma 4.2 in Sin (1998) and compute

$$\begin{aligned} E(\xi_T \mathbf{1}_{\tau > T}) &= E\big(\xi_T^{(n)} \mathbf{1}_{\tau_n > T}\big) \\ &= E_{\tilde{P}^n}(\mathbf{1}_{\tau_n > T}) \\ &= E_P(\mathbf{1}_{\tilde{\tau}_n > T}), \end{aligned}$$

where we recall that $\tilde{\tau}_n$ is defined in (17.1.3) and $\tilde{V}$ in (17.1.4). To justify the last equality, we note that by the Girsanov theorem, see Sect. 15.8, we obtain that the process

$$W_t^{(n)} = W_t + \rho \int_0^t \mathbf{1}_{u \le \tau_n} \sqrt{V_u}\, du$$

is a Brownian motion under $\tilde{P}^n$ and W and V satisfy

$$\begin{aligned} dW_t &= dW_t^{(n)} - \rho \mathbf{1}_{t \le \tau_n} \sqrt{V_t}\, dt, \\ dV_t &= \sigma V_t^p\, dW_t^{(n)} + \big(\kappa(\theta - V_t) - \rho \mathbf{1}_{t \le \tau_n} \sigma V_t^{p+\frac{1}{2}}\big)\, dt. \end{aligned}$$

Hence the stopped process $\tilde{V}_{t \wedge \tilde{\tau}_n}$ has the same law under P as the stopped process $V_{t \wedge \tau_n}$ under $\tilde{P}^n$. Now by Proposition 17.1.1, we have that V does not reach ∞ under P, hence

$$\begin{aligned} E(\xi_T) &= \lim_{n \to \infty} E(\xi_T \mathbf{1}_{\tau_n > T}) \\ &= \lim_{n \to \infty} E(\mathbf{1}_{\tilde{\tau}_n > T}) \\ &= P(\tilde{\tau}_\infty > T). \end{aligned}$$

The exchange of the limit and the expectation operator is justified by the monotone convergence theorem. This completes the proof of the first part of the result.

The second part follows immediately from Proposition 2.5 in Andersen and Piterbarg (2007). We replace the ρ in the statement of Proposition 2.5 in Andersen and Piterbarg (2007) by $-\rho$, as we consider the benchmarked savings account, which is essentially the inverse of the process considered in Andersen and Piterbarg (2007), and we compare Eq. (17.1.4) and Eq. (2.5) in Andersen and Piterbarg (2007). □

We have the following corollary to Lemma 17.1.2.

Corollary 17.1.3 *For important special cases, we obtain the following result*:

- *under the Heston model, which corresponds to* $p = \frac{1}{2}$, *the process* $\hat{B} = \{\hat{B}_t,\, t \ge 0\}$ *follows a martingale*;
- *under the* $\frac{3}{2}$ *model with linear drift, which corresponds to the case* $p = \frac{3}{2}$, $\hat{B} = \{\hat{B}_t,\, t \ge 0\}$ *follows a martingale for* $\rho \ge 0$ *and otherwise a strict local martingale*;

- *under a continuous limit of a GARCH model, which corresponds to* $p = 1$, $\hat{B} = \{\hat{B}_t,\ t \geq 0\}$ *follows a martingale for* $\rho \geq 0$, *otherwise a strict local martingale.*

For completeness, we present a general version of the result, which stems from Mijatovic and Urusov (2012). Consider the state space $J = (l, r)$, $-\infty \leq l < r \leq \infty$ and a J-valued diffusion $Y = \{Y_t,\ t \geq 0\}$ on $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ given by the SDE

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t, \tag{17.1.5}$$

where $Y_0 \in J$, W is a $\underline{\mathcal{A}}$-Brownian motion and $\mu, \sigma : J \to \Re$ are Borel functions satisfying the Engelbert-Schmidt conditions,

$$\sigma(x) \neq 0\ \forall x \in J; \quad \frac{1}{\sigma^2}, \frac{\mu}{\sigma^2} \in L^1_{loc}(J), \tag{17.1.6}$$

where $L^1_{loc}(J)$ denotes the class of locally integrable functions, i.e. mappings from J to $\Re$ that are integrable on compact subsets of J. The SDE (17.1.5) admits a unique in law weak solution that possibly exits its state space J at the exit time ζ. Following Mijatovic and Urusov (2012), we specify that if Y can exit its state space, i.e. $P(\zeta < \infty) > 0$, then Y stays at the boundary point of J at which it exits after the time ζ, so the boundary is absorbing. We introduce the stochastic exponential

$$\xi_t = \exp\left\{\int_0^{t\wedge\zeta} b(Y_u)\,dW_u - \frac{1}{2}\int_0^{t\wedge\zeta} b^2(Y_u)\,du\right\},$$

$t \geq 0$, and set $\xi_t := 0$ for $t \geq \zeta$ on the set $\{\zeta < \infty, \int_0^\zeta b^2(Y_u)\,du = \infty\}$. Also we assume that

$$\frac{b^2}{\sigma^2} \in L^1_{loc}(J). \tag{17.1.7}$$

Remark 17.1.4 In order to connect this discussion to the results in Proposition 17.1.1 and Lemma 17.1.2, set $Y = V$ and $b(x) = -\rho\sqrt{x}$.

We now consider an auxiliary J-valued diffusion $\tilde{Y}$, where

$$d\tilde{Y}_t = \mu(\tilde{Y}_t)\,dt + b(\tilde{Y}_t)\sigma(\tilde{Y}_t)\,dt + \sigma(\tilde{Y}_t)\,dW_t.$$

Then $\tilde{Y}$ admits a unique weak solution that possibly exits its state space at $\tilde{\zeta}$. Before we present Corollary 2.2 from Mijatovic and Urusov (2012), recall from Proposition 17.1.1 that V cannot reach infinity under P, and Lemma 17.1.2 states that $\hat{B}$ is a martingale if $\tilde{V}$ cannot reach infinity under P, i.e. $\hat{B}$ is martingale if the boundary behavior of V and $\tilde{V}$ coincides. Corollary 2.2 from Mijatovic and Urusov (2012) extends this to the general case.

Corollary 17.1.5 *Assume that Y does not exit its state space and assume that μ, σ and b satisfy the assumptions* (17.1.6) *and* (17.1.7). *Then ξ is a martingale if and only if $\tilde{Y}$ does not exit its state space.*

For a more general result, see Theorem 2.1 in Mijatovic and Urusov (2012). Finally, we recall an important remark from Mijatovic and Urusov (2012).

Remark 17.1.6 The conditions in Lemma 17.1.2 (resp. Corollary 17.1.5) are necessary and sufficient conditions for $\hat{B}$ (resp. ξ) to be a martingale on the time interval $(0, \infty)$. Furthermore, they are necessary and sufficient conditions for $\hat{B}$ (resp. ξ) to be a martingale on the time interval $[0, T]$ for any fixed $T \in (0, \infty)$.

Remark 17.1.6 can intuitively be explained as follows: the process $\hat{B}$ (resp. ξ) is a time-homogeneous diffusion process, hence the process cannot loose its martingale property over time, but would have to lose it immediately.

17.2 Multidimensional Extension

This section briefly illustrates how to extend the results from the previous section to a multidimensional setting. Assume that the dynamics of the benchmarked savings account are given by

$$d\hat{B}_t = -\hat{B}_t \sum_{k=1}^{d} \sqrt{V_t^k}\big(\rho^k\, dW_t^k + \big(\rho^k\big)^{\perp}\, dW_t^{d+k}\big), \tag{17.2.8}$$

where $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, W_t^2, \dots, W_t^{2d}),\ t \geq 0\}$ is a vector Brownian motion and

$$dV_t^k = \kappa^k\big(\theta^k - V_t^k\big)\, dt + \sigma^k\big(V_t^k\big)^{p^k}\, dW_t^k. \tag{17.2.9}$$

Here $V_0^k, \kappa^k, \theta^k, \sigma^k, p^k$ are positive parameters, $-1 \leq \rho^k \leq 1$, and

$$\big(\rho^k\big)^{\perp} = \sqrt{1 - \big(\rho^k\big)^2}.$$

Proposition 17.2.1 *Let $\hat{B}$ and V^k, for $k = 1, \dots, d$, be given by Eqs.* (17.2.8) *and* (17.2.9). *Denote by $\tilde{\tau}_\infty^k$ the explosion time for $\tilde{V}^k$,*

$$\tilde{\tau}_\infty^k = \lim_{n\to\infty} \tilde{\tau}_n^k, \qquad \tilde{\tau}_n^k = \inf\big\{t\colon\ \tilde{V}_t^k \geq n\big\},$$

where the dynamics of $\tilde{V}^k$ are given by

$$d\tilde{V}_t^k = \big(\kappa^k\big(\theta^k - \tilde{V}_t^k\big) - \rho^k\sigma^k\big(\tilde{V}_t^k\big)^{p^k+\frac{1}{2}}\big)\, dt + \sigma^k\big(\tilde{V}_t^k\big)^{p^k}\, dW_t^k. \tag{17.2.10}$$

Then

$$E(\hat{B}_T) = \hat{B}_0 \prod_{k=1}^{d} P\big(\tilde{\tau}_\infty^k > T\big).$$

Furthermore, $\hat{B}$ is a martingale if and only if for all $k \in \{1, \dots, d\}$, one of the following conditions holds:

- $p^k = \frac{1}{2}$ *or* $p^k > \frac{3}{2}$;

- $\frac{1}{2} < p^k < \frac{3}{2}$ *and* $\rho^k \geq 0$;
- $p^k = \frac{3}{2}$ *and* $\rho^k \geq -\frac{\sigma_k}{2}$.

As in Sect. 17.1, the conditions presented in Proposition 17.2.1 are necessary and sufficient for $\hat{B}$ to be a martingale on the time interval $[0, T]$ for any fixed $T \in (0, \infty)$.